

ABSTRACT

Title of dissertation: ON LAGRANGIAN FORMULATION
OF HIGHER SUPERSPIN IRREDUCIBLE
REPRESENTATIONS OF THE
SUPER-POINCARÉ GROUP

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Superspace techniques are used to construct lagrangians that describe massless and massive irreducible representations of the Super-Poincaré group. For the massless case we write the superspace action for an arbitrary superhelicity in terms of prepotentials. We use that to define components, derive the component action and the supersymmetric transformation laws for these components. The case of massive representations is examined by starting with the massless superspace action and adding mass corrections along with extra auxiliary superfields if necessary. In this manner we discover a new action for massive supergravity ($Y = 3/2$). In classical spin theory it is well understood how the massless states are organised to generate massive ones. Such a feature has been demonstrated for supersymmetric theories only on-shell and for the component formulation. We derive superspace actions that demonstrate this property for the case of superspin $Y = \frac{1}{2}$ and $Y = 1$.

ON LAGRANGIAN FORMULATION OF HIGHER SUPERSPIN
IRREDUCIBLE REPRESENTATIONS OF THE
SUPER-POINCARÉ GROUP

by

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Dedication

Αφιερωμένο στην Ανθή, στην Μαρία και στον Μίμη. Η αγάπη και η υποστήριξη τους αποδείχθηκε ανεκτίμητη για την εκπόνηση αυτής της διατριβής.

To my family. Their love and support proved to be invaluable.

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Chapter 1: Introduction

Higher spin field theory has a very rich history driving the developments of modern theoretical physics and after many decades still remains a very active subject. It started with Dirac [1] trying to generalize his celebrated spin- $\frac{1}{2}$ equation. His comment in that paper “the underlying theory is of considerable interest” still resonates. After the classical work by Fierz and Pauli [2] there was an increasing number of papers formulating the theory of a massive arbitrary spin in four dimensions [3, 4] as well as developments for the massless arbitrary helicities using the principle of gauge invariance [5, 6]. Since then there has been tremendous progress with generalizations of these results regarding irreducible representations of the little group in D -dimensions [7], derivations of the massive theories by means of dimensional reduction of the massless theories in $D + 1$ -dimensions [8], Stückelberg formulations [9], BRST [10], quantization and many other things.

Nowadays the interest in higher spin theories mostly has been generated from superstring theory. The spectrum of superstring theory includes an infinite tower of massive spin states. Therefore it is natural to expect that there is a field theory limit where superstring theory is formulated as an effective field theory of interacting spins. So in order to better understand the complexity of superstring theory one is

motivated to study higher spin supersymmetric field theories.

There are two approaches in studying supersymmetric theories. There is the field theory method, called component formulation. In this approach the theory is written in terms of fields, it is not manifestly supersymmetric and one has to check the invariance of the action and the closure of the algebra. The other method is the superspace formulation. The theory is staged in superspace and it is expressed in terms of superfields. In this approach supersymmetry is evident. There is a way to connect the two approaches, by starting with a superspace formulation and then recovering the corresponding component formulation.

This thesis will focus on both approaches. For example in chapter 3 we use superspace to describe massless irreducible representations of the Super-Poincaré group and then we present a new technique to define components, find the component action and the component transformation laws. Superspace constructions for massless theories exist [11, 12] already, but they are written in terms of constrained superfields. For our projection method to work we need to have the action written in terms of unconstrained superfields, called prepotentials. For that purpose we reconstruct the superspace actions in terms of free superfields. We use representation theory to determine the appropriate kind of superfields we need for the description of the massless representations and then we use them to build the massless action. We define the massless theory as the $m \rightarrow 0$ limit of the corresponding massive theory. This definition will force us to introduce redundancies (gauge symmetry) and it will uniquely determine the form of the gauge transformation. After the formulation of the superspace theory in terms of the new variables is complete, we use

the equations of motion and their properties, like the Bianchi identities, to define the auxiliary component structure of the theory. Then we let the superspace action guide us to the definition of the dynamical components. Having the entire component spectrum of the various massless theories we can do a counting of the off-shell degrees of freedom involved. That information will be useful because it will provide hints about whether theories that describe the same physical system are equivalent or not and if two $\mathcal{N} = 1$ theories can be combined to give an $\mathcal{N} = 2$ representation

In chapter 4 we discuss massive irreducible representations of the Super-Poincaré group. The massive problem is still unsolved for the arbitrary superspin case. For that reason we study case by case starting from small superspin values to higher ones, trying to build intuition and understand the mechanisms involved. Once again our derivation of the massless theories in terms of the unconstrained superfields seems to be relevant, because these are the objects that describe the physical degrees of freedom for the massive case. Our strategy is to start with the massless action (which was defined as the massless limit of the massive action) and then add mass corrections. In this way we get an action and a set of equations of motion that we use to determine all coefficients so the final theory describes the massive irreducible representation. If that is not possible then we introduce auxiliary superfields (in a way that in the massless limit they decouple) and repeat the process. In this manner we manage to provide a new superspace action that describes the superspin $Y = 3/2$ theory. This supermultiplet is very important because its spin content is $j = 2, j = 3/2, j = 3/2, j = 1/2$. It includes a massive state of spin 2. It is a well known fact that closed superstring theories when truncated to

four dimensions, must have a massive spin 2 state. So this new action can be used to make contact with the effective, low energy, field theoretic limit of superstring theory. Also it sheds some light on the underlying structure of auxiliary superfields required to describe massive supermultiplets. In addition we explore the idea of writing massive theories as a direct sum of massless theories. It is a well known and well understood feature of classical spin theory [9,13]. It has been demonstrated for supersymmetric theories only on-shell and in a component formulation [14]. We provide a set of new superspace actions that illustrate that feature completely off-shell for superspins $Y = 1/2, 1$.

The main results of this thesis are

1. A method for discovering the off-shell component structure of a supersymmetric theory
2. Application of this method to the three distinct classes of arbitrary superhelicity representations (one for integer superhelicity and two for half-integer) will give us results that have not appeared in the literature before:
 - (a) the number of the degrees of freedom in each theory (information needed for comparing theories and doing higher \mathcal{N} constructions)
 - (b) the component action
 - (c) the supersymmetric transformation laws for the components
3. A superspace action for massive supergravity (superspin $Y = \frac{3}{2}$)
4. A superspace action for massive gravitino $Y = 1$

5. A superspace action for massive gravitino $Y = 1$ in terms of massless $Y = 1$, $Y = \frac{1}{2}$, $Y = 0$
6. A superspace action for the massive vector multiplet (superspin $Y = \frac{1}{2}$) in terms of massless superhelicities $Y = \frac{1}{2}$ and $Y = 0$ (à la Stüeckelberg)

A few other results are worth mentioning:

1. A derivation of the appropriate gauge symmetry for the description of each class of massless representations (3.9, 3.61, 3.96)
2. The expression of the superspace actions for the massless representations in terms of unconstrained prepotentials (3.14, 3.64, 3.99)
3. The Bianchi identities (3.21, 3.66, 3.102)

The thesis start with a review of the mathematical tools and concepts that will be used. The review material is mostly based on [15–18]. The conventions that will be used along the thesis are the conventions of [15].

Chapter 2: Mathematical Background

This chapter presents the mathematical tools and concepts that will be used in the rest of the thesis. The focus points of this chapter are superspace, superfields and representation theory of the Super-Poincaré group. Nevertheless this review attempts to provide the minimum but complete material required so the interested reader can go through the technical chapters.

2.1 Clifford, Poincaré and left - right spinors

Consider the $\mathcal{C}_{(1,3)}$ Clifford algebra

$$e_m e_n + e_n e_m = 2\eta_{mn} \mathbb{I} \quad (2.1)$$

Representations of this algebra are the well known Γ_m matrices, $\Gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \bar{\sigma}_m & 0 \end{pmatrix}$ with $\sigma_m = (\mathbb{I}_{2 \times 2}, \vec{\sigma})$ and $\bar{\sigma}_m = (-\mathbb{I}_{2 \times 2}, \vec{\sigma})$. The 16-dimensional space can be spanned by the basis $\{\mathbb{I}_{4 \times 4}, \Gamma_m, \Gamma_{mn}, \Gamma \Gamma_m, \Gamma\}$ where $\Gamma = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ which squares to one and $\{\Gamma, \Gamma_m\} = 0$, $[\Gamma, \Gamma_{mn}] = 0$. It is easy to check that the objects $\{\Gamma_m^\dagger, \Gamma_m^*, \Gamma^T, -\Gamma_m\}$ satisfy the same algebra and since they have the same dimensionality as Γ_m , we expect them to be isomorphic. This means that there are

matrices A , B , C , D such that:

$$\begin{aligned}
A_i \Gamma_m A_i^{-1} &= \eta_{A_i} \Gamma_m^\dagger \\
B_i \Gamma_m B_i^{-1} &= \eta_{B_i} \Gamma_m^* \\
C_i \Gamma_m C_i^{-1} &= \eta_{C_i} \Gamma_m^T \\
D_i \Gamma_m D_i^{-1} &= \eta_{D_i} (-\Gamma_m)
\end{aligned} \tag{2.2}$$

where the η 's are phases. The index i takes two values and it is there because for each case we can find two different matrices that satisfy the equation above.

Explicitly

$$\begin{aligned}
A_1 &= \text{product of all hermitian} = \Gamma_1 \Gamma_2 \Gamma_3, \quad \eta_{A_1} = 1 \\
A_2 &= \text{product of all anti-hermitian} = \Gamma_0, \quad \eta_{A_2} = -1 \\
B_1 &= \text{product of all real} = \Gamma_0 \Gamma_1 \Gamma_3, \quad \eta_{B_1} = 1 \\
B_2 &= \text{product of all imaginary} = \Gamma_2, \quad \eta_{B_2} = -1 \\
C_1 &= \text{product of all symmetric} = \Gamma_1 \Gamma_3, \quad \eta_{C_1} = -1 \\
C_2 &= \text{product of all antisymmetric} = \Gamma_0 \Gamma_2, \quad \eta_{C_2} = 1 \\
D_1 &= \Gamma, \quad \eta_{D_1} = -1 \quad , \quad D_2 = \mathbb{I}, \quad \eta_{D_2} = -1
\end{aligned} \tag{2.3}$$

Because of the Clifford property, the object $\Sigma_{mn} = -\frac{i}{4} [\Gamma_m, \Gamma_n]$ satisfies the Lorentz algebra

$$i [\Sigma_{mn}, \Sigma_{rs}] = \eta_{ms} \Sigma_{nr} - \eta_{mr} \Sigma_{ns} - \eta_{ns} \Sigma_{mr} + \eta_{nr} \Sigma_{ms} \tag{2.4}$$

and together with Γ_m they complete the Poincaré algebra

$$i [\Sigma_{mn}, \Gamma_r] = \eta_{nr} \Gamma_m - \eta_{mr} \Gamma_n \quad (2.5)$$

The isomorphisms for the Γ 's give the isomorphisms for Σ_{mn}

$$A_i \Sigma_{mn} A_i^{-1} = \Sigma_{mn}^\dagger \quad , \quad B_i \Sigma_{mn} B_i^{-1} = -\Sigma_{mn}^*$$

$$C_i \Sigma_{mn} C_i^{-1} = -\Sigma_{mn}^T, \quad D_i \Sigma_{mn} D_i^{-1} = \Sigma_{mn}$$

The $\{\Sigma_{mn}^\dagger, -\Sigma_{mn}^*, -\Sigma_{mn}^T\}$ satisfy the same algebra and correspond to different representations called complex dual, complex and dual respectively [19, 20]. These representations are related with the initial one through the matrices A , B , C . For example, if V belongs in a representation \mathcal{S} of the Lorentz group then:

$$V' = U(\omega)V \Rightarrow V'^T C_i = V^T C_i U^{-1}(\omega) \quad (2.6)$$

$V^T C_i$ belongs in the dual representation and therefore we can use C_i to define scalar products like $V^T C_i V$.

Γ can be diagonalized. Because it squares to one it has eigenvalues (chirality) ± 1 and the corresponding eigenvectors can be written in terms of the two projection operators $P_\pm = \frac{1}{2} (\mathbb{I} \pm \Gamma)$

$$\Psi_L = P_+ \Psi \quad (left) \quad , \quad \Psi_R = P_- \Psi \quad (right)$$

with Ψ a general vector in the vector space on which the Γ 's act (spinors). Each one of these spaces has a dimensionality of 2 and therefore we assign an index α , $\dot{\alpha} = 1, 2$ to each eigenvector $\Psi_L \equiv \Psi_{L\alpha}$, $\Psi_R \equiv \Psi_{R\dot{\alpha}}$. Furthermore, since representations of Clifford algebra can build representations for the Poincaré algebra (through the Σ_{mn}

construction) each one of the left-right spaces is an irreducible representation of the Poincaré group

$$\begin{aligned} [\mathcal{J}_{mn}, \Psi_{L\alpha}] &= i(\sigma_{mn})_{\alpha}{}^{\beta} \Psi_{L\beta} \\ [\mathcal{J}_{mn}, \Psi_{R\dot{\alpha}}] &= i(\bar{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \Psi_{R\dot{\beta}} \end{aligned} \quad (2.7)$$

where $\sigma_{mn} \equiv \frac{1}{4}(\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m)$, $\bar{\sigma}_{mn} \equiv \frac{1}{4}(\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m)$. The index structure of the σ 's is $(\sigma_m)_{\alpha\dot{\alpha}}$, $(\bar{\sigma}_m)^{\dot{\alpha}\alpha}$ and the standard matrix multiplication rules apply.

As mentioned above, the scalar product can be defined as:

$$I = V^T C_i V = \begin{pmatrix} u^T & v^T \end{pmatrix} \begin{pmatrix} C_i^L & 0 \\ 0 & C_i^R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = u^T C_i^L u + v^T C_i^R v$$

so for the left $I^L = u_{\alpha} C_i^{L\alpha\beta} u_{\beta} = \left(-C_i^{L\beta\alpha}\right) u_{\alpha} u_{\beta}$ (C_i^L is antisymmetric).

We can define the following:

$$C_i^{\beta\alpha} \equiv -C_i^{L\beta\alpha} \quad (2.8)$$

$$u^{\beta} \equiv C_i^{\beta\alpha} u_{\alpha} \quad (\text{this is how we raise indices})$$

and the expression for the scalar product is $I^L = u^{\alpha} u_{\alpha}$

Similarly for the right space: $I^R = v^{\dot{\alpha}} C_{i\dot{\alpha}\dot{\beta}}^R v^{\dot{\beta}} = v^{\dot{\alpha}} v^{\dot{\beta}} \left(-C_{i\dot{\beta}\dot{\alpha}}^R\right)$

$$C_{i\dot{\beta}\dot{\alpha}} \equiv -C_{i\dot{\beta}\dot{\alpha}}^R \quad (2.9)$$

$$v_{\dot{\beta}} \equiv v^{\dot{\alpha}} C_{i\dot{\alpha}\dot{\beta}} \quad (\text{this is how we lower indices})$$

and $I^R = v^{\dot{\alpha}} v_{\dot{\alpha}}$. There are two choices for the metric C_i that will raise or lower the indices, one for each index value ($i = 1, 2$). It's a matter of choice and we choose to

work with $i = 2$ so we get

$$C^{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = C^{\dot{\alpha}\dot{\beta}}, \quad C_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = C_{\dot{\alpha}\dot{\beta}} \quad (2.10)$$

They have the property that

$$\begin{aligned} C^{\alpha\beta} C_{\gamma\rho} &= \delta^\alpha_\gamma \delta^\beta_\rho - \delta^\alpha_\rho \delta^\beta_\gamma \\ C^{\dot{\alpha}\dot{\beta}} C_{\dot{\gamma}\dot{\rho}} &= \delta^{\dot{\alpha}}_{\dot{\gamma}} \delta^{\dot{\beta}}_{\dot{\rho}} - \delta^{\dot{\alpha}}_{\dot{\rho}} \delta^{\dot{\beta}}_{\dot{\gamma}} \end{aligned} \quad (2.11)$$

To list a few other very useful properties

$$\begin{aligned} (\sigma_m)_{\alpha\dot{\alpha}} (\bar{\sigma}_n)^{\dot{\alpha}\beta} + (\sigma_n)_{\alpha\dot{\alpha}} (\bar{\sigma}_m)^{\dot{\alpha}\beta} &= 2\eta_{mn} \delta_\alpha^\beta \\ (\bar{\sigma}_m)^{\dot{\alpha}\alpha} (\sigma_n)_{\alpha\dot{\beta}} + (\bar{\sigma}_n)^{\dot{\alpha}\alpha} (\sigma_m)_{\alpha\dot{\beta}} &= 2\eta_{mn} \delta^{\dot{\alpha}}_{\dot{\beta}} \\ (\sigma^m)_{\alpha\dot{\alpha}} (\bar{\sigma}_m)^{\dot{\beta}\beta} &= 2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} \\ \frac{1}{2}\epsilon^{klmn} \sigma_{mn} &= -i\sigma^{kl}, \quad \frac{1}{2}\epsilon^{klmn} \bar{\sigma}_{mn} = i\bar{\sigma}^{kl} \\ \sigma^k \bar{\sigma}^{ln} &= \frac{1}{2}\eta^{lk} \sigma^n - \frac{1}{2}\eta^{nk} \sigma^l - \frac{i}{2}\epsilon^{klmn} \sigma_m \\ \bar{\sigma}^k \sigma^{ln} &= \frac{1}{2}\eta^{lk} \bar{\sigma}^n - \frac{1}{2}\eta^{nk} \bar{\sigma}^l + \frac{i}{2}\epsilon^{klmn} \bar{\sigma}_m \\ \sigma^{ln} \sigma^k &= -\frac{1}{2}\eta^{lk} \sigma^n + \frac{1}{2}\eta^{nk} \sigma^l - \frac{i}{2}\epsilon^{klmn} \sigma_m \\ \bar{\sigma}^{ln} \bar{\sigma}^k &= -\frac{1}{2}\eta^{lk} \bar{\sigma}^n + \frac{1}{2}\eta^{nk} \bar{\sigma}^l + \frac{i}{2}\epsilon^{klmn} \bar{\sigma}_m \end{aligned} \quad (2.12)$$

The $(\sigma_m)_{\alpha\dot{\alpha}}$, $(\bar{\sigma}_m)^{\dot{\alpha}\alpha}$ are the only objects that have all three different kinds of indices. For this reason they are very convenient for converting vector indices to left-right indices and vice versa. For example: $A_m = (\bar{\sigma}_m)^{\dot{\alpha}\alpha} A_{\alpha\dot{\alpha}} \rightsquigarrow A_{\alpha\dot{\alpha}} = \frac{1}{2}(\sigma^m)_{\alpha\dot{\alpha}} A_m$. An example of that would be the partial derivative, ∂_m . So let's define

$$\partial_{\alpha\dot{\alpha}} = (\sigma^m)_{\alpha\dot{\alpha}} \partial_m$$

We can show that it has the properties

$$\begin{aligned}\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\beta}} &= \delta^{\dot{\alpha}}_{\dot{\beta}}\square \quad , \quad \partial^{\dot{\alpha}\alpha}\partial_{\alpha\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}}\square \\ \partial^{\alpha}_{\dot{\beta}}\partial_{\beta\dot{\alpha}} - \partial^{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}} &= \delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}}\square\end{aligned}\tag{2.13}$$

The conversion of vector indices to spinorial ones, doesn't work just for vectors but for higher rank tensors as well. For example consider the case of a rank two tensor

$$A_{mn} = (\bar{\sigma}_m)^{\dot{\alpha}\alpha}(\bar{\sigma}_n)^{\dot{\beta}\beta}A_{\alpha\beta\dot{\alpha}\dot{\beta}}$$

$A_{\alpha\beta\dot{\alpha}\dot{\beta}}$ can be further decomposed by symmetrizing and anti-symmetrizing the undotted and dotted pair of indices.

$$A_{\alpha\beta\dot{\alpha}\dot{\beta}} = A_{\alpha\beta\dot{\alpha}\dot{\beta}}^{(S,S)} + C_{\alpha\beta}A_{\dot{\alpha}\dot{\beta}}^{(A,S)} + C_{\dot{\alpha}\dot{\beta}}A_{\alpha\beta}^{(S,A)} + C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}A^{(A,A)}$$

and we get that

$$\begin{aligned}A_{\alpha\beta\dot{\alpha}\dot{\beta}}^{(S,S)} &= \frac{1}{16}(\sigma^m)_{(\alpha(\dot{\alpha}(\sigma^n)_{\beta)\dot{\beta}})A_{mn} \\ A_{\dot{\alpha}\dot{\beta}}^{(A,S)} &= -\frac{1}{4}(\bar{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}A_{mn} \\ A_{\alpha\beta}^{(S,A)} &= \frac{1}{4}(\sigma^{mn})_{\alpha\beta}A_{mn} \\ A^{(A,A)} &= \frac{1}{8}\eta^{mn}A_{mn}\end{aligned}\tag{2.14}$$

From the above we can see that for a rank two antisymmetric tensor (like the generators of the Lorentz group, \mathcal{J}_{mn}) the completely symmetric and the scalar terms vanish ($A_{\alpha\beta\dot{\alpha}\dot{\beta}}^{(S,S)} = 0$, $A^{(A,A)} = 0$) and we get

$$\mathcal{J}_{mn} = 2(\sigma_{mn})^{\alpha\beta}\mathcal{J}_{\alpha\beta} - 2(\bar{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}\mathcal{J}_{\dot{\alpha}\dot{\beta}}\tag{2.15}$$

2.2 Conjugation and Reality

Because of (2.7), the hermiticity of \mathcal{J}_{mn} and the property $\sigma_{mn}^\dagger = -\bar{\sigma}_{mn}$ we get that:

$$\begin{aligned} [\mathcal{J}_{mn}, (\Psi_L^\dagger)^\dot{\alpha}] &= i(\bar{\sigma}_{mn})^\dot{\alpha}{}_{\dot{\beta}}(\Psi_L^\dagger)^{\dot{\beta}} \\ [\mathcal{J}_{mn}, (\Psi_R^\dagger)_\alpha] &= i(\sigma_{mn})_\alpha{}^\beta(\Psi_R^\dagger)_\beta \end{aligned} \quad (2.16)$$

The conclusion is that Ψ_L^\dagger transforms like Ψ_R and Ψ_R^\dagger transforms like Ψ_L . So we must have

$$(\Psi_{L\alpha})^\dagger = \eta_1 \Psi_{R\dot{\alpha}} \quad , \quad (\Psi_L^\alpha)^\dagger = \eta_2 \Psi_R^{\dot{\alpha}}$$

But $(\Psi_{L\alpha})^\dagger = (\Psi_L^\beta C_{\beta\alpha})^\dagger = -(\Psi_L^\beta)^\dagger C_{\beta\dot{\alpha}} = -\eta_2 \Psi_{R\dot{\alpha}} \rightsquigarrow \eta_1 = -\eta_2$. The convention we will use is

$$(\Psi_{L\alpha})^\dagger = -\Psi_{R\dot{\alpha}} \quad , \quad (\Psi_L^\alpha)^\dagger = \Psi_R^{\dot{\alpha}} \quad (2.17)$$

In general a (n, m) tensor (the tensor product of n left spinors and m right spinors) under conjugation will go to a (m, n) tensor

$$(n, m)^* \longrightarrow (m, n)$$

If $m = n$ then $(n, n)^* \longrightarrow (n, n)$ and we can impose the reality condition by equating the two sides. But if $m \neq n$ then $(n, m)^* \longrightarrow (m, n) \neq (n, m)$ and reality can not be imposed. In order to construct real objects we have to consider the direct sum $(n, m) \oplus (m, n)$. Then we see that $[(n, m) \oplus (m, n)]^* \longrightarrow (m, n) \oplus (n, m) = (n, m) \oplus (m, n)$ and we can demand reality.

2.3 Supersymmetry and the Super-Poincaré algebra

One of the things that motivated supersymmetry was the salvation from the Coleman-Mandula theorem, which restricted the possible non-trivial (not just direct sum) extensions of the Poincaré group. The theorem was considering only the possibility of a Lie algebra, therefore a natural way to avoid it was to consider graded algebras. A class of graded algebras are endowed with an anticommutative structure. They include generators which follow anti-commutation relations, so a good idea would be to start with the Poincaré algebra and add generators of spinorial nature. The simplest thing to do is to consider left or right spinors. But if we want to have real representations we need both.

