ABSTRACT

Title of dissertation: A PDE APPROACH TO NUMERICAL FRACTIONAL DIFFUSION

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Doctor of Philosophy, 2014

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This dissertation presents a decisive advance in the numerical solution and analysis of fractional diffusion, a relatively new but rapidly growing area of research. We exploit the cylindrical extension proposed and investigated by X. Cabré and J. Tan, in turn inspired by L. Caffarelli and L. Silvestre, to replace the intricate integral formulation of $(-\Delta)^s u = f$, $0 < s < 1$, in a bounded domain Ω, by the local elliptic PDE in one higher dimension y

$$\mathrm{div}(y^\alpha \nabla \mathscr{U}) = 0 \text{ in } \Omega \times (0,\infty) \qquad (\alpha = 1-2s) \qquad \mathscr{U} = 0 \text{ on } \partial\Omega \times (0,\infty),$$

with variable coefficient y^α; f enters as a natural boundary condition at $y = 0$. Inspired in the aforementioned localization results, we propose a simple strategy to study discretization and solution techniques for problems involving fractional powers of elliptic operators. We develop a complete and rigorous a priori and interpolation error analyses. We also design and study an efficient solver, and develop a suitable a posteriori error analysis. We conclude showing the flexibility of our approach by analyzing a fractional space-time parabolic equation.

A PDE APPROACH TO NUMERICAL FRACTIONAL DIFFUSION

by

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Dissertation submitted to the Faculty of the Graduate School of the University of Maryland, College Park in partial fulfillment of the requirements for the degree of Doctor of Philosophy
2014

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List of Abbreviations

$(-\Delta)^s$	fractional powers of the Dirichlet Laplace operator
$\mathcal{L}$	a general second order elliptic operator
$\mathcal{L}^s$	fractional powers of $\mathcal{L}$
I^σ	left Riemann-Liouville fractional integral of order σ
∂_t^γ	left-sided Caputo fractional derivative of order γ
K	simplex or n-rectangle
$T = K \times I$	quadrilateral prism
$\mathscr{T}$	mesh
$\mathscr{T}_\gamma$	tensor product mesh
$\mathbb{P}_m$	space of polynomials of total degree at most m
$\mathbb{Q}_m$	space of polynomials of degree not larger than m in each variable
AFEM	Adaptive Finite Element Method
FEM	Finite Element Method
PDE	Partial Differential Equations

Chapter 1: Introduction

Recently, a great deal of attention has been paid to the study of fractional and nonlocal operators, both from the point of view of pure mathematical research as well as motivated by several interesting applications where they constitute a fundamental part of the modeling and simulation of complex phenomena that span vastly different length scales.

Fractional and nonlocal operators can be found in a number of applications such as boundary control problems [73], finance [45, 165], electromagnetic fluids [122], image processing [85], materials science [20], optimization [73], porous media flow [59], turbulence [16], peridynamics [147], nonlocal continuum field theories [74] and others. From this it is evident that the particular type of operator appearing in applications can widely vary and that a unified analysis of their discretizations might be well beyond our reach. A more modest, but nevertheless quite ambitious, goal is to develop an analysis and approximation of a model operator that is representative of a particular class: the fractional powers of the Dirichlet Laplace operator, which for convenience we will simply call the fractional Laplacian.

The study of boundary value problems involving the fractional Laplacian is important in physical applications where long range or anomalous diffusion is considered. For instance, in the flow in porous media, it is used when modeling the transport of particles that experience very large transitions arising from high heterogeneity and very long spatial autocorrelation; see [22]. In the theory of stochastic processes, the fractional Laplacian is the infinitesimal generator of a stable Lévy process; see [25].

To make matters precise, in this work we shall be concerned with the following problem. Let Ω be an open and bounded subset of $\mathbb{R}^n$ $(n \geq 1)$, with boundary $\partial\Omega$. Given $s \in (0,1)$ and a smooth enough function f, find u such that

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Our approach, however, is by no means particular to the fractional Laplacian. It can also be applied to a general second order, symmetric and uniformly elliptic operator; see Section 3.6 and Chapter 7.

One of the main difficulties in the study of problem (1.1) is that the fractional Laplacian is a nonlocal operator; see [115, 43, 41]. To localize it, Caffarelli and Silvestre showed in [43] that any power of the fractional Laplacian in $\mathbb{R}^n$ can be realized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space $\mathbb{R}^{n+1}_+$. For a bounded domain Ω, the result by Caffarelli and Silvestre has been adapted in [44, 35, 155], thus obtaining an extension problem which is now posed on the semi-infinite cylinder

$\mathcal{C} = \Omega \times (0, \infty)$. This extension is the following mixed boundary value problem:

$$\begin{cases} \operatorname{div}\left(y^{\alpha} \nabla \mathscr{U}\right) = 0, & \text{in } \mathcal{C}, \\ \mathscr{U} = 0, & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial \mathscr{U}}{\partial \nu^{\alpha}} = d_s f, & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.2}$$

where $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$ denotes the lateral boundary of $\mathcal{C}$, and

$$\frac{\partial \mathscr{U}}{\partial \nu^{\alpha}} = -\lim_{y \to 0^+} y^{\alpha} \mathscr{U}_y, \tag{1.3}$$

is the the so-called conormal exterior derivative of $\mathscr{U}$ with ν being the unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$. The parameter α is defined as

$$\alpha = 1 - 2s \in (-1, 1). \tag{1.4}$$

Finally, d_s is a positive normalization constant which depends only on s; see [43] for details. We will call y the *extended variable* and the dimension $n+1$ in $\mathbb{R}^{n+1}_+$ the *extended dimension* of problem (1.2). The limit in (1.3) must be understood in the distributional sense; see [35, 41, 43] for more details. As noted in [43, 44, 155], the fractional Laplacian and the Dirichlet-to-Neumann operator of problem (1.2) are related by

$$d_s(-\Delta)^s u = \frac{\partial \mathscr{U}}{\partial \nu^{\alpha}} \quad \text{in } \Omega.$$

Using the aforementioned ideas, we propose the following simple strategy to find the solution of (1.1): given a sufficiently smooth function f we solve (1.2), thus obtaining a function $\mathscr{U} : (x', y) \in \mathcal{C} \mapsto \mathscr{U}(x', y) \in \mathbb{R}$. Setting $u : x' \in \Omega \mapsto u(x') = \mathscr{U}(x', 0) \in \mathbb{R}$, we obtain the solution of (1.1).

The main purpose of this dissertation is then the study of efficient discretization and solution techniques for problems involving fractional powers of elliptic operators based on the simple strategy described above, which is in turn inspired in the breakthrough by L. Caffarelli and L. Silvestre [43] to localize fractional powers of elliptic operators. Before proceeding with the thesis outine and its specific contributions, it is instructive to compare our proposed technique with those advocated in the literature.

1.1 State of the art

In this section, we review the state of the art involving numerical techniques to approximate problems involving fractional powers of elliptic operators.

In contrast to wavelets [99, 154], the use of finite element methods (FEM) is less understood. In fact, the integral formulation of fractional diffusion is notoriously difficult from the numerical standpoint, even in 1D [134], due to the presence of a kernel with non-integrable singularity.

An alternative approximation technique is based on the spectral decomposition of the operator $-\Delta$. For a general Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n > 1$), we may think about an algorithm for solving problem (1.1) inspired in this technique. However, to have a sufficiently good approximation, this requires the solution of a large number of eigenvalue problems which, in general, is very time consuming. In [103, 104] the authors studied computationally problem (1.1) in the one-dimensional case and introduced the so-called matrix transference technique (MTT). Basically, MTT computes a spatial discretization of the fractional Laplacian by first finding a matrix approximation, A, of the Laplace operator (via finite differences or finite elements) and then computing the s-th power of this matrix. This requires diagonalization of A which, again, amounts to the solution of a large number of eigenvalue problems. For the case $\Omega = (0,1)^2$ and $s \in (1/2, 1)$, [164] applies the MTT technique and avoids diagonalization of A by writing a numerical scheme in terms of the product of a function of the matrix and a vector, $f(A)b$, where b is a suitable vector. This product is then approximated by a preconditioned Lanczos method. Under the same setting, the work [40] makes a computational comparison of three techniques for the computation of $f(A)b$: the contour integral method, extended Krylov subspace methods and the pre-assigned poles and interpolation nodes method.

Recently two other papers that deal with the discretization of fractional powers of elliptic operators have appeared; [29] and [63]. Inspired in our work, and while this dissertation was in progress, Bonito and Pasciak developed in [29] an alternative approach, which is based on the integral formulation of fractional powers of self-adjoint operators [27, Chapter 10.4]. This yields a sequence of easily parallelizable uncoupled elliptics PDEs, and leads to quasi-optimal error estimates in the L^2-norm instead on the energy norm provided Ω is convex and $f \in \mathbb{H}^{2-2s}(\Omega)$. The results of Bonito and Pasciak, however, are not easy to extend to the energy norm, nor to time dependent problems and they are not suited for the treatment of nonlinear problems, which is something that can be somewhat easily achieved with our techniques [130]. The work by del Teso and Vázquez [63] studies the approximation of the α-harmonic extension problem, via a finite difference technique.

The authors consider a truncation of the Caffarelli-Silvestre extension problem to a $n+1$–rectangular domain in $\mathbb{R}^{n+1}_{+}$. This yields a truncation error decaying polynomially in each variable, not exponentially. The standard finite difference is done via a formal Taylor error analysis, which assumes sufficient but perhaps inconsistent regularity; no regularity results are presented for the authors. In contrast, this thesis examines the requisite regularity of the underlying problem, develops a general interpolation theory for anisotropic meshes, and applies it to fractional diffusion. It also addresses several related numerical issues as explained below.

1.2 Thesis outline and contributions

We develop PDE solution techniques for problems involving fractional powers of the Laplace operator in a bounded domain Ω with Dirichlet boundary conditions, i.e., $(-\Delta)^s u = f$. To overcome the inherent difficulty of nonlocality, we exploit the *cylindrical* extension proposed and investigated by X. Cabré and J. Tan [42], which is in turn inspired in the breakthrough by L. Caffarelli and L. Silvestre [43]. This leads to the (local) elliptic PDE (1.2) in one higher dimension y, with variable coefficient y^α, $\alpha = 1-2s$, which either degenerates ($s < 1/2$) or blows up ($s > 1/2$).

In order to study problems (1.1) and (1.2), we first introduce in Chapter 2 an appropiate notation and we recall some basic function spaces and functional analysis theory. Next, to approximate the solution of (1.2), we propose and analyze a discretization scheme, which is based on first degree tensor product finite elements and gives rise to the first contribution of this dissertation:

1. *A PDE approach to fractional diffusion in general domains: a priori error analysis.* Motivated by the rapid decay of the solution of problem (1.2), in Chapter 3, we propose a truncation that is suitable for numerical approximation. We discretize this truncation using first degree tensor product finite elements, and we derive a priori error estimates in weighted Sobolev spaces. For quasi-uniform meshes, these estimates exhibit optimal regularity but suboptimal order. We derive an almost-optimal a priori error analysis which combines asymptotic properties of Bessel functions with polynomial interpolation theory on weighted Sobolev spaces. The latter is valid for tensor product elements which may be graded in Ω and exhibit a large aspect ratio in y (anisotropy) to fit the behavior of $\mathscr{U}(x,y)$ with $x \in \Omega, y > 0$; this extends prior work of R. Durán and A. Lombardi [70]. The derived estimate is quasi-optimal in both order and regularity. We present numerical experiments to illustrate the method's performance.

   The discussion in this chapter is mainly based on the reference [129]:

   R.H. Nochetto, E. Otárola, and A.J. Salgado. A PDE approach to fractional diffusion in general domains: a priori error analysis. arXiv:1302.0698. Submitted to *Foundations of Computational Mathematics*, 2013

2. *Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications.*

   Upon realizing that the weight y^α belongs to the class A_2 of Muckemhoupt weights, we have been able to extend the interpolation theory of Chapter 3 to a general one in a Muckenhoupt weighted Sobolev space setting. This is of interest not only for the solution of problem (1.2) but also for that of nonuniformly elliptic problems in general. This brings to numerical analysis techniques and methods developed within harmonic analysis to deal with, for instance, maximal functions, Calderon Zygmund operators and weighted norm inequalities. This might serve as a starting point for the numerical analysis on homogeneous spaces.

In chapter 4, we develop a constructive piecewise polynomial approximation theory in weighted Sobolev spaces with Muckenhoupt weights for any polynomial degree. The main ingredients to derive optimal error estimates for an averaged Taylor polynomial are a suitable weighted Poincaré inequality, a cancellation property and a simple induction argument. We also construct a quasi-interpolation operator, built on local averages over stars, which is well defined for functions in L^1. We derive optimal error estimates for any polynomial degree on simplicial shape regular meshes. On rectangular meshes, these estimates are valid under the condition that neighboring elements have comparable size, which yields optimal anisotropic error estimates over n-rectangular domains. The interpolation theory extends to cases when the error and function regularity require different weights. We conclude with three applications: nonuniformly elliptic boundary value problems, elliptic problems with singular sources, and fractional powers of elliptic operators.

The discussion in this chapter is mainly based on the reference [132]:

R.H. Nochetto, E. Otárola, and A.J. Salgado. Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications. arXiv:1402.1916. Submitted to *Numerische Mathematik*, 2014

3. *Multilevel methods.*

Our PDE approach leads to the local problem (1.2). This advantage over integral techniques, however, comes at the expense of incorporating one more dimension to the problem, thereby raising the question of computational efficiency.

It is known that multilevel methods are among the most efficient techniques for the solution of discretized PDE. In Chapter 5 we study their applicability to (1.2), and develop a nearly uniformly convergent multilevel method. Our point of departure is the framework of Xu and Zikatanov [163] along with [160]. In view of the overwhelming evidence given in Chapter 3 that meshes must be highly anisotropic, we design a multilevel method with line smoothers in the y direction. Numerical experiments reveal a competitive performance of our method. Inspired in the interpolation theory developed in Chapter 4, we also develop and analyze multilevel methods for nonuniformly elliptic equations, deriving a nearly uniform convergence result.

We must point out that, in complete analogy to the unweighted case, removing the very weak (in fact logarithmical) dependence of the contraction factor in a multilevel method on the dimension of the problem, would require very fine properties of multilevel decompositions of functions in certain Muckenhoupt weighted

Besov spaces (see [101, 100, 135]).

The discussion in this chapter is mainly based on the reference [52]:

L. Chen, R.H. Nochetto, E. Otárola, and A.J. Salgado. Multigrid methods for nonuniformly elliptic operators. arXiv:1403.4278. Submitted to *Mathematics of Computations*, 2014

4. *A PDE approach to fractional diffusion: a posteriori error estimators and adaptivity.* Since the coefficient y^α in (1.2) either degenerates for $s < 1/2$ or blows up for $s > 1/2$, the usual residual estimators do not apply; integration by parts fails! In chapter 6, inspired in [14, 125], we deal with the natural anisotropy of the mesh in the extended variable y and the nonuniform coefficient y^α, upon considering local problems on *cylindrical stars*. The solutions of these local problems allow us to define a computable and anisotropic a posteriori error estimator which is equivalent to the error up to oscillations terms. In order to derive such a result, a computationally implementable geometric condition needs to be imposed on the mesh, which does not depend on the exact solution of problem (1.2). This approach is of value not only for (1.2), but in general for anisotropic problems since rigorous anisotropic a posteriori error estimators are not available in the literature.

5. *A PDE approach to space-time fractional parabolic problems.*

   In contrast to [29], our approach seems to be flexible enough to study other problems with fractional diffusion in space. One such problem is solution techniques for evolution equations with fractional diffusion and fractional time derivative.

We study solution techniques for evolution equations with fractional diffusion and fractional time derivative. The fractional time derivative, in the sense of Caputo, is discretized by a first order scheme and analyzed in a general Hilbert space setting. We show discrete stability estimates which yield an energy estimate for evolution problems with fractional time derivative. The spatial fractional diffusion is realized as the Dirichlet-to-Neumann map for a nonuniformly elliptic problem posed on a semi-infinite cylinder in one more spatial dimension. We write our evolution problem as a quasi-stationary elliptic problem with a dynamic boundary condition, and we analyze it in the framework of weighted Sobolev spaces. The rapid decay of the solution to this problem suggests a truncation that is suitable for numerical approximation. We propose and analyze a first order semi-implicit fully-discrete scheme to discretize the truncation: first degree tensor product finite elements in space and first order discretization in time. We prove stability and a near optimal a priori error estimate of the numerical scheme, in both order and regularity.

The discussion in this chapter is mainly based on the reference [131]:

R.H. Nochetto, E. Otárola, and A.J Salgado. A pde approach to space-time fractional parabolic problems. Submitted to *SIAM Journal on Numerical Analysis*, 2014

Chapter 2: Notations and Preliminaries

The purpose of this chapter is to establish the notation that shall be used in the subsequent chapters.

Throughout this work, Ω is an open, bounded and connected subset of $\mathbb{R}^n$, with $n \geq 1$. The boundary of Ω is denoted by $\partial\Omega$. Unless specified otherwise, we will assume that $\partial\Omega$ is Lipschitz.

The set of locally integrable functions on Ω is denoted by $L^1_{\mathrm{loc}}(\Omega)$. The Lebesgue measure of a measurable subset $E \subset \mathbb{R}^n$ is denoted by $|E|$. The mean value of a locally integrable function f over a set E is

$$\fint_E f \,\mathrm{d}x = \frac{1}{|E|} \int_E f \,\mathrm{d}x.$$

For a multi-index $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$ we denote its length by $|\kappa| = \kappa_1 + \dots + \kappa_n$, and, if $x \in \mathbb{R}^n$, we set $x^\kappa = x_1^{\kappa_1} \dots x_n^{\kappa_n} \in \mathbb{R}$, and

$$D^\kappa = \frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \cdots \frac{\partial^{\kappa_n}}{\partial x_n^{\kappa_n}}.$$

Given $p \in (1, \infty)$, we denote by p' the dual Lebesgue number, namely the real number such that $1/p + 1/p' = 1$, i.e., $p' = p/(p-1)$.

If X and Y are topological vector spaces, we write $X \hookrightarrow Y$ to denote that X is continuously embedded in Y. We denote by X' the dual of X. If X is normed, we denote by $\|\cdot\|_X$ its norm. The relation $a \lesssim b$ indicates that $a \leq Cb$, with a constant C that does not depend on either a or b, the value of C might change at each occurrence.

In order to study the so called α-harmonic extension problem (1.2), we define the semi-infinite cylinder

$$\mathcal{C} = \Omega \times (0, \infty), \tag{2.1}$$

and its lateral boundary

$$\partial_L \mathcal{C} = \partial\Omega \times [0, \infty). \tag{2.2}$$

Given $\mathcal{Y} > 0$, we define the truncated cylinder

$$\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y}). \tag{2.3}$$

The lateral boundary $\partial_L \mathcal{C}_{\mathcal{Y}}$ is defined accordingly.

Throughout our discussion we will be dealing with objects defined in $\mathbb{R}^{n+1}$ and it will be convenient to distinguish the extended dimension, as it plays a special role. A vector $x \in \mathbb{R}^{n+1}$, will be denoted by

$$x = (x^1, \ldots, x^n, x^{n+1}) = (x', x^{n+1}) = (x', y),$$

with $x^i \in \mathbb{R}$ for $i = 1, \ldots, n+1$, $x' \in \mathbb{R}^n$ and $y \in \mathbb{R}$. The upper half-space in $\mathbb{R}^{n+1}$ will be denoted by

$$\mathbb{R}^{n+1}_+ = \{x = (x', y) : x' \in \mathbb{R}^n \ y \in \mathbb{R},\ y > 0\}.$$

Let $\gamma, z \in \mathbb{R}^n$, the binary operation $\circ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\gamma \circ z = (\gamma_1 z_1, \gamma_2 z_2, \cdots, \gamma_n z_n) \in \mathbb{R}^n. \tag{2.4}$$

2.1 Fractional Sobolev spaces

Let us recall some fractional Sobolev spaces; for details the reader is referred to [66, 120, 123, 156]. For $0 < s < 1$, we introduce the so-called Gagliardo-Slobodeckiĭ seminorm

$$|w|^2_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|w(x_1') - w(x_2')|^2}{|x_1' - x_2'|^{n+2s}} \, \mathrm{d}x_1' \, \mathrm{d}x_2'.$$

The Sobolev space $H^s(\Omega)$ of order s is defined by

$$H^s(\Omega) = \left\{ w \in L^2(\Omega) : |w|_{H^s(\Omega)} < \infty \right\}, \tag{2.5}$$

which equipped with the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)} \right)^{\frac{1}{2}},$$

is a Hilbert space. An equivalent construction of $H^s(\Omega)$ is obtained by restricting functions in $H^s(\mathbb{R}^n)$ to Ω (cf. [156, Chapter 34]). The space $H^s_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$, i.e.,

$$H^s_0(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}. \tag{2.6}$$

If the boundary of Ω is smooth, an equivalent approach to define fractional Sobolev spaces is given by interpolation in [120, Chapter 1]. Set $H^0(\Omega) = L^2(\Omega)$, then Sobolev spaces with real index $0 \le s \le 1$ can be defined as interpolation spaces of index $\theta = 1 - s$ for the pair $[H^1(\Omega), L^2(\Omega)]$, that is

$$H^s(\Omega) = \left[H^1(\Omega), L^2(\Omega) \right]_\theta. \tag{2.7}$$

Analogously, for $s \in [0,1]\backslash\{\frac{1}{2}\}$, the spaces $H_0^s(\Omega)$ are defined as interpolation spaces of index $\theta = 1 - s$ for the pair $[H_0^1(\Omega), L^2(\Omega)]$, in other words

$$H_0^s(\Omega) = \left[H_0^1(\Omega), L^2(\Omega)\right]_\theta, \quad \theta \neq \tfrac{1}{2}. \tag{2.8}$$

The space $[H_0^1(\Omega), L^2(\Omega)]_{\frac{1}{2}}$ is the so-called *Lions-Magenes* space,

$$H_{00}^{\frac{1}{2}}(\Omega) = \left[H_0^1(\Omega), L^2(\Omega)\right]_{\frac{1}{2}},$$

which can be characterized as

$$H_{00}^{\frac{1}{2}}(\Omega) = \left\{ w \in H^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{w^2(x')}{\operatorname{dist}(x', \partial\Omega)} \, \mathrm{d}x' < \infty \right\}, \tag{2.9}$$

see [120, Theorem 11.7]. Moreover, we have the strict inclusion $H_{00}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega)$ because $1 \in H_0^{1/2}(\Omega)$ but $1 \notin H_{00}^{1/2}(\Omega)$. If the boundary of Ω is Lipschitz, the characterization (2.9) is equivalent to the definition via interpolation, and definitions (2.7) and (2.8) are also equivalent to definitions (2.5) and (2.6), respectively. To see this, it suffices to notice that when $\Omega = \mathbb{R}^n$ these definitions yield identical spaces and equivalent norms; see [4, Chapter 7]. Consequently, using the well-known extension result of Stein [151] for Lipschitz domains, we obtain the asserted equivalence (see [4, Chapter 7] for details).

When the boundary of Ω is Lipschitz, the space $C_0^\infty(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \leq \frac{1}{2}$, i.e., $H_0^s(\Omega) = H^s(\Omega)$. If $s > \frac{1}{2}$, we have that $H_0^s(\Omega)$ is strictly contained in $H^s(\Omega)$; see [120, Theorem 11.1]. In particular, we have the inclusions $H_{00}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega) = H^{1/2}(\Omega)$.

2.2 Weighted Sobolev spaces

We now introduce the class of Muckenhoupt weighted Sobolev spaces and refer to [79, 102, 108, 158] for details. We start with the definition of a weight.

Definition 2.1 (weight) *A weight is a function $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$.*

Every weight induces a measure, with density $\omega\,\mathrm{d}x$, over the Borel sets of $\mathbb{R}^n$. For simplicity, this measure will also be denoted by ω. For a Borel set $E \subset \mathbb{R}^n$ we define $\omega(E) = \int_E \omega\,\mathrm{d}x$.

We recall the definition of Muckenhoupt classes; see [79, 126, 158].

Definition 2.2 (Muckenhoupt class A_p) *Let ω be a weight and $1 < p < \infty$. We say $\omega \in A_p(\mathbb{R}^n)$ if there exists a positive constant $C_{p,\omega}$ such that*

$$\sup_B \left(\fint_B \omega \right) \left(\fint_B \omega^{1/(1-p)} \right)^{p-1} = C_{p,\omega} < \infty, \tag{2.10}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. In addition,

$$A_\infty(\mathbb{R}^n) = \bigcup_{p>1} A_p(\mathbb{R}^n), \qquad A_1(\mathbb{R}^n) = \bigcap_{p>1} A_p(\mathbb{R}^n).$$

If ω belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$, we say that ω is an A_p-weight, and we call the constant $C_{p,\omega}$ in (2.10) *the A_p-constant of ω.*

Remark 2.3 (characterization of the A_1-class) A useful characterization of the A_1-Muckenhoupt class is given in [152]: $\omega \in A_1(\mathbb{R}^N)$ if and only if

$$\sup_B \frac{\|\omega^{-1}\|_{L^\infty(B)}}{|B|} \int_B \omega = C_{1,\omega} < \infty. \tag{2.11}$$

A classical example is the function $|x|^\gamma$, which is an A_p-weight if and only if $-n < \gamma < n(p-1)$. Another important example is $d(x) = \mathsf{d}(x, \partial\Omega)^\alpha$, where for $x \in \Omega$, $\mathsf{d}(x, \partial\Omega)$ denotes the distance from the point x to the boundary $\partial\Omega$. The function d belongs to $A_2(\mathbb{R}^n)$ if and only if $-n < \alpha < n$. This function is used to define weighted Sobolev spaces which are important to study Poisson problems with singular sources; see [6, 60].

Throughout this work, we shall use some properties of the A_p-weights which, for completeness, we state and prove below.

Proposition 2.1 (properties of the A_p-class) *Let $1 < p < \infty$, and $\omega \in A_p(\mathbb{R}^n)$. Then, we have the following properties:*

(i) $\omega^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

(ii) $C_{p,\omega} \geq 1$.

(iii) If $1 < p < r < \infty$, then $A_p(\mathbb{R}^n) \subset A_r(\mathbb{R}^n)$, and $C_{r,\omega} \leq C_{p,\omega}$.

(iv) $\omega^{-1/(p-1)} \in A_{p'}(\mathbb{R}^n)$ and, conversely, $\omega^{-1/(p'-1)} \in A_p(\mathbb{R}^n)$. Moreover,

$$C_{p',\omega^{-1/(p-1)}} = C_{p,\omega}^{1/(p-1)}.$$

(v) The A_p-condition is invariant under translations and isotropic dilations, i.e., the weights $x \mapsto \omega(x + \mathbf{b})$ and $x \mapsto \omega(\mathbf{A}x)$, with $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} = a \cdot \mathbf{I}$ with $a \in \mathbb{R}$, both belong to $A_p(\mathbb{R}^n)$ with the same A_p-constant as ω.

Proof: Properties (i) and (iv) follow directly from the definition of the Muckenhoupt class $A_p(\mathbb{R}^n)$ given in (2.10). By writing $1 = \omega^{1/p}\omega^{-1/p}$ and the Hölder inequality, we obtain that for every ball $B \subset \mathbb{R}^n$,

$$1 = \fint_B \omega^{1/p}\omega^{-1/p} \leq \left(\fint_B \omega\right)^{1/p} \left(\fint_B \omega^{-1/(p-1)}\right)^{(p-1)/p},$$

which proves (ii). Using the Hölder inequality again, we obtain

$$\left(\fint_B \omega^{1/(1-r)}\right)^{r-1} \leq \left(\fint_B \omega^{1/(1-p)}\right)^{p-1},$$

which implies (iii). Finally, to prove property (v) we denote $\bar{\omega}(x) = \omega(\mathbf{A}x + \mathbf{b})$, and let B_r be a ball of radius r in $\mathbb{R}^n$. Using the change of variables $y = \mathbf{A}x + \mathbf{b}$, we obtain

$$\fint_{B_r} \bar{\omega}(x)\,\mathrm{d}x = \frac{1}{a^n|B_r|}\int_{B_{ar}} \omega(y)\,\mathrm{d}y, \tag{2.12}$$

which, since $a^n|B_r| = |B_{ar}|$, proves (v). □

From the A_p-condition and Hölder's inequality follows that an A_p-weight satisfies the so-called *strong doubling property*. The proof of this fact is standard and presented here for completeness; see [158, Proposition 1.2.7] for more details.

Proposition 2.2 (strong doubling property) *Let $\omega \in A_p(\mathbb{R}^n)$ with $1 < p < \infty$ and let $E \subset \mathbb{R}^n$ be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then*

$$\omega(B) \leq C_{p,\omega}\left(\frac{|B|}{|E|}\right)^p \omega(E). \tag{2.13}$$

Proof: Since $E \subset \mathbb{R}^n$ is measurable, we have that

$$\begin{aligned}|E| &\leq \left(\int_E \omega\,\mathrm{d}x\right)^{1/p}\left(\int_E \omega^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \leq \omega(E)^{1/p}|B|^{1/p'}\left(\fint_B \omega^{-p'/p}\right)^{1/p'} \\ &\leq C_{p,\omega}^{1/p}\,\omega(E)^{1/p}|B|^{1/p'}\left(\fint_B \omega\right)^{-1/p} = C_{p,\omega}^{1/p}\left(\frac{\omega(E)}{\omega(B)}\right)^{1/p}|B|.\end{aligned}$$

This completes the proof. $\square$

In particular, every A_p-weight satisfies a *doubling property*, i.e., there exists a positive constant C such that

$$\omega(B_{2r}) \le C\omega(B_r). \tag{2.14}$$

for every ball $B_r \subset \mathbb{R}^n$. The infimum over all constants C, for which (2.14) holds, is called the *doubling constant* of ω. The class of A_p-weights was introduced by B. Muckenhoupt [126], who proved that the A_p-weights are precisely those for which the Hardy-Littlewood maximal operator is bounded from $L^p(\omega, \mathbb{R}^n)$ to $L^p(\omega, \mathbb{R}^n)$, when $1 < p < \infty$. We now define weighted Lebesgue spaces as follows.

Definition 2.4 (weighted Lebesgue spaces) *Let $\omega \in A_p$, and let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. For $1 < p < \infty$, we define the weighted Lebesgue space $L^p(\omega, \Omega)$ as the set of measurable functions u on Ω equipped with the norm*

$$\|u\|_{L^p(\omega,\Omega)} = \left(\int_\Omega |u|^p \omega\right)^{1/p}. \tag{2.15}$$

An immediate consequence of $\omega \in A_p(\mathbb{R}^n)$ is that functions in $L^p(\omega, \Omega)$ are locally summable which, in fact, only requires that $\omega^{-1/(p-1)} \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Proposition 2.3 ($L^p(\omega, \Omega) \subset L^1_{\text{loc}}(\Omega)$) *Let Ω be an open set, $1 < p < \infty$ and ω be a weight such that $\omega^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega)$. Then, $L^p(\omega, \Omega) \subset L^1_{\text{loc}}(\Omega)$.*

Proof: Let $u \in L^p(\omega, \Omega)$, and let $B \subset \Omega$ be a ball. By Hölder's inequality, we have

$$\int_B |u| = \int_B |u|\omega^{1/p}\omega^{-1/p} \le \left(\int_B |u|^p \omega\right)^{1/p} \left(\int_B \omega^{-1/(p-1)}\right)^{(p-1)/p} \lesssim \|u\|_{L^p(\omega,\Omega)},$$

which concludes the proof. $\square$

Notice that when Ω is bounded we have $L^p(\omega, \Omega) \hookrightarrow L^1(\Omega)$. In particular, Proposition 2.3 shows that it makes sense to talk about weak derivatives of functions in $L^p(\omega, \Omega)$. We define weighted Sobolev spaces as follows.

Definition 2.5 (weighted Sobolev spaces) *Let ω be an A_p-weight with $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and $m \in \mathbb{N}$. The weighted Sobolev space $W_p^m(\omega, \Omega)$ is the set of functions $u \in L^p(\omega, \Omega)$ such that for any multi-index κ with $|\kappa| \leq m$, the weak derivatives $D^\kappa u \in L^p(\omega, \Omega)$, with seminorm and norm*

$$|u|_{W_p^m(\omega,\Omega)} = \left(\sum_{|\kappa|=m} \|D^\kappa u\|^p_{L^p(\omega,\Omega)} \right)^{1/p}, \quad \|u\|_{W_p^m(\omega,\Omega)} = \left(\sum_{j \leq m} |u|^p_{W_p^j(\omega,\Omega)} \right)^{1/p},$$

respectively. We also define $\mathring{W}_p^m(\omega, \Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W_p^m(\omega, \Omega)$.

Without any restriction on the weight ω, the space $W_p^m(\omega, \Omega)$ may not be complete. However, when $\omega^{-1/(p-1)}$ is locally integrable in $\mathbb{R}^n$, $W_p^m(\omega, \Omega)$ is a Banach space; see [109]. Properties of weighted Sobolev spaces can be found in classical references like [102, 108, 158]. It is remarkable that most of the properties of classical Sobolev spaces have a weighted counterpart and it is more so that this is not because of the specific form of the weight but rather due to the fact that the weight ω belongs to the Muckenhoupt class A_p; see [79, 87, 126]. In particular, we have the following results (cf. [158, Proposition 2.1.2, Corollary 2.1.6] and [87, Theorem 1]) .

Proposition 2.4 (properties of weighted Sobolev spaces) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain, $1 < p < \infty$, $\omega \in A_p(\mathbb{R}^n)$ and $m \in \mathbb{N}$. The spaces $W_p^m(\omega, \Omega)$ and $\mathring{W}_p^m(\omega, \Omega)$ are complete, and $W_p^m(\omega, \Omega) \cap C^\infty(\Omega)$ is dense in $W_p^m(\omega, \Omega)$.*

Chapter 3: Algorithm design and a priori error analysis

3.1 Introduction

The purpose of this work is the study of solution techniques for problems involving fractional powers of symmetric coercive elliptic operators in a bounded domain with Dirichlet boundary conditions. These operators can be realized as the Dirichlet to Neumann map for a nonuniformly elliptic problem posed on a semi-infinite cylinder, which we analyze in the framework of weighted Sobolev spaces. Motivated by the rapid decay of the solution of this problem, we propose a truncation that is suitable for numerical approximation. We discretize this truncation using first degree tensor product finite elements. We derive a priori error estimates in weighted Sobolev spaces. The estimates exhibit optimal regularity but suboptimal order for quasi-uniform meshes. For anisotropic meshes, instead, they are quasi-optimal in both order and regularity. We present numerical experiments to illustrate the method's performance.

The outline of this chapter is as follows. We recall the definition of the fractional Laplacian on a bounded domain via spectral theory in § 3.1.1 and in § 3.1.2 we introduce the functional framework that is suitable for the study of problems (1.1) and (1.2). We discuss the Caffarelli-Silvestre extension in § 3.1.3. In addition, in § 3.1.5 we study regularity of the solution to (1.2) via the asymptotic estimates fiven in § 3.1.4. The numerical analysis of (1.1) begins in § 3.2. Here we introduce a truncation of problem (1.2) and study some properties of its solution. Having understood the truncation we proceed, in § 3.3, to study its finite element approximation. We prove interpolation estimates in weighted Sobolev spaces, under mild shape regularity assumptions that allow us to consider anisotropic elements in the extended variable y. Based on the regularity results of § 3.1.5 we derive, in § 3.4, a priori error estimates for quasi-uniform meshes which exhibit optimal regularity but suboptimal order. To restore optimal decay, we resort to the so-called principle of error equidistribution and construct graded meshes in the extended variable y. They in turn capture the singular behavior of the solution to (1.2) and allow us to prove a quasi-optimal rate of convergence with respect to both regularity and degrees of freedom. In § 3.5, to illustrate the method's performance and theory, we provide several numerical experiments. Finally, in § 3.6 we show that our developments apply to general second order, symmetric and uniformly elliptic operators.

3.1.1 The fractional Laplace operator

It is important to mention that there is no unique way of defining a nonlocal operator related to the fractional Laplacian in a bounded domain. A first possibility is to suitably extend the functions to the whole space $\mathbb{R}^n$ and use Fourier transform

$$\mathcal{F}((-\Delta)^s w)(\xi') = |\xi'|^{2s}\mathcal{F}(w)(\xi').$$

After extension, the following point-wise formula also serves as a definition of the fractional Laplacian

$$(-\Delta)^s w(x') = C_{n,s}\text{p.v.}\int_{\mathbb{R}^n} \frac{w(x') - w(z')}{|x' - z'|^{n+2s}}\,\mathrm{d}z', \tag{3.1}$$

where p.v. stands for the Cauchy principal value and $C_{n,s}$ is a positive normalization constant that depends only on n and s which is introduced to guarantee that the symbol of the resulting operator is $|\xi'|^{2s}$. For details we refer the reader to [41, 115, 66] and, in particular, to [115, Section 1.1] or [66, Proposition 3.3] for a proof of the equivalence of these two definitions.

Even if we restrict ourselves to definitions that do not require extension, there is more than one possibility. For instance, the so-called regional fractional Laplacian ([92, 28]) is defined by restricting the Riesz integral to Ω, leading to an operator related to a Neumann problem. A different operator is obtained by using the spectral decomposition of the Dirichlet Laplace operator $-\Delta$, see [35, 42, 44]. This approach is also different to the integral formula (3.1). Indeed, the spectral definition depends on the domain Ω considered, while the integral one at any point is independent of the domain in which the equation is set. For more details see the discussion in [145].

The definition that we shall adopt is as in [35, 42, 44] and is based on the spectral theory of the Dirichlet Laplacian ([78, 84]) as we summarize below.

We define $-\Delta : L^2(\Omega) \to L^2(\Omega)$ with domain $\mathrm{Dom}(-\Delta) = \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\}$. This operator is positive, unbounded, closed and its inverse is compact. This implies that the spectrum of the operator $-\Delta$ is discrete, positive and accumulates at infinity. Moreover, there exist $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times H_0^1(\Omega)$ such that

$\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and, for $k \in \mathbb{N}$,

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k, & \text{in } \Omega, \\ \varphi_k = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Consequently, $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthogonal basis of $H_0^1(\Omega)$ and $\|\nabla_{x'}\varphi_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$.

With this spectral decomposition at hand, fractional powers of the Dirichlet Laplacian $(-\Delta)^s$ can be defined for $u \in C_0^\infty(\Omega)$ by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} u_k \lambda_k^s \varphi_k, \tag{3.3}$$

where the coefficients u_k are defined by $u_k = \int_\Omega u\varphi_k$. Therefore, if $f = \sum_{k=1}^{\infty} f_k\varphi_k$, and $(-\Delta)^s u = f$, then $u_k = \lambda_k^{-s} f_k$ for all $k \geq 1$.

By density the operator $(-\Delta)^s$ can be extended to the Hilbert space

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k\varphi_k \in L^2(\Omega) : \|w\|_{\mathbb{H}^s(\Omega)}^2 = \sum_{k=1}^{\infty} \lambda_k^s |w_k|^2 < \infty \right\}.$$

The theory of Hilbert scales presented in [120, Chapter 1] shows that

$$\left[H_0^1(\Omega), L^2(\Omega)\right]_\theta = \operatorname{Dom}(-\Delta)^{\frac{s}{2}},$$

where $\theta = 1 - s$. This implies the following characterization of the space $\mathbb{H}^s(\Omega)$,

$$\mathbb{H}^s(\Omega) = \begin{cases} H^s(\Omega), & s \in (0, \frac{1}{2}), \\ H_{00}^{1/2}(\Omega), & s = \frac{1}{2}, \\ H_0^s(\Omega), & s \in (\frac{1}{2}, 1). \end{cases} \tag{3.4}$$

We denote by $\mathbb{H}^{-s}(\Omega)$ the dual space of $\mathbb{H}^s(\Omega)$ for $0 < s < 1$.

3.1.2 Weighted Sobolev spaces

To exploit the Caffarelli-Silvestre extension [43], or its variants [35, 42, 44], we need to deal with a nonuniformly elliptic equation on $\mathbb{R}_+^{n+1}$. To this end, we consider weighted Sobolev spaces, with the specific weight $|y|^\alpha$ with $\alpha \in (-1, 1)$; see section 2.2.

Let $\mathcal{D} \subset \mathbb{R}^{n+1}$ be an open set and $\alpha \in (-1, 1)$. We define the weighted spaces $L^2(\mathcal{D}, |y|^\alpha)$ and $H^1(\mathcal{D}, |y|^\alpha)$ according to Definitions 2.4 and 2.5 respectively. The space $H^1(\mathcal{D}, |y|^\alpha)$ is equipped with the norm

$$\|w\|_{H^1(\mathcal{D},|y|^\alpha)} = \left(\|w\|^2_{L^2(\mathcal{D},|y|^\alpha)} + \|\nabla w\|^2_{L^2(\mathcal{D},|y|^\alpha)} \right)^{\frac{1}{2}}. \tag{3.5}$$

Notice that taking $\alpha = 0$ in the definition above, we obtain the classical $H^1(\mathcal{D})$.

Properties of this weighted Sobolev space can be found in classical references like [102, 108]. It is remarkable that most of the properties of classical Sobolev spaces have a weighted counterpart not so because of the specific form of the weight but rather due to the fact that the weight $|y|^\alpha$ belongs to the so-called Muckenhoupt class $A_2(\mathbb{R}^{n+1})$; see [79, 87, 126]. Since $\alpha \in (-1, 1)$ it is immediate that $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$, which implies the following important result; see Proposition 2.4.

Proposition 3.1 (Properties of weighted Sobolev spaces) *Let $\mathcal{D} \subset \mathbb{R}^{n+1}$ be an open set and $\alpha \in (-1, 1)$. Then $H^1(\mathcal{D}, |y|^\alpha)$, equipped with the norm* (3.5)*, is a Hilbert space. Moreover, the set $C^\infty(\mathcal{D}) \cap H^1(\mathcal{D}, |y|^\alpha)$ is dense in $H^1(\mathcal{D}, |y|^\alpha)$.*

Remark 3.1 (Weighted L^2 vs L^1) If $\mathcal{D}$ is a bounded domain and $\alpha \in (-1, 1)$

then, $L^2(\mathcal{D}, |y|^\alpha) \subset L^1(\mathcal{D})$. Indeed, since $|y|^{-\alpha} \in L^1_{loc}(\mathbb{R}^{n+1})$,

$$\int_{\mathcal{D}} |w| = \int_{\mathcal{D}} |w||y|^{\alpha/2}|y|^{-\alpha/2} \leq \left(\int_{\mathcal{D}} |w|^2|y|^\alpha\right)^{\frac{1}{2}} \left(\int_{\mathcal{D}} |y|^{-\alpha}\right)^{\frac{1}{2}} \lesssim \|w\|_{L^2(\mathcal{D},|y|^\alpha)}.$$

The following result is given in [108, Theorem 6.3]. For completeness we present here a version of the proof on the truncated cylinder $\mathcal{C}_\mathcal{Y}$, which will be important for the numerical approximation of problem (1.2).

Proposition 3.2 (Embeddings in weighted Sobolev spaces) *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\mathcal{Y} > 0$. Then*

$$H^1(\mathcal{C}_\mathcal{Y}) \hookrightarrow H^1(\mathcal{C}_\mathcal{Y}, y^\alpha), \qquad \textit{for } \alpha \in (0,1), \tag{3.6}$$

and

$$H^1(\mathcal{C}_\mathcal{Y}, y^\alpha) \hookrightarrow H^1(\mathcal{C}_\mathcal{Y}), \qquad \textit{for } \alpha \in (-1,0). \tag{3.7}$$

Proof: Let us prove (3.6), the proof of (3.7) being similar. Since $\alpha > 0$ we have $y^\alpha \leq \mathcal{Y}^\alpha$, whence $y^\alpha w^2 \leq \mathcal{Y}^\alpha w^2$ and $y^\alpha|\nabla w|^2 \leq \mathcal{Y}^\alpha|\nabla w|^2$ a.e. on $\mathcal{C}_\mathcal{Y}$ for all $w \in H^1(\mathcal{C}_\mathcal{Y})$. This implies $\|w\|_{H^1(\mathcal{C}_\mathcal{Y},y^\alpha)} \leq \sqrt{2}\mathcal{Y}^{\alpha/2}\|w\|_{H^1(\mathcal{C}_\mathcal{Y})}$, which is (3.6). $\square$

Define

$$\mathring{H}^1_L(\mathcal{C}, y^\alpha) = \left\{ w \in H^1(y^\alpha; \mathcal{C}) : w = 0 \text{ on } \partial_L\mathcal{C} \right\}. \tag{3.8}$$

This space can be equivalently defined as the set of measurable functions $w : \mathcal{C} \to \mathbb{R}$ such that $w \in H^1(\Omega \times (s,t))$ for all $0 < s < t < \infty$, $w = 0$ on $\partial_L\mathcal{C}$ and for which the following seminorm is finite

$$\|w\|^2_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} = \int_\mathcal{C} y^\alpha|\nabla w|^2; \tag{3.9}$$

see [44]. As a consequence of the usual Poincaré inequality, for any $k \in \mathbb{Z}$ and any function $w \in H^1(\Omega \times (2^k, 2^{k+1}))$ with $w = 0$ on $\partial\Omega \times (2^k, 2^{k+1})$, we have

$$\int_{\Omega \times (2^k, 2^{k+1})} y^\alpha w^2 \leq C_\Omega \int_{\Omega \times (2^k, 2^{k+1})} y^\alpha |\nabla w|^2, \tag{3.10}$$

where C_Ω denotes a positive constant that depends only on Ω. Summing up over $k \in \mathbb{Z}$, we obtain the following *weighted Poincaré inequality*:

$$\int_{\mathcal{C}} y^\alpha w^2 \lesssim \int_{\mathcal{C}} y^\alpha |\nabla w|^2. \tag{3.11}$$

Hence, the seminorm (3.9) is a norm on $\mathring{H}^1_L(\mathcal{C}, y^\alpha)$, equivalent to (3.5).

For a function $w \in H^1(\mathcal{C}, y^\alpha)$, we shall denote by $\operatorname{tr}_\Omega w$ its trace onto $\Omega \times \{0\}$. It is well known that $\operatorname{tr}_\Omega H^1(\mathcal{C}) = H^{1/2}(\Omega)$; see [4, 156]. In the subsequent analysis we need a characterization of the trace of functions in $H^1(\mathcal{C}, y^\alpha)$. For a smooth domain this was given in [42, Proposition 1.8] for $s = 1/2$ and in [44, Proposition 2.1] for any $s \in (0,1) \setminus \{\frac{1}{2}\}$. However, since the eigenvalue decomposition (3.3) of the Dirichlet Laplace operator holds true on a Lipschitz domain, we are able to extend this trace characterization to such domains. In summary, we have the following result.

Proposition 3.3 (Characterization of $\operatorname{tr}_\Omega \mathring{H}^1_L(\mathcal{C}, y^\alpha)$) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The trace operator tr_Ω satisfies $\operatorname{tr}_\Omega \mathring{H}^1_L(\mathcal{C}, y^\alpha) = \mathbb{H}^s(\Omega)$ and*

$$\|\operatorname{tr}_\Omega v\|_{\mathbb{H}^s(\Omega)} \lesssim \|v\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)} \qquad \forall\, v \in \mathring{H}^1_L(\mathcal{C}, y^\alpha),$$

where the space $\mathbb{H}^s(\Omega)$ is defined in (3.4) and $\alpha = 1 - 2s$.

3.1.3 The Caffarelli-Silvestre extension problem

It has been shown in [43] that any power of the fractional Laplacian in $\mathbb{R}^n$ can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem posed on $\mathbb{R}^{n+1}_+$. For a bounded domain, an analogous result has been obtained in [42] for $s = \frac{1}{2}$, and in [35, 44, 155] for any $s \in (0, 1)$.

Let us briefly describe these results. Consider a function u defined on Ω. We define the α-harmonic extension of u to the cylinder $\mathcal{C}$, as the function $\mathscr{U}$ that solves the boundary value problem

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathscr{U}) = 0, & \text{in } \mathcal{C}, \\ \mathscr{U} = 0, & \text{on } \partial_L \mathcal{C}, \\ \mathscr{U} = u, & \text{on } \Omega \times \{0\}. \end{cases} \tag{3.12}$$

From Proposition 3.3 and the Lax Milgram lemma we can conclude that this problem has a unique solution $\mathscr{U} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ whenever $u \in \mathbb{H}^s(\Omega)$. We define the *Dirichlet-to-Neumann* operator $\Gamma_{\alpha,\Omega} : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$

$$u \in \mathbb{H}^s(\Omega) \longmapsto \Gamma_{\alpha,\Omega}(u) = \frac{\partial \mathscr{U}}{\partial \nu^\alpha} \in \mathbb{H}^{-s}(\Omega),$$

where $\mathscr{U}$ solves (3.12) and $\frac{\partial \mathscr{U}}{\partial \nu^\alpha}$ is given in (1.3). The space $\mathbb{H}^{-s}(\Omega)$ can be characterized as the space of distributions $h = \sum_k h_k \varphi_k$ such that $\sum_k |h_k|^2 \lambda_k^{-s} < \infty$. The fundamental result of [43], see also [44, Lemma 2.2], is stated below.

Theorem 3.2 (Caffarelli–Silvestre extension) *If $s \in (0, 1)$ and $u \in \mathbb{H}^s(\Omega)$,* *then*

$$d_s(-\Delta)^s u = \Gamma_{\alpha,\Omega}(u),$$

in the sense of distributions. Here $\alpha = 1 - 2s$ *and* d_s *is given by*

$$d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{3.13}$$

It seems remarkable that the constant d_s does not depend on the dimension. This was proved originally in [43] and its precise value appears in several references, for instance [35, 41].

The relation between the fractional Laplacian and the extension problem is now clear. Given $f \in \mathbb{H}^{-s}(\Omega)$, a function $u \in \mathbb{H}^s(\Omega)$ solves (1.1) if and only if its α-harmonic extension $\mathscr{U} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ solves (1.2).

If $u = \sum_k u_k \varphi_k$, then, as shown in the proofs of [44, Proposition 2.1] and [35, Lemma 2.2], $\mathscr{U}$ can be expressed as

$$\mathscr{U}(x) = \sum_{k=1}^{\infty} u_k \varphi_k(x')\psi_k(y), \tag{3.14}$$

where the functions ψ_k solve

$$\begin{cases} \psi_k'' + \dfrac{\alpha}{y}\psi_k' - \lambda_k\psi_k = 0, & \text{in } (0,\infty), \\ \psi_k(0) = 1, & \displaystyle\lim_{y\to\infty}\psi_k(y) = 0. \end{cases} \tag{3.15}$$

If $s = \frac{1}{2}$, then clearly $\psi_k(y) = e^{-\sqrt{\lambda_k}y}$ (see [42, Lemma 2.10]). For $s \in (0,1)\setminus\{\frac{1}{2}\}$ instead (cf. [44, Proposition 2.1])

$$\psi_k(y) = c_s\left(\sqrt{\lambda_k}y\right)^s K_s(\sqrt{\lambda_k}y),$$

where K_s denotes the modified Bessel function of the second kind (see [1, Chapter 9.6]). Using the condition $\psi_k(0) = 1$, and formulas for small arguments of the function K_s (see for instance § 3.1.4) we obtain

$$c_s = \frac{2^{1-s}}{\Gamma(s)}.$$

The function $\mathscr{U} \in \mathring{H}_L^1(\mathcal{C}, y^\alpha)$ is the unique solution of

$$\int_{\mathcal{C}} y^\alpha \nabla \mathscr{U} \cdot \nabla \phi = d_s \langle f, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}, \quad \forall \phi \in \mathring{H}_L^1(\mathcal{C}, y^\alpha), \tag{3.16}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}$ denotes the duality pairing between $\mathbb{H}^s(\Omega)$ and $\mathbb{H}^{-s}(\Omega)$ which, in light of Proposition 3.3 is well defined for all $f \in \mathbb{H}^{-s}(\Omega)$ and $\phi \in \mathring{H}_L^1(\mathcal{C}, y^\alpha)$. This implies the following equalities (see [44, Proposition 2.1] for $s \in (0,1) \setminus \{\frac{1}{2}\}$ and [42, Proposition 2.1] for $s = \frac{1}{2}$):

$$\|\mathscr{U}\|^2_{\mathring{H}_L^1(\mathcal{C}, y^\alpha)} = d_s \|u\|^2_{\mathbb{H}^s(\Omega)} = d_s \|f\|^2_{\mathbb{H}^{-s}(\Omega)}. \tag{3.17}$$

Notice that for $s = \frac{1}{2}$, or equivalently $\alpha = 0$, problem (3.16) reduces to the weak formulation of the Laplace operator with mixed boundary conditions, which is posed on the classical Sobolev space $\mathring{H}_L^1(\mathcal{C})$. Therefore, the value $s = \frac{1}{2}$ becomes a special case for problem (3.16). In addition, $d_{1/2} = 1$, and $\|\mathscr{U}\|_{\mathring{H}_L^1(\mathcal{C})} = \|u\|_{H_{00}^{1/2}(\Omega)}$.

At this point it is important to give a precise meaning to the Dirichlet boundary condition in (1.1). For $s = \frac{1}{2}$, the boundary condition is interpreted in the sense of the Lions–Magenes space. If $\frac{1}{2} < s \leq 1$, there is a trace operator from $\mathbb{H}^s(\Omega)$ into $L^2(\partial\Omega)$ and the boundary condition can be interpreted in this sense. For $0 < s < 1/2$ this interpretation is no longer possible and thus, for an arbitrary $f \in \mathbb{H}^{-s}(\Omega)$ the boundary condition does not have a clear meaning. For instance, for every $s \in (0, \frac{1}{2})$, $f = (-\Delta)^s 1 \in \mathbb{H}^{-s}(\Omega)$ and the solution to (1.1) for this right hand side is $u = 1$. If $f \in H^\zeta(\Omega)$ with $\zeta > \frac{1}{2} - 2s > -s$, using that $(-\Delta)^s$ is a pseudo-differential operator of order $2s$ a shift-type result is valid, i.e., $u \in H^{\varrho}(\Omega)$ with $\varrho = \zeta + 2s > 1/2$. In this case, the trace of u on $\partial\Omega$ is well defined and the boundary condition is meaningful. Finally, we comment that it has been proved in [44, Lemma 2.10], that if $f \in L^{\infty}(\Omega)$ then the solution of (1.1) belongs to $C^{0,\varkappa}(\overline{\Omega})$ with $\varkappa \in (0, \min\{2s, 1\})$.

3.1.4 Asymptotic estimates

It is important to understand the behavior of the solution $\mathscr{U}$ of problem (1.2), given by (3.14). Consequently, it becomes necessary to recall some of the main properties of the modified Bessel function of the second kind $K_{\nu}(z)$, $\nu \in \mathbb{R}$; see [1, Chapter 9.6] for (i)-(iv) and [124, Theorem 5] for (v):

(i) For $\nu > -1$, $K_{\nu}(z)$ is real and positive.

(ii) For $\nu \in \mathbb{R}$, $K_{\nu}(z) = K_{-\nu}(z)$.

(iii) For $\nu > 0$,

$$\lim_{z \downarrow 0} \frac{K_{\nu}(z)}{\frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}} = 1. \tag{3.18}$$

(iv) For $k \in \mathbb{N}$,

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^{k} (z^{\nu} K_{\nu}(z)) = (-1)^{k} z^{\nu-k} K_{\nu-k}(z).$$

In particular, for $k = 1$ and $k = 2$, respectively, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{\nu} K_{\nu}(z)) = -z^{\nu} K_{\nu-1}(z) = -z^{\nu} K_{1-\nu}(z), \tag{3.19}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^{\nu} K_{\nu}(z)) = z^{\nu} K_{2-\nu}(z) - z^{\nu-1} K_{1-\nu}(z). \tag{3.20}$$

(v) For $z > 0$, $z^{\min\{\nu,1/2\}} e^z K_\nu(z)$ is a decreasing function.

As an application we obtain the following important properties of the function ψ_k, defined in (3.15). First, for $s \in (0,1)$, properties (ii), (iii) and (iv) imply

$$\lim_{y \downarrow 0^+} \frac{y^\alpha \psi_k'(y)}{d_s \lambda_k^s} = -1, \tag{3.21}$$

Property (v) provides the following asymptotic estimate for $s \in (0,1)$ and $y \geq 1$:

$$|y^\alpha \psi_k(y) \psi_k'(y)| \leq C(s) \lambda_k^s \left(\sqrt{\lambda_k} y\right)^{\left|s - \frac{1}{2}\right|} e^{-2\sqrt{\lambda_k} y}. \tag{3.22}$$

Multiplying the differential equation of problem (3.15) by $y^\alpha \psi_k(y)$ and integrating by parts yields

$$\int_a^b y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2\right) \mathrm{d}y = y^\alpha \psi_k(y) \psi_k'(y)\big|_a^b, \tag{3.23}$$

where a and b are real and positive constants.

Let us conclude this section with some remarks on the asymptotic behavior of the function $\mathscr{U}$ that solves (3.16). Using (3.14) we obtain

$$\mathscr{U}(x)|_{y=0} = \sum_{k=1}^{\infty} u_k \varphi_k(x') \psi_k(0) = \sum_{k=1}^{\infty} u_k \varphi_k(x') = u(x').$$

For $s \in (0,1)$, using formula (3.21) together with (3.3), we arrive at

$$\frac{\partial \mathscr{U}}{\partial \nu^\alpha}(x', 0) = -\lim_{y \downarrow 0} y^\alpha \mathscr{U}_y(x', y) = d_s f(x'), \quad \text{on } \Omega \times \{0\}. \tag{3.24}$$

Notice that, if $s = \frac{1}{2}$, then $\alpha = 0$, $d_{1/2} = 1$ and thus (3.24) reduces to

$$\left.\frac{\partial \mathscr{U}}{\partial \nu}\right|_{\Omega \times \{0\}} = f(x').$$

For $s \in (0,1) \setminus \{\frac{1}{2}\}$ the asymptotic behavior of the second derivative $\mathscr{U}_{yy}$ as $y \approx 0^+$ is a consequence of (3.20) applied to the function $\psi_k(y)$. For $s = \frac{1}{2}$ the behavior follows from $\psi_k(y) = e^{-\sqrt{\lambda_k}y}$. In conclusion, for $y \approx 0^+$, we have

$$\mathscr{U}_{yy} \approx y^{-\alpha-1} \quad \text{for } s \in (0,1) \setminus \{\tfrac{1}{2}\}, \qquad \mathscr{U}_{yy} \approx 1 \quad \text{for } s = \tfrac{1}{2}. \tag{3.25}$$

3.1.5 Regularity of the solution

Since we are interested in the approximation of the solution of problem (3.16), and this is closely related to its regularity, let us now study the behavior of its derivatives. According to (3.24), $\mathscr{U}_y \approx y^{-\alpha}$ for $y \approx 0^+$. This clearly shows the necessity of introducing the weight, as this behavior, together with the exponential decay given by (v) of § 3.1.4, imply that $\mathscr{U}_y \in L^2(\mathcal{C}, y^\alpha) \setminus L^2(\mathcal{C})$ for $s \in (0, 1/4]$.

However, the situation with second derivatives is much more delicate. To see this, let us first argue heuristically and compute how these derivatives scale with y. From the asymptotic formula (3.25), we see that, for $0 < \delta \ll 1$ and $s \in (0,1) \setminus \{\frac{1}{2}\}$,

$$\int_{\Omega \times (0,\delta)} y^\alpha \left|\mathscr{U}_{yy}\right|^2 \, \mathrm{d}x' \, \mathrm{d}y \approx \int_0^\delta y^\alpha y^{-2-2\alpha} \, \mathrm{d}y = \int_0^\delta y^{-2-\alpha} \, \mathrm{d}y, \tag{3.26}$$

which, since $\alpha \in (-1,1) \setminus \{0\}$, does not converge. However,

$$\int_{\Omega \times (0,\delta)} y^\beta \left|\mathscr{U}_{yy}\right|^2 \, \mathrm{d}x \, \mathrm{d}y \approx \int_0^\delta y^{\beta-2-2\alpha} \, \mathrm{d}y,$$

converges for $\beta > 2\alpha + 1$, hinting at the fact that $\mathscr{U} \in H^2(\mathcal{C}, y^\beta) \setminus H^2(\mathcal{C}, y^\alpha)$. The following result makes these considerations rigorous.

Theorem 3.3 (Global regularity of the α-harmonic extension) *Let the data $f \in \mathbb{H}^{1-s}(\Omega)$, where $\mathbb{H}^{1-s}(\Omega)$ is defined in (3.4) for $s \in (0,1)$. Let $\mathscr{U} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$*

solve (3.16) *with* f *as data. Then, for* $s \in (0,1) \setminus \{\frac{1}{2}\}$, *we have*

$$\|\Delta_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C},y^\alpha)} + \|\partial_y \nabla_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C},y^\alpha)} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}, \tag{3.27}$$

$$\|\mathscr{U}_{yy}\|_{L^2(\mathcal{C},y^\beta)} \lesssim \|f\|_{L^2(\Omega)}, \tag{3.28}$$

with $\beta > 2\alpha + 1$. *For the special case* $s = \frac{1}{2}$, *we obtain*

$$\|\mathscr{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1/2}(\Omega)}.$$

Remark 3.4 (Compatibility of f) It is possible to interpret the result of Theorem 3.3 as follows. Consider $s \in (\frac{1}{2}, 1)$, or equivalently $\alpha \in (-1, 0)$. Then the conormal exterior derivative condition for $\mathscr{U}$ gives us that $\mathscr{U}_y \approx -d_s y^{-\alpha} f$ as $y \approx 0^+$ on $\Omega \times \{0\}$, which in turn implies that $\mathscr{U}_y \to 0$ as $y \to 0^+$ on $\Omega \times \{0\}$. This is compatible with $\mathscr{U} = 0$ on $\partial_L \mathcal{C}$ since this implies $\mathscr{U}_y = 0$ on $\partial_L \mathcal{C}$. Consequently, we do not need any compatibility condition on the data $f \in H^{1-s}(\Omega)$ to avoid a jump on the derivative $\mathscr{U}_y$. On the other hand, when $\alpha \in (0,1)$, we have that, for a general f, $\mathscr{U}_y \nrightarrow 0$ as $y \to 0^+$ on $\Omega \times \{0\}$. To compensate this behavior we need the data f to vanish at the boundary $\partial\Omega$ at a certain rate. This condition is expressed by the requirement $f \in H_0^{1-s}(\Omega)$.

Proof of Theorem 3.3. Let us first consider $s = \frac{1}{2}$. In this case (3.16) reduces to the Poisson problem with mixed boundary conditions. In general, the solution of a mixed boundary value problem is not smooth, even for C^∞ data. The singular behavior occurs near the points of intersection between the Dirichlet and Neumann boundary. For instance, the solution $w = \sqrt{r}\sin(\theta/2)$ of $\Delta w = 0$ in $\mathbb{R}^2_+$, with $w_{x_2} = 0$ for $\{x_1 < 0,\ x_2 = 0\}$ and $w = 0$ for $\{x_1 \geq 0,\ x_2 = 0\}$ does not belong to $H^2(\mathbb{R}^2_+)$. To obtain more regular solutions, a compatibility condition between the data, the operator and the boundary must be imposed (see, for instance, [141]). Since in our case we have the representation (3.14), we can explicitly compute the second derivatives and, using that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $\{\varphi_k/\sqrt{\lambda_k}\}_{k\in\mathbb{N}}$ of $H^1_0(\Omega)$, it is not difficult to show that $f \in H^{1/2}_{00}(\Omega)$ implies $\mathscr{U} \in H^2(\mathcal{C})$, and $\|\mathscr{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{H^{1/2}_{00}(\Omega)}$.

In the general case $s \in (0,1)\setminus\{\frac{1}{2}\}$, i.e., $\alpha \in (-1,1)\setminus\{0\}$, using (3.23) as well as the asymptotic properties (3.21) and (3.22), we obtain

$$
\begin{aligned}
\|\Delta_{x'}\mathscr{U}\|^2_{L^2(\mathcal{C},y^\alpha)} + \|\partial_y\nabla_{x'}\mathscr{U}\|^2_{L^2(\mathcal{C},y^\alpha)} &= \sum_{k=1}^{\infty} u_k^2\lambda_k \int_0^\infty y^\alpha \left(\lambda_k\psi_k(y)^2 + \psi_k'(y)^2\right) \mathrm{d}y \\
&= d_s \sum_{k=1}^{\infty} u_k^2 \lambda_k^{1+s} = d_s \sum_{k=1}^{\infty} f_k^2 \lambda_k^{1-s} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)},
\end{aligned}
$$

which is exactly the regularity estimate given in (3.27). To obtain the regularity estimate on $\mathscr{U}_{yy}$ we, again, use the exact representation (3.14) and properties of Bessel functions to conclude that any derivative with respect to the extended variable y is smooth away from the Neumann boundary $\Omega\times\{0\}$. By virtue of (3.15) we deduce that the following partial differential equation holds in the strong sense

$$
\operatorname{div}(y^\alpha \nabla \mathscr{U}) = 0 \iff \mathscr{U}_{yy} = -\Delta_{x'}\mathscr{U} - \frac{\alpha}{y}\mathscr{U}_y. \tag{3.29}
$$

Consider sequences $\{a_k = 1/\sqrt{\lambda_k}\}_{k\geq 1}$, $\{b_k\}_{k\geq 1}$ and $\{\delta_k\}_{k\geq 1}$ with $0 < \delta_k \leq a_k \leq b_k$. Using (3.14) we have, for $k \geq 1$,

$$
\|\mathscr{U}_{yy}\|^2_{L^2(\mathcal{C},y^\beta)} = \sum_{k=1}^{\infty} u_k^2 \left(\lim_{\delta_k\downarrow 0} \int_{\delta_k}^{a_k} y^\beta |\psi_k''(y)|^2 \,\mathrm{d}y + \lim_{b_k\uparrow\infty} \int_{a_k}^{b_k} y^\beta |\psi_k''(y)|^2 \,\mathrm{d}y \right) \tag{3.30}
$$

Let us now estimate the first integral on the right hand side of (3.30). Formulas

(3.20) and (3.18) yield

$$\begin{aligned}\lim_{\delta_k \downarrow 0} \int_{\delta_k}^{a_k} y^\beta |\psi_k''(y)|^2 \, \mathrm{d}y &= c_s^2 \lambda_k^{2-\beta/2-1/2} \lim_{\delta_k \downarrow 0} \int_{\sqrt{\lambda_k}\delta_k}^{1} z^\beta \left| \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(z^s K_s(z) \right) \right|^2 \, \mathrm{d}z \\ &\lesssim c_s^2 \lambda_k^{2-\beta/2-1/2} \lim_{\delta_k \downarrow 0} \int_{\sqrt{\lambda_k}\delta_k}^{1} z^{\beta-2-2\alpha} \, \mathrm{d}z \approx \lambda_k^{2-\beta/2-1/2}\end{aligned} \tag{3.31}$$

where the integral converges because $\beta > 2\alpha + 1$. Let us now look at the second integral. Using property (v) of the modified Bessel functions, we have

$$\begin{aligned}\lim_{b_k \uparrow \infty} \int_{a_k}^{b_k} y^\beta |\psi_k''(y)|^2 \, \mathrm{d}y &= c_s^2 \lambda_k^{2-\beta/2-1/2} \lim_{b_k \uparrow \infty} \int_{1}^{\sqrt{\lambda_k} b_k} z^\beta \left| \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(z^s K_s(z) \right) \right|^2 \, \mathrm{d}z \\ &\lesssim c_s^2 \lambda_k^{2-\beta/2-1/2}.\end{aligned} \tag{3.32}$$

Replacing (3.31) and (3.32) into (3.30), and using that $u_k = \lambda_k^{-s} f_k$, we deduce

$$\|\mathscr{U}_{yy}\|^2_{L^2(\mathcal{C}, y^\beta)} \lesssim \sum_{k=1}^{\infty} \lambda_k^{2-\beta/2-1/2-2s} f_k^2 \leq \|f\|^2_{L^2(\Omega)},$$

because $2 - 2s - \frac{\beta}{2} - \frac{1}{2} = \frac{1}{2}(1 + 2\alpha - \beta) < 0$. This concludes the proof. $\square$

For the design of graded meshes later in § 3.4.2 we also need the following local regularity result in the extended variable.

Theorem 3.5 (Local regularity of the α-harmonic extension) *Let $\mathcal{C}(a,b) := \Omega \times (a,b)$ for $0 \leq a < b \leq 1$. The solution $\mathscr{U} \in \mathring{H}_L^1(\mathcal{C}, y^\alpha)$ of (3.16) satisfies for all a, b*

$$\|\Delta_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C}(a,b), y^\alpha)} + \|\partial_y \nabla_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C}(a,b), y^\alpha)} \lesssim (b-a) \, \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}, \tag{3.33}$$

and, with $\delta := \beta - 2\alpha - 1 > 0$,

$$\|\mathscr{U}_{yy}\|^2_{L^2(\mathcal{C}(a,b), y^\beta)} \lesssim \left(b^\delta - a^\delta \right) \|f\|^2_{L^2(\Omega)}. \tag{3.34}$$

Proof: To derive (3.33) we proceed as in Theorem 3.3. Since $0 \leq a < b \leq 1$, property (iii) of § 3.1.4, together with (3.21) imply that

$$|y^\alpha \psi_k(y)\psi_k'(y)| \lesssim \lambda_k^s.$$

This, together with (3.23) and the property $u_k = \lambda_k^{-s} f_k$, allows us to conclude

$$\begin{aligned}\|\Delta_{x'}\mathscr{U}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} + \|\partial_y \nabla_{x'}\mathscr{U}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} &= \sum_{k=1}^{\infty} u_k^2 \lambda_k \int_a^b y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2\right) \mathrm{d}y \\ &\lesssim (b-a)\sum_{k=1}^{\infty} u_k^2 \lambda_k^{1+s} = (b-a)\|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.\end{aligned}$$

To prove (3.34) we observe that the same argument used in (3.31) gives

$$\int_a^b y^\beta \left|\psi_k''(y)\right|^2 \mathrm{d}y \lesssim \lambda_k^{2-\beta/2-1/2}\left(b^\delta - a^\delta\right),$$

whence

$$\|\mathscr{U}_{yy}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} \lesssim \left(b^\delta - a^\delta\right) \sum_{k=1}^{\infty} f_k^2 \lambda_k^{2-\beta/2-1/2-2s} \lesssim \left(b^\delta - a^\delta\right) \|f\|^2_{L^2(\Omega)},$$

because $2 - 2s - \frac{\beta}{2} - \frac{1}{2} < 0$. □

Remark 3.6 (Domain and data regularity) The results of Theorem 3.3 and Theorem 3.5 are meaningful only if $f \in \mathbb{H}^{1-s}(\Omega)$ and the domain Ω is such that

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta_{x'} w\|_{L^2(\Omega)}, \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega),$$

holds. In the analysis that follows we will, without explicit mention, make this assumption. Let us, however, remark that our method works even when these conditions are not satisfied. We refer to § 3.5.3 for an illustration of that case.