We introduce one set of left (Q_α) and one set of right ($\bar{Q}^{\dot{\alpha}}$) spinors. The bar indicates that the right spinor is the hermitian conjugate of the left. Because they are left $(1/2, 0)$ and right $(0, 1/2)$ spinors their commutation with angular momentum is given by (2.7)

$$\begin{aligned} [\mathcal{J}_{mn}, Q_\alpha] &= i(\sigma_{mn})_\alpha{}^\beta Q_\beta \\ [\mathcal{J}_{mn}, \bar{Q}^{\dot{\alpha}}] &= i(\bar{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \end{aligned} \tag{2.18}$$

To complete the algebra we must find all the other (anti-)commutations. For example with the momentum operator P_m $(1/2, 1/2)$ the result has to be spinorial in nature and the possible results are $(1/2, 0) \otimes (1/2, 1/2) = (0, 1/2) \oplus (1, 1/2)$. There

is not an object like $(1, 1/2)$ so we are left with $(0, 1/2)$ which is $\bar{Q}_{\dot{\alpha}}$. So we have

$$[P_m, Q_\alpha] = f_{m\dot{\alpha}}{}^\alpha \bar{Q}_{\dot{\alpha}}$$

similarly we can get that

$$\begin{aligned} [P_m, \bar{Q}_{\dot{\alpha}}] &= g_{m\dot{\alpha}}{}^\alpha Q_\alpha \quad (g_{m\dot{\alpha}}{}^\alpha \sim (f_{m\alpha}{}^{\dot{\alpha}})^*) \\ \{Q_\alpha, Q_\beta\} &= f_{\alpha\beta}{}^{mn} \mathcal{J}_{mn} \\ \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= g_{\dot{\alpha}\dot{\beta}}{}^{mn} \mathcal{J}_{mn} \quad (g_{\dot{\alpha}\dot{\beta}}{}^{mn} \sim (f_{\alpha\beta}{}^{mn})^*) \\ \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= f_{\alpha\dot{\alpha}}{}^m P_m \end{aligned}$$

Compatibility of the above with the super-Jacobi identity will give

$$\begin{aligned} [P_m, \bar{Q}_{\dot{\alpha}}] &= 0 \quad , \quad \{Q_\alpha, Q_\beta\} = 0 \\ \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0 \quad , \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -ik\sigma^m{}_{\alpha\dot{\alpha}} \partial_m \end{aligned} \tag{2.19}$$

and k is completely unconstrained. In our conventions we scale the spinorial generators so

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = i\partial_{\alpha\dot{\alpha}}. \tag{2.20}$$

The general group element is

$$g(\omega^{mn}, a^m, \epsilon^\alpha, \bar{\epsilon}^{\dot{\alpha}}) = e^{\frac{i}{2}\omega^{mn} \mathcal{J}_{mn} - ia^m P_m + i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}}$$

2.4 Superspace, differential operators and covariant derivatives

Every time we have a group G which has a subgroup H , the most general group element of G can be decomposed (using the Baker-Campbell-Hausdorff identity) to the product of the general element of G/H times a group element of H

$$g = \Omega(x)h$$

That decomposition naturally defines a set of coordinates $\{x\}$ for every element of the coset G/H . We can use them to define the ‘spacetime’ the theory acts on. Also we can use this technique to find the transformation law of the coordinates under the action of the group:

- for the general group action: $\Omega(x) \longrightarrow g\Omega(x) = \Omega(x')h \rightsquigarrow \Omega(x') = g\Omega(x)h^{-1}$
- for $g \in G/H$: $\Omega(x) \longrightarrow g\Omega(x) = \Omega(x')$
- for $g \in H$: $\Omega(x) \longrightarrow h\Omega(x) = \Omega(x')h \rightsquigarrow \Omega(x') = h\Omega(x)h^{-1}$

The Super-Poincaré group has an algebra with the following structure:

$$[P_A, P_B] = f_{AB}{}^C P_C \quad , \quad P_A = \{Q_\alpha, \bar{Q}_{\dot{\alpha}}, P_m\} \quad , \quad A = \{\alpha, \dot{\alpha}, m\}$$

$$[\mathcal{J}, P_A] \sim P_A \quad , \quad [\mathcal{J}, \mathcal{J}] \sim \mathcal{J}$$

Obviously the Lorentz group (\mathcal{J}) is a subgroup, so the general group element can be written as

$$g(\omega, a, \epsilon, \bar{\epsilon}) = \Omega(x, \theta, \bar{\theta})h(\omega) \tag{2.21}$$

$$\Omega(x, \theta, \bar{\theta}) = e^{-ix^m P_m + i\theta^\alpha Q_\alpha + i\bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}} \quad , \quad h(\omega) = e^{\frac{i}{2}\omega^{mn} J_{mn}}$$

From the above we see that the coset theory lives on a ‘Superspace’ parametrized by eight real coordinates, four bosonic (x^m) and four fermionic ($\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$).

The transformation of these coordinates under *translations* ($g \in G/H$) is

$$\begin{aligned}\Omega(a, \epsilon, \bar{\epsilon})\Omega(x, \theta, \bar{\theta}) &= \Omega(x', \theta', \bar{\theta}') \\ x'^m &= x^m + a^m + \frac{i}{2}\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}} - \frac{i}{2}\epsilon^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \\ \theta'^\alpha &= \theta^\alpha + \epsilon^\alpha, \quad \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}\end{aligned}\tag{2.22}$$

Under *rotations* ($g \in H$)

$$\begin{aligned}\Omega(x', \theta', \bar{\theta}') &= h(\omega)\Omega(x, \theta, \bar{\theta})h^{-1}(\omega) \\ x'^m &= (e^\omega)^m_n x^n \\ \theta'^\alpha &= \theta^\beta \left(e^{-\frac{1}{2}\omega^{mn}\sigma_{mn}} \right)_\beta^\alpha, \quad \bar{\theta}'^{\dot{\alpha}} = \left(e^{\frac{1}{2}\omega^{mn}\bar{\sigma}_{mn}} \right)_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}}\end{aligned}\tag{2.23}$$

Supersymmetry transformations are a special case of translations, where the source of translation is fermionic $g(0, 0, \epsilon, \bar{\epsilon})$

$$\begin{aligned}x'^m &= x^m + \frac{i}{2}\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}} - \frac{i}{2}\epsilon^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \\ \theta'^\alpha &= \theta^\alpha + \epsilon^\alpha, \quad \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}\end{aligned}\tag{2.24}$$

All the above transformation laws contain information about the realization of the generators, that are responsible for the transformations, in terms of differential operators. The change of a field under supersymmetry is (indices ignored) $\Phi(x) \longrightarrow$

$\Phi'(x') = \Phi(x)$ and in the infinitesimal limit gives:

$$\begin{aligned}
\delta_S \Phi(x) &= \left(\frac{i}{2} \bar{\epsilon}^{\dot{\alpha}} \theta^\alpha (\sigma^m)_{\alpha\dot{\alpha}} + \frac{i}{2} \epsilon^\alpha (\sigma^m)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \right) \partial_m \Phi(x) \\
&\quad - \epsilon^\alpha \partial_\alpha \Phi(x) - \bar{\epsilon}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \Phi(x) \\
&= i\epsilon^\alpha [Q_\alpha, \Phi(x)] + i\bar{\epsilon}^{\dot{\alpha}} [\bar{Q}_{\dot{\alpha}}, \Phi(x)] \\
&= i\epsilon^\alpha Q_\alpha^d + i\bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}^d
\end{aligned}$$

So the expression for the realization of the spinorial generators are:

$$Q_\alpha^d = i\partial_\alpha + \frac{1}{2} \bar{\theta}^{\dot{\alpha}} (\sigma^m)_{\alpha\dot{\alpha}} \partial_m \quad , \quad \bar{Q}_{\dot{\alpha}}^d = i\bar{\partial}_{\dot{\alpha}} + \frac{1}{2} \theta^\alpha (\sigma^m)_{\alpha\dot{\alpha}} \partial_m \quad (2.25)$$

Also we want to define covariant derivatives. This need comes from the fact that we would like to impose differential constraints on fields and we want these constraints to remain true even after we perform transformations of the fields. The transformations are controlled by the the Q_α^d , $\bar{Q}_{\dot{\alpha}}^d$, so the covariant derivatives have to commute with both of them. Solving that constraint gives us expressions for the covariant derivatives up to an overall constant (scale). We will use the following expressions:

$$D_\alpha = \partial_\alpha + \frac{i}{2} \bar{\theta}^{\dot{\alpha}} \sigma_\alpha^m \dot{\alpha} \partial_m \quad , \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + \frac{i}{2} \theta^\alpha \sigma_\alpha^m \dot{\alpha} \partial_m \quad (2.26)$$

These will be the basic tools that we will use to formulate superspace lagrangians, for this reason we give a short list of conventions and properties:

$$\begin{aligned}
[D_\alpha, \bar{D}_{\dot{\alpha}}] &= i\partial_{\alpha\dot{\alpha}} \quad , \quad D^2 = \frac{1}{2} D^\alpha D_\alpha \quad , \quad D_\alpha D_\beta = -C_{\alpha\beta} D^2 \\
[D^2, \bar{D}_{\dot{\alpha}}] &= iD^\alpha \partial_{\alpha\dot{\alpha}} \quad , \quad D^2 \bar{D}_{\dot{\alpha}} D^2 = 0 \quad , \quad [D^\alpha, \bar{D}^{\dot{\alpha}}] \partial_{\alpha\dot{\alpha}} = 2i [D^2, \bar{D}^2] \\
D^\alpha \bar{D}_{\dot{\alpha}} D_\alpha &= -\bar{D}_{\dot{\alpha}} D^2 - D^2 \bar{D}_{\dot{\alpha}} \quad , \quad \{D^2, \bar{D}^2\} - D^\gamma \bar{D}^2 \bar{D}_\gamma = \square \quad , \quad (D_\alpha)^\dagger = -\bar{D}_{\dot{\alpha}}
\end{aligned} \quad (2.27)$$

2.5 Superfields, Berezin integrals and Components

The functions that take values in superspace $\Phi(x, \theta, \bar{\theta})$ are called super functions and are mappings from superspace to superspace. A special category of those are the ones that under a supersymmetric transformation, transform nicely

$$\delta_S \Phi = i [\epsilon^\alpha Q_\alpha, \Phi] + i [\bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}, \Phi]$$

and they are called superfields. Since they are functions of θ and $\bar{\theta}$ we can do a Taylor expansion of the superfield in terms of these variables. But because they are anticommuting objects and we have four of them (two θ and two $\bar{\theta}$) any term in the expansion with more than two θ -s or two $\bar{\theta}$ -s will identically vanish. Hence the expansion terminates and we have a finite number of terms.

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = & A(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}(x) + \theta^2 F(x) + \bar{\theta}^2 G(x) \\ & + \theta^\alpha \bar{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + \theta^2 \bar{\theta}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x) \end{aligned} \quad (2.28)$$

The coefficients in the expansion are functions of the bosonic coordinates, so they are fields in spacetime and are called components.

For superspace formulated theories, the action that describes the dynamics of the system will be given as the integral of a lagrangian density over the entire superspace

$$S = \int d^8 z \mathcal{L} = \int d^4 x d^2 \theta d^2 \bar{\theta} \mathcal{L}$$

To learn how to integrate over a grassmann variable (more properly called Berezin integral), we demand that it satisfies a few simple properties as does ordinary integration:

1. The integral of the sum of two functions is the sum of the integrals

$$\int d\theta (f + g) = \int d\theta f + \int d\theta g$$

2. A change of variable by a constant will not change the value of the integral

$$\int d\theta f(\theta + \epsilon) = \int d\theta f(\theta)$$

These requirements are enough to define the integral up to a normalization. We will use

$$\int d\theta = 0 \quad , \quad \int d\theta\theta = 1 \quad , \quad \int d\theta_\alpha\theta^\beta = \delta_\alpha^\beta \quad , \quad \int d\theta_\alpha f(\theta) = \partial_\alpha f(\theta) \quad (2.29)$$

Using the definitions for the covariant derivatives (2.26), we can rewrite the above in the following way

$$\int d\theta_\alpha f(\theta) = (D_\alpha f(\theta)) |_{\theta=0, \bar{\theta}=0} \quad (2.30)$$

So for our case

$$S = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{L} = \int d^4x \begin{cases} D^2 \bar{D}^2 \mathcal{L} |_{\theta=0, \bar{\theta}=0} \\ \bar{D}^2 D^2 \mathcal{L} |_{\theta=0, \bar{\theta}=0} \end{cases} \quad (2.31)$$

Although D^2 and \bar{D}^2 do not commute, under the x -integration sign they do, because their commutator consists of are partial derivative terms which can be integrated away. However choosing one form over the other can make a difference by simplifying things.

For the general superfield Φ (indices suppressed) the structure of the component expansion has the general form

$$\Phi \sim \theta^n \bar{\theta}^m \Phi_{(n,m)}$$

That means that the $\Phi_{(n,m)}$ is the coefficient of the term $\theta^n \bar{\theta}^m$ where n, m are integers between zero and two ($0 \leq n, m \leq 2$), at least for the $4D$, $\mathcal{N} = 1$ case we are considering. If we want to recover $\Phi_{(n,m)}$ starting from Φ all we have to do is take n ∂_α derivatives and m $\bar{\partial}_{\dot{\alpha}}$ derivatives and then set $\theta = \bar{\theta} = 0$

$$\Phi_{(n,m)} \sim \partial_\alpha^n \bar{\partial}_{\dot{\alpha}}^m \Phi|_{\theta=0, \bar{\theta}=0}$$

But again because of the connection between covariant derivatives and partial spinorial derivatives (2.26) we can rewrite that in the form $\Phi_{(n,m)} \sim f(D^n, \bar{D}^m) \Phi|_{\theta=0, \bar{\theta}=0}$, with f being a function of covariant derivatives. Specifically we have:

$$\begin{aligned} \Phi_{\alpha(n)\dot{\alpha}(m)}^{(0,0)} &= \Phi_{\alpha(n)\dot{\alpha}(m)}| & \Phi_{\alpha(n)\dot{\alpha}(m)}^{(0,2)} &= -\bar{D}^2 \Phi_{\alpha(n)\dot{\alpha}(m)}| \\ \Phi_{\beta\alpha(n)\dot{\alpha}(m)}^{(1,0)} &= D_\beta \Phi_{\alpha(n)\dot{\alpha}(m)}| & \Phi_{\alpha(n)\dot{\beta}\dot{\alpha}(m)}^{(2,1)} &= -\frac{1}{2} \{D^2, \bar{D}_{\dot{\beta}}\} \Phi_{\alpha(n)\dot{\alpha}(m)}| \\ \Phi_{\alpha(n)\dot{\beta}\dot{\alpha}(m)}^{(0,1)} &= \bar{D}_{\dot{\beta}} \Phi_{\alpha(n)\dot{\alpha}(m)}| & \Phi_{\beta\alpha(n)\dot{\alpha}(m)}^{(1,2)} &= -\frac{1}{2} \{\bar{D}^2, D_\beta\} \Phi_{\alpha(n)\dot{\alpha}(m)}| & (2.32) \\ \Phi_{\beta\alpha(n)\dot{\beta}\dot{\alpha}(m)}^{(1,1)} &= -\frac{1}{2} [D_\beta, \bar{D}_{\dot{\beta}}]| & \Phi_{\alpha(n)\dot{\alpha}(m)}^{(2,2)} &= \frac{1}{4} \square \Phi_{\alpha(n)\dot{\alpha}(m)}| + \frac{1}{2} D^\gamma \bar{D}^2 D_\gamma \Phi_{\alpha(n)\dot{\alpha}(m)}| \\ \Phi_{\alpha(n)\dot{\alpha}(m)}^{(2,0)} &= -D^2 \Phi_{\alpha(n)\dot{\alpha}(m)}| \end{aligned}$$

Also we can calculate the mass dimensions for each component

$$d_{(n,m)} = d_{[\Phi]} + \frac{1}{2}(n+m)$$

where $d_{[\Phi]}$ is the mass dimension of the superfield Φ . If we consider theories which are quadratic in the superfield then $2d_{[\Phi]} \leq [\mathcal{L}] \rightsquigarrow d_{[\Phi]} \leq 1$. For these superfields the mass dimensions of the components are :

$n + m$	Components	$d_{(n,m)}$		
		$d_{[\Phi]} = 0$	$d_{[\Phi]} = 1/2$	$d_{[\Phi]} = 1$
0	1:(0,0)	0	1/2	1
1	2:(1,0),(0,1)	1/2	1	3/2
2	3:(2,0),(1,1),(0,2)	1	3/2	2
3	2:(2,1),(1,2)	3/2	2	5/2
4	1:(2,2)	2	5/2	3

From that we see that the components that can play a dynamical role (dimension 1 for bosons and 3/2 for fermions) are very specific. The rest of the components included in the superfield are auxiliary fields with sole purpose to make supersymmetry manifest off-shell.

The index structure of the various dynamical components is different and therefore they can describe different spin representations of the Poincaré group. The one that can describe the highest possible spin is the one with the most indices. In the case of superfields with $d_{[\Phi]} = 0, 1/2$ that component is the completely symmetric piece of the (1,1) component.

The other thing we would like to know about all these components is how they transform under a supersymmetry transformation. We know how the superfield transforms: $\delta_S \Phi = i (\epsilon^\alpha Q_\alpha^d + \bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}^d) \Phi$. $\delta_S \Phi$ is a superfield in it's own right so we can perform a Taylor's expansion, $\delta_S \Phi \sim \theta^n \bar{\theta}^m \delta_s \Phi_{(n,m)}$ to reveal its components $\delta_s \Phi_{(n,m)}$. But these components are the change of the original component under

supersymmetry. To get them all we have to do is follow the discussion above. So:

$$\delta_S \Phi_{(n,m)} = f(D^n, \bar{D}^m) \delta_S \Phi | = f(D^n, \bar{D}^m) (i\epsilon^\alpha Q_\alpha^d + i\bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}^d) \Phi |$$

But by definition the covariant derivatives commute with the Q^d , \bar{Q}^d so

$$\delta_S \Phi_{(n,m)} = i (\epsilon^\alpha Q_\alpha^d + \bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}^d) f(D^n, \bar{D}^m) \Phi | \quad (2.33)$$

We can use the expressions for the Q^d , \bar{Q}^d (2.25) to express them in terms of the covariant derivatives

$$iQ_\alpha^d = -D_\alpha + i\bar{\theta}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^m \partial_m \quad , \quad i\bar{Q}_{\dot{\alpha}}^d = -\bar{D}^{\dot{\alpha}} + i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^m \partial_m$$

The terms proportional to θ and $\bar{\theta}$ will drop out when we $|_{\theta=0, \bar{\theta}=0}$ and we are left with the simple expression

$$\delta_S \Phi_{(n,m)} = - (\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}) f(D^n, \bar{D}^m) \Phi | \quad (2.34)$$

2.6 Representation theory of the Super-Poincaré group

The Super-Poincaré algebra in its full glory is

$$\begin{aligned} i[\mathcal{J}_{mn}, \mathcal{J}_{rs}] &= \eta_{ms} \mathcal{J}_{nr} - \eta_{mr} \mathcal{J}_{ns} - \eta_{ns} \mathcal{J}_{mr} + \eta_{nr} \mathcal{J}_{ms} \\ i[\mathcal{J}_{mn}, P_r] &= \eta_{mr} P_m - \eta_{nr} P_n \quad , \quad [\mathcal{J}_{mn}, Q_\alpha] = i(\sigma_{mn})_\alpha{}^\beta Q_\beta \\ [\mathcal{J}_{mn}, \bar{Q}^{\dot{\alpha}}] &= i(\bar{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \quad , \quad [P_m, P_n] = 0 \quad , \quad [P_m, Q_\alpha] = 0 \\ [P_m, \bar{Q}_{\dot{\alpha}}] &= 0 \quad , \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -(\sigma^m)_{\alpha\dot{\alpha}} P_m \quad , \quad \{Q_\alpha, Q_\beta\} = 0 \\ \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0 \end{aligned} \quad (2.35)$$

We want to find unitary irreducible representations of the above algebra. For that, we need to find the Cartan subalgebra and the Casimir operators and then diagonalize them. P_m and Q_α commute so they can be diagonalized simultaneously and we get a set of eigenvectors $|p_m, q_\alpha\rangle$ such that $P_m|p_m, q_\alpha\rangle = p_m|p_m, q_\alpha\rangle$, $Q_\alpha|p_m, q_\alpha\rangle = q_\alpha|p_m, q_\alpha\rangle$. It is obvious that it is impossible to have finite dimensional representations. The action of the group will generate infinitely many eigenstates with different eigenvalues, exactly as Wigner showed for the Poincare group. The way out is exactly analogous to the discussion for the Poincaré representation theory. Finiteness forces us to consider representations of the *little* group which respects P_m and Q_α and then we do rotations and boosts to cover everything else. The little group U , in this case will be defined through the properties

$$U^{-1}P_mU = P_m \quad , \quad U^{-1}Q_\alpha U = Q_\alpha \quad (2.36)$$

The most general solution for U is

$$U = e^{-i\eta^m Z_m - i\alpha^m P_m}, \quad Z_m = W_m - \frac{1}{4}(\bar{\sigma}_m)^{\dot{\alpha}\alpha}[Q_\alpha, \bar{Q}_{\dot{\alpha}}] \quad , \quad W^m = \frac{1}{2}\epsilon^{mnr s} \mathcal{J}_{nr} P_s \quad (2.37)$$

and the vector η_m that parametrizes the group element is not free but constrained to be perpendicular to the momentum P_m ($\eta^m P_m = 0$). W_m has exactly the same form as the Pauli-Lubanski vector, but it is not the same object. It is a supersymmetric extension of it because $\mathcal{J}_{mn} \neq \mathcal{J}_{mn}^{\text{Poincaré}}$. \mathcal{J}_{mn} is the generator of rotations for the entire superspace while $\mathcal{J}_{mn}^{\text{Poincaré}}$ is the generator of rotations just for spacetime, the ‘bosonic’ sector of superspace. In a differential realization of the two objects that is

obvious

$$\mathcal{J}_{mn} = ix_n \partial_m - ix_m \partial_n + i\bar{\theta}^{\dot{\beta}} (\bar{\sigma}_{mn})^{\dot{\alpha}}_{\dot{\beta}} \bar{\partial}_{\dot{\alpha}} - i\theta^{\beta} (\sigma_{mn})_{\beta}^{\alpha} \partial_{\alpha} - iM_{mn}$$

$$\mathcal{J}_{mn}^{\text{Poincaré}} = ix_n \partial_m - ix_m \partial_n - iM_{mn}$$

Z_m satisfies the following:

$$[Z_m, P_n] = 0 \quad , \quad [Z_m, Q_{\alpha}] = \frac{1}{2} Q_{\alpha} P_m \quad , \quad [Z_m, Z_n] = i\epsilon_{mnr s} Z^r P^s \quad (2.38)$$

which give

$$[Z_{[m} P_n], P_r] = 0 \quad , \quad [Z_{[m} P_n], Q_{\alpha}] = 0$$

We can identify a Casimir operator as

$$C = Z^{[m} P^{n]} Z_{[m} P_{n]} \quad (2.39)$$

and the other one is of course P^2 which distinguishes between massive and massless cases.