3.2 Truncation

The solution $\mathscr{U}$ of problem (3.16) is defined on the infinite domain $\mathcal{C}$ and, consequently, it cannot be directly approximated with finite element-like techniques. In this section we will show that $\mathscr{U}$ decays sufficiently fast – in fact exponentially – in the extended direction. This suggests truncating the cylinder $\mathcal{C}$ to $\mathcal{C}_{\mathcal{Y}}$, for a suitably defined $\mathcal{Y}$. The exponential decay is the content of the next result.

Proposition 3.4 (Exponential decay) *For every $\mathcal{Y} \geq 1$, the solution $\mathscr{U}$ of* (3.16) *satisfies*

$$\|\nabla \mathscr{U}\|_{L^2(\Omega\times(\mathcal{Y},\infty),y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2}\|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.35}$$

Proof: Recall that if $u \in \mathbb{H}^s(\Omega)$ has the decomposition $u = \sum_k u_k \varphi_k(x')$, the solution $\mathscr{U} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ to (3.16) has the representation $\mathscr{U} = \sum_k u_k \varphi(x')\psi_k(y)$, where the functions ψ_k solve (3.15).

Consider $s = \frac{1}{2}$. In this case $\psi_k(y) = e^{-\sqrt{\lambda_k}y}$. Using the fact that $\{\varphi_k\}_{k=1}^\infty$ are eigenfunctions of Dirichlet Laplacian on Ω, orthonormal in $L^2(\Omega)$ and orthogonal in $H^1_0(\Omega)$ with $\|\nabla_{x'}\varphi_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, we get

$$\int_{\mathcal{Y}}^\infty\int_\Omega |\nabla\mathscr{U}|^2 = \int_{\mathcal{Y}}^\infty\int_\Omega \left(|\nabla_{x'}\mathscr{U}|^2 + |\partial_y\mathscr{U}|^2\right) = \sum_{k=1}^\infty \lambda_k^{\frac{1}{2}}|u_k|^2 e^{-2\sqrt{\lambda_k}\mathcal{Y}} \leq e^{-2\sqrt{\lambda_1}\mathcal{Y}}\|u\|^2_{\mathbb{H}^{1/2}(\Omega)}.$$

Since $\|u\|_{\mathbb{H}^{1/2}(\Omega)} = \|f\|_{\mathbb{H}^{-1/2}(\Omega)}$, this implies (3.35).

Consider now $s \in (0,1)\setminus\{\frac{1}{2}\}$ and $\psi_k(y) = c_s\left(\sqrt{\lambda_k}y\right)^s K_s(\sqrt{\lambda_k}y)$. To be able to argue as before, we need the estimates on K_s and its derivative for sufficiently large arguments discussed in § 3.1.4. In fact, using (3.22) and (3.23), we obtain

$$
\begin{aligned}
\int_{\mathscr{Y}}^{\infty} \int_{\Omega} y^{\alpha} |\nabla \mathscr{U}|^2 &= \int_{\mathscr{Y}}^{\infty} y^{\alpha} \int_{\Omega} \left(|\nabla_{x'} \mathscr{U}|^2 + |\partial_y \mathscr{U}|^2 \right) \\
&= \sum_{k=1}^{\infty} |u_k|^2 \int_{\mathscr{Y}}^{\infty} y^{\alpha} \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2 \right) \, \mathrm{d}y \\
&= \sum_{k=1}^{\infty} |u_k|^2 y^{\alpha} \psi_k(y) \psi_k'(y) \Big|_{\mathscr{Y}}^{\infty} \lesssim e^{-\sqrt{\lambda_1}\mathscr{Y}} \|u\|_{\mathbb{H}^s(\Omega)}^2 .
\end{aligned}
$$

Again, since $\|u\|_{\mathbb{H}^s(\Omega)} = \|f\|_{\mathbb{H}^{-s}(\Omega)}$ we get (3.35). $\square$

Expression (3.35) motivates the approximation of $\mathscr{U}$ by a function v that solves

$$
\begin{cases}
\operatorname{div}(y^{\alpha} \nabla v) = 0, & \text{in } \mathcal{C}_{\mathscr{Y}}, \\
v = 0, & \text{on } \partial_L \mathcal{C}_{\mathscr{Y}} \cup \Omega \times \{\mathscr{Y}\}, \\
\dfrac{\partial v}{\partial \nu^{\alpha}} = d_s f, & \text{on } \Omega \times \{0\},
\end{cases}
\tag{3.36}
$$

with $\mathscr{Y}$ sufficiently large. Problem (3.36) is understood in the weak sense, i.e., we define the space

$$
\mathring{H}_L^1(\mathcal{C}_{\mathscr{Y}}, y^{\alpha}) = \left\{ v \in H^1(\mathcal{C}, y^{\alpha}) : v = 0 \text{ on } \partial_L \mathcal{C}_{\mathscr{Y}} \cup \Omega \times \{\mathscr{Y}\} \right\},
$$

and seek for $v \in \mathring{H}_L^1(\mathcal{C}_{\mathscr{Y}}, y^{\alpha})$ such that

$$
\int_{\mathcal{C}_{\mathscr{Y}}} y^{\alpha} \nabla v \cdot \nabla \phi = d_s \langle f, \operatorname{tr}_{\Omega} \phi \rangle, \quad \forall \phi \in \mathring{H}_L^1(\mathcal{C}_{\mathscr{Y}}, y^{\alpha}). \tag{3.37}
$$

Existence and uniqueness of v follows from the Lax-Milgram lemma.

Remark 3.7 (Zero extension) For every $\mathscr{Y} > 0$ we have the embedding

$$
\mathring{H}_L^1(\mathcal{C}_{\mathscr{Y}}, y^{\alpha}) \hookrightarrow \mathring{H}_L^1(\mathcal{C}, y^{\alpha}). \tag{3.38}
$$

To see this, it suffices to consider the extension by zero for $y > \mathscr{Y}$.

The next result shows the approximation properties of v, solution of (3.37) in $\mathcal{C}_{\mathscr{Y}}$.

Lemma 3.8 (Exponential convergence in $\mathscr{Y}$) *For any positive $\mathscr{Y} \geq 1$, we have*

$$\|\nabla(\mathscr{U} - v)\|_{L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathscr{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.39}$$

Proof: Given $\phi \in \mathring{H}^1_L(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$ denote by ϕ_e its extension by zero to $\mathcal{C}$. By Remark 3.7, $\phi_e \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$. Take ϕ_e and ϕ as test functions in (3.16) and (3.37), respectively. Subtract the resulting expressions to obtain

$$\int_{\mathcal{C}_{\mathscr{Y}}} y^\alpha (\nabla \mathscr{U} - \nabla v) \cdot \nabla \phi = 0 \quad \forall \phi \in \mathring{H}^1_L(\mathcal{C}_{\mathscr{Y}}, y^\alpha),$$

which implies that v is the best approximation of $\mathscr{U}$ in $\mathring{H}^1_L(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$, i.e.,

$$\|\nabla(\mathscr{U} - v)\|_{L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)} = \inf_{\phi \in \mathring{H}^1_L(\mathcal{C}_{\mathscr{Y}}, y^\alpha)} \|\nabla(\mathscr{U} - \phi)\|_{L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)}. \tag{3.40}$$

Let us construct explicitly a function $\phi_0 \in \mathring{H}^1_L(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$ to use in (3.40). Define

$$\rho(y) = \begin{cases} 1, & 0 \leq y \leq \mathscr{Y}/2, \\ \dfrac{2}{\mathscr{Y}}(\mathscr{Y} - y), & \mathscr{Y}/2 < y < \mathscr{Y}, \\ 0, & y \geq \mathscr{Y}. \end{cases} \tag{3.41}$$

Notice that $\rho \in W^1_\infty(0, \infty)$, $|\rho(y)| \leq 1$ and $|\rho'(y)| \leq 2/\mathscr{Y}$ for all $y > 0$. Set $\phi_0(x', y) = \mathscr{U}(x', y)\rho(y)$ for $x' \in \Omega$ and $y > 0$. A straightforward computation shows

$$|\nabla\left((1-\rho)\mathscr{U}\right)|^2 \leq 2\left(|\rho'|^2|\mathscr{U}|^2 + (1-\rho)^2|\nabla\mathscr{U}|^2\right) \leq 2\left(\frac{4}{\mathscr{Y}^2}\mathscr{U}^2 + |\nabla\mathscr{U}|^2\right),$$

so that

$$\|\nabla(\mathscr{U} - \phi_0)\|^2_{L^2(\mathcal{C}_\mathscr{Y}, y^\alpha)} \leq 2 \left(\frac{4}{\mathscr{Y}^2} \int_{\mathscr{Y}/2}^{\mathscr{Y}} \int_\Omega y^\alpha |\mathscr{U}|^2 + \int_{\mathscr{Y}/2}^{\mathscr{Y}} \int_\Omega y^\alpha |\nabla \mathscr{U}|^2 \right). \tag{3.42}$$

To estimate the first term on the right hand side of (3.42) we use the Poincaré inequality (3.10) over a dyadic partition that covers the interval $[\mathscr{Y}/2, \mathscr{Y}]$ (see the derivation of (3.11) in § 3.1.2), to obtain

$$\int_{\mathscr{Y}/2}^{\mathscr{Y}} y^\alpha \int_\Omega |\mathscr{U}|^2 \lesssim \int_{\mathscr{Y}/2}^{\mathscr{Y}} y^\alpha \int_\Omega |\nabla \mathscr{U}|^2. \tag{3.43}$$

To bound the second integral in (3.42) we use (3.23) as in the proof of Proposition 3.4:

$$\int_{\mathscr{Y}/2}^{\mathscr{Y}} y^\alpha \int_\Omega |\nabla \mathscr{U}|^2 = \sum_{k=1}^\infty |u_k|^2 y^\alpha \psi_k(y) \psi_k'(y) \Big|_{\mathscr{Y}/2}^{\mathscr{Y}} \lesssim e^{-\sqrt{\lambda_1}\mathscr{Y}/2} \|f\|^2_{\mathbb{H}^{-s}(\Omega)}.$$

Inserting these estimates into (3.40) implies (3.39). □

The following result is a direct consequence of Lemma 3.8.

Remark 3.9 (Stability) Let $\mathscr{Y} \geq 1$, then

$$\|\nabla v\|_{L^2(\mathcal{C}_\mathscr{Y}, y^\alpha)} \lesssim \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.44}$$

Indeed, by the triangle inequality

$$\|\nabla v\|_{L^2(\mathcal{C}_\mathscr{Y}, y^\alpha)} \leq \|\nabla(v - \mathscr{U})\|_{L^2(\mathcal{C}_\mathscr{Y}, y^\alpha)} + \|\nabla \mathscr{U}\|_{L^2(\mathcal{C}_\mathscr{Y}, y^\alpha)} \lesssim \left(e^{-\sqrt{\lambda_1}\mathscr{Y}/4} + 1\right) \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

The previous two results allow us to show a full approximation estimate.

Theorem 3.10 (Global exponential estimate) *Let $\mathscr{Y} \geq 1$, then*

$$\|\nabla(\mathscr{U} - v)\|_{L^2(\mathcal{C}, y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathscr{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.45}$$

In particular, for every $\epsilon > 0$, let

$$\mathscr{Y}_0 = \frac{2}{\sqrt{\lambda_1}} \left(\log C + 2 \log \frac{1}{\epsilon} \right),$$

where C depends only on s and Ω. Then, for $\mathscr{Y} \geq \max\{\mathscr{Y}_0, 1\}$, we have

$$\|\nabla(\mathscr{U} - v)\|_{L^2(\mathcal{C}, y^\alpha)} \leq \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.46}$$

Proof: Extending v by zero outside of $\mathcal{C}_{\mathscr{Y}}$ we obtain

$$\|\nabla(\mathscr{U} - v)\|^2_{L^2(\mathcal{C}, y^\alpha)} = \|\nabla(\mathscr{U} - v)\|^2_{L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)} + \|\nabla \mathscr{U}\|^2_{L^2(\Omega \times (\mathscr{Y}, \infty), y^\alpha)}.$$

Hence Lemma 3.8 and Proposition 3.4 imply

$$\|\nabla(\mathscr{U} - v)\|^2_{L^2(\mathcal{C}, y^\alpha)} \leq C e^{-\sqrt{\lambda_1}\mathscr{Y}/2} \|f\|^2_{\mathbb{H}^{-s}(\Omega)} \leq \epsilon^2 \|f\|^2_{\mathbb{H}^{-s}(\Omega)}, \tag{3.47}$$

for all $\mathscr{Y} \geq \max\{\mathscr{Y}_0, 1\}$. $\square$

3.3 Finite element discretization and interpolation estimates

In this section we prove error estimates for a piecewise $\mathbb{Q}_1$ interpolation operator on anisotropic elements in the extended variable y. We consider elements of the form $T = K \times I$, where $K \subset \mathbb{R}^n$ is an element isoparametrically equivalent to the unit cube $[0,1]^n$, via a $\mathbb{Q}_1$ mapping and, $I \subset \mathbb{R}$ is an interval. The anisotropic character of the mesh $\mathscr{T}_{\mathscr{Y}} = \{T\}$ will be given by the family of intervals I.

The error estimates are derived in the weighted Sobolev spaces $L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$ and $H^1(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$, and they are valid under the condition that neighboring elements have comparable size in the extended $(n+1)$–dimension (see [70]). This is a mild assumption that includes general meshes which do not satisfy the so-called shape-regularity assumption, i.e., mesh refinements for which the quotient between outer and inner diameter of the elements does not remain bounded (see [39, Chapter 4]).

Anisotropic or narrow elements are elements with disparate sizes in each direction. They arise naturally when approximating solutions of problems with a strong directional-dependent behavior since, using anisotropy, the local mesh size can be adapted to capture such features. Examples of this include boundary layers, shocks and edge singularities (see [70, 71]). In our problem, anisotropic elements are essential in order to capture the singular/degenerate behavior of the solution $\mathscr{U}$ to problem (3.16) at $y \approx 0^+$ given in (3.24). These elements will provide optimal error estimates, which cannot be obtained using shape-regular elements.

Error estimates for weighted Sobolev spaces have been obtained in several works; see, for instance, [8, 21, 70]. The type of weight considered in [8, 21] is related to the distance to a point or an edge, and the type of quasi-interpolators are modifications of the well known Clément [58] and Scott-Zhang [143] operators. These works are developed in 3D and 2D respectively, and the analysis developed in [8] allows for anisotropy. Our approach follows the work of Durán and Lombardi [70], and is based on a piecewise $\mathbb{Q}_1$ averaged interpolator on anisotropic elements. It allows us to obtain anisotropic interpolation estimates in the extended variable y and in weighted Sobolev spaces, using only that $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$, the Muckenhoupt class A_2 of Definition 2.2. We develop a general interpolation theory for weights of class A_p with $1 < p < \infty$ in Chapter 4; see also [132].

3.3.1 Finite element discretization

Let us now describe the discretization of problem (3.36). To avoid technical difficulties we assume that the boundary of Ω is polygonal. The difficulties inherent to curved boundaries could be handled, for instance, with the methods of [23] (see also [106, 107]). Let $\mathscr{T}_\Omega = \{K\}$ be a mesh of Ω made of isoparametric quadrilaterals K in the sense of Ciarlet [56] and Ciarlet and Raviart [57]. In other words, given $\hat{K} = [0,1]^n$ and a family of mappings $\{\mathcal{F}_K \in \mathbb{Q}_1(\hat{K})^n\}$ we have

$$K = \mathcal{F}_K(\hat{K}) \tag{3.48}$$

and

$$\bar{\Omega} = \bigcup_{K \in \mathscr{T}_\Omega} K, \qquad |\Omega| = \sum_{K \in \mathscr{T}_\Omega} |K|.$$

The collection of triangulations is denoted by $\mathbb{T}_\Omega$.

The mesh $\mathscr{T}_\Omega$ is assumed to be conforming or compatible, i.e., the intersection of any two isoparametric elements K and K' in $\mathscr{T}_\Omega$ is either empty or a common lower dimensional isoparametric element.

In addition, we assume that $\mathscr{T}_\Omega$ is shape regular (cf. [56, Chapter 4.3]). This means that $\mathcal{F}_K$ can be decomposed as $\mathcal{F}_K = \mathcal{A}_K + \mathcal{B}_K$, where $\mathcal{A}_K$ is affine and $\mathcal{B}_K$ is a perturbation map and, if we define $\tilde{K} = \mathcal{A}_K(\hat{K})$, $h_K = \mathrm{diam}(\tilde{K})$, ρ_K as the diameter of the largest sphere inscribed in $\tilde{K}$ and the shape coefficient of K as the ratio $\sigma_K = h_K/\rho_K$, then the following two conditions are satisfied:

(a) There exists a constant $\sigma_\Omega > 1$ such that for all $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$,

$$\max\{\sigma_K : K \in \mathscr{T}_\Omega\} \leq \sigma_\Omega.$$

(b) For all $K \in \mathscr{T}_\Omega$ the mapping $\mathcal{B}_K$ is Fréchet differentiable and

$$\|D\mathcal{B}_K\|_{L^\infty(\hat{K})} = \mathcal{O}(h_K^2),$$

for all $K \in \mathscr{T}_\Omega$ and all $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$.

As a consequence of these conditions, if h_K is small enough, the mapping $\mathcal{F}_K$ is one-to-one, its Jacobian $J_{\mathcal{F}_K}$ does not vanish, and

$$J_{\mathcal{F}_K} \lesssim h_K^n, \quad \|D\mathcal{F}_K\|_{L^\infty(\hat{K})} \lesssim h_K. \tag{3.49}$$

The set $\mathbb{T}_\Omega$ is called quasi-uniform if for all $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$,

$$\max\{\rho_K : K \in \mathscr{T}_\Omega\} \lesssim \min\{h_K : K \in \mathscr{T}_\Omega\}.$$

In this case, we define $h_{\mathscr{T}_\Omega} = \max_{K\in\mathscr{T}} h_K$.

We define $\mathscr{T}_\mathscr{Y}$ as a triangulation of $\mathcal{C}_\mathscr{Y}$ into cells of the form $T = K \times I$, where $K \in \mathscr{T}_\Omega$, and I denotes an interval in the extended dimension. Notice that each discretization of the truncated cylinder $\mathcal{C}_\mathscr{Y}$ depends on the truncation parameter $\mathscr{Y}$. The set of all such triangulations is denoted by $\mathbb{T}$. In order to obtain a global regularity assumption for $\mathbb{T}$ we assume the aforementioned conditions on $\mathbb{T}_\Omega$, besides the following weak regularity condition:

(c) There is a constant σ such that, for all $\mathscr{T}_\mathscr{Y} \in \mathbb{T}$, if $T_1 = K_1\times I_1, T_2 = K_2\times I_2 \in \mathscr{T}_\mathscr{Y}$ have nonempty intersection, then

$$\frac{h_{I_1}}{h_{I_2}} \le \sigma,$$

where $h_I = |I|$.

Notice that the assumptions imposed on $\mathbb{T}$ are weaker than the standard shape-regularity assumptions, since they allow for anisotropy in the extended variable (cf. [70]). It is also important to notice that, given the Cartesian product structure of the cells $T \in \mathscr{T}_{\mathcal{Y}}$, they are isoparametrically equivalent to $\hat{T} = [0,1]^{n+1}$. We will denote the corresponding mappings by $\mathcal{F}_T$. Then,

$$\mathcal{F}_T : \hat{x} = (\hat{x}', \hat{y}) \in \hat{T} \longmapsto x = (x', y) = (\mathcal{F}_K(\hat{x}'), \mathcal{F}_I(\hat{y})) \in T = K \times I,$$

where $\mathcal{F}_K$ is the bilinear mapping defined in (3.48) for K and, if $I = (c, d)$, $\mathcal{F}_I(y) = (y - c)/(d - c)$. From (3.49), we immediately conclude that

$$J_{\mathcal{F}_T} \lesssim h_K^n h_I, \quad \|D\mathcal{F}_T\|_{L^\infty(\hat{T})} \lesssim h_T, \tag{3.50}$$

for all elements $T \in \mathscr{T}_{\mathcal{Y}}$ where $h_T = \max\{h_K, h_I\}$.

Given $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$, we define the finite element space $\mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ by

$$\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in \mathcal{C}^0(\overline{\mathcal{C}_{\mathcal{Y}}}) : W|_T \in \mathbb{Q}_1(T) \ \forall T \in \mathscr{T}_{\mathcal{Y}},\ W|_{\Gamma_D} = 0 \right\}.$$

where $\Gamma_D = \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\}$ is called the Dirichlet boundary. The Galerkin approximation of (3.37) is given by the unique function $V_{\mathscr{T}_{\mathcal{Y}}} \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ such that

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \cdot \nabla W = d_s \langle f, \mathrm{tr}_\Omega W \rangle, \quad \forall W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}}). \tag{3.51}$$

Existence and uniqueness of $V_{\mathscr{T}_{\mathcal{Y}}}$ follows from $\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) \subset \mathring{H}^1_L(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$ and the Lax-Milgram lemma.

We define the space $\mathbb{U}(\mathscr{T}_\Omega) = \mathrm{tr}_\Omega\, \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, which is nothing more than a $\mathbb{Q}_1$ finite element space over the mesh $\mathscr{T}_\Omega$. The finite element approximation of $u \in \mathbb{H}^s(\Omega)$, solution of (1.1), is then given by

$$U_{\mathscr{T}_\Omega} = \mathrm{tr}_\Omega\, V_{\mathscr{T}_{\mathcal{Y}}} \in \mathbb{U}(\mathscr{T}_\Omega), \tag{3.52}$$

and we have the following result.

Theorem 3.11 (Energy error estimate) *Let v solve* (3.37) *with* $\mathscr{Y} \geq \max\{\mathscr{Y}_0, 1\}$. *If* $V_{\mathscr{T}_\mathscr{Y}} \in \mathbb{V}(\mathscr{T}_\mathscr{Y})$ *solves* (3.51) *and* $U_{\mathscr{T}_\Omega} \in \mathbb{U}(\mathscr{T}_\Omega)$ *is defined in* (3.52), *then we have*

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathscr{U} - V_{\mathscr{T}_\mathscr{Y}}\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)}, \tag{3.53}$$

and

$$\|\mathscr{U} - V_{\mathscr{T}_\mathscr{Y}}\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)} \lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|v - V_{\mathscr{T}_\mathscr{Y}}\|_{\mathring{H}^1_L(\mathcal{C}_\mathscr{Y}, y^\alpha)}. \tag{3.54}$$

Proof: Estimate (3.53) is just an application of the trace estimate of Proposition 3.3. Inequality (3.54) is obtained by the triangle inequality and (3.46). $\square$

By Galerkin orthogonality

$$\|v - V_{\mathscr{T}_\mathscr{Y}}\|_{\mathring{H}^1_L(\mathcal{C}_\mathscr{Y}, y^\alpha)} = \inf_{W \in \mathbb{V}(\mathscr{T}_\mathscr{Y})} \|v - W\|_{\mathring{H}^1_L(\mathcal{C}_\mathscr{Y}, y^\alpha)}.$$

Theorem 3.11 and Galerkin orthogonality imply that the approximation estimate (3.54) depends on the regularity of $\mathscr{U}$. To see this we introduce

$$\rho(y) = \begin{cases} 1, & 0 \leq y < \mathscr{Y}/2, \\ p, & \mathscr{Y}/2 \leq y \leq \mathscr{Y}, \end{cases} \tag{3.55}$$

where p is the unique cubic polynomial on $[\mathscr{Y}/2, \mathscr{Y}]$ defined by the conditions $p(\mathscr{Y}/2) = 1$, $p(\mathscr{Y}) = 0$, $p'(\mathscr{Y}/2) = 0$ and $p'(\mathscr{Y}) = 0$. Notice that $\rho \in W^2_\infty(0, \mathscr{Y})$, $|\rho(y)| \leq 1$, $|\rho'(y)| \lesssim 1$ and $|\rho''(y)| \lesssim 1$. Set $\mathscr{U}_0(x', y) = \rho(y)\mathscr{U}(x', y)$ for $x' \in \Omega$ and $y \in [0, \mathscr{Y}]$, and notice that $\mathscr{U}_0 \in \mathring{H}^1_L(\mathcal{C}_\mathscr{Y}, y^\alpha)$. With this construction at hand, repeating the arguments used in the proof of Lemma 3.8, we have that

$$\begin{aligned}
\|\Delta_{x'} \mathscr{U}_0\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} &\lesssim \|\Delta_{x'} \mathscr{U}\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)}, \\
\|\partial_y \nabla_{x'} \mathscr{U}_0\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} &\lesssim \|\partial_y \nabla_{x'} \mathscr{U}\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} + \|f\|_{\mathbb{H}^{-s}(\Omega)}, \\
\|\partial_{yy} \mathscr{U}_0\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\beta)} &\lesssim \|\partial_{yy} \mathscr{U}\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\beta)} + \|f\|_{\mathbb{H}^{-s}(\Omega)}.
\end{aligned} \tag{3.56}$$

In addition, if we assume that there is an operator

$$\Pi_{\mathscr{T}_{\mathcal{Y}}} : \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha) \to \mathbb{V}(\mathscr{T}_{\mathcal{Y}}),$$

that is stable, i.e., $\|\Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} \lesssim \|w\|_{\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)}$, for all $w \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$, then the following estimate holds

$$\|\mathscr{U} - V_{\mathscr{T}_{\mathcal{Y}}}\|_{\mathring{H}_L^1(\mathcal{C}, y^\alpha)} \lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|\mathscr{U}_0 - \Pi_{\mathscr{T}_{\mathcal{Y}}} \mathscr{U}_0\|_{\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)}. \tag{3.57}$$

To see this, we use (3.54), together with Galerkin orthogonality and the stability of the operator $\Pi_{\mathscr{T}_{\mathcal{Y}}}$, to obtain

$$\begin{aligned}
\|\mathscr{U} - V_{\mathscr{T}_{\mathcal{Y}}}\|_{\mathring{H}_L^1(\mathcal{C}, y^\alpha)} &\lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|v - \Pi_{\mathscr{T}_{\mathcal{Y}}} v\|_{\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} \\
&\lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|v - \mathscr{U}_0\|_{\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} + \|\mathscr{U}_0 - \Pi_{\mathscr{T}_{\mathcal{Y}}} \mathscr{U}_0\|_{\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)}.
\end{aligned}$$

The second term on the right hand side of the previous inequality is estimated as in Lemma 3.8. We leave the details to the reader.

Estimates for $\mathscr{U}_0 - \Pi_{\mathscr{T}_{\mathcal{Y}}} \mathscr{U}_0$ on weighted Sobolev spaces are derived in §3.3.2. Clearly, these depend on the regularity of $\mathscr{U}_0$ which, in light of (3.56), depends on the regularity of $\mathscr{U}$. For this reason, and to lighten the notation, we shall in the sequel write $\mathscr{U}$ and obtain interpolation error estimates for it, even though $\mathscr{U}$ does not vanish at $y = \mathcal{Y}$.

3.3.2 Interpolation estimates in weighted Sobolev spaces

Let us begin by introducing some notation and terminology. Given $\mathscr{T}_{\mathcal{Y}}$, we call $\mathcal{N}$ the set of its nodes and $\mathcal{N}_{\text{in}}$ the set of its interior and Neumann nodes. For each vertex $\mathtt{v} \in \mathcal{N}$, we write $\mathtt{v} = (\mathtt{v}', \mathtt{v}'')$, where $\mathtt{v}'$ corresponds to a node of $\mathscr{T}_{\Omega}$, and $\mathtt{v}''$ corresponds to a node of the discretization of the $n+1$–dimension. We define $h_{\mathtt{v}'} = \min\{h_K : \mathtt{v}' \text{ is a vertex of } K\}$, and $h_{\mathtt{v}''} = \min\{h_I : \mathtt{v}'' \text{ is a vertex of } I\}$.

Given $\mathtt{v} \in \mathcal{N}$, the *star* or patch around $\mathtt{v}$ is defined as

$$\omega_{\mathtt{v}} = \bigcup_{T \ni \mathtt{v}} T,$$

and for $T \in \mathscr{T}_{\mathcal{Y}}$ we define its *patch* as

$$\omega_T = \bigcup_{\mathtt{v} \in T} \omega_{\mathtt{v}}.$$

Let $\psi \in C^{\infty}(\mathbb{R}^{n+1})$ be such that $\int \psi = 1$ and $D := \operatorname{supp} \psi \subset B_r \times (0, r_{\mathcal{Y}})$, where B_r denotes the ball in $\mathbb{R}^n$ of radius r and centered at zero, and $r \leq 1/\sigma_{\Omega}$ and $r_{\mathcal{Y}} \leq 1/\sigma$. For $\mathtt{v} \in \mathcal{N}_{\text{in}}$, we rescale ψ as

$$\psi_{\mathtt{v}}(x) = \frac{1}{h_{\mathtt{v}'}^n h_{\mathtt{v}''}} \psi \left(\frac{x' - \mathtt{v}'}{h_{\mathtt{v}'}}, \frac{y - \mathtt{v}''}{h_{\mathtt{v}''}} \right),$$

and note that $\operatorname{supp} \psi_{\mathtt{v}} \subset \omega_{\mathtt{v}}$ and $\int_{\omega_{\mathtt{v}}} \psi_{\mathtt{v}} = 1$ for any interior and Neumann node $\mathtt{v}$.

Remark 3.12 (Boundary conditions of Neumann type) For an interior node $\mathtt{v}$, it would be natural to consider $B_r \times (-r_{\mathcal{Y}}, r_{\mathcal{Y}})$ as the support of the smooth function ψ. However, for a Neumann node $\mathtt{v}$, this choice would not provide the important properties $\operatorname{supp} \psi_{\mathtt{v}} \subset \omega_{\mathtt{v}}$ and $\int_{\omega_{\mathtt{v}}} \psi_{\mathtt{v}} = 1$. In order to treat both types of nodes indistinctly in the subsequent analysis, we have considered $\operatorname{supp}\psi \subset B_r \times (0, r_{\mathscr{Y}})$.

Given a function $w \in L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$ and a node $\mathtt{v}$ in $\mathcal{N}_{\text{in}}$ we define, following Durán and Lombardi [70], the regularized Taylor polynomial of first degree of w about $\mathtt{v}$ as

$$Q^1_{\mathtt{v}} w(z) = \int P^1(x,z)\psi_{\mathtt{v}}(x)\,\mathrm{d}x = \int_{\omega_{\mathtt{v}}} P^1(x,z)\psi_{\mathtt{v}}(x)\,\mathrm{d}x, \tag{3.58}$$

where P^1 denotes the Taylor polynomial of degree 1 in the variable z of the function w about the point x, i.e.,

$$P^1(x,z) = w(x) + \nabla w(x) \cdot (z - x). \tag{3.59}$$

As a consequence of Remark 3.1 and the fact that the averaged Taylor polynomial is defined for functions in $L^1(\mathcal{C}_{\mathscr{Y}})$ (cf. [39, Proposition 4.1.12]), we conclude that $Q^1_{\mathtt{v}}$ is well defined for any function in $L^2(\mathcal{C}_{\mathscr{Y}}, y^\alpha)$.

We define the averaged $\mathbb{Q}_1$ interpolant $\Pi_{\mathscr{T}_{\mathscr{Y}}} w$, as the unique piecewise $\mathbb{Q}_1$ function such that $\Pi_{\mathscr{T}_{\mathscr{Y}}} w(\mathtt{v}) = 0$ if $\mathtt{v}$ lies on the Dirichlet boundary Γ_D and $\Pi_{\mathscr{T}_{\mathscr{Y}}} w(\mathtt{v}) = Q^1_{\mathtt{v}} w(\mathtt{v})$ if $\mathtt{v} \in \mathcal{N}_{\text{in}}$. If $\phi_{\mathtt{v}}$ denotes the Lagrange basis function associated with node $\mathtt{v}$, then

$$\Pi_{\mathscr{T}_{\mathscr{Y}}} w = \sum_{\mathtt{v} \in \mathcal{N}_{\text{in}}} Q^1 w(\mathtt{v}) \phi_{\mathtt{v}}.$$

There are two principal reasons to consider averaged interpolation. First, we are interested in the approximation of singular functions and thus Lagrange interpolation cannot be used since point-wise values become meaningless. In fact, this motivated the introduction of averaged interpolation (see [58, 143]). In addition, averaged interpolation has better approximation properties when narrow elements are used (see [2]).

Finally, for $\mathtt{v} \in \mathcal{N}_{\text{in}}$, we define the weighted regularized average of w as

$$Q^0_{\mathtt{v}} w = \int w(x)\psi_{\mathtt{v}}(x)\,\mathrm{d}x = \int_{\omega_{\mathtt{v}}} w(x)\psi_{\mathtt{v}}(x)\,\mathrm{d}x. \tag{3.60}$$

3.3.2.1 Weighted Poincaré inequality

In order to obtain interpolation error estimates in $L^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$ and $H^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$, it is instrumental to have a weighted Poincaré-type inequality. Weighted Poincaré inequalities are particularly pertinent in the study of the nonlinear potential theory of degenerate elliptic equations, see [79, 102]. If the domain is a ball and the weight belongs to A_p, with $1 \leq p < \infty$, this result can be found in [79, Theorem 1.3 and Theorem 1.5]. However, to the best of our knowledge, such a result is not available in the literature for more general domains. For our specific weight we present here a constructive proof, i.e., not based on a compactness argument. This allows us to study the dependence of the constant on the domain.

Lemma 3.13 (Weighted Poincaré inequality I) *Let $\omega \subset \mathbb{R}^{n+1}$ be bounded, star-shaped with respect to a ball B, and $\operatorname{diam} \omega \approx 1$. Let $\chi \in C^0(\bar{\omega})$ with $\int_\omega \chi = 1$, and $\xi_\alpha(y) := |a|y| + b|^\alpha$ for $a, b \in \mathbb{R}$. If $w \in H^1(\omega, \xi_\alpha(y))$ is such that $\int_\omega \chi w = 0$, then*

$$\|w\|_{L^2(\omega,\xi_\alpha)} \lesssim \|\nabla w\|_{L^2(\omega,\xi_\alpha)}, \tag{3.61}$$

where the hidden constant depends only on χ, α and the radius r of B, but is independent of both a and b.

Proof: The fact that $\alpha \in (-1,1)$ implies $\xi_\alpha \in A_2(\mathbb{R}^{n+1})$ with a Muckenhoupt constant C_{2,ξ_α} in (2.10) uniform in both a and b. Define

$$\widetilde{w} = \xi_\alpha w - \left(\int_\omega \xi_\alpha w\right)\chi.$$

Clearly $\widetilde{w} \in L^1(\omega)$ and it has vanishing mean value by construction.

Since $\int_\omega \chi w = 0$ we obtain

$$\|w\|^2_{L^2(\omega,\xi_\alpha)} = \int_\omega w\widetilde{w} + \left(\int_\omega \xi_\alpha w\right)\int_\omega \chi w = \int_\omega w\widetilde{w}. \tag{3.62}$$

Consequently, given that ω is star shaped with respect to B, and $\xi_\alpha \in A_2(\mathbb{R}^{n+1})$, there exists $F \in H^1_0(\omega,\xi_\alpha)^{n+1}$ such that $-\text{div}F = \widetilde{w}$, and

$$\|F\|_{H^1_0(\omega,\xi_\alpha^{-1})^{n+1}} \lesssim \|\widetilde{w}\|_{L^2(\omega,\xi_\alpha^{-1})}, \tag{3.63}$$

where the hidden constant in (3.63) depends on r and the constant C_{2,ξ_α} from Definition 2.2 [72, Theorem 3.1].

Replacing $\widetilde{w}$ by $-\text{div}F$ in (3.62), integrating by parts and using (3.63), we get

$$\|w\|^2_{L^2(\omega,\xi_\alpha)} = -\int_\omega w\,\text{div}F = \int_\omega \nabla w \cdot F \lesssim \|\nabla w\|_{L^2(\omega,\xi_\alpha)}\|\widetilde{w}\|_{L^2(\omega,\xi_\alpha^{-1})}. \tag{3.64}$$

To estimate $\|\widetilde{w}\|_{L^2(\omega,\xi_\alpha^{-1})}$ we use the Cauchy-Schwarz inequality and the constant C_{2,ξ_α} from Definition 2.2 as follows:

$$\|\widetilde{w}\|^2_{L^2(\omega,\xi_\alpha^{-1})} \le 2\left(1 + \int_\omega \xi_\alpha \int_\omega \chi^2\xi_\alpha^{-1}\right)\|w\|^2_{L^2(\omega,\xi_\alpha)} \lesssim \|w\|^2_{L^2(\omega,\xi_\alpha)}.$$

Inserting the inequality above into (3.64), we obtain (3.61). □

We need a slightly more general form of the Poincaré inequality for the applications below. We now relax the geometric assumption on the domain ω and let the vanishing mean property hold just in a subdomain.

Corollary 3.14 (Weighted Poincaré inequality II) *Let* $\omega = \cup_{i=1}^{N} \omega_i \subset \mathbb{R}^{n+1}$ *be a connected domain and each* ω_i *be a star-shaped domain with respect to a ball* B_i*. Let* $\chi_i \in C^0(\bar{\omega}_i)$ *and* ξ_α *be as in Lemma 3.13. If* $w \in H^1(\omega, \xi_\alpha)$ *and* $w_i := \int_{\omega_i} w\chi_i$*, then*

$$\|w - w_i\|_{L^2(\omega,\xi_\alpha)} \lesssim \|\nabla w\|_{L^2(\omega,\xi_\alpha)} \qquad \forall 1 \leq i \leq N, \tag{3.65}$$

where the hidden constant depends on $\{\chi_i\}_{i=1}^{N}$*,* α*, the radius* r_i *of* B_i*, and the amount of overlap between the subdomains* $\{\omega_i\}_{i=1}^{N}$*, but is independent of both* a *and* b*.*

Proof: This is a consequence of Lemma 3.13 and [68, Theorem 7.1]. We sketch the proof here for completeness. It suffices to deal with two subdomains, ω_1, ω_2, and the overlapping region $B = \omega_1 \cap \omega_2$. We observe that

$$\|w - w_1\|_{L^2(\omega_2,\xi_\alpha)} \leq \|w - w_2\|_{L^2(\omega_2,\xi_\alpha)} + \|w_1 - w_2\|_{L^2(\omega_2,\xi_\alpha)},$$

together with $\|w_1 - w_2\|_{L^2(\omega_2,\xi_\alpha)} = \left(\frac{\int_{\omega_2} \xi_\alpha}{\int_B \xi_\alpha}\right)^{1/2} \|w_1 - w_2\|_{L^2(B,\xi_\alpha)}$ and

$$\|w_1 - w_2\|_{L^2(B,\xi_\alpha)} \lesssim \|w - w_1\|_{L^2(\omega_1,\xi_\alpha)} + \|w - w_2\|_{L^2(\omega_2,\xi_\alpha)},$$

imply $\|w - w_1\|_{L^2(\omega_2,\xi_\alpha)} \lesssim \|\nabla w\|_{L^2(\omega_1 \cup \omega_2,\xi_\alpha)}$. This, combined with (3.61), gives (3.65) for $i = 1$ with a stability constant depending on the ratio $\frac{\int_{\omega_2} \xi_\alpha}{\int_B \xi_\alpha}$. $\square$

3.3.2.2 Weighted L^2 interpolation estimates

Owing to the weighted Poincaré inequality of Corollary 3.14, we can adapt the proof of [70, Lemma 2.3] to obtain interpolation estimates in the weighted L^2-norm. These estimates allow a disparate mesh-size on the extended direction, relative to the coordinate directions x_i, $i = 1, \dots, n$, which may in turn be graded. This is the principal difference with [70, Lemma 2.3] where, however, the domain must be a cube.

Lemma 3.15 (Weighted L^2-based interpolation estimates) *Let* $\mathtt{v} \in \mathcal{N}_{\rm in}$. *Then, for all* $w \in H^1(\omega_{\mathtt{v}}, y^\alpha)$, *we have*

$$\|w - Q^0_{\mathtt{v}} w\|_{L^2(\omega_{\mathtt{v}}, y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} w\|_{L^2(\omega_{\mathtt{v}}, y^\alpha)} + h_{\mathtt{v}''} \|\partial_y w\|_{L^2(\omega_{\mathtt{v}}, y^\alpha)}, \tag{3.66}$$

and, for all $w \in H^2(\omega_{\mathtt{v}}, y^\alpha)$ *and* $j = 1, \dots, n+1$, *we have*

$$\|\partial_{x_j}(w - Q^1_{\mathtt{v}} w)\|_{L^2(\omega_{\mathtt{v}}, y^\alpha)} \lesssim h_{\mathtt{v}'} \sum_{i=1}^{n} \|\partial^2_{x_j x_i} w\|_{L^2(\omega_{\mathtt{v}}, y^\alpha)} + h_{\mathtt{v}''} \|\partial^2_{x_j y} w\|_{L^2(\omega_{\mathtt{v}}, y^\alpha)}, \tag{3.67}$$

where, in both inequalities, the hidden constant depends only on α, σ_Ω, σ *and* ψ.

Proof: Define by $\mathcal{F}_{\mathtt{v}} : (x', y) \to (\bar{x}', \bar{y})$ the scaling map

$$\bar{x}' = \frac{x' - \mathtt{v}'}{h_{\mathtt{v}'}}, \qquad \bar{y} = \frac{y - \mathtt{v}''}{h_{\mathtt{v}''}},$$

along with $\overline{\omega}_{\mathtt{v}} = \mathcal{F}_{\mathtt{v}}(\omega_{\mathtt{v}})$ and $\bar{w}(\bar{x}) = w(x)$. Define also $\bar{Q}^0 \bar{w} = \int \bar{w}\psi$, where ψ has been introduced in section 3.3.2. Since $\operatorname{supp} \psi \subset \overline{\omega}_{\mathtt{v}}$ integration takes place only over $\overline{\omega}_{\mathtt{v}}$, and $\int_{\overline{\omega}_{\mathtt{v}}} \psi = 1$. Then, $\bar{Q}^0 \bar{w}$ satisfies $\bar{Q}^0 \bar{w} = \int_{\overline{\omega}_{\mathtt{v}}} \bar{w}\psi = \int_{\omega_{\mathtt{v}}} w \psi_{\mathtt{v}} = Q^0_{\mathtt{v}} w$, and

$$\int_{\overline{\omega}_{\mathtt{v}}} (\bar{Q}^0 \bar{w} - \overline{w}) \psi \, \mathrm{d}\bar{x} = \bar{Q}^0 \bar{w} - \int_{\overline{\omega}_{\mathtt{v}}} \bar{w} \psi \, \mathrm{d}\bar{x} = 0. \tag{3.68}$$

Simple scaling, using the definition of the mapping $\mathcal{F}_{\mathtt{v}}$, yields

$$\int_{\omega_{\mathtt{v}}} y^\alpha |w - Q^0_{\mathtt{v}} w|^2 \, \mathrm{d}x = h^n_{\mathtt{v}'} h_{\mathtt{v}''} \int_{\overline{\omega}_{\mathtt{v}}} \xi_\alpha |\bar{w} - \bar{Q}^0 \bar{w}|^2 \, \mathrm{d}\bar{x}, \tag{3.69}$$

where $\xi_\alpha(y) := |\mathtt{v}'' + \bar{y} h_{\mathtt{v}''}|^\alpha$. By shape regularity, the mesh sizes $h_{\mathtt{v}'}, h_{\mathtt{v}''}$ satisfy $1/2\sigma \leq h_{\bar{\mathtt{v}}''} \leq 2\sigma$ and $1/2\sigma_\Omega \leq h_{\bar{\mathtt{v}}'} \leq 2\sigma_\Omega$, respectively, and $\operatorname{diam} \overline{\omega}_{\mathtt{v}} \approx 1$. In view of (3.68), we can apply Lemma 3.13 with the weight ξ_α and $\chi = \psi$, to $\omega = \overline{\omega}_{\mathtt{v}}$ to obtain

$$\|\bar{w} - \bar{Q}^0 \bar{w}\|_{L^2(\bar{\omega}_{\mathtt{v}},\xi_\alpha)} \lesssim \|\bar{\nabla}\bar{w}\|_{L^2(\bar{\omega}_{\mathtt{v}},\xi_\alpha)},$$

where the hidden constant depends only on α, σ_Ω, σ and ψ, but not on $\mathtt{v}''$ and $h_{\mathtt{v}''}$. Applying this to (3.69), together with a change of variables with $\mathcal{F}_{\mathtt{v}}^{-1}$, we get (3.66).

The proof of (3.67) is similar. Notice that

$$\begin{aligned} Q^1_{\mathtt{v}} w(z) &= \int_{\omega_{\mathtt{v}}} (w(x) + \nabla w(x) \cdot (z - x))\, \psi_{\mathtt{v}}(x) \,\mathrm{d}x \\ &= \int_{\overline{\omega}_{\mathtt{v}}} \big(\bar{w}(\bar{x}) + \bar{\nabla}\bar{w}(\bar{x}) \cdot (\bar{z} - \bar{x})\big)\, \psi(\bar{x}) \,\mathrm{d}\bar{x} =: \bar{Q}^1 \bar{w}(\bar{z}). \end{aligned}$$

Since $\partial_{\bar{z}_i} \bar{w}_0(\bar{z}) = \int_{\overline{\omega}_{\mathtt{v}}} \partial_{\bar{x}_i} \bar{w}(\bar{x}) \psi(\bar{x}) \,\mathrm{d}\bar{x}$ is constant, we have the vanishing mean value property

$$\int_{\overline{\omega}_{\mathtt{v}}} \partial_{\bar{z}_i} \left(\bar{w}(\bar{z}) - \bar{w}_0(\bar{z})\right) \psi(\bar{z}) \,\mathrm{d}\bar{z} = 0.$$

Applying Lemma 3.13 to $\partial_{\bar{x}_i} \left(\bar{w}(\bar{x}) - \bar{w}_0(\bar{x})\right)$, and scaling with $\mathcal{F}_{\mathtt{v}}$ we obtain (3.67).

□

By shape regularity, for all $\mathtt{v} \in \mathcal{N}_{\mathrm{in}}$ and $T \subset \omega_{\mathtt{v}}$, the quantities $h_{\mathtt{v}'}$ and $h_{\mathtt{v}''}$ are equivalent to h_K and h_I, up to a constant that depends only on σ_Ω and σ, respectively. This fact leads to the following result about interpolation estimates in the weighted L^2-norm on interior elements; we refer to § 3.3.2.4 for boundary elements.

Theorem 3.16 (Stability and local interpolation in the weighted L^2-norm) *For all $T \in \mathscr{T}_{\mathcal{Y}}$ such that $\partial T \cap \Gamma_D = \emptyset$, and $w \in L^2(\omega_T, y^\alpha)$ we have*

$$\|\Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \lesssim \|w\|_{L^2(\omega_T,y^\alpha)}. \tag{3.70}$$

If, in addition, $w \in H^1(\omega_T, y^\alpha)$

$$\|w - \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}. \tag{3.71}$$

The hidden constants in both inequalities depend only on σ_Ω, σ, ψ *and* α.

Proof: Let $T \in \mathscr{T}_\mathcal{Y}$ be an element such that $\partial T \cap \Gamma_D = \emptyset$. Assume, for the moment, that $\Pi_{\mathscr{T}_\mathcal{Y}}$ is uniformly bounded as a mapping from $L^2(\omega_T, y^\alpha)$ to $L^2(T, y^\alpha)$, i.e., (3.70).

Choose a node $\mathtt{v}$ of T. Since $Q^1_{\mathtt{v}} w$ is constant, we deduce $\Pi_{\mathscr{T}_\mathcal{Y}} Q^1_{\mathtt{v}} w = Q^1_{\mathtt{v}} w$, whence

$$\|w - \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} = \|(I - \Pi_{\mathscr{T}_\mathcal{Y}})(w - Q^1_{\mathtt{v}} w)\|_{L^2(T,y^\alpha)} \lesssim \|w - Q^1_{\mathtt{v}} w\|_{L^2(\omega_T,y^\alpha)},$$

so that (3.71) follows from Corollary 3.14.

It remains to show the local boundedness (3.70) of $\Pi_{\mathscr{T}_\mathcal{Y}}$. By definition,

$$\Pi_{\mathscr{T}_\mathcal{Y}} w = \sum_{i=1}^{n_T} Q^1_{\mathtt{v}_i}(\mathtt{v}_i) \phi_{\mathtt{v}_i},$$

where $\{\mathtt{v}_i\}_{i=1}^{n_T}$ denotes the set of interior vertices of T. By the triangle inequality

$$\|\Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \leq \sum_{i=1}^{n_T} \|Q^1_{\mathtt{v}_i}\|_{L^\infty(T)} \|\phi_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)}, \tag{3.72}$$

so that we need to estimate $\|Q^1_{\mathtt{v}_i}\|_{L^\infty(T)}$. This follows from (3.58) along with,

$$\left| \int_{\omega_{\mathtt{v}_i}} w \psi_{\mathtt{v}_i} \right| \leq \|w\|_{L^2(\omega_{\mathtt{v}_i},y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})}, \tag{3.73}$$

and, for $\ell = 1, \dots, n+1$,

$$\left| \int_{\omega_{\mathtt{v}_i}} \partial_{x_\ell} w(x)(z_\ell - x_\ell) \psi_{\mathtt{v}_i}(x) \, \mathrm{d}x \right| \lesssim \|w\|_{L^2(\omega_{\mathtt{v}_i},y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})}. \tag{3.74}$$

We get (3.74) upon integration by parts, and noticing that $\psi_{\mathtt{v}_i} = 0$ on $\partial\omega_{\mathtt{v}_i}$, and $|z_l - x_l||\partial_{x_l}\psi_{\mathtt{v}_i}| \lesssim 1$ for $1 \leq l \leq n+1$. Replacing (3.73) and (3.74) in (3.72), we get

$$\|\Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|w\|_{L^2(\omega_T,y^\alpha)} \sum_{i=1}^{n_T} \|\phi_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})} \lesssim \|w\|_{L^2(\omega_T,y^\alpha)},$$

where the last inequality is a consequence of $\phi_{\mathtt{v}_i}$ and ψ being bounded in $L^\infty(\omega_T)$,

$$\|\phi_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})} \lesssim |\omega_{\mathtt{v}_i}|^{-1} \left(\int_{\omega_{\mathtt{v}_i}} |y|^\alpha \int_{\omega_{\mathtt{v}_i}} |y|^{-\alpha} \right)^{1/2},$$

together with $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$; see (2.10). $\square$

3.3.2.3 Weighted H^1 interior interpolation estimates

Here we prove interpolation estimates on the first derivatives for interior elements. The, rather technical, proof is an adaption of [70, Theorem 2.5] to our particular geometric setting. In contrast to [70, Theorem 2.5], we do not have the symmetries of a cube. However, exploiting the Cartesian product structure of the elements $T = K \times I$, we are capable of handling the anisotropy in the extended variable y for general shape-regular graded meshes $\mathscr{T}_\mathcal{Y}$. This is the content of the following result.

Theorem 3.17 (Stability and local interpolation: interior elements) *Let $T \in \mathscr{T}_\mathcal{Y}$ be such that $\partial T \cap \Gamma_D = \emptyset$. For all $w \in H^1(\omega_T, y^\alpha)$ we have the stability bounds*

$$\|\nabla_{x'} \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|\nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)}, \tag{3.75}$$

$$\|\partial_y \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}, \tag{3.76}$$

and, for all $w \in H^2(\omega_T, y^\alpha)$ and $j = 1, \dots, n+1$ we have the error estimates

$$\|\partial_{x_j}(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} \partial_{x_j} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_y \partial_{x_j} w\|_{L^2(\omega_T,y^\alpha)}. \tag{3.77}$$

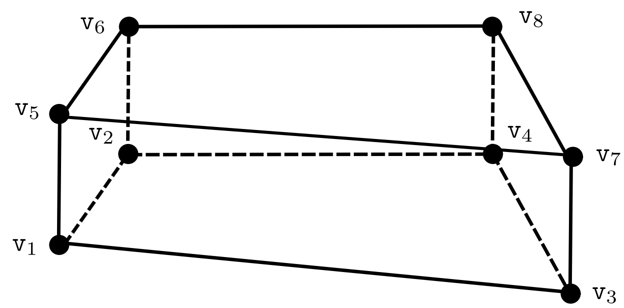

Figure 3.1: A generic element $T = K \times I$ in three dimensions: a quadrilateral prism.

Proof: To exploit the particular structure of T, we label its vertices in an appropriate way; see Figure 3.1 for the three-dimensional case. In general, if $T = K \times [a, b]$, we first assign a numbering $\{\mathtt{v}_k\}_{k=1,\dots,2^n}$ to the nodes that belong to $K \times \{a\}$. If $(\tilde{\mathtt{v}}', b)$ is a vertex in $K \times \{b\}$, then there is a $\mathtt{v}_k \in K \times \{a\}$ such that $\tilde{\mathtt{v}}' = \mathtt{v}_k'$, and we set $\mathtt{v}_{k+2^n} = \tilde{\mathtt{v}}$. We proceed in three steps.

$\boxed{1}$ *Derivative ∂_y in the extended dimension.* We wish to obtain a bound for the norm $\|\partial_y(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)}$. Since, $w - \Pi_{\mathscr{T}_\mathcal{Y}} w = (w - Q^1_{\mathtt{v}_1} w) + (Q^1_{\mathtt{v}_1} w - \Pi_{\mathscr{T}_\mathcal{Y}} w)$ and an estimate for the difference $w - Q^1_{\mathtt{v}_1} w$ is given in Lemma 3.15, it suffices to consider $q := Q^1_{\mathtt{v}_1} w - \Pi_{\mathscr{T}_\mathcal{Y}} w \in \mathbb{Q}_1$. Thanks to the special labeling of the nodes and the tensor product structure of the elements, i.e., $\partial_y \phi_{\mathtt{v}_{i+2^n}} = -\partial_y \phi_{\mathtt{v}_i}$, we get

$$\partial_y q = \sum_{i=1}^{2^{n+1}} q(\mathtt{v}_i) \partial_y \phi_{\mathtt{v}_i} = \sum_{i=1}^{2^n} (q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})) \partial_y \phi_{\mathtt{v}_i},$$

so that

$$\|\partial_y q\|_{L^2(T,y^\alpha)} \leq \sum_{i=1}^{2^n} |q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})| \|\partial_y \phi_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)}. \tag{3.78}$$

We now set $i = 1$ and proceed to estimate the difference $|q(\mathtt{v}_1) - q(\mathtt{v}_{1+2^n})|$.

By the definitions of $\Pi_{\mathscr{T}_\mathcal{Y}}$ and q, we have $\Pi_{\mathscr{T}_\mathcal{Y}} w(\mathtt{v}_1) = Q^1_{\mathtt{v}_1} w(\mathtt{v}_1)$, whence

$$\delta q(\mathtt{v}_1) := q(\mathtt{v}_1) - q(\mathtt{v}_{1+2^n}) = Q^1_{\mathtt{v}_{1+2^n}} w(\mathtt{v}_{1+2^n}) - Q^1_{\mathtt{v}_1} w(\mathtt{v}_{1+2^n}),$$

and by the definition (3.58) of the averaged Taylor polynomial we have

$$\delta q(\mathtt{v}_1) = \int_{\omega_{\mathtt{v}_{1+2^n}}} P^1(x, \mathtt{v}_{1+2^n})\psi_{\mathtt{v}_{1+2^n}}(x)\,\mathrm{d}x - \int_{\omega_{\mathtt{v}_1}} P^1(x, \mathtt{v}_{1+2^n})\psi_{\mathtt{v}_1}(x)\,\mathrm{d}x. \tag{3.79}$$

Recalling the operator $\odot$, introduced in (2.4), we notice that, for $h_{\mathtt{v}} = (h_{\mathtt{v}'}, h_{\mathtt{v}''})$ and $z \in \mathbb{R}^{n+1}$, the vector $h_{\mathtt{v}} \odot z$ is uniformly equivalent to $(h_K z', h_I z'')$ for all $T = K \times I$ in the star $\omega_{\mathtt{v}}$. Changing variables in (3.79) yields

$$\delta q(\mathtt{v}_1) = \int \left(P^1(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}} \odot z, \mathtt{v}_{1+2^n}) - P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z, \mathtt{v}_{1+2^n})\right) \psi(z)\,\mathrm{d}z. \tag{3.80}$$

To estimate this expression define

$$\theta = (0, \theta'') = \left(0, \mathtt{v}''_{1+2^n} - \mathtt{v}''_1 + (h_{\mathtt{v}''_1} - h_{\mathtt{v}''_{1+2^n}})z''\right), \tag{3.81}$$

and $F_z(t) = P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta, \mathtt{v}_{1+2^n})$. Using that $\mathtt{v}'_1 = \mathtt{v}'_{1+2^n}$ and $h_{\mathtt{v}'_1} = h_{\mathtt{v}'_{1+2^n}}$, we easily obtain

$$P^1(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}} \odot z, \mathtt{v}_{1+2^n}) - P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z, \mathtt{v}_{1+2^n}) = F_z(1) - F_z(0).$$

Consequently,

$$\delta q(\mathtt{v}_1) = \int \int_0^1 F'_z(t)\psi(z)\,\mathrm{d}t\,\mathrm{d}z = \int_0^1 \int F'_z(t)\psi(z)\,\mathrm{d}z\,\mathrm{d}t, \tag{3.82}$$

and since ψ is bounded in L^∞ and $\operatorname{supp}\psi = D \subset B_1 \times (-1, 1)$, we need to estimate the integral

$$I(t) = \int_D |F'_z(t)|\,\mathrm{d}z, \quad 0 \le t \le 1.$$

Invoking the definitions of F_z and $P^1(x, y)$, we deduce

$$F'_z(t) = \nabla_x P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta, \mathtt{v}_{1+2^n}) \cdot \theta,$$

and

$$\nabla_x P^1(x, \mathtt{v}) = D^2 w(x) \cdot (\mathtt{v} - x).$$

Using these two expressions, we arrive at

$$I(t) \leq \int_D \left(\left|\partial_{yy}^2 w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\right| \left|\mathtt{v}_{1+2^n}'' - \mathtt{v}_1'' + h_{\mathtt{v}_1''} z'' - t\theta''\right| \right.$$

$$\left. + \left|\partial_y \nabla_{x'} w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\right| \left|\mathtt{v}_{1+2^n}' - \mathtt{v}_1' + h_{\mathtt{v}_1'} z'\right| \right) |\theta''| \, \mathrm{d}z,$$

Now, since $|z'|, |z''| \leq 1$ and $0 \leq t \leq 1$, we see that

$$|\mathtt{v}_{1+2^n}' - \mathtt{v}_1' + h_{\mathtt{v}_1'} z'| \lesssim h_{\mathtt{v}_1'}, \qquad |\mathtt{v}_{1+2^n}'' - \mathtt{v}_1'' + h_{\mathtt{v}_1''} z'' - t\theta''| \lesssim h_{\mathtt{v}_1''}.$$

Consequently,

$$I(t) \lesssim \int_D \left(\left|\partial_{yy}^2 w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\right| h_{\mathtt{v}_1''}^2 + \left|\partial_y \nabla_{x'} w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\right| h_{\mathtt{v}_1'} h_{\mathtt{v}_1''} \right) \mathrm{d}z.$$

Changing variables, via $\tau = \mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta$, we obtain

$$I(t) \lesssim \int_{\omega_T} \left(\frac{h_{\mathtt{v}_1''}}{h_{\mathtt{v}_1'}^n} \left|\partial_{yy}^2 w(\tau)\right| + \frac{1}{h_{\mathtt{v}_1'}^{n-1}} \left|\partial_y \nabla_{x'} w(\tau)\right| \right) \mathrm{d}\tau, \tag{3.83}$$

because the support D of ψ is contained in $B_{1/\sigma_\Omega} \times (-1/\sigma_{\mathcal{Y}}, 1/\sigma_{\mathcal{Y}})$, and so is mapped into $\omega_{\mathtt{v}_1} \subset \omega_T$. Notice also that $h_{\mathtt{v}_1''} \lesssim (1-t) h_{\mathtt{v}_1''} + t h_{\mathtt{v}_{1+2^n}''}$. This implies

$$I(t) \lesssim \left(\frac{h_{\mathtt{v}_1''}}{h_{\mathtt{v}_1'}^n} \|\partial_{yy}^2 w\|_{L^2(\omega_T, y^\alpha)} + \frac{1}{h_{\mathtt{v}_1'}^{n-1}} \|\nabla_{x'} \partial_y w\|_{L^2(\omega_T, y^\alpha)} \right) \|1\|_{L^2(\omega_T, y^{-\alpha})}, \tag{3.84}$$

which, together with (3.82), yields

$$|\delta q(\mathtt{v}_1)| \|\partial_y \phi_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)} \lesssim \left(\frac{h_{\mathtt{v}_1''}}{h_{\mathtt{v}_1'}^n} \|\partial_{yy}^2 w\|_{L^2(\omega_T, y^\alpha)} + \frac{1}{h_{\mathtt{v}_1'}^{n-1}} \|\nabla_{x'} \partial_y w\|_{L^2(\omega_T, y^\alpha)} \right)$$

$$\cdot \|1\|_{L^2(\omega_T, y^{-\alpha})} \|\partial_y \phi_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)}. \tag{3.85}$$

Since $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$, we have

$$\|1\|_{L^2(\omega_T,y^{-\alpha})}\|\partial_y \phi_{\mathtt{v}_1}\|_{L^2(T,y^\alpha)} \lesssim h^n_{\mathtt{v}'_1}\frac{1}{h_{\mathtt{v}''_1}}\left(\int_I y^{-\alpha}\right)^{\frac{1}{2}}\left(\int_I y^{\alpha}\right)^{\frac{1}{2}} \lesssim h^n_{\mathtt{v}'_1}.$$

Replacing this into (3.85), we obtain

$$|\delta q(\mathtt{v}_1)|\|\partial_y \phi_{\mathtt{v}_1}\|_{L^2(T,y^\alpha)} \lesssim h'_{\mathtt{v}_1}\|\nabla_{x'}\partial_y w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''_1}\|\partial^2_{yy} w\|_{L^2(\omega_T,y^\alpha)}, \tag{3.86}$$

which, in this case, implies (3.77).

We now proceed to estimate the differences $|q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})|$ in (3.78) for $i = 2,\dots,2^n$. We employ the arguments presented in [70, Theorem 2.5] in conjunction with the techniques developed to get the estimate (3.86). We start by writing

$$\begin{aligned} q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n}) &= Q^1_{\mathtt{v}_1} w(\mathtt{v}_i) - Q^1_{\mathtt{v}_i}(\mathtt{v}_i) - (Q^1_{\mathtt{v}_1} w(\mathtt{v}_{i+2^n}) - Q^1_{\mathtt{v}_{i+2^n}}(\mathtt{v}_{i+2^n})) \\ &= Q^1_{\mathtt{v}_1} w(\mathtt{v}_i) - Q^1_{\mathtt{v}_1} w(\mathtt{v}_{i+2^n}) - (Q^1_{\mathtt{v}_i} w(\mathtt{v}_i) - Q^1_{\mathtt{v}_i} w(\mathtt{v}_{i+2^n})) \\ &\quad + (Q^1_{\mathtt{v}_{i+2^n}}(\mathtt{v}_{i+2^n}) - Q^1_{\mathtt{v}_i} w(\mathtt{v}_{i+2^n})) = I - II + III. \end{aligned}$$

Term III is identical to (3.79). The novelty here is the presence of terms I and II which, in view of (3.58) and the fact that $\mathtt{v}'_i = \mathtt{v}'_{i+2^n}$ for $i = 2,\dots,2^n$, can be rewritten as

$$\begin{aligned} I - II &= \int_{\omega_{\mathtt{v}_1}} (\mathtt{v}''_i - \mathtt{v}''_{i+2^n})\partial_y w(x)\psi_{\mathtt{v}_1}(x)\,\mathrm{d}x - \int_{\omega_{\mathtt{v}_i}} (\mathtt{v}''_i - \mathtt{v}''_{i+2^n})\partial_y w(x)\psi_{\mathtt{v}_i}(x)\,\mathrm{d}x \\ &= (\mathtt{v}''_i - \mathtt{v}''_{i+2^n})\int \left(\partial_y w(\mathtt{v}_1 - h_{\mathtt{v}_1}\odot z) - \partial_y w(\mathtt{v}_i - h_{\mathtt{v}_i}\odot z)\right)\psi(z)\,\mathrm{d}z. \end{aligned}$$

To estimate this expression, we define $\vartheta = (\mathtt{v}'_1 - \mathtt{v}'_i - (h_{\mathtt{v}'_1} - h_{\mathtt{v}'_i})z', 0)$, and the function $G_z(t) = \partial_y w(\mathtt{v}_i - h_{\mathtt{v}_i}\odot z + t\vartheta)$. Then, by using $\mathtt{v}''_1 = \mathtt{v}''_i$ and $h_{\mathtt{v}''_1} = h_{\mathtt{v}''_i}$ for $i = 2,\dots,2^n$, we arrive at

$$I - II = (\mathtt{v}''_i - \mathtt{v}''_{i+2^n})\int_0^1\int G'_z(t)\psi(z)\,\mathrm{d}z\,\mathrm{d}t.$$

Proceeding as in the case $i=1$, we obtain

$$|I - II| \|\partial_y \phi_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'_i} \|\partial_y \nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)}.$$

Collecting the estimates above for $i = 2, \dots, 2^n$, we finally get

$$|q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})| \|\partial_y \phi_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\partial_y \nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_{yy} w\|_{L^2(\omega_T,y^\alpha)}.$$

This together with (3.86) implies the desired estimate (3.77) for $j = n+1$.

$\boxed{2}$ *Derivatives* $\nabla_{x'}$ *in the domain* Ω. To prove an estimate for $\nabla_{x'}(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)$ we notice that, given a vertex $\mathtt{v}$, the associated basis function $\phi_{\mathtt{v}}$ can be written as $\phi_{\mathtt{v}}(x) = \Lambda_{\mathtt{v}'}(x')\mu_{\mathtt{v}''}(y)$, where $\Lambda_{\mathtt{v}'}$ is the canonical $\mathbb{Q}_1$ basis function on the variable x' associated to the node $\mathtt{v}'$ in the triangulation $\mathscr{T}_\Omega$, and $\mu_{\mathtt{v}''}$ corresponds to the piecewise $\mathbb{P}_1$ basis function associated to the node $\mathtt{v}''$. Recall that, by construction, the basis $\{\Lambda_i\}_{i=1}^{2^n}$ possesses the so-called partition of unity property, i.e.,

$$\sum_{i=1}^{2^n} \Lambda_{\mathtt{v}'_i}(x') = 1 \quad \forall x' \in K, \qquad \Longrightarrow \qquad \sum_{i=1}^{2^n} \nabla_{x'} \Lambda_{\mathtt{v}'_i}(x') = 0 \quad \forall x' \in K. \tag{3.87}$$

This implies that, for every $q \in \mathbb{Q}_1(T)$,

$$\begin{aligned}
\nabla_{x'} q &= \sum_{i=1}^{2^{n+1}} q(\mathtt{v}_i) \nabla_{x'} \phi_{\mathtt{v}_i} = \sum_{i=1}^{2^n} \Big(q(\mathtt{v}_i)\mu_{\mathtt{v}''_i}(y) + q(\mathtt{v}_{i+2^n})\mu_{\mathtt{v}''_{i+2^n}}(y) \Big) \nabla_{x'} \Lambda_{\mathtt{v}'_i}(x') \\
&= \sum_{i=1}^{2^n} \Big[(q(\mathtt{v}_i) - q(\mathtt{v}_1))\mu_{\mathtt{v}''_i}(y) + (q(\mathtt{v}_{i+2^n}) - q(\mathtt{v}_{1+2^n}))\mu_{\mathtt{v}''_{i+2^n}}(y) \Big] \nabla_{x'} \Lambda_{\mathtt{v}'_i}(x'),
\end{aligned}$$

whence, for $j = 1, \dots, n$,

$$\begin{aligned}
\|\partial_{x_j} q\|_{L^2(T,y^\alpha)} \lesssim& \sum_{i=1}^{2^n} |q(\mathtt{v}_i) - q(\mathtt{v}_1)| \|\mu_{\mathtt{v}''_i} \partial_{x_j} \Lambda_{\mathtt{v}'_i}\|_{L^2(T,y^\alpha)} \\
&+ \sum_{i=1}^{2^n} |q(\mathtt{v}_{1+2^n}) - q(\mathtt{v}_{i+2^n})| \|\mu_{\mathtt{v}''_{i+2^n}} \partial_{x_j} \Lambda_{\mathtt{v}'_i}\|_{L^2(T,y^\alpha)}.
\end{aligned}$$

This expression shows that the same techniques developed for the previous step lead to (3.77). In fact, we let $q = Q^1_{\mathtt{v}_1} w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w \in \mathbb{Q}_1$ and estimate $\delta q(\mathtt{v}_i) := q(\mathtt{v}_i) - q(\mathtt{v}_1)$ and $\delta q(\mathtt{v}_{i+2^n}) := q(\mathtt{v}_{i+2^n}) - q(\mathtt{v}_{1+2^n})$ for $i = 2, \dots, 2^n$ as follows; we deal with $\delta q(\mathtt{v}_i)$ only because the same argument applies to $\delta q(\mathtt{v}_{i+2^n})$. Using (3.58) and changing variables, we derive

$$\begin{aligned}\delta q(\mathtt{v}_i) &= Q^1_{\mathtt{v}_1} w(\mathtt{v}_i) - Q^1_{\mathtt{v}_i} w(\mathtt{v}_i) \\ &= \int \left(P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z, \mathtt{v}_i) - P^1(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z, \mathtt{v}_i) \right) \psi(z) \, \mathrm{d}z.\end{aligned}$$

Defining the vector $\varrho := (\varrho_1, 0) = (\mathtt{v}'_1 - \mathtt{v}'_i + (h'_{\mathtt{v}_1} - h'_{\mathtt{v}_i}) z', 0)$ and $H_z(t) := P(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\varrho, \mathtt{v}_i)$ yields

$$\delta q(\mathtt{v}_i) = \int_0^1 \int H'_z(t) \psi(z) \, \mathrm{d}z \, \mathrm{d}t.$$

Since ψ is bounded in L^∞ and $\operatorname{supp} \psi \subset D$, we next invoke the definitions of H_z and the polynomial P, to deduce

$$\begin{aligned}\int |H'_z(t) \psi(z)| \, \mathrm{d}z \lesssim \int_D &|\nabla_{x'} \partial_{x_j} w(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\varrho)| |h_{\mathtt{v}'_i} z' + t\varrho_1| |\varrho_1| \, \mathrm{d}z \\ &+ \int_D |\partial_y \partial_{x_j} w(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\varrho)| |h_{\mathtt{v}''_i} z''| |\varrho_1| \, \mathrm{d}z.\end{aligned}$$

Arguing as with the estimate (3.86), and using the scaling result

$$\|1\|_{L^2(\omega_T, y^\alpha)} \|\mu_{\mathtt{v}''_i} \partial_{x_j} \Lambda_{\mathtt{v}'_i}\|_{L^2(T, y^\alpha)} \lesssim h^{n-1}_{\mathtt{v}'_i} h_{\mathtt{v}''_i},$$

we infer that

$$|\delta q(\mathtt{v}_i)| \|\mu_{\mathtt{v}''_i} \partial_{x_j} \Lambda_{\mathtt{v}'_i}\|_{L^2(T, y^\alpha)} \lesssim h_{\mathtt{v}'_i} \|\nabla_{x'} \partial_{x_j} w\|_{L^2(\omega_T, y^\alpha)} + h_{\mathtt{v}''_i} \|\partial_y \partial_{x_j} w\|_{L^2(\omega_T, y^\alpha)}.$$

Finally, collecting the above estimates we obtain (3.77) for ∂_{x_j} with $j = 1, \dots, n$.

$\boxed{3}$ *Stability.* It remains to prove (3.75) and (3.76). By the triangle inequality,

$$\|\partial_y \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \le \|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} + \|\partial_y w\|_{L^2(T,y^\alpha)},$$

so that it suffices to estimate the first term. Add and subtract $Q^1_{\mathtt{v}_1} w$,

$$\|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} \le \|\partial_y (w - Q^1_{\mathtt{v}_1} w)\|_{L^2(T,y^\alpha)} + \|\partial_y (Q^1_{\mathtt{v}_1} w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)}. \tag{3.88}$$

Let us estimate the first term. The definition of $\psi_{\mathtt{v}_1}$, together with $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$ implies $\|\psi_{\mathtt{v}_1}\|_{L^2(\omega_{\mathtt{v}_1}, y^{-\alpha})} \|1\|_{L^2(\omega_{\mathtt{v}_1}, y^\alpha)} \lesssim 1$, whence invoking the definition (3.58) of the regularized Taylor polynomial $Q^1_{\mathtt{v}_1} w$ yields

$$\|\partial_y Q^1_{\mathtt{v}_1} w\|_{L^2(T,y^\alpha)} \le \|\partial_y w\|_{L^2(\omega_{\mathtt{v}_1}, y^\alpha)},$$

and

$$\|\partial_y (w - Q^1_{\mathtt{v}_1} w)\|_{L^2(T,y^\alpha)} \le \|\partial_y w\|_{L^2(T,y^\alpha) + \|\partial_y Q^1_{\mathtt{v}_1} w\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_{\mathtt{v}_1}, y^\alpha)}}. \tag{3.89}$$

To estimate the second term of the right hand side of (3.88), we repeat the steps used to obtain (3.77), starting from (3.79). We recall $\delta q(\mathtt{v}_i) = q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})$, and we proceed to estimate $\delta q(\mathtt{v}_1)$. Integrating by parts and using that $\psi_{\mathtt{v}_i} = 0$ on $\partial \omega_{\mathtt{v}_i}$, we get, for $\ell = 1, \dots, n+1$,

$$\begin{aligned} \int_{\omega_{\mathtt{v}_i}} \partial_{x_\ell} w(x)(z_\ell - x_\ell)\psi_{\mathtt{v}_i}(x) \,\mathrm{d}x &= \int_{\omega_{\mathtt{v}_i}} w(x)\psi_{\mathtt{v}_i}(x) \,\mathrm{d}x \\ &\quad - \int_{\omega_{\mathtt{v}_i}} w(x)(z_\ell - x_\ell)\partial_{x_\ell}\psi_{\mathtt{v}_i}(x) \,\mathrm{d}x, \end{aligned}$$

whence

$$\delta q(\mathtt{v}_1) = (n+2)\left(\int w(x)\psi_{\mathtt{v}_{1+2^n}}\,\mathrm{d}x - \int w(x)\psi_{\mathtt{v}_1}\,\mathrm{d}x\right)$$
$$- \int w(x)(\mathtt{v}_{1+2^n} - x)\cdot\nabla\psi_{\mathtt{v}_{1+2^n}}(x)\,\mathrm{d}x + \int w(x)(\mathtt{v}_1 - x)\cdot\nabla\psi_{\mathtt{v}_1}(x)\,\mathrm{d}x$$
$$= I_1 + I_2. \tag{3.90}$$

To estimate I_1 we consider the same change of variables used to obtain (3.80). Define $G_z(t) = (n+2)\cdot w(\mathtt{v}_1 - h_{\mathtt{v}_1}\odot z + t\theta)$, with θ as in (3.81), and observe that

$$I_1 = \int_0^1 \int G_z'(t)\psi(z)\,\mathrm{d}z\,\mathrm{d}t = (n+2)\int_0^1 \int \partial_y w(\mathtt{v}_1 - h_{\mathtt{v}_1}\odot z + t\theta)\theta''\psi(z)\,\mathrm{d}z\,\mathrm{d}t.$$

Introducing the change of variables $\tau = \mathtt{v}_1 - h_{\mathtt{v}_1}\odot z + t\theta$, we obtain

$$|I_1| \lesssim \int_{\omega_T} \frac{1}{h^n_{\mathtt{v}_1'}}|\partial_y w(\tau)|\,\mathrm{d}\tau \le \frac{1}{h^n_{\mathtt{v}_1'}}\|\partial_y w\|_{L^2(\omega_T,y^\alpha)}\|1\|_{L^2(\omega_T,y^{-\alpha})}. \tag{3.91}$$

We now estimate I_2. Changing variables,

$$I_2 = \int \big(w(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}}\odot z) - w(\mathtt{v}_1 - h_{\mathtt{v}_1}\odot z)\big)\, z'\cdot\nabla_{x'}\psi(z)\,\mathrm{d}z$$
$$+ \int \big(w(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}}\odot z)z'' - w(\mathtt{v}_1 - h_{\mathtt{v}_1}\odot z)(\vartheta + z'')\big)\,\partial_y\psi(z)\,\mathrm{d}z$$
$$= I_{2,1} + I_{2,2},$$

where $\vartheta = (\mathtt{v}''_{1+2^n} - \mathtt{v}''_1)/h_{\mathtt{v}''_1}$. Arguing as in the derivation of (3.91) we obtain

$$|I_{2,1}|, |I_{2,2}| \lesssim \int_{\omega_T} \frac{1}{h^n_{\mathtt{v}_1'}}|\partial_y w(\tau)|\,\mathrm{d}\tau \le \frac{1}{h^n_{\mathtt{v}_1'}}\|\partial_y w\|_{L^2(\omega_T,y^\alpha)}\|1\|_{L^2(\omega_T,y^{-\alpha})}. \tag{3.92}$$

Inserting (3.91) and (3.92) in (3.90) we deduce

$$|\delta q(\mathtt{v}_1)| \lesssim \frac{1}{h^n_{\mathtt{v}_1'}}\|\partial_y w\|_{L^2(\omega_T,y^\alpha)}\|1\|_{L^2(\omega_T,y^{-\alpha})},$$

whence

$$|\delta q(\mathtt{v}_1)| \|\partial_y \phi_{\mathtt{v}_1}\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}, \tag{3.93}$$

because $h_{\mathtt{v}_1'}^{-n} \|\partial_y \phi_{\mathtt{v}_1}\|_{L^2(\omega_T,y^\alpha)} \|1\|_{L^2(\omega_T,y^{-\alpha})} \leq C$. Replacing (3.93) in (3.78), we get

$$\|\partial_y (Q^1_{\mathtt{v}_1} w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T,y^\alpha)},$$

which, together with (3.88) and (3.89), imply the desired result (3.76) for $i = 1$. For $i = 2, \ldots, 2^n$, the estimates for $\delta q(\mathtt{v}_i)$ follow the same steps as in $\boxed{1}$. To prove the stability bound (3.75) we proceed as in $\boxed{2}$ to estimate the interpolation errors for the x'-derivatives, but we skip the details. $\square$

3.3.2.4 Weighted H^1 boundary interpolation estimates

Let us now extend the interpolation estimates of § 3.3.2.2 and § 3.3.2.3 to elements that intersect the Dirichlet boundary, where the functions to be approximated vanish. To do so, we start by adapting the results of [70, Theorem 3.1] to our particular case.

We consider, as in [70, Section 3], different cases according to the relative position of the element T in $\mathscr{T}_{\mathcal{Y}}$. We define the three sets

$$\mathcal{C}_1 = \left\{ T \in \mathscr{T}_{\mathcal{Y}} : \partial T \cap \Gamma_D = \emptyset \right\},$$

$$\mathcal{C}_2 = \left\{ T \in \mathscr{T}_{\mathcal{Y}} : \partial T \cap \partial_L \mathcal{C}_{\mathcal{Y}} \neq \emptyset \right\},$$

$$\mathcal{C}_3 = \left\{ T \in \mathscr{T}_{\mathcal{Y}} : \partial T \cap (\Omega \times \{\mathcal{Y}\}) \neq \emptyset \right\}.$$

The elements in $\mathcal{C}_1$ are interior, so the corresponding interpolation estimate is given in Theorem 3.17. Interpolation estimates on elements in $\mathcal{C}_3$ are a direct consequence of [70, Theorem 3.1] and Theorem 3.18 below. This is due to the fact that, since $\mathcal{Y} \geq 1$, the weight y^α over $\mathcal{C}_3$ is no longer singular nor degenerate. It remains only to provide interpolation estimates for elements in $\mathcal{C}_2$.

Theorem 3.18 (Weighted H^1 interpolation estimates over elements in $\mathcal{C}_2$) *Let $T \in \mathcal{C}_2$ and $w \in H^1(\omega_T, y^\alpha)$ vanish on $\partial T \cap \partial_L \mathcal{C}_\mathcal{Y}$. Then, we have the stability bounds*

$$\|\nabla_{x'} \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|\nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)}, \tag{3.94}$$

$$\|\partial_y \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}, \tag{3.95}$$

If, in addition, $w \in H^2(\omega_T, y^\alpha)$, then, for $j = 1, \ldots, n+1$,

$$\|\partial_{x_j}(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\partial_{x_j} \nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_{x_j y} w\|_{L^2(\omega_T,y^\alpha)}. \tag{3.96}$$

Proof: For simplicity we present the proof in two dimensions. Let $T = (0,a) \times (0,b) \in \mathcal{C}_2$ and let us label its vertices according to Figure 3.1: $\mathtt{v}_1 = (0,0)$, $\mathtt{v}_2 = (a,0)$, $\mathtt{v}_3 = (0,b)$, $\mathtt{v}_4 = (a,b)$. Notice that this is the worst situation because over such an element the weight becomes degenerate or singular; estimates over other elements of $\mathcal{C}_2$ are simpler. We proceed now to exploit the symmetry of T. By the definition of $\Pi_{\mathscr{T}_\mathcal{Y}}$ we have

$$\Pi_{\mathscr{T}_\mathcal{Y}} w|_T = Q^1_{\mathtt{v}_2}(\mathtt{v}_2)\phi_{\mathtt{v}_2} + Q^1_{\mathtt{v}_4}(\mathtt{v}_4)\phi_{\mathtt{v}_4}. \tag{3.97}$$

The proofs of (3.94) and (3.95) are similar to Step 3 of Theorem 3.17. To show (3.96), we write the local difference between a function and its interpolant as $(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T = (w - Q^1_{\mathtt{v}_2} w)|_T + (Q^1_{\mathtt{v}_2} w - \Pi_{\mathscr{T}_\mathcal{Y}})|_T$. Proceeding as in the proof of Lemma 3.15, we can bound $\partial_{x_j}(w - Q^1_{\mathtt{v}_2} w)|_T$ for $j = 1, 2$, in the $L^2(T, y^\alpha)$-norm, by the right hand side of (3.96) because this is independent of the trace of w. It remains then to derive a bound for $(Q^1_{\mathtt{v}_2} w - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T$, for which we consider two separate cases.

$\boxed{1}$ *Derivative in the extended direction.* We use that $Q^1_{\mathtt{v}_2} \in \mathbb{Q}_1$, (3.97) and $\Pi_{\mathscr{T}_\mathcal{Y}} w(\mathtt{v}_1) = \Pi_{\mathscr{T}_\mathcal{Y}} w(\mathtt{v}_3) = 0$, to write

$$\partial_y (Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T = \left(Q^1_{\mathtt{v}_2}(\mathtt{v}_3) - Q^1_{\mathtt{v}_2}(\mathtt{v}_1) \right) \partial_y \phi_{\mathtt{v}_3} + \left(Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_4) \right) \partial_y \phi_{\mathtt{v}_4}.$$

Since $w \equiv 0$ on $\{0\} \times (0, b)$, then $\partial_y w \equiv 0$ on $\{0\} \times (0, b)$. By the definition of the Taylor polynomial P^1, given in (3.59), and the fact that $\mathtt{v}'_1 = \mathtt{v}'_3$, we obtain

$$\begin{aligned} Q^1_{\mathtt{v}_2}(\mathtt{v}_3) - Q^1_{\mathtt{v}_2}(\mathtt{v}_1) &= (\mathtt{v}''_3 - \mathtt{v}''_1) \int_{\omega_{\mathtt{v}_2}} \partial_y w(x) \psi_{\mathtt{v}_2}(x) \, \mathrm{d}x \\ &= (\mathtt{v}''_3 - \mathtt{v}''_1) \int_{\omega_{\mathtt{v}_2}} \int_0^{x'} \partial_{x'y} w(\sigma, y) \psi_{\mathtt{v}_2}(x', y) \, \mathrm{d}\sigma \, \mathrm{d}x' \, \mathrm{d}y. \end{aligned}$$

Therefore

$$\begin{aligned} |Q^1_{\mathtt{v}_2}(\mathtt{v}_3) - Q^1_{\mathtt{v}_2}(\mathtt{v}_1)| &\lesssim h_{\mathtt{v}''_1} h_{\mathtt{v}'_1} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \|\psi_{\mathtt{v}_2}\|_{L^2(\omega_T, y^{-\alpha})} \\ &\lesssim h_{\mathtt{v}''_1} h_{\mathtt{v}'_1} \frac{h^{\frac{1}{2}}_{\mathtt{v}'_1}}{h_{\mathtt{v}'_2} h_{\mathtt{v}''_2}} \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)}. \end{aligned}$$

Since, in view of the weak shape regularity assumption on the mesh $\mathscr{T}_\mathcal{Y}$, $h_{\mathtt{v}'_1} \approx h_{\mathtt{v}'_2}$, $h_{\mathtt{v}''_1} = h_{\mathtt{v}''_2}$, and $y^\alpha \in A_2(\mathbb{R}^{n+1}_+)$, we conclude that

$$\begin{aligned} |Q^1_{\mathtt{v}_2}(\mathtt{v}_3) - Q^1_{\mathtt{v}_2}(\mathtt{v}_1)| \|\partial_y \phi_{\mathtt{v}_3}\|_{L^2(T, y^\alpha)} &\lesssim \frac{h_{\mathtt{v}'_1}}{h_{\mathtt{v}''_1}} \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \int_0^b y^\alpha \, \mathrm{d}y \right)^{\frac{1}{2}} \times \\ &\quad \times \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \\ &\lesssim h_{\mathtt{v}'_1} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)}. \end{aligned} \tag{3.98}$$

Finally, to bound $|Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_4)|$, we proceed as in Step 1 of the proof of Theorem 3.17, which is valid regardless of the trace of w, and deduce

$$|Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_4)| \|\partial_y \phi_{\mathtt{v}_3}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}_1'} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} + h_{\mathtt{v}_1''} \|\partial_{yy} w\|_{L^2(\omega_T, y^\alpha)}.$$

This, in conjunction with the previous estimate, yields (3.96) for the derivative in the extended direction.

$\boxed{2}$ *Derivative in the x' direction.* To estimate $\partial_{x'}(Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T$ we proceed as in Theorem 3.17 and [70, Theorem 3.1], but we cannot exploit the symmetry of the tensor product structure now. For brevity, we shall only point out the main technical differences. Using, again, that $(Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w) \in \mathbb{Q}_1$,

$$\begin{aligned}
\partial_{x'}(Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T &= Q^1_{\mathtt{v}_2}(\mathtt{v}_1)\partial_{x'}\phi_{\mathtt{v}_1} + Q^1_{\mathtt{v}_2}(\mathtt{v}_3)\partial_{x'}\phi_{\mathtt{v}_3} + (Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_4))\partial_{x'}\phi_{\mathtt{v}_4} \\
&= Q^1_{\mathtt{v}_2}(\mathtt{v}_1)\partial_{x'}\phi_{\mathtt{v}_1} + (Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_2}(\mathtt{v}_3))\partial_{x'}\phi_{\mathtt{v}_4} \\
&- (Q^1_{\mathtt{v}_4}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_3))\partial_{x'}\phi_{\mathtt{v}_4} - Q^1_{\mathtt{v}_4}(\mathtt{v}_3)\partial_{x'}\phi_{\mathtt{v}_4} \\
&= J(Q^1_{\mathtt{v}_2}, Q^1_{\mathtt{v}_4})\partial_{x'}\phi_{\mathtt{v}_4} + Q^1_{\mathtt{v}_2}(\mathtt{v}_1)\partial_{x'}\phi_{\mathtt{v}_1} - Q^1_{\mathtt{v}_4}(\mathtt{v}_3)\partial_{x'}\phi_{\mathtt{v}_4},
\end{aligned}$$

where

$$J(Q^1_{\mathtt{v}_2}, Q^1_{\mathtt{v}_4}) = \left(Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_2}(\mathtt{v}_3)\right) - \left(Q^1_{\mathtt{v}_4}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_3)\right).$$

Define $\theta = (0, \theta'') = (0, \mathtt{v}_4'' - \mathtt{v}_2'' - (h_{\mathtt{v}_4''} - h_{\mathtt{v}_2''})z'')$, and rewrite $J(Q^1_{\mathtt{v}_2}, Q^1_{\mathtt{v}_4})$ as follows:

$$\begin{aligned}
J(Q^1_{\mathtt{v}_2}, Q^1_{\mathtt{v}_4}) &= (\mathtt{v}_4' - \mathtt{v}_3') \int_D \left(\partial_{x'} w(\mathtt{v}_2 - h_{\mathtt{v}_2} \odot z) - \partial_{x'} w(\mathtt{v}_4 - h_{\mathtt{v}_4} \odot z)\right) \psi(z) \, \mathrm{d}z \\
&= -(\mathtt{v}_4' - \mathtt{v}_3') \int_D \int_0^1 \partial_{x'y} w(\mathtt{v}_2 - h_{\mathtt{v}_2} \odot z + \theta t)\theta'' \psi(z) \, \mathrm{d}t \, \mathrm{d}z,
\end{aligned}$$

where $D = \operatorname{supp} \psi$. Denote

$$I(t) = \int |\partial_{x'y} w(\mathtt{v}_2 - h_{\mathtt{v}_2} \odot z + \theta t)\theta''| \, \mathrm{d}z.$$

Using the change of variables $z \mapsto \tau = \mathtt{v}_2 - h_{\mathtt{v}_2} \odot z + \theta t$, results in

$$
\begin{aligned}
|I(t)| &\lesssim \frac{1}{h_{\mathtt{v}_2'}} \int_{\omega_T} |\partial_{x'y} w(\tau)| \psi(\tau) \, \mathrm{d}\tau \lesssim \frac{1}{h_{\mathtt{v}_2'}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \|1\|_{L^2(\omega_T, y^{-\alpha})} \\
&\lesssim h_{\mathtt{v}_2'}^{-\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}},
\end{aligned}
$$

whence $\left| J(Q^1_{\mathtt{v}_2}, Q^1_{\mathtt{v}_4}) \right| \lesssim h_{\mathtt{v}_2'}^{\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}}$. This implies

$$
\begin{aligned}
\|J(Q^1_{\mathtt{v}_2}, Q^1_{\mathtt{v}_4}) \partial_{x'} \phi_{\mathtt{v}_4}\|_{L^2(T, y^\alpha)} &\lesssim \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}} \left(\int_0^b y^{\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \\
&\lesssim h_{\mathtt{v}_2''} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)},
\end{aligned}
$$

which follows from the fact that $y^\alpha \in A_2(\mathbb{R}^+)$, and then (3.96) holds true.

The estimate of $Q^1_{\mathtt{v}_2}(\mathtt{v}_1) \partial_{x'} \phi_{\mathtt{v}_2}$ exploits the fact that the trace of w vanishes on $\partial_L \mathcal{C}_\mathscr{Y}$; the same happens with $Q^1_{\mathtt{v}_4}(\mathtt{v}_3) \partial_{x'} \phi_{\mathtt{v}_4}$. In fact, we can write

$$
\begin{aligned}
Q^1_{\mathtt{v}_2}(\mathtt{v}_1) = &\int_{\omega_{\mathtt{v}_2}} \int_0^{x'} \left(\partial_{x'} w(\tau, y) - \partial_{x'} w(x', y) \right) \psi_{\mathtt{v}_2}(x', y) \, \mathrm{d}\tau \, \mathrm{d}x' \, \mathrm{d}y \\
&+ \int_{\omega_{\mathtt{v}_2}} \left(\partial_y w(0, y) - \partial_y w(x', y) \right) y \psi_{\mathtt{v}_2}(x', y) \, \mathrm{d}x' \, \mathrm{d}y.
\end{aligned}
$$

To derive (3.96) we finally proceed as in the proofs of Theorem 3.17 and [70, Theorem 3.1]. We omit the details. □

We now conclude with a result involving weighted L^2 interpolation estimates on boundary elements. As in the weighted H^1 case, the elements in $\mathcal{C}_1$ are interior, and then, the interpolation estimates are given by Theorem (3.16). It remains, to analyze the interpolation estimates on the sets $\mathcal{C}_2$ and $\mathcal{C}_3$.

Theorem 3.19 (Weighted L^2 interpolation estimates in $\mathcal{C}_2$ and $\mathcal{C}_3$) *If $T \in \mathcal{C}_2 \cup \mathcal{C}_3$ and $w \in H^1(\omega_T, y^\alpha)$ vanish on $\partial T \cap \partial_L \mathcal{C}_\mathscr{Y}$ and $\partial T \cap (\{\Omega\} \times \mathscr{Y})$, then*

$$
\|w - \Pi_{\mathscr{T}_\mathscr{Y}} w\|_{L^2(T, y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} w\|_{L^2(\omega_T, y^\alpha)} + h_{\mathtt{v}''} \|\partial_y w\|_{L^2(\omega_T, y^\alpha)}. \tag{3.99}
$$

Proof: We consider $T \in \mathcal{C}_2$, and the same geometric setting as in the proof of Theorem 3.18; we skip the case $T \in \mathcal{C}_3$ as in Theorem 3.18. We write the difference $w - \Pi_{\mathscr{T}_\mathcal{Y}} w|_T = (w - Q^1_{\mathtt{v}_2})|_T + (Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T$. Applying Lemma 3.15, we can bound the term $(w - Q^1_{\mathtt{v}_2})|_T$ in the $L^2(T, y^\alpha)$-norm by the right hand side of (3.99). Then, it suffices to estimate $(Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T \in \mathbb{Q}_1(T)$. Writing

$$(Q^1_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T = Q^1_{\mathtt{v}_2}(\mathtt{v}_1)\phi_{\mathtt{v}_1} + Q^1_{\mathtt{v}_2}(\mathtt{v}_3)\phi_{\mathtt{v}_3} + (Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_4))\phi_{\mathtt{v}_4},$$

and using the fact that the trace of w vanishes on $\partial_L \mathcal{C}_\mathcal{Y}$, we see that

$$Q^1_(\mathtt{v}_1) = \int_{\omega_{\mathtt{v}_2}} \int_0^{x'} \partial_{x'} w(\sigma, y)\psi_{\mathtt{v}_2} \, \mathrm{d}\sigma \, \mathrm{d}x' \, \mathrm{d}y + \int_{\omega_{\mathtt{v}_2}} (\mathtt{v}_1 - x) \cdot \nabla w(x)\psi_{\mathtt{v}_2}(x) \, \mathrm{d}x; \quad (3.100)$$

the same argument holds for $Q^1_{\mathtt{v}_2}(\mathtt{v}_3)$. On the other hand, we handle $Q^1_{\mathtt{v}_2}(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}(\mathtt{v}_4)$ with the same techniques as in the proof of Theorem 3.17. $\square$

3.4 Error estimates

The estimates of § 3.3.2.3 and § 3.3.2.4 are obtained under the local assumption that $w \in H^2(\omega_T, y^\alpha)$. However, the solution $\mathscr{U}$ of (3.16) satisfies $\mathscr{U}_{yy} \in L^2(\mathcal{C}, y^\beta)$ only when $\beta > 2\alpha + 1$, according to Theorem 3.3. For this reason, in this section we derive error estimates for both quasi-uniform and graded meshes. The estimates of § 3.4.1 for quasi-uniform meshes are quasi-optimal in terms of regularity but suboptimal in terms of order. The estimates of § 3.4.2 for graded meshes are, instead, quasi-optimal in both regularity and order. Mesh anisotropy is able to capture the singular behavior of the solution and restore optimal decay rates.

3.4.1 Quasi-uniform meshes

We start with a simple one dimensional case ($n = 1$) and assume that we need to approximate over the interval $[0, \mathcal{Y}]$ the function $w(y) = y^{1-\alpha}$. Notice that $w_y(y) \approx y^{-\alpha}$ as $y \approx 0^+$ has the same behavior as the derivative in the extended direction of the α-harmonic extension $\mathscr{U}$.

Given $M \in \mathbb{N}$ we consider the uniform partition of the interval $[0, \mathcal{Y}]$

$$y_k = \frac{k}{M}\mathcal{Y}, \quad k = 0, \dots, M. \tag{3.101}$$

and the corresponding elements $I_k = [y_k, y_{k+1}]$ of size $h_k = h = \mathcal{Y}/M$ for $k = 0, \dots, M-1$.

We can adapt the definition of $\Pi_{\mathscr{T}_\mathcal{Y}}$ of § 3.3.2 to this setting, and bound the local interpolation errors $E_k = \|\partial_y(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(I_k, y^\alpha)}$. For $k = 2, \dots, M-1$, since $y \geq h$ and $\alpha < 2\alpha + 1 < \beta$, (3.77) implies

$$E_k^2 \lesssim h^2 \int_{\omega_{I_k}} y^\alpha |w_{yy}|^2 \, \mathrm{d}y \lesssim h^{2+\alpha-\beta} \int_{\omega_{I_k}} y^\beta |w_{yy}|^2 \, \mathrm{d}y, \tag{3.102}$$

because $\left(\frac{y}{h}\right)^\alpha \leq \left(\frac{y}{h}\right)^\beta$. The estimate for $E_0^2 + E_1^2$ follows from from the stability of the operator $\Pi_{\mathscr{T}_\mathcal{Y}}$ (3.76) and (3.95):

$$E_0^2 + E_1^2 \lesssim \int_0^{3h} y^\alpha |w_y|^2 \lesssim h^{1-\alpha}, \tag{3.103}$$

because $w(y) \approx y^{-\alpha}$ as $y \approx 0^+$. Using (3.102) and (3.103) in conjunction with $2 + \alpha - \beta < 1 - \alpha$, we obtain a global interpolation estimate

$$\|\partial_y(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2((0,\mathcal{Y}), y^\alpha)} \lesssim h^{(2+\alpha-\beta)/2}. \tag{3.104}$$

These ideas can be extended to prove an error estimate for $\mathscr{U}$ on uniform meshes.

Theorem 3.20 (Error estimate for quasi-uniform meshes) *Let $\mathscr{U}$ solve* (3.16), *and $V_{\mathscr{T}_\mathscr{Y}}$ be the solution of* (3.51), *constructed over a quasi-uniform mesh of size h. If $f \in \mathbb{H}^{1-s}(\Omega)$ and $\mathscr{Y} \approx |\log h|$, then for all $\epsilon > 0$*

$$\|\nabla(\mathscr{U} - V_{\mathscr{T}_\mathscr{Y}})\|_{L^2(\mathcal{C}_\mathscr{Y}, y^\alpha)} \lesssim h^{s-\varepsilon}\|f\|_{\mathbb{H}^{1-s}(\Omega)}, \tag{3.105}$$

where the hidden constant blows up if ε tends to 0.

Proof: Use first Theorem 3.10 and Theorem 3.11, combined with (3.57), to reduce the approximation error to the interpolation error of $\mathscr{U}$. Repeat next the steps leading to (3.102)–(3.103), but combining the interpolation estimates of Theorems 3.17 and 3.18 with the regularity results of Theorem 3.3, which are valid because $f \in \mathbb{H}^{1-s}(\Omega)$. $\square$

Remark 3.21 (Sharpness of (3.105) **for** $s \neq \frac{1}{2}$**)** According to (3.24) and (3.27), $\partial_y \mathscr{U} \approx y^{-\alpha}$, and this formally implies $\partial_y \mathscr{U} \in H^{s-\varepsilon}(\mathcal{C}, y^\alpha)$ for all $\varepsilon > 0$ provided $f \in \mathbb{H}^{1-s}(\Omega)$. In this sense (3.105) appears to be sharp with respect to regularity even though it does not exhibit the optimal rate. We verify this argument via a simple numerical illustration for dimension $n = 1$. We let $\Omega = (0, 1)$, $s = 0.2$, right hand side $f = \pi^{2s}\sin(\pi x)$, and note that $u(x) = \sin(\pi x)$, and the solution $\mathscr{U}$ to (1.2) is

$$\mathscr{U}(x, y) = \frac{2^{1-s}\pi^s}{\Gamma(s)}\sin(\pi x)K_s(\pi y).$$

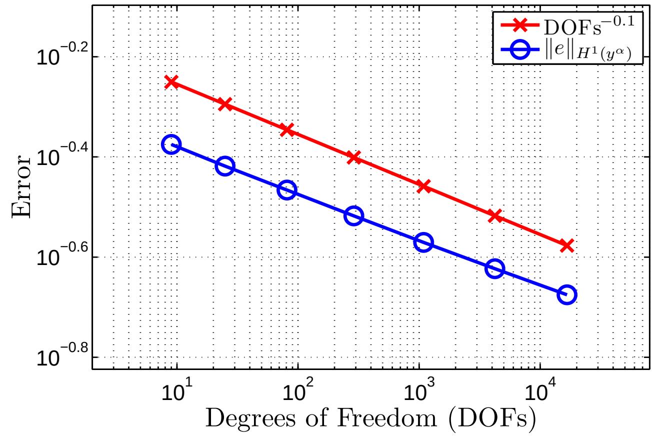

Figure 3.2: Computational rate of convergence $(\#\mathscr{T}_\mathcal{Y})^{-s/(n+1)}$ for quasi-uniform meshes $\mathscr{T}_\mathcal{Y}$, with $s = 0.2$ and $n = 1$.

Figure 3.2 shows the rate of convergence for the $H^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$-seminorm. Estimate (3.105) predicts a rate of $h^{-0.2-\varepsilon}$. We point out that for the α-harmonic extension we are solving a two dimensional problem and, since the mesh $\mathscr{T}_\mathcal{Y}$ is quasi-uniform, $\#\mathscr{T}_\mathcal{Y} \approx h^{-2}$. In other words the rate of convergence, when measured in terms of degrees of freedom, is $(\#\mathscr{T}_\mathcal{Y})^{-0.1-\varepsilon}$, which is what Figure 3.2 displays.

Remark 3.22 (Case $s = \frac{1}{2}$) Estimate (3.105) does not hold for $s = \frac{1}{2}$. In this case there is no weight and the scaling issues in (3.102) are no longer present, so that $E_k \lesssim h\|v\|_{H^2(I_k)}$. We thus obtain the optimal error estimate

$$\|\nabla(\mathscr{U} - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(\mathcal{C}_\mathcal{Y})} \lesssim h\|f\|_{H_{00}^{1/2}(\Omega)}.$$

3.4.2 Graded meshes

The estimate (3.105) can be written equivalently

$$\|\nabla(\mathscr{U} - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim (\#\mathscr{T}_\mathcal{Y})^{-\frac{s-\varepsilon}{n+1}}\|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

for quasi-uniform meshes in dimension $n+1$. We now show how to compensate the singular behavior in the extended variable y by anisotropic meshes and restore the optimal convergence rate $-1/(n+1)$.

As in § 3.4.1 we start the discussion in dimension $n=1$ with the function $w(y) = y^{1-\alpha}$ over $[0, \mathcal{Y}]$. We consider the graded partition $\mathscr{T}_{\mathcal{Y}}$ of the interval $[0, \mathcal{Y}]$

$$y_k = \left(\frac{k}{M}\right)^{\gamma} \mathcal{Y}, \quad k = 0, \dots, M, \tag{3.106}$$

where $\gamma = \gamma(\alpha) > 3/(1-\alpha) > 1$. If we denote by h_k the length of the interval

$$I_k = [y_k, y_{k+1}] = \left[\left(\frac{k}{M}\right)^{\gamma} \mathcal{Y}, \left(\frac{k+1}{M}\right)^{\gamma} \mathcal{Y}\right],$$

then

$$h_k = y_{k+1} - y_k \lesssim \frac{\mathcal{Y}}{M^{\gamma}} k^{\gamma-1}, \quad k = 1, \dots, M-1.$$

We again consider the operator $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ of § 3.3.2 on the one dimensional mesh $\mathscr{T}_{\mathcal{Y}}$ and wish to bound the local interpolation errors E_k of § 3.4.1. We apply estimate (3.77) to interior elements to obtain that, for $k = 2, \dots, M-1$,

$$\begin{aligned} E_k^2 &\lesssim h_k^2 \int_{\omega_{I_k}} y^{\alpha} |w_{yy}|^2 \,\mathrm{d}y \lesssim \mathcal{Y}^2 \frac{k^{2(\gamma-1)}}{M^{2\gamma}} \int_{\omega_{I_k}} y^{\alpha} |w_{yy}|^2 \,\mathrm{d}y \\ &\lesssim \mathcal{Y}^{2+\alpha-\beta} \frac{k^{2(\gamma-1)}}{M^{2\gamma}} \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \int_{\omega_{I_k}} y^{\beta} |w_{yy}|^2 \,\mathrm{d}y \lesssim \mathcal{Y}^{1-\alpha} \frac{k^{\gamma(1-\alpha)-3}}{M^{\gamma(1-\alpha)}}. \end{aligned} \tag{3.107}$$

because $y^{\alpha} \lesssim \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \mathcal{Y}^{\alpha-\beta} y^{\beta}$ and $w(y) = y^{1-\alpha}$ over $[0, \mathcal{Y}]$. Adding (3.107) over $k = 2, \dots, M-1$, and using that $\gamma(1-\alpha) > 3$, we arrive at

$$\|\partial_y (w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)\|^2_{L^2((y_2, \mathcal{Y}), y^{\alpha})} \lesssim \mathcal{Y}^{1-\alpha} M^{-2}. \tag{3.108}$$

For the errors E_0^2, E_1^2 we resort to the stability bounds (3.76) and (3.95) to write

$$\|\partial_y (w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)\|^2_{L^2((0, y_3), y^{\alpha})} \lesssim \int_0^{\left(\frac{3}{M}\right)^{\gamma} \mathcal{Y}} y^{-\alpha} \,\mathrm{d}y \lesssim \frac{\mathcal{Y}^{1-\alpha}}{M^{\gamma(1-\alpha)}}, \tag{3.109}$$

where we have used (3.106). Finally, adding (3.108) and (3.109) gives

$$\|\partial_y(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|^2_{L^2((0,\mathcal{Y}),y^\alpha)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2},$$

and shows that the interpolation error exhibits optimal decay rate.

We now apply this idea to the numerical solution of problem (3.37). We assume $\mathscr{T}_\Omega$ to be quasi-uniform in $\mathbb{T}_\Omega$ with $\#\mathscr{T}_\Omega \approx M^n$ and construct $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$ as the tensor product of $\mathscr{T}_\Omega$ and the partition given in (3.106), with $\gamma > 3/(1-\alpha)$. Consequently, $\#\mathscr{T}_\mathcal{Y} = M \cdot \#\mathscr{T}_\Omega \approx M^{n+1}$. Finally, we notice that since $\mathscr{T}_\Omega$ is shape regular and quasi-uniform, $h_{\mathscr{T}_\Omega} \approx (\#\mathscr{T}_\Omega)^{-1/n} \approx M^{-1}$.

Theorem 3.23 (Error estimate for graded meshes) *Let $V_\mathscr{T} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve* (3.51) *and $U_{\mathscr{T}_\Omega} \in \mathbb{U}(\mathscr{T}_\Omega)$ be defined as in* (3.52)*. If $f \in \mathbb{H}^{1-s}(\Omega)$, then*

$$\|\mathscr{U} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\mathbb{H}^{-s}(\Omega)} + \mathcal{Y}^{(1-\alpha)/2}(\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)}\|f\|_{\mathbb{H}^{1-s}(\Omega)}, \quad (3.110)$$

Proof: In light of (3.57), with $\epsilon \approx e^{-\sqrt{\lambda_1}\mathcal{Y}/4}$, it suffices to bound the interpolation error $\mathscr{U} - \Pi_{\mathscr{T}_\mathcal{Y}}\mathscr{U}$ on the mesh $\mathscr{T}_\mathcal{Y}$. To do so we, first of all, notice that if I_1 and I_2 are neighboring cells on the partition of $[0,\mathcal{Y}]$, then there is a constant $\sigma = \sigma(\gamma)$ such that $h_{I_1} \leq \sigma h_{I_2}$, whence the weak regularity condition (c) holds. We can thus apply the polynomial interpolation theory of § 3.3.2. We decompose the mesh $\mathscr{T}_\mathcal{Y}$ into the sets

$$\mathcal{T}_0 := \left\{T \in \mathscr{T}_\mathcal{Y} :\ \omega_T \cap (\bar{\Omega} \times \{0\}) = \emptyset\right\}, \quad \mathcal{T}_1 := \left\{T \in \mathscr{T}_\mathcal{Y} :\ \omega_T \cap (\bar{\Omega} \times \{0\}) \neq \emptyset\right\}.$$

We observe that for all $T = K \times I_k \in \mathcal{T}_0$ we have $k \geq 2$ and $y^\alpha \lesssim \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \mathcal{Y}^{\alpha-\beta} y^\beta$. Applying Theorem 3.17 and Theorem 3.18 to elements in $\mathcal{T}_0$ we obtain

$$\sum_{T \in \mathcal{T}_0} \|\nabla(\mathscr{U} - \Pi_{\mathscr{T}_\mathcal{Y}} \mathscr{U})\|^2_{L^2(T,y^\alpha)} \lesssim \sum_{T = K \times I \in \mathcal{T}_0} \Big(h_K^2 \|\nabla_{x'} \nabla \mathscr{U}\|^2_{L^2(\omega_T, y^\alpha)} \\ + h_I^2 \|\partial_y \nabla_{x'} \mathscr{U}\|^2_{L^2(\omega_T,y^\alpha)} + h_I^2 \|\partial_{yy} \mathscr{U}\|^2_{L^2(\omega_T, y^\beta)} \Big) = S_1 + S_2 + S_3.$$

We examine first the most problematic third term S_3, which we rewrite as follows:

$$S_3 \lesssim \sum_{k=2}^{M} \mathcal{Y}^{2+\alpha-\beta} \frac{k^{2(\gamma-1)}}{M^{2\gamma}} \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \int_{a_k}^{b_k} y^\beta \int_\Omega |\partial_{yy} \mathscr{U}|^2 \, \mathrm{d}x' \, \mathrm{d}y,$$

with $a_k = \left(\frac{k-1}{M}\right)^\gamma \mathcal{Y}$ and $b_k = \left(\frac{k+1}{M}\right)^\gamma \mathcal{Y}$. We now invoke the local estimate (3.34), as well as the fact that $b_k - a_k \lesssim \left(\frac{k}{M}\right)^{\gamma-1} \frac{\mathcal{Y}}{M}$, to end up with

$$S_3 \lesssim \sum_{k=2}^{M} \mathcal{Y}^{1-\alpha} \frac{k^{\gamma(1-\alpha)-3}}{M^{\gamma(1-\alpha)}} \|f\|^2_{L^2(\Omega)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2} \|f\|^2_{L^2(\Omega)}.$$

We now handle the middle term S_2 with the help of (3.33), which is valid for $b_k \leq 1$. This imposes the restriction $k \leq k_0 \leq M \mathcal{Y}^{-1/\gamma}$, whereas for $k > k_0$ we know that the estimate decays exponentially. We thus have

$$S_2 \lesssim \|f\|^2_{\mathbb{H}^{1-s}(\Omega)} \sum_{k=2}^{k_0} \left(\left(\frac{k}{M}\right)^{\gamma-1} \frac{\mathcal{Y}}{M} \right)^3 \lesssim \frac{\mathcal{Y}^{2/\gamma}}{M^2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)} \lesssim \frac{\mathcal{Y}^{1-\alpha}}{M^2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.$$

The first term S_1 is easy to estimate. Since $h_K \lesssim M^{-1}$ for all $K \in \mathscr{T}_\Omega$, we get

$$S_1 \lesssim M^{-2} \|\nabla_{x'} \nabla v\|^2_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim M^{-2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.$$

For elements in $\mathcal{T}_1$, we rely on the stability estimates (3.75), (3.76), (3.94) and (3.95) of $\Pi_{\mathscr{T}_\mathcal{Y}}$ and thus repeat the arguments used to derive (3.108) and (3.109). Adding the estimates for $\mathcal{T}_0$ and $\mathcal{T}_1$ we obtain the assertion. $\square$

Remark 3.24 (Choice of $\mathcal{Y}$) A natural choice of $\mathcal{Y}$ comes from equilibrating the two terms on the right-hand side of (3.110):

$$\epsilon \approx \#(\mathscr{T}_{\mathcal{Y}})^{-\frac{1}{n+1}} \quad \Leftrightarrow \quad \mathcal{Y} \approx \log(\#(\mathscr{T}_{\mathcal{Y}})).$$

This implies the near-optimal estimate

$$\|\mathscr{U} - V_{\mathscr{T}_{\mathcal{Y}}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}})|^s (\#\mathscr{T}_{\mathcal{Y}})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}. \tag{3.111}$$

Remark 3.25 (Estimate for u) In view of (3.53), we deduce the energy estimate

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}})|^s \, (\#\mathscr{T}_{\mathcal{Y}})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

We can rewrite this estimate in terms of regularity $u \in \mathbb{H}^{1+s}(\Omega)$ and $\#\mathscr{T}_\Omega$ as

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim |\log(\#\mathscr{T}_\Omega)|^s \, (\#\mathscr{T}_\Omega)^{-1/n} \|u\|_{\mathbb{H}^{1+s}(\Omega)}.$$

and realize that the order is near-optimal given the regularity shift from left to right. However, our PDE approach does not allow for a larger rate $(\#\mathscr{T}_\Omega)^{(2-s)/n}$ that would still be compatible with piecewise bilinear polynomials but not with (3.111).

Remark 3.26 (Computational complexity) The cost of solving the discrete problem (3.51) is related to $\#\mathscr{T}_{\mathcal{Y}}$, and not to $\#\mathscr{T}_\Omega$, but the resulting system is sparse. The structure of (3.51) is so that fast multilevel solvers can be designed with complexity proportional to $\#\mathscr{T}_{\mathcal{Y}}$; see Chapter 5 and [52]. On the other hand, using an integral formulation requires sparsification of an otherwise dense matrix with associated cost $(\#\mathscr{T}_\Omega)^2$.

Remark 3.27 (Fractional regularity) The function $\mathscr{U}$, solution of the α-harmonic extension problem, may also have singularities in the direction of the x'-variables and thus exhibit fractional regularity. This depends on Ω and the right hand side f (see Remark 3.4). The characterization of such singularities is as yet an open problem to us. The polynomial interpolation theory developed in § 3.3.2, however, applies to shape-regular but graded mesh $\mathscr{T}_\Omega$, which can resolve such singularities, provided we maintain the Cartesian structure of $\mathscr{T}_\mathscr{Y}$. The corresponding a posteriori error analysis is an entirely different but important direction; see Chapter 6.

Remark 3.28 (Simplicial elements) The approximation results presented and discussed in § 3.3.2.2, the interpolation theory developed in § 3.3.2.3 and § 3.3.2.4 and, consequently, the error estimates of this section hinge solely on the fact that the mesh $\mathscr{T}_\mathscr{Y}$ has a tensor product structure, i.e., it is composed of cells of the form $T = K \times I$. If we consider $\mathscr{T}_\Omega = \{K\}$ to be a mesh of $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ made of simplicial elements, together with the finite element space,

$$\mathbb{V}(\mathscr{T}_\mathscr{Y}) = \left\{ W \in C^0(\bar{\mathcal{C}}_\mathscr{Y}) : W|_T \in \mathbb{P}_1(K) \otimes \mathbb{P}_1(I) \ \forall T \in \mathscr{T}_\mathscr{Y}, \ W|_{\Gamma_D} = 0 \right\},$$

we can adapt, without major modifications, all the approximation, interpolation and convergence results of this work.

Remark 3.29 (Hanging nodes) It is important to notice that the assumption that the mesh is conforming was never explicitly used in the results of section 3.3 and that, actually, all that was required from the finite element space is the partition of unity property, i.e., (3.87). This observation allows us to generalize the results of section 3.3 to meshes that possess hanging nodes, which is important if one desires to use mesh adaptation to resolve possible singularities in the solution.

3.5 Numerical experiments for the fractional Laplacian

To illustrate the proposed techniques here we present a couple of numerical examples. The implementation has been carried out with the help of the `deal.II` library (see [17, 18]) which, by design, is based on tensor product elements and thus is perfectly suitable for our needs. The main concern while developing the code was correctness and, therefore, integrals are evaluated numerically with Gaussian quadratures of sufficiently high order and linear systems are solved using CG with ILU preconditioner with the exit criterion being that the ℓ^2-norm of the residual is less than 10^{-12}.

3.5.1 A square domain

Let $\Omega = (0,1)^2$. It is common knowledge that

$$\varphi_{m,n}(x_1,x_2) = \sin(m\pi x_1)\sin(n\pi x_2), \quad \lambda_{m,n} = \pi^2\left(m^2+n^2\right), \qquad m,n \in \mathbb{N}.$$

If $f(x_1,x_2) = (2\pi^2)^s \sin(\pi x_1)\sin(\pi x_2)$, by (3.3) we have

$$u(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2),$$

and, by (3.14),

$$\mathscr{U}(x_1,x_2,y) = \frac{2^{1-s}}{\Gamma(s)}(2\pi^2)^{s/2}\sin(\pi x_1)\sin(\pi x_2)y^s K_s(\sqrt{2}\pi y).$$

We construct a sequence of meshes $\{\mathscr{T}_{\mathcal{Y}_k}\}_{k\geq 1}$, where the triangulation of Ω is obtained by uniform refinement and the partition of $[0,\mathcal{Y}_k]$ is as in § 3.4.2, i.e., $[0,\mathcal{Y}_k]$ is divided with mesh points given by (3.106) with the election of the parameter

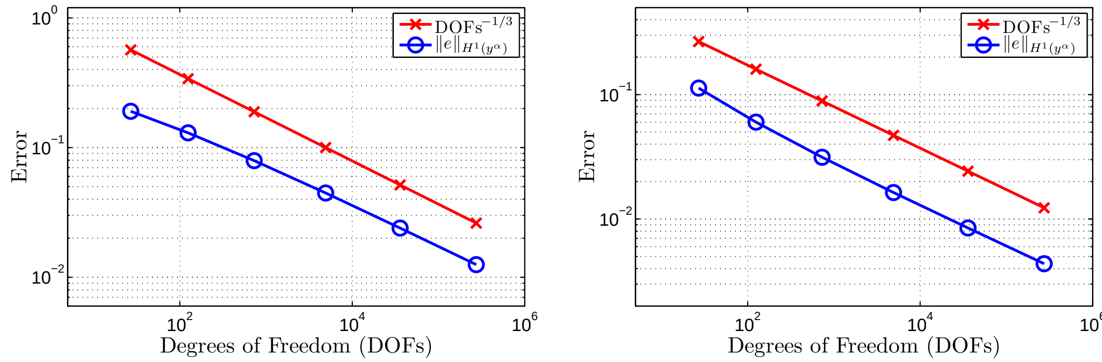

Figure 3.3: Computational rate of convergence for the approximate solution of the fractional Laplacian over a square with graded meshes on the extended dimension. The left panel shows the rate for $s = 0.2$ and the right one for $s = 0.8$. In both cases, the rate is $\approx (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3}$ in agreement with Theorem 3.23 and Remark 3.24

$\gamma > 3/(1-\alpha)$. On the basis of Theorem 3.10, for each mesh the truncation parameter $\mathcal{Y}_k$ is chosen so that $\epsilon \approx (\#\mathscr{T}_{\mathcal{Y}_{k-1}})^{-1/3}$. This can be achieved, for instance, by setting

$$\mathcal{Y}_k \geq \mathcal{Y}_{0,k} = \frac{2}{\sqrt{\lambda_1}}(\log C - \log \epsilon).$$

With this type of meshes,

$$\|u - U_{\mathscr{T}_{\Omega,k}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathscr{U} - V_{\mathscr{T}_{\mathcal{Y}_k}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}_k})|^s \cdot (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3},$$

which is near-optimal in $\mathscr{U}$ but suboptimal in u, since we should expect (see [39])

$$\|u - U_{\mathscr{T}_{\Omega,k}}\|_{\mathbb{H}^s(\Omega)} \lesssim h_{\mathscr{T}_\Omega}^{2-s} \lesssim (\#\mathscr{T}_{\mathcal{Y}_k})^{-(2-s)/3}.$$

Figure 3.3 shows the rates of convergence for $s = 0.2$ and $s = 0.8$ respectively. In both cases, we obtain the rate given by Theorem 3.23 and Remark 3.24.

3.5.2 A circular domain

Let $\Omega = \{|x'| \in \mathbb{R}^2 : |x'| < 1\}$. Using polar coordinates it can be shown that

$$\varphi_{m,n}(r,\theta) = J_m(j_{m,n}r)\left(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right), \tag{3.112}$$

where J_m is the m-th Bessel function of the first kind; $j_{m,n}$ is the n-th zero of J_m and $A_{m,n}$, $B_{m,n}$ are real normalization constants that ensure $\|\varphi_{m,n}\|_{L^2(\Omega)} = 1$ for all $m, n \in \mathbb{N}$. It is also possible to show that $\lambda_{m,n} = (j_{m,n})^2$.

If $f = (\lambda_{1,1})^s \varphi_{1,1}$, then (3.3) and (3.14) show that $u = \varphi_{1,1}$ and

$$\mathscr{U}(r,\theta,y) = \frac{2^{1-s}}{\Gamma(s)}(\lambda_{1,1})^{s/2}\varphi_{1,1}(r,\theta)y^s K_s(\sqrt{2}\pi y).$$

From [1, Chapter 9], we have that $j_{1,1} \approx 3.8317$.

We construct a sequence of meshes $\{\mathscr{T}_{\mathcal{Y}_k}\}_{k\geq 1}$, where the triangulation of Ω is obtained by quasi-uniform refinement and the partition of $[0, \mathcal{Y}_k]$ is as in § 3.4.2. The parameter $\mathcal{Y}_k$ is chosen so that $\epsilon \approx (\#\mathscr{T}_{\mathcal{Y}_{k-1}})^{-1/3}$. With these meshes

$$\|\mathscr{U} - V_{\mathscr{T}_{\mathcal{Y}_k}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}_k})|^s (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3}, \tag{3.113}$$

which is near-optimal.

Figure 3.4 shows the errors of $\|\mathscr{U} - V_{\mathscr{T}_{k,\mathcal{Y}}}\|_{H^1(y^\alpha,\mathcal{C}_{\mathcal{Y}_k})}$ for $s = 0.3$ and $s = 0.7$. The results, again, are in agreement with Theorem 3.23 and Remark 3.24.

3.5.3 Incompatible data for $s \in (0, 1)$

The computational results of previous paragraphs always entail $f \in \mathbb{H}^{1-s}(\Omega)$ and illustrate the error estimates of Theorem 3.23. Let us now consider a data f smooth

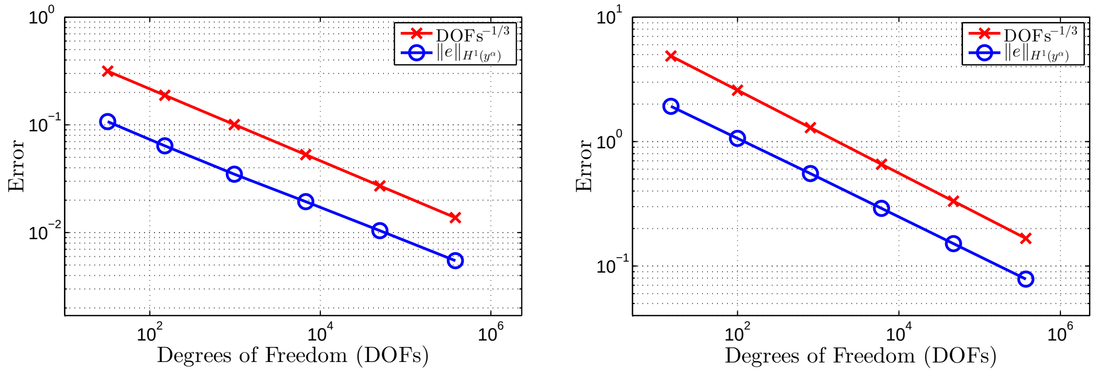

Figure 3.4: Computational rate of convergence for the approximate solution of the fractional Laplacian over a circle with graded meshes on the extended dimension. The left panel shows the rate for $s = 0.3$ and the right one for $s = 0.7$. In both cases, the rate is $\approx (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3}$ in agreement with Theorem 3.23 and Remark 3.24

but incompatible. Set $\Omega = (0, 1)$ and $f \equiv 1$. Notice that, if $s \leq \frac{1}{2}$ then $f \notin \mathbb{H}^{1-s}(\Omega)$ due to the fact that the function does not vanish at the boundary. In fact, we have that

$$\sum_{k=1}^{\infty} \lambda_k^{\sigma} |f_k|^2 < \infty \quad \Leftrightarrow \quad \sigma < \frac{1}{2},$$

in other words $f \in \mathbb{H}^{\sigma}(\Omega)$ if and only if $\sigma < \frac{1}{2}$. Since, the coefficients of the solution to (1.1) are given by $u_k = \lambda_k^{-s} f_k$, we can only expect that

$$\sum_{k=1}^{\infty} \lambda_k^{\mu} |u_k|^2 = \sum_{k=1}^{\infty} \lambda_k^{\mu-2s} |f_k|^2 < \infty \quad \Leftrightarrow \quad \mu - 2s < \frac{1}{2},$$

namely $u \in \mathbb{H}^{\mu}(\Omega)$ for $\mu < 2s + \frac{1}{2}$. In conclusion, full regularity is not possible but owing to the special character of the data some shift can be expected; see Remark 3.4 and the discussion at the end of § 3.1.3.

This heuristic argument is rather illuminating as it tells us that the best rate of convergence we can expect is

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \leq (\#\mathscr{T}_\Omega)^{-r}\|u\|_{\mathbb{H}^\mu(\Omega)},$$

with $r = \mu - s < s + \frac{1}{2}$. Since we are dealing with a one dimensional problem, the extension has two dimensions and, consequently, we expect

$$\|\mathscr{U} - V_{\mathscr{T}_\mathscr{Y}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim \begin{cases} (\#\mathscr{T}_\mathscr{Y})^{-\left(\frac{s}{2}+\frac{1}{4}\right)}, & s < \dfrac{1}{2}, \\ (\#\mathscr{T}_\mathscr{Y})^{-\frac{1}{2}}, & s > \dfrac{1}{2}. \end{cases} \tag{3.114}$$

Since $\lambda_k = \pi^2 k^2$ and $\varphi_k = \sqrt{2}\sin(\sqrt{\lambda_k}x')$, it is not difficult to show that $f_k = \sqrt{2}(1-(-1)^k)/\sqrt{\lambda_k}$, whence we can obtain an approximate solution $u_N = \sum_{k=1}^N \lambda_k^{-s} f_k \varphi_k$ with N sufficiently large. Figure 3.5 shows the norm of the difference between $V_{\mathscr{T}_\mathscr{Y}}$ and the α-harmonic extension of u_N for different values of s. The experimental rates of convergence seem to agree with (3.114): they are suboptimal for $s < \frac{1}{2}$.

To recover the optimal decay rate, we explore the a priori design of graded meshes in the x'-direction, which is within our theory of §3.3 and §3.4 (see Remark 3.27). Since $u \in \mathbb{H}^\mu(\Omega)$ with $\mu < 2s + \frac{1}{2}$, we expect that $u \approx r^{2s}$ as $r \to 0$, where r denotes the distance to the boundary. This, at least heuristically, can be figured out as follows: if $\partial_r^\mu r^{2s} \approx r^{2s-\mu}$, then

$$\|u\|^2_{\mathbb{H}^\mu(\Omega)} \approx \int_0^\varepsilon |\partial_r^\mu r^{2s}|^2 \,\mathrm{d}r < \infty \quad \Leftrightarrow \quad \mu < 2s + \frac{1}{2},$$

and $r^{2s} \in \mathbb{H}^\mu(\Omega)$ only for $\mu < 2s + \frac{1}{2}$.

Having guessed the nature of the singularity, we can apply the principle of error equidistribution as in § 3.4.2 to design an optimal graded mesh as x' approaches

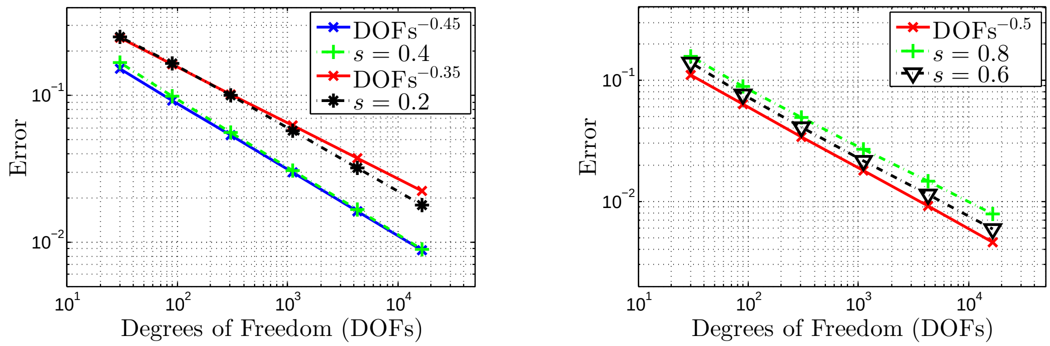

Figure 3.5: Computational rate of convergence for the approximate solution of the fractional Laplacian with incompatible data $f \equiv 1$. The domain Ω is the unit interval and the mesh is graded in the extended dimension. We show the $H^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$ norm of the difference between $V_{\mathscr{T}_\mathcal{Y}}$ and the harmonic extension of u_N with $N = 5 \cdot 10^4$. The left panel shows the rate for $s = 0.2$, 0.4 and the right one for $s = 0.6$, 0.8. As expected, the rate of convergence is optimal for values larger than $\frac{1}{2}$. On the other hand, if $s < \frac{1}{2}$ we see a reduction on the rate of convergence in accordance with (3.114).

either 0 or 1, with a grading parameter $\gamma > \frac{3}{2(1+s)}$ (compare with (3.106)). We proceed as follows: construct a quasi-uniform mesh of the interval $\Omega = (0,1)$ by bisection, and next transform the nodes $\mathtt{v}$ by the rule $\mathtt{v} \leftarrow \psi(\mathtt{v})$, where

$$\psi(\mathtt{v}) = \begin{cases} \dfrac{1}{4}\left(4\mathtt{v}\right)^\gamma, & \mathtt{v} \leq \dfrac{1}{4}, \\ \mathtt{v}, & \dfrac{1}{4} \leq \mathtt{v} \leq \dfrac{3}{4}, \\ 1 - \dfrac{1}{4}\left(4(1-\mathtt{v})\right)^\gamma, & \mathtt{v} \geq \dfrac{3}{4}. \end{cases} \tag{3.115}$$

We display in Figure 3.6 convergence plots for $s = 0.2$ and $s = 0.4$ over graded meshes in Ω which restore the optimal decay rate. The construction requires a priori knowledge of the solution, which is not obvious in higher dimensions. Adaptivity might provide a way to recover an optimal rate without such a knowledge (see

Remark 3.29 about hanging nodes).

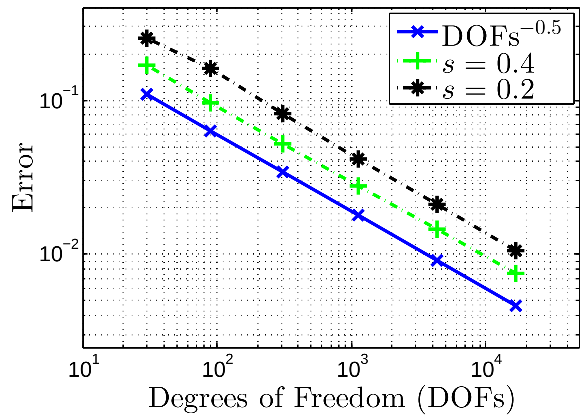

Figure 3.6: Computational rate of convergence for the approximate solution of the fractional Laplacian with incompatible data $f \equiv 1$ over meshes that are graded both in the x'- and y-directions. The domain Ω is the unit interval. The grading in the extended dimension obeys (3.106), whereas the one on the x'-direction is constructed using (3.115). We show the $H^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$ norm of the difference between $V_{\mathscr{T}_{\mathcal{Y}}}$ and the harmonic extension of u_N with $N = 5 \cdot 10^4$. An optimal rate of convergence can be recover irrespective of the fact that the solution does not possess full regularity.

3.6 Fractional powers of general second order elliptic operators

Let us now discuss how the methodology developed in previous sections extends to a general second order, symmetric and uniformly elliptic operator. This is an important property of our PDE approach. Recall that, in § 3.1.3, we discussed how the fractional Laplace operator can be realized as a Dirichlet to Neumann map via an extension problem posed on the semi-infinite cylinder $\mathcal{C}$. In the work of Stinga and

Torrea [155], the same type of characterization has been developed for the fractional powers of second order elliptic operators.

Let $\mathcal{L}$ be a second order symmetric differential operator of the form

$$\mathcal{L}w = -\mathrm{div}_{x'}(A\nabla_{x'}w) + cw, \tag{3.116}$$

where $c \in L^\infty(\Omega)$ with $c \geq 0$ almost everywhere, $A \in \mathcal{C}^{0,1}(\Omega, \mathsf{GL}(n, \mathbb{R}))$ is symmetric and positive definite, and Ω is Lipschitz. Given $f \in L^2(\Omega)$, the Lax-Milgram lemma shows that there is a unique $w \in H_0^1(\Omega)$ that solves

$$\mathcal{L}w = f \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega.$$

The operator $\mathcal{L}^{-1} : L^2(\Omega) \to L^2(\Omega)$ is positive, compact and symmetric, whence its spectrum is discrete, positive and accumulates at zero. Moreover, there exists $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times H_0^1(\Omega)$ such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and for, $k \in \mathbb{N}$,

$$\mathcal{L}\varphi_k = \lambda_k \varphi_k \text{ in } \Omega, \qquad \varphi_k = 0 \text{ on } \partial\Omega, \tag{3.117}$$

and $\lambda_k \to \infty$ as $k \to \infty$. For $u \in C_0^\infty(\Omega)$ we then define the fractional powers of $\mathcal{L}$ as

$$\mathcal{L}^s u = \sum_{k=1}^{\infty} u_k \lambda_k^s \varphi_k, \tag{3.118}$$

where $u_k = \int_\Omega u\varphi_k$. By density the operator $\mathcal{L}^s$ can be extended again to $\mathbb{H}^s(\Omega)$. This discussion shows that it is legitimate to study the following problem: given $s \in (0, 1)$ and $f \in \mathbb{H}^{-s}(\Omega)$, find $u \in \mathbb{H}^s(\Omega)$ such that

$$\mathcal{L}^s u = f \text{ in } \Omega. \tag{3.119}$$

To realize the operator $\mathcal{L}^s$ as the Dirichlet to Neumann map of an extension problem we use the generalization of the result by Caffarelli and Silvestre presented in [155]. We seek a function $\mathscr{U} : \mathcal{C} \to \mathbb{R}$ that solves

$$\begin{cases} \mathcal{L}\mathscr{U} - \dfrac{\alpha}{y}\partial_y \mathscr{U} - \partial_{yy}\mathscr{U} = 0, & \text{in } \mathcal{C}, \\ \mathscr{U} = 0, & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial \mathscr{U}}{\partial \nu^\alpha} = d_s f, & \text{on } \Omega \times \{0\}, \end{cases} \tag{3.120}$$

where the constant d_s is as in (3.13). In complete analogy to § 3.1.3 it is possible to show that

$$d_s \mathcal{L}^s u = \frac{\partial \mathscr{U}}{\partial \nu^\alpha} : \mathbb{H}^s(\Omega) \longmapsto \mathbb{H}^{-s}(\Omega).$$

Notice that the differential operator in (3.120) is

$$\operatorname{div}(y^\alpha \mathbf{A} \nabla \mathscr{U}) + y^\alpha c \mathscr{U},$$

where, for all $x \in \mathcal{C}$, $\mathbf{A}(x) = \operatorname{diag}\{A(x'), 1\} \in \mathsf{GL}(n+1, \mathbb{R})$.

It suffices now to notice that both $y^\alpha c$ and $y^\alpha \mathbf{A}$ are in $A_2(\mathbb{R}^{n+1}_+)$, to conclude that, given $f \in \mathbb{H}^{-s}(\Omega)$, there is a unique $\mathscr{U} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ that solves (3.120), [79]. In addition, $u = \mathscr{U}(\cdot, 0) \in \mathbb{H}^s(\Omega)$ solves (3.119) and we have the stability estimate

$$\|u\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla \mathscr{U}\|_{L^2(\mathcal{C}, y^\alpha)} \lesssim \|f\|_{\mathbb{H}^{-s}(\Omega)}, \tag{3.121}$$

where the hidden constants depend on A, c, C_{2,y^α} and Ω.

The representation (3.14) of $\mathscr{U}$ in terms of the Bessel functions is still valid. Consequently, we can show $\mathscr{U}_{yy} \in L^2(\mathcal{C}, y^\beta)$. We can also repeat the arguments in the proof of Theorem 3.10 to conclude that

$$\|\nabla \mathscr{U}\|_{L^2(\Omega \times (\mathscr{Y}, \infty), y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathscr{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)},$$

and introduce $v \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ — solution of a truncated version of (3.120) — and show that

$$\|\nabla(\mathscr{U} - v)\|_{L^2(\mathcal{C},y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.122}$$

Next, we define the finite element approximation of the solution to (3.120) as the unique function $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ that solves

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \mathbf{A}(x)\nabla V_{\mathscr{T}_\mathcal{Y}} \cdot \nabla W + y^\alpha c(x') V_{\mathscr{T}_\mathcal{Y}} W \,\mathrm{d}x' \,\mathrm{d}y = d_s\langle f, \mathrm{tr}_\Omega W\rangle, \quad \forall W \in \mathbb{V}(\mathscr{T}_\mathcal{Y}). \tag{3.123}$$

We construct, as in § 3.4.2, a shape regular triangulation $\mathscr{T}_\Omega$ of Ω, which we extend to $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$ with the partition given in (3.106), with $\gamma > 3/(1-\alpha)$. Following the proof of Theorem 3.23 we can also show the following error estimate.

Theorem 3.30 (Error estimate for general operators) *Let $V_\mathscr{T} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ be the solution of* (3.123) *and $U_{\mathscr{T}_\Omega} \in \mathbb{U}(\mathscr{T}_\Omega)$ be defined as in* (3.52). *If $\mathscr{U}$, solution of* (3.120)*, is such that $\mathcal{L}\mathscr{U},\ \partial_y\nabla\mathscr{U} \in L^2(\mathcal{C}, y^\alpha)$, then we have*

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathcal{U} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_\mathcal{Y})|^s(\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)}\|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

3.7 Conclusions

We develop PDE solution techniques for problems involving fractional powers of the Laplace operator $(-\Delta)^s u = f$ in a bounded domain Ω with Dirichlet boundary conditions. To overcome the inherent difficulty of nonlocality, we exploit the *cylindrical* extension proposed and investigated by X. Cabré and J. Tan [42], which is in turn inspired in the breakthrough by L. Caffarelli and L. Silvestre [43]. This leads to the (local) elliptic PDE (1.2) in one higher dimension y, with variable coefficient y^α, $\alpha = 1 - 2s$, which either degenerates ($s < 1/2$) or blows up ($s > 1/2$). Several remarks and comparisons with recent literature are now in order:

- *Regularity.* In § 3.1.5 we derive global and local regularity estimates for the solution of problem (1.2) in weighted Sobolev spaces.

- *Truncation.* In § 3.2 we propose the truncated problem (3.36), and show exponential convergence in the extended variable y to the solution of problem (1.2).

- *Tensor product meshes.* In § 3.3.1 we study a finite element strategy to approximate problem (1.2) which allows anisotropic elements in the extended dimension y.

- *Anisotropic interpolation theory.* In § 3.3.2 we extend the anisotropic interpolation estimates of [70] to the weighted Sobolev space $H^1(y^\alpha)$. This hinges on $y^\alpha \in A_2(\mathbb{R}^{n+1})$ and gives rise to a theory in Muckenhoupt weighted Sobolev spaces with a general weight in the class A_p ($1 < p < \infty$) along with applications [132].

- *Error analysis.* In § 3.4.1 we derive a priori error estimates for quasi-uniform meshes which exhibit optimal regularity, according to § 3.1.5, but suboptimal order. In § 3.4.2 we restore the optimal decay rate upon constructing suitably graded meshes in the extended variable y and applying the interpolation theory of § 3.3.2.

- *Assumptions on f and Ω.* We assume the regularity conditions of Remark 3.6 throughout solely for convenience. We could in fact compensate the lack of such regularity via graded but shape regular meshes in Ω, as illustrated in § 3.5.3, which are within our theory.

- *General operators.* In § 3.6 we extend our FEM and supporting theory to general linear second order, symmetric and uniformly elliptic operators.

Chapter 4: Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications

Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in inferring that he is an inexact man.

— Bertrand Russell.

4.1 Introduction

A fundamental tool in analysis, with both practical and theoretical relevance, is the approximation of a function by a simpler one. For continuous functions a foundational result in this direction was given by K. Weierstrass in 1885: continuous functions defined on a compact interval can be uniformly approximated as closely as desired by polynomials. Mollifiers, interpolants, splines and even Nevanlinna-Pick theory can be regarded as instances of this program; see, for instance, [5, 114]. For weakly differentiable functions, the approximation by polynomials is very useful when trying to understand their behavior. In fact, this idea goes back to S.L. Sobolev [150], who used a sort of averaged Taylor polynomial to discuss equivalent norms in Sobolev spaces.

The role of polynomial approximation and error estimation is crucial in numerical analysis: it is the basis of discretization techniques for partial differential equations (PDE), particularly the finite element method. For the latter, several constructions for standard Sobolev spaces W^1_p, with $1 \leq p \leq \infty$, and their properties are well studied; see [58, 68, 69, 70, 143].