2.6.1 Massive Case

For the massive case momentum takes the form $P_m = (-m, 0, 0, 0)$ which makes $\eta_m = (0, \vec{\eta})$ and the little group elements are

$$U = e^{-i\eta^i Z_i} e^{ia^0 m}$$

The generators of the group are the three Z_i and they satisfy

$$[Z_i, Z_j] = im\epsilon_{ijk} Z^k \quad , \quad [Z_i, P_m] = 0 \quad , \quad [Z_i, Q_{\alpha}] = 0$$

This is an $SU(2)$ algebra generated by $S_i = \frac{1}{m}Z_i$. The Cartan subalgebra, which will label the states is

$$\{P_m, Q_\alpha, S_3\}$$

and the Casimirs that label the representation are

$$P^2 = -m^2, \quad \vec{S}^2 = \frac{1}{m^2}\vec{Z}^2 = Y(Y+1)$$

\vec{S}^2 is the supersymmetric extension of the spin operator and is called superspin. Since C [2.39] is the expression for the Casimir, it must be related with \vec{S}^2 and they are. In the rest frame we get $C = -2m^4\vec{S}^2$. The eigenvectors of that representation are labelled as follows:

$$|m, Y, m_Y, q_\alpha\rangle$$

and the dimensionality of the representation is $4(2Y+1) = 8Y+4$.

2.6.2 Massless case

When the mass is zero, the momentum takes the form $P_m = (-E, 0, 0, E)$ and that introduces extra constraints through the supersymmetric algebra:

$$\{Q_2, \bar{Q}_2\} = 0 \rightsquigarrow Q_2 = 0, \quad \bar{Q}_2 = 0 \rightsquigarrow Q^2 = 0, \quad \bar{Q}^2 = 0$$

Because of them, the solution (2.37) for the group elements of the little group is no longer valid and we need to update it, taking into account these constraints. The correct answer for this case will be

$$U = e^{-i\eta^m Z_m - ia^m P_m}, \quad Z^m = W^m - \frac{1}{8}(\bar{\sigma}^m)^{\dot{\alpha}\alpha}[Q_\alpha, \bar{Q}_{\dot{\alpha}}] \quad (2.40)$$

The difference with the massive case is a change in the definition of Z_m and the vector parameter η_m is completely free now. The new Z_m has the following properties:

$$Z^m P_m = 0 \quad , \quad [Z_m, P_n] = 0 \quad , \quad [Z_m, Q_\alpha] = 0 \quad , \quad [Z_m, Z_n] = i\epsilon_{mnr s} Z^r P^s \quad (2.41)$$

The first one will force Z_m to take the form $Z_m = (-Z_3, Z_1, Z_2, Z_3)$ and the last one translates to $[Z^1, Z^2] = 0$, $[Z^1, Z^3] = -iEZ_2$, $[Z^2, Z^3] = iEZ_1$. This is exactly the $E2$ algebra and it has infinite dimensional representations. To avoid that we have to set $Z_1 = Z_2 = 0$. Therefore the final expression for Z_m is $Z_m = (-Z_3, 0, 0, Z_3)$ which makes it proportional to P_m . We define the proportionality constant such that

$$Z_m = (Y + \frac{1}{4})P_m \quad (2.42)$$

where Y is the superhelicity. The eigenvectors of the representation are $|q_1\rangle$ and the dimensionality of the representation is 2. For \mathcal{CPT} -invariant theories the number of states will be double because \mathcal{CPT} will flip the sign of superhelicity and therefore we have to include the states of superhelicity $-Y$. So the representation will have dimensionality 4.

Like in the representation theory of the Poincaré group, there is a discontinuity in the dimensionality of the massive vs massless representations. If we start with a massive irreducible representation and take a smooth limit $m \rightarrow 0$ the degrees of freedom will jump from $8Y + 4$ to 4. This basic fact is the root of the whole gauge invariance story. More on that will be discussed in the next chapter.

2.7 Superfield realization of representation theory

We want to use superfields to realize the irreducible representations of the Super-Poincaré group. For that to happen these superfields must diagonalize the Casimirs that were presented above. This means that they must satisfy a set of constraints and they must be expressed in terms of covariant derivatives D and \bar{D} so they will not change under a supersymmetric transformation.

2.7.1 Massive Case

The Casimir operator \vec{S}^2 written in terms of the covariant derivatives is:

$$\vec{S}^2 = \frac{W_{\text{Poincaré}}^2}{m^2} + \frac{3}{4}P_0 + B \quad (2.43)$$

$$P_0 = -\frac{1}{m^2}D^\gamma\bar{D}^2D_\gamma = -\frac{1}{m^2}\bar{D}^{\dot{\gamma}}D^2\bar{D}_{\dot{\gamma}} \quad , \quad B = \frac{1}{2m^2}W_{\text{Poincaré}}^m(\bar{\sigma}_m)^{\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}]$$

where $W_{\text{Poincaré}}^m$ is the Poincaré Pauli-Lubanski vector. P_0 is a projection operator and together with two other projection operators $P_+ = \frac{1}{m^2}D^2\bar{D}^2$, $P_- = \frac{1}{m^2}\bar{D}^2D^2$ they span the entire space:

$$P_+ + P_- + P_0 = \mathbb{I} \quad , \quad P_i P_j = \delta_{ij} P_i \quad , \quad i, j = \{+, -, 0\}$$

B has the properties:

$$BP_+ = 0 \quad , \quad BP_- = 0 \quad , \quad BP_0 = B$$

$$B^2 + B = \frac{W_{\text{Poincaré}}^2}{m^2}P_0 \quad (2.44)$$

The first three constraints force B to be proportional to P_0 : $B = \lambda P_0$ and the last one fixes the proportionality constant. So the expression for the Casimir is

$$\vec{S}^2 = \frac{W_{\text{Poincaré}}^2}{m^2} + \left(\frac{3}{4} + \lambda \right) P_0 \quad (2.45)$$

Now let's consider a superfield with n undotted and m dotted indices $\Phi_{\alpha(n)\dot{\alpha}(m)}$. In order for this to describe a massive superspin Y irreducible representation it must satisfy the condition

$$\vec{S}^2 \Phi_{\alpha(n)\dot{\alpha}(m)} = Y(Y + 1) \Phi_{\alpha(n)\dot{\alpha}(m)}$$

For that to happen it must diagonalize both $W_{\text{Poincaré}}^2$ and P_0 .

P_0 as a projection operator has two eigenvalues 0 and 1

$$P_0 \Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad \rightsquigarrow \quad \left\{ \begin{array}{l} \bar{D}_{\dot{\beta}} \Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad (\text{chiral}) \\ \text{or} \\ D_{\beta} \Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad (\text{anti-chiral}) \end{array} \right.$$

$$P_0 \Phi_{\alpha(n)\dot{\alpha}(m)} = \Phi_{\alpha(n)\dot{\alpha}(m)} \quad \rightsquigarrow \quad \left\{ \begin{array}{l} D^2 \Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \\ \text{and} \\ \bar{D}^2 \Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \end{array} \right.$$

Diagonalization of $W_{\text{Poincaré}}^2$ demands:

1. $\Phi_{\alpha(n)\dot{\alpha}(m)}$ is independently symmetrized in both undotted and dotted set of indices: $\Phi_{\alpha(n)\dot{\alpha}(m)} = \frac{1}{n!m!} \Phi_{(\alpha(n))(\dot{\alpha}(m))}$
2. $\partial^{\beta\dot{\beta}} \Phi_{\beta\alpha(n-1)\dot{\beta}\dot{\alpha}(m-1)} = 0$

and the eigenvalue is :

$$W_{\text{Poincaré}}^2 \Phi_{\alpha(n)\dot{\alpha}(m)} = j(j + 1) \Phi_{\alpha(n)\dot{\alpha}(m)} \quad , \quad j = \frac{n + m}{2}$$

which gives as well $\lambda = j$ or $\lambda = -j - 1$.

For the first value of λ we have $B\Phi_{\alpha(n)\dot{\alpha}(m)} = jP_0\Phi_{\alpha(n)\dot{\alpha}(m)} \rightsquigarrow D^\gamma\Phi_{\gamma\alpha(n-1)\dot{\alpha}(m)} = 0$

and for the second one $B\Phi_{\alpha(n)\dot{\alpha}(m)} = -(j+1)P_0\Phi_{\alpha(n)\dot{\alpha}(m)} \rightsquigarrow D_{(\gamma}\Phi_{\alpha(n))\dot{\alpha}(m)} = 0$.

Putting everything together we get:

$\Phi_{\alpha(n)\dot{\alpha}(m)}$ must have symmetrized dotted and undotted indices and the various representations are organized as follows

1. Chiral superfield

$$\begin{aligned} \bar{D}_{\dot{\gamma}}\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad \partial^{\gamma\dot{\gamma}}\Phi_{\gamma\alpha(n-1)\dot{\gamma}\dot{\alpha}(m-1)} = 0 \quad , \quad \square\Phi_{\alpha(n)\dot{\alpha}(m)} = m^2\Phi_{\alpha(n)\dot{\alpha}(m)} \\ Y = \frac{n+m}{2} \end{aligned} \quad (2.46)$$

2. Anti-chiral superfield

$$\begin{aligned} D_\gamma\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad \partial^{\gamma\dot{\gamma}}\Phi_{\gamma\alpha(n-1)\dot{\gamma}\dot{\alpha}(m-1)} = 0 \quad , \quad \square\Phi_{\alpha(n)\dot{\alpha}(m)} = m^2\Phi_{\alpha(n)\dot{\alpha}(m)} \\ Y = \frac{n+m}{2} \end{aligned} \quad (2.47)$$

3. Highest superspin linear superfield

$$\begin{aligned} D^2\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad \bar{D}^2\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad D^\gamma\Phi_{\gamma\alpha(n-1)\dot{\alpha}(m)} = 0 \quad , \\ \partial^{\gamma\dot{\gamma}}\Phi_{\gamma\alpha(n-1)\dot{\gamma}\dot{\alpha}(m-1)} = 0 \quad , \quad \square\Phi_{\alpha(n)\dot{\alpha}(m)} = m^2\Phi_{\alpha(n)\dot{\alpha}(m)} \\ Y = \frac{n+m+1}{2} \end{aligned} \quad (2.48)$$

4. Lowest superspin linear superfield

$$\begin{aligned} D^2\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad \bar{D}^2\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad D_{(\gamma}\Phi_{\alpha(n))\dot{\alpha}(m)} = 0 \quad , \\ \partial^{\gamma\dot{\gamma}}\Phi_{\gamma\alpha(n-1)\dot{\gamma}\dot{\alpha}(m-1)} = 0 \quad , \quad \square\Phi_{\alpha(n)\dot{\alpha}(m)} = m^2\Phi_{\alpha(n)\dot{\alpha}(m)} \\ Y = \frac{n+m-1}{2} \end{aligned} \quad (2.49)$$

The $()$ means symmetrization of the indices with weight 1

A useful observation is that for the highest superspin case we can solve the constraints if

$$\Phi_{\alpha(n)\dot{\alpha}(m)} = -\frac{1}{m^2} D^{\alpha_{n+1}} W_{\alpha(n+1)\dot{\alpha}(m)} \quad , \quad W_{\alpha(n+1)\dot{\alpha}(m)} = \frac{1}{(n+1)!} \bar{D}^2 D_{(\alpha_{n+1}} \Phi_{\alpha(n)\dot{\alpha}(m)}$$

where $W_{\alpha(n+1)\dot{\alpha}(m)}$ is chiral and it satisfies $\partial^{\beta\dot{\beta}} W_{\beta\alpha(n)\dot{\beta}\dot{\alpha}(m-1)} = 0$. These are exactly the constraints for a chiral superfield to describe a superspin $Y = \frac{n+m+1}{2}$ representation. So we see an equivalence between $\Phi_{\alpha(n)\dot{\alpha}(m)}$ and $W_{\alpha(n+1)\dot{\alpha}(m)}$. A similar statement can be made for the lowest superspin case where

$$\Phi_{\alpha(n)\dot{\alpha}(m)} = \frac{1}{m^2} \frac{n}{(n+1)!} D_{(\alpha_n} W_{\alpha(n-1)\dot{\alpha}(m)} \quad , \quad W_{\alpha(n-1)\dot{\alpha}(m)} = \bar{D}^2 D^{\alpha_n} \Phi_{\alpha(n)\dot{\alpha}(m)}$$

with $\partial^{\beta\dot{\beta}} W_{\beta\alpha(n-2)\dot{\beta}\dot{\alpha}(m-1)} = 0$

From the connection between the superspin Casimir operator (\vec{S}^2) and the spin Casimir operator ($W_{\text{Poincaré}}^2$) we get that the spin content of a superspin Y irreducible representation is

$$j = Y + \frac{1}{2}, \quad j = Y, \quad j = Y, \quad j = Y - \frac{1}{2} \quad (2.50)$$

2.7.2 Massless Case

For the massless case the expression for Z_m in terms of the covariant derivatives is:

$$Z_m = W_m^{\text{Poincaré}} + \frac{1}{8} (\bar{\sigma}_m)^{\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] \quad (2.51)$$

and we have to make it proportional to P_m

$$Z_m \Phi_{\alpha(n)\dot{\alpha}(m)} = \left(Y + \frac{1}{4}\right) P_m \Phi_{\alpha(n)\dot{\alpha}(m)}$$

That happens in two different ways

1. Highest superhelicity chiral superfield

$$\bar{D}_{\dot{\gamma}}\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad D^{\beta}\Phi_{\beta\alpha(n-1)\dot{\alpha}(m)} = 0 \quad , \quad \partial_{\gamma}^{\dot{\beta}}\Phi_{\alpha(n)\dot{\beta}\dot{\alpha}(m-1)} = 0 \quad (2.52)$$

$$Y = \frac{n-m}{2}$$

2. Lowest superhelicity anti-chiral superfield

$$D_{\gamma}\Phi_{\alpha(n)\dot{\alpha}(m)} = 0 \quad , \quad \bar{D}^{\dot{\beta}}\Phi_{\alpha(n)\dot{\beta}\dot{\alpha}(m-1)} = 0 \quad , \quad \partial^{\beta}_{\dot{\gamma}}\Phi_{\beta\alpha(n-1)\dot{\alpha}(m)} = 0 \quad (2.53)$$

$$Y = \frac{n-m-1}{2}$$

and the helicity content of a superhelicity Y representation is

$$\lambda = Y + \frac{1}{2}, \quad \lambda = Y$$

2.8 Real Representations

We showed earlier that conjugation maps a superfield of type (n, m) to a superfield of type (m, n)

$$\bar{\Phi}_{\alpha(m)\dot{\alpha}(n)} = (-1)^{n+m} (\Phi_{\alpha(n)\dot{\alpha}(m)})^*$$

But there is another way to exchange all the undotted indices to dotted ones and vice versa. That is to use the $\partial_{\alpha\dot{\alpha}}$ to convert one type of index to another. Because we don't want to change the mass dimension of the superfield we define the operator

$$\Delta_{\alpha\dot{\alpha}} = -i \frac{\partial_{\alpha\dot{\alpha}}}{\sqrt{\square}}$$

which has all the properties of the partial derivative, like

$$\Delta_\alpha^{\dot{\beta}} \Delta_{\dot{\alpha}}^\beta = \Delta_{\alpha\dot{\alpha}} \Delta^{\gamma\dot{\gamma}} + \delta^\gamma_\alpha \delta^{\dot{\gamma}}_{\dot{\alpha}}$$

We can act with a sequence of these operators on a (n, m) superfield and convert it to a (m, n) superfield

$$\Phi_{\alpha(n)\dot{\alpha}(m)} \rightarrow \tilde{\Phi}_{\alpha(m)\dot{\alpha}(n)} = \Delta_{\alpha_1}^{\dot{\gamma}_1} \dots \Delta_{\alpha_m}^{\dot{\gamma}_m} \Delta^{\gamma_1}_{\dot{\alpha}_1} \dots \Delta^{\gamma_n}_{\dot{\alpha}_n} \Phi_{\gamma(n)\dot{\gamma}(m)} \quad (2.54)$$

In a real representation we should have

$$\bar{\Phi}_{\alpha(m)\dot{\alpha}(n)} = \tilde{\Phi}_{\alpha(m)\dot{\alpha}(n)}$$

For a massive representation where $\Delta^{\gamma\dot{\gamma}} \Phi_{\gamma\alpha(n-1)\dot{\gamma}\dot{\alpha}(m-1)} = 0$, we get

$$\text{for } n = m, \quad \bar{\Phi}_{\alpha(n)\dot{\alpha}(n)} = \Phi_{\alpha(n)\dot{\alpha}(n)} \quad (\text{reality})$$

$$\text{for } n = m + 1, \quad i\partial_{\alpha_n}^{\dot{\alpha}_n} \bar{\Phi}_{\alpha(n-1)\dot{\alpha}(n)} + m\Phi_{\alpha(n)\dot{\alpha}(n-1)} = 0 \quad (\text{Dirac equation})$$

2.9 Superfields for (Half)Integer superspin / superhelicity representations

Having all the above in mind the question remains what are the proper superfields to describe an irreducible superspin / superhelicity Y representation. We showed that a superfield with a specific index structure can describe different representations depending on the constraints we impose. So let's focus on the highest possible superspin / superhelicity a superfield can describe.

For the massive case that means that the total number of indices must be $n + m = 2Y - 1$. There is a finite list of possible superfields with that feature

$(2Y - 1, 0)$, $(2Y - 2, 1)$, \dots , $(1, 2Y - 2)$, $(0, 2Y - 1)$ but all of them are equivalent choices and can be connected to each other through the $\Delta^{\alpha}_{\dot{\alpha}}$ operator. More specifically:

1. An integer superspin $Y = s$, has as a highest spin a fermion ($j = s + 1/2$) therefore we would like to recover the Dirac equation. That is exactly the reality condition for the case of $m = n - 1$. For this reason we should build the theory based on a spinorial superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$. Also based on the mass dimensions discussion above, we conclude that it must have mass dimension $1/2$
2. A half-integer superspin $Y = s + 1/2$, has as a highest spin component a boson ($j = s + 1$) which suggests that we can impose a reality condition directly. That can be done if $n = m$. Thus the construction of this theory must be based on a real bosonic superfield $H_{\alpha(s)\dot{\alpha}(s)}$ and its mass dimension must be 0

We do the same for the massless case. If we want to describe a superhelicity Y and it is the highest superhelicity we can describe then the superfield we should use must have an index structure such that $n - m = 2Y$. In this case there is an infinite list of possible superfields that have this feature: $(2Y, 0)$, $(2Y + 1, 1)$, \dots . But all of them can be generated by $(2Y, 0)$. For example the $(2Y + n, n)$ can be written as the action of $n \Delta_{\alpha\dot{\alpha}}$ on $(2Y, 0)$ with all the indices symmetrized. So we conclude that the massless superhelicity Y will be described by a chiral $F_{\alpha(2Y)}$ superfield. Its conjugate $\bar{F}_{\dot{\alpha}(2Y)}$ will describe the $-Y$ superhelicity, so we will have a \mathcal{CPT} complete theory.

Chapter 3: Lagrangians for massless representations

In this chapter we present the superspace and component lagrangians that will dynamically generate the constraints needed to describe on-shell a massless irreducible representation.

3.1 Massless representations as the limit of massive representations

The construction of all massless theories will be based on the demand that they are the massless limit of a massive irreducible superspin theory. In other words, assuming that we have a lagrangian that describes a massive irreducible representation of superspin Y then we must be able to take the massless limit of it. We want this limit to give the massless irreducible theory of a superhelicity with the same value Y , plus possibly other things that decouple. So if $S_Y^{(m)}$ is the action that describes a massive irreducible representation of superspin Y and S_Y is the action for a massless irreducible representation of superhelicity Y , then we want to have

$$\lim_{m \rightarrow 0} S_Y^{(m)} = S_Y + \text{other things} \quad (3.1)$$

This simple demand will introduce non-trivial connections between the superfields used to describe the two theories.

3.2 Integer superhelicity $Y = s$

A theory of massive integer superspin $Y = s$ must be constructed in terms of a fermionic superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$, or equivalently a chiral superfield $W_{\alpha(s+1)\dot{\alpha}(s-1)} \sim \bar{D}^2 D_{(\alpha_{s+1}} \Psi_{\alpha(s))\dot{\alpha}(s-1)}$. On the other hand the theory of massless integer superhelicity must be described in terms of a chiral superfield $F_{\alpha(2s)}$. But then if we use the above definition (3.1) of the massless theory, there must be a way to get $F_{\alpha(2s)}$ out of $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ or $W_{\alpha(s+1)\dot{\alpha}(s-1)}$

Given the chirality properties of F and W and their index structures we could guess a map between the two.

$$F_{\alpha(2s)} \sim \partial_{(\alpha_{2s}}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+2}}^{\dot{\alpha}_1} \bar{D}^2 D_{\alpha_{s+1}} \Psi_{\alpha(s))\dot{\alpha}(s-1)} \quad (3.2)$$

However this identification can not be valid as it is. The problem is that the natural variable (F) for the description of the massless theory and the physical degrees of freedom it carries, seems to be defined in terms of Ψ . That suggests that, Ψ is the fundamental object and not F . But the whole representation theory discussion says otherwise. Also F as defined above seems to have the on-shell degrees of freedom of Ψ which is more than needed. If this is going to work we have to find a way to 1) remove the physical (observable) status of Ψ and 2) remove its extra degrees of freedom.

There is a mechanism that can do both of them at the same time. That is to introduce a redundancy. We identify $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ with $\Psi_{\alpha(s)\dot{\alpha}(s-1)} + R_{\alpha(s)\dot{\alpha}(s-1)}$ and

instead of talking about $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ we talk about equivalence classes

$$\Psi_{\alpha(s)\dot{\alpha}(s-1)} \sim \Psi_{\alpha(s)\dot{\alpha}(s-1)} + R_{\alpha(s)\dot{\alpha}(s-1)}$$

If that's the case then the redundancy has to respect the physical/propagating degrees of freedom of F . Hence

$$\partial_{(\alpha_{2s}}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+2}}^{\dot{\alpha}_1} \bar{D}^2 D_{\alpha_{s+1}} R_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad (3.3)$$

The most general solution to that is

$$R_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s!} D_{(\alpha_s} K_{\alpha(s-1))\dot{\alpha}(s-1)} + \frac{1}{(s-1)!} \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2)})$$

From the above it is obvious that this redundancy will be what we call gauge symmetry. Then it is obvious that this symmetry at least from the representation point of view is not something deeply fundamental but only a choice that we make. The choice we make is to describe the massless systems in terms of the variables that describe the massive system and we do that so there is a smooth transition between the two when we take the mass to zero limit.

3.2.1 The superspace action

Using the equivalence class of Ψ and the idea of redundancy we attempt to construct an action that will describe the irreducible representation of integer superhelicity. Because Ψ has mass dimensions 1/2 and appears quadratically, the action must involve two covariant derivatives D or \bar{D} .

The most general action that we can write is

$$\begin{aligned}
S = \int d^8z & a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_3 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_4 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}
\end{aligned} \tag{3.4}$$

and it has to depend only on the equivalence class of Ψ . The *c.c.* stands for the complex (hermitian) conjugate of the corresponding term, so the superspace action is real.

The goal is to get $\delta_G S = 0$ which is the start of the gauge invariance story. The strategy to do that is to pick the free parameters in a special way. If this is not possible then we introduce auxiliary superfields, called compensators and/or put constraints on the parameters of the redundancy (gauge parameters). It is reasonable to expect that the compensators introduced, if necessary, will not introduce degrees of freedom with spin higher than or equal to the one we want to describe. Therefore they must be superfields with lower rank.