On the other hand, many applications lead to boundary value problems for nonuniformly elliptic equations. The ellipticity distortion can be caused by degenerate/singular behavior of the coefficients of the differential operator or by singularities in the domain. For such equations it is natural to look for solutions in weighted Sobolev spaces [6, 21, 41, 43, 60, 72, 79, 80, 108, 158] and to study the regularity properties of the solution in weighted spaces as well [110]. Of particular importance are weighted Sobolev spaces with a weight belonging to the so-called Muckenhoupt class A_p [126]; see also [79, 101, 158]. However, the literature focusing on polynomial approximation in this type of Sobolev spaces is rather scarce; we refer the reader to [6, 8, 11, 21, 60, 83, 88, 116] for some partial results. Most of these results focus on a particular degenerate elliptic equation and exploit the special structure of the coefficient to derive polynomial interpolation results.

To fix ideas, consider the following nonuniformly elliptic boundary value problem: let Ω be an open and bounded subset of $\mathbb{R}^n$ $(n \geq 1)$ with boundary $\partial\Omega$. Given a function f, find u that solves

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x)\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathcal{A} : \Omega \to \mathbb{R}^{n\times n}$ is symmetric and satisfies the following nonuniform ellipticity condition

$$\omega(x)|\xi|^2 \lesssim \xi^\intercal \mathcal{A}(x)\xi \lesssim \omega(x)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad a.e.\ \Omega. \tag{4.2}$$

Here the relation $a \lesssim b$ indicates that $a \leq Cb$, with a constant C and ω is a weight function, i.e., a nonnegative and locally integrable measurable function, which might vanish, blow up, and possess singularities. Examples of this type of equations are the harmonic extension problem related with the fractional Laplace operator [41, 43], elliptic problems involving measures [6, 60], elliptic PDE in an axisymmetric three dimensional domain with axisymmetric data [21, 88], and equations modeling the motion of particles in a central potential field in quantum mechanics [11]. Due to the nature of the coefficient $\mathcal{A}$, the classical Sobolev space $H^1(\Omega)$ is not appropriate for the analysis and approximation of this problem.

Nonuniformly elliptic equations of the type (4.1)–(4.2), with ω in the so-called Muckenhoupt class A_2, have been studied in [79]: for $f \in L^2(\omega^{-1}, \Omega)$, there exists a unique solution in $H_0^1(\omega, \Omega)$ [79, Theorem 2.2] (see § 2.2 for notation). Consider the discretization of (4.1) with the finite element method. Let $\mathscr{T}$ be a conforming triangulation of Ω and let $\mathbb{V}(\mathscr{T})$ be a finite element space. The Galerkin approximation of the solution to (4.1) is given by the unique function $U_{\mathscr{T}} \in \mathbb{V}(\mathscr{T})$ that solves

$$\int_\Omega \mathcal{A}\nabla U_{\mathscr{T}} \cdot \nabla W = \int_\Omega fW, \quad \forall W \in \mathbb{V}(\mathscr{T}). \tag{4.3}$$

Invoking Galerkin orthogonality, we deduce

$$\|u - U_{\mathscr{T}}\|_{H_0^1(\omega,\Omega)} \lesssim \inf_{W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})} \|u - W\|_{H_0^1(\omega,\Omega)}. \tag{4.4}$$

In other words, the numerical analysis of this boundary value problem reduces to a result in approximation theory: the distance between the exact solution u and its approximation $U_{\mathscr{T}}$ in a finite element space is bounded by the best approximation error in the finite element space with respect to an appropriate weighted Sobolev norm. A standard way of obtaining bounds for the approximation error is by considering $W = \Pi_{\mathscr{T}} v$ in (4.4), where $\Pi_{\mathscr{T}}$ is a suitable interpolation operator.

The purpose of this work is twofold. We first go back to the basics, and develop an elementary constructive approach to piecewise polynomial interpolation in weighted Sobolev spaces with Muckenhoupt weights. We consider an averaged version of the Taylor polynomial and, upon using an appropriate weighted Poincaré inequality and a cancellation property, we derive optimal approximation estimates for constant and linear approximations. We extend these results to any polynomial degree m ($m \geq 0$), by a simple induction argument.

The functional framework considered is weighted Sobolev spaces with weights in the Muckenhoupt class $A_p(\mathbb{R}^n)$, thereby extending the classical polynomial approximation theory in Sobolev spaces

The second main contribution of this work is the construction of a quasi-interpolation operator $\Pi_{\mathscr{T}}$, built on local averages over stars and thus well defined for functions in $L^1(\Omega)$ as those in [58, 143]. The ensuing polynomial approximation theory in weighted Sobolev spaces with Muckenhoupt weights allows us to obtain optimal and local interpolation estimates for the quasi-interpolant $\Pi_{\mathscr{T}}$. On simplicial discretizations, these results hold true for any polynomial degree $m \geq 0$, and they are derived in the weighted W^k_p-seminorm ($0 \leq k \leq m+1$). The key ingredient is an invariance property of the quasi-interpolant $\Pi_{\mathscr{T}}$ over the finite element space.

On the other hand, on rectangular discretizations, we only assume that neighboring cells in $\mathscr{T}$ have comparable size, as in [70, 129]. This mild assumption enables us also to obtain anisotropic error estimates for domains that can be decomposed into n–rectangles. These estimates are derived in the weighted W_p^1-semi-norm and the weighted L^p-norm, the latter being a new result even for the unweighted setting. For $m = 0, 1$, we also derive interpolation estimates in the space $W_q^m(\rho, \Omega)$ when the smoothness is measured in the space $W_p^{m+1}(\omega, \Omega)$, with different weights $\omega \neq \rho$ and Lebesgue exponents $1 < p \leq q$, provided $W_p^{m+1}(\omega, \Omega) \hookrightarrow W_q^m(\rho, \Omega)$.

The outline of this Chapter is as follows. Section 4.2 is dedicated to an important weighted L^p-based Poincaré inequality over star-shaped domains and domains that can be written as the finite union of star-shaped domains. In section 4.3, we consider an averaged version of the Taylor polynomial, and we develop a constructive theory of piecewise polynomial interpolation in weighted Sobolev spaces with Muckenhoupt weights. We discuss the quasi-interpolation operator $\Pi_{\mathscr{T}}$ and its properties in section 4.4. We derive optimal approximation properties in the weighted W_p^k-seminorm for simplicial triangulations in § 4.4.1. In § 4.4.2 we derive anisotropic error estimates on rectangular discretizations for a $\mathbb{Q}_1$ quasi-interpolant operator assuming that Ω is an n-rectangle. Section 4.5 is devoted to derive optimal and local interpolation estimates for different metrics (i.e., $p \leq q$, $\omega \neq \rho$). Finally, in section 4.6 we present applications of our interpolation theory to nonuniformly elliptic equations (4.1), elliptic equations with singular sources, and fractional powers of elliptic operators.

4.2 A weighted Poincaré inequality

In order to obtain interpolation error estimates in $L^p(\omega,\Omega)$ and $W^1_p(\omega,\Omega)$, it is instrumental to have a weighted Poincaré-like inequality [70, 129]. A pioneering reference is the work by Fabes, Kenig and Serapioni [79], which shows that, when the domain is a ball and the weight belongs to A_p $1 < p < \infty$, a weighted Poincaré inequality holds [79, Theorem 1.3 and Theorem 1.5]. For generalizations of this result see [82, 98]. For a star-shaped domain, and a specific A_2-weight, we have proved a weighted Poincaré inequality in §3.3.2.1 (see also [129, Lemma 4.2]). In this section we extend this result to a general exponent p and a general weight $\omega \in A_p(\mathbb{R}^n)$. Our proof is constructive and not based on a compactness argument. This allows us to trace the dependence of the stability constant on the domain geometry.

Lemma 4.1 (weighted Poincaré inequality I) *Let $S \subset \mathbb{R}^n$ be bounded, star-shaped with respect to a ball $\hat{B}$, with $\operatorname{diam} S \approx 1$. Let χ be a continuous function on S with $\|\chi\|_{L^1(S)} = 1$. Given $\omega \in A_p(\mathbb{R}^n)$, we define $\mu(x) = \omega(\mathbf{A}x + \mathbf{b})$, for $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} = a \cdot \mathbf{I}$, with $a \in \mathbb{R}$. If $v \in W^1_p(\mu, S)$ is such that $\int_S \chi v = 0$, then*

$$\|v\|_{L^p(\mu,S)} \lesssim \|\nabla v\|_{L^p(\mu,S)}, \tag{4.5}$$

where the hidden constant depends only on χ, $C_{p,\omega}$ and the radius $\hat{r}$ of $\hat{B}$, but is independent of $\mathbf{A}$ and $\mathbf{b}$.

Proof: Property (v) of Proposition 2.1 shows that $\mu \in A_p(\mathbb{R}^n)$ and $C_{\mu,p} = C_{\omega,p}$. Given $v \in W_p^1(\mu, S)$, we define

$$\tilde{v} = \operatorname{sign}(v)|v|^{p-1}\mu - \left(\int_S \operatorname{sign}(v)|v|^{p-1}\mu\right)\chi.$$

Hölder's inequality yields

$$\int_S \mu|v|^{p-1} = \int_S \mu^{1/p'}|v|^{p-1}\mu^{1/p} \le \left(\int_S \mu|v|^p\right)^{1/p'}\left(\int_S \mu\right)^{1/p} \lesssim \|v\|_{L^p(\mu,S)}^{p-1}, \qquad (4.6)$$

which implies that $\tilde{v} \in L^1(S)$ and $\|\tilde{v}\|_{L^1(S)} \lesssim \|v\|_{L^p(\mu,S)}^{p-1}$. Notice, in addition, that since $\int_S \chi = 1$, the function $\tilde{v}$ has vanishing mean value.

Given $1 < p < \infty$, we define $q = -p'/p$, and we notice that $q + p' = 1$ and $p'(p-1) = p$. We estimate $\|\tilde{v}\|_{L^{p'}(\mu^q,S)}$ as follows:

$$\begin{aligned}\left(\int_S \mu^q|\tilde{v}|^{p'}\right)^{1/p'} &= \left(\int_S \mu^q \left|\operatorname{sign}(v)|v|^{p-1}\mu - \left(\int_S \operatorname{sign}(v)|v|^{p-1}\mu\right)\chi\right|^{p'}\right)^{1/p'}\\ &\le \left(\int_S \mu^{q+p'}|v|^{p'(p-1)}\right)^{1/p'} + \left(\int_S |v|^{p-1}\mu\right)\|\chi\|_{L^{p'}(\mu^q,S)}\\ &\lesssim \|v\|_{L^p(\mu,S)}^{p-1},\end{aligned}$$

where we have used (4.6) together with the fact that $\mu \in A_p(\mathbb{R}^n)$ implies $\mu^q \in L^1_{\text{loc}}(\mathbb{R}^n)$ (see Proposition 2.1 (i)), whence $\|\chi\|_{L^{p'}(\mu^q,S)} \le \|\chi\|_{L^\infty(S)}\mu^q(S)^{1/p'} \lesssim 1$.

Properties $\mu^q \in A_{p'}(\mathbb{R}^n)$, that S is star-shaped with respect to $\hat{B}$ and $\tilde{v} \in L^{p'}(\mu^q, S)$ has vanishing mean value, suffice for the existence of $\mathbf{u} \in \left[\mathring{W}_{p'}^1(\mu^q, S)\right]^n$ satisfying

$$\operatorname{div} \mathbf{u} = \tilde{v},$$

and,

$$\|\nabla \mathbf{u}\|_{\left[W_{p'}^1(\mu^q,S)\right]^n} \lesssim \|\tilde{v}\|_{L^{p'}(\mu^q,S)}, \qquad (4.7)$$

where the hidden constant depends on C_{p',μ^q} and the radius r of $\hat{B}$; see [72, Theorem 3.1].

Finally, since $\int_S \chi v = 0$, the definition of $\tilde{v}$ implies

$$\|v\|^p_{L^p(\mu,S)} = \int_S v\tilde{v} + \left(\int \operatorname{sign}(v)|v|^{p-1}\mu\right)\int_S \chi v = \int_S v\tilde{v}.$$

Replacing $\tilde{v}$ by $-\operatorname{div}\mathbf{u}$, integrating by parts and using (4.7), we conclude

$$\begin{aligned}\|v\|^p_{L^p(\mu,S)} = -\int_S \nabla v\cdot\mathbf{u} &\le \left(\int_S \mu|\nabla v|^p\right)^{1/p}\left(\int_S \mu^q|\mathbf{u}|^{p'}\right)^{1/p'}\\ &\lesssim \|\nabla v\|_{L^p(\mu,S)}\|\tilde{v}\|_{L^{p'}(\mu^q,S)}.\end{aligned}$$

Invoking $\|\tilde{v}\|_{L^{p'}(\mu^q,S)} \lesssim \|v\|^{p-1}_{L^p(\mu,S)}$ yields the desired inequality. $\square$

In section 4.4 we construct an interpolation operator based on local averages. Consequently, the error estimates on an element T depend on the behavior of the function over a so-called *patch* of T, which is not necessarily star shaped. Then, we need to relax the geometric assumptions on the domain S and let the vanishing mean property hold just in a subdomain. The following result is an adaptation of Corollary 4.2 (see also [129, Corollary 4.4]).

Corollary 4.2 (weighted Poincaré inequality II) *Let $S = \cup_{i=1}^N S_i \subset \mathbb{R}^n$ be a connected domain and each S_i be star-shaped with respect to a ball B_i. Let $\chi_i \in C^0(\bar{S}_i)$ and μ be as in Lemma 4.1. If $v \in W^1_p(\mu,S)$ and $v_i = \int_{\omega_i} v\chi_i$, then*

$$\|v - v_i\|_{L^p(\mu,S)} \lesssim \|\nabla v\|_{L^p(\mu,S)} \qquad \forall 1 \le i \le N, \tag{4.8}$$

where the hidden constant depends on $\{\chi_i\}_{i=1}^N$, the radii r_i of B_i, and the amount of overlap between the subdomains $\{S_i\}_{i=1}^N$, but is independent of $\mathbf{A}$ and $\mathbf{b}$.

Proof: This is an easy consequence of Lemma 4.1 and [68, Theorem 7.1]. For completeness, we sketch the proof. It suffices to deal with two subdomains S_1, S_2 and the overlapping region $D = S_1 \cap S_2$. We start from

$$\|v - v_1\|_{L^p(\mu,S_2)} \leq \|v - v_2\|_{L^p(\mu,S_2)} + \|v_1 - v_2\|_{L^p(\mu,S_2)}.$$

Since v_1 and v_2 are constant

$$\|v_1 - v_2\|_{L^p(\mu,S_2)} = \left(\frac{\mu(S_2)}{\mu(D)}\right)^{1/p} \|v_1 - v_2\|_{L^p(\mu,D)},$$

which together with

$$\|v_1 - v_2\|_{L^p(\mu,D)} \leq \|v - v_1\|_{L^p(\mu,S_1)} + \|v - v_2\|_{L^p(\mu,S_2)},$$

and (4.5) imply $\|v - v_1\|_{L^p(\mu,S_2)} \lesssim \|\nabla v\|_{L^p(\mu,S_1 \cup S_2)}$. This and (4.5) give (4.8) for $i = 1$, with a stability constant depending on the ratio $\frac{\mu(S_2)}{\mu(D)}$. □

4.3 Approximation theory in weighted Sobolev spaces

In this section, we introduce an averaged version of the Taylor polynomial and study its approximation properties in Muckenhoupt weighted Sobolev spaces. Our results are optimal and are used to obtain error estimates for the quasi-interpolation operator defined in section 4.4 on simplicial and rectangular discretizations. The interpolation operator is built on local averages over stars, and so is similar to the one introduced in [68]. The main difference is that it is directly defined on the given mesh instead of using a reference element. This idea is fundamental in order to relax the regularity assumptions on the elements, which is what allows us to derive the anisotropic estimates on rectangular elements presented in § 4.4.2.

4.3.1 Discretization

We start with some terminology and describe the construction of the underlying finite element spaces. In order to avoid technical difficulties we shall assume $\partial\Omega$ is polyhedral. We denote by $\mathscr{T} = \{T\}$ a partition, or mesh, of Ω into elements T (simplices or cubes) such that

$$\bar{\Omega} = \bigcup_{T \in \mathscr{T}} T, \qquad |\Omega| = \sum_{T \in \mathscr{T}} |T|.$$

The mesh $\mathscr{T}$ is assumed to be conforming or compatible: the intersection of any two elements is either empty or a common lower dimensional element.

The collection of all conforming meshes is denoted by $\mathbb{T}$. We say that $\mathbb{T}$ is *shape regular* if there exists a constant $\sigma > 1$ such that, for all $\mathscr{T} \in \mathbb{T}$,

$$\max\{\sigma_T : T \in \mathscr{T}\} \leq \sigma, \tag{4.9}$$

where $\sigma_T = h_T/\rho_T$ is the shape coefficient of T. In the case of simplices, $h_T = \mathrm{diam}(T)$ and ρ_T is the diameter of the sphere inscribed in T; see, for instance, [39]. For the definition of h_T and ρ_T in the case of n-rectangles see [56].

We assume the collection of meshes $\mathbb{T}$ to be conforming and satisfying the regularity assumption (4.9). In § 4.4.2, we consider rectangular discretizations of the domain $\Omega = (0,1)^n$ which satisfy a weaker regularity assumption and thus allow for anisotropy in each coordinate direction (cf. [70]).

Given a conforming mesh $\mathscr{T} \in \mathbb{T}$, we define the finite element space of continuous piecewise polynomials of degree $m \geq 1$

$$\mathbb{V}(\mathscr{T}) = \left\{ W \in \mathcal{C}^0(\bar{\Omega}) : W_{|T} \in \mathcal{P}(T) \ \forall T \in \mathscr{T}, \ W_{|\partial\Omega} = 0 \right\}, \tag{4.10}$$

where, for a simplicial element T, $\mathcal{P}(T)$ corresponds to $\mathbb{P}_m$ — the space of polynomials of total degree at most m. If T is an n-rectangle, then $\mathcal{P}(T)$ stands for $\mathbb{Q}_m$ — the space of polynomials of degree not larger than m in each variable.

Given an element $T \in \mathscr{T}$, we denote by $\mathcal{N}(T)$ and $\mathring{\mathcal{N}}(T)$ the set of nodes and interior nodes of T, respectively. We set $\mathcal{N}(\mathscr{T}) := \cup_{T\in\mathscr{T}}\mathcal{N}(T)$ and $\mathring{\mathcal{N}}(\mathscr{T}) := \mathcal{N}(\mathscr{T}) \cap \partial\Omega$. Then, any discrete function $V \in \mathbb{V}(\mathscr{T})$ is characterized by its nodal values on the set $\mathring{\mathcal{N}}(\mathscr{T})$. Moreover, the functions $\phi_z \in \mathbb{V}(\mathscr{T})$, $z \in \mathring{\mathcal{N}}(\mathscr{T})$, such that $\phi_z(y) = \delta_{yz}$ for all $y \in \mathcal{N}(\mathscr{T})$ are the canonical basis of $\mathbb{V}(\mathscr{T})$, and

$$V = \sum_{z\in\mathring{\mathcal{N}}(\mathscr{T})} V(z)\phi_z.$$

The functions $\{\phi_z\}_{z\in\mathring{\mathcal{N}}(\mathscr{T})}$ are the so called *shape functions*.

Given $z \in \mathcal{N}(\mathscr{T})$, the *star* or patch around z is $S_z := \bigcup_{z\in T} T$, and, for $T \in \mathscr{T}$, its *patch* is $S_T := \bigcup_{z\in T} S_z$. For each $z \in \mathcal{N}(\mathscr{T})$, we define $h_z := \min\{h_T : z \in T\}$.

4.3.2 The averaged interpolation operator

We now develop an approximation theory in Muckenhoupt weighted Sobolev spaces, which is instrumental in section 4.4. We define an averaged Taylor polynomial, built on local averages over stars and thus well defined for $L^p(\omega, \Omega)$-functions. Exploiting the weighted Poincaré inequality derived in section 4.2, we show optimal error estimates for constant and linear approximations. These results are the basis to extend these estimates to any polynomial degree via a simple induction argument in section 4.3.4.

Let $\psi \in C^\infty(\mathbb{R}^n)$ be such that $\int \psi = 1$ and $\operatorname{supp}\psi \subset B$, where B denotes the ball in $\mathbb{R}^n$ of radius r and centered at zero and $r \leq 1/\sigma$, with σ defined as in (4.9). For $z \in \mathring{\mathcal{N}}(\mathscr{T})$, we define the rescaled smooth function

$$\psi_z(x) = \frac{1}{h_z^n}\psi\left(\frac{z-x}{h_z}\right). \tag{4.11}$$

Owing to the regularity assumption (4.9) and $r \leq 1/\sigma$, we have $\operatorname{supp}\psi_z \subset S_z$.

Given a smooth function v, we denote by $P^m v(x,y)$ the Taylor polynomial of order m in the variable y about the point x, i.e.,

$$P^m v(x,y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha v(x)(y-x)^\alpha. \tag{4.12}$$

For $z \in \mathring{\mathcal{N}}(\mathscr{T})$, and $v \in W_p^m(\omega,\Omega)$, we define the corresponding *averaged* Taylor polynomial of order m of v about the node z as

$$Q_z^m v(y) = \int P^m v(x,y)\psi_z(x)\,\mathrm{d}x. \tag{4.13}$$

Integration by parts shows that $Q_z^m v$ is well-defined for functions in $L^1(\Omega)$ [39, Proposition 4.1.12]. Proposition 2.3 then allows us to conclude that (4.13) is well defined for $v \in L^p(\omega,\Omega)$. Since $\operatorname{supp}\psi_z \subset S_z$, the integral appearing in (4.13) can be also written over S_z. Moreover, we have the following properties of $Q_z^m v$:

- $Q_z^m v$ is a polynomial of degree less or equal than m in the variable y (cf. [39, Proposition 4.1.9]).

- $Q_z^m v = Q_z^m Q_z^m v$, i.e., Q_z^m is invariant over $\mathbb{P}_m$.

- For any α such that $|\alpha| \leq m$,

$$D^\alpha Q_z^m v = Q_z^{m-|\alpha|} D^\alpha v \qquad \forall v \in W_1^{|\alpha|}(B), \tag{4.14}$$

(cf. [39, Proposition 4.1.17]). As a consequence of $\omega \in A_p(\mathbb{R}^n)$, together with Proposition 2.3, we have that (4.14) holds for v in $W_1^{|\alpha|}(\omega, B)$.

The following stability result is important in the subsequent analysis.

Lemma 4.3 (stability of Q_z^m) *Let $\omega \in A_p(\mathbb{R}^n)$ and $z \in \mathring{\mathscr{N}}(\mathscr{T})$. If $v \in W_p^k(\omega, S_z)$, with $0 \leq k \leq m$, we have the following stability result*

$$\|Q_z^m v\|_{L^\infty(S_z)} \lesssim \|\psi_z\|_{L^{p'}(\omega^{-p'/p}, S_z)} \sum_{l=0}^{k} h_z^l |v|_{W_p^l(\omega, S_z)}. \tag{4.15}$$

Proof: Using the definition of the averaged Taylor polynomial (4.13), we arrive at

$$\|Q_z^m v\|_{L^\infty(S_z)} \lesssim \sum_{|\alpha| \leq m} \left\| \int_{S_z} D^\alpha v(x)(y-x)^\alpha \psi_z(x) \, \mathrm{d}x \right\|_{L^\infty(S_z)}.$$

This implies estimate (4.15) if $k = m$. Otherwise, integration by parts on the higher derivatives $D^\alpha v$ with $k < |\alpha| \leq m$, together with $\psi_z = 0$ on ∂S_z, and the estimate $|y - x| \lesssim h_z$ for all $x, y \in S_z$, together with Hölder's inequality, yields (4.15). $\square$

Given $\omega \in A_p(\mathbb{R}^n)$ and $v \in W_p^{m+1}(\omega, \Omega)$ with $m \geq 0$, in the next section we derive approximation properties of the averaged Taylor polynomial $Q_z^m v$ in the weighted $W_p^k(\omega, \Omega)$-norm, with $0 \leq k \leq m$, via a weighted Poincaré inequality and a simple induction argument. Consequently, we must first study the approximation properties of $Q_z^0 v$, the weighted average of $v \in L^p(\omega, \Omega)$, which for $z \in \mathring{\mathscr{N}}(\mathscr{T})$ reads

$$Q_z^0 v = \int_{S_z} v(x) \psi_z(x) \, \mathrm{d}x. \tag{4.16}$$

4.3.3 Weighted L^p-based error estimates

We start by adapting the proofs of [70, Lemma 2.3] and [129, Lemma 4.5] to obtain local approximation estimates in the weighted L^p-norm for the polynomials $Q_z^0 v$ and

$Q_z^1 v$.

Lemma 4.4 (weighted L^p-based error estimates) *Let $z \in \mathring{\mathcal{N}}(\mathscr{T})$. If the function $v \in W_p^1(\omega, S_z)$, then we have*

$$\|v - Q_z^0 v\|_{L^p(\omega,S_z)} \lesssim h_z \|\nabla v\|_{L^p(\omega,S_z)}. \tag{4.17}$$

If $v \in W_p^2(\omega, S_z)$ instead, the following estimate holds

$$\|\partial_{x_j}(v - Q_z^1 v)\|_{L^p(\omega,S_z)} \lesssim h_z \|\partial_{x_j} \nabla v\|_{L^p(\omega,S_z)}, \tag{4.18}$$

for $j = 1, \ldots, n$. In both inequalities, the hidden constants depend only on $C_{p,\omega}$, σ and ψ.

Proof: Define the mapping $\mathcal{F}_z : x \mapsto \bar{x}$ by

$$\bar{x} = \frac{z - x}{h_z},$$

the star $\bar{S}_z = \mathcal{F}_z(S_z)$ and the function $\bar{v}(\bar{x}) = v(x)$. Set $\bar{Q}^0 \bar{v} = \int \bar{v}\psi \,\mathrm{d}\bar{x}$, where ψ is the smooth function introduced in section 4.3.2.

Notice that $\operatorname{supp} \psi \subset \bar{S}_z$. Consequently, in the definition of $\bar{Q}^0 \bar{v}$, integration takes place over $\bar{S}_z$ only. Using the mapping $\mathcal{F}_z$, we have

$$Q_z^0 v = \int_{S_z} v \psi_z \,\mathrm{d}x = \int_{\bar{S}_z} \bar{v}\psi \,\mathrm{d}\bar{x} = \bar{Q}^0 \bar{v},$$

and, since $\int_{\bar{S}_z} \psi \,\mathrm{d}\bar{x} = 1$,

$$\int_{\bar{S}_z} (\bar{v} - \bar{Q}^0 \bar{v}) \psi \,\mathrm{d}\bar{x} = \int_{\bar{S}_z} \bar{v}\psi \,\mathrm{d}\bar{x} - \bar{Q}^0 \bar{v} = 0. \tag{4.19}$$

Define the weight $\bar{\omega}_z = \omega \circ \mathcal{F}_z^{-1}$. In light of property (v) in Proposition 2.1 we have $\bar{\omega}_z \in A_p(\mathbb{R}^n)$ and $C_{p,\bar{\omega}_z} = C_{p,\omega}$. Changing variables we get

$$\int_{S_z} \omega |v - Q_z^0 v|^p \,\mathrm{d}x = h_z^n \int_{\bar{S}_z} \bar{\omega}_z |\bar{v} - \bar{Q}^0 \bar{v}|^p \,\mathrm{d}\bar{x}. \tag{4.20}$$

As a consequence of the shape regularity assumption (4.9), $\operatorname{diam} \bar{S}_z \approx 1$. Then, in view of (4.19), we can apply Lemma 4.1 to $\bar{v} - \bar{Q}^0\bar{v}$ over $S = \bar{S}_z$, with $\mu = \bar{\omega}_z$ and $\chi = \psi$, to conclude

$$\|\bar{v} - \bar{Q}^0 \bar{v}\|_{L^p(\bar{\omega}_z, \bar{S}_z)} \lesssim \|\bar{\nabla} \bar{v}\|_{L^p(\bar{\omega}_z, \bar{S}_z)},$$

where the hidden constant depends only on σ, $C_{p,\bar{\omega}_z}$ and ψ. Inserting this estimate into (4.20) and changing variables with $\mathcal{F}_z^{-1}$ to get back to $\bar{S}_z$ we get (4.17).

In order to prove (4.18), we define

$$\bar{Q}^1 \bar{v}(\bar{y}) = \int_{\bar{S}_z} \big(\bar{v}(\bar{x}) + \bar{\nabla}\bar{v}(\bar{x}) \cdot (\bar{y} - \bar{x})\big)\, \psi(\bar{x}) \,\mathrm{d}\bar{x},$$

and observe that $Q_z^1 v(y) = \bar{Q}^1 \bar{v}(\bar{y})$, where $Q_z^1 v$ is defined by (4.13). Since $\partial_{\bar{y}_i} \bar{Q}^1 \bar{v}(\bar{y}) = \int_{\bar{S}_z} \partial_{\bar{x}_i} \bar{v}(\bar{x}) \psi(\bar{x}) \,\mathrm{d}\bar{x}$ is constant for $i \in \{1, \cdots, n\}$, we have the vanishing mean value property

$$\int_{\bar{S}_z} \partial_{\bar{x}_i} \Big(\bar{v}(\bar{x}) - \bar{Q}^1 \bar{v}(\bar{x}) \Big) \psi(\bar{x}) \,\mathrm{d}\bar{x} = 0.$$

This, together with Lemma 4.1, leads to (4.18). $\square$

The following result is an optimal error estimate in the L^p-weighted norm for the averaged Taylor polynomial $Q_z^1 v$, which is instrumental to study $Q_z^m v$ $(m \geq 0)$.

Lemma 4.5 (weighted L^p-based error estimate for Q_z^1) *Let $z \in \mathring{\mathcal{N}}(\mathscr{T})$. If $v \in W_p^2(\omega, S_z)$, then the following estimate holds*

$$\|v - Q_z^1 v\|_{L^p(\omega, S_z)} \lesssim h_z^2 |v|_{W_p^2(\omega, S_z)}, \tag{4.21}$$

where the hidden constant depends only on $C_{p,\omega}$, σ and ψ.

Proof: Since

$$v - Q_z^1 v = (v - Q_z^1 v) - Q_z^0(v - Q_z^1 v) - Q_z^0(Q_z^1 v - v),$$

and $\nabla(v - Q_z^1 v) = \nabla v - Q_z^0 \nabla v$ from (4.14), we can apply (4.17) twice to obtain

$$\|(v - Q_z^1 v) - Q_z^0(v - Q_z^1 v)\|_{L^p(\omega,S_z)} \lesssim h_z \|\nabla(v - Q_z^1 v)\|_{L^p(\omega,S_z)} \lesssim h_z^2 \|D^2 v\|_{L^p(\omega,S_z)}.$$

So it remains to estimate the term $R_z^1(v) := Q_z^0(Q_z^1 v - v)$. Since $Q_z^0 v = Q_z^0 Q_z^0 v$, we notice that $R_z^1(v) = Q_z^0(Q_z^1 v - Q_z^0 v)$. Then, using the definition of the averaged Taylor polynomial given by (4.13), we have

$$R_z^1(v) = \int_{S_z} \left(\int_{S_z} \nabla v(x) \cdot (y - x) \psi_z(x) \, \mathrm{d}x \right) \psi_z(y) \, \mathrm{d}y.$$

We exploit the crucial *cancellation property* $R_z^1(p) = 0$ for all $p \in \mathbb{P}_1$ as follows: $R_z^1(v) = R_z^1(v - Q_z^1 v) = 0$. This yields

$$\|R_z^1(v)\|_{L^p(\omega,S_z)}^p = \int_{S_z} \omega \left| \int_{S_z} \left(\int_{S_z} \nabla(v(x) - Q_z^1 v(x)) \cdot (y - x) \psi_z(x) \, \mathrm{d}x \right) \psi_z(y) \, \mathrm{d}y \right|^p$$

Applying Hölder inequality to the innermost integral $I(y)$ leads to

$$|I(y)|^p \lesssim h_z^p \left(\int_{S_z} \omega |\nabla(v(x) - Q_z^1 v(x))|^p \, \mathrm{d}x \right) \left(\int_{S_z} \omega^{-p'/p} \psi_z(x)^{p'} \, \mathrm{d}x \right)^{p/p'}.$$

This is combined with $\int_{S_z} \psi_z(y) \, \mathrm{d}y = 1$ and $\|\psi_z\|_{L^{p'}(\omega^{-p'/p},S_z)} \|1\|_{L^p(\omega,S_z)} \lesssim 1$, which follows from the definition of ψ_z and the definition (2.10) of the A_p-class, to arrive at

$$\|R_z^1(v)\|_{L^p(\omega,S_z)}^p \lesssim h_z^{2p} \int_{S_z} \omega |D^2 v|^p. \tag{4.22}$$

This yields the desired estimate (4.21). $\square$

4.3.4 Induction argument

In order to derive approximation properties of the averaged Taylor polynomial $Q_z^m v$ for any $m \geq 0$, we apply an induction argument. We assume the following estimate as *induction hypothesis:*

$$\|v - Q_z^{m-1} v\|_{L^p(\omega,S_z)} \lesssim h_z^m |v|_{W_p^m(\omega,S_z)}. \tag{4.23}$$

Notice that, for $m = 1$, the induction hypothesis is exactly (4.18), while for $m = 2$ is given by Lemma 4.5. We have the following general result for any $m \geq 0$.

Lemma 4.6 (weighted L^p-based error estimate for Q_z^m) *Let $z \in \mathring{\mathcal{N}}(\mathscr{T})$ and $m \geq 0$. If $v \in W_p^{m+1}(\omega, S_z)$, then we have the following approximation result*

$$\|v - Q_z^m v\|_{L^p(\omega,S_z)} \lesssim h_z^{m+1} |v|_{W_p^{m+1}(\omega,S_z)}, \tag{4.24}$$

where the hidden constant depends only on $C_{p,\omega}$, σ, ψ and m.

Proof: We proceed as in the proof of Lemma 4.5. Notice, first of all, that

$$v - Q_z^m v = (v - Q_z^m v) - Q_z^{m-1}(v - Q_z^m v) - Q_z^{m-1}(Q_z^m v - v).$$

The induction hypothesis (4.23) yields

$$\|(v - Q_z^m v) - Q_z^{m-1}(v - Q_z^m v)\|_{L^p(\omega,S_z)} \lesssim h_z^m |v - Q_z^m v|_{W_p^m(\omega,S_z)}.$$

Since $D^\alpha Q_z^m v = Q_z^0 D^\alpha v$ for all $|\alpha| = m$, according to property (4.14), the estimate (4.17) yields $|v - Q_z^m v|_{W_p^m(\omega,S_z)} \lesssim h_z |v|_{W_p^{m+1}(\omega,S_z)}$, and then

$$\|(v - Q_z^m v) - Q_z^{m-1}(v - Q_z^m v)\|_{L^p(\omega,S_z)} \lesssim h_z^{m+1} |v|_{W_p^{m+1}(\omega,S_z)}.$$

It thus remains to bound the term

$$R_z^m(v) := Q_z^{m-1}(Q_z^m v - v).$$

Since $Q_z^{m-1}Q_z^{m-1}v = Q_z^{m-1}v$, writing $Q_z^m = Q_z^{m-1} + \sum_{|\beta|=m} T_z^\beta$ with

$$T_z^\beta(v) = \frac{1}{\beta!}\int_{S_z} D^\beta v(\zeta)(x-\zeta)^\beta \psi_z(\zeta)\,\mathrm{d}\zeta,$$

we obtain

$$R_z^m(v) = \sum_{|\beta|=m} Q_z^{m-1}T_z^\beta(v).$$

This representation allows us to write

$$R_z^m(v)(y) = \sum_{|\alpha|<m, |\beta|=m} I_{\alpha,\beta}v(y),$$

with

$$\begin{aligned} I_{\alpha,\beta}v(y) &= \frac{1}{\alpha!}\int_{S_z} \psi_z(x) D_x^\alpha T_z^\beta v(x)(y-x)^\alpha \,\mathrm{d}x \\ &= \frac{1}{\alpha!}\int_{S_z} \psi_z(x)\frac{1}{(\beta-\alpha)!}\int_{S_z} D_\zeta^\beta v(\zeta)(x-\zeta)^{\beta-\alpha}\psi_z(\zeta)\,\mathrm{d}\zeta (y-x)^\alpha\,\mathrm{d}x. \end{aligned}$$

Finally, we notice the following *cancellation property*: $Q_z^m p = p$ for all $p \in \mathbb{P}_m$, whence $R_z^m(p) = 0$. Consequently $R_z^m(v) = R_z^m(v - Q_z^m v)$ implies

$$\|I_{\alpha,\beta}v\|_{L^p(\omega,S_z)}^p \lesssim h_z^{mp}\int_{S_z}\omega(y)\left|\int_{S_z}\psi_z(x)\int_{S_z} D_\zeta^\beta(v - Q_z^m v)(\zeta)\psi_z(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}x\right|^p \mathrm{d}y.$$

Combining the identity $D^\beta Q_z^m v = Q_z^0 D^\beta v$, with (4.17) and the bound

$$\|\psi_z\|_{L^{p'}(\omega^{-p'/p},S_z)}\|1\|_{L^p(\omega,S_z)} \lesssim 1,$$

we infer that

$$\begin{aligned} \|R_z^m v\|_{L^p(\omega,S_z)}^p &\lesssim h_z^{mp}\|1\|_{L^p(\omega,S_z)}^p\|D^m v - D^m Q_z^m v\|_{L^p(\omega,S_z)}^p\|\psi_z\|_{L^p(\omega^{-p'/p},S_z)}^p \\ &\lesssim h_z^{(m+1)p}|v|_{W_p^{m+1}(\omega,S_z)}^p. \end{aligned}$$

This concludes the proof. $\square$

The following corollary is a simple consequence of Lemma 4.6.

Corollary 4.7 (weighted W_p^k-based error estimate for Q_z^m) *Let $z \in \mathring{\mathcal{N}}(\mathscr{T})$. If $v \in W_p^{m+1}(\omega, S_z)$ with $m \geq 0$, then*

$$|v - Q_z^m v|_{W_p^k(\omega,S_z)} \lesssim h_z^{m+1-k} |v|_{W_p^{m+1}(\omega,S_z)}, \quad k = 0, 1, \ldots, m+1, \tag{4.25}$$

where the hidden constant depends only on $C_{p,\omega}$, σ, ψ and m.

Proof: For $k = 0$, the estimate (4.25) is given by Lemma 4.6, while for $k = m+1$,

$$|v - Q_z^m v|_{W_p^{m+1}(\omega,S_z)} = |v|_{W_p^{m+1}(\omega,S_z)}.$$

For $0 < k < m+1$, we employ property (4.14) of $D^\alpha Q_z^m v$ with $|\alpha| = k$ to write

$$|v - Q_z^m v|_{W_p^k(\omega,S_z)} = \left(\sum_{|\alpha|=k} \|D^\alpha v - Q_z^{m-k} D^\alpha v\|_{L^p(\omega,S_z)}^p \right)^{1/p}.$$

Therefore, applying estimate (4.24) to $\|D^\alpha v - Q_z^{m-k} D^\alpha v\|_{L^p(\omega,S_z)}$, we obtain

$$|v - Q_z^m v|_{W_p^k(\omega,S_z)} \lesssim h_z^{m+1-k} |v|_{W_p^{m+1}(\omega,S_z)},$$

which is the asserted estimate. $\square$

4.4 Weighted interpolation error estimates

In this section we construct a quasi-interpolation operator $\Pi_{\mathscr{T}}$, based on local averages over stars. This construction is well defined for functions in $L^1(\Omega)$, and thus for functions in the weighted space $L^p(\omega, \Omega)$. It is well known that this type of quasi-interpolation operator is important in the approximation of nonsmooth functions without point values because the Lagrange interpolation operator is not even defined [58, 143]. Moreover, averaged interpolation has better approximation properties than the Lagrange interpolation for anisotropic elements [2]. We refer the reader to [19, 70, 129] for applications of quasi-interpolantion.

The construction of $\Pi_{\mathscr{T}}$ is based on the averaged Taylor polynomial defined in (4.13). In § 4.4.1, using the approximation estimates derived in section 4.3 together with an invariance property of $\Pi_{\mathscr{T}}$ over the space of polynomials, we derive optimal error estimates for $\Pi_{\mathscr{T}}$ in Muckenhoupt weighted Sobolev norms on simplicial discretizations. The case of rectangular discretizations is considered in § 4.4.2.

Given $\omega \in A_p(\mathbb{R}^n)$ and $v \in L^p(\omega, \Omega)$, we recall that $Q_z^m v$ is the averaged Taylor polynomial of order m of v over the node z; see (4.13). We define the quasi-interpolant $\Pi_{\mathscr{T}} v$ as the unique function of $\mathbb{V}(\mathscr{T})$ that satisfies $\Pi_{\mathscr{T}} v(z) = Q_z^m v(z)$ if $z \in \mathring{\mathcal{N}}(\mathscr{T})$, and $\Pi_{\mathscr{T}} v(z) = 0$ if $z \in \mathcal{N}(\mathscr{T}) \cap \partial\Omega$, i.e.,

$$\Pi_{\mathscr{T}} v = \sum_{z \in \mathring{\mathcal{N}}(\mathscr{T})} Q_z^m v(z)\, \phi_z. \tag{4.26}$$

Optimal error estimates for $\Pi_{\mathscr{T}}$ rely on its stability, which follows from the stability of Q_z^m obtained in Lemma 4.3.

Lemma 4.8 (stability of $\Pi_{\mathscr{T}}$) *Let $v \in W_p^k(\omega, S_T)$ with $0 \leq k \leq m+1$ and $T \in \mathscr{T}$. Then, the quasi-interpolant operator $\Pi_{\mathscr{T}}$ defined by (4.26) satisfies the following local stability bound*

$$|\Pi_{\mathscr{T}} v|_{W_p^k(\omega,T)} \lesssim \sum_{l=0}^{k} h_T^{l-k} |v|_{W_p^l(\omega,S_T)}. \tag{4.27}$$

Proof: Using the definition of $\Pi_{\mathscr{T}}$ given by (4.26), we have

$$|\Pi_{\mathscr{T}} v|_{W^k_p(\omega,T)} \leq \sum_{z \in \mathring{\mathcal{N}}(T)} \|Q^m_z v\|_{L^\infty(S_z)} \; |\phi_z|_{W^k_p(\omega,T)}.$$

We resort to Lemma 4.3 to derive

$$|\Pi_{\mathscr{T}} v|_{W^k_p(\omega,T)} \lesssim \sum_{z \in \mathring{\mathcal{N}}(T)} |\phi_z|_{W^k_p(\omega,T)} \|\psi_z\|_{L^{p'}(\omega^{-p'/p},S_z)} \sum_{l=0}^{k} h^l_z |v|_{W^l_p(\omega,S_z)}.$$

Since ϕ_z and ψ_z are bounded in $L^\infty(S_T)$ and $\omega \in A_p(\mathbb{R}^n)$, we obtain

$$|\phi_z|_{W^k_p(\omega,T)} \|\psi_z\|_{L^{p'}(\omega^{-p'/p},\, S_z)} \lesssim \frac{h_z^{-k}}{h_z^n} \left(\int_{S_z} \omega \right)^{1/p} \left(\int_{S_z} \omega^{-p'/p} \right)^{1/p'} \lesssim h_z^{-k},$$

which, given the definition of S_T, the shape regularity of $\mathscr{T}$, and the finite overlapping property of stars imply (4.27). $\square$

4.4.1 Interpolation error estimates on simplicial discretizations

The quasi-interpolant operator $\Pi_{\mathscr{T}}$ is invariant over the space of polynomials of degree m on simplicial meshes: $\Pi_{\mathscr{T}} v|_{S_z} = v$ for $v \in \mathbb{P}_m(S_z)$ and $z \in \mathring{\mathcal{N}}(\mathscr{T})$ such that $\partial S_z \cap \partial\Omega = \emptyset$. Consequently,

$$\Pi_{\mathscr{T}} Q^m_z \phi = Q^m_z \phi. \quad \forall\, \phi \in L^1(\omega, S_z). \tag{4.28}$$

This property, together with (4.7), yields optimal interpolation estimates for $\Pi_{\mathscr{T}}$.

Theorem 4.9 (interpolation estimate on interior simplices) *Given $T \in \mathscr{T}$ such that $\partial T \cap \partial\Omega = \emptyset$ and $v \in W^{m+1}_p(\omega, S_T)$, we have the following interpolation error estimate*

$$|v - \Pi_{\mathscr{T}} v|_{W^k_p(\omega,T)} \lesssim h_T^{m+1-k} |v|_{W^{m+1}_p(\omega,S_T)}, \quad k = 0, 1, \ldots, m+1, \tag{4.29}$$

where the hidden constant depends only on $C_{p,\omega}$, σ, ψ and m.

Proof: Given $T \in \mathscr{T}$, choose a node $z \in \mathring{\mathcal{N}}(T)$. Property (4.28) yields,

$$|v - \Pi_{\mathscr{T}} v|_{W_p^k(\omega,T)} \leq |v - Q_z^m v|_{W_p^k(\omega,T)} + |\Pi_{\mathscr{T}}(Q_z^m v - v)|_{W_p^k(\omega,T)}.$$

Combining the stability of $\Pi_{\mathscr{T}}$ given by (4.27) together with (4.25) implies

$$|v - \Pi_{\mathscr{T}} v|_{W_p^k(\omega,T)} \lesssim \sum_{l=0}^{k} h_T^{l-k} |v - Q_z^m v|_{W_p^l(\omega,S_T)} \lesssim h_T^{m+1-k} |v|_{W_p^{m+1}(\omega,S_T)},$$

which is exactly (4.29). $\square$

By using the fact that, $v \in W_p^{m+1}(\omega,\Omega) \cap \mathring{W}_p^1(\omega,\Omega)$ implies $\Pi_{\mathscr{T}} v_{|\partial\Omega} = 0$ we can extend the results of Theorem 4.9 to boundary elements. The proof is an adaption of standard techniques and, in order to deal with the weight, those of the aforementioned Theorem 4.9. See also Theorem 4.17 below.

Theorem 4.10 (interpolation estimates on Dirichlet simplices) *Let the function $v \in \mathring{W}_p^1(\omega,\Omega) \cap W_p^{m+1}(\omega,\Omega)$. If $T \in \mathscr{T}$ is a boundary simplex, then (4.29) holds with a constant that depends only on $C_{p,\omega}$, σ and ψ.*

We are now in the position to write a global interpolation estimate. To this end, it is convenient to introduce the meshsize function $h \in L^\infty(\Omega)$ given by

$$h_{|T} = h_T, \quad \forall T \in \mathscr{T}.$$

Theorem 4.11 (global interpolation estimate over simplicial meshes) *Given $\mathscr{T} \in \mathbb{T}$ and $v \in W_p^{m+1}(\omega,\Omega)$, we have the following global interpolation error estimate*

$$\left(\sum_{T \in \mathscr{T}} |v - \Pi_{\mathscr{T}} v|^2_{W_p^k(\omega,T)} \right)^{1/p} \lesssim |h^{m+1-k} v|_{W_p^{m+1}(\omega,\Omega)}, \quad k = 0, 1, \ldots, m+1, \quad (4.30)$$

where the hidden constant depends only on $C_{p,\omega}$, σ, ψ and m.

Proof: Raise (4.29) to the p-th power and add over all $T \in \mathscr{T}$. The finite overlapping property of stars of $\mathscr{T}$ yields the result. $\square$

4.4.2 Anisotropic interpolation estimates on rectangular meshes

Narrow or anisotropic elements are those with disparate sizes in each direction. They are necessary, for instance, for the optimal approximation of functions with a strong directional-dependent behavior such as line and edge singularities, boundary layers, and shocks (see [70, 71, 129]).

Inspired by [70], here we derive interpolation error estimates assuming only that neighboring elements have comparable sizes, thus obtaining results which are valid for a rather general family of anisotropic meshes. Since symmetry is essential, we assume that $\Omega = (0,1)^n$, or that Ω is any domain which can be decomposed into n-rectangles. We use below the notation introduced in [70].

We assume that the mesh $\mathscr{T}$ is composed of rectangular elements R, with sides parallel to the coordinate axes. By $\mathtt{v} \in \mathscr{N}(\mathscr{T})$ we denote a node or vertex of the triangulation $\mathscr{T}$ and by $S_{\mathtt{v}}$, S_R the associated patches; see § 4.3.1. Given $R \in \mathscr{T}$, we define h_R^i as the length of R in the i-th direction and, if $\mathtt{v} \in \mathscr{N}(\mathscr{T})$, we define $h_{\mathtt{v}}^i = \min\{h_R^i : \mathtt{v} \in R\}$ for $i = 1, \cdots, n$. The finite element space is defined by (4.10) with $\mathcal{P} = \mathbb{Q}_1$.

We assume the following weak shape regularity condition: there exists a constant $\sigma > 1$, such that if $R, S \in \mathscr{T}$ are neighboring elements, we have

$$\frac{h_R^i}{h_S^i} \leq \sigma, \qquad i = 1, \ldots, n. \tag{4.31}$$

Whenever $\mathtt{v}$ is a vertex of R the shape regularity assumption (4.31) implies that $h_{\mathtt{v}}^i$ and h_R^i are equivalent up to a constant that depends only on σ. We define

$$\psi_{\mathtt{v}}(x) = \frac{1}{h_{\mathtt{v}}^1 \dots h_{\mathtt{v}}^n} \psi\left(\frac{\mathtt{v}_1 - x_1}{h_{\mathtt{v}}^1}, \dots, \frac{\mathtt{v}_n - x_n}{h_{\mathtt{v}}^n}\right),$$

which, owing to (4.31) and $r \leq 1/\sigma$, satisfies $\operatorname{supp} \psi_{\mathtt{v}} \subset S_{\mathtt{v}}$. Notice that this function incorporates a different length scale on each direction x_i, which will prove useful in the study of anisotropic estimates.

Given $\omega \in A_p(\mathbb{R}^n)$, and $v \in L^p(\omega, \Omega)$, we define $Q_{\mathtt{v}}^1 v$, the first degree regularized Taylor polynomial of v about the vertex $\mathtt{v}$ as in (4.13). We also define the quasi-interpolation operator $\Pi_{\mathscr{T}}$ as in (4.26), i.e., upon denoting by $\phi_{\mathtt{v}}$ the Lagrange nodal basis function of $\mathbb{V}(\mathscr{T})$, $\Pi_{\mathscr{T}} v$ reads

$$\Pi_{\mathscr{T}} v := \sum_{\mathtt{v} \in \mathring{\mathscr{N}}(\mathscr{T})} Q_{\mathtt{v}}^1 v(\mathtt{v}) \phi_{\mathtt{v}}. \tag{4.32}$$

The finite element space $\mathbb{V}(\mathscr{T})$ is not invariant under the operator defined in (4.32). Consequently, we cannot use the techniques for simplicial meshes developed in § 4.4.1. This, as the results below show, is not a limitation to obtain interpolation error estimates.

Lemma 4.12 (anisotropic L^p-weighted error estimates I) *Let $\mathtt{v} \in \mathring{\mathscr{N}}(\mathscr{T})$. If $v \in W_p^1(\omega, S_{\mathtt{v}})$, then we have*

$$\|v - Q_{\mathtt{v}}^0 v\|_{L^p(\omega, S_{\mathtt{v}})} \lesssim \sum_{i=1}^n h_{\mathtt{v}}^i \|\partial_{x_i} v\|_{L^p(\omega, S_{\mathtt{v}})}. \tag{4.33}$$

If $v \in W_p^2(\omega, S_{\mathtt{v}})$ instead, then the following estimate holds

$$\|\partial_{x_j}(v - Q_{\mathtt{v}}^1 v)\|_{L^p(\omega, S_{\mathtt{v}})} \lesssim \sum_{i=1}^n h_{\mathtt{v}}^i \|\partial_{x_i} \partial_{x_j} v\|_{L^p(\omega, S_{\mathtt{v}})}, \tag{4.34}$$

for $j = 1, \ldots, n$. In both inequalities, the hidden constants depend only on $C_{p,\omega}$, σ and ψ.

Proof: To exploit the symmetry of the elements we define the map

$$\mathcal{F}_{\mathtt{v}} : x \mapsto \bar{x}, \qquad \bar{x}_i = \frac{\mathtt{v}_i - x_i}{h_{\mathtt{v}}^i}, \qquad i = 1, \ldots, n, \tag{4.35}$$

and proceed exactly as in the proof of Lemma 4.4. $\square$

Lemma 4.12, in conjunction with the techniques developed in Lemma 4.5 give rise the second order anisotropic error estimates in the weighted L^p-norm.

Lemma 4.13 (anisotropic L^p-weighted error estimate II) *Let $\mathtt{v} \in \mathring{\mathscr{N}}(\mathscr{T})$. If $v \in W_p^2(\omega, S_{\mathtt{v}})$, then we have*

$$\|v - Q_{\mathtt{v}}^1 v\|_{L^p(\omega, S_{\mathtt{v}})} \lesssim \sum_{i,j=1}^{n} h_{\mathtt{v}}^i h_{\mathtt{v}}^j \|\partial_{x_i} \partial_{x_j} v\|_{L^p(\omega, S_{\mathtt{v}})}, \tag{4.36}$$

where the hidden constant in the inequality above depends only on $C_{p,\omega}$, σ and ψ.

Proof: Recall that, if $R_{\mathtt{v}}^1(v) = Q_{\mathtt{v}}^0(Q_{\mathtt{v}}^1 v - v)$, then we can write

$$v - Q_{\mathtt{v}}^1 v = (v - Q_{\mathtt{v}}^1 v) - Q_{\mathtt{v}}^0(v - Q_{\mathtt{v}}^1 v) - R_{\mathtt{v}}^1(v).$$

Applying estimates (4.33) and (4.34) successively, we see that

$$\begin{aligned} \|(v - Q_{\mathtt{v}}^1 v) - Q_{\mathtt{v}}^0(v - Q_{\mathtt{v}}^1 v)\|_{L^p(\omega, S_{\mathtt{v}})} &\lesssim \sum_{i=1}^{n} h_{\mathtt{v}}^i \|\partial_{x_i}(v - Q_{\mathtt{v}}^1 v)\|_{L^p(\omega, S_{\mathtt{v}})} \\ &\lesssim \sum_{i,j=1}^{n} h_{\mathtt{v}}^i h_{\mathtt{v}}^j \|\partial_{x_i} \partial_{x_j} v\|_{L^p(\omega, S_{\mathtt{v}})}. \end{aligned}$$

It remanins then to bound $R_{\mathtt{v}}^1(v)$. We proceed as in the proof of (4.22) in Lemma 4.5. The definition (4.13) of the averaged Taylor polynomial, together with the cancellation property $R^1_{\mathtt{v}}(v) = R^1_{\mathtt{v}}(v - Q^1_{\mathtt{v}}v)$, implies

$$\|R^1_{\mathtt{v}}(v)\|^p_{L^p(\omega,S_{\mathtt{v}})} \lesssim \sum_{i=1}^n (h^i_{\mathtt{v}})^p \|\partial_{x_i}(v - Q^1_{\mathtt{v}}v)\|^p_{L^p(\omega,S_{\mathtt{v}})} \|1\|^p_{L^p(\omega,S_{\mathtt{v}})} \|\psi_{\mathtt{v}}\|^p_{L^{p'}(\omega^{-p'/p},S_{\mathtt{v}})}$$

Combining (4.34) with the inequality $\|\psi_{\mathtt{v}}\|_{L^{p'}(\omega^{-p'/p},S_{\mathtt{v}})} \|1\|_{L^p(\omega,S_{\mathtt{v}})} \lesssim 1$, which follows from the the definition of $\psi_{\mathtt{v}}$ and the definition (2.10) of the A_p-class, yields

$$\|R^1_{\mathtt{v}}(v)\|_{L^p(\omega,S_{\mathtt{v}})} \lesssim \sum_{i,j=1}^n h^i_{\mathtt{v}} h^j_{\mathtt{v}} \|\partial_{x_i}\partial_{x_j} v\|_{L^p(\omega,S_{\mathtt{v}})},$$

and leads to the asserted estimate (4.36). □

The anisotropic error estimate (4.33) together with the weighted L^p stability of the interpolation operator $\Pi_{\mathscr{T}}$, enables us to obtain anisotropic weighted L^p interpolation estimates, as shown in the following Theorem.

Theorem 4.14 (anisotropic L^p-weighted interpolation estimate I) *Let $\mathscr{T}$ satisfy* (4.31) *and $R \in \mathscr{T}$. If $v \in L^p(\omega, S_R)$, we have*

$$\|\Pi_{\mathscr{T}} v\|_{L^p(\omega,R)} \lesssim \|v\|_{L^p(\omega,S_R)}. \tag{4.37}$$

If, in addition, $w \in W^1_p(\omega, S_R)$ and $\partial R \cap \partial\Omega = \emptyset$, then

$$\|v - \Pi_{\mathscr{T}} v\|_{L^p(\omega,R)} \lesssim \sum_{i=1}^n h^i_R \|\partial_{x_i} v\|_{L^p(\omega,S_R)}. \tag{4.38}$$

The hidden constants in both inequalities depend only on $C_{p,\omega}$, σ and ψ.

Proof: The local stability (4.37) of $\Pi_{\mathscr{T}}$ follows from Lemma 4.8 with $k = 0$. Let us now prove (4.38). Choose a node $\mathtt{v} \in \mathring{\mathcal{N}}(R)$. Since $Q^0_{\mathtt{v}}v$ is constant, and $\partial R \cap \partial\Omega = \emptyset$, $\Pi_{\mathscr{T}} Q^0_{\mathtt{v}} v = Q^0_{\mathtt{v}} v$ over R. This, in conjunction with estimate (4.37), allows us to write

$$\|v - \Pi_{\mathscr{T}} v\|_{L^p(\omega,R)} = \|(I - \Pi_{\mathscr{T}})(v - Q^0_{\mathtt{v}} v)\|_{L^p(\omega,R)} \lesssim \|v - Q^0_{\mathtt{v}} v\|_{L^p(\omega,S_R)}.$$

The desired estimate (4.38) now follows from Corollary 4.2. $\square$

To prove interpolation error estimates on the first derivatives for interior elements we follow [70, Theorem 2.6] and use the symmetries of a cube, thus handling the anisotropy in every direction separately. We start by studying the case of *interior elements.*

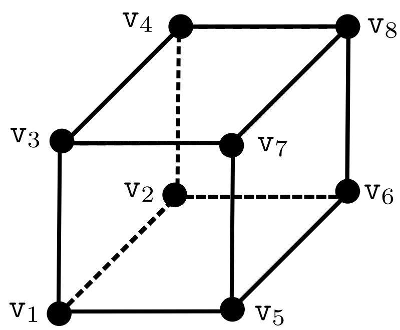

Figure 4.1: An anisotropic cube with sides parallel to the coordinate axes and the labeling of its vertices. The numbering of the vertices proceeds recursively as follows: a cube in dimension m is obtained as the Cartesian product of an $(m-1)$-dimensional cube with vertices $\{\mathtt{v}_i\}_{i=1}^{2^{m-1}}$ and an interval, and the new vertices are $\{\mathtt{v}_{i+2^{m-1}}\}_{i=1}^{2^{m-1}}$.

Theorem 4.15 (anisotropic W_p^1-weighted interpolation estimates) *Let $R \in \mathscr{T}$ be such that $\partial R \cap \partial\Omega = \emptyset$. If $v \in W_p^1(\omega, S_R)$ we have the stability bound*

$$\|\nabla \Pi_{\mathscr{T}} v\|_{L^p(\omega,R)} \lesssim \|\nabla v\|_{L^p(\omega,S_R)}. \tag{4.39}$$

If, in addition, $v \in W_p^2(\omega, S_R)$ we have, for $j = 1, \cdots, n$,

$$\|\partial_{x_j}(v - \Pi_{\mathscr{T}} v)\|_{L^p(\omega,R)} \lesssim \sum_{i=1}^{n} h_R^i \|\partial_{x_j}\partial_{x_i} v\|_{L^p(\omega,S_R)}. \tag{4.40}$$

The hidden constants in the inequalities above depend only on $C_{p,\omega}$, σ and ψ.

Proof: Let us bound the derivative with respect to the first argument x_1. The other ones follow from similar considerations. As in [70, Theorem 2.5], to exploit the geometry of R, we label its vertices in an appropriate way: vertices that differ only in the first component are denoted $\mathtt{v}_i$ and $\mathtt{v}_{i+2^{n-1}}$ for $i = 1, \ldots, 2^{n-1}$; see Figure 4.1 for the three-dimensional case.

Clearly $v - \Pi_{\mathscr{T}} v = (v - Q^1_{\mathtt{v}_1} v) + (Q^1_{\mathtt{v}_1} v - \Pi_{\mathscr{T}} v)$, and the difference $v - Q^1_{\mathtt{v}_1} v$ is estimated by Lemma 4.12. Consequently, it suffices to consider $q = Q^1_{\mathtt{v}_1} v - \Pi_{\mathscr{T}} v \in \mathbb{Q}_1(R)$. Thanks to the special labeling of the vertices we have that $\partial_{x_1} \phi_{\mathtt{v}_{i+2^{n-1}}} = -\partial_{x_1} \phi_{\mathtt{v}_i}$. Therefore

$$\partial_{x_1} q = \sum_{i=1}^{2^n} q(\mathtt{v}_i) \partial_{x_1} \phi_{\mathtt{v}_i} = \sum_{i=1}^{2^{n-1}} (q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^{n-1}})) \partial_{x_1} \phi_{\mathtt{v}_i},$$

so that

$$\|\partial_{x_1} q\|_{L^p(\omega,R)} \leq \sum_{i=1}^{2^{n-1}} |q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^{n-1}})| \|\partial_{x_1} \phi_{\mathtt{v}_i}\|_{L^p(\omega,R)}. \tag{4.41}$$

This shows that it suffices to estimate $\delta q(\mathtt{v}_1) = q(\mathtt{v}_1) - q(\mathtt{v}_{1+2^{n-1}})$. The definitions of $\Pi_{\mathscr{T}}$, q, and the averaged Taylor polynomial (4.13), imply that

$$\delta q(\mathtt{v}_1) = \int P^1(x, \mathtt{v}_{1+2^{n-1}}) \psi_{\mathtt{v}_{1+2^{n-1}}}(x) \, \mathrm{d}x - \int P^1(x, \mathtt{v}_{1+2^{n-1}}) \psi_{\mathtt{v}_1}(x) \, \mathrm{d}x, \tag{4.42}$$

whence employing the operation $\circ$ defined in (2.4) and changing variables, we get

$$\int \left(P^1(\mathtt{v}_{1+2^{n-1}} - h_{\mathtt{v}_{1+2^{n-1}}} \circ z, \mathtt{v}_{1+2^{n-1}}) - P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z, \mathtt{v}_{1+2^{n-1}}) \right) \psi(z) \, \mathrm{d}z.$$

Define

$$\theta_1 = \mathtt{v}^1_{1+2^{n-1}} - \mathtt{v}^1_1 + (h^1_{\mathtt{v}_1} - h^1_{\mathtt{v}_{1+2^{n-1}}}) z_1,$$

$\theta = (\theta_1, 0, \ldots, 0)$ and, for $t \in [0,1]$, the function $F_z(t) = P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z + t\theta, \mathtt{v}_{1+2^{n-1}})$.

Since for $i = 1, \cdots, n-1$ we have that $h^i_{\mathtt{v}_1} = h^i_{\mathtt{v}_{1+2^{n-1}}}$ and $\mathtt{v}^i_1 = \mathtt{v}^i_{1+2^{n-1}}$ we obtain

$$P^1(\mathtt{v}_{1+2^{n-1}} - h_{\mathtt{v}_{1+2^{n-1}}} \circ z, \mathtt{v}_{1+2^{n-1}}) - P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z, \mathtt{v}_{1+2^{n-1}}) = F_z(1) - F_z(0),$$

and consequently

$$\delta q(\mathtt{v}_1) = \int (F_z(1) - F_z(0))\psi(z)\,\mathrm{d}z = \int_0^1 \int F_z'(t)\psi(z)\,\mathrm{d}z\,\mathrm{d}t.$$

Since ψ is bounded and $B = \operatorname{supp}\psi \subset B(0,1)$, it suffices to bound the integral

$$I(t) = \int_B |F_z'(t)|\,\mathrm{d}z.$$

Invoking the definition of F_z, we get $F_z'(t) = \nabla P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z + t\theta, \mathtt{v}_{1+2^{n-1}}) \cdot \theta$, which, together with the definition of the polynomial P^1 given by (4.12), yields

$$\begin{aligned} I(t) &\lesssim \int_B |\partial^2_{x_1} v(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z + t\theta)|\,|\mathtt{v}^1_{1+2^{n-1}} - \mathtt{v}^1_1 + h^1_{\mathtt{v}_1} z_1 - t\theta_1|\,|\theta_1|\,\mathrm{d}z \\ &+ \lesssim \sum_{i=2}^n \int_B |\partial^2_{x_i x_1} v(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z + t\theta)|\,|\mathtt{v}^i_{1+2^{n-1}} - \mathtt{v}^i_1 + h^i_{\mathtt{v}_1} z_i|\,|\theta_1|\,\mathrm{d}z. \end{aligned}$$

Now, using that $|z| \le 1$, $0 \le t \le 1$, and the definition of θ, we easily see that $|\theta| = |\theta_1| \lesssim h^1_{\mathtt{v}_1}$ as well as $|\mathtt{v}^1_{1+2^{n-1}} - \mathtt{v}_1 + h^1_{\mathtt{v}_1} z_1 - t\theta_1| \lesssim h^1_{\mathtt{v}_1}$ and $|\mathtt{v}^i_{1+2^{n-1}} - \mathtt{v}^i_1 - h^i_{\mathtt{v}_1} z_i| \lesssim h^i_{\mathtt{v}_1}$ for $i = 2, \ldots n$, whence

$$I(t) \lesssim \sum_{i=1}^n h_{\mathtt{v}^1_1} h_{\mathtt{v}^i_1} \int_B |\partial^2_{x_i x_1} v(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z + t\theta)|\,\mathrm{d}z.$$

Changing variables via $y = \mathtt{v}_1 - h_{\mathtt{v}_1} \circ z + t\theta$, we obtain

$$I(t) \lesssim \frac{1}{h_{\mathtt{v}^2_1} \ldots h_{\mathtt{v}^n_1}} \sum_{i=1}^n h_{\mathtt{v}^i_1} \int_{S_R} |\partial^2_{x_i x_1} v(y)|\,\mathrm{d}y,$$

where we have used that the support of ψ is mapped into $S_{\mathtt{v}_1} \subset S_R$. Hölder's inequality implies

$$I(t) \lesssim \frac{1}{h_{\mathtt{v}^2_1} \ldots h_{\mathtt{v}^n_1}} \|1\|_{L^{p'}(\omega^{-p'/p}, S_R)} \sum_{i=1}^n h_{\mathtt{v}^i_1} \|\partial^2_{x_i x_1} v\|_{L^p(\omega, S_R)},$$

which combined with $\|\partial_{x_1}\phi_{\mathtt{v}_1}\|_{L^p(\omega,R)}\|1\|_{L^{p'}(\omega^{-p'/p},S_R)} \lesssim h^2_{\mathtt{v}_1}\dots h^n_{\mathtt{v}_1}$, because $\omega \in A_p(\mathbb{R}^n)$, gives the following bound for the first term in (4.41)

$$\delta q(\mathtt{v}_1)\|\partial_{x_1}\phi_{\mathtt{v}_1}\|_{L^p(\omega,R)} \lesssim \sum_{i=1}^{n} h_{\mathtt{v}_1^i}\|\partial^2_{x_ix_1}v\|_{L^p(\omega,S_R)}.$$

This readily yields (4.40).

The estimate (4.39) follows along the same arguments as in [129, Theorem 4.7]. In fact, by the triangle inequality

$$\|\nabla\Pi_{\mathscr{T}}v\|_{L^p(\omega,R)} \leq \|\nabla Q^1_{\mathtt{v}_1}v\|_{L^p(\omega,R)} + \|\nabla(Q^1_{\mathtt{v}_1}v - \Pi_{\mathscr{T}}v)\|_{L^p(\omega,R)}. \tag{4.43}$$

The estimate of the first term on the right hand side of (4.43) begins by noticing that the definition of $\psi_{\mathtt{v}_1}$ and the definition (2.10) of the A_p class imply

$$\|\psi_{\mathtt{v}_1}\|_{L^{p'}(\omega^{-p'/p},S_R)}\|1\|_{L^p(\omega,S_R)} \lesssim 1.$$

This, together with the definition (4.13) of regularized Taylor polynomial $Q^1_{\mathtt{v}_1}v$, yields

$$\|\nabla Q^1_{\mathtt{v}_1}v\|_{L^p(\omega,R)} \leq \|\nabla v\|_{L^p(\omega,S_R)}\|\psi_{\mathtt{v}_1}\|_{L^{p'}(\omega^{-p'/p},S_R)}\|1\|_{L^p(\omega,S_R)} \lesssim \|\nabla v\|_{L^p(\omega,S_R)}.$$

To estimate the second term of the right hand side of (4.43), we integrate by parts (4.42), using that $\psi_{\mathtt{v}_i} = 0$ on $\partial\omega_{\mathtt{v}_i}$ for $i = 1, \dots, n$, to get

$$\begin{aligned}\delta q(\mathtt{v}_1) = (n+1)&\left(\int v(x)\psi_{\mathtt{v}_{1+2^{n-1}}}(x)\,\mathrm{d}x - \int v(x)\psi_{\mathtt{v}_1}(x)\,\mathrm{d}x\right)\\ &- \int v(x)(\mathtt{v}_{1+2^{n-1}} - x)\cdot\nabla\psi_{\mathtt{v}_{1+2^{n-1}}}(x)\,\mathrm{d}x + \int v(x)(\mathtt{v}_1 - x)\cdot\nabla\psi_{\mathtt{v}_1}(x)\,\mathrm{d}x.\end{aligned}$$

In contrast to (4.42), we have now created differences which involved $v(x)$ instead of $\nabla v(x)$. However, the same techniques used to derive (4.40) yield

$$|\delta q(\mathtt{v}_1)| \lesssim \frac{1}{h^2_{\mathtt{v}_1}\dots h^n_{\mathtt{v}_1}}\|\nabla v\|_{L^p(\omega,S_R)}\|1\|_{L^{p'}(\omega^{-p'/p},S_R)},$$

which, since $\|\partial_{x_1}\phi_{\mathtt{v}_1}\|_{L^{p'}(\omega^{-p'/p},S_R)}\|1\|_{L^p(\omega,S_R)} \lesssim h^2_{\mathtt{v}_1}\dots h^n_{\mathtt{v}_1}$, results in

$$|\delta q(\mathtt{v}_1)|\|\partial_{x_1}\phi_{\mathtt{v}_1}\|_{L^p(\omega,R)} \lesssim \|\nabla v\|_{L^p(\omega,S_R)}.$$

Replacing this estimate in (4.41), we get

$$\|\nabla(Q^1_{\mathtt{v}_1}v - \Pi_{\mathscr{T}}v)\|_{L^p(\omega,R)} \lesssim \|\nabla v\|_{L^p(\omega,S_R)},$$

which implies the desired result (4.39). This completes the proof. □

Let us now derive a second order anisotropic interpolation error estimates for the weighted L^p-norm, which is novel even for unweighted norms. For the sake of simplicity, and because the arguments involved are rather technical (as in Theorem 4.15), we prove the result in two dimensions. However, analogous results can be obtained in three and more dimensions by using similar arguments.

Theorem 4.16 (anisotropic L^p-weighted interpolation estimate II) *Let $\mathscr{T}$ satisfy* (4.31) *and $R \in \mathscr{T}$ such that $\partial R \cap \partial\Omega = \emptyset$. If $v \in W^2_p(\omega, S_R)$, then we have*

$$\|v - \Pi_{\mathscr{T}}v\|_{L^p(\omega,R)} \lesssim \sum_{i,j=1}^{n} h^i_R h^j_R \|\partial_{x_i}\partial_{x_j}v\|_{L^p(\omega,S_R)}, \tag{4.44}$$

where the hidden constant in the inequality above depends only on $C_{p,\omega}$, σ and ψ.

Proof: To exploit the symmetry of R, we label its vertices of R according to Figure 4.1: $\mathtt{v}_2 = \mathtt{v}_1 + (a, 0), \mathtt{v}_3 = \mathtt{v}_1 + (0, b), \mathtt{v}_4 = \mathtt{v}_1 + (a, b)$. We write $v - \Pi_{\mathscr{T}}v = (v - Q^1_{\mathtt{v}_1}v) + (Q^1_{\mathtt{v}_1}v - \Pi_{\mathscr{T}}v)$. The difference $v - Q^1_{\mathtt{v}_1}v$ is estimated by Lemma 4.13. Consequently, it suffices to estimate $q = Q^1_{\mathtt{v}_1}v - \Pi_{\mathscr{T}}v$.

Since $q \in \mathbb{V}(\mathscr{T})$,

$$q = \sum_{i=1}^{4} q(\mathtt{v}_i)\phi_{\mathtt{v}_i} \implies \|q\|_{L^p(\omega,R)} \le \sum_{i=1}^{4} |q(\mathtt{v}_i)|\|\phi_{\mathtt{v}_i}\|_{L^p(\omega,R)}, \tag{4.45}$$

and we only need to deal with $q(\mathtt{v}_i)$ for $i = 1, \ldots, 4$. Since $q(\mathtt{v}_1) = 0$, in accordance with the definition (4.32) of $\Pi_{\mathscr{T}}$, we just consider $i = 2$. Again, by (4.32), we have

$$q(\mathtt{v}_2) = Q^1_{\mathtt{v}_1} v(\mathtt{v}_2) - Q^1_{\mathtt{v}_2} v(\mathtt{v}_2)$$

which, together with the definition of the averaged Taylor polynomial (4.13) and a change of variables, yields

$$q(\mathtt{v}_2) = \int \big(P^1(\mathtt{v}_1 - h_{\mathtt{v}_1} \circ z, \mathtt{v}_2) - P^1(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z, \mathtt{v}_2)\big)\, \psi(z)\, \mathrm{d}z.$$

To estimate this integral, we define $\theta = (\theta_1, 0)$, where $\theta_1 = \mathtt{v}^1_1 - \mathtt{v}^1_2 + (h^1_{\mathtt{v}_2} - h^1_{\mathtt{v}_1}) z_1$, and the function $F_z(t) = P(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta, \mathtt{v}_2)$. Exploiting the symmetries of R, i.e., using that $\mathtt{v}^2_1 = \mathtt{v}^2_2$ and $h^2_{\mathtt{v}_1} = h^2_{\mathtt{v}_2}$, we arrive at

$$q(\mathtt{v}_2) = \int \big(F_z(1) - F_z(0)\big)\psi(z)\, \mathrm{d}z = \int_0^1 \int F'_z(t)\psi(z)\, \mathrm{d}z\, \mathrm{d}t.$$

By using the definition of the Taylor polynomial P^1 given in (4.12), we obtain

$$F'_z(t) = \theta D^2 v(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta)(h_{\mathtt{v}_2} \circ z - t\theta)$$

which, together with the definition of θ and the inequalities $|\theta_1| \lesssim h^1_{\mathtt{v}_2}$, $|h^1_{\mathtt{v}_2} z_1 - t\theta_1| \lesssim h^1_{\mathtt{v}_2}$ and $|h^2_{\mathtt{v}_2} z_2| \lesssim h^2_{\mathtt{v}_2}$, implies

$$\begin{aligned}
\int F'_z(t)\psi(z)\, \mathrm{d}z &\leq \int |\partial_{x_1x_1} v(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta)|\, |h^1_{\mathtt{v}_2} z_1 - t\theta_1|\, |\theta_1|\, |\psi(z)|\, \mathrm{d}z \\
&\quad + \int |\partial_{x_2x_1} v(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta)|\, |h^2_{\mathtt{v}_2} z_2|\, |\theta_1|\, |\psi(z)|\, \mathrm{d}z \\
&\lesssim h^1_{\mathtt{v}_2} h^1_{\mathtt{v}_2} \int |\partial_{x_1x_1} v(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta)|\, |\psi(z)|\, \mathrm{d}z \\
&\quad + h^2_{\mathtt{v}_2} h^1_{\mathtt{v}_2} \int |\partial_{x_2x_1} v(\mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta)|\, |\psi(z)|\, \mathrm{d}z.
\end{aligned}$$

The change of variables $y = \mathtt{v}_2 - h_{\mathtt{v}_2} \circ z + t\theta$ yields

$$\int F_z'(t)\psi(z)\,\mathrm{d}z \lesssim \left(\frac{h^1_{\mathtt{v}_2}}{h^2_{\mathtt{v}_2}}\|\partial_{x_1x_1}v\|_{L^p(\omega,S_R)} + \|\partial_{x_2x_1}v\|_{L^p(\omega,S_R)}\right)\|1\|_{L^{p'}(\omega^{-p'/p},S_R)},$$

where we used Hölder inequality, that the support of ψ is mapped into S_R, and $\psi \in L^\infty(\mathbb{R}^n)$. Finally, using the A_p-condition, we conclude

$$|q(\mathtt{v}_2)|\|\phi_{\mathtt{v}_2}\|_{L^p(\omega,R)} \lesssim (h^1_{\mathtt{v}_2})^2\|\partial_{x_1x_1}v\|_{L^p(\omega,S_R)} + h^1_{\mathtt{v}_2}h^2_{\mathtt{v}_2}\|\partial_{x_2x_1}v\|_{L^p(\omega,S_R)}.$$

The same arguments above apply to the remaining terms in (4.45). For the term labeled $i = 3$, we obtain

$$|q(\mathtt{v}_3)|\|\phi_{\mathtt{v}_3}\|_{L^p(\omega,R)} \lesssim (h^2_{\mathtt{v}_3})^2\|\partial_{x_2x_2}v\|_{L^p(\omega,S_R)} + h^1_{\mathtt{v}_3}h^2_{\mathtt{v}_3}\|\partial_{x_1x_2}v\|_{L^p(\omega,S_R)},$$

whereas for the term labeled $i = 4$, rewritten first in the form

$$q(\mathtt{v}_4) = \big(Q^1_{\mathtt{v}_1}v(\mathtt{v}_4) - Q^1_{\mathtt{v}_3}v(\mathtt{v}_4)\big) + \big(Q^1_{\mathtt{v}_3}v(\mathtt{v}_4) - Q^1_{\mathtt{v}_4}v(\mathtt{v}_4)\big),$$

we deduce

$$|q(\mathtt{v}_4)|\|\phi_{\mathtt{v}_4}\|_{L^p(\omega,R)} \lesssim \sum_{i,j=1}^{2} h^i_{\mathtt{v}_4}h^j_{\mathtt{v}_4}\|\partial_{x_i}\partial_{x_j}v\|_{L^p(\omega,S_R)}.$$

Finally, replacing the previous estimates back into (4.45), and using the shape regularity properties $h^j_{\mathtt{v}_i} \approx h^j_R$ for $i = 1, \ldots, 4$ and $j = 1, 2$, which result from (4.31), shows the desired anisotropic estimate (4.44). $\square$

Let us comment on the extension of the interpolation estimates of Theorem 4.15 to elements that intersect the Dirichlet boundary, where the functions to be approximated vanish. The proof is very technical and is an adaptation of the arguments of [70, Theorem 3.1] and [129, Theorem 4.7], together with the ideas involved in the proof of Theorem 4.15 to deal with the Muckenhoupt weight $\omega \in A_p(\mathbb{R}^n)$.

Theorem 4.17 (stability and local interpolation: Dirichlet elements) *Let $R \in \mathscr{T}$ be a boundary element. If $v \in W^1_p(\omega, S_R)$ and $v = 0$ on $\partial R \cap \partial\Omega$, then we have*

$$\|\nabla \Pi_{\mathscr{T}} v\|_{L^p(\omega,R)} \lesssim \|\nabla v\|_{L^p(\omega,S_R)}. \tag{4.46}$$

Moreover, if $v \in W^2_p(\omega, S_R)$, then

$$\|\partial_{x_j}(v - \Pi_{\mathscr{T}} v)\|_{L^p(\omega,R)} \lesssim \sum_{i=1}^{n} h^i_R \|\partial_{x_j}\partial_{x_i} v\|_{L^p(\omega,S_R)}. \tag{4.47}$$

for $j = 1, \dots, n$. The hidden constants in both inequalities depend only on $C_{p,\omega}$, σ and ψ.

4.5 Interpolation estimates for different metrics

Given $v \in W^1_p(\omega, S_T)$ with $\omega \in A_p(\mathbb{R}^n)$ and $p \in (1, \infty)$, the goal of this section is to derive local interpolation estimates for v in the space $L^q(\rho, T)$, with weight $\rho \neq \omega$ and Lebesgue exponent $q \neq p$. To derive such an estimate, it is necessary to ensure that the function v belongs to $L^q(\rho, T)$, that is we need to discuss embeddings between weighted Sobolev spaces with different weights and Lebesgue exponents.

Embedding results in spaces of weakly differentiable functions are fundamental in the analysis of partial differential equations. They provide some basic tools in the study of existence, uniqueness and regularity of solutions. To the best of our knowledge, the first to prove such a result was S.L. Sobolev in 1938 [149]. Since then, a great deal of effort has been devoted to studying and improving such inequalities; see, for instance, [26, 128, 166]. In the context of weighted Sobolev spaces, there is an abundant literature that studies the dependence of this result on the properties of the weight; see [82, 87, 93, 97, 98, 100, 101].

Let us first recall the embedding results in the classical case, which will help draw an analogy for the weighted case. We recall the *Sobolev number* of $W_p^m(\Omega)$

$$\mathrm{sob}(W_p^m) = m - \frac{n}{p},$$

which governs the scaling properties of the seminorm $|v|_{W_p^m(\Omega)}$: the change of variables $\hat{x} = x/h$ transforms Ω into $\hat{\Omega}$ and v into $\hat{v}$, while the seminorms scale as

$$|\hat{v}|_{W_p^m(\hat{\Omega})} = h^{\mathrm{sob}(W_p^m)} |v|_{W_p^m(\Omega)}.$$

With this notation classical embeddings [84, Theorem 7.26] can be written in a concise way: if Ω denotes an open and bounded domain with Lipschitz boundary, $1 \leq p < n$ and $\mathrm{sob}(W_p^1) \geq \mathrm{sob}(L^q)$, then $\mathring{W}_p^1(\Omega) \hookrightarrow L^q(\Omega)$ and

$$\|v\|_{L^q(\Omega)} \lesssim \mathrm{diam}(\Omega)^{\mathrm{sob}(W_p^1) - \mathrm{sob}(L^q)} \|\nabla v\|_{L^p(\Omega)} \tag{4.48}$$

for all $v \in \mathring{W}_p^1(\Omega)$. When $\mathrm{sob}(W_p^1) > \mathrm{sob}(L^q)$ the embedding is compact. Results analogous to (4.48) in the weighted setting have been studied in [49, 82, 121, 136] for $n > 1$. For $n = 1$, if $\Omega = (0, a)$, $v \in W_p^1(\omega, \Omega)$, and $\omega \in A_p(\mathbb{R}^n)$, Proposition 2.3 yields $v \in W_1^1(\Omega)$. Consequently $v \in L^\infty(\Omega)$, and then $v \in L^q(\rho, \Omega)$ for any weight ρ and $q \in (1, \infty)$. However, to gain intuition on the explicit dependence of the embbedding constant in terms of the weights and the Lebesgue measure of the domain, let us consider the trivial case $n = 1$ in more detail. To simplify the discussion assume that $v(0) = v(a) = 0$. We thus have

$$\begin{aligned}\int_0^a |v(x)|^q \rho(x) \, \mathrm{d}x &= \int_0^a \rho(x) \left| \int_0^x v'(s) \omega(s)^{1/p} \omega(s)^{-1/p} \, \mathrm{d}s \right|^q \mathrm{d}x \\ &\leq \int_0^a \rho(x) \left(\int_0^x \omega(s) |v'(s)|^p \, \mathrm{d}s \right)^{q/p} \left(\int_0^x \omega(s)^{-p'/p} \, \mathrm{d}s \right)^{q/p'} \mathrm{d}x\end{aligned}$$

whence invoking the definition of the the Muckenhoupt class (2.10) we realize that

$$\int_0^a |v(x)|^q \rho(x)\,\mathrm{d}x \lesssim \|v'\|_{L^p(\omega,\Omega)}^q |\Omega|^q \rho(\Omega)\omega(\Omega)^{-q/p}.$$

The extension of this result to the n-dimensional case has been studied in [49, 82, 121] and is reported in the next two theorems; see [49] for a discussion.

Theorem 4.18 (embeddings in weighted spaces) *Let $\omega \in A_p(\mathbb{R}^n)$, $p \in (1, q]$, and ρ be a weight that satisfies the strong doubling property* (2.13)*. Let the pair (ρ, ω) satisfy the compatibility condition*

$$\frac{r}{R}\left(\frac{\rho(B(x,r))}{\rho(B(x,R))}\right)^{1/q} \leq C_{\rho,\omega}\left(\frac{\omega(B(x,r))}{\omega(B(x,R))}\right)^{1/p}, \tag{4.49}$$

for all $x \in \Omega$ and $r \leq R$. If $v \in \mathring{W}_p^1(\omega,\Omega)$, then $v \in L^q(\rho,\Omega)$ and

$$\|v\|_{L^q(\rho,\Omega)} \lesssim \operatorname{diam}(\Omega)\rho(\Omega)^{1/q}\omega(\Omega)^{-1/p}\|\nabla v\|_{L^p(\omega,\Omega)}, \tag{4.50}$$

where the hidden constant depends on the quotient between the radii of the balls inscribed and circumscribed in Ω.

Proof: Given $v \in \mathring{W}_p^1(\omega,\Omega)$ we denote by $\tilde{v}$ its extension by zero to a ball B_R of radius R containing Ω such that $R \leq 2\operatorname{diam}(\Omega)$. We then apply [49, Theorem 1.5] if $p < q$, or [121, Corollary 2.1] if $p = q$, to conclude

$$\|\tilde{v}\|_{L^q(\rho,B_R)} \lesssim R\rho(B_R)^{1/q}\omega(B_R)^{-1/p}\|\nabla\tilde{v}\|_{L^p(\omega,B_R)}.$$

By assumption ρ satisfies the strong doubling property (2.13) and so, for $B_r \subset \Omega \subset \bar{\Omega} \subset B_R$, we have $\rho(B_R) \lesssim \rho(B_r) \leq \rho(\Omega)$ with a constant that only depends on R/r. Applying this property, together with $\omega(\Omega) \leq \omega(B_R)$, we derive (4.50). $\square$

Theorem 4.19 (Poincaré inequality) *Let $p \in (1, q]$, ρ be a weight that satisfies the strong doubling property (2.13), and $\omega \in A_p(\mathbb{R}^n)$, and let the pair (ρ, ω) satisfy (4.49). If $v \in W_p^1(\omega, \Omega)$, then there is a constant v_Ω such that*

$$\|v - v_\Omega\|_{L^q(\rho,\Omega)} \lesssim \operatorname{diam}(\Omega)\rho(\Omega)^{1/q}\omega(\Omega)^{-1/p}\|\nabla v\|_{L^p(\omega,\Omega)}, \tag{4.51}$$

where the hidden constant depends on the quotient between the radii of the balls inscribed and circumscribed in Ω.

Proof: Since Ω is open and bounded, we can choose $0 < r < R$ such that $\bar{B}_r \subset \Omega \subset \bar{\Omega} \subset B_R$, where B_δ is a ball of radius δ. The extension theorem on weighted Sobolev spaces proved in [55, Theorem 1.1] shows that there exists $\tilde{v} \in W_p^1(\omega, B_R)$ such that $\tilde{v}_{|\Omega} = v$ and

$$\|\nabla\tilde{v}\|_{L^p(\omega,B_R)} \lesssim \|\nabla v\|_{L^p(\omega,\Omega)}, \tag{4.52}$$

where the hidden constant does not depend on v. If $p < q$, then we invoke [82, Theorem 1] and [49, Theorem 1.3] to show that inequality (4.51) holds over B_R with v_Ω being a weighted mean of $\tilde{v}$ in B_R. If $p = q$ instead, we appeal to [121, Remark 2.3] and arrive at the same conclusion. Consequently, we have

$$\|\tilde{v} - v_\Omega\|_{L^q(\rho,\Omega)} \leq \|\tilde{v} - v_\Omega\|_{L^q(\rho,B_R)} \lesssim R\rho(B_R)^{1/q}\omega(B_R)^{-1/p}\|\nabla\tilde{v}\|_{L^p(\omega,B_R)}.$$

The strong doubling property $\rho(B_R) \lesssim \rho(\Omega)$ and $\omega(\Omega) \leq \omega(B_R)$ yield

$$\|\tilde{v} - v_\Omega\|_{L^q(\rho,\Omega)} \lesssim \operatorname{diam}(\Omega)\rho(\Omega)^{1/q}\omega(\Omega)^{-1/p}\|\nabla\tilde{v}\|_{L^p(\omega,B_R)}.$$

Employing (4.52) we finally conclude (4.51). $\square$

Inequalities (4.50) and (4.51) are generalizations of several classical results. We first consider $\omega = \rho \equiv 1$, for which an easy manipulation shows that (4.49) holds if $\mathrm{sob}(W^1_p) \geq \mathrm{sob}(L^q)$, whence (4.51) reduces to (4.48). We next consider $\rho = \omega \in A_p(\mathbb{R}^n)$, for which (4.49) becomes

$$\omega(B(x,R)) \lesssim \left(\frac{R}{r}\right)^{pq/(q-p)} \omega(B(x,r)).$$

This is a consequence of the strong doubling property (2.13) for ω in conjunction with $|B_R| \approx R^n$, provided the restriction $q \leq pn/(n-1)$ between q and p is valid. Moreover, owing to the so-called *open ended property* of the Muckenhoupt classes [126]: if $\omega \in A_p(\mathbb{R}^n)$, then $\omega \in A_{p-\epsilon}(\mathbb{R}^n)$ for some $\epsilon > 0$, we conclude that $q \leq pn/(n-1) + \delta$ for some $\delta > 0$, thus recovering the embedding results proved by Fabes, Kenig and Serapioni [79, Theorem 1.3] and [79, Theorem 1.5]; see [49] for details.

The embedding result of Theorem 4.19 allows us to obtain polynomial interpolation error estimates in $L^q(\rho, T)$ for functions in $W^1_p(\omega, S_T)$.

Theorem 4.20 (interpolation estimates for different metrics I) *Let $\mathscr{T}$ be a simplicial mesh and $\mathcal{P} = \mathbb{P}_1$ in (4.10). Let the pair $(\rho, \omega) \in A_q(\mathbb{R}^n) \times A_p(\mathbb{R}^n)$ satisfy (4.49). If $v \in W^1_p(\omega, S_T)$ for any $T \in \mathscr{T}$, then then*

$$\|v - \Pi_{\mathscr{T}} v\|_{L^q(\rho,T)} \lesssim h_T \rho(S_T)^{1/q} \omega(S_T)^{-1/p} \|\nabla v\|_{L^p(\omega,S_T)}, \tag{4.53}$$

where the hidden constant depends only on σ, ψ, $C_{p,\omega}$ and $C_{\rho,\omega}$.

Proof: Given an interior element $T \in \mathscr{T}$, let us denote v_T the constant such that the estimate (4.51) holds true on S_T. Since v_T is constant over S_T, we have that

$\Pi_{\mathscr{T}} v_T = v_T$ in T. This, together with the stability bound (4.27) for the operator $\Pi_{\mathscr{T}}$, implies

$$\|v - \Pi_{\mathscr{T}} v\|_{L^q(\rho,T)} = \|(I - \Pi_{\mathscr{T}})(v - v_T)\|_{L^q(\rho,T)} \lesssim \|v - v_T\|_{L^q(\rho,S_T)}.$$

The Poincaré inequality (4.51) and the mesh regularity assumption (4.31) yield

$$\|v - \Pi_{\mathscr{T}} v\|_{L^q(\rho,T)} \lesssim \|v - v_T\|_{L^q(\rho,S_T)} \lesssim h_T \rho(S_T)^{1/q} \omega(S_T)^{-1/p} \|\nabla v\|_{L^p(\omega,S_T)}$$

which is (4.53). A similar argument yields (4.53) on boundary elements. $\square$

A trivial but important consequence of Theorem 4.20 is the standard, unweighted, interpolation error estimate in Sobolev spaces; see [56, Theorem 3.1.5].

Corollary 4.21 (L^q-based interpolation estimate) *If $p < n$ and $\operatorname{sob}(W_p^1) > \operatorname{sob}(L^q)$, then for all $T \in \mathscr{T}$ and $v \in W_p^1(S_T)$, we have the local error estimate*

$$\|v - \Pi_{\mathscr{T}} v\|_{L^q(T)} \lesssim h_T^{\operatorname{sob}(W_p^1) - \operatorname{sob}(L^q)} \|\nabla v\|_{L^p(S_T)}, \tag{4.54}$$

where the hidden constant depends only on σ and ψ.

For simplicial meshes, the invariance property of $\Pi_{\mathscr{T}}$ and similar arguments to those used in § 4.4.1 enable us to obtain other interpolation estimates. We illustrate this in the following.

Theorem 4.22 (interpolation estimates for different metrics II) *Let $\mathscr{T}$ be a simplicial mesh and $\mathcal{P} = \mathbb{P}_1$ in (4.10). Given $p \in (1, q]$, let the pair $(\omega, \rho) \in A_p(\mathbb{R}^n) \times A_q(\mathbb{R}^n)$ satisfy (4.49). Then, for every $T \in \mathscr{T}$ and every $v \in W_p^2(\omega, S_T)$ we have*

$$\|\nabla(v - \Pi_{\mathscr{T}} v)\|_{L^q(\rho,T)} \lesssim h_T \rho(S_T)^{1/q} \omega(S_T)^{-1/p} \|D^2 v\|_{L^p(\omega,S_T)}, \tag{4.55}$$

where the hidden constant depends only on σ, ψ, $C_{p,\omega}$ and $C_{\rho,\omega}$.

Proof: Let, again, $T \in \mathscr{T}$ be an interior element, the proof for boundary elements follows from similar arguments. Denote by $\mathtt{v}$ a vertex of T. Since the pair of weights (ω, ρ) satisfies (4.49) the embedding $W^2_p(\omega, S_T) \hookrightarrow W^1_q(\rho, S_T)$ holds and it is legitimate to write

$$\|\nabla(v - \Pi_{\mathscr{T}} v)\|_{L^q(\rho,T)} \leq \|\nabla v - \nabla Q^1_{\mathtt{v}} v\|_{L^q(\rho,T)} + \|\nabla(Q^1_{\mathtt{v}} v - \Pi_{\mathscr{T}} v)\|_{L^q(\rho,T)}$$

In view of (4.27), we have $\|\nabla(Q^1_{\mathtt{v}} v - \Pi_{\mathscr{T}} v)\|_{L^q(\rho,T)} \lesssim \|\nabla v - \nabla Q^1_{\mathtt{v}} v\|_{L^q(\rho,T)}$. We now recall (4.14), namely $\nabla Q^1_{\mathtt{v}} v = Q^0_{\mathtt{v}} \nabla v$, to end up with

$$\|\nabla(v - \Pi_{\mathscr{T}} v)\|_{L^q(\rho,T)} \lesssim \|\nabla v - Q^0_{\mathtt{v}} \nabla v\|_{L^q(\rho,S_T)} \lesssim \|\nabla v - (\nabla v)_T\|_{L^q(\rho,S_T)},$$

because $Q^0_{\mathtt{v}} c = c$ for any constant c. Applying (4.51) finally implies (4.55). $\square$

4.6 Applications

We now present some immediate applications of the interpolation error estimates developed in the previous sections. We recall that $\mathbb{V}(\mathscr{T})$ denotes the finite element space over the mesh $\mathscr{T}$, $\Pi_{\mathscr{T}}$ the quasi-interpolation operator defined in (4.26), and $U_{\mathscr{T}}$ the Galerkin solution to (4.3).

4.6.1 Nonuniformly elliptic boundary value problems

We first derive novel error estimates for the finite element approximation of solutions of a *nonuniformly* elliptic boundary value problem. Let Ω be a polyhedral domain in $\mathbb{R}^n$ with Lipschitz boundary, $\omega \in A_2(\mathbb{R}^n)$ and f be a function in $L^2(\omega^{-1}, \Omega)$. Consider problem (4.1) with $\mathcal{A}$ as in (4.2). The natural space to seek a solution u of problem (4.1) is the weighted Sobolev space $H^1_0(\omega, \Omega)$.

Since Ω is bounded and $\omega \in A_2(\mathbb{R}^n)$, Proposition 2.4 shows that $H_0^1(\omega, \Omega)$ is Hilbert. The Poincaré inequality proved in [79, Theorem 1.3] and the Lax-Milgram lemma then imply the existence and uniqueness of a solution to (4.1) as well as (4.3). The following result establishes a connection between u and $U_{\mathscr{T}}$.

Corollary 4.23 (error estimates for nonuniformly elliptic PDE) *Let the weight $\omega \in A_2(\mathbb{R}^n)$ and $\mathbb{V}(\mathscr{T})$ consist of simplicial elements of degree $m \geq 1$ or rectangular elements of degree $m = 1$. If the solution u of (4.1) satisfies $u \in H_0^1(\omega, \Omega) \cap H^{k+1}(\omega, \Omega)$ for some $1 \leq k \leq m$, then we have the following global error estimate*

$$\|\nabla(u - U_{\mathscr{T}})\|_{L^2(\omega,\Omega)} \lesssim \|h^k D^{k+1} u\|_{L^2(\omega,\Omega)}, \tag{4.56}$$

where h denotes the local mesh-size function of $\mathscr{T}$.

Proof: By Galerkin orthogonality we have

$$\|\nabla(u - U_{\mathscr{T}})\|_{L^2(\omega,\Omega)} \lesssim \inf_{V \in \mathbb{V}(\mathscr{T})} \|\nabla(u - V)\|_{L^2(\omega,\Omega)}.$$

Consider $V = \Pi_{\mathscr{T}} u$ and use the local estimates of either Theorem 4.11 or Theorems 4.15 and 4.17, depending on the discretization. This concludes the proof. $\square$

Remark 4.24 (regularity assumption) We assumed that $u \in H^{m+1}(\omega, \Omega)$ in Corollary 4.23. Since the coefficient matrix $\mathcal{A}$ is not smooth but rather satisfies (4.2), it is natural to ponder whether $u \in H^{m+1}(\omega, \Omega)$ holds. References [48, 53] provide sufficient conditions on $\mathcal{A}, \Omega$ and f for this result to be true for $m = 1$.

Remark 4.25 (multilevel methods) Multilevel methods are known to exhibit linear complexity for the solution of the ensuing algebraic systems. We refer to [90] for weights of class A_1 and [52] for weights of class A_2 (including fractional diffusion).

4.6.2 Elliptic problems with Dirac sources

Dirac sources arise in applications as diverse as modeling of pollutant transport, degradation in an aquatic medium [10] and problems in fractured domains [60]. The analysis of the finite element method applied to such problems is not standard, since in general the solution does not belong to $H^1(\Omega)$ for $n \geq 1$. A priori error estimates in the $L^2(\Omega)$-norm have been derived in the literature using different techniques. In a two dimensional setting and assuming that the domain is smooth, Babuška [12] derived almost optimal a priori error estimates of order $\mathcal{O}(h^{1-\epsilon})$, for an arbitrary $\epsilon > 0$. Scott [142] improved these estimates by removing the ϵ and thus obtaining an optimal error estimate of order $\mathcal{O}(h^{2-n/2})$ for $n = 2, 3$. It is important to notice, as pointed out in [144, Remark 3.1], that these results leave a "regularity gap". In other words, the results of [142] require a $\mathcal{C}^\infty$ domain yet the triangulation is assumed to consist of simplices. Using a different technique, Casas [47] obtained the same result for polygonal or polyhedral domains and general regular Borel measures on the right-hand side.

In the context of weighted Sobolev spaces, interpolation estimates and a priori error estimates have been developed in [6, 60] for such problems. We now show how to apply our polynomial interpolation theory to obtain similar results.

Let Ω be a convex polyhedral domain in $\mathbb{R}^n$ with Lipschitz boundary, and x_0 be an interior point of Ω. Consider the following elliptic boundary value problem:

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \mathbf{b} \cdot \nabla u + cu = \delta_{x_0}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.57}$$

where $\mathcal{A} \in L^\infty(\Omega)$ is a piecewise smooth and uniformly symmetric positive definite matrix, $\mathbf{b} \in W^{1,\infty}(\Omega)^n$, $c \in L^\infty(\Omega)$, and δ_{x_0} denotes the Dirac delta supported at $x_0 \in \Omega$. Existence and uniqueness of u in weighted Sobolev spaces follows from [6, Theorem 1.3] and Lemma 4.29 below, and its asymptotic behavior near x_0 is dictated by that of the Laplacian

$$\nabla u(x) \approx |x - x_0|^{1-n}. \tag{4.58}$$

Denote by $d = \operatorname{diam}(\Omega)$ the diameter of Ω and by $\mathsf{d}_{x_0}(x)$ the scaled Euclidean distance $\mathsf{d}_{x_0}(x) = |x - x_0|/(2d)$ to x_0. Define the weight

$$\varpi(x) = \begin{cases} \dfrac{\mathsf{d}_{x_0}(x)^{n-2}}{\log^2 \mathsf{d}_{x_0}(x)}, & 0 < \mathsf{d}_{x_0}(x) < \dfrac{1}{2}, \\ \dfrac{2^{2-n}}{\log^2 2}, & \mathsf{d}_{x_0}(x) \geq \dfrac{1}{2}. \end{cases} \tag{4.59}$$

We now study two important properties of ϖ: $\nabla u \in L^2(\varpi, \Omega)$ and $\varpi \in A_2(\mathbb{R}^n)$.

Lemma 4.26 (regularity of ∇u) *The solution u of (4.57) satisfies $\nabla u \in L^2(\varpi, \Omega)$.*

Proof: Since $\Omega \subset B$, the ball of radius d centered at x_0, we readily have from (4.58)

$$\int_\Omega |\nabla u|^2 \varpi \lesssim \int_B \mathsf{d}_{x_0}(x)^{2(1-n)} \frac{\mathsf{d}_{x_0}(x)^{n-2}}{\log^2 \mathsf{d}_{x_0}(x)} \, \mathrm{d}x \lesssim \int_0^{\frac{1}{2}} \frac{1}{r \log^2 r} \, \mathrm{d}r = \frac{1}{\log 2},$$

which is the asserted result. $\square$

Lemma 4.27 ($\varpi \in A_2(\mathbb{R}^n)$) *The weight ϖ belongs to the Muckenhoupt class $A_2(\mathbb{R}^n)$ with constant $C_{2,\varpi}$ only depending on d.*

Proof: Let $x_0 = 0$ for simplicity, let $B_r = B_r(y)$ be a ball in $\mathbb{R}^n$ of radius r and center y, and denote $\varpi(B_r) = \int_{B_r} \varpi$ and $\varpi^{-1}(B_r) = \int_{B_r} \varpi^{-1}$. We must show

$$\varpi(B_r)\,\varpi^{-1}(B_r) \lesssim r^{2n} \qquad \forall\, r > 0, \tag{4.60}$$

with a hidden constant depending solely on d. We split the proof into two cases.

1. Case $|y| < 2r$: Since $B_r(y) \subset B_{3r}(0)$ we infer that

$$\varpi(B_r) \lesssim \int_{B_{3r}(0)} \frac{\left(\frac{|x|}{2d}\right)^{n-2}}{\log^2 \frac{|x|}{2d}}\,\mathrm{d}x \lesssim \int_0^{\frac{3r}{2d}} \frac{s^{2n-3}}{\log^2 s}\,\mathrm{d}s \approx \frac{\left(\frac{3r}{2d}\right)^{2n-2}}{\log^2 \frac{3r}{2d}}$$

   and

$$\varpi^{-1}(B_r) \lesssim \int_{B_{3r}(0)} \left(\frac{|x|}{2d}\right)^{2-n} \log^2\left(\frac{|x|}{2d}\right) \mathrm{d}x \lesssim \int_0^{\frac{3r}{2d}} s \log^2 s\,\mathrm{d}s \approx \left(\frac{3r}{2d}\right)^2 \log^2 \frac{3r}{2d},$$

   provided $3r < d$. The equivalences $\approx$ can be checked via L'Hôpital's rule for $r \to 0$. If $3r \geq d$, then both $\varpi(B_r)$ and $\varpi^{-1}(B_r)$ are bounded by constants depending only on d. Therefore, this yields (4.60).

2. Case $|y| \geq 2r$: Since all $x \in B_r(y)$ satisfy $\frac{1}{2}|y| \leq |x| \leq \frac{3}{2}|y|$ we deduce

$$\varpi \leq \min\left\{ \frac{\left(\frac{3|y|}{4d}\right)^{n-2}}{\log^2 \frac{3|y|}{4d}}, \frac{2^{2-n}}{\log^2 2} \right\}, \quad \varpi^{-1} \leq \max\left\{ \left(\frac{|y|}{4d}\right)^{2-n} \log^2 \frac{|y|}{4d}, 2^{n-2}\log^2 2 \right\},$$

   whence $\varpi(B_r)\,\varpi^{-1}(B_r)$ satisfies again (4.60).

This completes the proof. $\square$

The fact that the weight $\varpi \in A_2(\mathbb{R}^n)$ is the key property for the analysis of discretizations of problem (4.57). Let us apply the results of Theorem 4.18 to this particular weight.

Lemma 4.28 **($H^1(\Omega) \hookrightarrow L^2(\varpi^{-1}, \Omega)$)** *Let ϖ be defined in* (4.59)*. If $n < 4$, then the following embedding holds:*

$$H^1(\Omega) \hookrightarrow L^2(\varpi^{-1}, \Omega).$$

Proof: This is an application of Theorem 4.18. We must show when condition (4.49) holds with $p = q = 2$, $\omega = 1$ and $\rho = \varpi^{-1}$. In other words, we need to verify

$$\Lambda(r, R) := \frac{r^{2-n}}{R^{2-n}} \frac{\varpi^{-1}(B_r)}{\varpi^{-1}(B_R)} \lesssim 1, \quad \forall r \in (0, R],$$

where both B_r and B_R are centered at $y \in \mathbb{R}^n$. We proceed as in Lemma 4.27 and consider now three cases.

1. $|y| < 2r$. We know from Lemma 4.27 that $\varpi^{-1}(B_r) \lesssim \left(\frac{3r}{2d}\right)^2 \log^2\left(\frac{3r}{2d}\right)$. Moreover, every $x \in B_R(y)$ satisfies $|x| < |y| + R \leq 3R$ whence

$$\varpi^{-1}(B_R) \geq \int_{B_R} \left(\frac{3|x|}{2d}\right)^{2-n} \log^2\left(\frac{3|x|}{2d}\right) \mathrm{d}x \approx \int_0^{\frac{3R}{2d}} s \log^2 s \, \mathrm{d}s \approx \left(\frac{3R}{2d}\right)^2 \log^2\left(\frac{3R}{2d}\right).$$

If $n < 4$, then this shows

$$\Lambda(r, R) \lesssim \frac{r^{4-n}}{R^{4-n}} \frac{\log^2\left(\frac{3r}{2d}\right)}{\log^2\left(\frac{3R}{2d}\right)} \lesssim 1.$$

2. $2r \leq |y| < 2R$. We learn from Lemma 4.27 that

$$\varpi^{-1}(B_r) \lesssim |B_r| \left(\frac{|y|}{4d}\right)^{2-n} \log^2\left(\frac{|y|}{4d}\right) \lesssim \left(\frac{r}{2d}\right)^2 \log^2\left(\frac{r}{2d}\right).$$

In addition, any $x \in B_R$ satisfies $|x| \leq |y| + R \leq 3R$ and the same bound as in Case 1 holds for $\varpi^{-1}(B_R)$. Consequently, $\Lambda(r, R) \lesssim 1$ again for $n < 4$.

3. $|y| \geq 2R$. Since still $|y| > 2r$ we have for $\varpi^{-1}(B_r)$ the same upper bound as in Case 2. On the other hand, for all $x \in B_R$ we realize that $|x| \leq |y| + R \leq \frac{3}{2}|y|$ and $\varpi^{-1}(x) \geq \varpi^{-1}(\frac{3}{2}y)$. Therefore, we deduce

$$\left(\frac{3R}{d}\right)^2 \log^2 \frac{3R}{d} \lesssim R^n \left(\frac{3|y|}{2d}\right)^{2-n} \log^2 \left(\frac{3|y|}{2d}\right) \lesssim \varpi^{-1}(B_R),$$

which again leads to $\Lambda(r,R) \lesssim 1$ for $n < 4$.

This concludes the proof. □

The embedding of Lemma 4.28 allows us to develop a general theory for equations of the form (4.57) on weighted spaces. To achieve this, define

$$a(w,v) = \int_\Omega \mathcal{A}\nabla w \cdot \nabla v + \mathbf{b} \cdot \nabla w v + cwv. \tag{4.61}$$

The following results follow [60, 6].

Lemma 4.29 (inf–sup conditions) *The bilinear form a, defined in* (4.61)*, satisfies*

$$1 \lesssim \inf_{w \in H_0^1(\varpi,\Omega)} \sup_{v \in H_0^1(\varpi^{-1},\Omega)} \frac{a(w,v)}{\|\nabla w\|_{L^2(\varpi,\Omega)} \|\nabla v\|_{L^2(\varpi^{-1},\Omega)}}, \tag{4.62}$$

$$1 \lesssim \inf_{v \in H_0^1(\varpi^{-1},\Omega)} \sup_{w \in H_0^1(\varpi,\Omega)} \frac{a(w,v)}{\|\nabla w\|_{L^2(\varpi,\Omega)} \|\nabla v\|_{L^2(\varpi^{-1},\Omega)}}. \tag{4.63}$$

Proof: We divide the proof into several steps:

1. We first obtain an orthogonal decomposition of $L^2(\varpi^{-1},\Omega)^n$ [60, Lemma 2.1]: for every $\mathbf{q} \in L^2(\varpi^{-1},\Omega)^n$ there is a unique couple $(\boldsymbol{\sigma}, v) \in \mathbb{X} := L^2(\varpi^{-1},\Omega)^n \times H_0^1(\varpi^{-1},\Omega)$ such that

$$\mathbf{q} = \boldsymbol{\sigma} + \nabla v, \qquad \int_\Omega \mathcal{A}\boldsymbol{\sigma} \cdot \nabla w = 0, \qquad \forall w \in H_0^1(\varpi,\Omega), \tag{4.64}$$

$$\|\boldsymbol{\sigma}\|_{L^2(\varpi^{-1},\Omega)^n} + \|\nabla v\|_{L^2(\varpi^{-1},\Omega)} \lesssim \|\mathbf{q}\|_{L^2(\varpi^{-1},\Omega)^n}. \tag{4.65}$$

To see this, we let $\mathbb{Y} := L^2(\varpi,\Omega)^n \times H_0^1(\varpi,\Omega)$, write (4.64) in mixed form

$$\mathcal{B}[(\boldsymbol{\sigma},v),(\boldsymbol{\tau},w)] := \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_\Omega \nabla v\cdot\boldsymbol{\tau} + \int_\Omega \mathcal{A}\boldsymbol{\sigma}\cdot\nabla w = \int_\Omega \mathbf{q}\cdot\boldsymbol{\tau} \quad \forall\,(\boldsymbol{\tau},w)\in\mathbb{Y},$$

and apply the generalized Babuška-Brezzi inf–sup theory [24, Theorem 2.1], [60, Lemma 2.1]. This requires only that $\mathcal{A}$ be positive definite along with the trivial fact that $\phi \in L^2(\varpi^{-1},\Omega)$ implies $\varpi^{-1}\phi \in L^2(\varpi,\Omega)$.

2. Set $|\mathbf{b}| = c = 0$ and let $w \in H_0^1(\varpi,\Omega)$ be given. According to Step 1 we can decompose $\mathbf{q} = \varpi\nabla w \in L^2(\varpi^{-1},\Omega)$ into $\mathbf{q} = \boldsymbol{\sigma} + \nabla v$. Invoking (4.64), as in [60, Corollary 2.2] and [6, Proposition 1.1], we infer that

$$\int_\Omega \mathcal{A}\nabla w\cdot\nabla v = \int_\Omega \mathcal{A}\nabla w\cdot\mathbf{q} - \int_\Omega \mathcal{A}\nabla w\cdot\sigma = \int_\Omega \varpi\mathcal{A}\nabla w\cdot\nabla w \approx \int_\Omega \varpi|\nabla w|^2,$$

whence, using (4.65) in the form $\|\nabla v\|_{L^2(\varpi^{-1},\Omega)} \lesssim \|\nabla w\|_{L^2(\varpi,\Omega)}$, we deduce the inf–sup condition (4.62).

3. As in [6], we show that for every $F \in H_0^1(\varpi^{-1},\Omega)'$ the problem

$$w \in H_0^1(\varpi,\Omega): \quad a(w,v) = \langle F, v\rangle, \quad \forall v \in H_0^1(\varpi^{-1},\Omega),$$

is well posed. To this end, we decompose $w = w_1 + w_2 \in H_0^1(\varpi,\Omega)$, with

$$w_1 \in H_0^1(\varpi,\Omega): \quad \int_\Omega \mathcal{A}\nabla w_1\cdot\nabla v = \langle F, v\rangle, \quad \forall v \in H_0^1(\varpi^{-1},\Omega), \tag{4.66}$$

$$w_2 \in H_0^1(\Omega): \quad a(w_2,v) = -\int_\Omega (\mathbf{b}\cdot\nabla w_1 + cw_1)\,v, \quad \forall v \in H_0^1(\Omega). \tag{4.67}$$

In fact, if problems (4.66) and (4.67) have a unique solution, then we obtain

$$\begin{aligned} a(w,v) &= a(w_1+w_2,v) \\ &= \int_\Omega \mathcal{A}\nabla w_1\cdot\nabla v + \int_\Omega (\mathbf{b}\cdot\nabla w_1 + cw_1)\,v + a(w_2,v) = \langle F, v\rangle, \end{aligned}$$

for any $v \in H_0^1(\varpi^{-1}, \Omega) \subset H_0^1(\Omega)$. The conclusion of Step 2 shows that (4.66) is well posed. The Cauchy-Schwarz inequality and Lemma 4.28 yield

$$\int_\Omega (\mathbf{b} \cdot \nabla w_1 + c w_1)\, v \lesssim \|w_1\|_{H^1(\varpi,\Omega)} \|v\|_{L^2(\varpi^{-1},\Omega)} \lesssim \|F\|_{H_0^1(\varpi^{-1},\Omega)'} \|\nabla v\|_{L^2(\varpi^{-1},\Omega)},$$

which combines with the fact that $a(\cdot,\cdot)$ satisfies the inf–sup condition in $H_0^1(\Omega)$ [13, Theorem 5.3.2 - Part I] to show that (4.67) is well posed as well.

Finally, the general inf–sup theory [75] [133, Theorem 2] guarantees the validity of the two inf–sup conditions (4.62) and (4.63). This concludes the proof. $\square$

We also have the following discrete counterpart of Lemma 4.29. We refer to [60, Lemma 3.3] and [6, Theorem 2.1] for similar results which, however, do not exploit the Muckenhoupt structure of the weight ϖ.

Lemma 4.30 (discrete inf–sup conditions) *Let $\mathscr{T}$ be a quasi-uniform mesh of size h consisting of simplices. If $\mathbb{V}(\mathscr{T})$ is made of piecewise linears, then the bilinear form a, defined in (4.61), satisfies:*

$$1 \lesssim \inf_{W \in \mathbb{V}(\mathscr{T})} \sup_{V \in \mathbb{V}(\mathscr{T})} \frac{a(W,V)}{\|\nabla W\|_{L^2(\varpi,\Omega)} \|\nabla V\|_{L^2(\varpi^{-1},\Omega)}},$$
$$1 \lesssim \inf_{V \in \mathbb{V}(\mathscr{T})} \sup_{W \in \mathbb{V}(\mathscr{T})} \frac{a(W,V)}{\|\nabla W\|_{L^2(\varpi,\Omega)} \|\nabla V\|_{L^2(\varpi^{-1},\Omega)}}.$$

where the hidden constants depend on $C_{2,\varpi}$ but not on h.

Proof: We proceed as in Lemma 4.29. We define the spaces of piecewise constants

$$\mathbb{V}_0(\mathscr{T}) = \mathbb{W}_0(\mathscr{T}) = \left\{ \mathbf{Q} \in L^\infty(\Omega)^n : \mathbf{Q}_{|T} \in \mathbb{R}^n,\ \forall T \in \mathscr{T} \right\},$$

those of piecewise linears $\mathbb{V}_1(\mathscr{T}) = \mathbb{W}_1(\mathscr{T}) = \mathbb{V}(\mathscr{T})$, and endow the product spaces $\mathbb{V}_0(\mathscr{T}) \times \mathbb{V}_1(\mathscr{T})$ and $\mathbb{W}_0(\mathscr{T}) \times \mathbb{W}_1(\mathscr{T})$ with the norms of $\mathbb{X}$ and $\mathbb{Y}$ respectively, the latter spaces being defined in Lemma 4.29. Given $\mathbf{Q} \in \mathbb{V}_0(\mathscr{T})$, we need the following orthogonal decomposition — a discrete counterpart of (4.64)-(4.65): find $\Sigma \in \mathbb{V}_0(\mathscr{T}), V \in \mathbb{V}_1(\mathscr{T})$ so that

$$\mathbf{Q} = \mathbf{\Sigma} + \nabla V, \qquad \int_\Omega \mathcal{A}\mathbf{\Sigma} \cdot \nabla W = 0, \qquad \forall W \in \mathbb{W}_1(\mathscr{T}), \tag{4.68}$$

$$\|\mathbf{\Sigma}\|_{L^2(\varpi^{-1},\Omega)^n} + \|\nabla V\|_{L^2(\varpi^{-1},\Omega)} \lesssim \|\mathbf{Q}\|_{L^2(\varpi^{-1},\Omega)^n}. \tag{4.69}$$

We first have to verify that the bilinear form $\mathcal{B}$ satisfies a discrete inf–sup condition, as in Step 1 of Lemma 4.29. We just prove the most problematic inf–sup

$$\|\nabla W\|_{L^2(\varpi,\Omega)} \lesssim \sup_{\mathbf{T}\in\mathbb{V}_0(\mathscr{T})} \frac{\int_\Omega \mathcal{A}\mathbf{T}\cdot\nabla W}{\|\mathbf{T}\|_{L^2(\varpi^{-1},\Omega)^n}}.$$

We let $\mathbf{T} = \varpi_{\mathscr{T}} \nabla W \in \mathbb{V}_0(\mathscr{T})$, where $\varpi_{\mathscr{T}}$ is the piecewise constant weight defined on each element $T \in \mathscr{T}$ as $\varpi_{\mathscr{T}}|_T = |T|^{-1}\int_T \varpi$. Since $\nabla W \in \mathbb{V}_0(\mathscr{T})$, we get

$$\int_\Omega \mathcal{A}\mathbf{T}\cdot\nabla W = \int_\Omega \varpi_{\mathscr{T}} \mathcal{A}\nabla W \cdot \nabla W \approx \int_\Omega \varpi_{\mathscr{T}} \nabla W \cdot \nabla W = \int_\Omega \varpi |\nabla W|^2,$$

and

$$\int_\Omega \varpi^{-1}|\mathbf{T}|^2 = \sum_{T\in\mathscr{T}} \int_T |T|^{-2}\varpi^{-1}\left(\int_T \varpi\right)^2 |\nabla W_{|T}|^2 \le C_{2,\varpi}\int_\Omega \varpi|\nabla W|^2.$$

We employ a similar calculation to perform Step 2 of Lemma 4.29, and the rest is exactly the same as in Lemma 4.29. The proof is thus complete. □

The numerical analysis of a finite element approximation to the solution of problem (4.57) is now a consequence of the interpolation estimates developed in section 4.5.

Corollary 4.31 (error estimate for elliptic problems with Dirac sources) *Assume that $n < 4$ and let $u \in H_0^1(\varpi,\Omega)$ be the solution of* (4.57) *and $U_{\mathscr{T}} \in \mathbb{V}(\mathscr{T})$*

be the finite element solution to (4.57)*. If $\mathscr{T}$ is simplicial, quasi-uniform and of size h, we have the following error estimate*

$$\|u - U_{\mathscr{T}}\|_{L^2(\Omega)} \lesssim h^{2-n/2}|\log h|\|\nabla u\|_{L^2(\varpi,\Omega)}. \tag{4.70}$$

Proof: We employ a duality argument. Let $\varphi \in H_0^1(\Omega)$ be the solution of

$$a(v,\varphi) = \int_\Omega (u - U_{\mathscr{T}})v \quad \forall\, v \in H_0^1(\Omega), \tag{4.71}$$

which is the adjoint of (4.57). Since Ω is convex and polyhedral, and the coefficients $\mathcal{A}, \mathbf{b}, c$ are sufficiently smooth, we have the standard regularity pick-up [84]:

$$\|\varphi\|_{H^2(\Omega)} \lesssim \|u - U_{\mathscr{T}}\|_{L^2(\Omega)}. \tag{4.72}$$

This, together with Lemma 4.28, allows us to conclude that, if $n < 4$,

$$\varphi \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow H_0^1(\varpi^{-1},\Omega).$$

Moreover, Theorem 4.22 yields the error estimate

$$\|\nabla(\varphi - \Pi_{\mathscr{T}}\varphi)\|_{L^2(\varpi^{-1},\Omega)} \lesssim \sigma(h)\|\varphi\|_{H^2(\Omega)}.$$

with

$$\sigma(h) = h\big(\varpi^{-1}(B_h)\big)^{\frac{1}{2}}|B_h|^{-\frac{1}{2}} \lesssim h^{2-\frac{n}{2}}|\log h|.$$

Let $\Phi_{\mathscr{T}} \in \mathbb{V}(\mathscr{T})$ be the Galerkin solution to (4.71). Galerkin orthogonality and the continuity of the form a on $H_0^1(\varpi,\Omega) \times H_0^1(\varpi^{-1},\Omega)$ yield

$$\|u - U_{\mathscr{T}}\|^2_{L^2(\Omega)} = a(u, \varphi - \Phi_{\mathscr{T}}) \lesssim \|\nabla u\|_{L^2(\varpi,\Omega)}\|\nabla(\varphi - \Phi_{\mathscr{T}})\|_{L^2(\varpi^{-1},\Omega)}. \tag{4.73}$$

The discrete inf–sup conditions of Lemma 4.30 and and the continuity of the form a allow us to conclude that

$$\|\nabla(\varphi - \Phi_{\mathscr{T}})\|_{L^2(\varpi^{-1},\Omega)} \lesssim \|\nabla(\varphi - \Pi_{\mathscr{T}}\varphi)\|_{L^2(\varpi^{-1},\Omega)}.$$

Combining this bound with (4.72) and (4.73) results in

$$\|u - U_{\mathscr{T}}\|_{L^2(\Omega)}^2 \lesssim \sigma(h)\|\nabla u\|_{L^2(\varpi,\Omega)}\|u - U_{\mathscr{T}}\|_{L^2(\Omega)},$$

which is the asserted estimate (4.70) in disguise. $\square$

Remark 4.32 (an interpolation result) For any $\beta \in (-n, n)$ we can consider the weight $\mathsf{d}_{x_0}(x)^\beta$, which belongs to the $A_2(\mathbb{R}^n)$ Muckenhoupt class. Theorem 4.11 and Theorems 4.15 and 4.17 show that

$$\|u - \Pi_{\mathscr{T}}u\|_{L^2(\mathsf{d}_{x_0}^\beta,\Omega)} \lesssim \|h\nabla u\|_{L^2(\mathsf{d}_{x_0}^\beta,\Omega)}.$$

This extends the interpolation error estimates of [6, Proposition 4.6], which are valid for $\beta \in (-n, 0)$ only.

4.6.3 Fractional powers of uniformly elliptic operators

We finally comment on finite element approximations of solutions to fractional differential equations. Let Ω be a polyhedral domain in $\mathbb{R}^n$ ($n \geq 1$), with boundary $\partial\Omega$. Given a piecewise smooth and uniformly symmetric positive definite matrix $\mathcal{A} \in L^\infty(\Omega)$ and a nonnegative function $c \in L^\infty(\Omega)$, define the differential operator

$$\mathcal{L}w = -\mathrm{div}(\mathcal{A}\nabla w) + cw.$$

Given $f \in H^{-1}(\Omega)$, the problem of finding $u \in H_0^1(\Omega)$ such that $\mathcal{L}u = f$ has a unique solution. Moreover, the operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2(\Omega) \to L^2(\Omega)$ with domain $\mathcal{D}(\mathcal{L}) = H^2(\Omega) \cap H_0^1(\Omega)$ has a compact inverse [91, Theorem 2.4.2.6]. Therefore, there exists a sequence of eigenpairs $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$, with $\lambda_k > 0$, such that

$$\mathcal{L}\varphi_k = \lambda_k \varphi_k, \text{ in } \Omega \qquad \varphi_{k|\partial\Omega} = 0.$$

The sequence $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$.

In this case, for $s \in (0,1)$, we define the fractional powers of $\mathcal{L}_0$ (where the sub-index is used to indicate the homogeneous Dirichlet boundary conditions) by

$$w = \sum_k w_k \varphi_k \quad \Longrightarrow \quad \mathcal{L}_0^s w = \sum_k \lambda_k^s w_k \varphi_k.$$

We are interested in finding numerical solutions to the following fractional differential equation: given $s \in (0,1)$ and a function $f \in \mathbb{H}^s(\Omega)'$, find u such that

$$\mathcal{L}_0^s u = f. \tag{4.74}$$

The fractional operator $\mathcal{L}_0^s$ is *nonlocal* (see [115, 43, 41]). To localize it, we consider the Caffarelli and Silvestre [43] and its extensions [155, 44] to replace the nonlocal problem (4.74) by the local problem

$$-\mathrm{div}\big(y^\alpha \mathbf{A} \nabla \mathscr{U}\big) + y^\alpha c \mathscr{U} = 0$$

with $\alpha := 1 - 2s$, $\mathbf{A} = \mathrm{diag}\{\mathcal{A}, 1\} \in \mathbb{R}^{(n+1)\times(n+1)}$, posed in the semi-infinite cylinder $\mathcal{C}$, and subject to a Neumann condition at $y = 0$ involving f. Since $\mathcal{C}$ is an unbounded domain, this problem cannot be directly approximated with finite-element-like techniques. However, as Proposition 3.4 shows, the solution to this problem decays exponentially in the extended variable y so that, by truncating the cylinder $\mathcal{C}$ to $\mathcal{C}_{\mathscr{Y}} = \Omega \times (0, \mathscr{Y})$, and setting a vanishing Dirichlet condition on the upper boundary $y = \mathscr{Y}$, we only incur in an exponentially small error in terms of $\mathscr{Y}$; see Theorem 3.10. Define

$$\mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}}) = \left\{ v \in H^1(y^\alpha, \mathcal{C}_{\mathscr{Y}}) : v = 0 \text{ on } \partial_L \mathcal{C}_{\mathscr{Y}} \cup \Omega \times \{\mathscr{Y}\} \right\}.$$

Then, the aforementioned problem reads: find $\mathscr{U} \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}})$ such that for all $v \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}})$

$$\int_{\mathcal{C}_{\mathscr{Y}}} y^\alpha \left((\mathbf{A} \nabla \mathscr{U}) \cdot \nabla v + c \mathscr{U} v \right) = d_s \langle f, \operatorname{tr}_\Omega v \rangle_{\mathbb{H}^s(\Omega) \times \mathbb{H}^s(\Omega)'}, \tag{4.75}$$

We construct a mesh over $\mathcal{C}_{\mathscr{Y}}$ with cells of the form $T = K \times I$ with $K \subset \Omega$ being an element that is isoparametrically equivalent either to $[0,1]^n$ or the unit simplex in $\mathbb{R}^n$ and $I \subset \mathbb{R}$ is an interval. In view of the regularity estimate (3.28) it is necessary to measure the regularity of $\mathscr{U}_{yy}$ with a stronger weight and thus compensate with a graded mesh in the extended dimension. This makes anisotropic estimates essential. We consider the graded partition of the interval $[0, \mathscr{Y}]$ with mesh points

$$y_k = \left(\frac{k}{M} \right)^\gamma \mathscr{Y}, \quad k = 0, \dots, M, \tag{4.76}$$

where $\gamma > 3/(1-\alpha)$, along with a quasi-uniform triangulation $\mathscr{T}_\Omega$ of the domain Ω. We construct the mesh $\mathscr{T}_{\mathscr{Y}}$ as the tensor product of $\mathscr{T}_\Omega$ and the partition given in (4.76); hence $\#\mathscr{T} = M \, \#\mathscr{T}_\Omega$. Assuming that $\#\mathscr{T}_\Omega \approx M^n$ we have $\#\mathscr{T}_{\mathscr{Y}} \approx M^{n+1}$. Finally, since $\mathscr{T}_\Omega$ is shape regular and quasi-uniform, $h_{\mathscr{T}_\Omega} \approx (\#\mathscr{T}_\Omega)^{-1/n}$. All these considerations allow us to obtain the following result.

Corollary 4.33 (error estimate for fractional powers of elliptic operators) *Let $\mathscr{T}$ be a graded tensor product grid, which is quasi-uniform in Ω and graded in the extended variable so that (4.76) hold. If $\mathbb{V}(\mathscr{T})$ is made of bilinear elements, then the solution of (4.75) and its Galerkin approximation $U_{\mathscr{T}} \in \mathbb{V}(\mathscr{T})$ satisfy*

$$\|\mathscr{U} - U_{\mathscr{T}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}})|^s(\#\mathscr{T}_{\mathcal{Y}})^{-1/(n+1)}\|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

where $\mathcal{Y} \approx \log(\#\mathscr{T}_{\mathcal{Y}})$. Alternatively, if u denotes the solution of (4.74), then

$$\|u - U_{\mathscr{T}}(\cdot,0)\|_{\mathbb{H}^s(\Omega)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}})|^s(\#\mathscr{T}_{\mathcal{Y}})^{-1/(n+1)}\|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

Proof: First of all, notice that $y^\alpha \in A_2(\mathbb{R}^{n+1})$ for $\alpha \in (-1,1)$. Owing to the exponential decay of $\mathscr{U}$, and the choice of the parameter $\mathcal{Y}$, it suffices to estimate $\mathscr{U} - \Pi_{\mathscr{T}_{\mathcal{Y}}}\mathscr{U}$ on the mesh $\mathscr{T}_{\mathcal{Y}}$; see § 3.3.1. To do so, we notice that if I_1 and I_2 are neighboring cells on the partition of $[0,\mathcal{Y}]$, then the weak regularity condition (4.31) holds. Thus, we decompose the mesh $\mathscr{T}_{\mathcal{Y}}$ into the sets

$$\mathcal{T}_0 = \left\{T \in \mathscr{T}_{\mathcal{Y}} :\ S_T \cap (\bar{\Omega} \times \{0\}) = \emptyset\right\}, \quad \mathcal{T}_1 = \left\{T \in \mathscr{T}_{\mathcal{Y}} :\ S_T \cap (\bar{\Omega} \times \{0\}) \neq \emptyset\right\},$$

and apply our interpolation theory developed in Theorems 4.15 and 4.17 for interior and boundary elements respectively, together with the local regularity estimates for the function $\mathscr{U}$ derived in § 3.1.5 (see also [129, Theorem 2.8]). $\square$

The error estimates with graded meshes are quasi-optimal in both regularity and order. Error estimates for quasi-uniform meshes are suboptimal in terms of order [129, Section 5]. Mesh anisotropy is thus able to capture the singular behavior of the solution $\mathscr{U}$ and restore optimal decay rates.

Chapter 5: Multilevel methods for nonuniformly elliptic operators

5.1 Introduction

The main advantage of the algorithm proposed in Chapter 3, is that we are solving the local problem (1.2) instead of dealing with the nonlocal operator $(-\Delta)^s$ of problem (1.1). However, this comes at the expense of incorporating one more dimension to the problem, thus raising the question of how computationally efficient this approach is. A quest for the answer is the motivation for the study of multilevel methods, since it is known that they are the most efficient techniques for the solution of discretizations of partial differential equations, see [36, 37, 94, 161]. Multigrid methods for equations of the type (1.2), however, are not very well understood.

The purpose of this work is twofold and hinges on the multilevel framework developed in [32, 161] and the Xu-Zikatanov identity [163]. First, we show nearly uniform convergence of a multilevel method for a class of general nonuniformly elliptic equations on quasi-uniform meshes. Second, we derive an almost uniform convergence of a multilevel method for the local problem that arises from our PDE approach to the fractional Laplacian (1.2) on anisotropic meshes [129, 132]. The former result assumes that the weight ω in the differential operator belongs to the so-called Muckenhoupt class A_2; see Definition 2.2 for details. A somewhat related work is [90] where the authors show a uniform norm equivalence for a multilevel space decomposition under the assumption that the weight belongs to the smaller class A_1. Their results and techniques, however, do not apply to our setting since, simply put, an A_1-weight is "almost bounded", which is too restrictive; see Remark 2.3 for details. We make no regularity assumption on the weight ω and show that our estimates solely depend on the A_2-constant $C_{2,\omega}$. However, our results depend on the number J of levels, and thus logarithmically on the meshsize, which seems unavoidable without further regularity assumptions. For the fractional Laplacian, Chapter 3 shows that a quasi-uniform mesh cannot yield quasi-optimal error estimates and, consequently, the mesh in the extended dimension must be graded towards the bottom of the cylinder thus becoming anisotropic. We apply line smoothers over vertical lines in the extended domain and prove that the corresponding multigrid $\mathcal{V}$-cycle converges almost uniformly.

We propose an algorithm with complexity $\mathcal{O}(M^{n+1} \log M)$ for computing a nearly optimal approximation of the fractional Laplacian problem (1.1) in $\mathbb{R}^n$, where M denotes the number of degrees of freedom in each direction. Notice that using the intrinsic integral formulation of the fractional Laplacian [41, 43], a discretization would result in a dense matrix with $\mathcal{O}(M^{2n})$. Special techniques such as fast multipole methods [89], the $\mathcal{H}$-matrix methods [96] or wavelet methods [99, 154] might be applied to reduce the complexity of storage and manipulation of the dense matrix as well as the complexity of solvers.

The outline of this paper is as follows. Section 5.2 contains the salient results about the finite element approximation of nonuniformly elliptic equations including the fractional Laplacian on anisotropic meshes. Here we also collect the relevant properties of a quasi-interpolant which are crucial to obtain the convergence analysis of our multilevel methods. In section 5.3, we recall the theory of subspace corrections [161] and the Xu-Zikatanov identity [163]. We present multigrid algorithms for nonuniformly elliptic equations discretized on quasi-uniform meshes in section 5.4 and prove their nearly uniform convergence. We adapt the algorithms and analysis of section 5.4 to the fractional Laplacian discretized on anisotropic meshes in section 5.5. This requires a line smoother along the extended direction. Finally, to illustrate the performance of our methods and the sharpness of our results, we present a series of numerical experiments in section 5.6.

5.2 Finite element discretization of nonuniformly elliptic equations

In order to keep this Chapter self-contained, we recall some elements and results of Chapters 3 and 4. Let D be an open and bounded subset of $\mathbb{R}^N$ ($N \geq 1$) with boundary ∂D and let $f \in L^2(\omega^{-1}, D)$. In this section, we focus on the study of a finite element method for the following nonuniformly elliptic boundary value problem: find $u \in H_0^1(\omega, D)$ that solves

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x)\nabla u) = f, & \text{in } D, \\ u = 0, & \text{on } \partial D, \end{cases} \tag{5.1}$$

where $\mathcal{A} : D \to \mathbb{R}^{N\times N}$ is symmetric and satisfies the following nonuniform ellipticity condition

$$\omega(x)|\xi|^2 \lesssim \xi^{\intercal}\mathcal{A}(x)\xi \lesssim \omega(x)|\xi|^2, \qquad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } x \in D, \tag{5.2}$$

and ω belongs to the Muckenhoupt class A_2, which is defined by (2.2). We define the bilinear form

$$a(u,v) = \int_D \mathcal{A}\nabla u \cdot \nabla v \,\mathrm{d}x, \tag{5.3}$$

which is clearly continuous and coercive in $H_0^1(\omega, D)$. Then, a weak formulation of problem (5.1) reads: find $u \in H_0^1(\omega, D)$ such that

$$a(u,v) = \int_D f v \,\mathrm{d}x, \quad \forall v \in H_0^1(\omega, D). \tag{5.4}$$

5.2.1 Finite element approximation on quasi-uniform meshes

We recall the construction of the underlying finite element spaces given in § 4.3.1. To avoid technical difficulties, we assume D to be a polyhedral domain. Let $\mathscr{T} = \{T\}$ be a mesh of D into elements T (simplices or cubes) such that

$$\bar{D} = \bigcup_{T\in\mathscr{T}} T, \qquad |D| = \sum_{T\in\mathscr{T}} |T|.$$

The partition $\mathscr{T}$ is assumed to be conforming or compatible. We denote by $\mathbb{T}$ the collection of all conforming meshes. We say that $\mathbb{T}$ is *shape regular*, i.e., if there exists a constant $\sigma > 1$ such that, for all $\mathscr{T} \in \mathbb{T}$,

$$\max\{\sigma_T : T \in \mathscr{T}\} \leq \sigma, \tag{5.5}$$

where $\sigma_T := h_T/\rho_T$ is the shape coefficient of T.

We assume that the collection of meshes $\mathbb{T}$ is conforming and satisfies the regularity assumption (5.5), which says that the element shape does not degenerate with refinement. A refinement method generating meshes satisfying the shape regular condition (5.5) will be called *isotropic refinement*. A particular instance of an isotropic refinement is the so called quasi-uniform refinement. We recall that $\mathbb{T}$ is quasi-uniform if it is shape regular and for all $\mathscr{T} \in \mathbb{T}$ we have

$$\max\left\{h_T : T \in \mathscr{T}\right\} \lesssim \min\left\{h_T : T \in \mathscr{T}\right\},$$

where the hidden constant is independent of $\mathscr{T}$. In this case, all the elements on the same refinement level are of comparable size. We define $h_{\mathscr{T}} = \max_{T\in\mathscr{T}} h_T$.

Given a mesh $\mathscr{T} \in \mathbb{T}$, we define the finite element space of continuous piecewise polynomials of degree one

$$\mathbb{V}(\mathscr{T}) = \left\{W \in \mathcal{C}^0(\bar{D}) : \ W|_T \in \mathcal{P}(T) \ \forall T \in \mathscr{T}, \ W|_{\partial\Omega} = 0\right\}, \tag{5.6}$$

where for a simplicial element T, $\mathcal{P}(T)$ corresponds to the space of polynomials of total degree at most one, i.e., $\mathbb{P}_1(T)$, and for n-rectangles, $\mathcal{P}(T)$ stands for the space of polynomials of degree at most one in each variable, i.e., $\mathbb{Q}_1(T)$.

The finite element approximation of u, solution of problem (5.1), is defined as the unique discrete function $U_{\mathscr{T}} \in \mathbb{V}(\mathscr{T})$ satisfying

$$a(U_{\mathscr{T}}, W) = \int_D fW, \quad \forall W \in \mathbb{V}(\mathscr{T}). \tag{5.7}$$

5.2.2 Quasi-interpolation operator

Let us recall the main properties of the quasi-interpolation operator $\Pi_{\mathscr{T}}$ introduced and analyzed in Chapter 4 (see also [132]). This operator is based on local averages over stars, and then it is well defined for functions in $L^p(\omega, D)$. We summarize its construction and its approximation properties as follows.

Given a mesh $\mathscr{T} \in \mathbb{T}$ and $T \in \mathscr{T}$, we denote by $\mathcal{N}(T)$ the set of nodes of T. We set $\mathcal{N}(\mathscr{T}) := \cup_{T\in\mathscr{T}}\mathcal{N}(T)$ and $\mathring{\mathcal{N}}(\mathscr{T}) := \mathcal{N}(\mathscr{T}) \cap D$. Then, any discrete function

$W \in \mathbb{V}(\mathscr{T})$ is characterized by its nodal values on the set $\mathring{\mathcal{N}}(\mathscr{T})$. Moreover, the functions $\phi_{\mathtt{v}} \in \mathbb{V}(\mathscr{T})$, $\mathtt{v} \in \mathring{\mathcal{N}}(\mathscr{T})$, such that $\phi_{\mathtt{v}}(\mathtt{w}) = \delta_{\mathtt{vw}}$ for all $\mathtt{w} \in \mathcal{N}(\mathscr{T})$ are the canonical basis of $\mathbb{V}(\mathscr{T})$, and

$$W = \sum_{\mathtt{v} \in \mathring{\mathcal{N}}(\mathscr{T})} W(\mathtt{v})\phi_{\mathtt{v}}.$$

Given a vertex $\mathtt{v} \in \mathcal{N}(\mathscr{T})$, we define the star or patch around $\mathtt{v}$ as $S_{\mathtt{v}} = \cup_{T \ni \mathtt{v}} T$, and for $T \in \mathscr{T}$ we define its patch as $S_T = \cup_{\mathtt{v} \in T} S_{\mathtt{v}}$. For each vertex $\mathtt{v} \in \mathcal{N}(\mathscr{T})$, we define $h_{\mathtt{v}} = \min\{h_T : \mathtt{v} \in T\}$.

Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\int \psi = 1$ and $\operatorname{supp} \psi \subset B$, where B denotes the ball in $\mathbb{R}^N$ of radius r centered at zero with $r \leq 1/\sigma$, with σ defined by (5.5). For $\mathtt{v} \in \mathring{\mathcal{N}}(\mathscr{T})$, we define the rescaled smooth function

$$\psi_{\mathtt{v}}(x) = \frac{1}{h_{\mathtt{v}}^N}\psi\left(\frac{\mathtt{v} - x}{h_{\mathtt{v}}}\right).$$

Given a smooth function v, we denote by $P^1 v(x, z)$ the Taylor polynomial of degree one of the function v in the variable z about the point x, i.e.,

$$P^1 v(x, z) = v(x) + \nabla v(x) \cdot (z - x).$$

Then, given $\mathtt{v} \in \mathring{\mathcal{N}}(\mathscr{T})$ and a function $v \in W^1_p(\omega, D)$, we define the corresponding averaged Taylor polynomial of first degree of v about the vertex $\mathtt{v}$ as

$$Q^1_{\mathtt{v}} v(z) = \int P^1 v(x, z)\psi_{\mathtt{v}}(x)\,\mathrm{d}x. \tag{5.8}$$

Since $\operatorname{supp} \psi_{\mathtt{v}} \subset S_{\mathtt{v}}$, the integral appearing in (5.8) can be written over $S_{\mathtt{v}}$. Moreover, integration by parts shows that $Q^1_{\mathtt{v}}$ is well defined for functions in $L^1(D)$; see [27, Proposition 4.1.12]. Consequently, Proposition 2.3 implies that $Q^1_{\mathtt{v}}$ is also well defined for functions in $L^p(\omega, D)$ with $\omega \in A_p(\mathbb{R}^N)$.

Given $\omega \in A_p(\mathbb{R}^N)$ and $v \in L^p(\omega, D)$, we define the quasi-interpolant $\Pi_{\mathscr{T}} v$ as the unique function $\Pi_{\mathscr{T}} v \in \mathbb{V}(\mathscr{T})$ that satisfies $\Pi_{\mathscr{T}} v(\mathtt{v}) = Q_{\mathtt{v}}^1(\mathtt{v})$ if $\mathtt{v} \in \mathcal{N}(\mathscr{T})$, and $\Pi_{\mathscr{T}} v(\mathtt{v}) = 0$ if $\mathtt{v} \in \mathcal{N}(\mathscr{T}) \cap \partial\Omega$, i.e.,

$$\Pi_{\mathscr{T}} v(\mathtt{v}) = \sum_{\mathtt{v} \in \mathring{\mathcal{N}}(\mathscr{T})} Q_{\mathtt{v}}^1(\mathtt{v}) \phi_{\mathtt{v}}.$$

For this operator, Chapter 4 (see also [132, Section 5]) proves stability and interpolation error estimates in the weighted L^p-norm and W_p^1-seminorm. We recall these results for completeness.

Proposition 5.1 (weighted stability and local error estimate I) *Let $T \in \mathscr{T}$, $\omega \in A_p(\mathbb{R}^N)$ and $v \in L^p(\omega, S_T)$. Then, we have the following local stability bound*

$$\|\Pi_{\mathscr{T}} v\|_{L^p(\omega,T)} \lesssim \|v\|_{L^p(\omega,S_T)}. \tag{5.9}$$

If, in addition, $v \in W_p^1(\omega, S_T)$, then we have the local interpolation error estimate

$$\|v - \Pi_{\mathscr{T}} v\|_{L^p(\omega,T)} \lesssim h_{\mathtt{v}} \|\nabla v\|_{L^p(\omega,S_T)}. \tag{5.10}$$

The hidden constants in both inequalities depend only on $C_{p,\omega}$, ψ and σ.

Proposition 5.2 (weighted stability and local error estimate II) *Let $T \in \mathscr{T}$, $\omega \in A_p(\mathbb{R}^N)$ and $v \in W_p^1(\omega, S_T)$. Then, we have the following local stability bound*

$$\|\nabla \Pi_{\mathscr{T}} v\|_{L^p(\omega,T)} \lesssim \|\nabla v\|_{L^p(\omega,S_T)}. \tag{5.11}$$

If, in addition, $v \in W_p^2(\omega, S_T)$, then

$$\|\nabla(v - \Pi_{\mathscr{T}} v)\|_{L^p(\omega,T)} \lesssim h_{\mathtt{v}} \|D^2 v\|_{L^p(\omega,S_T)}. \tag{5.12}$$

The hidden constants in both inequalities depend only on $C_{p,\omega}$, ψ and σ.

5.2.3 Finite element approximation on anisotropic meshes

Let us now focus our attention on the finite element discretization of problem (1.2). Estimates (3.27)-(3.28) motivate the construction of a mesh over $\mathcal{C}_\mathcal{Y}$ with cells of the form $T = K \times I$, where $K \subset \mathbb{R}^n$ is an element that is isoparametrically equivalent either to the unit cube $[0,1]^n$ or the unit simplex in $\mathbb{R}^n$ and $I \subset \mathbb{R}$ is an interval. To be precise, let $\mathscr{T}_\Omega = \{T\}$ be a conforming and shape regular mesh of Ω. In order to obtain a global regularity assumption for $\mathscr{T}_\mathcal{Y}$, we assume that there is a constant $\sigma_\mathcal{Y}$ such that if $T_1 = K_1 \times I_1$ and $T_2 = K_2 \times I_2 \in \mathscr{T}_\mathcal{Y}$ have nonempty intersection, then

$$\frac{h_{I_1}}{h_{I_2}} \leq \sigma_\mathcal{Y}, \tag{5.13}$$

where $h_I = |I|$. Exploiting the Cartesian structure of the mesh it is possible to handle anisotropy in the extended variable and obtain estimates of the form

$$\|v - \Pi_{\mathscr{T}_\mathcal{Y}} v\|_{L^2(y^\alpha,T)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} v\|_{L^2(y^\alpha,S_T)} + h_{\mathtt{v}''} \|\partial_y v\|_{L^2(y^\alpha,S_T)},$$

$$\|\partial_{x_j}(v - \Pi_{\mathscr{T}_\mathcal{Y}} v)\|_{L^2(y^\alpha,T)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} \partial_{x_j} v\|_{L^2(y^\alpha,S_T)} + h_{\mathtt{v}''} \|\partial_y \partial_{x_j} v\|_{L^2(y^\alpha,S_T)},$$

with $j = 1, \ldots, n+1$, where $h_{\mathtt{v}'} = \min\{h_K : \mathtt{v}' \in K\}$, $h_{\mathtt{v}''} = \min\{h_I : \mathtt{v}'' \in I\}$ and v is the solution of problem (3.37); see § 3.3.2.3 and § 3.3.2.4 for details. However, since $\mathscr{U}_{yy} \approx y^{-\alpha-1}$ as $y \approx 0$, we realize that $\mathscr{U} \notin H^2(y^\alpha, \mathcal{C})$ and the second estimate is not meaningful for $j = n+1$. In view of the regularity estimate (3.28) it is necessary to measure the regularity of $\mathscr{U}_{yy}$ with a different weight and thus compensate with a graded mesh in the extended dimension. This makes anisotropic estimates essential.

In order to simplify the analysis and implementation of multilevel techniques, we consider a sequence of nested discretizations. We construct such meshes as follows. First, we introduce a sequence of nested uniform partitions of the unit interval $\{\mathcal{T}_k\}$, with mesh points $\widehat{y}_{l,k}$, for $l = 0, \dots, M_k$ and $k = 0, \dots, J$. Then, we obtain a family of meshes of the interval $[0, \mathcal{Y}]$ given by the mesh points

$$y_{l,k} = \mathcal{Y}\widehat{y}_{l,k}^{\gamma}, \quad l = 0, \dots, M_k, \tag{5.14}$$

where $\gamma > 3/(1-\alpha)$. Then, for $k = 0, \dots, J$, we consider a quasi-uniform triangulation $\mathscr{T}_{\Omega,k}$ of the domain Ω and construct the mesh $\mathscr{T}_{\mathcal{Y},k}$ as the tensor product of $\mathscr{T}_{\Omega,k}$ and the partition given in (5.14); hence $\#\mathscr{T}_{\mathcal{Y},k} = M_k \, \#\mathscr{T}_{\Omega,k}$. Assuming that $\#\mathscr{T}_{\Omega,k} \approx M_k^n$ we have $\#\mathscr{T}_{\mathcal{Y},k} \approx M_k^{n+1}$. Finally, since $\mathscr{T}_{\Omega,k}$ is shape regular and quasi-uniform, $h_{\mathscr{T}_{\Omega,k}} \approx (\#\mathscr{T}_{\Omega,k})^{-1/n}$. All these considerations allow us to obtain the following result; see § 3.4.

Theorem 5.1 (error estimate) *Denote by $V_{\mathscr{T}_{\mathcal{Y},k}} \in \mathbb{V}(\mathscr{T}_{\mathcal{Y},k})$ the Galerkin approximation of problem* (3.37) *with first degree tensor product elements. Then,*

$$\|\nabla(\mathscr{U} - V_{\mathscr{T}_{\mathcal{Y},k}})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y},k})|^s (\#\mathscr{T}_{\mathcal{Y},k})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

where $\mathcal{Y} \approx \log(\#\mathscr{T}_{\mathcal{Y},k})$.

We notice that the anisotropic meshes of the cylinder $\mathcal{C}_{\mathcal{Y}}$ considered above are semi-structured by construction. They are generated as the tensor product of an unstructured grid $\mathscr{T}_\Omega$ together with the structured mesh $\mathcal{T}_k$. Figure 5.1 shows an example of this type of meshes in three dimensions.

Notice that the approximation estimates (5.9)-(5.12) are local and thus valid under the weak shape regularity condition (5.13). Owing to the tensor product structure of the mesh, we have the following anisotropic error estimate.

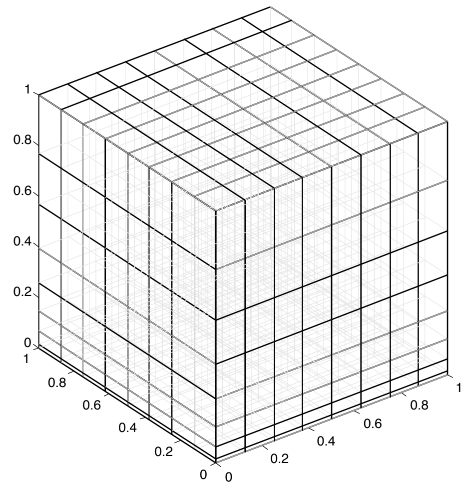

Figure 5.1: A three dimensional graded mesh of the cylinder $(0,1)^2 \times (0,\mathscr{Y})$ with 392 degrees of freedom. The mesh is constructed as a tensor product of a quasi-uniform mesh of $(0,1)^2$ with cardinality 49 and the image of the quasi-uniform partition of the interval $(0,1)$ with cardinality 8 under the mapping (5.14).

Lemma 5.2 (weighted L^2 anisotropic error estimate) *Let $v \in \mathring{H}^1_L(y^\alpha,\mathcal{C}_\mathscr{Y})$ be the solution of problem* (3.37)*. Then, the quasi-interpolation operator $\Pi_{\mathscr{T}_\mathscr{Y}}$ satisfies the following error estimate*

$$\|v - \Pi_{\mathscr{T}_\mathscr{Y}} v\|_{L^2(y^\alpha,\mathcal{C}_\mathscr{Y})} \lesssim \#\mathscr{T}_\mathscr{Y}^{-1/(n+1)} \left(\|\nabla_{x'} v\|_{L^2(y^\alpha,\mathcal{C}_\mathscr{Y})} + \|\partial_y v\|_{L^2(y^\alpha,\mathcal{C}_\mathscr{Y})} \right).$$

Proof: This is a direct consequence of the results from Chapter 4 together with the Cartesian structure of the mesh $\mathscr{T}_\mathscr{Y}$. $\square$

A simple application of the mean value theorem yields

$$y_{l+1,k} - y_{l,k} = \frac{\mathscr{Y}}{M_k^\gamma}\left((l+1)^\gamma - l^\gamma\right) \leq \gamma \frac{\mathscr{Y}}{M_k}\left(\frac{l+1}{M_k}\right)^{\gamma-1} \leq \gamma \frac{\mathscr{Y}}{M_k}, \tag{5.15}$$

for every $l = 0,\dots,M_k - 1$, where $\gamma > 3/(1-\alpha) = 3/(2s)$ according to (5.14). In other words, since the meshsize of the quasi-uniform mesh $\mathscr{T}_{\Omega,k}$ is $\mathcal{O}(M_k^{-1})$, the size of the partitions in the extended variable y can be uniformly controlled by $h_{\mathscr{T}_{\Omega,k}}$ for $k = 0, \ldots, J$. However, γ blows up as $s \downarrow 0$.

5.3 Multilevel space decomposition and multigrid methods

In this section, we present a $\mathcal{V}$-cycle multigrid algorithm based on the method of subspace corrections [32, 161], and we present the key identity of Xu and Zikatanov [163] in order to analyze the convergence of the proposed multigrid algorithm.

5.3.1 Multilevel decomposition

We follow [30, 31] to present a multilevel decomposition of the space $\mathbb{V}(\mathscr{T})$. Assume that we have an initial conforming mesh $\mathscr{T}_0$ made of simplices or cubes, and a nested sequence of discretizations $\{\mathscr{T}_k\}_{k=0}^J$ where, for $k > 0$, $\mathscr{T}_k$ is obtained by uniform refinement of $\mathscr{T}_{k-1}$. We then obtain a nested sequence, in the sense of trees, of quasi-uniform meshes

$$\mathscr{T}_0 \leq \mathscr{T}_1 \leq \cdots \leq \mathscr{T}_J = \mathscr{T}.$$

Denoting by $h_k := h_{\mathscr{T}_k}$ the meshsize of the mesh $\mathscr{T}_k$, we have that $h_k \eqsim \rho^k$ for some $\rho \in (0, 1)$, and then $J \eqsim |\log h_J|$. Let $\mathbb{V}_k := \mathbb{V}(\mathscr{T}_k)$ denote the corresponding finite element space over $\mathscr{T}_k$ defined by (5.6). We thus get a sequence of nested spaces

$$\mathbb{V}_0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_J = \mathbb{V},$$

and a macro space decomposition

$$\mathbb{V} = \sum_{k=0}^{J} \mathbb{V}_k.$$

Note the redundant overlapping of the multilevel decomposition above; in particular, the sum is not direct. We now introduce a space micro-decomposition. We start by defining $\mathcal{N}_k := \mathcal{N}(\mathscr{T}_k) = \dim \mathbb{V}_k$, i.e., the number of interior vertices of the mesh $\mathscr{T}_k$. In order to deal with point and line Gauss-Seidel smoothers, we introduce the following sets of indices: For $j = 1, \ldots, \mathcal{M}_k$ we denote by $\mathcal{I}_{k,j}$ a subset of the index set $\{1, 2, \ldots, \mathcal{N}_k\}$, and assume $\mathcal{I}_{k,j}$ satisfies

$$\bigcup_{j=1}^{\mathcal{M}_k} \mathcal{I}_{k,j} = \{1, 2, \ldots, \mathcal{N}_k\}.$$

The sets $\mathcal{I}_{k,j}$ may overlap, i.e., given $0 < j_1, j_2 \leq \mathcal{M}_k$ such that $j_1 \neq j_2$, we may have $\mathcal{I}_{k,j_1} \cap \mathcal{I}_{k,j_2} \neq \emptyset$. This overlap, however, is finite and independent of J and $\mathcal{N}_k$.

Upon denoting the standard nodal basis of $\mathbb{V}_k$ by $\phi_{k,i}$, $i = 1, \ldots, \mathcal{N}_k$, we define $\mathbb{V}_{k,j} = \text{span}\{\phi_{k,i} : i \in \mathcal{I}_{k,j}\}$ and we have the space decomposition

$$\mathbb{V} = \sum_{k=0}^{J} \sum_{j=1}^{\mathcal{M}_k} \mathbb{V}_{k,j}. \tag{5.16}$$

5.3.2 Multigrid algorithm

We now describe the multigrid algorithm for the non-uniformly elliptic problem (5.1). We start by introducing several auxiliary operators. For $k = 0, \ldots, J$, we define the operator $A_k : \mathbb{V}_k \to \mathbb{V}_k$ by

$$(A_k v_k, w_k)_{L^2(\omega,D)} = a(v_k, w_k), \quad \forall v_k, w_k \in \mathbb{V}_k,$$

where the bilinear form a is defined in (5.3). Notice that this operator is symmetric and positive definite with respect to the weighted L^2-inner product. The projection operator $P_k : \mathbb{V}_J \to \mathbb{V}_k$ in the a-inner product is defined by

$$a(P_k v, w_k) = a(v, w_k), \quad \forall w_k \in \mathbb{V}_k,$$

and the weighted L^2-projection $Q_k : \mathbb{V}_J \to \mathbb{V}_k$ is defined by

$$(Q_k v, w_k)_{L^2(\omega,D)} = (v, w_k)_{L^2(\omega,D)}, \quad w_k \in \mathbb{V}_k.$$

We define, analogously, the operators $A_{k,j} : \mathbb{V}_{k,j} \to \mathbb{V}_{k,j}$, $P_{k,j} : \mathbb{V}_k \to \mathbb{V}_{k,j}$ and $Q_{k,j} : \mathbb{V}_k \to \mathbb{V}_{k,j}$. The operator $A_{k,j}$ can be regarded as the restriction of A_k to the subspace $\mathbb{V}_{k,j}$, and its matrix representation, which is the sub-matrix of A_k obtained by deleting the indices $i \notin \mathcal{I}_{k,j}$, is symmetric and positive definite. On the other hand, the operators $P_{k,j}$ and $Q_{k,j}$ denote the projections with respect to the a- and the weighted L^2-inner products into $\mathbb{V}_{k,j}$, respectively. We also remark that the matrix representation of the operator $Q_{k,j}$ is the so-called restriction operator, and the prolongation operator $Q_{k,j}^T$ corresponds to the natural embedding $\mathbb{V}_{k,j} \hookrightarrow \mathbb{V}_k$. The following property, which is of fundamental importance, will be used frequently in the paper

$$A_{k,j} P_{k,j} = Q_{k,j} A_k. \tag{5.17}$$

With this notation we define a symmetric $\mathcal{V}$-cycle multigrid method as in Algorithm 1. When $m = 1$, it is equivalent to the application of successive subspace corrections (SSC) to the decomposition (5.16) with exact sub-solvers $A_{k,j}^{-1}$ so that the $\mathcal{V}$-cycle multigrid method has a smoother at each level of block Gauss-Seidel type [31, 161]. In particular, if we consider a nodal decomposition $\mathcal{I}_{k,j} = \{j\}$ we obtain a point-wise Gauss-Seidel smoother. On the other hand, if the indices in $\mathcal{I}_{k,j}$

are such that the corresponding vertices lie on a straight line, we obtain the so-called line Gauss-Seidel smoother, which will be essential to efficiently solve problem (1.2) with anisotropic elements.

5.3.3 Analysis of the multigrid method

In order to prove the nearly uniform convergence of the symmetric $\mathcal{V}$-cycle multigrid method without any assumptions, we rely on the following fundamental identity developed by Xu and Zikatanov [163]; see also [51, 54] for alternative proofs.

Theorem 5.3 (XZ Identity) *Let $\mathbb{V}$ be a Hilbert space with inner product $(\cdot,\cdot)_A$ and norm $\|\cdot\|_A$. Let $\mathbb{V}_j \subset \mathbb{V}$ be a closed subspace of $\mathbb{V}$ for $j = 0, \ldots, J$, satisfying*

$$\mathbb{V} = \sum_{j=0}^{J} \mathbb{V}_j.$$

Denote by $P_j : \mathbb{V} \to \mathbb{V}_j$ the orthogonal projection in the inner product $(\cdot,\cdot)_A$ onto $\mathbb{V}_j$. Then, the following identity holds

$$\left\| \prod_{j=0}^{J} (I - P_j) \right\|_A^2 = 1 - \frac{1}{1 + c_0},$$

where

$$c_0 = \sup_{\|\nu\|_A = 1} \inf_{\sum_{i=0}^{J} \nu_i = \nu} \sum_{i=0}^{J} \left\| P_i \sum_{j=i+1}^{J} \nu_j \right\|_A^2. \tag{5.18}$$

The XZ identity given by Theorem 5.3, the properties of the interpolation operator $\Pi_{\mathscr{T}}$ defined in §5.2.2, the stability of the nodal decomposition stated in Lemma 5.4 below, and the weighted inverse inequality proved in Lemma 5.5 below, will allow us to obtain the nearly uniform convergence of the symmetric $\mathcal{V}$-cycle

Algorithm 1: Symmetric $\mathcal{V}$-cycle multigrid method

$e = \mathsf{MG}(r, k, m)$

input : $r \in \mathbb{V}_k$ — residual;

$k \in \{0, \dots J\}$ — level;

$m \in \mathbb{N}$ — number of smoothing steps.

output : $e \in \mathbb{V}_k$ — an approximate solution of $A_k e = r$.

if $k = 0$ **then**

  $e = A_0^{-1} r$;

`// pre-smoothing:` m `steps`

$u^0 = 0$;

for $l \leftarrow 1$ **to** m **do**

  $v \leftarrow u^{l-1}$;

  for $j \leftarrow 1$ **to** $\mathcal{M}_k$ **do**

    $v \leftarrow v + A_{k,j}^{-1} Q_{k,j}(r - A_k v)$;

  $u^l \leftarrow v$;

`// coarse grid correction`

$u^{m+1} = u^m + \mathsf{MG}\left(Q_{k-1}(r - A_k u^m), k-1, m\right)$;

`// post-smoothing:` m `steps`

for $l \leftarrow m+2$ **to** $2m+1$ **do**

  $v \leftarrow u^{l-1}$;

  for $j \leftarrow \mathcal{M}_k$ **to** 1 **do**

    $v \leftarrow v + A_{k,j}^{-1} Q_{k,j}(r - A_k v)$;

  $u^l \leftarrow v$;

`// output`

$e = u^{2m+1}$;

multigrid method described in Algorithm 1, without resorting to any regularity assumptions on the solution. To see how this is possible we recall the basic ingredients in the analysis of multilevel methods; see [30, 31, 33, 161] for details. We introduce, for $k = 1, 2, \ldots, J$, the operator

$$\begin{aligned} K_k &= (I - A_{k,\mathcal{M}_k}^{-1} Q_{k,\mathcal{M}_k} A_k) \cdots (I - A_{k,1}^{-1} Q_{k,1} A_k) \\ &= (I - P_{k,\mathcal{M}_k}) \cdots (I - P_{k,1}) = \prod_{j=1}^{\mathcal{M}_k} (I - P_{k,j}), \end{aligned}$$

where we used (5.17) to obtain the second equality. With this notation, Algorithm 1 can then be recast as a two-layer iterative scheme for the solution of $A_J u = f$ of the form

$$u^{\ell+1} = u^{\ell} + B_J(f - A_J u^{\ell}),$$

where the iterator B_J satisfies

$$I - B_J A_J = (K_J^m)^* \cdots (K_1^m)^* (I - P_0) K_1^m \cdots K_J^m,$$

with M^* denoting the adjoint operator of M with respect to the a–inner product. Notice that $I - B_J A_J$ is the so-called error transfer operator i.e.,

$$u - u^{\ell+1} = (I - B_J A_J) \left(u - u^{\ell} \right).$$

Consequently, to show convergence of our scheme we must show that this operator is a contraction with a contraction factor, ideally, independent of J. Owing to the fact that

$$\|K_k^m\|_A \leq \|K_k\|_A^m \leq \|K_k\|_A,$$

it suffices to consider the case $m = 1$, where, given an operator S, we denote by $\|S\|_A$ the operator norm induced by the bilinear form a. Therefore

$$\|I - B_J A_J\|_A \leq \left\| \prod_{k=0}^{J} \prod_{j=1}^{\mathcal{M}_k} (I - P_{k,j}) \right\|_A^2, \tag{5.19}$$

because P_0 is an exact solve and thus a projection, whence $(I - P_0)^2 = I - P_0$. Notice that the right hand side of (5.19) is precisely the quantity that the XZ identity provides a value for. In conclusion, based on Theorem 5.3, to prove the convergence of the symmetric $\mathcal{V}$-cycle multigrid method described in Algorithm 1, we must obtain an estimate for the constant c_0 given by (5.18), which will be the content of the next two sections.

5.4 Analysis of multigrid methods on quasi-uniform grids

In this section we consider the $\mathcal{V}$-cycle multigrid method described in Algorithm 1 applied to solve the weighted discrete problem (5.7) on quasi-uniform meshes. We consider standard pointwise Gauss-Seidel smoothers and prove the convergence of Algorithm 1 with a nearly optimal rate up to a factor $J \approx |\log h_J|$. Our main contribution is the extension of the standard multigrid analysis [36, 37, 94, 161] to include weights belonging to the Muckenhoput class $A_2(\mathbb{R}^N)$. An optimal result for weights in the $A_1(\mathbb{R}^N)$-class is derived in [90]. Nevertheless, since our main motivation is the fractional Laplacian, and the weight $y^\alpha \in A_2(\mathbb{R}^N) \setminus A_1(\mathbb{R}^N)$, we need to consider the larger class $A_2(\mathbb{R}^N)$.

5.4.1 Stability of the nodal decomposition in the weighted L^2-norm

The following result states that the nodal decomposition is stable in the weighted L^2-norm or, equivalently, the mass matrix for this inner product is spectrally equivalent to its diagonal.

Lemma 5.4 (stability of the nodal decomposition) *Let $\mathscr{T} \in \mathbb{T}$ be a quasi-uniform mesh, and let $v \in \mathbb{V}(\mathscr{T})$. Then, we have the following norm equivalence*

$$\sum_{i=1}^{\mathcal{N}(\mathscr{T})} \|v_i\|_{L^2(\omega,D)}^2 \lesssim \|v\|_{L^2(\omega,D)}^2 \lesssim \sum_{i=1}^{\mathcal{N}(\mathscr{T})} \|v_i\|_{L^2(\omega,D)}^2, \tag{5.20}$$

where $v = \sum_{i=1}^{\mathcal{N}(\mathscr{T})} v_i$ denotes the nodal decomposition for v, and the hidden constants in each inequality above only depend on the dimension and the A_2-constant of the weight ω.

Proof: Let $\widehat{T} \subset \mathbb{R}^N$ be a reference element and $\{\widehat{\phi}_1, \ldots, \widehat{\phi}_{\mathcal{N}_{\widehat{T}}}\}$ be its local shape functions, where $\mathcal{N}_{\widehat{T}}$ is the number of vertices of $\widehat{T}$. A standard argument shows

$$\widehat{c}_1 \left(\int_{\hat{T}} \widehat{\omega} \right) \sum_{i=1}^{\mathcal{N}_{\widehat{T}}} \widehat{V}_i^2 \leq \|\widehat{v}\|_{L^2(\widehat{\omega},\widehat{T})}^2 \leq \widehat{c}_2 \left(\int_{\hat{T}} \widehat{\omega} \right) \sum_{i=1}^{\mathcal{N}_{\widehat{T}}} \widehat{V}_i^2,$$

where $0 < \widehat{c}_1 \leq \widehat{c}_2$, $\widehat{v} = \sum_{i=1}^{\mathcal{N}_{\widehat{T}}} \widehat{V}_i \widehat{\phi}_i$ and $\widehat{\omega}$ is a weight; see [75, Lemma 9.7]. Now, given $T \in \mathscr{T}$, we denote by $F_T : \widehat{T} \to T$ the mapping such that $\hat{v} = v \circ F_T$. Since the A_2 class is invariant under isotropic dilations Proposition 2.1, a scaling argument shows

$$\left(\int_{T} \omega \right) \sum_{i=1}^{\mathcal{N}_T} V_i^2 \lesssim \|v\|_{L^2(\omega,T)}^2 \lesssim \left(\int_{T} \omega \right) \sum_{i=1}^{\mathcal{N}_T} V_i^2.$$

It remains thus to show that $\int_T \omega \approx \int_T \omega\phi_i^2$. The fact that $0 \le \phi_i \le 1$ yields immediately

$$\int_T \omega\phi_i^2 \le \int_T \omega.$$

The converse inequality follows from the *strong doubling property* of ω given in Proposition 2.2. In fact, setting $E = \{x \in T : \phi_i^2 \ge \frac{1}{2}\} \subset T$, we have

$$\int_T \omega\phi_i^2 \ge \int_E \omega\phi_i^2 \ge \frac{1}{2}\int_E \omega \ge \frac{1}{2C_{2,\omega}}\left(\frac{|E|}{|T|}\right)^2 \int_T \omega.$$

Finally, notice that the supports of the nodal basis functions $\{\phi_i\}_{i=1}^{\mathcal{N}(\mathscr{T})}$ have a finite overlap which is independent of the refinement level, i.e., for every $i = 1, \dots, \mathcal{N}(\mathscr{T})$, the number $n(i) = \#\{j : \operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j \ne \emptyset\}$ is uniformly bounded. We arrive at (5.20) summing over all the elements $T \in \mathscr{T}$. $\square$

With the aid of the stability of the nodal decomposition, we now show a weighted inverse inequality.

Lemma 5.5 (weighted inverse inequality) *Let $\mathscr{T} \in \mathbb{T}$ be a quasi-uniform mesh, and let $T \in \mathscr{T}$ and $v \in \mathbb{V}(\mathscr{T})$. Then, we have the following inverse inequality*

$$\|\nabla v\|_{L^2(\omega,T)} \lesssim h_{\mathscr{T}}^{-1}\|v\|_{L^2(\omega,T)}. \tag{5.21}$$

Proof: Since $\mathscr{T}$ is quasi-uniform with meshsize $h_{\mathscr{T}}$, we have $|\nabla\phi_i| \lesssim h_{\mathscr{T}}^{-1}$, and

$$\int_T \omega|\nabla v|^2 \lesssim h_{\mathscr{T}}^{-2}\sum_{i=1}^{\mathcal{N}_T} V_i^2 \int_T \omega,$$

where, we have used the nodal decomposition of $v = \sum_{i=1}^{\mathcal{N}_T} V_i\phi_i$. As in the proof of Lemma 5.4, the strong doubling property of ω yields

$$\int_T \omega \lesssim C_{2,\omega}\int_T \omega\phi_i^2$$

so that we obtain

$$\int_T \omega|\nabla v|^2 \lesssim C_{2,\omega} h_{\mathscr{T}}^{-2} \sum_{i=1}^{\mathcal{N}} V_i^2 \int_T \omega\phi_i^2 \lesssim C_{2,\omega} h_{\mathscr{T}}^{-2} \int_T \omega v^2,$$

where, in the last step, we have used (5.20). This concludes the proof. □

5.4.2 Convergence analysis

We now present a convergence analysis of Algorithm 1 applied to solve the weighted discrete problem (5.7) over quasi-uniform meshes and with standard pointwise Gauss-Seidel smoothers i.e., $\mathcal{M}_k = \mathcal{N}_k$ and $\mathcal{I}_{k,j} = \{j\}$ for $j = 1, \dots \mathcal{N}_k$. The main ingredients in such analysis are the stability of the nodal decomposition obtained in Lemma 5.4, the weighted inverse inequality of Lemma 5.5, and the properties of the quasi-interpolant introduced in section 5.2. We follow [160, 162].

Theorem 5.6 (convergence of symmetric $\mathcal{V}$-cycle multigrid) *Algorithm 1 with point-wise Gauss-Seidel smoother is convergent with a contraction rate*

$$\delta \leq 1 - \frac{1}{1 + CJ},$$

where C is independent of the meshsize, and it depends on the weight ω only through the constant $C_{2,\omega}$ defined in (2.10).

Proof: By the XZ identity stated in Theorem 5.3, we only need to estimate

$$c_0 = \sup_{\|v\|_{H_0^1(\omega,D)}=1} \inf_{\sum_{k=0}^{J} \sum_{i=1}^{\mathcal{N}_k} v_{k,i}=v} \sum_{k=0}^{J} \sum_{i=1}^{\mathcal{N}_k} \left\| \nabla \left(P_{k,i} \sum_{(l,j)\succ(k,i)} v_{l,j} \right) \right\|_{L^2(\omega,D)}^2, \qquad (5.22)$$

where $\succ$ stands for the so called *lexicographic ordering*, i.e.,

$$(l,j) \succ (k,i) \Leftrightarrow \begin{cases} l > k, \\ l = k \quad \& \quad j > i. \end{cases}$$

We recall that $k = 0, \cdots J$, $j = 1, \ldots, \mathcal{N}_k$ and the operator $P_{k,i} : \mathbb{V}_k \to \mathbb{V}_{k,i}$ is the projection with respect to the bilinear form a. For $k = 0, \cdots J$ we denote by $\Pi_{\mathscr{T}_k}$ the quasi-interpolation operator defined in §5.2.2 over the mesh $\mathscr{T}_k$. Next, we introduce the telescopic multilevel decomposition

$$v = \sum_{k=0}^{J} v_k, \qquad v_k = (\Pi_{\mathscr{T}_k} - \Pi_{\mathscr{T}_{k-1}})v, \qquad \Pi_{\mathscr{T}_{-1}} v := 0, \tag{5.23}$$

along with the nodal decomposition

$$v_k = \sum_{i=1}^{\mathcal{N}_k} v_{k,i},$$

for each level k. Consequently, the right hand side of (5.22) can be rewritten by using the telescopic multilevel decomposition (5.23) as follows:

$$\begin{aligned} V_{k,i} := \sum_{(l,j)\succ(k,i)} v_{l,j} &= \sum_{l=k+1}^{J} \sum_{j=1}^{\mathcal{N}_k} v_{l,j} + \sum_{j=i+1}^{\mathcal{N}_k} v_{k,j} \\ &= \sum_{l=k+1}^{J} v_l + \sum_{j=i+1}^{\mathcal{N}_k} v_{k,j} = v - \Pi_{\mathscr{T}_k} v + \sum_{j=i+1}^{\mathcal{N}_k} v_{k,j}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\nabla P_{k,i} V_{k,i}\|^2_{L^2(\omega,D)} &\lesssim \|\nabla P_{k,i}(v - \Pi_{\mathscr{T}_k} v)\|^2_{L^2(\omega,D)} + \left\| \nabla P_{k,i} \sum_{j=i+1}^{\mathcal{N}_k} v_{k,j} \right\|^2_{L^2(\omega,D)} \\ &\lesssim \|\nabla(v - \Pi_{\mathscr{T}_k} v)\|^2_{L^2(\omega,D_{k,i})} + \sum_{\substack{j=i+1 \\ D_{k,i}\cap D_{k,j}\neq\emptyset}}^{\mathcal{N}_k} \|\nabla v_{k,j}\|^2_{L^2(\omega,D)}, \end{aligned}$$

where $D_{k,i} = \operatorname{supp} \phi_{k,i}$. Adding over $i = 1, \ldots, \mathcal{N}_k$, and using the finite overlapping property of the sets $D_{k,i}$, yields

$$\sum_{i=1}^{\mathcal{N}_k} \sum_{\substack{j=i+1 \\ D_{k,i} \cap D_{k,j} \neq \emptyset}}^{\mathcal{N}_k} \|\nabla v_{k,j}\|^2_{L^2(\omega,D)} \lesssim \sum_{i=1}^{\mathcal{N}_k} \|\nabla v_{k,i}\|^2_{L^2(\omega,D)},$$

whence, the weighted inverse inequality (5.21) gives

$$\sum_{i=1}^{\mathcal{N}_k} \|\nabla P_{k,i} V_{k,i}\|^2_{L^2(\omega,D)} \lesssim \|\nabla(v - \Pi_{\mathscr{T}_k} v)\|^2_{L^2(\omega,D)} + \sum_{i=1}^{\mathcal{N}_k} h_k^{-2} \|v_{k,i}\|^2_{L^2(\omega,D)}.$$

We resort to the stability of the operator $\Pi_{\mathscr{T}_k}$, Proposition 5.2, and the stability of the micro decomposition, Lemma 5.4, to arrive at

$$\sum_{i=1}^{\mathcal{N}_k} \|\nabla P_{k,i} V_{k,i}\|^2_{L^2(\omega,D)} \lesssim \|\nabla v\|^2_{L^2(\omega,D)} + h_k^{-2} \|v_k\|^2_{L^2(\omega,D)}.$$

Since $v_k = (\Pi_{\mathscr{T}_k} - \Pi_{\mathscr{T}_{k-1}})v$, we utilize the approximation properties of $\Pi_{\mathscr{T}_k}$, given in Proposition 3.16, to deduce

$$\|v_k\|_{L^2(\omega,D)} \leq \|v - \Pi_{\mathscr{T}_k} v\|_{L^2(\omega,D)} + \|v - \Pi_{\mathscr{T}_{k-1}} v\|_{L^2(\omega,D)} \lesssim h_k \|\nabla v\|_{L^2(\omega,D)}.$$

This implies $\sum_{i=1}^{\mathcal{N}_k} \|\nabla P_{k,i} V_{k,i}\|^2_{L^2(\omega,D)} \lesssim \|\nabla v\|^2_{L^2(\omega,D)}$, and adding over k from 0 to J yields $c_0 \lesssim J$, which completes the proof. $\square$

5.5 A multigrid method for the fractional Laplace operator

As we explained in § 5.2.3, the regularity estimate (3.28) implies the necessity of graded meshes in the extended variable y. This allows us to recover an almost-optimal error estimate for the finite element approximation of problem (1.2) [129, Theorem 5.4]. In fact, finite elements on quasi-uniform meshes have *poor* approximation properties for small values of the parameter s. The isotropic error estimates of [129, Theorem 5.1] are not optimal, which makes anisotropic estimates essential. For this reason, in this section we develop a multilevel theory for problem (1.2) having in mind anisotropic partitions in the extended variable y and the multilevel setting described in section 5.3 for the nonuniformly elliptic equation (4.1). We shall obtain nearly uniform convergence of a $\mathcal{V}$-cycle multilevel method for the problem (1.2) without any regularity assumptions. We consider line Gauss-Seidel smoothers. The analysis is an adaptation of the results presented in [160] for anisotropic elliptic equations, and it is again based on the XZ identity [163].

5.5.1 A multigrid algorithm with line smoothers

As W. Hackbusch rightfully explains [95]: "*the multigrid method cannot be understood as a fixed algorithm. Usually, the components of the multigrid iteration should be adapted to the given problem, [...] being the smoothing iteration the most delicate part of the multigrid process*".

The success of multigrid methods for uniformly elliptic operators is due to the fact that the smoothers are effective in reducing the nonsmooth (high frequency) components of the error and the coarse grid corrections are effective in reducing the smooth (low frequency) components. However, the effectiveness of both strategies depends crucially on several factors such as the anisotropy of the mesh. A key ingredient in the design and analysis of a multigrid method on anisotropic meshes is the use of the so called line smoothers; see [9, 34, 95, 153].

Intuitively, when solving the α-harmonic extension (1.2) on graded meshes, the approximation from the coarse grid is dominated by the larger meshsize in the x-direction and thus the coarse grid correction cannot capture the smaller scale in the y-direction. One possible solution is the use of semi-coarsening, i.e., coarsening only the y-direction until the meshsizes in both directions are comparable. Another solution is the use of line smoothing, i.e., solving sub-problems restricted to one vertical line. We shall use the latter approach which is relatively easy to implement for tensor-product meshes.

Let us describe the decomposition of $\mathbb{V}_J = \mathbb{V}(\mathscr{T}_{\mathcal{Y}_J})$ that we shall use. To do so, we follow the notation of §5.3.1. We set $\mathcal{M}_k$ to be the number of interior nodes of $\mathscr{T}_{\Omega,k}$ and define, for $j = 1, \dots, \mathcal{M}_k$, the set $\mathcal{I}_{k,j}$ as the collection of indices for the vertices that lie on the line $\{\mathtt{v}'_j\} \times (0, \mathcal{Y})$ at the level k. The decomposition is then given by (5.16). This decomposition is also stable, which allows us to obtain the appropriate anisotropic inverse inequalities; see Lemma 5.7 below.

Owing to the nature of the decomposition, the smoother requires the evaluation of $A_{k,j}^{-1}$ which corresponds to the action of the operator over a vertical line. This can be efficiently realized since the corresponding matrix is tri-diagonal.

Lemma 5.7 (nodal stability and anisotropic inverse inequalities) *Let $\mathscr{T}_{\mathcal{Y}}$ be a graded tensor product grid, which is quasi-uniform in Ω and graded in the extended variable so that* (5.14) *holds. If $v \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ can be decomposed as $v = \sum_{j=1}^{\mathcal{M}_J} v_j$, then*

$$\sum_{j=1}^{\mathcal{M}_J} \|v_j\|^2_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \lesssim \|v\|^2_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \lesssim \sum_{j=1}^{\mathcal{M}_J} \|v_j\|^2_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})}. \tag{5.24}$$

Moreover, we have the following inverse inequalities

$$\|\nabla_{x'} v\|_{L^2(y^\alpha,T)} \lesssim h_K^{-1} \|v\|_{L^2(y^\alpha,T)}, \qquad \|\partial_y v\|_{L^2(y^\alpha,T)} \lesssim h_I^{-1} \|v\|_{L^2(y^\alpha,T)}, \tag{5.25}$$

where $T = K \times I$ is a generic element of $\mathscr{T}_{\mathcal{Y}}$.

Proof: The nodal stability (5.24) follows along the same lines of Lemma 5.4 upon realizing that the functions $v_j = v_j(x', y)$ are defined on the vertical lines $(\mathtt{v}'_j, y)$ with $y \in (0, \mathcal{Y})$ and the index j corresponds to a nodal decomposition in Ω. Moreover, noticing that $|\nabla_{x'} \phi_i| \lesssim h_K^{-1}$ and $|\partial_y \phi_i| \lesssim h_I^{-1}$, we derive (5.25) inspired in Lemma 5.5. $\square$

We examine Algorithm 1 applied to the decomposition (5.16) with exact subsolvers on $\mathbb{V}_{k,j}$, i.e., with line smoothers; see [33, §III.12] and [160]. A key observation in favor of subspaces $\{\mathbb{V}_{k,j}\}_{j=1}^{\mathcal{M}_k}$ follows.

Lemma 5.8 (nodal stability of y-derivatives) *Under the same assumptions of Lemma 5.7 we have*

$$\sum_{j=1}^{\mathcal{M}_J} \|\partial_y v_j\|^2_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim \|\partial_y v\|^2_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim \sum_{j=1}^{\mathcal{M}_J} \|\partial_y v_j\|^2_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})}. \tag{5.26}$$

Proof: We just proceed as in Lemma (5.4) with v replaced by $\partial_y v = \sum_{j=1}^{\mathcal{M}_J} \partial_y v_j$. $\square$

Exploiting Theorem 5.3, the properties of the quasi-interpolation operator $\Pi_{\mathscr{T}_k}$ defined in §5.2.2, and Lemmas 5.7 and 5.8, we obtain the nearly uniform convergence of the symmetric $\mathcal{V}$-cycle multigrid method. We follow [160, 162].

Theorem 5.9 (convergence of multigrid methods with line smoothers) *The symmetric $\mathcal{V}$-cycle multigrid method with line smoothing converges with a contraction rate*

$$\delta \leq 1 - \frac{1}{1 + CJ},$$

where C is independent of the number of degrees of freedom. The constant C depends on the weight y^α only through the constant C_{2,y^α}, and on s like $C \approx \gamma$, where γ is the parameter that defines the graded mesh (5.14).

Proof: We use the XZ identity (5.3) and modify the arguments in the proof of Theorem 5.6. We introduce the telescopic multilevel decomposition

$$v = \sum_{k=0}^{J} v_k, \qquad v_k = (\Pi_{\mathscr{T}_{\mathcal{Y},k}} - \Pi_{\mathscr{T}_{\mathcal{Y},k-1}})v, \qquad \Pi_{\mathscr{T}_{\mathcal{Y},-1}} v := 0, \tag{5.27}$$

along with the line decomposition

$$v_k = \sum_{j=1}^{\mathcal{M}_k} v_{k,j}.$$

Following the same arguments developed in the proof of Theorem 5.6, and denoting $V_{k,i} = \sum_{(l,j)\succ(k,i)} v_{l,j}$, we arrive at the inequality

$$\sum_{i=1}^{\mathcal{M}_k} \|\nabla P_{k,i} V_{k,i}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \|\nabla(v - \Pi_{\mathscr{T}_{\mathcal{Y},k}} v)\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \sum_{j=1}^{\mathcal{M}_k} \|\nabla v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}, \tag{5.28}$$

where we have used the finite overlapping property of the sets $\mathcal{I}_{k,j}$; see § 5.3.1. It remains to estimate both terms in (5.28). The stability of the quasi-interpolant $\Pi_{\mathscr{T}_{\mathcal{Y},k}}$ stated in (5.11) (see also [129, Theorems 4.7 and 4.8] and [132, Lemma 5.1]) yields

$$\|\nabla(v - \Pi_{\mathscr{T}_{\mathcal{Y},k}} v)\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \|\nabla v\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}. \tag{5.29}$$

To estimate the second term in (5.28) we begin by noticing that

$$\sum_{j=1}^{\mathcal{M}_k} \|\nabla v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} = \sum_{j=1}^{\mathcal{M}_k} \|\nabla_{x'} v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \sum_{j=1}^{\mathcal{M}_k} \|\partial_y v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}. \tag{5.30}$$

The first term is estimated via the first weighted inverse inequality (5.25) and the stability of the nodal decomposition (5.24), that is

$$\sum_{j=1}^{\mathcal{M}_k} \|\nabla_{x'} v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \sum_{j=1}^{\mathcal{M}_k} {h'_k}^{-2} \|v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim {h'_k}^{-2} \|v_k\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}, \tag{5.31}$$

where h'_k denotes the meshsize in the x' direction at level k. The approximation property of $\Pi_{\mathscr{T}_\mathcal{Y},k}$ stated in Lemma 5.2 (see also [132, Theorem 5.7]) and the definition of v_k yield

$$\begin{aligned}\|v_k\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} &\leq \|v - \Pi_{\mathscr{T}_\mathcal{Y},k} v\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \|v - \Pi_{\mathscr{T}_\mathcal{Y},k-1} v\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \\ &\lesssim h'_k \|\nabla_{x'} v\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + h''_k \|\partial_y v\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}\end{aligned}$$

where h''_k denotes the *maximal* meshsize in the y direction at level k. Using (5.15) we see that $h''_k \lesssim \gamma h'_k$, and replacing the estimate above in (5.31), we obtain

$$\sum_{j=1}^{\mathcal{M}_k} \|\nabla_{x'} v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \|\nabla v\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}, \tag{5.32}$$

which bounds the first term in (5.30). To estimate the second term, we resort to Lemma 5.8, namely

$$\sum_{j=1}^{\mathcal{M}_k} \|\partial_y v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \|\partial_y v_k\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}. \tag{5.33}$$

Finally, inequalities (5.32) and (5.33) allow us to conclude

$$\sum_{j=1}^{\mathcal{M}_k} \|\nabla v_{k,j}\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \|\nabla v\|^2_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})},$$

which together with (5.29) yields the desired result after summing over k. $\square$

Remark 5.10 (dependence on s) We point out the use of (5.15), which in turn implies $h''_k \lesssim \gamma h'_k$, to derive (5.32). This translates into $C \approx \gamma$ in Theorem 5.9 and, since $\gamma > 3/(1-\alpha) = 3/(2s)$, in deterioration of the contraction factor as $s \downarrow 0$. We explore a remedy in § 5.6.3.

5.6 Numerical Illustrations

In this section, we present numerical experiments to support our theoretical findings. We consider two examples:

$$(5.6.1) \qquad n = 1, \quad \Omega = (0,1), \quad u = \sin(3\pi x),$$

$$(5.6.2) \qquad n = 2, \quad \Omega = (0,1)^2, \quad u = \sin(2\pi x_1)\sin(2\pi x_2),$$

and $\mathscr{Y} = 1$. The length $\mathscr{Y}$ of the cylinder in the extended direction is fixed, as discussed in Chapter 3 (see also[129]), so that it captures the exponential decay of the solution. All of our algorithms are implemented based on the MATLAB© software package *i*FEM [50].

5.6.1 Multigrid with line smoothers on graded meshes

We partition Ω into a uniform grid of size $h_{\mathscr{T}_\Omega}$, and we construct a graded mesh in the extended direction using the mapping (5.14) with parameter $\gamma = \frac{3}{2s} + 0.1$ and $M = \frac{1}{h}$. Some sample meshes are shown in Figure 5.2. The mesh points are ordered column-wise so that the indices associated to vertical lines are easily accessible. Starting from $h_{\mathscr{T}_0} = \frac{1}{4}$ we obtain a sequence of meshes by halving the meshsize of Ω and applying the mapping (5.14) in the extended direction with double number of mesh points.

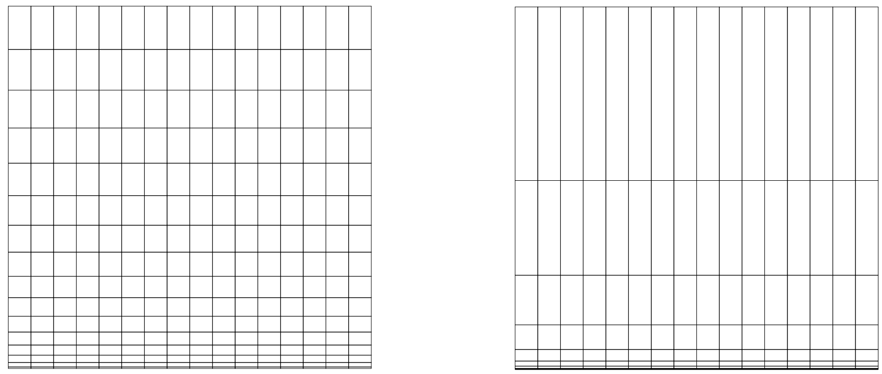

Figure 5.2: Graded meshes for the cylinder $\mathcal{C}_{\mathcal{Y}} = (0,1) \times (0,1)$. In both cases, the mesh in Ω is uniform and of size $h_{\mathscr{T}_0} = \frac{1}{16}$. The meshes in the extended direction are graded according to (5.14) with $M = \frac{1}{h}$ and $\gamma = \frac{3}{2s} + 0.1$. The *left* mesh is for $s = 0.8$ and the *right* for $s = 0.15$.

We assemble the matrix corresponding to the finite element discretization of (5.7) on each level. The natural embedding $\mathbb{V}(\mathscr{T}_k) \to \mathbb{V}(\mathscr{T}_{k+1})$ for $k = 0, \dots, J-1$ gives us the prolongation matrix between two consecutive levels. Notice that the prolongation in the x'-direction is obtained by standard averaging, while in the extended direction the weights must be modified to take into account the grading of the mesh. The restriction matrix is taken as the transpose of the prolongation matrix.

As discussed in section 5.5 we must use vertical line smoothers to attain efficiency of the multigrid method. The tri-diagonal sub-matrix corresponding to one vertical line is inverted exactly by using the built-in direct solver in MATLAB©. Red-black ordering of the indices in the x'-direction is used to further improve the efficiency of the line smoothers. We perform three pre- and post-smoothing steps, i.e., $m = 3$. We start with the zero initial guess and use as exit criterion that the

ℓ^2-norm of the relative residual is smaller than 10^{-7}.

Tables 5.1 and 5.2 show the number of iterations for the implemented multigrid method for the one and two dimensional problems, respectively. As we see, the method converges almost uniformly with respect to the number of degrees of freedom. Notice that the number of iterations for $s = 0.15$ is significantly larger than that for the remaining tested cases. This can be explained by the fact that, as Theorem 5.9 states, the contraction factor depends on $\gamma \approx \frac{1}{s}$ and thus, we observe a preasymptotic regime where the number of iterations grows. This is exactly the case for the one dimensional problem and we would expect a similar behavior in the two dimensional case. However, since the extended problem is now in three dimensions, the size of the problems grows rather quickly and thus our computational resources were not sufficient to deal with the cases $h_{\mathscr{T}_\Omega} = \frac{1}{256}$ and $h_{\mathscr{T}_\Omega} = \frac{1}{512}$. In §5.6.3 we will propose a modification of the graded mesh in the extended direction to address this issue.

We also tested a point Gauss-Seidel smoother for the one dimensional case $\Omega = (0, 1)$. Except for the trivial case $h_{\mathscr{T}_\Omega} = 1/16$, the corresponding $\mathcal{V}$-cycle is not able to achieve the desired accuracy in 200 iterations.

5.6.2 Multigrid methods on quasi-uniform meshes

Even though the approximation of the Caffarelli-Silvestre extension of the fractional Laplace operator on quasi-uniform meshes in the extended direction is suboptimal, let us use this problem to illustrate the convergence properties of the multilevel

$h_{\mathscr{T}_\Omega}$	DOFs	$s = 0.15$	$s = 0.3$	$s = 0.6$	$s = 0.8$
$\frac{1}{16}$	289	7	6	5	5
$\frac{1}{32}$	1,089	13	9	6	6
$\frac{1}{64}$	4,225	25	10	6	6
$\frac{1}{128}$	16,641	33	11	6	6
$\frac{1}{256}$	66,049	37	10	6	6
$\frac{1}{512}$	263,169	38	10	6	7

Table 5.1: Number of iterations for a multigrid method for the one dimensional fractional Laplacian using a line smoother in the extended direction. The mesh in Ω is uniform of size $h_{\mathscr{T}_\Omega}$. The mesh in the extended direction is graded according to (5.14).

$h_{\mathscr{T}_\Omega}$	DOFs	$s = 0.15$	$s = 0.3$	$s = 0.6$	$s = 0.8$
$\frac{1}{16}$	4,913	10	7	6	5
$\frac{1}{32}$	35,937	19	8	6	6
$\frac{1}{64}$	274,625	34	9	6	6
$\frac{1}{128}$	2,146,689	47	9	6	6

Table 5.2: Number of iterations for a multigrid method for the two dimensional fractional Laplacian using a line smoother in the extended direction. The mesh in Ω is uniform of size $h_{\mathscr{T}_\Omega}$. The mesh in the extended direction is graded according to (5.14).

method, developed in section 5.4, for general A_2 weights. The setting is the same as in the previous subsection but we use a point-wise Gauss-Seidel smoother. Tables 5.3 and 5.4 show the number of iterations with respect to the number of degrees of freedom and s. We see that the convergence is almost uniform with respect to the number of unknowns as well as the parameter $s \in (0, 1)$.

$h_{\mathscr{T}_\Omega}$	DOFs	$s = 0.15$	$s = 0.3$	$s = 0.6$	$s = 0.8$
$\frac{1}{16}$	289	12	13	13	14
$\frac{1}{32}$	1,089	15	15	15	17
$\frac{1}{64}$	4,225	15	16	16	17
$\frac{1}{128}$	16,641	15	16	16	18
$\frac{1}{256}$	66,049	15	15	16	18
$\frac{1}{512}$	263,169	15	15	16	18

Table 5.3: Number of iterations for a multigrid method with point-wise Gauss-Seidel smoothers on uniform meshes for the one dimensional fractional Laplacian.

$h_{\mathscr{T}_\Omega}$	DOFs	$s = 0.15$	$s = 0.3$	$s = 0.6$	$s = 0.8$
$\frac{1}{16}$	4,913	13	12	13	15
$\frac{1}{32}$	35,937	15	15	15	17
$\frac{1}{64}$	274,625	15	16	16	18
$\frac{1}{128}$	2,146,689	15	16	16	19

Table 5.4: Number of iterations for a multigrid method with point-wise Gauss-Seidel smoothers on uniform meshes for the two dimensional fractional Laplacian.

5.6.3 Modified mesh grading

Examining the proof of Theorem 5.9, we realize that the critical step (5.32) consists in the application of inequality (5.15), namely $h_k'' \lesssim \gamma h_k'$, which deteriorates as s becomes small because $\gamma > 3/(1-\alpha) = 3/(2s)$. Numerically, this effect can be seen in Tables 5.1 and 5.2 where, for instance, the number of iterations needed for $s = 0.15$ is significantly larger than that for all the other tested values; see the right mesh for $s = 0.15$ in Figure 5.2. As a result, the contraction rate of Theorem 5.9 becomes $1 - 1/(1 + C\gamma J)$. Here we explore computationally how to overcome this issue. We construct a mesh such that the maximum meshsize in the extended direction is uniformly bounded, with respect to s, by the uniform meshsize in the x'-direction without changing the ratio of degrees of freedom in Ω and the extended direction by more than a constant.

Let us begin with some heuristic motivation. In order to control the aspect ratio h_k''/h_k' uniformly on $s \in (0,1)$, we may apply some extra refinements to the largest elements in the y direction, increasing the number of degrees of freedom of $\mathscr{T}_{\mathcal{Y}}$ just by a constant. We denote by $\tilde{\mathscr{T}}_{\mathcal{Y}}$ the resulting mesh and we notice that $\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) \subset \mathbb{V}(\tilde{\mathscr{T}}_{\mathcal{Y}})$. Thus, Galerkin orthogonality implies

$$
\begin{aligned}
\|\nabla(v - V_{\tilde{\mathscr{T}}_{\mathcal{Y}}})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} &= \inf\left\{ \|\nabla(v - W)\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} : W \in \mathbb{V}(\tilde{\mathscr{T}}_{\mathcal{Y}}) \right\} \\
&\leq \|\nabla(v - V_{\mathscr{T}_{\mathcal{Y}}})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \lesssim (\#\mathscr{T}_{\mathcal{Y}})^{-\frac{1}{n+1}} \approx (\#\tilde{\mathscr{T}}_{\mathcal{Y}})^{-\frac{1}{n+1}}.
\end{aligned}
$$

We build on this idea through a modification of the mapping function below.

Let $F : (0,1) \to (0, \mathcal{Y})$ be an increasing and differentiable function such that

$F(0) = 0$ and $F(1) = \mathcal{Y}$. By mapping a uniform grid of $(0, 1)$ via the function F, we can construct a graded mesh with mesh points given by $y_l = F(l/M)$ for $l = 1, ..., M$; for instance, $F(\xi) = \mathcal{Y}\xi^\gamma$ yields (5.14). The mean value theorem implies

$$y_{l+1} - y_l = \frac{F'(c_l)}{M} \leq \frac{1}{M} \max \left\{ |F'(\xi)| : \xi \in \left[\frac{l}{M}, \frac{l+1}{M} \right] \right\},$$

which shows that the map of (5.14) is not uniformly bounded with respect to s.

For this reason, we instead consider the following construction: Let $(\xi_\star, y_\star) \in (0, 1)^2$, which we will call the *transition point*, and define the mapping

$$F(\xi) = \begin{cases} y_\star \mathcal{Y} \left(\dfrac{\xi}{\xi_\star} \right)^\gamma, & 0 < \xi \leq \xi_\star, \\ \mathcal{Y} \left(\dfrac{1 - y_\star}{1 - \xi_\star} (\xi - \xi_\star) + y_\star \right), & \xi_\star < \xi < 1. \end{cases}$$

Over the interval $(0, \xi_\star)$ the mapping F defines the same type of graded mesh but, over $(\xi_\star, 1)$ it defines a uniform mesh. Let us now choose the transition point to obtain a bound on the derivative of F. We have

$$F'(\xi) = \begin{cases} \gamma \mathcal{Y} \dfrac{y_\star}{\xi_\star} \left(\dfrac{\xi}{\xi_\star} \right)^{\gamma-1}, & 0 < \xi \leq \xi_\star, \\ \mathcal{Y} \dfrac{1 - y_\star}{1 - \xi_\star}, & \xi_\star < \xi < 1, \end{cases} \tag{5.34}$$

so that

$$\mathcal{F} := \max_{\xi \in [0,1]} |F'(\xi)| = \mathcal{Y} \max \left\{ \gamma \frac{y_\star}{\xi_\star}, \frac{1 - y_\star}{1 - \xi_\star} \right\}.$$

Given $\xi_\star$ we choose $y_\star$ to have $\gamma \frac{y_\star}{\xi_\star} = \frac{1-y_\star}{1-\xi_\star}$, i.e.,

$$y_\star = \frac{1}{1 + \gamma \frac{1-\xi_\star}{\xi_\star}}.$$

this immediately yields $F \in \mathcal{C}^1([0, 1])$ and, more importantly,

$$\mathcal{F} = \gamma \mathcal{Y} \frac{y_\star}{\xi_\star} = \mathcal{Y} \frac{\gamma}{\xi_\star + (1 - \xi_\star)\gamma} \leq \mathcal{Y} \frac{1}{1 - \xi_\star}.$$

We can now choose $\xi_\star$ to gain control of $\mathcal{F}$. For instance, $\xi_\star = 0.5$ gives us that $\mathcal{F} \leq 2$ and $\xi_\star = 0.75$ that $\mathcal{F} \leq 4$. In the experiments presented below we choose $\xi_\star = 0.75$. The theory presented in § 5.5 still applies.

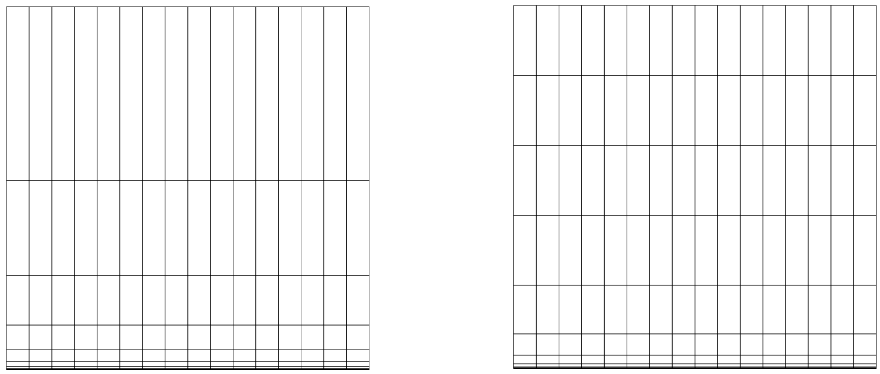

Figure 5.3: Graded meshes for the extended domain $\mathcal{C}_\mathcal{Y} = (0,1) \times (0,1)$, $h_{\mathscr{T}_\Omega} = \frac{1}{16}$ and $s = 0.15$. *Left*: The grading is according to (5.14). *Right*: The grading is given by the map (5.34).

To better visualize the effect of this modification Figures 5.3 and 5.4, show the original graded mesh, defined by (5.14), and the modified one obtained using (5.34), in two and three dimensions, respectively. The modified graded meshes have asymptotically the same distribution of points near the bottom part of the cylinder and so they are also capable of capturing the singular behavior of the solution $\mathscr{U}$. However, near the top part, the aspect ratio is uniformly controlled by a factor 4. The modified mesh is only applied for $\gamma > 4$. For $s = 0.3$, 0.6 and 0.8, no modification is needed in the original mesh.

Upon constructing a mesh with this modification, we can develop a $\mathcal{V}$-cycle multigrid solver with vertical line smoothers. Comparisons of this approach with the setting of § 5.6.1 are shown in Tables 5.5 and 5.6. From them we can conclude that

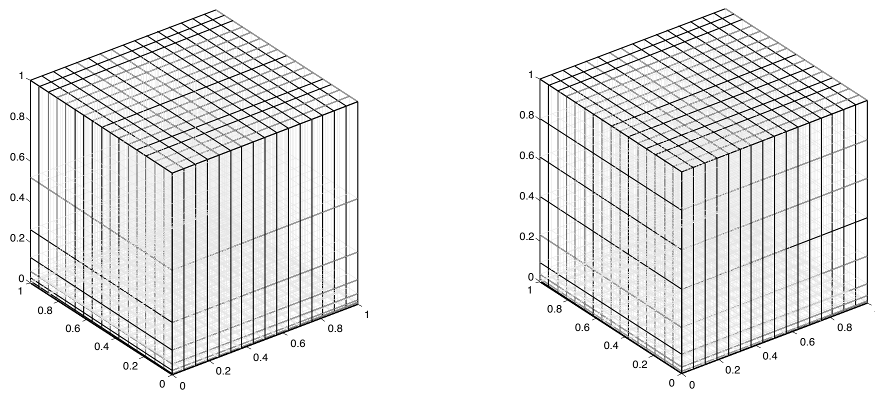

Figure 5.4: Graded meshes for the extended domain $\mathcal{C}_{\mathscr{Y}} = (0,1)^2 \times (0,1)$, $h_{\mathscr{T}_\Omega} = \frac{1}{16}$ and $s = 0.15$. *Left*: The grading is according to (3.106). *Right*: The grading is given by the map (5.34).

the strong anisotropic behavior of the mesh grading (5.14) affects the performance of the $\mathcal{V}$-cycle multigrid with vertical line smoothers. For the original graded meshes, there is a preasymptotic regime where the number of iterations increases faster than $\log J$. The modification of the mesh proposed in (5.34) allows us to obtain an almost uniform number of iterations for all problem sizes without sacrificing the accuracy of the method. This is also evidenced by the computational time required to solve a problem with a fixed number of degrees of freedom.

$h_{\mathscr{T}_\Omega}$	DOFs	I(o)	I(m)	E(o)	E(m)	CPU(o)	CPU(m)
$\frac{1}{16}$	289	7	7	0.1556	0.1739	0.0209	0.0554
$\frac{1}{32}$	1,089	13	9	0.0828	0.0937	0.0664	0.0985
$\frac{1}{64}$	4,225	25	10	0.0426	0.0485	0.2337	0.2720
$\frac{1}{128}$	16,641	33	10	0.0216	0.0246	0.9041	0.4496
$\frac{1}{256}$	66,049	37	11	0.0109	0.0124	4.8168	1.7051
$\frac{1}{512}$	263,169	38	11	0.0055	0.0062	25.1351	7.3439

Table 5.5: Comparison of the multilevel solver with vertical line smoother over two graded meshes for the one dimensional fractional Laplacian, $s = 0.15$. *Legend*: The original mesh, given by (3.106) is denoted by o, whereas the modification proposed in (5.34) is denoted by m. I – number of iterations, E – error in the energy norm, CPU – cpu time (s).

$h_{\mathscr{T}_\Omega}$	DOFs	I(o)	I(m)	E(o)	E(m)	CPU(o)	CPU(m)
$\frac{1}{16}$	4,913	10	8	0.1070	0.1198	0.41	0.31
$\frac{1}{32}$	35,937	19	11	0.0570	0.0646	4.76	2.95
$\frac{1}{64}$	274,625	34	12	0.0294	0.0334	82.56	31.48
$\frac{1}{128}$	2,146,689	47	13	0.0149	0.0170	892.65	269.63

Table 5.6: Comparison of the multilevel solver with vertical line smoother over two graded meshes for the two dimensional fractional Laplacian, $s = 0.15$. *Legend*: The original mesh, given by (3.106) is denoted by o, whereas the modification proposed in (5.34) is denoted by m. I – number of iterations, E – error in the energy norm, CPU – cpu time (s).

Chapter 6: A posteriori error analysis

6.1 Introduction

We are interested in the derivation and the analysis of a computable a posteriori error estimator for problems involving the fractional powers of the Dirichlet Laplace operator $(-\Delta)^s$.

The main advantage of the algorithm described and analyzed in chapter 3, is that we are solving the local problem (1.2) instead of dealing with the nonlocal operator $(-\Delta)^s$ of problem (1.1). However, this comes at the expense of incorporating one more dimension to the problem, thus raising the question of how computationally efficient this approach is. A quest for the answer is the motivation for the study of a posteriori error estimators and adaptivity.

In this work we derive a computable a posteriori error estimator, which is the basic tool to solve problem (1.2) via an adaptive procedure. In view of the overwhelming evidence given in chapter 3 that meshes must be highly anisotropic in the extended dimension y, we design an a posteriori error estimator which is able to deal with such anisotropic behavior. Before proceeding with our analysis, it is instructive to comment about the anisotropic a posteriori error estimator theory advocated in the literature.

A posteriori error estimators are computable quantities, depending on the computed solution and data, that provide information about the quality of approximation. They are problem-dependent and may be used to make judicious mesh refinement. For isotropic discretizations, i.e., meshes where the aspect ratio of the cells is bounded, the theory of a posteriori error estimation is well understood. Starting with the pioneering work [15] of Babuška and Rheinbolt, a great deal of work has been devoted to its study. We refer to [7, 159] for an overview of the state-of the-art. However, the theory of a posteriori error estimation on anisotropic discretizations, i.e., meshes where the cells have disparate sizes in each direction, is still at its infancy.

To the best of the author's knowledge the first work that attempts to deal with anisotropic a posteriori error estimation is [146]. In this work, a residual a posteriori error estimator is introduced and analyzed on anisotropic meshes. However, such analysis relies on assumptions on the exact and discrete solutions and on the mesh, which are neither proved nor there is a way to enforce them in the course of computations; [146, § 6, Remark 3]. Later, in [111] the concept of *matching function* is introduced in order to derive anisotropic a posteriori error indicators. The correct alignment of the grid with the exact solution is crucial to derive an upper bound for the error. Indeed, this upper bound involves the matching function, which depends on the error itself and then it does not provide a *real computable* quantity; see [111, Theorem 2]. For similar works in this direction see [113, 112, 127]. In [137], the anisotropic interpolation estimates derived in [81] are used to derive a Zienkiewicz–Zhu type of a posteriori error estimator. However, as [137, Proposition 2.3] states, the derived upper bound for the error depends on the error itself, and then, it is not *computable*.

In our case, since the coefficient y^α in (1.2) either degenerates for $s < 1/2$ or blows up for $s > 1/2$, the usual residual estimators do not apply; integration by parts fails! In chapter 6, inspired in [14, 125], we deal with the natural anisotropy of the mesh in the extended variable y and the nonuniform coefficient y^α, upon considering local problems on *cylindrical stars*. The solutions of these local problems allow us to define a computable and anisotropic a posteriori error estimator which is equivalent to the error up to oscillations terms. In order to derive such a result, a computationally implementable geometric condition needs to be imposed on the mesh, which does not depend on the exact solution of problem (1.2). This approach is of value not only for (1.2), but in general for anisotropic problems since rigorous anisotropic a posteriori error estimators are not available in the literature.

The outline of this Chapter is as follows. Section 6.2 recalls some elements of the a priori theory developed in Chapter 3, and motivates our a posteriori error estimator. Section 6.3 is dedicated to the study of an ideal error estimator which set the stage for our analysis. The former is not computable but it is equivalent to the error on anisotropic meshes. In §6.4, we introduce and study the anisotropic and computable a posteriori error estimator $\mathscr{E}_{\mathscr{T}_\mathcal{Y}}$, based on local star indicators $\mathscr{E}_{z'}$. We prove the equivalence between the error and the estimator up to oscillation terms on anisotropic meshes.

6.2 Preliminaries

We start recalling some elements of the a priori error analysis developed in chapter 3 (see also [129]).

6.2.1 A priori error analysis

Estimates (3.27)-(3.28) motivate the construction of a mesh over $\mathcal{C}_{\mathcal{Y}}$ as follows. We first consider a graded partition $\mathcal{I}_{\mathcal{Y}}$ of the interval $[0, \mathcal{Y}]$ with mesh points

$$y_k = \left(\frac{k}{M}\right)^{\gamma} \mathcal{Y}, \quad k = 0, \dots, M, \tag{6.1}$$

where $\gamma > 3/(1-\alpha)$. We also consider $\mathscr{T}_{\Omega} = \{K\}$ to be a conforming and shape regular mesh of Ω, where $K \subset \mathbb{R}^n$ is an element that is isoparametrically equivalent either to the unit cube $[0,1]^n$ or the unit simplex in $\mathbb{R}^n$. We construct the mesh $\mathscr{T}_{\mathcal{Y}}$ as the tensor product triangulation of $\mathscr{T}_{\Omega}$ and $\mathcal{I}_{\mathcal{Y}}$. In order to obtain a global regularity assumption for $\mathscr{T}_{\mathcal{Y}}$, we assume that there is a constant $\sigma_{\mathcal{Y}}$ such that if $T_1 = K_1 \times I_1$ and $T_2 = K_2 \times I_2 \in \mathscr{T}_{\mathcal{Y}}$ have nonempty intersection, then

$$\frac{h_{I_1}}{h_{I_2}} \leq \sigma_{\mathcal{Y}}, \tag{6.2}$$

where $h_I = |I|$, which allows for anisotropy in the extended variable (cf. [70, 129]). The set of all such triangulations is denoted by $\mathbb{T}$.