The deformation of the action under the gauge transformation of Ψ is

$$\begin{aligned}
\delta_G S = \int d^8 z \left[\right. & \left\{ -2a_1 D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} \right. \\
& \left. + a_4 \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right\} D^\beta \bar{D}_{\dot{\alpha}_{s-1}} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \\
& + \left\{ -a_3 \left[\frac{s-1}{s} \right] \bar{D}_{\dot{\alpha}_s} D_{\alpha_{s-1}} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right. \\
& \left. + \left[-a_3 + \frac{s+1}{s} a_4 \right] D_{\alpha_{s-1}} \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right\} D^\beta K_{\beta\alpha(s-2)\dot{\alpha}(s-1)} \\
& + \left\{ 2a_2 D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} - a_3 \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right\} K_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& \left. + c.c. \right] \tag{3.5}
\end{aligned}$$

We see that there is no non-trivial choice for the free parameters in order to make all the coefficients vanish. A possible exception for that will be the $s = 1$ case that will be discussed later. Also we can not introduce compensators to cancel some of these terms because either they will have to be the same rank as Ψ or have an algebraic term in their transformation which means that we can gauge it away. There is one option left, to constrain the gauge parameters. The only possible constraint will be $D^\beta K_{\beta\alpha(s-2)\dot{\alpha}(s-1)} = 0$ because everything else will drastically reduce the gauge symmetry of Ψ . If we choose:

$$\begin{aligned}
a_1 = a_4 = 0 \\
D^\beta K_{\beta\alpha(s-2)\dot{\alpha}(s-1)} = 0 \rightarrow K_{\alpha(s-1)\dot{\alpha}(s-1)} = D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} \\
2a_2 = -a_3
\end{aligned} \tag{3.6}$$

where $L_{\alpha(s)\dot{\alpha}(s-1)}$ is a completely free superfield, then the change of the action takes

the following form:

$$\delta_G S = -a_3 \int d^8 z D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \left(D^\beta L_{\beta\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\beta}} \bar{L}_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} \right) \quad (3.7)$$

$+c.c.$

This suggests to introduce a real bosonic compensator $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ which transforms like $\delta_G V_{\alpha(s-1)\dot{\alpha}(s-1)} = D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)}$ and couples to the real piece of $D^{\alpha_s} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)}$.

In order to get invariance we add to the action a few new terms, the coupling term of V with Ψ and the kinetic energy term for V , in other words the most general quadratic action for V . The full action becomes

$$\begin{aligned} S = \int d^8 z \left\{ -\frac{1}{2} a_3 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right. \\ + a_3 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\ - a_3 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^{\alpha_s} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ + b_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ + b_2 V^{\alpha(s-1)\dot{\alpha}(s-1)} \{D^2, \bar{D}^2\} V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ + b_3 V^{\alpha(s-1)\dot{\alpha}(s-1)} D_{\alpha_{s-1}} \bar{D}^2 D^\gamma V_{\gamma\alpha(s-2)\dot{\alpha}(s-1)} + c.c. \\ \left. + b_4 V^{\alpha(s-1)\dot{\alpha}(s-1)} D_{\alpha_{s-1}} \bar{D}^{\dot{\alpha}_{s-1}} D^\gamma \bar{D}^{\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} + c.c. \right\} \quad (3.8) \end{aligned}$$

and it has to be invariant under

$$\begin{aligned} \delta_G \Psi_{\alpha(s)\dot{\alpha}(s-1)} &= -D^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \left[\frac{1}{(s-1)!} \right] \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2)} \quad (3.9) \\ \delta_G V_{\alpha(s-1)\dot{\alpha}(s-1)} &= D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} \end{aligned}$$

The equations of motion of the superfields are the variation of the action with

respect to the corresponding superfield,

$$\begin{aligned}
T_{\alpha(s)\dot{\alpha}(s-1)} &= \frac{\delta S}{\delta \Psi^{\alpha(s)\dot{\alpha}(s-1)}} , \quad G_{\alpha(s-1)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta V^{\alpha(s-1)\dot{\alpha}(s-1)}} \\
T_{\alpha(s)\dot{\alpha}(s-1)} &= -a_3 \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + \frac{a_3}{s!} \bar{D}^{\dot{\alpha}_s} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} + \frac{a_3}{s!} \bar{D}^2 D_{(\alpha_s} V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
G_{\alpha(s-1)\dot{\alpha}(s-1)} &= -a_3 (D^{\alpha_s} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} D^2 \bar{\Psi}_{\alpha(s-1)\alpha(s)}) \\
&\quad + 2b_1 D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} + 2b_2 \{D^2, \bar{D}^2\} V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&\quad + \frac{2b_3}{(s-1)!} (D_{(\alpha_{s-1}} \bar{D}^2 D^\gamma V_{\gamma\alpha(s-2)\dot{\alpha}(s-1)} + \bar{D}_{(\dot{\alpha}_{s-1}} D^2 \bar{D}^{\dot{\gamma}} V_{\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)})) \\
&\quad + \frac{2b_4}{(s-1)!(s-1)!} D_{(\alpha_{s-1}} \bar{D}_{(\dot{\alpha}_{s-1}} D^\gamma \bar{D}^{\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)}) \\
&\quad + \frac{2b_4}{(s-1)!(s-1)!} \bar{D}_{(\dot{\alpha}_{s-1}} D_{(\alpha_{s-1}} \bar{D}^{\dot{\gamma}} D^\gamma V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)})
\end{aligned} \tag{3.10}$$

The invariance of the action will give two Bianchi identities, one for each gauge parameter:

$$\begin{aligned}
\delta_G S &= \int d^8 z \left\{ \delta_G \Psi^{\alpha(s)\dot{\alpha}(s-1)} \frac{\delta S}{\delta \Psi^{\alpha(s)\dot{\alpha}(s-1)}} + c.c. \right. \\
&\quad \left. + \delta_G V^{\alpha(s-1)\dot{\alpha}(s-1)} \frac{\delta S}{\delta V^{\alpha(s-1)\dot{\alpha}(s-1)}} \right\} \\
&= \int d^8 z \left\{ L^{\alpha(s)\dot{\alpha}(s-1)} \left[D^2 T_{\alpha(s)\dot{\alpha}(s-1)} + \frac{1}{s!} D_{(\alpha_s} G_{\alpha(s-1)\dot{\alpha}(s-1)} \right] + c.c. \right. \\
&\quad \left. + \Lambda^{\alpha(s)\dot{\alpha}(s-2)} \left[\bar{D}^{\dot{\alpha}_{s-1}} T_{\alpha(s)\dot{\alpha}(s-1)} \right] + c.c. \right\} \\
&= 0 \quad \forall L_{\alpha(s-1)\dot{\alpha}(s-1)}, \Lambda_{\alpha(s)\dot{\alpha}(s-2)}
\end{aligned} \tag{3.11}$$

therefore we must have

$$D^2 T_{\alpha(s)\dot{\alpha}(s-1)} + \frac{1}{s!} D_{(\alpha_s} G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \tag{3.12a}$$

$$\bar{D}^{\dot{\alpha}_{s-1}} T_{\alpha(s)\dot{\alpha}(s-1)} = 0 \tag{3.12b}$$

These Bianchi identities will be proven to be extremely powerful and they contain the full information about the system, not only at the superfield level but

also in components as we will see. For now we use them to determine all the coefficients. The satisfaction of the Bianchi identities fixes all the coefficients

$$b_1 = \frac{1}{2}a_3, \quad b_3 = 0, \quad b_2 = 0, \quad b_4 = 0 \quad (3.13)$$

and the final expression for the superspace action is

$$\begin{aligned} S = \int d^8z \left\{ -\frac{1}{2}c\Psi^{\alpha(s)\dot{\alpha}(s-1)}\bar{D}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right. \\ \left. + c\Psi^{\alpha(s)\dot{\alpha}(s-1)}\bar{D}^{\dot{\alpha}_s}D_{\alpha_s}\bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \right. \\ \left. - cV^{\alpha(s-1)\dot{\alpha}(s-1)}D^{\alpha_s}\bar{D}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right. \\ \left. + \frac{1}{2}cV^{\alpha(s-1)\dot{\alpha}(s-1)}D^\gamma\bar{D}^2D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} \quad (3.14) \end{aligned}$$

The equations of motion are

$$\begin{aligned} T_{\alpha(s)\dot{\alpha}(s-1)} &= -c\bar{D}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)} + \frac{c}{s!}\bar{D}^{\dot{\alpha}_s}D_{(\alpha_s}\bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\ &\quad + \frac{c}{s!}\bar{D}^2D_{(\alpha_s}V_{\alpha(s-1)\dot{\alpha}(s-1)} \quad (3.15a) \end{aligned}$$

$$\begin{aligned} G_{\alpha(s-1)\dot{\alpha}(s-1)} &= -c\left(D^{\alpha_s}\bar{D}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s}D^2\bar{\Psi}_{\alpha(s-1)\alpha(s)}\right) \\ &\quad + cD^\gamma\bar{D}^2D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \quad (3.15b) \end{aligned}$$

where c is an overall unconstrained parameter which can be absorbed into the definition of Ψ . We leave it as it is for now and fix it later in the component discussion.

3.2.2 On-shell equations of motion

Now we must check that this action on-shell describes an irreducible integer superhelicity $Y = s$. That means that we must find a chiral superfield $F_{\alpha(2s)}$, in

agreement with the map (3.2) and it has to satisfy on-shell the required constraints (2.52). For that purpose we use the equations of motion to generate terms like the right hand side of (3.2) and we get:

$$\begin{aligned}
& \frac{1}{(2s-1)!} \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_s}_{\dot{\alpha}_{2s-1}} T_{\alpha(s)\dot{\alpha}(s-1)} = \\
& = -\frac{c}{(2s-1)!} \bar{D}^2 \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& + \frac{ic}{(2s)!} \bar{D} \dot{\gamma} D^2 \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s-1)\dot{\gamma}} \\
& + \frac{ic(2s-1)}{(2s)!} \bar{D}_{(\dot{\alpha}_{2s-1}} D^2 X_{\dot{\alpha}(2s-2)} \\
& - \frac{ic}{(2s-1)!} \bar{D}^2 D^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \frac{ic}{(2s-1)!} \bar{D}^2 D^2 \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} V_{\alpha(s-1)\dot{\alpha}(s-1)}
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
& \frac{1}{(2s-1)!} \bar{D}^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)} = \\
& = -\frac{c}{(2s-1)!} \bar{D}^2 D^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \frac{ic}{(2s-1)!} \bar{D}^2 \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& + \frac{c}{(2s-1)!} \bar{D}^2 D^2 \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} V_{\alpha(s-1)\dot{\alpha}(s-1)}
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
& \frac{1}{(2s-1)!} \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} G_{\alpha(s-1)\dot{\alpha}(s-1)} = \\
& = -\frac{ic}{(2s-1)!} \bar{D}^2 \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& + \frac{c}{(2s-1)!} \bar{D}^2 D^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - \frac{c}{(2s-1)!} \bar{D}^2 D^2 \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} V_{\alpha(s-1)\dot{\alpha}(s-1)}
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
& \frac{1}{(2s-1)!} \bar{D}_{(\dot{\alpha}_{2s-1}} D^\gamma \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} T_{\gamma\alpha(s-1)\dot{\alpha}(s-1)}) = \\
& = -\frac{ic}{(2s-1)!} \bar{D}^2 \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)}) \\
& \quad + \frac{c}{(2s-1)!} \bar{D}^2 D^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}) \\
& \quad - \frac{c(2s-1)}{s(2s-1)!} \bar{D}_{(\dot{\alpha}_{2s-1}} D^2 X_{\dot{\alpha}(2s-2)}) \\
& \quad - \frac{c}{(2s-1)!} \bar{D}^2 D^2 \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} V_{\alpha(s-1)\dot{\alpha}(s-1)})
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
X_{\dot{\alpha}(2s-2)} &= \frac{1}{(2s-1)!} \bar{D}^{\dot{\gamma}} \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)}) \\
& \quad + \frac{s-1}{(2s-1)!} \partial^{\gamma\dot{\gamma}} \bar{D}_{(\dot{\alpha}_{2s-2}} \partial^{\alpha_{s-2}}_{\dot{\alpha}_{2s-3}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} \bar{\Psi}_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-1)})
\end{aligned}$$

Now we can take a linear combination of the above equations:

$$\begin{aligned}
& \frac{A}{(2s-1)!} \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} T_{\alpha(s)\dot{\alpha}(s-1)}) \\
& + \frac{B}{(2s-1)!} \bar{D}^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)}) \\
& + \frac{\Gamma}{(2s-1)!} \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} G_{\alpha(s-1)\dot{\alpha}(s-1)}) \\
& + \frac{\Delta}{(2s-1)!} \bar{D}_{(\dot{\alpha}_{2s-1}} D^\gamma \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} T_{\gamma\alpha(s-1)\dot{\alpha}(s-1)}) = \\
& = \frac{c}{(2s-1)!} (-A + iB - i\Gamma - i\Delta) \bar{D}^2 \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)}) \\
& \quad + \frac{ic}{(2s-1)!} (-A + iB - i\Gamma - i\Delta) \bar{D}^2 D^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}) \\
& \quad - \frac{ic}{(2s-1)!} (-A + iB - i\Gamma - i\Delta) \bar{D}^2 D^2 \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} V_{\alpha(s-1)\dot{\alpha}(s-1)}) \\
& \quad + \frac{c}{(2s-1)!} \left(iA \frac{2s-1}{2s} - \Delta \frac{2s-1}{s} \right) \bar{D}_{(\dot{\alpha}_{2s-1}} D^2 X_{\dot{\alpha}(2s-2)}) \\
& \quad + icA \bar{D}^{\dot{\alpha}_{2s}} \bar{F}_{\dot{\alpha}(2s)}
\end{aligned} \tag{3.20}$$

From this it is obvious that if we choose $\Gamma = \frac{i}{2}A + B$, $\Delta = \frac{i}{2}A$, $icA = 1$

then we get the identity

$$\begin{aligned}
\bar{D}^{\dot{\alpha}_{2s}} \bar{F}_{\dot{\alpha}(2s)} = & -\frac{i}{(2s-1)!c} \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}) \\
& + \frac{B}{(2s-1)!} \bar{D}^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)}) \\
& + \frac{1+2cB}{(2s-1)!2c} \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} G_{\alpha(s-1)\dot{\alpha}(s-1)}) \\
& + \frac{1}{(2s-1)!2c} \bar{D}_{(\dot{\alpha}_{2s-1}} D^{\alpha_s} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)})
\end{aligned} \tag{3.21}$$

where

$$\bar{F}_{\dot{\alpha}(2s)} = \frac{1}{(2s)!} D^2 \bar{D}_{(\dot{\alpha}_{2s}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-1}} \cdots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)})$$

That proves that on-shell ($T_{\alpha(s)\dot{\alpha}(s-1)} = 0$, $G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0$) we get

$$D^{\alpha_{2s}} F_{\alpha(2s)} = 0$$

and by definition, $\bar{D}_{\dot{\beta}} F_{\alpha(2s)} = 0$. Therefore it describes an integer superhelicity $Y = s$ system. In the above expression B is a completely free parameter and can be set to zero or any other value.

3.2.3 A two parameter family of superspace actions

The action (3.14) is not unique but a representative of a two parameter family of equivalent theories. The mass dimensions and index structure of $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ and $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ allow us to make the following superfield redefinitions:

$$\Psi_{\alpha(s)\dot{\alpha}(s-1)} \rightarrow \Psi_{\alpha(s)\dot{\alpha}(s-1)} + \frac{z}{s!} D_{(\alpha_s} V_{\alpha(s-1)\dot{\alpha}(s-1)} \tag{3.22}$$

where z is a complex parameter. This operation will generate an entire class of actions and transformation laws which all are related by the above redefinition.

The general action takes the form:

$$\begin{aligned}
S = & \int d^8 w \left\{ -\frac{1}{2} c \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right. \\
& + c \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + c(z + \bar{z} - 1) V^{\alpha(s-1)\dot{\alpha}(s-1)} D^{\alpha_s} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + c\bar{z} V^{\alpha(s-1)\dot{\alpha}(s-1)} \bar{D}^2 D^{\alpha_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& - \left[\frac{s-1}{s} \right] c\bar{z} V^{\alpha(s-1)\dot{\alpha}(s-1)} \bar{D}_{\dot{\alpha}_{s-1}} D^{\beta} \bar{D}^{\dot{\beta}} \Psi_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} + c.c. \\
& + \frac{1}{2} c(z + \bar{z} - 1)^2 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^{\gamma} \bar{D}^2 D_{\gamma} V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \left[\frac{1}{s} \right] cz\bar{z} V^{\alpha(s-1)\dot{\alpha}(s-1)} \{D^2, \bar{D}^2\} V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \left[\frac{s-1}{2s} \right] cz(z + 2\bar{z} - 2) V^{\alpha(s-1)\dot{\alpha}(s-1)} D_{\alpha_{s-1}} \bar{D}^2 D^{\gamma} V_{\gamma\alpha(s-2)\dot{\alpha}(s-1)} + c.c. \\
& \left. - \left[\frac{(s-1)^2}{2s^2} \right] cz\bar{z} V^{\alpha(s-1)\dot{\alpha}(s-1)} D_{\alpha_{s-1}} \bar{D}_{\dot{\alpha}_{s-1}} D^{\gamma} \bar{D}^{\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} + c.c. \right\} \tag{3.23}
\end{aligned}$$

and the transformation laws are

$$\begin{aligned}
\delta_G \Psi_{\alpha(s)\dot{\alpha}(s-1)} = & (z-1) D^2 L_{\alpha(s)\dot{\alpha}(s-1)} - \frac{z}{s!} D_{(\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \left[\frac{1}{(s-1)!} \right] \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2)} \tag{3.24a}
\end{aligned}$$

$$\delta_G V_{\alpha(s-1)\dot{\alpha}(s-1)} = D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} \tag{3.24b}$$

The equations of motion are

$$\begin{aligned}
T_{\alpha(s)\dot{\alpha}(s-1)} = & -c\bar{D}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)} + \frac{c}{s!}\bar{D}^{\dot{\alpha}s}D_{(\alpha_s}\bar{\Psi}_{\alpha(s-1))\dot{\alpha}(s)} \\
& -\frac{c}{s!}[z+\bar{z}-1]\bar{D}^2D_{(\alpha_s}V_{\alpha(s-1))\dot{\alpha}(s-1)} \\
& -\frac{c}{s!}[\bar{z}]D_{(\alpha_s}\bar{D}^2V_{\alpha(s-1))\dot{\alpha}(s-1)} \\
& +\frac{c}{s!(s-1)!s}[\bar{z}]\bar{D}_{(\dot{\alpha}_{s-1}}D_{(\alpha_s}\bar{D}^{\dot{\gamma}}V_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
G_{\alpha(s-1)\dot{\alpha}(s-1)} = & c[z+\bar{z}-1](D^{\alpha_s}\bar{D}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}s}D^2\bar{\Psi}_{\alpha(s-1)\alpha(s)}) \\
& +c[\bar{z}]D^2D^{\alpha_s}\Psi_{\alpha(s)\dot{\alpha}(s-1)} + c[z]\bar{D}^2\bar{D}^{\dot{\alpha}s}\bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& -c\frac{s-1}{s!}[\bar{z}]\bar{D}_{\dot{\alpha}_{s-1}}D^{\alpha_s}\bar{D}^{\dot{\gamma}}\Psi_{\alpha(s)\dot{\gamma}\dot{\alpha}(s-2)} \\
& -c\frac{s-1}{s!}[z]D_{\alpha_{s-1}}\bar{D}^{\dot{\alpha}s}D^{\gamma}\bar{\Psi}_{\gamma\alpha(s-2))\dot{\alpha}(s)} \\
& +2\frac{c}{s}[z\bar{z}]\{D^2, \bar{D}^2\}V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& +c[z^2+\bar{z}^2+2z\bar{z}-2z-2\bar{z}+1]D^{\gamma}\bar{D}^2D_{\gamma}V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& +c\frac{s-1}{s!}[z^2+2z\bar{z}-2z]D_{(\alpha_{s-1}}\bar{D}^2D^{\gamma}V_{\gamma\alpha(s-2))\dot{\alpha}(s-1)} \\
& +c\frac{s-1}{s!}[\bar{z}^2+2z\bar{z}-2\bar{z}]\bar{D}_{(\dot{\alpha}_{s-1}}D^2\bar{D}^{\dot{\gamma}}V_{\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)} \\
& -c\frac{(s-1)^2}{s!s!}[z\bar{z}]D_{(\alpha_{s-1}}\bar{D}_{(\dot{\alpha}_{s-1}}D^{\gamma}\bar{D}^{\dot{\gamma}}V_{\gamma\alpha(s-2))\dot{\gamma}\dot{\alpha}(s-2)} \\
& -c\frac{(s-1)^2}{s!s!}[z\bar{z}]\bar{D}_{(\dot{\alpha}_{s-1}}D_{(\alpha_{s-1}}\bar{D}^{\dot{\gamma}}D^{\dot{\gamma}}V_{\gamma\alpha(s-2))\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned} \tag{3.26}$$

3.2.4 Field spectrum and components action

Although superspace was developed so we can study supersymmetric theories in a more efficient, compact and clear way, there are still some reasons why we would like to study the off-shell component structure of the theory.

1. There are cases where two theories on-shell describe the same physical system. Therefore from the path integral point of view the theories are equivalent. Nevertheless the off-shell structure of the two theories might be completely different. Knowledge of the component formulation of the two theories will help us decide if they are different theories with the same on-shell description or they are the same and one theory can be translated to the other.
2. The off-shell component structure of a supersymmetric theory will give us clues about which theories can be used to realize higher \mathcal{N} and higher D representations
3. To make contact with the effective field theoretic limit of superstring theory

This section focuses on the analysis of the theory's off-shell field spectrum.

Let's start by thinking about what kind of fields we get and how the final theory may depend on them. The discussion in section 2.5 makes it obvious that a superfield has three different sets of fields:

1. Set \mathcal{D} : The Dynamical fields

We know from representation theory that an irreducible representation of superhelicity Y will contain an irreducible representation of helicity $\lambda = Y$ and one of helicity $\lambda = Y + 1/2$. So we are expecting a set of dynamical fields (mass dimension 1 or $3/2$) which are exactly the fields needed to describe the above theories. These fields must have a specific gauge transformation and we can use them to build field strengths which are gauge invariant.

2. Set \mathcal{A}_l : The low mass dimension Auxiliary fields

These are auxiliary fields that exist so that the supersymmetry algebra within superfields closes. They have low mass dimension (0 or 1/2) and for that reason their kinetic energy terms have more than two derivatives. Also because of their low mass dimension their gauge transformations include algebraic terms. They appear in the action and the susy-transformation laws but the theory doesn't depend on them. This means that we can go to a gauge (use the algebraic terms in their transformations) to gauge them away. This gauge is called the Wess-Zumino (W-Z) gauge. Alternatively instead of picking a gauge we can use them to redefine other fields and in this way eliminate them from the action and the susy-transformation laws.

3. Set \mathcal{A}_h : The high mass dimension Auxiliary fields

These are auxiliary fields like the previous kind but they have higher mass dimensionality (3/2 or 2 or 5/2). They must be present off-shell so the action is supersymmetric invariant and the algebra closes without the need for any constraints. Because of their higher mass dimension their kinetic energy terms are algebraic. Also on-shell they must vanish, so that the only fields left are the dynamical ones. We can use them to do two kinds of redefinitions. One type of redefinition includes derivatives acting on elements of \mathcal{A}_l and is responsible for the elimination of the \mathcal{A}_l 's from the action (and susy-transformation laws) as described above. The other type of redefinition involves elements of \mathcal{A}_h and derivatives acting on elements of \mathcal{D} and its purpose is to remove from the

action any cross terms and bring the component action in a *diagonal* form.

The *diagonal* form of the action is when it takes the following form:

$$S \int d^4x \left\{ \mathcal{L}_{\lambda=Y} + \mathcal{L}_{\lambda=Y+\frac{1}{2}} + \text{quadratic monomials of auxiliary fields} \right\}$$

The action is the sum of the component lagrangian that describes helicity Y , the component lagrangian that describes helicity $Y + \frac{1}{2}$ and the sum of algebraic terms that involve only auxiliary fields, such that each auxiliary field appears in exactly one and only one term. For example if A and B are auxiliary fields, acceptable terms for the quadratic monomials can be A^2 or AB but not $A^2 + AB$. The reason why something like that is desirable and useful is because it does three things:

- (a) It makes it obvious that on-shell the theory describes helicities Y and $Y + 1/2$
- (b) It makes obvious the auxiliary status of the auxiliary fields (they vanish on-shell $A = 0$)
- (c) It makes the auxiliary fields to be gauge invariant ($\delta_G A = 0$). The gauge invariance of the auxiliary fields is desirable because it will make the counting of the degrees of freedom extremely easy, since the dynamical fields will be the only ones that have gauge transformations.

The standard ‘algorithm’ to find the components of the theory and it’s component action is the following

1. We define the components as the coefficients in the Taylor expansion of the

superfields that participate in the action, as in (2.32).