For $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$, we define the finite element space

$$\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in \mathcal{C}^0(\overline{\mathcal{C}_{\mathcal{Y}}}) : W|_T \in \mathcal{P}_1(K) \otimes \mathbb{P}_1(I) \ \forall T \in \mathscr{T}_{\mathcal{Y}}, \ W|_{\Gamma_D} = 0 \right\}. \tag{6.3}$$

where the space $\mathcal{P}_1(K)$ corresponds to $\mathcal{P}_1$ for a simplicial element K, and to $\mathbb{Q}_1$ for a n-rectangle K. We also define $\mathbb{U}(\mathscr{T}_\Omega) = \operatorname{tr}_\Omega \mathbb{V}(\mathscr{T}_\mathcal{Y})$, i.e., a $\mathcal{P}_1$ finite element space over the mesh $\mathscr{T}_\Omega$.

The Galerkin approximation of (3.37) is given by the unique function $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solution of (3.51).

Exploiting the Cartesian structure of the mesh it is possible to handle anisotropy in the extended variable and obtain estimates of the form

$$\|v - \Pi_{\mathscr{T}} v\|_{L^2(y^\alpha,T)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} v\|_{L^2(y^\alpha,S_T)} + h_{\mathtt{v}''} \|\partial_y v\|_{L^2(y^\alpha,S_T)},$$

$$\|\partial_{x_j}(v - \Pi_{\mathscr{T}} v)\|_{L^2(y^\alpha,T)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} \partial_{x_j} v\|_{L^2(y^\alpha,S_T)} + h_{\mathtt{v}''} \|\partial_y \partial_{x_j} v\|_{L^2(y^\alpha,S_T)},$$

with $j = 1, \dots, n+1$, where $h_{\mathtt{v}'} = \min\{h_K : \mathtt{v}' \in K\}$, and $h_{\mathtt{v}''} = \min\{h_I : \mathtt{v}'' \in I\}$; see chapter 3 and [129, Theorem 4.5] for details. However, since $\mathscr{U}_{yy} \approx y^{-\alpha-1}$ as $y \approx 0$, we have that $\mathscr{U} \notin H^2(y^\alpha, \mathcal{C}_\mathcal{Y})$ and the second estimate is not meaningful for $j = n+1$. In view of the regularity estimate (3.28) it is necessary to measure the regularity of $\mathscr{U}_{yy}$ with a stronger weight and thus compensate with a graded mesh in the extended dimension. This makes anisotropic estimates essential.

Finally, notice that $\#\mathscr{T}_\mathcal{Y} = M \, \#\mathscr{T}_\Omega$. Assuming that $\#\mathscr{T}_\Omega \approx M^n$ we have $\#\mathscr{T}_\mathcal{Y} \approx M^{n+1}$. If $\mathscr{T}_\Omega$ is shape regular and quasi-uniform, we have $h_{\mathscr{T}_\Omega} \approx (\#\mathscr{T}_\Omega)^{-1/n}$. All these considerations allow us to obtain the following result; see [129, Theorem 5.4] and [129, Corollary 7.11].

Corollary 6.1 (a priori error estimate) *Let $\mathscr{T}_\mathcal{Y}$ be a graded tensor product grid, which is quasi-uniform in Ω and graded in the extended variable so that* (6.1) *holds. If Ω satisfies*

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta w\|_{L^2(\Omega)}, \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega), \tag{6.4}$$

then the solution $\mathscr{U}$ *of* (3.37) *and its Galerkin approximation* $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ *satisfy*

$$\|\mathscr{U} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_\mathcal{Y})|^s (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

where $\mathcal{Y} \approx \log(\#\mathscr{T}_\mathcal{Y})$*. Alternatively, if* u *denotes the solution of* (1.1)*, then*

$$\|u - U_{\mathscr{T}}\|_{\mathbb{H}^s(\Omega)} \lesssim |\log(\#\mathscr{T}_\mathcal{Y})|^s (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

Remark 6.2 (Domain and data regularity) *The results of Theorem 6.1 are meaningful only if* $f \in \mathbb{H}^{1-s}(\Omega)$ *and the domain* Ω *is such that* (6.4) *holds.*

6.2.2 Motivation

The function $\mathscr{U}$, solution of the α-harmonic extension problem (1.2), has a singular behaviour on the extended variable y, which is compensated by considering anisotropic meshes in such a direction dictated by (6.1). However, the solution $\mathscr{U}$, may also have singularities in the direction of the x'-variables and thus exhibit fractional regularity, which would not allow us to attain the almost optimal rate of convergence given by Corollary 6.1. In fact, as Remark 6.2 shows, in order to have an almost optimal rate of convergence, it is necessary $f \in \mathbb{H}^{1-s}(\Omega)$ and property (6.4). If any of these two conditions is not satisfied, we may have singularities, whose characterization is as yet an open problem; see § 3.5.3 in Chapter 3 and [129, § 6.3] for an illustration of this situation. In this Chapter, we plan to derive a computable a posteriori error estimator for the finite element approximation of problem (3.37), which can resolve such singularities via an adaptive algorithm.

In order to derive such a posteriori error estimator a computationally implementable geometric condition needs to be imposed on the mesh. The latter, allow us to consider graded meshes in Ω in order to compensate some possible singularities in the direction of the x'-variables. We thus assume the following condition over the family of meshes $\mathbb{T}$: there exists a positive constant $C_{\mathbb{T}}$ such that for every mesh $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$

$$h_{\mathcal{Y}} \leq C_{\mathbb{T}}\, h_{z'}, \tag{6.5}$$

for all the interior nodes z' of $\mathscr{T}_{\Omega}$, where $h_{\mathcal{Y}}$ denotes the *maximal* mesh size in the y direction, and $h_{z'} \approx |S_{z'}|^{1/n}$; see below for the precise definition of $h_{z'}$ and $S_{z'}$. We remark that this condition is satisfied in the case of quasi-uniform refinement in the variable x', which is a consequence of the convexity of the function involved in (6.1). In fact, a simple computation shows

$$h_{\mathcal{Y}} = y_M - y_{M-1} = \frac{\mathcal{Y}}{M^{\gamma}}\big((M)^{\gamma} - (M-1)^{\gamma}\big) \leq \gamma \frac{\mathcal{Y}}{M}, \tag{6.6}$$

where $\gamma > 3/(1-\alpha) = 3/(2s)$.

We now consider some terminology and notation. Given a node z on the mesh $\mathscr{T}_{\mathcal{Y}}$, we exploit the tensor product structure of $\mathscr{T}_{\mathcal{Y}}$, and we write $z = (z', z'')$ where z' and z'' are nodes on the meshes $\mathscr{T}_{\Omega}$ and $\mathcal{I}_{\mathcal{Y}}$ respectively.

Given a cell $K \in \mathscr{T}_{\Omega}$, we denote by $\mathcal{N}(K)$ and $\mathring{\mathcal{N}}(K)$ the set of nodes and interior and Neumann nodes of K respectively. We set

$$\mathcal{N}(\mathscr{T}_{\Omega}) := \bigcup_{K \in \mathscr{T}_{\Omega}} \mathcal{N}(K) \qquad \mathring{\mathcal{N}}(\mathscr{T}_{\Omega}) := \bigcup_{K \in \mathscr{T}_{\Omega}} \mathring{\mathcal{N}}(K).$$

Given $T \in \mathscr{T}_{\mathcal{Y}}$, we define $\mathcal{N}(T)$, $\mathring{\mathcal{N}}(T)$ accordingly, and consequently $\mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$ and $\mathcal{N}(\mathscr{T}_{\mathcal{Y}})$. Then, any discrete function $V \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ is characterized by its nodal values on the set $\mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$. Moreover, the functions $\phi_z \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, $z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$, such that

$\phi_z(\mathtt{w}) = \delta_{z\mathtt{w}}$ for all $\mathtt{w} \in \mathcal{N}(\mathscr{T}_{\mathcal{Y}})$ are the canonical basis of $\mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, and

$$V = \sum_{z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})} V(z)\phi_z.$$

The functions $\{\phi_z : z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})\}$ are the so-called *shape functions* of $\mathbb{V}(\mathscr{T}_{\mathcal{Y}})$.

Analogously, given a node $z' \in \mathring{\mathcal{N}}(\mathscr{T}_{\Omega})$, we also consider the discrete functions $\varphi_{z'} \in \mathbb{U}(\mathscr{T}_{\Omega}) = \operatorname{tr}_{\Omega} \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ defined by $\varphi_{z'}(\mathtt{w}') = \delta_{z'\mathtt{w}'}$ for all $\mathtt{w}' \in \mathcal{N}(\mathscr{T}_{\Omega})$. The set $\{\varphi_{z'} : z' \in \mathring{\mathcal{N}}(\mathscr{T}_{\Omega})\}$ is the canonical basis of $\mathbb{U}(\mathscr{T}_{\Omega})$.

We have the following important properties associated with the set of functions $\{\phi_z : z \in \mathcal{N}(\mathscr{T}_{\mathcal{Y}})\}$. First, we have the so-called partition of union property, i.e.,

$$\sum_{z \in \mathcal{N}(\mathscr{T}_{\mathcal{Y}})} \phi_z = 1 \quad \text{in} \quad \mathcal{C}_{\mathcal{Y}}. \tag{6.7}$$

Second, for any interior or Neumann node $z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$, the corresponding basis function $\phi_z \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ and then we have the so called Galerkin orthogonality, i.e.,

$$d_s\langle f, \operatorname{tr}_{\Omega} \phi_z \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^{s}(\Omega)} - \int_{\mathcal{C}_{\mathcal{Y}}} y^{\alpha} \nabla V_{\mathscr{T}_{\mathcal{Y}}} \nabla \phi_z = 0. \tag{6.8}$$

We also have a partition of unity property for the canonical basis $\{\varphi_{z'} : z' \in \mathcal{N}(\mathscr{T}_{\Omega})\}$. Now, given $z' \in \mathcal{N}(\mathscr{T}_{\Omega})$ and the associated basis function $\varphi_{z'}(x')$, we define the extended basis function $\tilde{\varphi}_{z'}(x', y) = \varphi_{z'}(x')\mathbb{1}_{(0,\mathcal{Y})}$. Consequently, we have the following partition of unity property

$$\sum_{z' \in \mathcal{N}(\mathscr{T}_{\Omega})} \tilde{\varphi}_{z'} = 1 \quad \text{in} \quad \mathcal{C}_{\mathcal{Y}}. \tag{6.9}$$

Given $z' \in \mathcal{N}(\mathscr{T}_{\Omega})$, we define the *star* around z' as

$$S_{z'} := \bigcup_{K \ni z'} K \subset \Omega,$$

and the *cylindrical star* around z' as

$$\mathcal{C}_{z'} := \bigcup \{T \in \mathscr{T}_{\mathcal{Y}} : T = K \times I, K \ni z'\} = S_{z'} \times (0, \mathcal{Y}) \subset \mathcal{C}_{\mathcal{Y}}.$$

Given an element $K \in \mathscr{T}_\Omega$ we define its *patch* as $S_K := \bigcup_{z' \in K} S_{z'}$, and then, given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ we define its the *cylindrical patch* as $\mathcal{D}_{z'} = \bigcup_{K \ni z'} S_K \times (0, \mathcal{Y}) \subset \mathcal{C}_{\mathcal{Y}}$. Finally, we define, for each $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, $h_{z'} := \min\{h_K : z' \in K\}$.

6.2.3 Local weighted Sobolev spaces

In order to define the local a posteriori error estimatores considered in our work, we first need to define some local weighted Sobolev spaces.

Definition 6.3 *Given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, for each cylindrical star $\mathcal{C}_{z'}$, we define*

$$\mathbb{W}(\mathcal{C}_{z'}) := \left\{ w \in H^1(y^\alpha, \mathcal{C}_{z'}) : w = 0 \text{ on } \partial\mathcal{C}_{z'} \setminus \Omega \times \{0\} \right\}.$$

The space $\mathbb{W}(\mathcal{C}_{z'})$ defined above is a Hilbert space due to the fact that weight $|y|^\alpha$ belongs to the so-called Muckenhoupt class $A_2(\mathbb{R}^{n+1})$; see [79, 87, 109, 126]. Moreover, as it is shown in the following proposition, a weighted Poincaré-type inequality holds true, and consequently the semi-norm $|||w|||_{\mathcal{C}_{z'}} := \|\nabla w\|_{L^2(y^\alpha, \mathcal{C}_{z'})}$ defines a norm on $\mathbb{W}(\mathcal{C}_{z'})$; see also [129, §2.3].

Proposition 6.1 (weighted Poincaré inequality) *Let $z' \in \mathcal{N}(\mathscr{T}_\Omega)$. If the function $w \in \mathbb{W}(\mathcal{C}_{z'})$, then we have*

$$\|w\|_{L^2(y^\alpha, \mathcal{C}_{z'})} \lesssim \|\partial_y w\|_{L^2(y^\alpha, \mathcal{C}_{z'})}, \tag{6.10}$$

Proof: By density [158, Corollary 2.1.6], it suffices to reduce the considerations to a function w, which is smooth. Given $x' \in S_{z'}$, we have that $w(x', \mathscr{Y}) = 0$ so that

$$w(x', y) = -\int_y^{\mathscr{Y}} \partial_y w(x', \xi) \,\mathrm{d}\xi.$$

Multiplying the expression above by $|y|^\alpha$, integrating over $\mathcal{C}_{z'}$, and using the Cauchy Schwarz inequality, we arrive at

$$\begin{aligned}
\int_{\mathcal{C}_{z'}} |y|^\alpha |w(x', y)|^2 \,\mathrm{d}x' \,\mathrm{d}y &\leq \int_{\mathcal{C}_{z'}} |y|^\alpha \left(\int_0^{\mathscr{Y}} |\xi|^\alpha |\partial_y w(x', \xi)|^2 \,\mathrm{d}\xi \int_0^{\mathscr{Y}} |\xi|^{-\alpha} \,\mathrm{d}\xi \right) \mathrm{d}x' \,\mathrm{d}y \\
&= \int_0^{\mathscr{Y}} |y|^\alpha \,\mathrm{d}y \int_0^{\mathscr{Y}} |\xi|^{-\alpha} \,\mathrm{d}\xi \int_{\mathcal{C}_{z'}} |\xi|^\alpha |\partial_y w(x', \xi)|^2 \,\mathrm{d}x' \,\mathrm{d}\xi \\
&\leq C_{2,|y|^\alpha} |\mathscr{Y}|^2 \int_{\mathcal{C}_{z'}} |y|^\alpha |\partial_y w(x', y)|^2 \,\mathrm{d}x' \,\mathrm{d}y,
\end{aligned}$$

where in the third inequality we have used that y^α belongs to the Muckenhoupt class $A_2(\mathbb{R}^{n+1})$; see Definition (2.2). In conclusion, we have derived the anisotropic Poincaré inequality

$$\|w\|_{L^2(y^\alpha, \mathcal{C}_{z'})} \lesssim \mathscr{Y} \|\partial_y w\|_{L^2(y^\alpha, \mathcal{C}_{z'})},$$

which concludes the proof of (6.10). $\square$

Remark 6.4 (anisotropic weighted Poincaré inequality) Let $z' \in \mathcal{N}(\mathscr{T}_\Omega)$. If the function $w \in \mathbb{W}(\mathcal{C}_{z'})$, then by extending the one-dimensional argument developed above to a n-dimensional setting, we are also able to derive

$$\|w\|_{L^2(y^\alpha, \mathcal{C}_{z'})} \lesssim h_{z'} \|\nabla_{x'} w\|_{L^2(y^\alpha, \mathcal{C}_{z'})}.$$

6.3 An ideal a posteriori error estimator

In this subsection, we define an *ideal* a posteriori error estimator on anisotropic meshes which is not *computable*. However, it provides the intuition and establishes the basis to define a discrete and computable error indicator. We prove that this ideal error estimator is equivalent to the error without any oscillation term, which relies on the assumption (6.5) imposed on the family of discretizations $\mathbb{T}$.

Inspired by [14, 46, 125] we now define $\zeta_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$ to be the solution of

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla \zeta_{z'} \nabla \psi = d_s \langle f, \operatorname{tr}_\Omega \psi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla \psi, \tag{6.11}$$

for all $\psi \in \mathbb{W}(\mathcal{C}_{z'})$. The existence and uniqueness of $\zeta_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$ is guaranteed by the Lax–Milgram Lemma and the weighted Poincaré inequality of Proposition 6.1. The continuity of the right hand side of (6.11), as a linear functional in $\mathbb{W}(\mathcal{C}_{z'})$, follows from Proposition 3.3 and the Cauchy-Schwarz inequality. We then define the global error estimator

$$\tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}} = \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \tilde{\mathscr{E}}_{z'}^2 \right)^{\frac{1}{2}}, \tag{6.12}$$

in terms of the local error indicators

$$\tilde{\mathscr{E}}_{z'} = |\!|\!| \, \zeta_{z'} \, |\!|\!|_{\mathcal{C}_{z'}} . \tag{6.13}$$

The properties of this ideal estimator are as follows.

Proposition 6.2 (ideal estimator) *Let $v \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve* (3.37) *and* (3.51) *respectively. Then, the ideal estimator $\tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}}$, defined in* (6.12)–(6.13)*, satisfies*

$$\|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \lesssim \tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}}, \tag{6.14}$$

and

$$\tilde{\mathscr{E}}_{z'} \leq \|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_{z'})}, \tag{6.15}$$

for all $z' \in \mathcal{N}(\mathscr{T}_\Omega)$.

Proof: Denote the error by $e_{\mathscr{T}_{\mathcal{Y}}} = v - V_{\mathscr{T}_{\mathcal{Y}}}$, then for any $\phi \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ we have

$$\begin{aligned}
\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla e_{\mathscr{T}_{\mathcal{Y}}} \nabla \phi &= d_s \langle f, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \nabla \phi \\
&= \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} d_s \langle f, \operatorname{tr}_\Omega \phi \tilde{\varphi}_{z'} \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \nabla (\phi \tilde{\varphi}_{z'}) \\
&= \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} d_s \langle f, \operatorname{tr}_\Omega (\phi - W) \tilde{\varphi}_{z'} \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \nabla \left((\phi - W) \tilde{\varphi}_{z'} \right)
\end{aligned}$$

for any $W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$. To derive the expresion above, we have used the partition of the unity property (6.9), and Galerkin orthogonality (6.8).

Now, we first notice that for each $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ the function $(\phi - W)\tilde{\varphi}_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$. In fact, if z' is an interior node,

$$(\phi - W)\tilde{\varphi}_{z'}|_{\partial \mathcal{C}_{z'} \setminus \Omega \times \{0\}} = 0 \tag{6.16}$$

because of the vanishing property of the basis function $\varphi_{z'}$ on $\partial S_{z'}$ together with the fact that $\phi = W = 0$ on $\Omega \times \{\mathcal{Y}\}$. On the other hand, if z' is a Dirichlet node similar arguments allow us to derive (6.16).

Second, by setting $W = \Pi_{\mathscr{T}_{\mathcal{Y}}} \phi$, the quasi-interpolant introduced in [132, § 5], we obtain boundedness of $|||(\phi - W)\tilde{\varphi}_{z'}|||_{\mathcal{C}_{z'}}$ as follows:

$$|||(\phi - W)\tilde{\varphi}_{z'}|||^2_{\mathcal{C}_{z'}} \lesssim \int_{\mathcal{C}_{z'}} y^\alpha |\nabla(\phi - \Pi_{\mathscr{T}_{\mathcal{Y}}}\phi)|^2 \tilde{\varphi}^2_{z'} + \int_{\mathcal{C}_{z'}} y^\alpha |\phi - \Pi_{\mathscr{T}_{\mathcal{Y}}}\phi|^2 |\nabla_{x'} \tilde{\varphi}_{z'}|^2 \lesssim |||\phi|||^2_{\mathcal{D}_{z'}}.$$

The fist term of the expression above is estimated via the local stability of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ [132, Lemma 5.1] together with the fact that $0 \le \tilde{\varphi}_{z'} \le 1$ for all $x \in \mathcal{C}_{\mathcal{Y}}$. The second term is estimated via the local approximation properties of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ [129, Theorems 4.7 and

4.8]. In fact,

$$\int_{\mathcal{C}_{z'}} y^\alpha |\phi - \Pi_{\mathscr{T}_\mathcal{Y}}\phi|^2 |\nabla_{x'}\tilde{\varphi}_{z'}|^2 \lesssim \frac{1}{h_{z'}^2}\left(h_{z'}^2 \|\nabla_{x'}\phi\|^2_{L^2(\mathcal{D}_{z'},y^\alpha)} + h_{\mathcal{Y}}^2 \|\partial_y \phi\|^2_{L^2(\mathcal{D}_{z'},y^\alpha)} \right)$$

$$\lesssim |||\phi|||^2_{\mathcal{D}_{z'}}, \tag{6.17}$$

where we have used that $|\nabla_{x'}\tilde{\varphi}_{z'}| = |\nabla_{x'}\varphi_{z'}| \lesssim h_{z'}^{-1}$ together with (6.5).

Consequently, setting $\psi_{z'} = (\phi - \Pi_{\mathscr{T}_\mathcal{Y}}\phi)\tilde{\varphi}_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$ as test function in problem (6.11) we obtain

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \phi = \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{S_{z'}} y^\alpha \nabla \zeta_{z'} \nabla \psi_{z'} \lesssim \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} |||\zeta_{z'}|||_{\mathcal{C}_{z'}} \, |||\phi|||_{\mathcal{D}_{z'}}$$

$$\lesssim \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} |||\zeta_{z'}|||^2_{\mathcal{C}_{z'}} \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)} = \tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}} \|\nabla\phi\|_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)},$$

where we have used that $|||\psi|||_{\mathcal{C}_{z'}} \lesssim |||\phi|||_{\mathcal{D}_{z'}}$ and the finite overlapping property of the stars $S_{z'}$.

To obtain (6.14), set $\phi = e_{\mathscr{T}_\mathcal{Y}} \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$.

Finally, inequality (6.15) is immediate:

$$\tilde{\mathscr{E}}_{z'}^2 = |||\zeta_{z'}|||^2_{\mathcal{C}_{z'}} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla\zeta_{z'}\nabla\zeta_{z'} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \zeta_{z'} \leq \|\nabla e_{\mathscr{T}_\mathcal{Y}}\|_{L^2(\mathcal{C}_{z'},y^\alpha)} \, |||\, \zeta_{z'}|||_{\mathcal{C}_{z'}},$$

which concludes the proof. □

Remark 6.5 (Anisotropic meshes) Examining the proof of Proposition 6.2, we realize that a critical part of (6.17) consists in the application of inequality (6.5), namely $h_\mathcal{Y} \leq C_\mathbb{T}\, h_{z'}$ for all $z' \in \mathcal{N}(\mathscr{T}_\Omega)$. Therefore, Proposition 6.2 shows how the resolution of local problems on cylindrical stars allows us to consider anisotropic meshes on the extended variable y and graded meshes in Ω. The latter allows us to compensate possible singularities in the x' variables.

6.4 A computable a posteriori error estimator

Notice that the estimator has an important drawback: for each node z', it requires the exact solution $\zeta_{z'}$ to the local problem (6.11) in the infinite dimensional space $\mathbb{W}(\mathcal{C}_{z'})$, therefore it is not computable. However, it provides intuition and establishes the basis to define a discrete and computable error indicator. We now define local discrete spaces and local computable error indicators, which will allow us to write a global error indicator.

Definition 6.6 (discrete local spaces) *For $z' \in \mathscr{N}(\mathscr{T}_\Omega)$, define the discrete space*

$$\mathcal{W}(\mathcal{C}_{z'}) = \left\{ W \in \mathcal{C}^0(\overline{\mathcal{C}_\mathcal{Y}}) : W|_T \in \mathcal{P}_2(K) \otimes \mathbb{P}_2(I) \ \forall T \in \mathcal{C}_{z'},\ W|_{\partial \mathcal{C}_{z'} \setminus \Omega \times \{0\}} = 0 \right\}.$$

where, if K is a simplex, $\mathcal{P}_2(K)$ corresponds to $\mathbb{P}_2(K)$ and, if K is quadrilateral, $\mathcal{P}_2(K)$ stands for $\mathbb{Q}_2(K)$.

We then define the discrete local problems: For each cylindrical star $\mathcal{C}_{z'}$, we define $\eta_{z'} \in \mathcal{W}(\mathcal{C}_{z'})$ to be the solution of

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla W = \langle f, \operatorname{tr}_\Omega W \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla W, \tag{6.18}$$

for all $W \in \mathcal{W}(\mathcal{C}_{z'})$. We also define the global error estimator

$$\mathscr{E}_{\mathscr{T}_\mathcal{Y}} = \left(\sum_{z' \in \mathscr{N}(\mathscr{T}_\Omega)} \mathscr{E}_{z'}^2 \right)^{\frac{1}{2}}, \tag{6.19}$$

in terms of the local error indicators

$$\mathscr{E}_{z'} = |||\eta_{z'}|||_{\mathcal{C}_{z'}}. \tag{6.20}$$

Let us now prove the equivalence, up to oscillation terms, of the error and the a posteriori error estimator (6.19). We first prove a local lower bound for the error without any oscillation term and free of any constant.

Theorem 6.7 (localized lower bound) *Let* $v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathscr{Y})$ *and* $V_{\mathscr{T}_\mathscr{Y}} \in \mathbb{V}(\mathscr{T}_\mathscr{Y})$ *solve* (3.37) *and* (3.51) *respectively. Then, for any* $z' \in \mathcal{N}(\mathscr{T}_\Omega)$*, we have*

$$\mathscr{E}_{z'} \leq \|\nabla(v - V_{\mathscr{T}_\mathscr{Y}})\|_{L^2(y^\alpha, \mathcal{C}_{z'})}. \tag{6.21}$$

Proof: The proof repeats the arguments used to obtain inequality (6.15). Let $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, and let $\eta_{z'}$ and $\mathscr{E}_{z'}$ as in (6.18) and (6.20). Then,

$$\mathscr{E}_{z'}^2 = |||\eta_{z'}|||^2_{\mathcal{C}_{z'}} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla\eta_{z'} \nabla\eta_{z'} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathscr{Y}} \nabla\eta_{z'} \leq \|\nabla e_{\mathscr{T}_\mathscr{Y}}\|_{L^2(y^\alpha, \mathcal{C}_{z'})} \, ||| \, \eta_{z'}|||_{\mathcal{C}_{z'}},$$

which concludes the proof. □

Remark 6.8 (data oscillation free bound) This data oscillation free lower bound implies a strong concept of reliability: the relative size of $\mathscr{E}_{z'}$ dictates mesh refinement regardless of fine structure of the data f, and thus works even in the pre-asymptotic regime.

In order to derive an upper bound, we need to introduce the so called *oscillation terms* and the following operator. Given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, we define $\mathcal{M}_{z'}$: $\mathbb{W}(\mathcal{C}_{z'}) \to \mathcal{W}(\mathcal{C}_{z'})$ such that for any $\psi \in \mathbb{W}(\mathcal{C}_{z'})$ the following conditions hold:

$$\int_F (\psi - \mathcal{M}_{z'}\psi) = 0, \tag{6.22}$$

where F is either an internal face of $\mathcal{C}_{z'}$ or a boundary face that lies on $\Omega \times \{0\}$,

$$\int_T (\psi - \mathcal{M}_{z'}\psi) = 0, \tag{6.23}$$

for every $T \subset \mathcal{C}_{z'}$, and

$$\int_{S_{z'} \times I} (\psi - \mathcal{M}_{z'}\psi) = 0. \tag{6.24}$$

for every cell $I \subset \mathcal{I}_{\mathscr{Y}}$. In the literature, the operator considered above is known as an *operator defined via moments*, and it is a slight modification of the one introduced and studied in [3, 86]. The properties of this operator are as follows.

Proposition 6.3 (continuity of $\mathcal{M}_{z'}$) *For every $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ the operator $\mathcal{M}_{z'}$ is well defined and continuous, that is*

$$|||\, \mathcal{M}_{z'}\psi|||_{S_{z'}} \lesssim |||\psi|||_{\mathcal{D}_{z'}}, \quad \forall \psi \in \mathbb{W}(\mathcal{C}_{z'}), \tag{6.25}$$

where the hidden constant is independent of z'.

Proof: To show that $\mathcal{M}_{z'}$ is well defined it suffices to consider $\psi \equiv 0$ and prove that $\mathcal{M}_{z'}\psi \equiv 0$ is the unique solution of conditions (6.22)-(6.24) on $\mathcal{W}(\mathcal{C}_{z'})$. Since such a result relies on standard finite element arguments, we skip the details.

In order to prove the stability estimate (6.25), we shall be inspired in [67, 38], and derive the boundedness of $\mathcal{M}_{z'}$ from the fact that it is local, bounded in W^1_1 and an appropiate inverse inequality. In fact, since the definition of the operator $\mathcal{M}_{z'}$ is based on the moment conditions (6.22)-(6.24), we have that, for every $T \in \mathcal{C}_{z'}$ ([3, Lemma 1])

$$\|\nabla \mathcal{M}_{z'}\psi\|_{L^1(T)} \lesssim \|\nabla\psi\|_{L^1(\tilde{S}(T))},$$

where $\tilde{S}(T) = S(T) \cap \mathcal{C}_{z'}$. On the other hand, if $T \in \mathscr{T}_\mathcal{Y}$ and $\psi \in \mathbb{W}(\mathcal{C}_{z'})$, by using that $\mathcal{M}_{z'}\psi$ is a discrete function, we have

$$\|\nabla \mathcal{M}_{z'}\psi\|_{L^\infty(T)} \lesssim \fint_T |\nabla \mathcal{M}_{z'}\psi|.$$

Therefore, collecting the estimates above, we derive

$$\int_T y^\alpha |\nabla \mathcal{M}_{z'}\psi|^2 \lesssim \int_T y^\alpha \left(\fint_T |\nabla \mathcal{M}_{z'}\psi| \right)^2 \lesssim \frac{1}{|T|^2} \int_T y^\alpha \left(\int_{\tilde{S}(T)} |\nabla \psi| \right)^2.$$

which, together with Definition 2.2, yields

$$\int_T y^\alpha |\nabla \mathcal{M}_{z'}\psi| \lesssim \frac{1}{|T|^2} \left(\int_T y^\alpha \right) \left(\int_{\tilde{S}(T)} y^{-\alpha} \right) \int_{tildeS(T)} y^\alpha |\nabla \psi|^2 \lesssim \int_{\tilde{S}(T)} y^\alpha |\nabla \psi|^2.$$

Adding over $T \in \mathcal{C}_{z'}$, using that $|T| \approx \tilde{S}(T)$ and using the finite overlapping property of stars concludes the proof. $\square$

Finally, for every $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, let us define the local data oscillation as

$$\mathsf{osc}_{z'}(f, y^\alpha, W)^2 = \|f - f_{z'}\|^2_{L^2(S_{z'})} + \|y^\alpha \nabla W - \boldsymbol{\sigma}_{z'}\|^2_{L^2(\mathcal{C}_{z'}, y^{-\alpha})}, \tag{6.26}$$

where $f_{z'}|_K \in \mathbb{R}$ and $\boldsymbol{\sigma}'_z|_T \in \mathbb{R}^{n+1}$, that is, they are piecewise constant over $S_{z'}$ and $\mathcal{C}_{z'}$ respectively, and defined by

$$f_{z'}|_K = \fint_K f, \qquad \boldsymbol{\sigma}_{z'}|_T = \fint_T y^\alpha \nabla W, \tag{6.27}$$

for $K \in S_{z'}$ and $T \in \mathcal{C}_{z'}$. We also define the global data oscillation as

$$\mathsf{osc}_{\mathscr{T}_\mathcal{Y}}(f, y^\alpha, W)^2 = \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \mathsf{osc}_z(f, y^\alpha, W)^2. \tag{6.28}$$

With the aid of the operators $\mathcal{M}_{z'}$ we can bound, up to oscillation terms, the error by the estimator.

Theorem 6.9 (global upper bound) *Let $v \in \mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$ and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve* (3.37) *and* (3.51), *respectively. Then, the estimator $\mathscr{E}_{\mathscr{T}_\mathcal{Y}}$, defined in* (6.19)–(6.20) *satisfies*

$$\|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \mathscr{E}_{\mathscr{T}_\mathcal{Y}} + \mathsf{osc}_{\mathscr{T}_\mathcal{Y}}(f, y^\alpha, V_{\mathscr{T}_\mathcal{Y}}), \tag{6.29}$$

where the oscillation terms are defined in (6.26)

Proof: Denote the error by $e_{\mathscr{T}_\mathcal{Y}} = v - V_{\mathscr{T}_\mathcal{Y}}$. Given any function $\phi \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ we define $\psi_{z'} = (\phi - \Pi_{\mathscr{T}_\mathcal{Y}}\phi)\tilde{\varphi}_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$, where $\Pi_{\mathscr{T}_\mathcal{Y}}$ is the quasi-interpolant introduced in [132, § 5], and we recall the estimate $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||\phi|||_{\mathcal{D}_{z'}}$. Then, as in the proof of Proposition 6.2, we have

$$\begin{aligned}\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \phi &= \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \psi_{z'} \\ &= \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \mathcal{M}_{z'}\psi_{z'} - \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla (\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}).\end{aligned}$$

We now examine each term separately:

1. First, for every $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ we have $\mathcal{M}_{z'}\psi_{z'} \in \mathcal{W}(\mathcal{C}_{z'})$, and then the definition of the discrete local problem (6.18) yields

$$\begin{aligned}\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \mathcal{M}_{z'}\psi_{z'} &= \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla \mathcal{M}_{z'}\psi_{z'} \\ &\le \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \mathscr{E}_{z'}^2 \right)^{\frac{1}{2}} \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} |||\psi_{z'}|||_{\mathcal{C}_{z'}}^2 \right)^{\frac{1}{2}} \lesssim \mathscr{E}_{\mathscr{T}_\mathcal{Y}} \|\nabla \phi\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})},\end{aligned}$$

   where in the last inequality we have used the continuity property of the operator $\mathcal{L}_s$ given in Proposition 6.3, the inequality $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||\phi|||_{\mathcal{D}_{z'}}$, and the finite overlapping property of the stars $S_{z'}$.

2. Second, for any $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, we use conditions (6.22)–(6.24) which define the operator $\mathcal{M}_{z'}$, to derive

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) = d_s \int_{S_{z'}} (f - f_{z'}) \operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) \\ - \int_{\mathcal{C}_{z'}} (y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} - \boldsymbol{\sigma}_{z'}) \nabla(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}),$$

and consequently

$$\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) \lesssim \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \Big(\|f - f_{z'}\|_{L^2(S_{z'})} \\ |||\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}|||_{\mathcal{C}_{z'}} + \|y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} - \boldsymbol{\sigma}_{z'}\|_{L^2(\mathcal{C}_{z'}, y^{-\alpha})} |||\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}|||_{\mathcal{C}_{z'}} \Big) \\ \lesssim \mathsf{osc}_{\mathscr{T}_\mathcal{Y}}(f, y^\alpha, V_{\mathscr{T}_\mathcal{Y}}) \|\nabla \phi\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)},$$

where we applied the trace inequality of Proposition (3.3), the continuity of the operator $\mathcal{M}_{z'}$, the bound $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||\phi|||_{\mathcal{D}_{z'}}$, and the finite overlapping property of the stars $S_{z'}$.

Collecting the estimates derived in Steps 1 and 2, we derive the desired global upper bound (6.29). $\square$

Chapter 7: Space-time fractional parabolic problems

7.1 Introduction

We are interested in the numerical approximation of an initial boundary value problem for a space-time fractional parabolic equation. To be concrete, let Ω be an open and bounded subset of $\mathbb{R}^n$ $(n \geq 1)$, with boundary $\partial\Omega$. Given $s \in (0,1)$, $\gamma \in (0,1]$, a function f, and an initial datum u_0, the problem reads as follows: find u such that

$$\begin{cases} \partial_t^\gamma \mathsf{u} + \mathcal{L}^s \mathsf{u} = f, & \text{in } \Omega,\ t \in (0,T), \\ \mathsf{u}(0) = \mathsf{u}_0, & \text{in } \Omega, \\ \mathsf{u} = 0, & \text{on } \partial\Omega,\ t \in (0,T). \end{cases} \tag{7.1}$$

The operator $\mathcal{L}^s$, with $s \in (0,1)$ denotes the fractional powers of a general second order, symmetric and uniformly elliptic operator $\mathcal{L}$, supplemented with homogeneous Dirichlet boundary conditions, defined by

$$\mathcal{L}w = -\mathrm{div}_{x'}(A\nabla_{x'}w) + cw, \tag{7.2}$$

where $c \in L^\infty(\Omega)$ with $c \geq 0$ almost everywhere, and $A \in \mathcal{C}^{0,1}(\Omega, \mathsf{GL}(n,\mathbb{R}))$ is symmetric and positive definite.

The fractional derivative in time ∂_t^γ for $\gamma \in (0,1)$ is understood as *the left-sided*

Caputo fractional derivative of order γ with respect to t, which is defined by

$$\partial_t^\gamma \mathsf{u}(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{1}{(t-r)^\gamma} \frac{\partial \mathsf{u}(x,r)}{\partial r} \,\mathrm{d}r, \tag{7.3}$$

where Γ is the Gamma function. For $\gamma = 1$, we consider the usual derivative ∂_t.

One of the main difficulties in the study of problem (7.1) is given by the nonlocality of the fractional time derivative and the fractional space operator (see [43, 44, 138, 140, 155]). A possible approach to overcome the nonlocality in space is given by the seminal result of Caffarelli and Silvestre in $\mathbb{R}^n$ [43] and its extensions to bounded domains [42, 44, 155]. Fractional powers of the spatial operator $\mathcal{L}$ can be realized as an operator that maps a Dirichlet boundary condition to a Neumann condition via an extension problem on the semi-infinite cylinder $\mathcal{C} = \Omega \times (0,\infty)$. This extension is the following mixed boundary value problem (see [43, 155] for details):

$$\begin{cases} \mathcal{L}\mathscr{U} - \dfrac{\alpha}{y}\partial_y \mathscr{U} - \partial_{yy}\mathscr{U} = 0, \quad \text{in } \mathcal{C}, \\ \mathscr{U} = 0, \quad \text{on } \partial_L \mathcal{C}, \qquad\qquad \dfrac{\partial \mathscr{U}}{\partial \nu^\alpha} = d_s f, \quad \text{on } \Omega \times \{0\}, \end{cases} \tag{7.4}$$

where $\partial_L \mathcal{C} = \partial\Omega \times [0,\infty)$ is the lateral boundary of $\mathcal{C}$, and d_s is a positive normalization constant that depends only on s. The parameter α is defined as

$$\alpha = 1 - 2s \in (-1,1), \tag{7.5}$$

and the so-called conormal exterior derivative of $\mathscr{U}$ at $\Omega \times \{0\}$ is

$$\frac{\partial \mathscr{U}}{\partial \nu^\alpha} = -\lim_{y\to 0^+} y^\alpha \mathscr{U}_y. \tag{7.6}$$

We will call y the *extended variable* and the dimension $n+1$ in $\mathbb{R}^{n+1}_+$ the *extended dimension* of problem (7.4). The limit in (7.6) must be understood in the distributional sense; see [43, 155]. As noted in [42, 43, 44, 155], we can relate the fractional powers of the operator $\mathcal{L}$ with the Dirichlet-to-Neumann map of problem (7.4): $d_s \mathcal{L}^s u = \frac{\partial \mathscr{U}}{\partial \nu^\alpha}$ in Ω. Notice that the differential operator in (7.4) is $-\mathrm{div}\,(y^\alpha \mathbf{A} \nabla \mathscr{U}) + y^\alpha c \mathscr{U}$ where, for all $(x', y) \in \mathcal{C}$, $\mathbf{A}(x', y) = \mathrm{diag}\{A(x'), 1\} \in \mathcal{C}^{0,1}(\mathcal{C}, \mathsf{GL}(n+1, \mathbb{R}))$.

Recently, the Caffarelli-Silvestre extension has been also employed for the study of evolution equations with space fractional diffusion. For instance, by using this technique, interior and Hölder estimates for the fractional heat equation and a drift equation with fractional diffusion have been proved in [148]. In [61, 62], existence, uniqueness and regularity results have been derived for a porous medium equation with fractional diffusion. Inspired in these techniques, we shall use the Caffarelli-Silvestre extension to rewrite problem (7.1) as a quasi-stationary elliptic problem with dynamic boundary condition:

$$\begin{cases} -\mathrm{div}\,(y^\alpha \mathbf{A} \nabla \mathcal{U}) + y^\alpha c \mathcal{U} = 0, & \text{in } \mathcal{C},\ t \in (0,T), \\ \mathcal{U} = 0, & \text{on } \partial_L \mathcal{C},\ t \in (0,T), \\ d_s \partial_t^\gamma \mathcal{U} + \dfrac{\partial \mathcal{U}}{\partial \nu^\alpha} = d_s f, & \text{on } \Omega \times \{0\},\ t \in (0,T), \\ \mathcal{U} = \mathsf{u}_0, & \text{on } \Omega \times \{0\},\ t = 0. \end{cases} \tag{7.7}$$

Before proceeding with the description and analysis of our method, let us give an overview of those advocated in the literature. The design of an efficient technique to treat numerically the left-sided Caputo fractional derivative of order γ is not an easy task. The main difficulty is given by the nonlocality of the operator ∂_t^γ. There are several approaches via finite differences, finite elements and spectral methods. For instance, a finite difference scheme is proposed and analyzed in [118, 119]. The truncation error is $\mathcal{O}(\tau^{2-\gamma})$, where τ denotes the time step. Approaches via finite elements and spectral approximations have been studied in [64, 65, 76, 118], and references therein. In this Chapter, we employ the finite difference scheme of [118, 119], improve on the truncation error, and show discrete stability estimates. The latter lead to novel energy estimates for evolution problems with fractional time derivative in a general Hilbert space setting, which are written in terms of a fractional integral of a norm of the solution.

In Chapter 3 (see also [129]) we used the Caffarelli-Silvestre extension to discretize the fractional space operator and obtained near-optimal error estimates in weighted Sobolev spaces for the extension. An alternative approach has been developed in [29], which is based on the integral formulation of fractional powers of self-adjoint operators [27, Chapter 10.4]. This yields a sequence of easily parallelizable uncoupled elliptic PDEs, and leads to quasi-optimal error estimates in the L^2-norm instead of the energy norm provided Ω is convex and $f \in \mathbb{H}^{2-2s}(\Omega)$. However, the extension of [29] to the evolution case with fractional diffusion is not completely evident, even for the heat equation with fractional diffusion, i.e., $\gamma = 1$ in (7.1). In this Chapter, we will adapt the approach developed in Chapter 3 to the evolution case, and refer to Chapter 1 for an overview of the existing numerical techniques to solve problems involving fractional diffusion.

We use the extension problem (7.7) to propose a strategy to find the solution of (7.1): given a function f and an initial datum u_0, we solve (7.7), thus obtaining a function $\mathcal{U} : (x,t) \in \mathcal{C} \times (0,T) \mapsto \mathcal{U}(x,t) \in \mathbb{R}$. Setting $\mathsf{u} : (x',t) \in \Omega \times (0,T) \mapsto$

$\mathsf{u}(x',t) = \mathcal{U}(x',0,t) \in \mathbb{R}$, we obtain the solution of (7.1). The main objective of this work is to describe and analyze a fully discrete scheme for problem (7.7). We use finite differences for time discretization [118, 119], and first degree tensor product finite elements for space discretization.

The outline of this Chapter is as follows. In section 7.2 we introduce some terminology used throughout this work. We recall the definition of the fractional powers of elliptic operators on a bounded domain via spectral theory in §7.2.2, and in §7.2.3 we introduce the functional framework that is suitable to study problems (7.1) and (7.7). In §7.2.4, we derive a representation for the solution of problem (7.4). Regularity results are discussed in §7.2.5. The time discretization of problem (7.1) is analyzed in section 7.3: the case $\gamma = 1$ is discretized by the standard backward Euler scheme whereas, for $\gamma \in (0,1)$, we consider the finite difference approximation of [118, 119]. For both cases we derive stability results and a novel energy estimate for evolution problems with fractional time derivative in a general Hilbert space setting. We discuss error estimates for semi-discrete schemes in §7.3.4. The space discretization of problem (7.7) begins in section 7.4: in §7.4.1, we introduce a truncation of the domain $\mathcal{C}$ and study some properties of the solution of a truncated problem; in §7.4.2 we present the finite element approximation to the solution of (7.7) in a bounded domain and in §7.4.3 we study a weighted elliptic projector and its properties. In §7.5, we introduce fully discrete schemes and derive near optimal error estimates in time and space for all $\gamma \in (0,1]$.

7.2 Notation and preliminaries

Let $T > 0$ be a fixed time, and let ϕ be a function defined on $\mathcal{D} \times (0,T)$, with $\mathcal{D}$ being an open domain in $\mathbb{R}^N$, $N \geq 1$. As it is standard in time dependent problems, we consider ϕ as a function of t with values in a Banach space $\mathcal{X}$

$$\phi : (0,T) \ni t \longmapsto \phi(t) \equiv \phi(\cdot, t) \in \mathcal{X}.$$

For $1 \leq p \leq \infty$, $L^p(0,T;\mathcal{X})$ is the space of $\mathcal{X}$-valued functions whose norm in $\mathcal{X}$ is in $L^p(0,T)$. This is a Banach space for the norm

$$\|\phi\|_{L^p(0,T;\mathcal{X})} = \left(\int_0^T \|\phi(t)\|_{\mathcal{X}}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|\phi\|_{L^\infty(0,T;\mathcal{X})} = \sup_{t \in (0,T)} \|\phi(t)\|_{\mathcal{X}}.$$

Wwe recall that we adopt the left-sided Caputo fractional derivative, defined in (7.3), as the fractional derivative ∂_t^γ in problem (7.1). Indeed, there are three definitions, not completely equivalent, of fractional derivatives: Riemann Liouville derivative, Caputo derivative and Grünwald-Letnikov derivative. A comprehensive survey of these three different definitions for fractional derivatives and their properties, is given in [105, 138, 140].

7.2.1 Fractional integrals

We recall an important element from fractional calculus, which will be fundamental in our analysis. Given a function g, the left Riemann-Liouville fractional integral $I^\sigma g$ of order $\sigma > 0$ is defined by [105, 138, 140]

$$(I^\sigma g)(t) = \frac{1}{\Gamma(\sigma)} \int_0^t \frac{g(r)}{(t-r)^{1-\sigma}} \,\mathrm{d}r. \tag{7.8}$$

The following result yields the continuity of a general class of integral operators.

Lemma 7.1 (continuity) *If $g \in L^2(0,T)$ and $\phi \in L^1(0,T)$, then the operator*

$$\Phi(t) = \int_0^t \phi(t-r) g(r) \, \mathrm{d}r$$

is continuous from $L^2(0,T)$ into itself and $\|\Phi\|_{L^2(0,T)} \leq \|\phi\|_{L^1(0,T)} \|g\|_{L^2(0,T)}$.

Proof: We first express Φ as an integral over $\mathbb{R}$ and change variables to obtain

$$|\Phi(t)| \leq \int_{\mathbb{R}} |\phi(z)| \chi_{[0,T]}(z) |g(t-z)| \chi_{[0,T]}(t-z) \, \mathrm{d}z.$$

We next write the L^2-norm of Φ also as an integral over $\mathbb{R}$ and apply Minkowski inequality to get

$$\|\Phi\|_{L^2(0,T)} \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\phi(z)|^2 \chi_{[0,T]}(z) |g(t-z)|^2 \chi_{[0,T]}(t-z) \, \mathrm{d}t \right\}^{\frac{1}{2}} \mathrm{d}z.$$

Reordering the integrals and noticing that $z, t-z \in [0,T]$, we arrive at

$$\|\Phi\|_{L^2(0,T)} \leq \int_0^T |\phi(z)| \, \mathrm{d}z \left\{ \int_0^T |g(t)|^2 \, \mathrm{d}t \right\}^{\frac{1}{2}} = \|\phi\|_{L^1(0,T)} \|g\|_{L^2(0,T)},$$

which concludes the proof. $\square$

Lemma 7.1 yields immediately the continuity of the fractional operator I^σ.

Corollary 7.2 (continuity of I^σ) *The left Riemann-Liouville fractional integral $I^\sigma g$ is continuous from $L^2(0,T)$ into itself for any $\sigma > 0$ and*

$$\|I^\sigma g\|_{L^2(0,T)} \leq \frac{T^\sigma}{\Gamma(\sigma+1)} \|g\|_{L^2(0,T)} \quad \forall g \in L^2(0,T).$$

7.2.2 Fractional powers of general second order elliptic operators

Our definition is based on spectral theory. For any $f \in L^2(\Omega)$, the Lax Milgram Lemma provides the existence and uniqueness of $w \in H_0^1(\Omega)$ that solves

$$\mathcal{L}w = f \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega.$$

The operator $\mathcal{L}^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact, symmetric and positive, whence its spectrum $\{\lambda_k^{-1}\}_{k\in\mathbb{N}}$ is discrete, real, positive and accumulates at zero. Moreover, there exists $\{\varphi_k\}_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ which is an orthonormal basis of $L^2(\Omega)$ and satisfies

$$\mathcal{L}\varphi_k = \lambda_k\varphi_k \text{ in } \Omega, \qquad \varphi_k = 0 \text{ on } \partial\Omega, \tag{7.9}$$

for all $k \in \mathbb{N}$. Fractional powers of the operator $\mathcal{L}$ can be defined for $w \in C_0^\infty(\Omega)$ by

$$\mathcal{L}^s w := \sum_{k=1}^{\infty} \lambda_k^s w_k \varphi_k, \tag{7.10}$$

where $w_k = \int_\Omega w\varphi_k$. By density $\mathcal{L}^s$ can be extended to the space

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k\varphi_k : \sum_{k=1}^{\infty} \lambda_k^s w_k^2 < \infty \right\} = \begin{cases} H^s(\Omega), & s \in (0, \frac{1}{2}), \\ H_{00}^{1/2}(\Omega), & s = \frac{1}{2}, \\ H_0^s(\Omega), & s \in (\frac{1}{2}, 1). \end{cases} \tag{7.11}$$

The characterization given by the second equality is shown in [120, Chapter 1]. For $s \in (0,1)$ we denote by $\mathbb{H}^{-s}(\Omega)$ the dual space of $\mathbb{H}^s(\Omega)$.

7.2.3 The Caffarelli-Silvestre extension problem

To exploit the Caffarelli-Silvestre result [43], or its variants [42, 44, 155], we need to deal with a nonuniformly elliptic equation. To this end, we consider weighted

Sobolev spaces with the weight $|y|^\alpha$, $\alpha \in (-1,1)$. Let $\mathcal{D} \subset \mathbb{R}^{n+1}$ be an open set and $\alpha \in (-1,1)$. We define the weighted spaces $L^2(\mathcal{D}, |y|^\alpha)$ and $H^1(\mathcal{D}, |y|^\alpha)$ according to Definitions 2.4 and 2.5 respectively. The space $H^1(\mathcal{D}, |y|^\alpha)$ is equipped with the norm

$$\|w\|_{H^1(\mathcal{D},|y|^\alpha)} = \left(\|w\|^2_{L^2(\mathcal{D},|y|^\alpha)} + \|\nabla w\|^2_{L^2(\mathcal{D},|y|^\alpha)} \right)^{\frac{1}{2}}. \tag{7.12}$$

Since $\alpha \in (-1,1)$ we have that $|y|^\alpha$ belongs to the so-called Muckenhoupt class $A_2(\mathbb{R}^{n+1})$; see [87, 158]. This, in particular, implies that $H^1(\mathcal{D}, |y|^\alpha)$ equipped with the norm (7.12) is a Hilbert space. Moreover, the set $C^\infty(\mathcal{D}) \cap H^1(\mathcal{D}, |y|^\alpha)$ is dense in $H^1(\mathcal{D}, |y|^\alpha)$ (cf. [158, Proposition 2.1.2, Corollary 2.1.6] and [87, Theorem 1]).

To study problem (7.7) we define the weighted Sobolev space

$$\mathring{H}^1_L(\mathcal{C}, y^\alpha) := \left\{ w \in H^1(\mathcal{C}, y^\alpha) : w = 0 \text{ on } \partial_L \mathcal{C} \right\}. \tag{7.13}$$

As Chapter 3 shows, the following *weighted Poincaré inequality* holds:

$$\int_{\mathcal{C}} y^\alpha w^2 \lesssim \int_{\mathcal{C}} y^\alpha |\nabla w|^2, \quad \forall w \in \mathring{H}^1_L(\mathcal{C}, y^\alpha). \tag{7.14}$$

Then, the seminorm on $\mathring{H}^1_L(\mathcal{C}, y^\alpha)$ is equivalent to the norm (7.12). For $w \in H^1(\mathcal{C}, y^\alpha)$, we denote by $\operatorname{tr}_\Omega w$ its trace onto $\Omega \times \{0\}$, and we recall that the trace operator tr_Ω satisfies (see Chapter 3 and [44, Proposition 2.1])

$$\operatorname{tr}_\Omega \mathring{H}^1_L(\mathcal{C}, y^\alpha) = \mathbb{H}^s(\Omega), \qquad \|\operatorname{tr}_\Omega w\|_{\mathbb{H}^s(\Omega)} \leq C_{\operatorname{tr}_\Omega} \|w\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)}. \tag{7.15}$$

Let us now describe the Caffarelli-Silvestre result and its extension to second order operators; [43, 155]. Let u be the solution of $\mathcal{L}^s u = f$ in Ω. We define the α-harmonic extension of u to the cylinder $\mathcal{C}$ as the function $\mathscr{U}$, solution of problem

(7.4), namely

$$d_s \mathcal{L}^s u = \frac{\partial \mathscr{U}}{\partial \nu^\alpha} \quad \text{in } \Omega, \quad \text{where} \quad d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}.$$

To make the above considerations precise, we define

$$\mathbb{W} := \{w \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbb{H}^s(\Omega)) : \partial_t^\gamma w \in L^2(0,T;\mathbb{H}^{-s}(\Omega))\},$$

$$\mathbb{V} := \{w \in L^2(0,T;\mathring{H}_L^1(\mathcal{C},y^\alpha)) : \partial_t^\gamma \operatorname{tr}_\Omega w \in L^2(0,T;\mathbb{H}^{-s}(\Omega))\}.$$

Given $f \in L^2(0,T;\mathbb{H}^{-s}(\Omega))$, a function $\mathsf{u} \in \mathbb{W}$ solves (7.1) if and only if the harmonic extension $\mathcal{U} \in \mathbb{V}$ solves (7.7). A possible weak formulation of problem (7.7) reads: seek $\mathcal{U} \in \mathbb{V}$ such that for a.e. $t \in (0,T)$,

$$\begin{cases} \langle \operatorname{tr}_\Omega \partial_t^\gamma \mathcal{U}, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} + a(\mathcal{U},\phi) = \langle f, \operatorname{tr}_\Omega \phi\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} \\ \operatorname{tr}_\Omega \mathcal{U}(0) = \mathsf{u}_0, \end{cases} \tag{7.16}$$

for all $\phi \in \mathring{H}_L^1(\mathcal{C},y^\alpha)$, where

$$a(w,\phi) = \frac{1}{d_s}\int_{\mathcal{C}} y^\alpha \mathbf{A}(x)\nabla w \cdot \nabla \phi + y^\alpha c(x') w\phi \tag{7.17}$$

and $\langle \cdot,\cdot \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}$ denotes the duality pairing between $\mathbb{H}^s(\Omega)$ and $\mathbb{H}^{-s}(\Omega)$, which, as a consequence of (7.15), is well defined for $f \in L^2(0,T;\mathbb{H}^{-s}(\Omega))$ and $\phi \in \mathring{H}_L^1(\mathcal{C},y^\alpha)$.

Remark 7.3 (equivalent seminorm) Notice that the regularity assumed of the coefficients A and c, together with the weighted Poincaré inequality (7.14), imply that the bilinear form a, defined in (7.17), is bounded and coercive in $\mathring{H}_L^1(\mathcal{C},y^\alpha)$. In what follows we shall use repeatedly the fact that $a(w,w)^{1/2}$ is a norm, equivalent to the seminorm in $\mathring{H}_L^1(\mathcal{C},y^\alpha)$.

Remark 7.4 (dynamic boundary condition) Problem (7.16) corresponds to a weak formulation of an elliptic problem with the dynamic boundary condition

$$\langle f - \operatorname{tr}_\Omega \partial_t^\gamma \mathcal{U}, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)},$$

posed on the bottom part of the cylinder boundary. As a consequence, the analysis of problem (7.16) is slightly different from the standard theory for parabolic equations.

Remark 7.5 (initial datum) The initial datum u_0 of problem (7.1) defines only $\mathcal{U}(0)$ on $\Omega \times \{0\}$ in a trace sense. However, in the subsequent analysis it is necessary to consider its extension to the whole cylinder $\mathcal{C}$. Thus, we define $\mathcal{U}(0)$ to be the solution of problem (7.4) with the Neumann condition replaced by the Dirichlet condition $\mathcal{U} = \mathsf{u}_0$, and then we have the estimate [44]

$$\|\mathcal{U}(0)\|_{\mathring{H}_L^1(\mathcal{C},y^\alpha)} \lesssim \|\mathsf{u}_0\|_{\mathbb{H}^s(\Omega)}.$$

7.2.4 Solution representation

Here we give a representation of the solution of problem (7.7) using the eigenpairs $\{\lambda_k, \varphi_k\}$ defined in (7.9). Let the solution to (7.1) be given by $\mathsf{u}(x',t) = \sum_k \mathsf{u}_k(t)\varphi_k(x')$. The solution $\mathcal{U}$ of problem (7.7) can then be written as

$$\mathcal{U}(x,t) = \sum_{k=1}^{\infty} \mathsf{u}_k(t)\varphi_k(x')\psi_k(y), \tag{7.18}$$

where ψ_k solves

$$\psi_k'' + \frac{\alpha}{y}\psi_k' - \lambda_k\psi_k = 0, \quad \psi_k(0) = 1, \quad \lim_{y\to\infty}\psi_k(y) = 0. \tag{7.19}$$

If $s = \frac{1}{2}$, then clearly $\psi_k(y) = e^{-\sqrt{\lambda_k} y}$. For $s \in (0,1) \setminus \{\frac{1}{2}\}$ we have that if $c_s = \frac{2^{1-s}}{\Gamma(s)}$, then [44, Proposition 2.1]

$$\psi_k(y) = c_s \left(\sqrt{\lambda_k} y\right)^s K_s(\sqrt{\lambda_k} y),$$

where K_s is the modified Bessel function of the second kind [1, Chapter 9.6].

To write an equation for $\mathsf{u}_k(t)$, we first recall some formulas from Chapter 3. For $s \in (0,1)$, we have

$$\lim_{y \downarrow 0^+} \frac{y^\alpha \psi_k'(y)}{d_s \lambda_k^s} = -1, \tag{7.20}$$

and

$$\int_a^b y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2\right) \, \mathrm{d}y = y^\alpha \psi_k(y) \psi_k'(y)\big|_a^b, \tag{7.21}$$

where $a, b \in \mathbb{R}^+$. Then, using the dynamic boundary condition on problem (7.7), and the asymptotic formula (7.20) together with the definitions (7.6) and (7.18), we have

$$\begin{aligned} d_s f(x) = \frac{\partial \mathcal{U}}{\partial \nu^\alpha}(x', 0) + d_s \operatorname{tr}_\Omega \partial_t^\gamma \mathcal{U}(x, 0) &= -\lim_{y \downarrow 0} y^\alpha \mathcal{U}_y(x', y) + d_s \operatorname{tr}_\Omega \partial_t^\gamma \mathcal{U}(x, 0) \\ &= d_s \sum_{k=1}^\infty \varphi_k(x') \big(\lambda_k^s \mathsf{u}_k(t) + \partial_t^\gamma \mathsf{u}_k(t)\big). \end{aligned}$$

The equation above, together with the initial condition $\mathsf{u}(x', 0) = \mathsf{u}_0(x')$ gives us the following fractional initial value problem for $\mathsf{u}_k(t)$:

$$\partial_t^\gamma \mathsf{u}_k(t) + \lambda_k^s \mathsf{u}_k(t) = f_k(t), \qquad \mathsf{u}_k(0) = \mathsf{u}_{0,k}, \tag{7.22}$$

with $\mathsf{u}_{0,k} = (\mathsf{u}_0, \varphi_k)_{L^2(\Omega)}$, and $f_k = (f, \varphi_k)_{L^2(\Omega)}$. According to the existence theory for fractional ordinary differential equations [138, 140], there exists a unique function $\mathsf{u}_k(t)$ satisfying problem (7.22).

Notice that, using (7.18) and (7.19), we obtain

$$\mathcal{U}(x,t)|_{y=0} = \sum_{k=1}^{\infty} \mathsf{u}_k(t)\varphi_k(x')\psi_k(0) = \sum_{k=1}^{\infty} \mathsf{u}_k(t)\varphi_k(x') = \mathsf{u}(x',t).$$

Moreover, by using Remark 7.3, together with formulas (7.20) and (7.21), we have that for almost every $t \in (0,T)$

$$\|\nabla \mathcal{U}(t)\|_{L^2(\mathcal{C},y^\alpha)}^2 \lesssim \sum_{k=1}^{\infty} \mathsf{u}_k(t)^2 \int_0^\infty y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2\right) = d_s \|\mathsf{u}(t)\|_{\mathbb{H}^s(\Omega)}^2. \quad (7.23)$$

We now turn our attention to the solution of problem (7.22).

7.2.4.1 Case $\gamma = 1$: The exponential function

If $\gamma = 1$, then problem (7.22) reduces to a standard first order initial value problem. We introduce the operator

$$E(t)w = \sum_{k=1}^{\infty} e^{-\lambda_k^s t} w_k \varphi_k,$$

where $w_k = \int_\Omega w\varphi_k$. This is the solution operator of (7.1) with $f \equiv 0$. For the nonhomogeneous equation, Duhamel's principle gives the solution u of problem (7.1):

$$\mathsf{u}(x',t) = E(t)\mathsf{u}_0 + \int_0^t E(t-r) f(x',r)\,\mathrm{d}r.$$

7.2.4.2 Case $\gamma \in (0,1)$: The Mittag-Leffler function

In order to explore (7.22), we introduce some preliminary elements from fractional calculus such as the Mittag-Leffler function and recall some of its main properties; see [105, 138, 140]. For $\gamma > 0$ and $\mu \in \mathbb{R}$, we define the Mittag Leffler function $E_{\gamma,\mu}(z)$ as

$$E_{\gamma,\mu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \mu)}, \quad z \in \mathbb{C}.$$

It can be shown that $E_{\gamma,\mu}(z)$ is an entire function of $z \in \mathbb{C}$. The two most important members of this family are $E_{\gamma,1}(z)$, and $E_{\gamma,\gamma}(z)$, which are essential to write the solution operator of problem (7.1). There are several important properties of the Mittag-Leffler function. We recall the differentiation formula

$$\partial_t^\gamma E_{\gamma,1}(-\lambda t^\gamma) = -\lambda E_{\gamma,1}(-\lambda t^\gamma), \tag{7.24}$$

which holds true for $\lambda > 0$, $\gamma > 0$, and $t > 0$ [105, Lemma 2.23].

Following [139], we construct a representation of the solution to problem (7.1). We introduce the solution operator of (7.1) with a homogeneous right-hand side $f \equiv 0$, i.e., $G_\gamma(t)\mathsf{u}_0 = \mathsf{u}$, where

$$G_\gamma(t)w = \sum_{k=1}^{\infty} E_{\gamma,1}(-\lambda_k^s t^\gamma) w_k \varphi_k. \tag{7.25}$$

This follows from the eigenfunction expansion and property (7.24) of the Mittag-Leffler function. For the non-homogeneous equation with vanishing initial datum $\mathsf{u}_0 \equiv 0$, we use the operator defined by

$$F_\gamma(t)w = \sum_{k=1}^{\infty} t^{\gamma-1} E_{\gamma,\gamma}(-\lambda_k^s t^\gamma) w_k \varphi_k. \tag{7.26}$$

These operators are used to represent the solution $\mathsf{u}(x',t)$ of (7.1):

$$\mathsf{u}(x',t) = G_\gamma(t)\mathsf{u}_0 + \int_0^t F_\gamma(t-r) f(x',r)\,\mathrm{d}r; \tag{7.27}$$

see [139, Theorem 2.2] for details. We have thus the following result about existence and uniqueness of solutions of problems (7.1) and (7.7).

Theorem 7.6 (existence and uniqueness of u and $\mathcal{U}$) *Given $s \in (0,1)$, $\gamma \in (0,1]$, $f \in L^2(0,T;\mathbb{H}^{-s}(\Omega))$ and $\mathsf{u}_0 \in L^2(\Omega)$, problems (7.1) and (7.7) have a unique solution.*

Proof: Existence and uniqueness of problem (7.1) can be obtained modifying the spectral decomposition approach studied in [139] based on the solution representation (7.27); see [139, Theorems 2.1 and 2.2]. Similar arguments apply to conclude the well-posedness of problem (7.7). For brevity, we leave the details to the reader. We refer to §7.3 for energy estimates (see also [139]). $\square$

7.2.5 Regularity

We have shown that problem (7.1), for every $\gamma \in (0,1]$ and $s \in (0,1)$, always has a unique solution. Let us now discuss some results about the regularity of the solution, both in space and time.

We begin by describing the regularity in space. As a consequence of the asymptotic behavior $\mathcal{U}_{yy}(t) \approx y^{-\alpha-1}$ as $y \approx 0+$, we conclude $\mathcal{U} \notin H^2(\mathcal{C}, y^\alpha)$. In fact, [129, Theorem 2.6] shows, for the elliptic problem (7.4), that

$$\|\mathcal{L}\mathscr{U}\|_{L^2(\mathcal{C},y^\alpha)} + \|\partial_y \nabla_{x'} \mathscr{U}\|_{L^2(\mathcal{C},y^\alpha)} \lesssim \|f\|_{\mathbb{H}^{1-s}(\Omega)}, \tag{7.28}$$

$$\|\mathscr{U}_{yy}\|_{L^2(\mathcal{C},y^\beta)} \lesssim \|f\|_{L^2(\Omega)}, \tag{7.29}$$

for $\beta > 2\alpha + 1$. Estimate (7.28), however, requires $f \in \mathbb{H}^{1-s}(\Omega)$, which might be too strong an assumption since it does not allow for meaningful duality arguments. For this reason, here we present an improvement over (7.28), in which we weaken the regularity of f, at the expense of strengthening the weight from y^α to y^β as in (7.29), which is already needed to control the term $\mathscr{U}_{yy}$. Concerning the domain Ω, in the analysis that follows we will tacitly assume

$$\|w\|_{H^2(\Omega)} \lesssim \|\mathcal{L}w\|_{L^2(\Omega)}, \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega).$$

Theorem 7.7 (global regularity of the harmonic extension) *Let* $f \in L^2(\Omega)$ *and* $\mathscr{U} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ *solve* (7.4) *with* f *as data. Then, for* $s \in (0,1) \setminus \{\frac{1}{2}\}$, *we have*

$$\|\mathscr{U}\|_{H^2(\mathcal{C}, y^\beta)} \lesssim \|f\|_{L^2(\Omega)}, \tag{7.30}$$

with $\beta > 2\alpha + 1$. *If* $s = \frac{1}{2}$, *then*

$$\|\mathscr{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}.$$

Proof: The critical case $s = \frac{1}{2}$ as well as the estimate for the term $\mathscr{U}_{yy}$ with $s \in (0,1) \setminus \{\frac{1}{2}\}$ are both studied in [129, Theorem 2.6]. It thus remains to analyze the terms $\|\mathcal{L}\mathscr{U}\|_{L^2(\mathcal{C}, y^\beta)}$ and $\|\partial_y \nabla_{x'} \mathscr{U}\|_{L^2(\mathcal{C}, y^\beta)}$ in (7.30). First, using the fact that $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ satisfying (7.9), we obtain

$$\|\mathcal{L}_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C}, y^\beta)} \lesssim \sum_{k=1}^{\infty} u_k^2 \lambda_k^2 \int_0^\infty y^\beta |\psi_k(y)|^2 \, \mathrm{d}y.$$

By considering the sequence $\{a_k = 1/\sqrt{\lambda_k}\}_{k \geq 1}$, we can write

$$\|\mathcal{L}\mathscr{U}\|^2_{L^2(\mathcal{C}, y^\beta)} \lesssim \sum_{k=1}^{\infty} u_k^2 \lambda_k^2 \left(\int_0^{a_k} y^\beta |\psi_k(y)|^2 \, \mathrm{d}y + \int_{a_k}^\infty y^\beta |\psi_k(y)|^2 \, \mathrm{d}y \right).$$

We estimate the two terms on the right hand side separately. Since $z^s K_s(z) \approx 1$ as $z \downarrow 0$ [1, Chapter 9.6], we get

$$\int_0^{a_k} y^\beta |\psi_k(y)|^2 \, \mathrm{d}y = c_s^2 \lambda_k^{-\beta/2 - 1/2} \int_0^1 z^{\beta + 2s} K_s(z)^2 \, \mathrm{d}z \approx \lambda_k^{-\beta/2 - 1/2},$$

where the integral converges because $\beta > 2\alpha + 1 > -1$. On the other hand, exploiting the exponential decay of $K_s(z)$ as $z \uparrow \infty$, the second term above can be bounded similarly. This, together with the fact that $u_k = f_k \lambda_k^{-s}$ and $2 - 2s - \frac{\beta}{2} - \frac{1}{2} = \frac{1}{2}(1 + 2\alpha - \beta) < 0$, allows us to deduce

$$\|\mathcal{L}\mathscr{U}\|^2_{L^2(\mathcal{C}, y^\beta)} \lesssim \sum_{k=1}^{\infty} f_k^2 \lambda_k^{2 - 2s - \beta/2 - 1/2} \leq \|f\|^2_{L^2(\Omega)}.$$

Estimating $\partial_y \nabla_{x'} \mathscr{U}$ follows along the same lines. In fact, we have

$$\|\partial_y \nabla_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C},y^\beta)} \lesssim \sum_{k=1}^{\infty} u_k^2 \lambda_k \left(\int_0^{a_k} y^\beta |\psi_k'(y)|^2 \, \mathrm{d}y + \int_{a_k}^{\infty} y^\beta |\psi_k'(y)|^2 \, \mathrm{d}y \right).$$

We utilize $\frac{\mathrm{d}}{\mathrm{d}z}(z^s K_s(z)) = -z^s K_{1-s}(z)$ [1, Chapter 9.6] to estimate the first integral on the right hand side as follows

$$\begin{aligned} \int_0^{a_k} y^\beta |\psi_k'(y)|^2 \, \mathrm{d}y &= c_s^2 \lambda_k^{1/2-\beta/2} \int_0^1 z^{\beta+2s} K_{1-s}^2(z) \, \mathrm{d}z \\ &\lesssim \lambda_k^{1/2-\beta/2} \int_0^1 z^{\beta+4s-2} \, \mathrm{d}z \approx \lambda_k^{1/2-\beta/2}, \end{aligned}$$

where the integral converges because $\beta + 4s - 2 = \beta - 2\alpha > 1$. We obtain a similar estimate for the second integral above that again exploits the exponential decay of $K_{1-s}(z)$ as $z \uparrow 0$. Replacing the estimates back we derive

$$\|\partial_y \nabla_{x'} \mathscr{U}\|^2_{L^2(\mathcal{C},y^\beta)} \lesssim \sum_{k=1}^{\infty} f_k^2 \lambda_k^{1-2s+1/2-\beta/2} \le \|f\|^2_{L^2(\Omega)},$$

because $1 - 2s + \frac{1}{2} - \frac{\beta}{2} = \frac{1}{2}(1 + 2\alpha - \beta) < 0$. This concludes the proof. $\square$

Having just discussed the regularity in space, let us briefly elaborate on the regularity in time. Since our problem is linear, we could simply demand sufficient regularity (in time) of the right-hand side along with compatibility conditions for the initial datum u_0. However, we express the requisite regularity directly in terms of $\mathcal{U}$ for all $\gamma \in (0, 1]$:

$$\partial_{tt} \operatorname{tr}_\Omega \mathcal{U} \in L^2(0, T; \mathbb{H}^{-s}(\Omega)). \tag{7.31}$$

7.3 Time discretization

Let $\mathcal{K} \in \mathbb{N}$ denote the number of time steps. We define the time step as $\tau = T/\mathcal{K} > 0$, and set $t^k = k\tau$ for $0 \le k \le \mathcal{K}$. If E is a normed space equipped with the norm

$\|\cdot\|_E$, then for any time dependent function $\phi \in C([0,T],E)$, we denote $\phi^k = \phi(t^k)$ and $\phi^\tau = \{\phi^k\}_{k=0}^{\mathcal{K}}$. Moreover, we define

$$\|\phi^\tau\|_{\ell^\infty(E)} = \max_{0\le k\le \mathcal{K}} \|\phi^k\|_E, \qquad \|\phi^\tau\|_{\ell^2(E)}^2 = \sum_{k=1}^{\mathcal{K}} \tau \|\phi^k\|_E^2.$$

For a sequence of time-discrete functions $W^\tau \subset E$ we define, for $k = 0, \ldots, \mathcal{K}-1$,

$$\delta^1 W^{k+1} = \frac{W^{k+1} - W^k}{\tau}. \tag{7.32}$$

7.3.1 Time discretization for $\gamma = 1$

We apply the usual backward Euler scheme to problem (7.16) with $\gamma = 1$, which computes $V^\tau = \{V^k\}_{k=0}^{\mathcal{K}} \subset \mathring{H}_L^1(\mathcal{C}, y^\alpha)$ as follows. The first step is the initialization

$$\operatorname{tr}_\Omega V^0 = \mathsf{u}_0. \tag{7.33}$$

Then, for $k = 0, \ldots, \mathcal{K}-1$, we find $V^{k+1} \in \mathring{H}_L^1(\mathcal{C}, y^\alpha)$ solution of

$$\left(\delta^1 \operatorname{tr}_\Omega V^{k+1}, \operatorname{tr}_\Omega W\right)_{L^2(\Omega)} + a(V^{k+1}, W) = \left\langle f^{k+1}, \operatorname{tr}_\Omega W \right\rangle_{\mathbb{H}^{-s}(\Omega)\times \mathbb{H}^s(\Omega)}, \tag{7.34}$$

for all $W \in \mathring{H}_L^1(\mathcal{C}, y^\alpha)$, where $f^{k+1} = f(t^{k+1})$. By defining

$$U^k := \operatorname{tr}_\Omega V^k, \tag{7.35}$$

we obtain a sequence $U^\tau = \{U^k\}_{k=0}^{\mathcal{K}} \subset \mathbb{H}^s(\Omega)$, which is a piecewise constant approximation of u, solution to problem (7.1).