2. We find the gauge transformation laws of these components.
3. We identify the ones that have algebraic terms in their transformation law and therefore can be eliminated in a W-Z gauge. These fields can be gauged away and the rest are the off-shell field spectrum of the theory.
4. The definition of the component action is in equation (2.31): $\bar{D}^2 D^2 \mathcal{L}|$ or $D^2 \bar{D}^2 \mathcal{L}|$ where \mathcal{L} is the superspace action.
5. We distribute the covariant derivatives and use the above definitions of fields to write the component action
6. Do redefinitions of the fields that appear algebraically in the action, to bring it in the *diagonal* form

This process is straightforward but cumbersome. Just the projection of the superspace action to components, as they have been defined in the θ expansion of the superfields in W-Z gauge is quite bulky. All this complexity arises because we are doing a very naive and brute force expansion of the superfields and plug this information to the action which is quadratic to the superfields. This generates a very large amount of terms which then by doing redefinitions we try to repackage them in a different way. We propose an alternative technique that will illuminate a more natural way to define the component structure and make the entire process of finding the component action and susy-transformation laws more efficient. This

technique is based on the equations of motion and their properties, such as the Bianchi identities.

Since we want the auxiliary fields of the final action to be gauge invariant it might be smart to define them using objects that are already gauge invariant. But the superspace action itself provides us with gauge invariant objects, the equations of motion. There is also the superfield strength $F_{\alpha(2s)}$ but because of mass dimensional reasons we can not use it to write the component action. So consider the following superfields:

$$T_{\alpha(s)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta \Psi^{\alpha(s)\dot{\alpha}(s-1)}} , \quad [T_{\alpha(s)\dot{\alpha}(s-1)}] = \frac{3}{2} \quad (3.27a)$$

$$G_{\alpha(s-1)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta V^{\alpha(s-1)\dot{\alpha}(s-1)}} , \quad [G_{\alpha(s-1)\dot{\alpha}(s-1)}] = 2 \quad (3.27b)$$

$$G_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{G}_{\alpha(s-1)\dot{\alpha}(s-1)}$$

Because they are gauge invariant, if we expand them to components, each one of them will be gauge invariant. Furthermore because they vanish on-shell each one of these components will vanish as well. So it looks like the ideal place to look for the auxiliary structure. These superfields satisfy a big list of identities, that we will discover as we go along. At the top of the list we have the Bianchi identities and their consequences:

$$D^2 T_{\alpha(s)\dot{\alpha}(s-1)} + \frac{1}{s!} D_{(\alpha_s} G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \rightsquigarrow D^2 G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \quad (3.28a)$$

$$\bar{D}^2 G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \quad (\text{reality})$$

$$\bar{D}^{\dot{\alpha}_{s-1}} T_{\alpha(s)\dot{\alpha}(s-1)} = 0 \rightsquigarrow \bar{D}^2 T_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad (3.28b)$$

The Bianchi identities force most of the components in the expansion of T and G to vanish and we are left with very few, that can be associated with auxiliary fields. For example the bosonic auxiliary fields (mass dimension 2) have to be related to $\bar{D}_{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}|$, $D^{\alpha_s} T_{\alpha(s)\dot{\alpha}(s-1)}|$, $D_{(\alpha(s+1)} T_{\alpha(s)\dot{\alpha}(s-1)}|$, $G_{\alpha(s-1)\dot{\alpha}(s-1)}|$ and the fermionic ones (dimension $\frac{3}{2}$, $\frac{5}{2}$) will have to be related to $T_{\alpha(s)\dot{\alpha}(s-1)}|$, $D^2 T_{\alpha(s)\dot{\alpha}(s-1)}|$. So by just looking at the Bianchi identities we get for free the spectrum of the auxiliary fields of the action. We can play a similar game for the dynamical fields using $F_{\alpha(2s)}$. Since we can use them to define field strengths and it is logical to expect that these field strengths will be components of $F_{\alpha(2s)}$ there is connection there that can give us an idea about the proper definition of the dynamical components. Instead we will let the action, the equations of motion and their properties to dictate their definition.

Following the idea that the equations of motion, and not the superfields, are the proper objects to define the components, we must be able to express the superspace action in terms of them, because the superspace action is the starting point of the projection story. That can be easily done by using the definitions of T and G (as the variation of the action) to rewrite the action in the following way:

$$\begin{aligned}
S &= \int d^8 z \left\{ \frac{1}{2} \Psi^{\alpha(s)\dot{\alpha}(s-1)} T_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right. \\
&\quad \left. + \frac{1}{2} V^{\alpha(s-1)\dot{\alpha}(s-1)} G_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} \\
&= \int d^4 x \left[\frac{1}{2} D^2 \bar{D}^2 (\Psi^{\alpha(s)\dot{\alpha}(s-1)} T_{\alpha(s)\dot{\alpha}(s-1)}) + c.c. \right. \\
&\quad \left. + \frac{1}{2} D^2 \bar{D}^2 (V^{\alpha(s-1)\dot{\alpha}(s-1)} G_{\alpha(s-1)\dot{\alpha}(s-1)}) \right]
\end{aligned} \tag{3.29}$$

This is the crucial difference between the equations of motion and the superfield strength $F_{\alpha(2s)}$. For the general superhelicity case dimensionality forbids us to use F to express the action.

Now we can go on with the process and distribute the covariant derivatives. To illustrate the structures appearing in the distribution of the covariant derivatives we give the following general formulas. In an abstract way the superspace action is the sum of quadratic terms like AB . Then the component lagrangian will be the sum of terms like this

$$\begin{aligned}
D^2\bar{D}^2(AB)| &= D^2\bar{D}^2 A| B| + (-1)^{\epsilon(A)} D^\rho\bar{D}^2 A| D_\rho B| + \bar{D}^2 A| D^2 B| \\
&+ (-1)^{\epsilon(A)} D^2\bar{D}^\rho A| \bar{D}_\rho B| - D^\rho\bar{D}^\rho A| D_\rho\bar{D}_\rho B| + (-1)^{\epsilon(A)}\bar{D}^\rho A| D^2\bar{D}_\rho B| \\
&+ D^2 A| \bar{D}^2 B| + (-1)^{\epsilon(A)} D^\rho A| D_\rho\bar{D}^2 B| + A| D^2\bar{D}^2 B| \\
&= \mathcal{L}_1 + \mathcal{L}_2
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
\mathcal{L}_1 &= D^2\bar{D}^2 A| B| + \bar{D}^2 A| D^2 B| - D^\rho\bar{D}^\rho A| D_\rho\bar{D}_\rho B| \\
&+ D^2 A| \bar{D}^2 B| + A| D^2\bar{D}^2 B| \\
\mathcal{L}_2 &= (-1)^{\epsilon(A)} D^\rho\bar{D}^2 A| + (-1)^{\epsilon(A)} D^2\bar{D}^\rho A| \\
&+ (-1)^{\epsilon(A)}\bar{D}^\rho A| D^2\bar{D}_\rho B| + (-1)^{\epsilon(A)} D^\rho A| D_\rho\bar{D}^2 B|
\end{aligned} \tag{3.31}$$

and depending on the nature of superfields involved (fermionic, $\epsilon(A) = 1$ or bosonic $\epsilon(A)=0$) one of them is the lagrangian for bosons and the other one for fermions.

3.2.4.1 Fermionic components lagrangian

Let's focus on the lagrangian for the fermions first. After the distribution of D 's and the usage of Bianchi identities we get the following expression:

$$\begin{aligned}
\mathcal{L}_F = & \frac{1}{2} D^2 \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} |T_{\alpha(s)\dot{\alpha}(s-1)}| \\
& + \frac{1}{2} \left(\bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} - \frac{1}{s!} \bar{D}^2 D^{(\alpha_s} V^{\alpha(s-1))\dot{\alpha}(s-1)} \right) |D^2 T_{\alpha(s)\dot{\alpha}(s-1)}| \\
& - \frac{1}{2} \frac{1}{(s+1)!s!} D^{(\alpha_{s+1}} \bar{D}^{(\dot{\alpha}_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)}) | \frac{1}{(s+1)!s!} D^{(\alpha_{s+1}} \bar{D}^{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}) | \\
& + \frac{1}{2} \frac{s}{s+1} \frac{1}{s!} D_\gamma \bar{D}^{(\dot{\alpha}_s} \Psi^{\gamma\alpha(s-1)\dot{\alpha}(s-1)}) | \frac{1}{s!} D^{\alpha_s} \bar{D}^{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}) | \\
& - \frac{s-1}{2s} \bar{D}^2 D_\gamma V^{\gamma\alpha(s-2)\dot{\alpha}(s-1)} | D^{\alpha_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-1)} | \\
& + c.c.
\end{aligned} \tag{3.32}$$

At this point we can show that T and G satisfy a few more identities:

$$\begin{aligned}
\frac{1}{(s+1)!s!} D^{(\alpha_{s+1}} \bar{D}^{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}) & = \\
= -\frac{ic}{(s+1)!} \partial_{(\alpha_{s+1}}^{\dot{\alpha}_{s+1}} \left[\frac{1}{(s+1)!s!} \bar{D}^{(\dot{\alpha}_{s+1}} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1))\dot{\alpha}(s)} \right] & \tag{3.33} \\
+ \frac{ic}{(s+1)!s!} \frac{s}{s+1} \partial_{(\alpha_{s+1}}^{\dot{\alpha}_s} \left[\frac{1}{s!} \bar{D}^{\dot{\gamma}} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)} \right] &
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s!} D^{\alpha_s} \bar{D}^{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}) & = \frac{i}{s!} \frac{s+1}{s} \partial^{\alpha_s}_{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)}) \\
& + \frac{s+1}{s} \bar{D}^2 \bar{T}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - \frac{ic}{s!(s+1)!} \partial^{\alpha_s \dot{\alpha}_{s+1}} \bar{D}^{(\dot{\alpha}_{s+1}} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1))\dot{\alpha}(s)} \\
& - \frac{ic}{s!s!} \frac{2s+1}{s(s+1)} \partial^{\alpha_s}_{(\dot{\alpha}_s} \bar{D}^{\dot{\gamma}} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)}) \\
& - \frac{ic}{s!s!} \frac{s^2-1}{s} \partial_{(\alpha_{s-1}}^{\dot{\alpha}_s} \bar{D}^2 D^\gamma V_{\gamma\alpha(s-2)\dot{\alpha}(s-1)})
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
D^{\alpha_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-1)} &= i \partial^{\alpha_{s-1}\dot{\alpha}_s} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)} \\
&\quad - \frac{ic}{s!} \partial^{\alpha_{s-1}\dot{\alpha}_s} D^\gamma \bar{D}_{(\dot{\alpha}_s} \Psi_{\gamma\alpha(s-1)\dot{\alpha}(s-1)})} \\
&\quad - ic \frac{s-1}{s!} \partial^{\alpha_{s-1}(\dot{\alpha}_{s-1}} D^2 \bar{D}^{\dot{\gamma}} V_{\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)})}
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\bar{D}^2 \bar{T}_{\alpha(s-1)\dot{\alpha}(s)} + \frac{i}{s!} \partial^{\alpha_s} (\dot{\alpha}_s T_{\alpha(s)\dot{\alpha}(s-1)}) &= \\
&= \frac{ic}{s!s!} \partial^{\alpha_s(\dot{\alpha}_s} \bar{D}^{\dot{\gamma}} D_{(\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)})} \\
&\quad - c \bar{D}^2 D^2 \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
&\quad + ic \frac{(s-1)}{s!s!} \partial_{(\alpha_{s-1}(\dot{\alpha}_s} \bar{D}^2 D^\gamma V_{\gamma\alpha(s-2)\dot{\alpha}(s-1)})}
\end{aligned} \tag{3.36}$$

We notice that in all the above there are some combinations that appear again and again. So let's define the following fields

$$\begin{aligned}
\frac{1}{s!(s+1)!} D_{(\alpha_{s+1}} \bar{D}_{(\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)})} &\equiv N_1 \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
\frac{1}{s!} \bar{D}^{\dot{\alpha}_s} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)})} &\equiv N_2 \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
D^2 \bar{D}^{\dot{\alpha}_{s-1}} V_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv N_3 \psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned} \tag{3.37}$$

Putting everything together we get for the lagrangian

$$\begin{aligned}
\mathcal{L}_F &= -\frac{1}{2c} T^{\alpha(s)\dot{\alpha}(s-1)} \left(2D^2 T_{\alpha(s)\dot{\alpha}(s-1)} + \frac{i}{s!} \partial_{(\alpha_s}^{\dot{\alpha}_s} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)})} \right) + c.c. \\
&\quad - ic |N_1|^2 \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
&\quad - ic \frac{s}{s+1} N_1 N_2 \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
&\quad + ic \frac{2s+1}{(s+1)^2} |N_2|^2 \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s \dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
&\quad + ic \frac{s-1}{s} N_2 N_3 \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
&\quad + ic \left(\frac{s-1}{s} \right)^2 |N_3|^2 \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1} \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned} \tag{3.38}$$

The first term in the lagrangian is the algebraic term of two auxiliary fields and the rest of the terms have exactly the structure of a theory that describes helicity $\lambda = s + 1/2$ (A.2). To have an exact match we choose coefficients

$$c = -1, N_2 = 1$$

$$N_1 = 1, N_3 = -\frac{s}{s-1}$$

So the fields that appear in the fermionic action are defined as:

$$\begin{aligned} \rho_{\alpha(s)\dot{\alpha}(s-1)} &\equiv T_{\alpha(s)\dot{\alpha}(s-1)}| \\ \beta_{\alpha(s)\dot{\alpha}(s-1)} &\equiv D^2 T_{\alpha(s)\dot{\alpha}(s-1)}| + \frac{i}{2s!} \partial_{(\alpha_s}^{\dot{\alpha}_s} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)}| \\ \psi_{\alpha(s+1)\dot{\alpha}(s)} &\equiv \frac{1}{s!(s+1)!} D_{(\alpha_{s+1}} \bar{D}_{\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)}| \\ \psi_{\alpha(s)\dot{\alpha}(s-1)} &\equiv \frac{1}{s!} \bar{D}^{\dot{\alpha}_s} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}| \\ \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv -\frac{s-1}{s} D^2 \bar{D}^{\dot{\alpha}_{s-1}} V_{\alpha(s-1)\dot{\alpha}(s-1)}| \end{aligned} \quad (3.39)$$

The lagrangian is

$$\begin{aligned} \mathcal{L}_F &= \rho^{\alpha(s)\dot{\alpha}(s-1)} \beta_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ &+ i \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1} \dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\ &+ i \left[\frac{s}{s+1} \right] \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ &- i \left[\frac{2s+1}{(s+1)^2} \right] \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s \dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\ &+ i \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\ &- i \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1} \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \end{aligned} \quad (3.40)$$

and the gauge transformations of the fields are

$$\begin{aligned}
\delta_G \rho_{\alpha(s)\dot{\alpha}(s-1)} &= 0, \quad \delta_G \psi_{\alpha(s+1)\dot{\alpha}(s)} = \frac{1}{s!(s+1)!} \partial_{(\alpha_{s+1}(\dot{\alpha}_s \xi_{\alpha(s)})\dot{\alpha}(s-1))} \\
\delta_G \beta_{\alpha(s)\dot{\alpha}(s-1)} &= 0, \quad \delta_G \psi_{\alpha(s)\dot{\alpha}(s-1)} = -\frac{1}{s!} \partial_{(\alpha_s \dot{\alpha}_s \bar{\xi}_{\alpha(s-1)})\dot{\alpha}(s)} \\
\delta_G \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \frac{s-1}{s} \partial^{\alpha_s \dot{\alpha}_{s-1}} \xi_{\alpha(s)\dot{\alpha}(s-1)} \\
&\text{with } \xi_{\alpha(s)\dot{\alpha}(s-1)} = -iD^2 L_{\alpha(s)\dot{\alpha}(s-1)} |
\end{aligned} \tag{3.41}$$

3.2.4.2 Bosonic components lagrangian

For the lagrangian of the bosons we follow exactly the same procedure as was presented for the fermionic sector. The fields that appear in the action are defined as:

$$\begin{aligned}
U_{\alpha(s+1)\dot{\alpha}(s-1)} &\equiv \frac{1}{(s+1)!} D_{(\alpha(s+1)} T_{\alpha(s)\dot{\alpha}(s-1)} | \\
u_{\alpha(s)\dot{\alpha}(s)} &\equiv \frac{1}{2s!} \{ \bar{D}_{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)} - D_{(\alpha_s} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)} \} | \\
v_{\alpha(s)\dot{\alpha}(s)} &\equiv -\frac{i}{2s!} \{ \bar{D}_{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1)} + D_{(\alpha_s} \bar{T}_{\alpha(s-1)\dot{\alpha}(s)} \} | \\
A_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv G_{\alpha(s-1)\dot{\alpha}(s-1)} | - \frac{s}{2s+1} (D^{\alpha_s} T_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{T}_{\alpha(s)\dot{\alpha}(s-1)}) | \\
S_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2} \{ D^{\alpha_s} T_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{T}_{\alpha(s)\dot{\alpha}(s-1)} \} | \\
P_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv -\frac{i}{2} \{ D^{\alpha_s} T_{\alpha(s)\dot{\alpha}(s-1)} - \bar{D}^{\dot{\alpha}_s} \bar{T}_{\alpha(s)\dot{\alpha}(s-1)} \} | \\
h_{\alpha(s)\dot{\alpha}(s)} &\equiv \frac{1}{\sqrt{2}} \left\{ \frac{1}{s!} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} - \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} \right. \\
&\quad \left. - \frac{1}{2s!s!} [D_{(\alpha_s}, \bar{D}_{(\dot{\alpha}_s}] V_{\alpha(s-1)\dot{\alpha}(s-1)}) \right\} | \\
h_{\alpha(s-2)\dot{\alpha}(s-2)} &\equiv -\frac{1}{2\sqrt{2}} \frac{s-1}{s^2} [D^{\alpha_{s-1}}, \bar{D}^{\dot{\alpha}_{s-1}}] V_{\alpha(s-1)\dot{\alpha}(s-1)} |
\end{aligned} \tag{3.42}$$

The gauge transformations are

$$\begin{aligned}
\delta_G U_{\alpha(s+1)\dot{\alpha}(s-1)} &= 0, & \delta_G A_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0 \\
\delta_G u_{\alpha(s)\dot{\alpha}(s)} &= 0, & \delta_G S_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0 \\
\delta_G V_{\alpha(s)\dot{\alpha}(s)} &= 0, & \delta_G P_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0 \\
\delta_G h_{\alpha(s)\dot{\alpha}(s)} &= \frac{1}{s!s!} \partial_{(\alpha_s(\dot{\alpha}_s \zeta_{\alpha(s-1)})\dot{\alpha}(s-1))} \\
\delta_G h_{\alpha(s-2)\dot{\alpha}(s-2)} &= \frac{s-1}{s^2} \partial^{\alpha_{s-1}\dot{\alpha}_{s-1}} \zeta_{\alpha(s-1)\dot{\alpha}(s-1)} \\
\text{where } \zeta_{\alpha(s-1)\dot{\alpha}(s-1)} &= \frac{i}{2\sqrt{2}} \left(D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} - \bar{D}^{\dot{\alpha}_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} \right) |
\end{aligned} \tag{3.43}$$

and the lagrangian is

$$\begin{aligned}
\mathcal{L}_B &= -\frac{1}{2} U^{\alpha(s+1)\dot{\alpha}(s-1)} U_{\alpha(s+1)\dot{\alpha}(s-1)} + c.c. \\
&+ u^{\alpha(s)\dot{\alpha}(s)} u_{\alpha(s)\dot{\alpha}(s)} \\
&+ v^{\alpha(s)\dot{\alpha}(s)} v_{\alpha(s)\dot{\alpha}(s)} \\
&- \left[\frac{2s+1}{4s} \right] A^{\alpha(s-1)\dot{\alpha}(s-1)} A_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \left[\frac{s^2}{(2s+1)(s+1)} \right] S^{\alpha(s-1)\dot{\alpha}(s-1)} S_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \left[\frac{s^2}{s+1} \right] P^{\alpha(s-1)\dot{\alpha}(s-1)} P_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&+ h^{\alpha(s)\dot{\alpha}(s)} \square h_{\alpha(s)\dot{\alpha}(s)} \\
&- \frac{s}{2} h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \\
&+ s(s-1) h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&- s(2s-1) h^{\alpha(s-2)\dot{\alpha}(s-2)} \square h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&- \left[\frac{s(s-2)^2}{2} \right] h^{\alpha(s-2)\dot{\alpha}(s-2)} \partial_{\alpha_{s-2}\dot{\alpha}_{s-2}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-3)\dot{\gamma}\dot{\alpha}(s-3)}
\end{aligned} \tag{3.44}$$

The first six terms are the algebraic terms for the auxiliary fields and the last five terms make-up the lagrangian for an integer superhelicity $\lambda = s$ (A.1), exactly as expected.

3.2.5 Off-shell degrees of freedom

Let's count the bosonic degrees of freedom:

<i>fields</i>	<i>d.o.f</i>	<i>redundancy</i>	<i>net</i>
$h_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	s^2	$s^2 + 2$
$h_{\alpha(s-2)\dot{\alpha}(s-2)}$	$(s-1)^2$		
$u_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$v_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$A_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
$U_{\alpha(s+1)\dot{\alpha}(s-1)}$	$2(s+2)s$	0	$2(s+2)s$
$S_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
$P_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
		<i>Total</i>	$8s^2 + 8s + 4$

Table 3.1: Off-shell bosonic degrees of freedom for an integer superhelicity

and the same counting for the fermionic degrees of freedom:

<i>fields</i>	<i>d.o.f</i>	<i>redundancy</i>	<i>net</i>
$\psi_{\alpha(s+1)\dot{\alpha}(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2 + 4s + 4$
$\psi_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot{\alpha}(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
$\beta_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
		<i>Total</i>	$8s^2 + 8s + 4$

Table 3.2: Off-shell fermionic degrees of freedom for an integer superhelicity

therefore the theory of integer superhelicity $Y = s$ is an $8s^2 + 8s + 4 \mid 8s^2 + 8s + 4$ system

3.2.6 Supersymmetric transformations for the components

Since we want to study the off-shell component structure of a supersymmetric theory and we have expressions for the component action, we would like to have explicit expressions for the symmetries this component action has. For that reason we will calculate the supersymmetric transformation laws of the components that keep the above action invariant. The component transformation under supersymmetry can be easily calculated using equation (2.34)

$$\delta_S \text{Component} = - \left(\epsilon^\beta D_\beta + \bar{\epsilon}^{\dot{\beta}} \bar{D}_{\dot{\beta}} \right) \text{Component} |$$

So the calculation of the transformation laws is a matter of acting on the definitions of the fields with the above operator and use the algebra of the D's and the properties of T, G .

But the fields are not all on an equal footing. The dynamical ones ($\in \mathcal{D}$) are treated as equivalence classes, in other words they have a gauge transformation of the form $\{\mathcal{D}\} \sim \{\mathcal{D}\} + \partial(\zeta)$. Hence when we do the susy-transformation they will get an extra kick in the gauge parameter space

$$\delta_S\{\mathcal{D}\} \sim \delta_S\{\mathcal{D}\} + \partial(\delta_S\zeta)$$

Because we identify the two classes, we can ignore any terms in the transformation law of the dynamical fields that have the same structure as their gauge transformation.

3.2.6.1 Transformation laws for Fermions

With all that in mind we get for the transformation of the fermionic fields the following expressions:

$$\begin{aligned} \delta_S \rho_{\alpha(s)\dot{\alpha}(s-1)} = & -\epsilon^{\alpha_{s+1}} U_{\alpha(s+1)\dot{\alpha}(s-1)} \\ & + \frac{s}{(s+1)!} \epsilon_{(\alpha_s} [S_{\alpha(s-1))\dot{\alpha}(s-1)} + iP_{\alpha(s-1))\dot{\alpha}(s-1)}] \\ & - \bar{\epsilon}^{\dot{\alpha}_s} [u_{\alpha(s)\dot{\alpha}(s)} + iv_{\alpha(s)\dot{\alpha}(s)}] \end{aligned} \quad (3.45)$$

$$\begin{aligned}
\delta_S \beta_{\alpha(s)\dot{\alpha}(s-1)} = & -i\bar{\epsilon}^{\dot{\beta}} \partial^{\alpha_{s+1}}{}_{\dot{\beta}} U_{\alpha(s+1)\dot{\alpha}(s-1)} \\
& - \frac{i}{2s!} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial_{(\alpha_s}{}^{\dot{\alpha}_s} \bar{U}_{\alpha(s-1))\dot{\alpha}(s+1)} \\
& + \frac{i}{2s!} \epsilon^{\beta} \partial_{(\alpha_s}{}^{\dot{\alpha}_s} [u_{\beta\alpha(s-1))\dot{\alpha}(s)} - iv_{\beta\alpha(s-1))\dot{\alpha}(s)}] \\
& + \frac{i}{2} \frac{1}{s!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} A_{\alpha(s-1))\dot{\alpha}(s-1))} \\
& + \frac{i}{2} \left[\frac{2s^2 - 1}{(s+1)(2s+1)} \right] \frac{1}{s!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} S_{\alpha(s-1))\dot{\alpha}(s-1))} \\
& + \frac{1}{2} \left[\frac{2s^2 - 2s - 1}{s+1} \right] \frac{1}{s!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} P_{\alpha(s-1))\dot{\alpha}(s-1))} \tag{3.46} \\
& - \frac{i}{2} \left[\frac{(s-1)^2}{s(s+1)} \right] \frac{1}{s!(s-1)!} \bar{\epsilon}^{(\dot{\alpha}_{s-1}} \partial_{(\alpha_s}{}^{\dot{\gamma}} S_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)} \\
& + \frac{1}{2} \left[\frac{(s-1)(3s+1)}{s(s+1)} \right] \frac{1}{s!(s-1)!} \bar{\epsilon}^{(\dot{\alpha}_{s-1}} \partial_{(\alpha_s}{}^{\dot{\gamma}} P_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)} \\
& - \sqrt{2} \bar{\epsilon}^{\dot{\alpha}_s} \square h_{\alpha(s)\dot{\alpha}(s)} \\
& + \frac{s}{\sqrt{2}} \frac{1}{s!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1))} \\
& - \frac{s(s-1)}{\sqrt{2}} \frac{1}{s!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} h_{\alpha(s-2))\dot{\alpha}(s-2)}
\end{aligned}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s+1)\dot{\alpha}(s)} = & - \frac{1}{s!} \bar{\epsilon}^{(\dot{\alpha}_s} U_{\alpha(s+1)\dot{\alpha}(s-1))} \\
& - \frac{1}{(s+1)!} \epsilon_{(\alpha_{s+1}} [u_{\alpha(s))\dot{\alpha}(s)} - iv_{\alpha(s))\dot{\alpha}(s)}] \\
& + \frac{i\sqrt{2}}{(s+1)!} \bar{\epsilon}^{\dot{\beta}} \partial_{(\alpha_{s+1}\dot{\beta}} h_{\alpha(s))\dot{\alpha}(s)} \tag{3.47}
\end{aligned}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s)\dot{\alpha}(s-1)} = & \bar{\epsilon}^{\dot{\alpha}_s} \left[u_{\alpha(s)\dot{\alpha}(s)} + i v_{\alpha(s)\dot{\alpha}(s)} \right] \\
& - \frac{1}{s!} \frac{s}{2s+1} \epsilon_{(\alpha_s S_{\alpha(s-1)})\dot{\alpha}(s-1)} \\
& - \frac{i s}{s!} \epsilon_{(\alpha_s P_{\alpha(s-1)})\dot{\alpha}(s-1)} \\
& + \frac{1}{s!} \frac{s+1}{2s} \epsilon_{(\alpha_s A_{\alpha(s-1)})\dot{\alpha}(s-1)} \\
& + i \frac{s-1}{\sqrt{2}} \epsilon^\beta \partial_\beta^{\dot{\alpha}_s} h_{\alpha(s)\dot{\alpha}(s)} \\
& + i \frac{(s+1)s(s-1)}{\sqrt{2}s!s!} \epsilon_{(\alpha_s \partial_{\alpha_{s-1}(\dot{\alpha}_{s-1})} h_{\alpha(s-2)})\dot{\alpha}(s-2)}
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s-1)\dot{\alpha}(s-2)} = & \frac{1}{2} \frac{(s-1)(2s+1)}{s^2} \bar{\epsilon}^{\dot{\alpha}_{s-1}} A_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \frac{i}{\sqrt{2}} \frac{(s-1)^2}{s} \frac{1}{(s-1)!^2} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1}) h_{\alpha(s-2)})\dot{\alpha}(s-2)} \\
& - i \sqrt{2} \frac{(s-1)^2}{s} \frac{1}{(s-1)!^2} \partial_{(\alpha_{s-1}^{\dot{\alpha}_{s-1}} \bar{\epsilon}_{(\dot{\alpha}_{s-1})} h_{\alpha(s-2)})\dot{\alpha}(s-2)}
\end{aligned} \tag{3.49}$$

3.2.6.2 Transformation laws for Bosons

The susy-transformation laws for the bosonic fields are:

$$\begin{aligned}
\delta_S U_{\alpha(s+1)\dot{\alpha}(s-1)} = & \frac{1}{(s+1)!} \epsilon_{(\alpha_{s+1} \beta_{\alpha(s)})\dot{\alpha}(s-1)} \\
& - \frac{i}{2} \frac{1}{(s+1)!} \epsilon_{(\alpha_{s+1} \partial_{\alpha_s}^{\dot{\alpha}_s} \bar{\rho}_{\alpha(s-1)})\dot{\alpha}(s)} \\
& - \frac{i}{(s+1)!} \bar{\epsilon}^{\dot{\beta}} \partial_{(\alpha_{s+1} \dot{\beta} \rho_{\alpha(s)})\dot{\alpha}(s-1)} \\
& - \frac{i}{(s+1)!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_{s+1}^{\dot{\alpha}_{s+1}} \bar{\psi}_{\alpha(s)})\dot{\alpha}(s+1)} \\
& - i \frac{s}{s+1} \frac{1}{(s+1)!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_{s+1}(\dot{\alpha}_s \psi_{\alpha(s)})\dot{\alpha}(s-1))}
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
\delta_S (u_{\alpha(s)\dot{\alpha}(s)} + iv_{\alpha(s)\dot{\alpha}(s)}) &= \frac{i}{(s+1)!} \epsilon^{\alpha_{s+1}} \partial_{(\alpha_{s+1}}^{\dot{\alpha}_{s+1}} \bar{\psi}_{\alpha(s)\dot{\alpha}(s+1)}) \\
&\quad - i \frac{s}{s+1} \frac{1}{s!} \epsilon_{(\alpha_s} \partial^{\gamma \dot{\alpha}_{s+1}} \bar{\psi}_{\gamma \alpha(s-1)\dot{\alpha}(s+1)}) \\
&\quad + i \frac{s}{s+1} \frac{1}{(s+1)! s!} \epsilon^{\alpha_{s+1}} \partial_{(\alpha_{s+1}(\dot{\alpha}_s \psi_{\alpha(s)\dot{\alpha}(s-1)}) \\
&\quad + i \frac{2s+1}{(s+1)^2} \frac{1}{s! s!} \epsilon_{(\alpha_s} \partial^{\gamma} (\dot{\alpha}_s \psi_{\gamma \alpha(s-1)\dot{\alpha}(s-1)}) \quad (3.51) \\
&\quad + i \frac{1}{s! s!} \epsilon_{(\alpha_s} \partial_{\alpha_{s-1}(\dot{\alpha}_s \bar{\psi}_{\alpha(s-2)\dot{\alpha}(s-1)}) \\
&\quad + \frac{1}{s!} \epsilon_{(\alpha_s} \bar{\beta}_{\alpha(s-1)\dot{\alpha}(s)}) \\
&\quad + \frac{i}{2} \frac{1}{s! s!} \epsilon_{(\alpha_s} \partial^{\gamma} (\dot{\alpha}_s \rho_{\gamma \alpha(s-1)\dot{\alpha}(s-1)})
\end{aligned}$$

$$\begin{aligned}
\delta_S A_{\alpha(s-1)\dot{\alpha}(s-1)} &= -\frac{i}{2s+1} \frac{1}{s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_s} (\dot{\alpha}_s \rho_{\alpha(s)\dot{\alpha}(s-1)}) + c.c. \\
&\quad + i \frac{(s-1)(s+1)}{s(2s+1)} \frac{1}{(s-1)!} \bar{\epsilon}_{(\dot{\alpha}_{s-1}} \partial^{\alpha_s \dot{\gamma}} \rho_{\alpha(s)\dot{\gamma}\dot{\alpha}(s-2)}) + c.c. \\
&\quad + i \frac{s}{2s+1} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_s \dot{\alpha}_{s+1}} \bar{\psi}_{\alpha(s)\dot{\alpha}(s+1)} + c.c. \\
&\quad - \frac{i}{s+1} \frac{1}{s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_s} (\dot{\alpha}_s \psi_{\alpha(s)\dot{\alpha}(s-1)}) + c.c. \quad (3.52) \\
&\quad + i \frac{s-1}{s!} \epsilon_{(\alpha_{s-1}} \partial^{\gamma \dot{\alpha}_s} \bar{\psi}_{\gamma \alpha(s-2)\dot{\alpha}(s)}) + c.c. \\
&\quad + i \frac{s+1}{2s+1} \frac{1}{(s-1)! s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_{s-1}(\dot{\alpha}_s \bar{\psi}_{\alpha(s-2)\dot{\alpha}(s-1)}) + c.c. \\
&\quad - i \frac{s-1}{s!(s-1)!} \epsilon_{(\alpha_{s-1}} \partial^{\gamma} (\dot{\alpha}_{s-1} \psi_{\gamma \alpha(s-2)\dot{\alpha}(s-2)}) + c.c.
\end{aligned}$$

$$\begin{aligned}
\delta_S (S_{\alpha(s-1)\dot{\alpha}(s-1)} + iP_{\alpha(s-1)\dot{\alpha}(s-1)}) &= \\
&= \epsilon^{\alpha_s} \beta_{\alpha(s)\dot{\alpha}(s-1)} \\
&\quad + \frac{s+1}{s} \bar{\epsilon}^{\dot{\alpha}_s} \bar{\beta}_{\alpha(s-1)\dot{\alpha}(s)} \\
&\quad - \frac{i}{2s!} \epsilon^{\alpha_s} \partial_{(\alpha_s} \bar{\rho}_{\alpha(s-1)\dot{\alpha}(s)} \\
&\quad - i \frac{s-1}{2s} \frac{1}{s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_s}_{(\dot{\alpha}_s} \rho_{\alpha(s)\dot{\alpha}(s-1)}) \\
&\quad + i \frac{s-1}{s!} \bar{\epsilon}_{(\dot{\alpha}_{s-1}} \partial^{\alpha_s \dot{\gamma}} \rho_{\alpha(s)\dot{\gamma}\dot{\alpha}(s-2)}) \\
&\quad - i \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_s \dot{\alpha}_{s+1}} \bar{\psi}_{\alpha(s)\dot{\alpha}(s+1)} \\
&\quad + i \frac{2s+1}{s(s+1)} \frac{1}{s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_s}_{(\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)}) \\
&\quad + i \frac{s+1}{s} \frac{1}{(s-1)!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_{s-1}(\dot{\alpha}_s} \bar{\psi}_{\alpha(s-2)\dot{\alpha}(s-1)})
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
\delta_S h_{\alpha(s)\dot{\alpha}(s)} &= \frac{1}{\sqrt{2s!}} \epsilon_{(\alpha_s} \bar{\rho}_{\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
&\quad + \frac{1}{\sqrt{2}} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \bar{\psi}_{\alpha(s)\dot{\alpha}(s+1)} + c.c. \\
&\quad - \frac{1}{\sqrt{2}(s+1)} \frac{1}{s!} \bar{\epsilon}_{(\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)}) + c.c.
\end{aligned} \tag{3.54}$$

$$\delta_S h_{\alpha(s-2)\dot{\alpha}(s-2)} = -\frac{1}{\sqrt{2s}} \epsilon^{\alpha_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \tag{3.55}$$

3.3 Half-Integer superhelicity $Y = s + \frac{1}{2}$ (I)

Now that we have presented in detail the construction of theories that describe the highest integer superhelicity, we repeat the process for the half-integer representations. We will discover that, unlike the integer case there are two different theories (different off-shell structures) that describe the same physical system on-shell.

The starting point is the same, the requirement that the massless limit of the massive superspin $Y = s + 1/2$ theory give the massless theory of superhelicity

$Y = s + 1/2$ (and other things that will decouple and we can ignore). As before, the superfields that describe the massive and the massless theory are completely different. The massive $Y = s + 1/2$ theory must be based on a real bosonic field $H_{\alpha(s)\dot{\alpha}(s)}$ or equivalently by a chiral superfield $W_{\alpha(s+1)\dot{\alpha}(s)} \sim \bar{D}^2 D_{\alpha_{s+1}} H_{\alpha(s)\dot{\alpha}(s)}$. On the other side the massless $Y = s + 1/2$ theory is described by a chiral superfield $F_{\alpha(2s+1)}$. Our demand to define the massless theory as the massless limit of the massive one suggests that we can generate $F_{\alpha(2s+1)}$ out of $H_{\alpha(s)\dot{\alpha}(s)}$ or $W_{\alpha(s+1)\alpha(s)}$. The chirality of both F and W and their index structure suggests the identification

$$F_{\alpha(2s+1)} \sim \partial_{(\alpha_{2s+1}}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+2}}^{\dot{\alpha}_1} \bar{D}^2 D_{\alpha_{s+1}} H_{\alpha(s)\dot{\alpha}(s)} \quad (3.56)$$

Therefore once again to make sense out of this identification (F is the fundamental object and not H and kill the extra degrees of freedom in H) we must treat $H_{\alpha(s)\dot{\alpha}(s)}$ as an equivalence class and identify $H_{\alpha(s)\dot{\alpha}(s)}$ with $H_{\alpha(s)\dot{\alpha}(s+)} + R_{\alpha(s)\dot{\alpha}(s)}$, where $R_{\alpha(s)\dot{\alpha}(s)}$ is real. The invariance under the equivalence of the physical (propagating) degrees of freedom of F give

$$\partial_{(\alpha_{2s+1}}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+2}}^{\dot{\alpha}_1} \bar{D}^2 D_{\alpha_{s+1}} R_{\alpha(s)\dot{\alpha}(s)} = 0 \quad (3.57)$$

The solution of the above will set R to

$$R_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s!} D_{(\alpha_s} \bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1)} \quad (3.58)$$

3.3.1 The superspace action

To construct a superspace action for the highest irreducible representation of half-integer superhelicity which is quadratic to H (H has mass dimension zero)

we must use four covariant derivatives. The most general action is

$$\begin{aligned}
S = \int d^8 z & a_1 H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
& + a_2 H^{\alpha(s)\dot{\alpha}(s)} \{D^2, \bar{D}^2\} H_{\alpha(s)\dot{\alpha}(s)} \\
& + a_3 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
& + a_4 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} D^\gamma \bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c.
\end{aligned} \tag{3.59}$$

The deformation of the action under the equivalence of H is

$$\begin{aligned}
\delta_G S = \int d^8 z & \left\{ \left[(-2a_1 + 2\frac{s+1}{s}a_3 + 2a_4) D^2 \bar{D}_{\dot{\alpha}_s} H^{\alpha(s)\dot{\alpha}(s)} \right. \right. \\
& \left. \left. + (-2a_3 - \frac{s+1}{s}a_4) D^{\alpha_s} \bar{D}_{\dot{\gamma}} D_\gamma H^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right] (\bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} \right. \\
& \left. \left. + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}) \right. \\
& + 2a_2 H^{\alpha(s)\dot{\alpha}(s)} D^2 \bar{D}^2 D_{\alpha_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - 2a_4 \bar{D}_{\dot{\beta}} D_\gamma \bar{D}_{\dot{\gamma}} H^{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)} \left[\bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} \right. \\
& \left. + \frac{s-1}{s} D^{\alpha_s} \bar{D}^{\dot{\alpha}_{s-1}} L_{\alpha(s)\dot{\alpha}(s-1)} \right. \\
& \left. \left. + \bar{D}_{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)} \right] \right. \\
& \left. + c.c. \right\}
\end{aligned} \tag{3.60}$$

Notice the presence of two new terms: $D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$ and $\bar{D}_{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)}$. Because of the D-algebra these terms identically vanish and they don't effect the result.

Obviously we can not set the variation of the action to zero just by picking values for the a 's without setting them all to zero. But we can introduce compensators with proper mass dimension and index structure. There are two different ways to do that:

I Choose coefficients to kill the last two terms ($a_2 = a_4 = 0$) and introduce a compensator that cancels the first term

II Choose coefficients to kill the first two terms

($-2a_1 + 2\frac{s+1}{s}a_3 + 2a_4 = 0$, $-2a_3 - \frac{s+1}{s}a_4$, $a_2 = 0$) and introduce a compensator to cancel the last term

These two different approaches will lead to the two different formulations for half-integer superhelicity, mentioned above. In this section we focus on case (I).

So for case (I) we get: $a_2 = a_4 = 0$

$$\delta_G S = \int d^8 z \left[\left(-2a_1 + 2\frac{s+1}{s}a_3 \right) D^2 \bar{D}_{\dot{\alpha}s} H^{\alpha(s)\dot{\alpha}(s)} - 2a_3 D^{\alpha_s} \bar{D}_{\dot{\gamma}} D_{\gamma} H^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right] \left(\bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \right)$$

This suggests that we introduce a fermionic compensator $\chi_{\alpha(s)\dot{\alpha}(s-1)}$ which transforms like $\delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} = \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$. We add to the action the coupling term of H with χ and the kinetic energy terms for χ . The full action

takes the form

$$\begin{aligned}
S = \int d^8z & a_1 H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
& + a_3 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
& - (2a_1 - 2\frac{s+1}{s}a_3) H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + 2a_3 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} D^\gamma \chi_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& + b_1 \chi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + b_2 \chi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + b_3 \chi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}_{\dot{\alpha}_s} D_{\alpha_s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + b_4 \chi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}
\end{aligned}$$

and it has to be invariant under

$$\begin{aligned}
\delta_G H_{\alpha(s)\dot{\alpha}(s)} &= \frac{1}{s!} D_{(\alpha_s} \bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1)}) \quad (3.61) \\
\delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} &= \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}
\end{aligned}$$

The equations of motion for the superfields are the variation of the action with respect the superfields

$$T_{\alpha(s)\dot{\alpha}(s)} = \frac{\delta S}{\delta H^{\alpha(s)\dot{\alpha}(s)}}, \quad G_{\alpha(s)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta \chi^{\alpha(s)\dot{\alpha}(s-1)}} \quad (3.62)$$

and the invariance of the action gives the following Bianchi Identities

$$\bar{D}_{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s)} - \bar{D}^2 G_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad (3.63a)$$

$$\frac{1}{(s+1)!} D_{(\alpha_{s+1}} G_{a(s)\dot{\alpha}(s-1)}) = 0 \quad (3.63b)$$

The Bianchi identities fix all the coefficients

$$\begin{aligned}
a_3 &= 0, & b_3 &= 0 \\
b_1 &= -\frac{s+1}{s}a_1, & b_4 &= 2a_1 \\
b_2 &= 0
\end{aligned}$$

and the final form of the action is:

$$\begin{aligned}
S = \int d^8z \left\{ & c H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \right. \\
& - 2c H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& - \frac{s+1}{s} c \chi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& \left. + 2c \chi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \right\}
\end{aligned} \tag{3.64}$$

The expression for the equations of motion are:

$$\begin{aligned}
T_{\alpha(s)\dot{\alpha}(s)} &= 2c D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
&+ \frac{2c}{s!} \left(D_{(\alpha_s} \bar{D}^2 \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} - \bar{D}_{(\dot{\alpha}s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)}) \right)
\end{aligned} \tag{3.65a}$$

$$\begin{aligned}
G_{\alpha(s)\dot{\alpha}(s-1)} &= -2c D^2 \bar{D}^{\dot{\alpha}s} H_{\alpha(s)\dot{\alpha}(s)} - 2c \frac{s+1}{s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} \\
&+ \frac{2c}{s!} D_{(\alpha_s} \bar{D}^{\dot{\alpha}s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}
\end{aligned} \tag{3.65b}$$

where c is a free overall parameter that can be absorbed in the definition of the superfields but we will fix it later when we define the components.

3.3.2 On-shell equations of motion

We constructed an action, but we still have to prove that this action describes a superhelicity $Y = s + \frac{1}{2}$ system. We must find a chiral superfield like $F_{\alpha(2s+1)}$ that on-shell will satisfy all the necessary constraints set by representation theory. In a very similar way as in section (3.2.2) we can prove the following identity:

$$\begin{aligned}
D^{\alpha_{2s+1}} F_{\alpha(2s+1)} &= \\
&= \frac{1}{2c} \frac{1}{(2s)!} \partial_{(\alpha_{2s}}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+1}}^{\dot{\alpha}_1} T_{\alpha(s)\dot{\alpha}(s)} \\
&+ \frac{i}{2c} \frac{s}{2s+1} \frac{B}{B+\Delta} \frac{1}{(2s)!} D_{(\alpha_{2s}} \bar{D}^2 \partial_{\alpha_{2s-1}}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+1}}^{\dot{\alpha}_1} G_{\alpha(s)\dot{\alpha}(s-1)} \\
&+ \frac{1}{2c} \frac{s}{2s+1} \frac{1}{(2s)!} D_{(\alpha_{2s}} \partial_{\alpha_{2s-1}}^{\dot{\alpha}_s} \dots \partial_{\alpha_s}^{\dot{\alpha}_1} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)} \\
&+ \frac{i}{2c} \frac{s}{2s+1} \frac{\Delta}{B+\Delta} \frac{1}{(2s)!} D_{(\alpha_{2s}} \bar{D}^{\dot{\alpha}_s} \partial_{\alpha_{2s-1}}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+1}}^{\dot{\alpha}_1} T_{\alpha(s)\dot{\alpha}(s)}
\end{aligned} \tag{3.66}$$

where

$$F_{\alpha(2s+1)} = \frac{1}{(2s+1)!} \bar{D}^2 D_{(\alpha_{2s+1}} \partial_{\alpha_{2s}}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+1}}^{\dot{\alpha}_1} H_{\alpha(s)\dot{\alpha}(s)}$$

and that proves that on-shell ($T_{\alpha(s)\dot{\alpha}(s)} = 0$, $G_{\alpha(s)\dot{\alpha}(s-1)} = 0$) we get the desired constraints to describe a superhelicity $Y = s + \frac{1}{2}$ system

$$D^{\alpha_{2s+1}} F_{\alpha(2s+1)} = 0, \quad \bar{D}_{\dot{\gamma}} F_{\alpha(2s+1)} = 0$$

The constants B and Δ are only constrained by $B + \Delta \neq 0$.

3.3.3 A two parameter family of superspace actions

Like the integer superhelicity case, this action is a member of a two parameter family of equivalent theories. Dimensionality and index structure allow us to perform

the following redefinition of χ

$$\chi_{\alpha(s)\dot{\alpha}(s-1)} \rightarrow \chi_{\alpha(s)\dot{\alpha}(s-1)} + z\bar{D}^{\dot{\alpha}s} H_{\alpha(s)\dot{\alpha}(s)} \quad (3.67)$$

where z is a complex parameter. This operation will generate an entire class of actions and transformation laws which all are related by the above redefinition.