Remark 7.8 (initial datum) Step (7.33) does not require an extension of u_0.

Remark 7.9 (dynamic boundary condition) Problem (7.16) is an elliptic problem with a dynamic boundary condition, and so is problem (7.33)–(7.34). As a consequence, the stability and error analyses are slightly different from the standard theory for, say, the heat equation.

Remark 7.10 (locality) The main advantage of scheme (7.33)–(7.34) is its local nature, thereby mimicking that of problem (7.16).

The stability of this scheme is rather elementary as the following result shows.

Lemma 7.11 (unconditional stability for $\gamma = 1$) *The semi-discrete scheme* (7.33)–(7.34) *is unconditionally stable, namely*

$$\| \operatorname{tr}_\Omega V^\tau \|^2_{\ell^\infty(L^2(\Omega))} + \| V^\tau \|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C},y^\alpha))} \lesssim \| \mathsf{u}_0 \|^2_{L^2(\Omega)} + \| f^\tau \|^2_{\ell^2(\mathbb{H}^{-s}(\Omega))}. \tag{7.36}$$

Proof: Choose $W = 2\tau V^{k+1}$ in (7.34) and use the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$. The trace estimate (7.15) and Young's inequality yield

$$\| \operatorname{tr}_\Omega V^{k+1} \|^2_{L^2(\Omega)} + \tau \| V^{k+1} \|^2_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim \| \operatorname{tr}_\Omega V^k \|^2_{L^2(\Omega)} + \tau \| f^{k+1} \|^2_{\mathbb{H}^{-s}(\Omega)}.$$

Adding this inequality over k yields (7.36). □

7.3.2 Time discretization for $\gamma \in (0, 1)$

We now discretize in time the nonlocal operator given by the fractional derivative ∂_t^γ of order $\gamma \in (0,1)$. We consider the finite difference scheme proposed in [118, 119]. By using the definition of the left-sided Caputo fractional derivative of order γ given in (7.3) and the Taylor formula with integral remainder we have, for $0 \le k \le \mathcal{K} - 1$,

$$\begin{aligned}
\partial_t^\gamma u(\cdot, t^{k+1}) &= \frac{1}{\Gamma(1-\gamma)} \int_0^{t^{k+1}} \frac{\partial_t u(\cdot, t)}{(t^{k+1} - t)^\gamma} \, \mathrm{d}t \\
&= \frac{1}{\Gamma(1-\gamma)} \sum_{j=0}^{k} \frac{u(\cdot, t^{j+1}) - u(\cdot, t^j)}{\tau} \int_{t^j}^{t^{j+1}} \frac{\mathrm{d}t}{(t^{k+1} - t)^\gamma} + \mathsf{r}_\gamma^{k+1}(\cdot) \qquad (7.37) \\
&= \frac{1}{\Gamma(2-\gamma)} \sum_{j=0}^{k} a_j \frac{u(\cdot, t^{k+1-j}) - u(\cdot, t^{k-j})}{\tau^\gamma} + \mathsf{r}_\gamma^{k+1}(\cdot),
\end{aligned}$$

where

$$a_j = (j+1)^{1-\gamma} - j^{1-\gamma}, \tag{7.38}$$

and

$$\mathsf{r}_\gamma^{k+1} = \frac{1}{\Gamma(1-\gamma)} \sum_{j=0}^{k} \int_{t^j}^{t^{j+1}} \frac{1}{(t^{k+1}-t)^\gamma} R(\cdot,t)\,\mathrm{d}t,$$

denotes the remainder, with the function R defined by

$$R(\cdot,t) = \partial_t u(\cdot,t) - \frac{1}{\tau}\left(u(\cdot,t^{j+1}) - u(\cdot,t^j)\right) \qquad \forall t \in (t^j, t^{j+1}). \tag{7.39}$$

Notice that from definition (7.38), we deduce that $a_j > 0$ for all $j \geq 0$ and

$$1 = a_0 > a_1 > a_2 > \cdots > a_j, \qquad \lim_{j\to\infty} a_j = 0.$$

7.3.2.1 Consistency estimate

We now estimate the residual r_γ^τ by exploiting some cancellation property. We first observe that the function R defined in (7.39) has vanishing mean in (t^j, t^{j+1}) for all $j \in \{0, \ldots, \mathcal{K}-1\}$, whence we can write

$$\mathsf{r}_\gamma^{k+1} = \frac{1}{\Gamma(1-\gamma)} \sum_{i=0}^{k} \int_{t^j}^{t^{j+1}} (\psi(t) - \bar{\psi}^j) R(\cdot,t)\,\mathrm{d}t, \tag{7.40}$$

where $\psi(t) = \frac{1}{(t^{k+1}-t)^\gamma}$ and $\bar{\psi}^j = ⨍_{t^j}^{t^{j+1}} \psi(t)\,\mathrm{d}t$ is its mean. The following result gives an estimate for $\|R^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))}$.

Lemma 7.12 (estimate for R^τ) *The term R defined by* (7.39) *satisfies*

$$\|R^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))} \leq \tau \|\partial_{tt} u\|_{L^2(0,T;\mathbb{H}^{-s}(\Omega))}.$$

Proof: For $t \in (t^j, t^{j+1})$, from the definition of R in (7.39), we have

$$\|R(\cdot,t)\|_{\mathbb{H}^{-s}(\Omega)} \le \frac{1}{\tau}\int_{t^j}^{t^{j+1}} \|\partial_t u(\cdot,t) - \partial_t u(\cdot,r)\|_{\mathbb{H}^{-s}(\Omega)}\,\mathrm{d}r \le \int_{t^j}^{t^{j+1}} \|\partial_{tt} u(\cdot,z)\|_{\mathbb{H}^{-s}(\Omega)}\,\mathrm{d}z,$$

whence

$$\|R(\cdot,t)\|^2_{\mathbb{H}^{-s}(\Omega)} \le \tau \int_{t^j}^{t^{j+1}} \|\partial_{tt} u(\cdot,z)\|^2_{\mathbb{H}^{-s}(\Omega)}\,\mathrm{d}z.$$

Finally,

$$\|R^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))} \le \left(\sum_{j=1}^{\mathcal{K}} \tau^2 \int_{t^{j-1}}^{t^j} \|\partial_{tt} u(\cdot,z)\|^2_{\mathbb{H}^{-s}(\Omega)} \right)^{\frac{1}{2}} \le \tau \|\partial_{tt} u\|_{L^2(0,T;\mathbb{H}^{-s}(\Omega))},$$

which concludes the proof. $\square$

Now, we compute the L^1-norm of $\psi - \bar{\psi}^\tau$. We start with the interval (t^j, t^{j+1}):

$$\begin{aligned}\int_{t^j}^{t^{j+1}} |\psi(t) - \bar{\psi}^j|\,\mathrm{d}t &= \frac{1}{\tau}\int_{t^j}^{t^{j+1}} \left| \int_{t^j}^{t^{j+1}} (\psi(t)-\psi(r))\,\mathrm{d}r \right| \mathrm{d}t \le \tau \int_{t^j}^{t^{j+1}} |\psi'(t)|\,\mathrm{d}t \\ &= \tau\gamma \int_{t^j}^{t^{j+1}} \frac{1}{(t^{k+1}-t)^{\gamma+1}}\,\mathrm{d}t = \tau^{1-\gamma}\left[\frac{1}{(k-j)^\gamma} - \frac{1}{(k-j+1)^\gamma} \right],\end{aligned}$$

which is valid for all $0 \le j < k$. For $j = k$, we take $\bar{\psi}_k = 0$ and simply compute

$$\int_{t^k}^{t^{k+1}} \psi(t)\,\mathrm{d}t = \int_{t^k}^{t^{k+1}} \frac{1}{(t^{k+1}-t)^\gamma}\,\mathrm{d}t = \frac{\tau^{1-\gamma}}{1-\gamma}.$$

Consequently,

$$\begin{aligned}\|\psi - \bar{\psi}^\tau\|_{L^1(0,T)} &= \sum_{j=0}^{k} \int_{t^j}^{t^{j+1}} |\psi(t) - \bar{\psi}^j|\,\mathrm{d}t \\ &\le \tau^{1-\gamma}\left(\frac{1}{1-\gamma} + \sum_{j=0}^{k-1} \left[\frac{1}{(k-j)^\gamma} - \frac{1}{(k-j+1)^\gamma} \right] \right) \\ &= \tau^{1-\gamma}\left(\frac{1}{1-\gamma} + 1 - \frac{1}{(k+1)^\gamma} \right) \le \frac{2-\gamma}{1-\gamma}\tau^{1-\gamma}.\end{aligned}$$

We thus have the following result.

Lemma 7.13 (kernel estimate) *The kernel ψ satisfies*

$$\|\psi - \bar{\psi}^\tau\|_{L^1(0,T)} \leq \frac{2-\gamma}{1-\gamma}\tau^{1-\gamma}.$$

We now derive an estimate for r_γ^τ, which is an improvement over [118, (3.4)].

Proposition 7.1 (consistency) *The fractional residual $\mathsf{r}_\gamma^\tau = \{\mathsf{r}_j^k\}_{k=0}^{\mathcal{K}-1}$ satisfies*

$$\|\mathsf{r}_\gamma^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))} \lesssim \tau^{2-\gamma}\|\partial_{tt}u\|_{L^2(0,T;\mathbb{H}^{-s}(\Omega))}. \tag{7.41}$$

Proof: Setting $g(t) = R(\cdot, t)$ and $\phi(t) = \psi(t) - \bar{\psi}^\tau$, we apply Lemma 7.1 to r_γ^τ:

$$\|\mathsf{r}_\gamma^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))} \leq \|\psi - \bar{\psi}\|_{L^1(0,T)}\|R^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))},$$

which, together with Lemmas 7.12 and 7.13 concludes the proof. $\square$

7.3.2.2 Stability and energy estimates

To fix the ideas concerning the application of the discretization (7.37), we present an abstract approach within a general Hilbert space setting. Given a Gelfand triple $\mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}' \subset \mathcal{V}'$, with $\mathcal{V}$ dense in $\mathcal{H}$, let $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ be a linear, continuous and coercive operator. If $(\cdot, \cdot)_{\mathcal{H}}$ is the inner product in $\mathcal{H}$, set

$$\|U\|_{\mathcal{H}} = (U, U)_{\mathcal{H}}^{1/2}, \qquad \|U\|_{\mathcal{V}} = \langle \mathcal{A}U, U\rangle^{1/2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$. Given $f \in L^2(0, T; \mathcal{V}')$ and $\mathsf{u}_0 \in \mathcal{H}$, we study a time discretization scheme for the fractional evolution problem

$$\partial_t^\gamma \mathsf{u} + \mathcal{A}\mathsf{u} = f, \qquad \mathsf{u}(0) = \mathsf{u}_0. \tag{7.42}$$

For $\phi^\tau = \{\phi^k\}_{k=0}^{\mathcal{K}} \subset \mathcal{V}$, we define the discrete fractional operator

$$\delta^\gamma \phi^{k+1} := \frac{1}{\Gamma(2-\gamma)} \sum_{j=0}^{k} a_j \frac{\phi^{k+1-j} - \phi^{k-j}}{\tau^\gamma}, \quad k = 0, \dots \mathcal{K} - 1, \tag{7.43}$$

for $\gamma \in (0,1)$, which, using that $a_0 = 1$, is equivalent to

$$\delta^\gamma \phi^{k+1} := \frac{1}{\Gamma(2-\gamma)\tau^\gamma} \left(\phi^{k+1} - \sum_{j=0}^{k-1} (a_j - a_{j+1}) \phi^{k-j} - a_k \phi^0 \right),$$

for $0 \le k \le \mathcal{K} - 1$ provided the sum for $k = 0$ is defined to be zero. The relations (7.37) motivate the following semi-discrete scheme to solve (7.42). Let $U^0 = \mathsf{u}_0$ and, for $k = 0, \dots, \mathcal{K} - 1$, compute $U^{k+1} \in \mathcal{V}$ as the solution of

$$(\delta^\gamma U^{k+1}, W)_{\mathcal{H}} + \langle \mathcal{A} U^{k+1}, W \rangle = \langle f^{k+1}, W \rangle, \qquad \forall W \in \mathcal{V}. \tag{7.44}$$

We have the following stability result.

Theorem 7.14 (unconditional stability for $\gamma \in (0,1)$) *The semi-discrete scheme* (7.44) *is unconditionally stable and satisfies*

$$I^{1-\gamma} \|U^\tau\|_{\mathcal{H}}^2 + \|U^\tau\|_{\ell^2(\mathcal{V})}^2 \le I^{1-\gamma} \|U^0\|_{\mathcal{H}}^2 + \|f^\tau\|_{\ell^2(\mathcal{V}')}^2. \tag{7.45}$$

Proof: Denote $\kappa = \Gamma(2-\gamma)\tau^\gamma$ and set $W = 2\kappa U^{k+1}$ in (7.44). We obtain

$$\begin{aligned} &2\|U^{k+1}\|_{\mathcal{H}}^2 + 2\kappa \|U^{k+1}\|_{\mathcal{V}}^2 \\ &\qquad = 2 \sum_{j=0}^{k-1} (a_j - a_{j+1}) (U^{k-j}, U^{k+1})_{\mathcal{H}} + 2a_k (U^0, U^{k+1})_{\mathcal{H}} + 2\kappa \langle f^{k+1}, U^{k+1} \rangle, \end{aligned}$$

for $0 \le k \le \mathcal{K} - 1$ provided the sum vanishes for $k = 0$. Using the Cauchy-Schwarz inequality, the fact that $a_j - a_{j+1} > 0$, and the telescopic property of the sum

$\sum_{j=0}^{k-1}(a_j - a_{j+1}) = 1 - a_k$, we obtain for $0 \leq k \leq \mathcal{K}-1$

$$\begin{aligned}(2-(1-a_k)-a_k)\,\|U^{k+1}\|^2_{\mathcal{H}} + \kappa\|U^{k+1}\|^2_{\mathcal{V}} \\ \leq \sum_{j=0}^{k-1}(a_j - a_{j+1})\|U^{k-j}\|^2_{\mathcal{H}} + a_k\|U^0\|^2_{\mathcal{H}} + \kappa\|f^{k+1}\|^2_{\mathcal{V}'}.\end{aligned}$$

A simple manipulation of the left-hand side of the inequality above yields

$$\sum_{j=0}^{k} a_j\|U^{k+1-j}\|^2_{\mathcal{H}} + \kappa\|U^{k+1}\|^2_{\mathcal{V}} \leq \sum_{j=0}^{k-1} a_j\|U^{k-j}\|^2_{\mathcal{H}} + a_k\|U^0\|^2_{\mathcal{H}} + \kappa\|f^{k+1}\|^2_{\mathcal{V}'},$$

where the sum on the right-hand side vanishes for $k = 0$. Adding the inequality above over k, for $0 \leq k \leq \mathcal{K}-1$, we get

$$\sum_{j=0}^{\mathcal{K}-1} a_j\|U^{\mathcal{K}-j}\|^2_{\mathcal{H}} + \kappa\sum_{k=1}^{\mathcal{K}}\|U^k\|^2_{\mathcal{V}} \leq \left(\sum_{k=0}^{\mathcal{K}-1} a_k\right)\|U^0\|^2_{\mathcal{H}} + \kappa\sum_{k=1}^{\mathcal{K}}\|f^k\|^2_{\mathcal{V}'}.$$

Multiplying this inequality by $\tau^{1-\gamma}/\Gamma(2-\gamma)$, we obtain

$$\frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=0}^{\mathcal{K}-1} a_j\|U^{\mathcal{K}-j}\|^2_{\mathcal{H}} + \|U^\tau\|^2_{\ell^2(\mathcal{V})} \leq \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\|U^0\|^2_{\mathcal{H}} + \|f^\tau\|^2_{\ell^2(\mathcal{V}')}. \tag{7.46}$$

Now, changing the summation index and using the definition (7.38), we obtain

$$\begin{aligned}\sum_{j=0}^{\mathcal{K}-1} a_j\|U^{\mathcal{K}-j}\|^2_{\mathcal{H}} &= \frac{1}{\tau^{1-\gamma}}\sum_{l=1}^{\mathcal{K}}\left((T-t^{l-1})^{1-\gamma} - (T-t^l)^{1-\gamma}\right)\|U^l\|^2_{\mathcal{H}} \\ &= \frac{1-\gamma}{\tau^{1-\gamma}}\sum_{l=1}^{\mathcal{K}}\int_{t^{l-1}}^{t^l}\frac{\|U^\tau(r)\|^2_{\mathcal{H}}}{(T-r)^\gamma}\,\mathrm{d}r,\end{aligned}$$

whence,

$$\frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=0}^{\mathcal{K}-1} a_j\|U^{\mathcal{K}-j}\|^2_{\mathcal{H}} = I^{1-\gamma}\|U^\tau\|^2_{\mathcal{H}},$$

which together with (7.46) yields the desired estimate (7.45). □

Deducing an energy estimate for problem (7.42) is nontrivial due to the nonlocality of the fractional time derivative. The main technical difficulty lies on the fact that a key ingredient in deriving such a result is an integration by parts formula, which for a function u not vanishing at $t = 0$ and $t = T$ involves boundary terms and these need to be estimated; for a step in this direction see [77, 117]. In this sense, the discrete energy estimate (7.45) has an important consequence at the continuous level.

Corollary 7.15 (fractional energy estimate for u) *Let $\gamma \in (0,1)$ and u be the solution of problem* (7.42). *Then, the following estimate holds*

$$I^{1-\gamma}\|\mathsf{u}\|_{\mathcal{H}}^2 + \|\mathsf{u}\|_{L^2(0,T;\mathcal{V})}^2 \le I^{1-\gamma}\|\mathsf{u}_0\|_{\mathcal{H}}^2 + \|f\|_{L^2(0,T;\mathcal{V}')}^2. \tag{7.47}$$

Proof: Given that the estimate (7.45) is uniform in τ and $\|\mathsf{r}_\gamma^{k+1}\|_{\mathcal{V}'} \lesssim \tau^{2-\gamma}$, we easily derive (7.47) by taking $\tau \downarrow 0$. $\square$

Remark 7.16 (limiting case) Given $g \in L^p(0,T)$, we have $I^\sigma g \to g$ in $L^p(0,T)$ as $\sigma \downarrow 0$; see [140, Theorem 2.6]. This implies that, taking the limit as $\gamma \uparrow 1$ in (7.47), we recover the well known stability result for a parabolic equation, i.e.,

$$\|\mathsf{u}\|_{L^\infty(0,T;\mathcal{H})}^2 + \|\mathsf{u}\|_{L^2(0,T;\mathcal{V})}^2 \le \|\mathsf{u}_0\|_{\mathcal{H}}^2 + \|f\|_{L^2(0,T;\mathcal{V}')}^2. \tag{7.48}$$

Notice that Remark 7.16 in conjunction with Theorem 7.14, allows us to unify the fractional energy estimate given in Corollary 7.15 to $\gamma \in (0,1]$.

7.3.3 Discrete stability

We now apply the ideas developed in §7.3.1 and §7.3.2 to problem (7.1), i.e., we consider $\mathcal{A} = \mathcal{L}^s$. As it was discussed in §7.2.3, we realize the nonlocal spatial operator $\mathcal{L}^s$ with the Caffarelli-Silvestre extension and look for solutions of the extended problem (7.16). In view of (7.34) and (7.44), we propose the following *semi-discrete* numerical scheme to approximate problem (7.16) for $\gamma \in (0,1]$:

Set $\operatorname{tr}_\Omega V^0 = \mathsf{u}_0$. For $k = 0, \ldots, \mathcal{K}-1$ find $V^{k+1} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$, solution of

$$(\delta^\gamma \operatorname{tr}_\Omega V^{k+1}, \operatorname{tr}_\Omega W)_{L^2(\Omega)} + a(V^{k+1}, W) = \langle f^{k+1}, \operatorname{tr}_\Omega W \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \tag{7.49}$$

for all $W \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$, where a is the bilinear form defined in (7.17), and δ^γ is defined by (7.43) for $\gamma \in (0,1)$, and (7.32) for $\gamma = 1$. We have the following stability result.

Corollary 7.17 (unconditional stability for $0 < \gamma \le 1$) *The semi-discrete scheme* (7.49) *is unconditionally stable and satisfies*

$$I^{1-\gamma}\|\operatorname{tr}_\Omega V^\tau\|_{\mathbb{H}^s(\Omega)} + \|V^\tau\|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C},y^\alpha))} \le I^{1-\gamma}\|\mathsf{u}_0\|_{\mathbb{H}^s(\Omega)} + \|f^\tau\|^2_{\ell^2(\mathbb{H}^{-s}(\Omega))}. \tag{7.50}$$

Proof: The desired estimate (7.50) is a direct consequence of Theorem 7.14 for $\gamma \in (0,1)$ and Lemma 7.11 for $\gamma = 1$. $\square$

7.3.4 Error Estimates

We present the following semi-discrete error estimate.

Theorem 7.18 (error estimates for semi-discrete schemes) *Let $\mathcal{U}$ solve* (7.16) *and V^τ solve* (7.49). *If $\mathcal{U}$ satisfies* (7.41), *then we have for $\gamma \in (0,1]$*

$$I^{1-\gamma}\|\operatorname{tr}_\Omega(\mathcal{U}^\tau - V^\tau)\|^2_{L^2(\Omega)} + \|\mathcal{U}^\tau - V^\tau\|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C},y^\alpha))} \lesssim \tau^{2(2-\gamma)}\|\partial_{tt}\mathcal{U}\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))},$$

where the hidden constants depend on T, f and u_0 but not on $\mathcal{U}$.

Proof: In view of (7.16) and (7.49), together with the residual estimate (7.41), the equation for the error $E^k := \mathcal{U}^k - V^k$ reads

$$(\delta^\gamma \operatorname{tr}_\Omega E^{k+1}, \operatorname{tr}_\Omega W)_{L^2(\Omega)} + a(E^{k+1}, W) = -\langle r_\gamma^{k+1}, \operatorname{tr}_\Omega W\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}.$$

Apply now either (7.36) or (7.50) in conjunction with (7.41) to conclude the proof. $\square$

7.4 Space Discretization

7.4.1 Truncation

Given that $\mathcal{C}$ is an infinite cylinder, problem (7.7) cannot be directly approximated with finite element-like techniques. A first step towards the discretization is to truncate the domain $\mathcal{C}$. Since, for a.e. $t \in (0,T)$, $\mathcal{U}(t)$ decays exponentially in the extended direction y, we truncate the cylinder $\mathcal{C}$ to $\mathcal{C}_{\mathscr{Y}} = \Omega \times (0, \mathscr{Y})$ for a suitable $\mathscr{Y}$ and seek solutions in this bounded domain; see [129, §3]. The next result is an adaptation of [129, Proposition 3.1] and shows the exponential decay of $\mathcal{U}$, solution of problem (7.16). To write such a result, we first define for $\gamma \in (0,1]$

$$\Lambda_\gamma^2(\mathsf{u}_0, f) := I^{1-\gamma}\|\mathsf{u}_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))}, \tag{7.51}$$

where I^0 is the identity according to Remark 7.16 (case $\gamma = 1$).

Proposition 7.2 (exponential decay) *For every $\gamma \in (0,1]$, $s \in (0,1)$ and $\mathscr{Y} > 1$, the solution $\mathcal{U}$ of (7.16) satisfies*

$$\|\nabla \mathcal{U}\|_{L^2(0,T;L^2(\Omega\times(\mathscr{Y},\infty),y^\alpha))} \lesssim e^{-\sqrt{\lambda_1}\mathscr{Y}/2}\Lambda_\gamma(\mathsf{u}_0, f). \tag{7.52}$$

Proof: Recall from (7.18) that $\mathcal{U}(x,t) = \sum_k \mathsf{u}_k(t)\varphi_k(x')\psi_k(y)$ solves (7.16). Since $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ that satisfies (7.9) we have

$$\int_0^T \int_{\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}}} y^\alpha |\nabla \mathcal{U}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim \int_0^T \sum_{k=1}^\infty \mathsf{u}_k(t)^2 \int_{\mathcal{Y}}^\infty y^\alpha \left(\lambda_k\psi_k(y)^2 + \psi_k'(y)^2\right)\,\mathrm{d}y\,\mathrm{d}t$$
$$= \sum_{k=1}^\infty |\mathcal{Y}^\alpha \psi_k(\mathcal{Y})\psi_k'(\mathcal{Y})| \int_0^T \mathsf{u}_k(t)^2\,\mathrm{d}t.$$

where we have used (7.21). Since $|\mathcal{Y}^\alpha\psi_k(\mathcal{Y})\psi_k'(\mathcal{Y})| \lesssim \lambda_k^s e^{-\sqrt{\lambda_k}\mathcal{Y}}$, according to [129, (2.32)], we deduce

$$\int_0^T \int_{\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}}} y^\alpha |\nabla \mathcal{U}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}} \|\mathsf{u}\|^2_{L^2(0,T;\mathbb{H}^s(\Omega))}.$$

Finally, by setting $\mathcal{V} = \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$ and $\mathcal{H} = L^2(\Omega)$, the estimate (7.52) follows from either (7.47) for $\gamma \in (0,1)$ or (7.48) for $\gamma = 1$. $\square$

As a consequence of Proposition 7.2, we can consider the truncated problem

$$\begin{cases} -\mathrm{div}\,(y^\alpha \mathbf{A}\nabla v) + y^\alpha c v = 0, & \text{in } \mathcal{C}_{\mathcal{Y}},\ t \in (0,T), \\ v = 0, & \text{on } \partial_L\mathcal{C}_{\mathcal{Y}} \cup \Omega\times\{\mathcal{Y}\},\ t\in(0,T), \\ d_s\partial_t^\gamma v + \dfrac{\partial v}{\partial \nu^\alpha} = d_s f, & \text{on } \Omega\times\{0\},\ t\in(0,T), \\ \mathrm{tr}_\Omega\, v(0) = \mathsf{u}_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{7.53}$$

with $\mathcal{Y}$ sufficiently large. In order to obtain a weak formulation of (7.53), we define

$$\mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathcal{Y}}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}_{\mathcal{Y}}) : w = 0 \text{ on } \partial_L\mathcal{C}_{\mathcal{Y}} \cup \Omega\times\{\mathcal{Y}\}\right\},$$

and

$$\mathbb{V}_{\mathcal{Y}} := \{w \in L^2(0,T;\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha)) : \partial_t^\gamma\,\mathrm{tr}_\Omega\, w \in L^2(0,T;\mathbb{H}^{-s}(\Omega))\}. \tag{7.54}$$

Problem (7.53) is understood as follows: seek $v \in \mathbb{V}_{\mathcal{Y}}$ such that, for a.e. $t \in (0,T)$,

$$\langle \partial_t^\gamma\,\mathrm{tr}_\Omega\, v, \mathrm{tr}_\Omega\,\phi\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} + a_{\mathcal{Y}}(v,\phi) = \langle f, \mathrm{tr}_\Omega\,\phi\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \tag{7.55}$$

for all $\phi \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ and $\operatorname{tr}_\Omega v(0) = \mathsf{u}_0$. Here

$$a_\mathcal{Y}(w,\phi) = \frac{1}{d_s} \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \mathbf{A}(x) \nabla w \cdot \nabla \phi + y^\alpha c(x') w \phi. \tag{7.56}$$

Remark 7.19 (initial datum) As in Remark 7.5, we define $v(0) \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ to be the solution of the truncated problem associated with (7.4) with the Neumann condition replaced by the Dirichlet condition $v = \mathsf{u}_0$. The following estimate holds [129, Remark 3.4])

$$\|v(0)\|_{\mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim \|\mathsf{u}_0\|_{\mathbb{H}^s(\Omega)}.$$

Moreover, if $\beta > 2\alpha + 1$, then the proof of Theorem 7.7 yields

$$\|v(0)\|_{\mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\beta)} \lesssim \|\mathsf{u}_0\|_{\mathbb{H}^{2s}(\Omega)}.$$

Lemma 7.20 (exponential convergence) *For every $\gamma \in (0,1]$ and $\mathcal{Y} \geq 1$, we have*

$$I^{1-\gamma} \| \operatorname{tr}_\Omega (\mathcal{U} - v) \|^2_{L^2(\Omega)} + \|\nabla(\mathcal{U} - v)\|^2_{L^2(0,T;L^2(\mathcal{C}_\mathcal{Y}, y^\alpha))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}} \Lambda^2_\gamma(\mathsf{u}_0, f). \tag{7.57}$$

Proof: Let $w(x,t) := \mathcal{U}(x', y, t) - \mathcal{U}(x', \mathcal{Y}, t) \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ be a modification of $\mathcal{U}$ with vanishing trace at $y = \mathcal{Y}$. We observe that w satisfies

$$\begin{aligned}\langle \operatorname{tr}_\Omega \partial_t^\gamma w, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} + a_\mathcal{Y}(w,\phi) &= \langle f, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}\setminus\mathcal{C}_\mathcal{Y}} y^\alpha \nabla \mathcal{U} \nabla \phi \\ &- \langle \operatorname{tr}_\Omega \partial_t^\gamma \mathcal{U}(\cdot, \mathcal{Y}, \cdot), \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - a_\mathcal{Y}(\mathcal{U}(\cdot,\mathcal{Y},\cdot), \phi).\end{aligned}$$

Therefore, the error $e := v - w$ satisfies

$$\begin{aligned}\langle \operatorname{tr}_\Omega \partial_t^\gamma e, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} + a_\mathcal{Y}(e,\phi) &= \int_{\mathcal{C}\setminus\mathcal{C}_\mathcal{Y}} y^\alpha \nabla \mathcal{U} \nabla \phi + a_\mathcal{Y}(\mathcal{U}(\cdot,\mathcal{Y},\cdot), \phi) \\ &+ \langle \operatorname{tr}_\Omega \partial_t^\gamma \mathcal{U}(\cdot, \mathcal{Y}, \cdot), \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}.\end{aligned}$$

Setting $\mathcal{V} = \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ and $\mathcal{H} = L^2(\Omega)$, the assertion is a consequence of Corollary 7.15 for $\gamma < 1$ and Remark 7.16 for $\gamma = 1$, provided we can estimate the right-hand side of the previous expression and $e(\cdot, 0) = \mathcal{U}(\cdot, \mathcal{Y}, 0)$. We estimate the four terms in question separately upon exploiting the expression (7.18), namely

$$\mathcal{U}(x,t) = \sum_{k=1}^{\infty} \mathsf{u}_k(t)\varphi_k(x')\psi_k(y),$$

and Proposition 7.2. We start by noticing that (7.52) implies

$$\left| \int_{\mathcal{C}\setminus\mathcal{C}_\mathcal{Y}} y^\alpha \nabla\mathcal{U}\nabla\phi \right| \leq e^{-\sqrt{\lambda_1}\mathcal{Y}/2}\Lambda_\gamma(\mathsf{u}_0, f)\|\phi\|_{\mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha)}.$$

For the second term, we use $|a_\mathcal{Y}(\mathcal{U}(\cdot,\mathcal{Y},\cdot),\phi)| \lesssim \|\mathcal{U}(\cdot,\mathcal{Y},\cdot)\|_{\mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha)}\|\phi\|_{\mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha)}$, and

$$\|\nabla\mathcal{U}(\cdot,\mathcal{Y},\cdot)\|^2_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)} = \frac{1}{\alpha+1}\sum_{k=1}^{\infty}\lambda_k \mathsf{u}_k^2(t)\mathcal{Y}^{1+\alpha}\psi_k^2(\mathcal{Y}).$$

Now, since $|\psi_k(y)| \lesssim (\sqrt{\lambda_k}y)^s e^{-\sqrt{\lambda_k}y}$ for $y \geq 1$, we easily see that

$$\begin{aligned}\|\nabla\mathcal{U}(\cdot,\mathcal{Y},\cdot)\|^2_{L^2(0,T;L^2(\mathcal{C}_\mathcal{Y},y^\alpha))} &\lesssim \mathcal{Y}^{2(1-s)}\sum_{k=1}^{\infty}\lambda_k\int_0^T \mathsf{u}_k^2(t)\,\mathrm{d}t(\sqrt{\lambda_k}\mathcal{Y})^{2s}e^{-2\sqrt{\lambda_k}\mathcal{Y}} \\ &\lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}}\sum_{k=1}^{\infty}\lambda_k^s\int_0^T \mathsf{u}_k^2(t)\,\mathrm{d}t = e^{-\sqrt{\lambda_1}\mathcal{Y}}\|\mathsf{u}\|^2_{L^2(0,T;\mathbb{H}^s(\Omega))}.\end{aligned}$$

For the third term, we have $\partial_t^\gamma\mathcal{U}(\cdot,\mathcal{Y},t) = \sum_{k=1}^\infty \partial_t^\gamma \mathsf{u}_k(t)\varphi_k\psi_k(\mathcal{Y})$, whence

$$\|\partial_t^\gamma\mathcal{U}(\cdot,\mathcal{Y},t)\|_{\mathbb{H}^{-s}(\Omega)} = \sum_{k=1}^{\infty}|\partial_t^\gamma\mathsf{u}_k(t)|^2\lambda_k^{-s}|\psi_k(\mathcal{Y})|^2 \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}}\sum_{k=1}^{\infty}|\partial_t^\gamma\mathsf{u}_k(t)|^2\lambda_k^{-s}.$$

On the other hand, in light of (7.22), we deduce

$$\sum_{k=1}^{\infty}|\partial_t^\gamma\mathsf{u}_k(t)|^2\lambda_k^{-s} \lesssim \sum_{k=1}^{\infty}\mathsf{u}_k^2(t)\lambda_k^s + f_k^2(t)\lambda_k^{-s} = \|\mathsf{u}(t)\|^2_{\mathbb{H}^s(\Omega)} + \|f(t)\|^2_{\mathbb{H}^{-s}(\Omega)}.$$

Finally,

$$\|\mathcal{U}(\cdot,\mathcal{Y},0)\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty}\mathsf{u}_k^2(0)\psi_k^2(\mathcal{Y}) \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}}\|\mathsf{u}_0\|_{L^2(\Omega)}.$$

Collecting the previous estimates, we deduce

$$I^{1-\gamma}\|\operatorname{tr}_\Omega e\|^2_{L^2(\Omega)} + \|\nabla e\|^2_{L^2(0,T;L^2(\mathcal{C}_\mathcal{Y},y^\alpha))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}}\Lambda^2_\gamma(\mathsf{u}_0,f), \tag{7.58}$$

where we have used the stability bounds (7.47) and (7.48) for u. Moreover, we have

$$I^{1-\gamma}\|\operatorname{tr}_\Omega \mathcal{U}(\cdot,\mathcal{Y},\cdot,)\|^2_{L^2(\Omega)} + \|\mathcal{U}(\cdot,\mathcal{Y},\cdot)\|^2_{L^2(0,T;\mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}}\Lambda^2_\gamma(\mathsf{u}_0,f),$$

which together with (7.58) implies the desired estimate (7.57). $\square$

Finally, as in §7.3, we consider a semi-discrete approximation of (7.55). Given the initialization $\mathcal{V}^0 = \operatorname{tr}_\Omega \mathsf{u}_0$, for $k = 0,\dots,\mathcal{K}-1$, let $\mathcal{V}^{k+1} \in \mathring{H}^1_L(\mathcal{C},y^\alpha)$ solve

$$(\delta^\gamma \operatorname{tr}_\Omega \mathcal{V}^{k+1}, \operatorname{tr}_\Omega W)_{L^2(\Omega)} + a_\mathcal{Y}(\mathcal{V}^{k+1},W) = \langle f^{k+1},W\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \tag{7.59}$$

for all $W \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha)$. The stability of this scheme is a direct consequence of Lemma 7.11 for $\gamma = 1$ and Theorem 7.14 for $\gamma \in (0,1)$. In addition, one can also prove a result analogous to Theorem 7.18, but for brevity we skip these details.

7.4.2 Finite element methods

We follow Chapter 3 but summarize here the main ideas and results. To avoid technical difficulties we assume that the boundary of Ω is polygonal. Let $\mathscr{T}_\Omega = \{K\}$ be a partition, or mesh, of Ω into elements K (simplices or n-rectangles) such that

$$\bar{\Omega} = \bigcup_{K\in\mathscr{T}_\Omega} K, \qquad |\Omega| = \sum_{K\in\mathscr{T}_\Omega} |K|.$$

The mesh $\mathscr{T}_\Omega$ is assumed to be conforming or compatible and shape regular (see [56, Chapter 4.3] and [39, Chapter 4]). The collection of such triangulations is denoted by $\mathbb{T}_\Omega$. If $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$ we define $h_{\mathscr{T}_\Omega} = \max_{K\in\mathscr{T}_\Omega} h_K$.

We define $\mathscr{T}_{\mathcal{Y}}$ to be a partition of $\mathcal{C}_{\mathcal{Y}}$ into cells of the form $T = K \times I$, where $K \in \mathscr{T}_\Omega$, and I denotes an interval in the extended dimension. The intervals $I = (y_{k-1}, y_k)$ form a partition of $(0, \mathcal{Y})$ and, for them, we consider two cases: either they are uniform, i.e., $y_k = k\mathcal{Y}/M$, or graded and given by the formula

$$y_k = \left(\frac{k}{M}\right)^{\mu} \mathcal{Y}, \quad k = 0, \dots, M, \tag{7.60}$$

where $\mu = \mu(\alpha) > 3/(1-\alpha) > 1$. Notice that each discretization of the truncated cylinder $\mathcal{C}_{\mathcal{Y}}$ depends on the truncation parameter $\mathcal{Y}$. The set of all such triangulations is denoted by $\mathbb{T}$. In addition, if we assume that $\mathscr{T}_\Omega$ is shape regular and the partitions in the extended direction are given by (7.60), the following weak regularity condition is valid: there is a constant σ such that, for all $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$, if $T_1 = K_1 \times I_1, T_2 = K_2 \times I_2 \in \mathscr{T}_{\mathcal{Y}}$ have nonempty intersection, then $h_{I_1}/h_{I_2} \leq \sigma$, where $h_I = |I|$; see [70, 129].

The main motivation to consider elements as in (7.60) is to capture the singular behavior of the solution $\mathcal{U}$ of problem (7.16) as $y \approx 0^+$. In fact, it is well known that the numerical approximation of functions with a strong directional-dependent behavior needs anisotropic elements in order to recover (quasi)optimal error estimates. In our setting, anisotropic elements of tensor product structure are essential.

Given $\mathscr{T}_{\mathcal{Y}}$, we call $\mathcal{N}$ the set of its nodes and $\mathcal{N}_{\text{in}}$ the set of its interior and Neumann nodes, and denote by $N = \#\mathcal{N}_{\text{in}}$ the number of degrees of freedom of $\mathscr{T}_{\mathcal{Y}}$. For each vertex $\mathtt{v} \in \mathcal{N}$, we write $\mathtt{v} = (\mathtt{v}', \mathtt{v}'')$, where $\mathtt{v}'$ corresponds to a node of $\mathscr{T}_\Omega$, and $\mathtt{v}''$ corresponds to a node of the discretization of the extended dimension. We define $h_{\mathtt{v}'} = \min\{h_K : \mathtt{v}' \text{ is a vertex of } K\}$, and $h_{\mathtt{v}''} = \min\{h_I : \mathtt{v}'' \text{ is a vertex of } I\}$.

The *star* or patch around $\mathtt{v}$ is the set $S_{\mathtt{v}} = \bigcup_{\mathtt{v} \in T} T$, whereas for $T \in \mathscr{T}_{\mathcal{Y}}$ its *patch* is the set $S_T = \bigcup_{\mathtt{v} \in T} S_{\mathtt{v}}$.

For $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$, we define the finite element space

$$\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in \mathcal{C}^0(\overline{\mathcal{C}_{\mathcal{Y}}}) : W|_T \in \mathcal{P}_1(K) \ \forall T \in \mathscr{T}_{\mathcal{Y}}, \ W|_{\Gamma_D} = 0 \right\},$$

where $\Gamma_D = \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\}$ is called the Dirichlet boundary; the space $\mathcal{P}_1(K)$ corresponds to $\mathbb{P}_1$ for a simplicial element K, and to $\mathbb{Q}_1$ for a n-rectangle K. We also define $\mathbb{U}(\mathscr{T}_{\Omega}) = \operatorname{tr}_{\Omega} \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, i.e., a $\mathcal{P}_1$ finite element space over the mesh $\mathscr{T}_{\Omega}$.

The graded meshes described by (7.60) yield near optimal estimates both in regularity and order for the elliptic case investigated in [129].

7.4.3 Weighted elliptic projector and properties

This subsection is dedicated to the definition of a *weighted elliptic projector*, which is fundamental in the error analysis of the fully-discrete schemes introduced below. This projector is the operator $P_{\mathscr{T}_{\mathcal{Y}}} : \mathring{H}^1_L(\mathcal{C}_{\mathcal{Y}}, y^{\alpha}) \to \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ such that, for $w \in \mathring{H}^1_L(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$, is given by

$$a_{\mathcal{Y}}\left(P_{\mathscr{T}_{\mathcal{Y}}} w, W\right) = a_{\mathcal{Y}}(w, W), \quad \forall W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}}). \tag{7.61}$$

To easily describe the properties of the weighted elliptic projection operator $P_{\mathscr{T}_{\mathcal{Y}}}$ we introduce the mesh-size functions $h', h'' \in L^{\infty}(\mathcal{C}_{\mathcal{Y}})$ given by

$$h'_{|T} = h_K, \quad h''_{|T} = h_I \quad \forall T = K \times I \in \mathscr{T}_{\mathcal{Y}}.$$

We have the following result.

Proposition 7.3 (weighted elliptic projector) *If $w \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$, then the weighted elliptic projector $P_{\mathscr{T}_\mathcal{Y}}$ is stable, i.e.,*

$$\|\nabla P_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim \|\nabla w\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)}. \tag{7.62}$$

If, in addition, $w \in H^2(\mathcal{C}_\mathcal{Y}, y^\alpha)$, then $P_{\mathscr{T}_\mathcal{Y}}$ has the following approximation property

$$\|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim \|h' \partial_{x'} \nabla w\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} + \|h'' \partial_y \nabla w\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)}. \tag{7.63}$$

Proof: To show stability set $W = P_{\mathscr{T}_\mathcal{Y}} w$ in (7.61), use Cauchy-Schwarz inequality and the equivalence of $a_\mathcal{Y}(w, w)$ with $\|\nabla w\|^2_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)}$ (see Remark 7.3).

Obtaining the estimate (7.63) hinges on Galerkin orthogonality, namely

$$\begin{aligned} \|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|^2_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} &\lesssim a_\mathcal{Y}\left(w - P_{\mathscr{T}_\mathcal{Y}} w, w - P_{\mathscr{T}_\mathcal{Y}} w\right) \\ &= a_\mathcal{Y}\left(w - P_{\mathscr{T}_\mathcal{Y}} w, w - \Pi_{\mathscr{T}_\mathcal{Y}} w\right) \end{aligned}$$

where $\Pi_{\mathscr{T}_\mathcal{Y}}$ is the interpolation operator defined in Chapter 3. The result then follows from the anisotropic interpolation estimates of Theorems 3.17 and 3.18 in Chapter 3. $\square$

In order to apply estimate (7.63) to v, solution of problem (7.55), we need $v \in H^2(\mathcal{C}_\mathcal{Y}, y^\alpha)$, which is not a valid assumption. However, as it is explained in Chapter 3, the regularity estimates (7.28) and (7.29), together with the graded mesh (7.60), allow us to capture the singular behavior of v and, consequently, derive near-optimal error estimates. Before we write these estimates we briefly comment on the regularity of v in terms of $\mathcal{U}$.

Remark 7.21 (regularity of v vs $\mathcal{U}$) We recall $w \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ defined in the proof Theorem (7.20): $w(x,t) = \mathcal{U}(x', y, t) - \mathcal{U}(x', \mathcal{Y}, t)$. Applying now the stability of $P_{\mathscr{T}_\mathcal{Y}}$ we obtain

$$
\begin{aligned}
\|\nabla(v - P_{\mathscr{T}_\mathcal{Y}} v)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} &\leq \|\nabla(v - w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} + \|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \\
&\quad + \|\nabla(P_{\mathscr{T}_\mathcal{Y}} w - P_{\mathscr{T}_\mathcal{Y}} v)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \\
&\lesssim \|\nabla(v - w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} + \|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \\
&\lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}} \Lambda_\gamma(\mathsf{u}_0, f) + \|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)},
\end{aligned}
\tag{7.64}
$$

where we have used the estimate for $\|\nabla(v - w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)}$ in the proof of Theorem (7.20). Consequently, the estimate above depends on the regularity of $\mathcal{U}$.

Using the graded mesh (7.60), we derive near-optimal approximation results for the elliptic projector.

Lemma 7.22 (error estimates for the elliptic projector) *Let v be the solution of* (7.55)*, and $P_{\mathscr{T}_\mathcal{Y}}$ the weighted elliptic projector defined in* (7.61)*. Then, given $f \in L^2(\Omega)$, we have the following near optimal estimates*

$$
\|\nabla(v - P_{\mathscr{T}_\mathcal{Y}} v)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim |\log N|^s N^{-1/(n+1)} \|\mathcal{U}(t)\|_{H^2(\mathcal{C}, y^\beta)}, \tag{7.65}
$$

and

$$
\|\operatorname{tr}_\Omega(v - P_{\mathscr{T}_\mathcal{Y}} v)\|_{\mathbb{H}^s(\Omega)} \lesssim |\log N|^s N^{-1/(n+1)} \|\mathcal{U}(t)\|_{H^2(\mathcal{C}, y^\beta)}. \tag{7.66}
$$

Proof: The proof of (7.65) is a direct consequence of estimates (7.63) and (7.64), the regularity estimates (7.30) and Theorem 3.23 in Chapter 3, where the graded mesh (7.60) on the extended variable y is essential to recover near optimality.

The proof of (7.66) is a consequence of the trace estimate (7.15). $\square$

Using the regularity result of Theorem 7.7 we can obtain L^2 approximation properties for the trace of the elliptic projection via duality.

Proposition 7.4 ($L^2(\Omega)$-approximation) *If* $w \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha) \cap H^2(\mathcal{C}_\mathcal{Y}, y^\beta)$ *with* $\beta > 2\alpha + 1$, *and the mesh* $\mathscr{T}_\mathcal{Y}$ *is quasiuniform, then*

$$\| \operatorname{tr}_\Omega(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\Omega)} \lesssim h_{\mathscr{T}_\mathcal{Y}}^{2+\alpha-\beta} \|w\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)}. \tag{7.67}$$

If $w \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha) \cap H^2(\mathcal{C}_\mathcal{Y}, y^\beta)$ *and the mesh is graded as in* (7.60), *then*

$$\| \operatorname{tr}_\Omega(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\Omega)} \lesssim |\log N|^{2s} N^{-2/(n+1)} \|w\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)}. \tag{7.68}$$

Proof: We argue by duality. Let $z \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ solve the adjoint problem

$$a_\mathcal{Y}(\phi, z) = \langle \operatorname{tr}_\Omega(w - P_{\mathscr{T}_\mathcal{Y}} w), \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}, \quad \forall \phi \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha). \tag{7.69}$$

The regularity for z is given in Theorem 7.7: $\|z\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)} \lesssim \| \operatorname{tr}_\Omega(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\Omega)}$.

Set $\phi = w - P_{\mathscr{T}_\mathcal{Y}} w$ in (7.69). By definition of the the elliptic projection we have

$$\begin{aligned} \| \operatorname{tr}_\Omega(w - P_{\mathscr{T}_\mathcal{Y}} w)\|^2_{L^2(\Omega)} &= a_\mathcal{Y}\left(w - P_{\mathscr{T}_\mathcal{Y}} w, z - P_{\mathscr{T}_\mathcal{Y}} z\right) \\ &\lesssim \|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \|\nabla(z - P_{\mathscr{T}_\mathcal{Y}} z)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)}. \end{aligned}$$

It remains to estimate the two terms in the right hand side of this inequality. The approximation result (7.63), together with an improvement over [129, Theorem 5.1] based on Theorem 7.7, allows us to obtain

$$\|\nabla(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim h^\varrho_{\mathscr{T}_\mathcal{Y}} \|w\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)}, \tag{7.70}$$

where $\varrho = (2 + \alpha - \beta)/2$. The regularity estimates for z, together with (7.63), yield

$$\|\nabla(z - P_{\mathscr{T}_\mathcal{Y}} z)\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim h^\varrho_{\mathscr{T}_\mathcal{Y}} \|z\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)} \lesssim h^\varrho_{\mathscr{T}_\mathcal{Y}} \| \operatorname{tr}_\Omega(w - P_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(\Omega)}. \tag{7.71}$$

This implies (7.67) for $\mathscr{T}_{\mathcal{Y}}$ quasiuniform. If $\mathscr{T}_{\mathcal{Y}}$ is graded according to (7.60), then we can prove the following improvement over [129, Theorem 5.4] based on Theorem 7.7:

$$\|\nabla(w - P_{\mathscr{T}_{\mathcal{Y}}} w)\|_{L^2(\mathcal{C}_{\mathcal{Y}},y^\alpha)} \lesssim |\log N|^s N^{-1/(n+1)} \|w\|_{H^2(\mathcal{C}_{\mathcal{Y}},y^\beta)}.$$

Using this estimate in conjunction with the previous argument yields (7.68). $\square$

Remark 7.23 (duality) If the functions w, z satisfy $w, z \in H^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)$ and $\mathscr{T}_{\mathcal{Y}}$ is quasiuniform, then the above analysis gives the usual estimate

$$\| \operatorname{tr}_\Omega(w - P_{\mathscr{T}_{\mathcal{Y}}} w)\|_{L^2(\Omega)} \lesssim h^2_{\mathscr{T}_{\mathcal{Y}}} \|w\|_{H^2(\mathcal{C}_{\mathcal{Y}},y^\alpha)}.$$

7.5 A fully discrete scheme for $\gamma \in (0, 1]$

Let us now describe the fully discrete first order numerical scheme to solve problem (7.55). The discretization in space is given via truncation and the finite element method discussed in §7.4; the discretization in time uses the backward Euler method for $\gamma = 1$, and the finite difference scheme proposed in (7.3.2) for $\gamma \in (0, 1)$.

The scheme computes $V^\tau_{\mathscr{T}_{\mathcal{Y}}} \subset \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, an approximation of the solution to problem (7.55) at each time step. We initialize the scheme by setting

$$\operatorname{tr}_\Omega V^0_{\mathscr{T}_{\mathcal{Y}}} = P_{\mathscr{T}_\Omega} \mathsf{u}_0, \tag{7.72}$$

where $P_{\mathscr{T}_\Omega}$ denotes an appropriate interpolation or projection operator into the space $\mathbb{U}(\mathscr{T}_\Omega)$; we let $\mathfrak{e}_{\mathscr{T}_\Omega}(\mathsf{u}_0) = \|\mathsf{u}_0 - P_{\mathscr{T}_\Omega}\mathsf{u}_0\|_{L^2(\Omega)}$. Notice that the initial datum u_0 is approximated in the space $\mathbb{U}_{\mathscr{T}_\Omega}$ via the operator $P_{\mathscr{T}_\Omega}$, so no extension is needed.

We define a *first order fully-discrete scheme* to approximate the solution of (7.55) as follows: given $V^0_{\mathscr{T}_{\mathcal{Y}}}$ satisfying (7.72), for $k = 0, \ldots, \mathcal{K}-1$, let $V^{k+1}_{\mathscr{T}_{\mathcal{Y}}} \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$

solve

$$(\delta^\gamma \operatorname{tr}_\Omega V_{\mathscr{T}_\mathcal{Y}}^{k+1}, \operatorname{tr}_\Omega W)_{L^2(\Omega)} + a_\mathcal{Y}(V_{\mathscr{T}_\mathcal{Y}}^{k+1}, W) = \left\langle f^{k+1}, \operatorname{tr}_\Omega W \right\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \tag{7.73}$$

for all $W \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$, where the discrete operator δ^γ is defined in (7.43) for $\gamma \in (0,1)$ and by (7.32) for $\gamma = 1$. To obtain an approximation of the solution u to problem (7.1) we define the sequence $U_{\mathscr{T}_\Omega}^\tau \subset \mathbb{V}(\mathscr{T}_\Omega)$ by

$$U_{\mathscr{T}_\Omega}^\tau = \operatorname{tr}_\Omega V_{\mathscr{T}_\mathcal{Y}}^\tau. \tag{7.74}$$

Remark 7.24 (dynamic condition) Problem (7.72)-(7.73) is a discrete elliptic problem with a dynamic boundary condition. Consequently, its stability and error analyses are slightly different than the standard theory for the heat equation.

Remark 7.25 (computational efficiency) The main advantage of scheme (7.73) is that $U_{\mathscr{T}_\Omega}^\tau$ is obtained as an approximation of the local problem (7.55). The numerical scheme is simple to implement and is such that multilevel methods can be designed with complexity proportional to N; see Chapter 5.

We have the following unconditional stability result.

Lemma 7.26 (unconditional stability) *The discrete scheme* (7.72)-(7.73) *is unconditionally stable for all* $\gamma \in (0,1]$, *i.e.,*

$$I^{1-\gamma}\|\operatorname{tr}_\Omega V_{\mathscr{T}_\mathcal{Y}}^\tau\|_{L^2(\Omega)}^2 + \|V_{\mathscr{T}_\mathcal{Y}}^\tau\|_{\ell^2(\mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha))}^2 \lesssim I^{1-\gamma}\|\mathsf{u}_0\|_{L^2(\Omega)}^2 + \|f^\tau\|_{\ell^2(\mathbb{H}^{-s}(\Omega))}^2. \tag{7.75}$$

Proof: Set $W = 2\tau V_{\mathscr{T}_\mathcal{Y}}^{k+1}$ for $\gamma = 1$ and $W = 2\Gamma(2-\gamma)\tau^\gamma V_{\mathscr{T}_\mathcal{Y}}^{k+1}$ for $0 < \gamma < 1$ in (7.73) and proceed as in Lemma 7.11 and Theorem 7.14, respectively. $\square$

Let us now obtain an error estimate for the fully discrete scheme (7.73). This estimate relies on the properties of the elliptic projector studied in §7.4.3. We split the error into the so-called interpolation and approximation errors [75, 157]:

$$v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}} = \left(v^\tau - P_{\mathscr{T}_\mathcal{Y}} v^\tau\right) + \left(P_{\mathscr{T}_\mathcal{Y}} v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}}\right) = \eta^\tau + E^\tau_{\mathscr{T}_\mathcal{Y}}.$$

Property (7.65) of the elliptic projection implies that η^τ is controlled near-optimally in energy

$$\|\nabla \eta^\tau\|_{\ell^2(L^2(\mathcal{C}_\mathcal{Y}, y^\alpha))} \lesssim |\log N|^s N^{-1/(n+1)} \|v^\tau\|_{\ell^2(H^2(\mathcal{C}_\mathcal{Y}, y^\beta))}, \tag{7.76}$$

and, by (7.68), we have super-approximation in the L^2-norm of the trace

$$I^{1-\gamma} \| \operatorname{tr}_\Omega \eta^\tau \|_{L^2(\Omega)} \lesssim |\log N|^{2s} N^{-2/(n+1)} I^{1-\gamma} \|v^\tau\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)}. \tag{7.77}$$

Therefore, to obtain an error estimate it suffices to bound $E^\tau_{\mathscr{T}_\mathcal{Y}}$. To do that, given a function w, let us introduce

$$\Phi^k(w) = \frac{1}{\tau} \int_{t^k}^{t^{k+1}} \|\partial_t w(s)\|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)} \, \mathrm{d}s, \qquad \forall k = 1, \dots, \mathcal{K} - 1 \tag{7.78}$$

and denote

$$\mathfrak{C} = \mathfrak{C}(v, \mathsf{u}_0, f, \gamma) = I^{1-\gamma} \|v^\tau\|^2_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)} + I^{1-\gamma} \|\mathsf{u}_0\|^2_{\mathbb{H}^{2s}(\Omega)} + \|\Phi^\tau(v)\|^2_{\ell^2},$$

and

$$\mathfrak{D} = \mathfrak{D}(v, \mathsf{u}_0, f, \gamma) = \|v\|^2_{\ell^2(H^2(\mathcal{C}_\mathcal{Y}, y^\beta))} + I^{1-\gamma} \|\mathsf{u}_0\|^2_{\mathbb{H}^{2s}(\Omega)} + \|\Phi^\tau(v)\|^2_{\ell^2}.$$

With this notation the error estimates for scheme (7.72)-(7.73) read as follows.

Theorem 7.27 (error estimates) *Let* $\gamma \in (0,1]$, v *and* $V^\tau_{\mathscr{T}_\mathscr{Y}}$ *solve* (7.55) *and* (7.72)-(7.73), *respectively. If* $\mathscr{T}_\mathscr{Y}$ *is graded according* (7.60), *then we have*

$$I^{1-\gamma}\|\operatorname{tr}_\Omega(v^\tau - V^\tau_{\mathscr{T}_\mathscr{Y}})\|^2_{L^2(\Omega)} \lesssim I^{1-\gamma}\mathfrak{e}^2_{\mathscr{T}_\Omega}(\mathsf{u}_0) + \tau^{2(2-\gamma)}\|\operatorname{tr}_\Omega \partial_{tt} v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{C}|\log N|^{4s} N^{\frac{-4}{n+1}}, \tag{7.79}$$

and

$$\|v^\tau - V^\tau_{\mathscr{T}_\mathscr{Y}}\|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C}_\mathscr{Y},y^\alpha))} \lesssim I^{1-\gamma}\mathfrak{e}^2_{\mathscr{T}_\Omega}(\mathsf{u}_0) + \tau^{2(2-\gamma)}\|\operatorname{tr}_\Omega \partial_{tt} v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{D}|\log N|^{2s} N^{\frac{-2}{n+1}}. \tag{7.80}$$

Proof: Using the continuous problem (7.55), the discrete equation (7.73), and the definition (7.61) of the weighted elliptic projector $P_{\mathscr{T}_\mathscr{Y}}$, we arrive at the equation that controls the error,

$$(\delta^\gamma \operatorname{tr}_\Omega E^{k+1}_{\mathscr{T}_\mathscr{Y}}, \operatorname{tr}_\Omega W)_{L^2(\Omega)} + a_\mathscr{Y}(E^{k+1}_{\mathscr{T}_\mathscr{Y}}, W) = \left\langle \operatorname{tr}_\Omega \omega^{k+1}, \operatorname{tr}_\Omega W\right\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \tag{7.81}$$

for all $W \in \mathbb{V}(\mathscr{T}_\mathscr{Y})$, where $\omega^{k+1} = \delta^\gamma P_{\mathscr{T}_\mathscr{Y}} v(t^{k+1}) - \partial^\gamma_t v(t^{k+1})$. The stability estimate (7.75) applied to (7.81) yields

$$I^{1-\gamma}\|\operatorname{tr}_\Omega E^\tau_{\mathscr{T}_\mathscr{Y}}\|^2_{L^2(\Omega)} + \|E^\tau_{\mathscr{T}_\mathscr{Y}}\|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C}_\mathscr{Y},y^\alpha))} \lesssim I^{1-\gamma}\|\operatorname{tr}_\Omega E^0_{\mathscr{T}_\mathscr{Y}}\|^2_{L^2(\Omega)} + \|\operatorname{tr}_\Omega \omega^\tau\|^2_{\ell^2(\mathbb{H}^{-s}(\Omega))},$$

for all $\gamma \in (0,1]$. Now, (7.68) together with (7.72) implies

$$\begin{aligned}\|\operatorname{tr}_\Omega E^0_{\mathscr{T}_\mathscr{Y}}\|_{L^2(\Omega)} &\le \|\operatorname{tr}_\Omega(P_{\mathscr{T}_\mathscr{Y}} v(0) - v(0))\|_{L^2(\Omega)} + \|\operatorname{tr}_\Omega v(0) - P_{\mathscr{T}_\Omega}\mathsf{u}_0\|_{L^2(\Omega)} \\ &\lesssim |\log N|^{2s} N^{-2/(n+1)}\|v(0)\|_{H^2(\mathcal{C}_\mathscr{Y},y^\beta)} + \|\mathsf{u}_0 - P_{\mathscr{T}_\Omega}\mathsf{u}_0\|_{L^2(\Omega)}.\end{aligned}$$

Remark 7.19 implies $\|v(0)\|_{H^2(\mathcal{C}_\mathscr{Y},y^\beta)} \lesssim \|\mathsf{u}_0\|_{\mathbb{H}^{2s}(\Omega)}$, whence

$$I^{1-\gamma}\|\operatorname{tr}_\Omega E^0_{\mathscr{T}_\mathscr{Y}}\|^2_{L^2(\Omega)} \lesssim |\log N|^{4s} N^{-4/(n+1)} I^{1-\gamma}\|\mathsf{u}_0\|^2_{\mathbb{H}^{2s}(\Omega)} + I^{1-\gamma}\mathfrak{e}^2_{\mathscr{T}_\Omega}(\mathsf{u}_0).$$

To control the term involving ω^τ we decompose it as

$$\omega^{k+1} = \left(\partial_t^\gamma v(t^{k+1}) - \delta^\gamma v(t^{k+1})\right) + \delta^\gamma \left(v(t^{k+1}) - P_{\mathscr{T}_\mathcal{Y}} v(t^{k+1})\right) := \omega_1^{k+1} + \omega_2^{k+1}.$$

The first term is controlled by using Proposition 7.1

$$\| \operatorname{tr}_\Omega \omega_1^\tau \|_{\ell^2(\mathbb{H}^{-s}(\Omega))} \lesssim \tau^{2-\gamma} \| \operatorname{tr}_\Omega \partial_{tt} v \|_{L^2(0,T;\mathbb{H}^{-s}(\Omega))}.$$

To deal with ω_2^{k+1} we utilize (7.43) to write

$$\omega_2^{k+1} = \frac{1}{\Gamma(2-\gamma)} \sum_{j=0}^{k} \frac{a_j}{\tau^\gamma} \int_{t^{k-j}}^{t^{k+1-j}} (I - P_{\mathscr{T}_\mathcal{Y}}) \partial_t v(s) \, \mathrm{d}s,$$

and estimate this as follows

$$\| \operatorname{tr}_\Omega \omega_2^{k+1} \|_{\mathbb{H}^{-s}(\Omega)} \lesssim \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} |\log N|^{2s} N^{\frac{-2}{n+1}} \sum_{j=0}^{k} a_j \int_{t^{k-j}}^{t^{k+1-j}} \| \partial_t v(s) \|_{H^2(\mathcal{C}_\mathcal{Y}, y^\beta)} \, \mathrm{d}s$$

because of (7.68) in Proposition 7.4. In view of definition (7.78) of Φ and (7.8) of the fractional integral, as well as the fact that all terms in the sum are positive, we get

$$\begin{aligned} \| \operatorname{tr}_\Omega \omega_2^{k+1} \|_{\mathbb{H}^{-s}(\Omega)} &\lesssim \frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)} |\log N|^{2s} N^{\frac{-2}{n+1}} \sum_{j=0}^{k} a_j \Phi^{k-j}(v) \\ &= |\log N|^{2s} N^{\frac{-2}{n+1}} (I^{1-\gamma} \Phi^\tau(v))(t^k). \end{aligned}$$

Using the continuity of $I^{1-\gamma}$ from $L^2(0,T)$ into itself (Corollary 7.2), we deduce

$$\| \operatorname{tr}_\Omega \omega_2^\tau \|_{\ell^2(\mathbb{H}^{-s}(\Omega))} \lesssim |\log N|^{2s} N^{\frac{-2}{n+1}} \| \Phi^\tau(v) \|_{\ell^2}.$$

Collecting all the previous estimates together with (7.76) and (7.77), allows us to obtain the desired results. $\square$

Remark 7.28 (smooth initial data) If $\mathsf{u}_0 \in H^2(\Omega)$, then we can take $P_{\mathscr{T}_\Omega}$ in (7.72) to be the quasi-interpolation operator introduced in [129, 132], which yields the error estimate $\mathfrak{e}^2_{\mathscr{T}_\Omega}(\mathsf{u}_0) \lesssim N^{-2/(n+1)}\|\mathsf{u}_0\|_{H^2(\Omega)}$. In this case, the estimates (7.79) and (7.80) read

$$I^{1-\gamma}\|\operatorname{tr}_\Omega(v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}})\|^2_{L^2(\Omega)} \lesssim \tau^{2(2-\gamma)}\|\partial_{tt}v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{C}|\log N|^{4s}N^{\frac{-4}{n+1}},$$

and

$$\|v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}}\|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha))} \lesssim \tau^{2(2-\gamma)}\|\partial_{tt}v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{D}|\log N|^{2s}N^{\frac{-2}{n+1}}.$$

where the term $I^{1-\gamma}\|\mathsf{u}_0\|_{\mathbb{H}^{2s}(\Omega)}$ in $\mathfrak{C}$ and $\mathfrak{D}$ is replaced by $I^{1-\gamma}\|\mathsf{u}_0\|_{H^2(\Omega)}$.

Remark 7.29 (limiting case $\gamma = 1$) In the framework of Remark 7.28, if $\gamma = 1$, we recover the standard error estimates for the heat equation (see [75, 157])

$$\|\operatorname{tr}_\Omega(v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}})\|^2_{\ell^\infty(L^2(\Omega))} \lesssim \tau^2\|\partial_{tt}v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{C}|\log N|^{4s}N^{\frac{-4}{n+1}},$$

and

$$\|v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}}\|^2_{\ell^2(\mathring{H}^1_L(\mathcal{C}_\mathcal{Y},y^\alpha))} \lesssim \tau^2\|\partial_{tt}v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{D}|\log N|^{2s}N^{\frac{-2}{n+1}},$$

where $\mathfrak{C} = \mathfrak{D} = \|v^\tau\|^2_{\ell^2(H^2(\mathcal{C}_\mathcal{Y},y^\beta))} + \|\mathsf{u}_0\|^2_{H^2(\Omega)} + \|\partial_t v^\tau\|^2_{\ell^2(H^2(\mathcal{C}_\mathcal{Y},y^\beta))}$.

Remark 7.30 (Estimate for u) In the framework of Remark 7.28 and in view of the estimates (7.79) and (7.80), we deduce the following error estimates for u

$$I^{1-\gamma}\|\mathsf{u}^\tau - U^\tau\|^2_{L^2(\Omega)} \lesssim \tau^{2(2-\gamma)}\|\partial_{tt}v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{C}|\log N|^{4s}N^{\frac{-4}{n+1}},$$

and

$$\|\mathsf{u}^\tau - U^\tau\|^2_{\ell^2(\mathbb{H}^s(\Omega))} \lesssim \tau^{2(2-\gamma)}\|\partial_{tt}v\|^2_{L^2(0,T;\mathbb{H}^{-s}(\Omega))} + \mathfrak{D}|\log N|^{2s}N^{\frac{-2}{n+1}}.$$

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