The generalized action is

$$\begin{aligned} S = \int d^8w \quad & cH^{\alpha(s)\dot{\alpha}(s)}D^\gamma\bar{D}^2D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\ & -2c\left[1 + \frac{s+1}{s}z\right] H^{\alpha(s)\dot{\alpha}(s)}\bar{D}_{\dot{\alpha}s}D^2\chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ & -2c\bar{z} H^{\alpha(s)\dot{\alpha}(s)}D_{\alpha_s}\bar{D}_{\dot{\alpha}s}D^\gamma\chi_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\ & -2c\bar{z}\left[1 + \frac{s+1}{s}\bar{z}\right] H^{\alpha(s)\dot{\alpha}(s)}D_{\alpha_s}\bar{D}^2D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \quad (3.68) \\ & -c|z|^2 H^{\alpha(s)\dot{\alpha}(s)}D_{\alpha_s}\bar{D}_{\dot{\alpha}s}D^\gamma\bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}(s-1)} + c.c. \\ & -\frac{s+1}{s}c \chi^{\alpha(s)\dot{\alpha}(s-1)}D^2\chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ & +2c \chi^{\alpha(s)\dot{\alpha}(s-1)}D_{\alpha_s}\bar{D}^{\dot{\alpha}s}\bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \end{aligned}$$

and the generalized transformation laws are

$$\delta_G H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!}D_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!}\bar{D}_{(\dot{\alpha}s}L_{\alpha(s)\dot{\alpha}(s-1)}) \quad (3.69a)$$

$$\begin{aligned} \delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} = & \left[1 + \frac{s+1}{s}z\right] \bar{D}^2L_{\alpha(s)\dot{\alpha}(s-1)} - \frac{z}{s!}\bar{D}^{\dot{\alpha}s}D_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} \quad (3.69b) \\ & +D^{\alpha_{s+1}}\Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \end{aligned}$$

The equations of motion are

$$\begin{aligned}
T_{\alpha(s)\dot{\alpha}(s)} &= 2cD^\gamma\bar{D}^2D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
&\quad - \frac{2c}{s!}[\bar{z}(1 + \frac{s+1}{s}\bar{z})]D_{(\alpha_s}\bar{D}^2D^\gamma H_{\gamma\alpha(s-1))\dot{\alpha}(s)} \\
&\quad - \frac{2c}{s!}[z(1 + \frac{s+1}{s}z)]\bar{D}_{(\dot{\alpha}_s}D^2\bar{D}^\gamma H_{\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)} \\
&\quad - \frac{c}{s!s!}[z\bar{z}]D_{(\alpha_s}\bar{D}_{(\dot{\alpha}_s}D^\gamma\bar{D}^\gamma H_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)} \\
&\quad - \frac{c}{s!s!}[z\bar{z}]\bar{D}_{(\dot{\alpha}_s}D_{(\alpha_s}\bar{D}^\gamma D^\gamma H_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)} \\
&\quad + \frac{2c}{s!}[1 + \frac{s+1}{s}\bar{z}]D_{(\alpha_s}\bar{D}^2\bar{\chi}_{\alpha(s-1))\dot{\alpha}(s)} \\
&\quad - \frac{2c}{s!}[1 + \frac{s+1}{s}z]\bar{D}_{(\dot{\alpha}_s}D^2\chi_{\alpha(s)\dot{\alpha}(s-1)} \\
&\quad - \frac{2c}{s!s!}[\bar{z}]D_{(\alpha_s}\bar{D}_{(\dot{\alpha}_s}D^\gamma\chi_{\gamma\alpha(s-1))\dot{\alpha}(s-1)} \\
&\quad + \frac{2c}{s!s!}[z]\bar{D}_{(\dot{\alpha}_s}D_{(\alpha_s}\bar{D}^\gamma\bar{\chi}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)}
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
G_{\alpha(s)\dot{\alpha}(s-1)} &= -2c[1 + \frac{s+1}{s}z]D^2\bar{D}^{\dot{\alpha}_s}H_{\alpha(s)\dot{\alpha}(s)} \\
&\quad - \frac{2c}{s!}[\bar{z}]D_{(\alpha_s}\bar{D}^\gamma D^\gamma H_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)} \\
&\quad - 2c\frac{s+1}{s}D^2\chi_{\alpha(s)\dot{\alpha}(s-1)} \\
&\quad + \frac{2c}{s!}D_{(\alpha_s}\bar{D}^{\dot{\alpha}_s}\bar{\chi}_{\alpha(s-1))\dot{\alpha}(s)}
\end{aligned} \tag{3.71}$$

3.3.4 Field spectrum and component lagrangian

To find the field spectrum and the component action we use the same technique as presented in section 3.2.4. The Bianchi identities and their consequences are:

$$\bar{D}^{\dot{\alpha}s} T_{\alpha(s)\dot{\alpha}(s)} - \bar{D}^2 G_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad \rightsquigarrow \quad \bar{D}^2 T_{\alpha(s)\dot{\alpha}(s)} = 0 \quad (3.72a)$$

$$D^2 T_{\alpha(s)\dot{\alpha}(s)} = 0 \quad (\text{reality})$$

$$\frac{1}{(s+1)!} D_{(\alpha_{s+1}} G_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad \rightsquigarrow \quad D^2 G_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad (3.72b)$$

The above constrain most of the components of the superfields T and G and the rest of them will be associated with the off-shell auxiliary components of the theory:

$$\bar{D}^{\dot{\alpha}s-1} G_{\alpha(s)\dot{\alpha}(s-1)}|, \bar{D}_{(\dot{\alpha}s} G_{\alpha(s)\dot{\alpha}(s-1)})|, T_{\alpha(s)\dot{\alpha}(s)}|, D^{\alpha s} G_{\alpha(s)\dot{\alpha}(s-1)}| \text{ for bosons}$$

$$G_{\alpha(s)\dot{\alpha}(s-1)}|, D_{(\alpha_s} \bar{D}^{\dot{\alpha}s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)})| \text{ for fermions}$$

Now for the component action. Step number one is to express the action in terms of T and G

$$\begin{aligned} S &= \int d^8 z \left\{ \frac{1}{2} H^{\alpha(s)\dot{\alpha}(s)} T_{\alpha(s)\dot{\alpha}(s)} \right. \\ &\quad \left. + \frac{1}{2} \chi^{\alpha(s)\dot{\alpha}(s-1)} G_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right\} \\ &= \int d^4 x \left[\frac{1}{2} \bar{D}^2 D^2 (H^{\alpha(s)\dot{\alpha}(s)} T_{\alpha(s)\dot{\alpha}(s)}) \right. \\ &\quad \left. + \frac{1}{2} \bar{D}^2 D^2 (\chi^{\alpha(s)\dot{\alpha}(s-1)} G_{\alpha(s)\dot{\alpha}(s-1)}) + c.c. \right] \end{aligned} \quad (3.73)$$

and then we distribute the $D(\bar{D})$'s.

3.3.4.1 Fermionic component lagrangian

After the distribution of D's and the usage of Bianchi identities we get for the fermionic lagrangian:

$$\begin{aligned}
\mathcal{L}_F = & \frac{1}{2} D^2 \bar{D}^{\dot{\alpha}_{s+1}} H^{\alpha(s)\dot{\alpha}(s)} \Big| \frac{1}{(s+1)!} \bar{D}_{(\dot{\alpha}_{s+1}} T_{\alpha(s)\dot{\alpha}(s)} \Big| \\
& + \frac{1}{2} \left(-\frac{s}{s+1} D^2 \bar{D}_{\dot{\gamma}} H^{\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)} + D^2 \chi^{\alpha(s)\dot{\alpha}(s-1)} \right) \Big| \bar{D}^{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s)} \Big| \\
& + \frac{1}{2} \frac{s}{s+1} \bar{D}^{\dot{\alpha}_s} D_{\gamma} \chi^{\gamma\alpha(s-1)\dot{\alpha}(s-1)} \Big| \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} D^{\alpha_s} G_{\alpha(s)\dot{\alpha}(s-1)} \Big| \\
& - \frac{1}{2} \frac{s-1}{s+1} \bar{D}_{\dot{\gamma}} D_{\gamma} \chi^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)} \Big| \bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} G_{\alpha(s)\dot{\alpha}(s-1)} \Big| \\
& + \frac{1}{2} \bar{D}^2 D^2 \chi^{\alpha(s)\dot{\alpha}(s-1)} \Big| G_{\alpha(s)\dot{\alpha}(s-1)} \Big| \\
& + c.c.
\end{aligned} \tag{3.74}$$

T and G satisfy a few more identities:

$$\begin{aligned}
& \frac{1}{(s+1)!} \bar{D}_{(\dot{\alpha}_{s+1}} T_{\alpha(s)\dot{\alpha}(s)} = \\
& = \frac{2ic}{(s+1)!^2} \partial^{\alpha_{s+1}}{}_{(\dot{\alpha}_{s+1}} \bar{D}^2 D_{(\alpha_{s+1}} H_{\alpha(s)\dot{\alpha}(s)} \\
& - \frac{2ic}{(s+1)!s!} \frac{s}{s+1} \partial_{(\alpha_s(\dot{\alpha}_{s+1}} \left[\bar{D}^2 D^{\gamma} H_{\gamma\alpha(s-1)\dot{\alpha}(s)} - \frac{s+1}{s} \bar{D}^2 \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \right]
\end{aligned}$$

$$\begin{aligned}
\bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} G_{\alpha(s)\dot{\alpha}(s-1)} = & i \frac{s+1}{s} \partial^{\alpha_s \dot{\alpha}_{s-1}} \left[G_{\alpha(s)\dot{\alpha}(s-1)} + 2c D^2 \bar{D}^{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} \right. \\
& \left. + 2c \frac{s+1}{s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} \right] \\
& - \frac{2ic}{(s-1)!} \frac{s^2-1}{s^2} \partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_{s-1}} D^{\gamma} \bar{D}^{\dot{\gamma}} \bar{\chi}_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-1)}
\end{aligned}$$

$$\begin{aligned}
\bar{D}^{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s)} = & \frac{2ic}{(s+1)!} \partial^{\alpha_{s+1}\dot{\alpha}_s} \bar{D}^2 D_{(\alpha_{s+1})\dot{\alpha}(s)} H_{\alpha(s)\dot{\alpha}(s)} \\
& + \frac{2ic}{s!} \frac{2s+1}{s(s+1)} \partial_{(\alpha_s \dot{\alpha}_s} \left[\bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} - \frac{s+1}{s} \bar{D}^2 \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \right] \\
& + \frac{2ic}{s!(s-1)!} \frac{s^2-1}{s^2} \partial_{(\alpha_s(\dot{\alpha}_{s-1})} \bar{D}^{\dot{\gamma}} D^\gamma \chi_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)} \\
& + \frac{1}{s!} D_{(\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - \frac{i}{s!} \frac{s+1}{s} \partial_{(\alpha_s \dot{\alpha}_s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)}
\end{aligned}$$

We observe that specific combinations appear again and again. So let's define

$$\begin{aligned}
\frac{1}{(s+1)!} \bar{D}^2 D_{(\alpha_{s+1})\dot{\alpha}(s)} H_{\alpha(s)\dot{\alpha}(s)} & | \equiv N_1 \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
\left\{ D^2 \bar{D}^{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} + \frac{s+1}{s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} \right\} & | \equiv N_2 \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
\bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} \chi_{\alpha(s)\dot{\alpha}(s-1)} & | \equiv N_3 \psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned} \tag{3.75}$$

Putting everything together we get for the lagrangian

$$\begin{aligned}
\mathcal{L}_F = & G^{\alpha(s)\dot{\alpha}(s-1)} \left(-\frac{1}{2c} \frac{s}{s+1} \frac{1}{s!} D_{(\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)} \right. \\
& \left. + \frac{i}{4c} \frac{1}{s!} \partial_{(\alpha_s \dot{\alpha}_s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)} \right) | + c.c. \\
& + 2ic |N_1|^2 \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& - 2ic \frac{s}{s+1} N_1 N_2 \bar{\psi}^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& - 2ic \frac{2s+1}{(s+1)^2} |N_2|^2 \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s \dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& + 2ic \frac{s-1}{s} N_2 N_3 \bar{\psi}^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
& - 2ic \left(\frac{s-1}{s} \right)^2 |N_3|^2 \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1} \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned} \tag{3.76}$$

The first term in the lagrangian is the algebraic kinetic energy term of two auxiliary fields and the rest of the terms are exactly the structure of a theory that describes helicity $\lambda = s + 1/2$ (A.2). To have an exact match we choose coefficients

$$c = 1, \quad N_2 = -\frac{1}{\sqrt{2}}$$

$$N_1 = \frac{1}{\sqrt{2}}, \quad N_3 = -\frac{1}{\sqrt{2}} \frac{s}{s-1}$$

The fields that appear in the fermionic action are defined as:

$$\begin{aligned} \rho_{\alpha(s)\dot{\alpha}(s-1)} &\equiv G_{\alpha(s)\dot{\alpha}(s-1)} | \\ \beta_{\alpha(s)\dot{\alpha}(s-1)} &\equiv -\frac{1}{2s!} \left\{ \frac{s}{s+1} D_{(\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)} - \frac{i}{2} \partial_{(\alpha_s}^{\dot{\alpha}_s} \bar{G}_{\alpha(s-1)\dot{\alpha}(s)} \right\} | \\ \psi_{\alpha(s+1)\dot{\alpha}(s)} &\equiv \frac{\sqrt{2}}{(s+1)!} \bar{D}^2 D_{(\alpha_{s+1}} H_{\alpha(s)\dot{\alpha}(s)} | \\ \psi_{\alpha(s)\dot{\alpha}(s-1)} &\equiv -\sqrt{2} \left\{ D^2 \bar{D}^{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} + \frac{s+1}{s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} \right\} | \\ \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv -\sqrt{2} \frac{(s-1)}{s} \bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} \chi_{\alpha(s)\dot{\alpha}(s-1)} | \end{aligned} \quad (3.77)$$

and the final expression for the lagrangian of the fermions is

$$\begin{aligned} \mathcal{L}_F &= \rho^{\alpha(s)\dot{\alpha}(s-1)} \beta_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ &+ i \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}}_{\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\ &+ i \left[\frac{s}{s+1} \right] \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ &- i \left[\frac{2s+1}{(s+1)^2} \right] \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s}_{\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\ &+ i \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\ &- i \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \end{aligned} \quad (3.78)$$

which is invariant under the following the gauge transformations:

$$\begin{aligned}
\delta_G \rho_{\alpha(s)\dot{\alpha}(s-1)} &= 0, \quad \delta_G \psi_{\alpha(s+1)\dot{\alpha}(s)} = \frac{1}{s!(s+1)!} \partial_{(\alpha_{s+1}(\dot{\alpha}_s \xi_{\alpha(s)})\dot{\alpha}(s-1))} \\
\delta_G \beta_{\alpha(s)\dot{\alpha}(s-1)} &= 0, \quad \delta_G \psi_{\alpha(s)\dot{\alpha}(s-1)} = -\frac{1}{s!} \partial_{(\alpha_s \dot{\alpha}_s \bar{\xi}_{\alpha(s-1)})\dot{\alpha}(s)} \\
\delta_G \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \frac{s-1}{s} \partial^{\alpha_s \dot{\alpha}_{s-1}} \xi_{\alpha(s)\dot{\alpha}(s-1)} \\
&\text{with } \xi_{\alpha(s)\dot{\alpha}(s-1)} = -i\sqrt{2} \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} |
\end{aligned} \tag{3.79}$$

3.3.4.2 Bosonic components lagrangian

For the bosonic lagrangian we follow exactly the same procedure. The fields that appear in the action are defined as:

$$\begin{aligned}
U_{\alpha(s)\dot{\alpha}(s-2)} &\equiv \bar{D}^{\dot{\alpha}_{s-1}} G_{\alpha(s)\dot{\alpha}(s-1)} | \\
u_{\alpha(s)\dot{\alpha}(s)} &\equiv \frac{1}{2s!} \{ D_{(\alpha_s} \bar{G}_{\alpha(s-1))\dot{\alpha}(s)} - \bar{D}_{(\dot{\alpha}_s} G_{\alpha(s)\dot{\alpha}(s-1)} \} | \\
v_{\alpha(s)\dot{\alpha}(s)} &\equiv -\frac{i}{2s!} \{ D_{(\alpha_s} \bar{G}_{\alpha(s-1))\dot{\alpha}(s)} + \bar{D}_{(\dot{\alpha}_s} G_{\alpha(s)\dot{\alpha}(s-1)} \} | \\
A_{\alpha(s)\dot{\alpha}(s)} &\equiv T_{\alpha(s)\dot{\alpha}(s)} | + \frac{s}{2s+1} \frac{1}{s!} (D_{(\alpha_s} \bar{G}_{\alpha(s-1))\dot{\alpha}(s)} - \bar{D}_{(\dot{\alpha}_s} G_{\alpha(s)\dot{\alpha}(s-1)}) | \\
S_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2} \{ D^{\alpha_s} G_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{G}_{\alpha(s)\dot{\alpha}(s-1)} \} | \\
P_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv -\frac{i}{2} \{ D^{\alpha_s} G_{\alpha(s)\dot{\alpha}(s-1)} - \bar{D}^{\dot{\alpha}_s} \bar{G}_{\alpha(s)\dot{\alpha}(s-1)} \} | \\
h_{\alpha(s+1)\dot{\alpha}(s+1)} &\equiv \frac{1}{2} \frac{1}{(s+1)!^2} [D_{(\alpha_{s+1}}, D_{(\dot{\alpha}_{s+1}}] H_{\alpha(s)\dot{\alpha}(s)} | \\
h_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2} \frac{s}{(s+1)^2} [D^{\alpha_s}, \bar{D}^{\dot{\alpha}_s}] H_{\alpha(s)\dot{\alpha}(s)} | \\
&\quad + \frac{1}{s+1} (D^{\alpha_s} \chi_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}) |
\end{aligned} \tag{3.80}$$

the gauge transformations are

$$\begin{aligned}
\delta_G U_{\alpha(s)\dot{\alpha}(s-2)} &= 0, & \delta_G A_{\alpha(s)\dot{\alpha}(s)} &= 0 \\
\delta_G u_{\alpha(s)\dot{\alpha}(s)} &= 0, & \delta_G S_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0 \\
\delta_G V_{\alpha(s)\dot{\alpha}(s)} &= 0, & \delta_G P_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0 \\
\delta_G h_{\alpha(s+1)\dot{\alpha}(s+1)} &= \frac{1}{(s+1)!^2} \partial_{(\alpha_{s+1}(\dot{\alpha}_{s+1} \zeta_{\alpha(s)\dot{\alpha}(s)})} \\
\delta_G h_{\alpha(s-1)\dot{\alpha}(s-1)} &= \frac{s}{(s+1)^2} \partial^{\alpha_s \dot{\alpha}_s} \zeta_{\alpha(s)\dot{\alpha}(s)} \\
\text{where } \zeta_{\alpha(s)\dot{\alpha}(s)} &= \frac{i}{2s!} \left(D_{(\alpha_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} + \bar{D}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1)}) \right) |
\end{aligned} \tag{3.81}$$

and the lagrangian

$$\begin{aligned}
\mathcal{L}_B &= \frac{1}{4} \left[\frac{s-1}{s+1} \right] U^{\alpha(s)\dot{\alpha}(s-2)} U_{\alpha(s)\dot{\alpha}(s-2)} + c.c. \\
&+ \left[\frac{s}{2} \right] u^{\alpha(s)\dot{\alpha}(s)} u_{\alpha(s)\dot{\alpha}(s)} \\
&- \frac{1}{2} \left[\frac{s}{2s+1} \right] v^{\alpha(s)\dot{\alpha}(s)} v_{\alpha(s)\dot{\alpha}(s)} \\
&+ \frac{1}{8} \left[\frac{2s+1}{s+1} \right] A^{\alpha(s)\dot{\alpha}(s)} A_{\alpha(s)\dot{\alpha}(s)} \\
&- \frac{1}{2} \left[\frac{s^2}{(s+1)^2} \right] S^{\alpha(s-1)\dot{\alpha}(s-1)} S_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \frac{1}{2} \left[\frac{s^2}{(s+1)^2} \right] P^{\alpha(s-1)\dot{\alpha}(s-1)} P_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&+ h^{\alpha(s+1)\dot{\alpha}(s+1)} \square h_{\alpha(s+1)\dot{\alpha}(s+1)} \\
&- \left[\frac{s+1}{2} \right] h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha_{s+1} \dot{\alpha}_{s+1}} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s) \dot{\gamma} \dot{\alpha}(s)} \\
&+ [s(s+1)] h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha_{s+1} \dot{\alpha}_{s+1}} \partial_{\alpha_s \dot{\alpha}_s} h_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- [(s+1)(2s+1)] h^{\alpha(s-1)\dot{\alpha}(s-1)} \square h_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \left[\frac{(s+1)(s-1)^2}{2} \right] h^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1} \dot{\alpha}_{s-1}} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s-2) \dot{\gamma} \dot{\alpha}(s-2)}
\end{aligned} \tag{3.82}$$

gives rise to the theory of helicity $\lambda = s + 1$ (A.1) as expected.

3.3.5 Off-shell degrees of freedom

Let's count the bosonic degrees of freedom:

<i>fields</i>	<i>d.o.f</i>	<i>redundancy</i>	<i>net</i>
$h_{\alpha(s+1)\dot{\alpha}(s+1)}$	$(s+2)^2$	$(s+1)^2$	$s^2 + 2s + 3$
$h_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2		
$u_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$v_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$A_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$U_{\alpha(s)\dot{\alpha}(s-2)}$	$2(s+1)(s-1)$	0	$2(s+1)(s-1)$
$S_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
$P_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
		<i>Total</i>	$8s^2 + 8s + 4$

Table 3.3: Off-shell bosonic degrees of freedom for a half-integer(I) superhelicity

The counting for the fermionic degrees of freedom is

<i>fields</i>	<i>d.o.f</i>	<i>redundancy</i>	<i>net</i>
$\psi_{\alpha(s+1)\dot{\alpha}(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2 + 4s + 4$
$\psi_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot{\alpha}(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
$\beta_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
		<i>Total</i>	$8s^2 + 8s + 4$

Table 3.4: Off-shell fermionic degrees of freedom for a half-integer(I) superhelicity

Hence this theory of half-integer superhelicity $Y = s+1/2$ is a $8s^2+8s+4 \mid 8s^2+8s+4$ system. Immediately we observe that this theory has exactly the same number of degrees of freedom as the theory of integer superhelicity, as it was presented in 3.2.5. This simple observation provides a very important clue about the construction of an $\mathcal{N} = 2$ representation. We will get back on that once we complete the analysis of massless representations.

3.3.6 Supersymmetric transformations for the components

Here we calculate the supersymmetric transformations of the components of the theory in the same way as in 3.2.6. We must keep in mind that all components are not treated equally. The dynamical fields are equivalence classes so we can ignore

$$\begin{aligned}
\delta_S \psi_{\alpha(s+1)\dot{\alpha}(s)} &= \frac{\sqrt{2}i}{(s+1)!} \epsilon^\gamma \partial_{(\alpha_{s+1}} \dot{\gamma} h_{\gamma\alpha(s))\dot{\gamma}\dot{\alpha}(s)} \\
&\quad - \frac{i}{\sqrt{2}(s+1)!} \epsilon_{(\alpha_{s+1}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s))\dot{\gamma}\dot{\alpha}(s)} \\
&\quad + \frac{1}{2\sqrt{2}} \frac{2s+1}{s+1} \frac{1}{(s+1)!} \epsilon_{(\alpha_{s+1}} A_{\alpha(s))\dot{\alpha}(s)}
\end{aligned} \tag{3.85}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s)\dot{\alpha}(s-1)} &= -\frac{1}{2\sqrt{2}} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_s} A_{\alpha(s)\dot{\alpha}(s)} \\
&\quad + \frac{1}{\sqrt{2}} \frac{s+1}{2s+1} \bar{\epsilon}^{\dot{\alpha}_s} u_{\alpha(s)\dot{\alpha}(s)} \\
&\quad - i \frac{s+1}{\sqrt{2}} \bar{\epsilon}^{\dot{\alpha}_s} v_{\alpha(s)\dot{\alpha}(s)} \\
&\quad + \frac{1}{\sqrt{2}} \frac{s-1}{s!} \bar{\epsilon}_{(\dot{\alpha}_{s-1}} U_{\alpha(s)\dot{\alpha}(s-2)}) \\
&\quad - \frac{is}{\sqrt{2}} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s)} \\
&\quad - \frac{i}{s!^2} \frac{s(s+2)}{\sqrt{2}} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} h_{\alpha(s-1))\dot{\alpha}(s-1))}
\end{aligned} \tag{3.86}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= -\frac{1}{\sqrt{2}} \frac{s-1}{s+1} \epsilon^{\alpha_s} U_{\alpha(s)\dot{\alpha}(s-2)} \\
&\quad - \frac{1}{\sqrt{2}} \frac{s-1}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} [S_{\alpha(s-1)\dot{\alpha}(s-1)} - iP_{\alpha(s-1)\dot{\alpha}(s-1)}] \\
&\quad + i\sqrt{2} \frac{s-1}{s!} \epsilon^{\alpha_s} \partial_{(\alpha_s} \dot{\alpha}_{s-1} h_{\alpha(s-1))\dot{\alpha}(s-1)}
\end{aligned} \tag{3.87}$$

3.3.6.2 Transformation laws for bosons

For the bosonic fields we get:

$$\begin{aligned}
\delta_S A_{\alpha(s)\dot{\alpha}(s)} = & -\frac{i\sqrt{2}}{(s+1)!} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial^{\alpha_{s+1}} (\dot{\alpha}_{s+1} \psi_{\alpha(s+1)\dot{\alpha}(s)}) + c.c. \\
& + \frac{i\sqrt{2}}{s!} \frac{s^2}{(2s+1)(s+1)} \bar{\epsilon}_{(\dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} \psi_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)}) + c.c. \\
& - \frac{i}{s!(s+1)!} \frac{s}{2s+1} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial_{(\alpha_s(\dot{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1))\dot{\alpha}(s)}) + c.c. \\
& - \frac{i}{s!^2} \frac{s}{(2s+1)(s+1)} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s} \dot{\gamma} \bar{\rho}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)}) + c.c. \quad (3.88) \\
& + \frac{i\sqrt{2}}{s!(s+1)!} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial_{(\alpha_s(\dot{\alpha}_{s+1} \bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)}) + c.c. \\
& + \frac{i\sqrt{2}}{s!^2} \frac{s}{(s+1)^2} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s} \dot{\gamma} \bar{\psi}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)}) + c.c. \\
& - \frac{i\sqrt{2}}{s!^2} \frac{s}{2s+1} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1))\dot{\alpha}(s-2)}) + c.c.
\end{aligned}$$

$$\begin{aligned}
\delta_S (u_{\alpha(s)\dot{\alpha}(s)} + i v_{\alpha(s)\dot{\alpha}(s)}) = & -\frac{i\sqrt{2}}{s!} \epsilon_{(\alpha_s} \partial^{\gamma\dot{\gamma}} \bar{\psi}_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s)} \\
& + \frac{i\sqrt{2}}{s!^2} \frac{2s+1}{s(s+1)} \epsilon_{(\alpha_s} \partial^{\gamma} (\dot{\alpha}_s \psi_{\gamma\alpha(s-1))\dot{\alpha}(s-1)}) \\
& + \frac{i\sqrt{2}}{s!^2} \frac{s+1}{s} \epsilon_{(\alpha_s} \partial_{\alpha_{s-1}} (\dot{\alpha}_s \bar{\psi}_{\alpha(s-2))\dot{\alpha}(s-1)}) \\
& - \frac{2}{s!} \frac{s+1}{s} \epsilon_{(\alpha_s} \bar{\beta}_{\alpha(s-1))\dot{\alpha}(s)} \quad (3.89) \\
& + \frac{2}{s!} \bar{\epsilon}_{(\dot{\alpha}_s} \beta_{\alpha(s)\dot{\alpha}(s-1)}) \\
& - \frac{i}{s!^2} \frac{s+1}{2s} \epsilon_{(\alpha_s} \partial^{\gamma} \dot{\alpha}_s \rho_{\gamma\alpha(s-1))\dot{\alpha}(s-1)}) \\
& - \frac{i}{s!(s+1)!} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial_{(\alpha_s(\dot{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1))\dot{\alpha}(s)}) \\
& + \frac{i}{s!^2} \frac{s-1}{2(s+1)} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s} \dot{\gamma} \bar{\rho}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)})
\end{aligned}$$

$$\begin{aligned}
\delta_S U_{\alpha(s)\dot{\alpha}(s-2)} &= i\sqrt{2}\bar{\epsilon}^{\dot{\alpha}s-1}\partial^{\alpha s+1\dot{\alpha}s}\psi_{\alpha(s+1)\dot{\alpha}(s)} \\
&+ \frac{i\sqrt{2}}{s!}\frac{2s+1}{s(s+1)}\bar{\epsilon}^{\dot{\alpha}s-1}\partial_{(\alpha_s}{}^{\dot{\alpha}s}\bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)} \\
&+ \frac{i\sqrt{2}}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot{\gamma}}\psi_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)} \\
&+ \frac{i\sqrt{2}}{s!}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}}{}^{\dot{\alpha}s-1}\bar{\psi}_{\alpha(s-2))\dot{\alpha}(s-1)} \\
&- \frac{i\sqrt{2}}{s!(s-1)!}\frac{s+1}{s}\bar{\epsilon}^{\dot{\alpha}s-1}\partial_{(\alpha_s(\dot{\alpha}_{s-1}}\psi_{\alpha(s-1))\dot{\alpha}(s-2)} \\
&- \frac{i}{s!}\frac{1}{s+1}\epsilon_{(\alpha_s}\partial^{\gamma\dot{\gamma}}\rho_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)} \\
&- \frac{i}{(s+1)!}\epsilon^{\alpha s+1}\partial_{(\alpha_{s+1}}{}^{\dot{\alpha}s-1}\rho_{\alpha(s))\dot{\alpha}(s-1)} \\
&- \frac{i}{s!}\frac{s+1}{2s}\bar{\epsilon}^{\dot{\alpha}s-1}\partial_{(\alpha_s}{}^{\dot{\alpha}s}\bar{\rho}_{\alpha(s-1))\dot{\alpha}(s)} \\
&- 2\frac{s+1}{s}\bar{\epsilon}^{\dot{\alpha}s-1}\beta_{\alpha(s)\dot{\alpha}(s-1)}
\end{aligned} \tag{3.90}$$

$$\begin{aligned}
\delta_S (S_{\alpha(s-1)\dot{\alpha}(s-1)} + iP_{\alpha(s-1)\dot{\alpha}(s-1)}) &= \\
&= 2\frac{s+1}{s}\bar{\epsilon}^{\dot{\alpha}s}\bar{\beta}_{\alpha(s-1)\dot{\alpha}(s)} \\
&- \frac{i}{s!}\frac{s+1}{2s}\bar{\epsilon}^{\dot{\alpha}s}\partial^{\alpha s}(\dot{\alpha}_s\rho_{\alpha(s)\dot{\alpha}(s-1)}) \\
&+ \frac{i}{(s-1)!}\frac{(s-1)(s+1)}{s^2}\bar{\epsilon}_{(\dot{\alpha}_{s-1}}\partial^{\gamma\dot{\gamma}}\rho_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)} \\
&- \frac{i\sqrt{2}}{(s-1)!}\frac{(s-1)(s+1)}{s^2}\bar{\epsilon}_{(\dot{\alpha}_{s-1}}\partial^{\gamma\dot{\gamma}}\psi_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-2)} \\
&- \frac{i\sqrt{2}}{(s-1)!^2}\frac{(s-1)(s+1)}{s^2}\bar{\epsilon}_{(\dot{\alpha}_{s-1}}\partial_{(\alpha_{s-1}}{}^{\dot{\gamma}}\bar{\psi}_{\alpha(s-2))\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned} \tag{3.91}$$

$$\delta_S h_{\alpha(s+1)\dot{\alpha}(s+1)} = \frac{1}{\sqrt{2}(s+1)!}\epsilon_{(\alpha_{s+1}}\bar{\psi}_{\alpha(s))\dot{\alpha}(s+1)} + c.c. \tag{3.92}$$

$$\begin{aligned}
\delta_S h_{\alpha(s-1)\dot{\alpha}(s-1)} &= \frac{1}{\sqrt{2}(s+1)^2}\epsilon^{\alpha s}\psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
&- \frac{1}{2(s+1)}\epsilon^{\alpha s}\rho_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
&- \frac{1}{\sqrt{2}(s+1)}\frac{1}{(s-1)!}\bar{\epsilon}_{(\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c.
\end{aligned} \tag{3.93}$$

3.4 Half-integer superhelicity $Y = s + \frac{1}{2}$ (II)

When examining the deformation of the superspace action under the gauge symmetry of the superfield $H_{\alpha(s)\dot{\alpha}(s)}$ (3.60) we found that there are two different approaches we can take. In this section we follow the second one. It will give us another formulation for half-integer superhelicity systems that has a different off-shell structure.

3.4.1 The superspace action

We start at the same point where the most general action is

$$\begin{aligned}
S = \int d^8 z & a_1 H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
& + a_2 H^{\alpha(s)\dot{\alpha}(s)} \{D^2, \bar{D}^2\} H_{\alpha(s)\dot{\alpha}(s)} \\
& + a_3 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
& + a_4 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} D^\gamma \bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c.
\end{aligned} \tag{3.94}$$

and leads to the following deformation under the gauge transformation of H :

$$\begin{aligned}
\delta_G S = \int d^8 z & \left\{ \left[(-2a_1 + 2\frac{s+1}{s}a_3 + 2a_4) D^2 \bar{D}_{\dot{\alpha}_s} H^{\alpha(s)\dot{\alpha}(s)} \right. \right. \\
& \left. \left. + (-2a_3 - \frac{s+1}{s}a_4) D^{\alpha_s} \bar{D}_{\dot{\gamma}} D_\gamma H^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right] (\bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)}) \right. \\
& \left. + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \right\} + c.c. \\
& + 2a_2 H^{\alpha(s)\dot{\alpha}(s)} D^2 \bar{D}^2 D_{\alpha_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
& - 2a_4 \bar{D}_{\dot{\beta}} D_\gamma \bar{D}_{\dot{\gamma}} H^{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)} \left\{ \bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)} \right. \\
& \left. + \frac{s-1}{s} D^{\alpha_s} \bar{D}^{\dot{\alpha}_{s-1}} L_{\alpha(s)\dot{\alpha}(s-1)} \right\} + c.c. \left. \right\}
\end{aligned} \tag{3.95}$$

Now for case (II) we choose

$$a_1 = c, \quad a_2 = 0, \quad a_3 = \frac{s(s+1)}{2s+1}c, \quad a_4 = -\frac{s^2}{2s+1}c$$

and we have to introduce a fermionic compensator $\chi_{\alpha(s-1)\dot{\alpha}(s-2)}$ which transforms like

$$\begin{aligned} \delta_G \chi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s} D^{\alpha_s} \bar{D}^{\dot{\alpha}_{s-1}} L_{\alpha(s)\dot{\alpha}(s-1)} \\ &\quad + \bar{D}^{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)} \end{aligned}$$

and couples to the term $\bar{D}^{\dot{\beta}} D^\gamma \bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)}$. Notice the index structure of the fermionic compensator and how it is different from the fermionic compensator in case (I).

Therefore we add to the action the coupling term of H with χ and the kinetic energy terms for χ . The full action takes the form

$$\begin{aligned} S &= \int d^8 z \, c \, H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\ &\quad + \frac{s(s+1)}{2s+1} c \, H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\ &\quad - \frac{s^2}{2s+1} c \, H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} D^\gamma \bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c. \\ &\quad - \frac{2s^2}{2s+1} c \, H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D_{\alpha_s} \bar{D}_{\dot{\alpha}_{s-1}} \chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\ &\quad + b_1 \chi^{\alpha(s-1)\dot{\alpha}(s-2)} D^2 \chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\ &\quad + b_2 \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \bar{D}^2 \chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\ &\quad + b_3 \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \bar{D}^{\dot{\alpha}_{s-1}} D_{\alpha_{s-1}} \bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)} \\ &\quad + b_4 \chi^{\alpha(s-1)\dot{\alpha}(s-2)} D_{\alpha_{s-1}} \bar{D}^{\dot{\alpha}_{s-1}} \bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)} \end{aligned}$$

and it has to be invariant under

$$\begin{aligned}
\delta_G H_{\alpha(s)\dot{\alpha}(s)} &= \frac{1}{s!} D_{(\alpha_s} \bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1)})} \quad (3.96) \\
\delta_G \chi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s} D^{\alpha_s} \bar{D}^{\dot{\alpha}_{s-1}} L_{\alpha(s)\dot{\alpha}(s-1)} \\
&\quad + \bar{D}^{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)}
\end{aligned}$$

The equations of motion of the superfields defined as the variations of the action with respect to the superfields

$$T_{\alpha(s)\dot{\alpha}(s)} = \frac{\delta S}{\delta H^{\alpha(s)\dot{\alpha}(s)}}, \quad G_{\alpha(s-1)\dot{\alpha}(s-2)} = \frac{\delta S}{\delta \chi^{\alpha(s-1)\dot{\alpha}(s-2)}} \quad (3.97)$$

satisfy the following Bianchi Identities

$$\begin{aligned}
\bar{D}^{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s)} + \frac{1}{s!(s-1)!} D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}} G_{\alpha(s-1))\dot{\alpha}(s-2)})} \\
+ \left[\frac{s-1}{s} \right] \frac{1}{s!(s-1)!} \bar{D}_{(\dot{\alpha}_{s-1}} D_{(\alpha_s} G_{\alpha(s-1))\dot{\alpha}(s-2)})} = 0 \quad (3.98a)
\end{aligned}$$

$$\bar{D}^{\dot{\alpha}_{s-2}} G_{\alpha(s-1)\dot{\alpha}(s-2)} = 0 \quad (3.98b)$$

The Bianchi identities will fix uniquely all the coefficients to

$$\begin{aligned}
b_1 = 0, \quad b_2 &= \frac{s^2(s+1)}{(2s+1)(s-1)} c \\
b_4 = 0, \quad b_3 &= \frac{2s^2}{2s+1} c
\end{aligned}$$

The final action is

$$\begin{aligned}
S = \int d^8 z \, c \, H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[\frac{s(s+1)}{2s+1} \right] c \, H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
- \left[\frac{s^2}{2s+1} \right] c \, H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} D^\gamma \bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c. \\
- \left[\frac{2s^2}{2s+1} \right] c \, H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D_{\alpha_s} \bar{D}_{\dot{\alpha}_{s-1}} \chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \quad (3.99) \\
+ \left[\frac{s^2(s+1)}{(2s+1)(s-1)} \right] c \, \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \bar{D}^2 \chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
+ \left[\frac{2s^2}{2s+1} \right] c \, \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \bar{D}^{\dot{\alpha}_{s-1}} D_{\alpha_{s-1}} \bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)}
\end{aligned}$$

and the equations of motion are

$$\begin{aligned}
T_{\alpha(s)\dot{\alpha}(s)} &= 2c D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
&+ \frac{2c}{s!} \left[\frac{s(s+1)}{2s+1} \right] D_{(\alpha_s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} \\
&+ \frac{2c}{s!} \left[\frac{s(s+1)}{2s+1} \right] \bar{D}_{(\dot{\alpha}_s} D^2 \bar{D}^{\dot{\gamma}} H_{\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)} \\
&- \frac{2c}{s!s!} \left[\frac{s^2}{2s+1} \right] D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_s} D^\gamma \bar{D}^{\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \\
&- \frac{2c}{s!s!} \left[\frac{s^2}{2s+1} \right] \bar{D}_{(\dot{\alpha}_s} D_{(\alpha_s} \bar{D}^{\dot{\gamma}} D^\gamma H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \quad (3.100) \\
&- \frac{2c}{s!s!} \left[\frac{s^2}{2s+1} \right] \bar{D}_{(\dot{\alpha}_s} D_{(\alpha_s} \bar{D}_{\dot{\alpha}_{s-1}} \chi_{\alpha(s-1)\dot{\alpha}(s-2)} \\
&- \frac{2c}{s!s!} \left[\frac{s^2}{2s+1} \right] D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_s} D_{\alpha_{s-1}} \bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)}
\end{aligned}$$

$$\begin{aligned}
G_{\alpha(s-1)\dot{\alpha}(s-2)} &= 2c \left[\frac{s^2}{2s+1} \right] \bar{D}^{\dot{\alpha}_{s-1}} D^{\alpha_s} \bar{D}^{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} \\
&+ 2c \left[\frac{s^2(s+1)}{(2s+1)(s-1)} \right] \bar{D}^2 \chi_{\alpha(s-1)\dot{\alpha}(s-2)} \quad (3.101) \\
&+ \frac{2c}{(s-1)!} \left[\frac{s^2}{2s+1} \right] \bar{D}^{\dot{\alpha}_{s-1}} D_{(\alpha_{s-1}} \bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)}
\end{aligned}$$

3.4.2 On-shell equations of motion

To prove that this action indeed describes an $Y = s + 1/2$ superhelicity we use superfield T to show the following identity:

$$D^{\alpha_{2s+1}} F_{\alpha(2s+1)} = \frac{1}{2c} \frac{1}{(2s)!} \partial_{(\alpha_{2s}}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+1}}^{\dot{\alpha}_1} T_{\alpha(s)\dot{\alpha}(s)} \quad (3.102)$$

where

$$F_{\alpha(2s+1)} = \frac{1}{(2s+1)!} \bar{D}^2 D_{(\alpha_{2s+1}} \partial_{\alpha_{2s}}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+1}}^{\dot{\alpha}_1} H_{\alpha(s)\dot{\alpha}(s)}$$

Therefore when we go on-shell ($T_{\alpha(s)\dot{\alpha}(s)} = 0$) the chiral superfield $F_{\alpha(2s+1)}$ will satisfy all the necessary conditions set by representation theory, in order to describe the highest superhelicity $Y = s + 1/2$ irreducible representation.

3.4.3 Unique action

For the previous two cases (integer and half-integer (I)) the superspace action that described the system was not unique but a member of an infinite family of equivalent actions. This is not true in this case. The reason is that, mass dimensions and index structure of $H_{\alpha(s)\dot{\alpha}(s)}$, $\chi_{\alpha(s-1)\dot{\alpha}(s-2)}$ does not permit any local redefinitions.

3.4.4 Field spectrum and component lagrangian

Here we present the components and the component action for this case. First we write the action in terms of the equations of motion:

$$\begin{aligned}
S &= \int d^8z \left\{ \frac{1}{2} H^{\alpha(s)\dot{\alpha}(s)} T_{\alpha(s)\dot{\alpha}(s)} \right. \\
&\quad \left. + \frac{1}{2} \chi^{\alpha(s-1)\dot{\alpha}(s-2)} G_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \right\} \\
&= \int d^4x \frac{1}{2} D^2 \bar{D}^2 (H^{\alpha(s)\dot{\alpha}(s)} T_{\alpha(s)\dot{\alpha}(s)}) \\
&\quad + \frac{1}{2} D^2 \bar{D}^2 (\chi^{\alpha(s-1)\dot{\alpha}(s-2)} G_{\alpha(s-1)\dot{\alpha}(s-2)}) + c.c.
\end{aligned} \tag{3.103}$$

and then we distribute the $D(\bar{D})$ -s to find the lagrangian for the fermionic components and the lagrangian for the bosonic components

3.4.4.1 Fermionic component lagrangian

After the distribution of the covariant derivatives and the usage of Bianchi identities we get for the fermionic lagrangian:

$$\begin{aligned}
\mathcal{L}_F &= \\
&= \frac{1}{2} \frac{1}{(s+1)!} D^2 \bar{D}^{(\dot{\alpha}_{s+1}} H^{\alpha(s)\dot{\alpha}(s)} \left| \frac{1}{(s+1)!} \bar{D}_{(\dot{\alpha}_{s+1}} T_{\alpha(s)\dot{\alpha}(s)} \right| \\
&+ \left(\frac{1}{2} \frac{1}{s+1} D^2 \bar{D}_{\dot{\gamma}} H^{\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)} - \frac{1}{2} \frac{1}{s!(s-1)!} D^{(\alpha_s} \bar{D}^{(\dot{\alpha}_{s-1}} \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \right) \\
&\quad - \frac{i}{2} \frac{1}{s!} D^{(\alpha_s} \partial_{\gamma\dot{\gamma}} H^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \left| \frac{1}{s!(s-1)!} D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} \right| \\
&+ \left(\frac{1}{2} \frac{s-1}{s} \frac{1}{(s-1)!} D_{\gamma} \bar{D}^{(\dot{\alpha}_{s-1}} \chi^{\gamma\alpha(s-2)\dot{\alpha}(s-2)} \right. \\
&\quad \left. + \frac{i}{2} \frac{s-1}{s} D_{\beta} \partial_{\gamma\dot{\gamma}} H^{\beta\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-1)} \right) \left| \frac{1}{(s-1)!} D^{\alpha_{s-1}} \bar{D}_{(\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} \right| \\
&+ \left(-\frac{i}{2} \frac{s-1}{s+1} \partial_{\alpha_s \dot{\alpha}_{s-1}} D^2 \bar{D}_{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} \right. \\
&\quad \left. + \frac{1}{2} D^2 \bar{D}^2 \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \right) \left| G_{\alpha(s-1)\dot{\alpha}(s-2)} \right| \\
&+ \frac{1}{2} \bar{D}^2 \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \left| D^2 G_{\alpha(s-1)\dot{\alpha}(s-2)} \right| \\
&+ c.c.
\end{aligned} \tag{3.104}$$

We can prove the following identities for T and G :

$$\begin{aligned}
& \frac{1}{(s+1)!} \bar{D}_{(\dot{\alpha}_{s+1}} T_{\alpha(s)\dot{\alpha}(s)} = \\
& = \frac{2ic}{(s+1)!} \partial^{\alpha_{s+1}}_{(\dot{\alpha}_{s+1}} \left\{ \frac{1}{(s+1)!} \bar{D}^2 D_{(\alpha_{s+1}} H_{\alpha(s)\dot{\alpha}(s)} \right\} \\
& - \frac{2ic}{(s+1)! s!} \frac{s^2}{(2s+1)(s+1)} \partial_{(\alpha_s(\dot{\alpha}_{s+1}} \left\{ \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} \right. \\
& \quad \left. + \frac{i(s+1)}{s!} \bar{D}_{\dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right. \\
& \quad \left. + \frac{s+1}{s!(s-1)!} \bar{D}_{(\dot{\alpha}_s} D_{(\alpha_{s-1}} \bar{\chi}_{\alpha(s-2))\dot{\alpha}(s-1)} \right\}
\end{aligned} \tag{3.105}$$

$$\begin{aligned}
& \frac{1}{s!(s-1)!} D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} = \\
& = -\frac{2ic}{s!} \frac{s^2}{(2s+1)(s+1)} \partial_{(\alpha_s}^{\dot{\alpha}_s} \left\{ \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} \right. \\
& \quad \left. + \frac{i(s+1)}{s!} \bar{D}_{(\dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right. \\
& \quad \left. + \frac{s+1}{s!(s-1)!} \bar{D}_{(\dot{\alpha}_s} D_{(\alpha_{s-1}} \bar{\chi}_{\alpha(s-2))\dot{\alpha}(s-1)} \right\} \\
& - 2ic \frac{s^2}{2s+1} \partial^{\alpha_{s+1}\dot{\alpha}_s} \left\{ \frac{1}{(s+1)!} \bar{D}^2 D_{(\alpha_{s+1}} H_{\alpha(s)\dot{\alpha}(s)} \right\} \\
& + \frac{2ic}{s!(s-1)!} \frac{s(s-1)}{2s+1} \partial_{(\alpha_s(\dot{\alpha}_{s-1}} \left\{ i \bar{D}^{\dot{\beta}} \partial^{\gamma\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)} \right. \\
& \quad \left. + \frac{1}{(s-1)!} \bar{D}^{\dot{\gamma}} D_{(\alpha_{s-1}} \bar{\chi}_{\alpha(s-2))\dot{\gamma}\dot{\alpha}(s-2)} \right\}
\end{aligned} \tag{3.106}$$

$$\begin{aligned}
& \frac{1}{(s-1)!} \bar{D}^{\dot{\alpha}_{s-1}} D_{(\alpha_{s-1}) \dot{\alpha}(s-1)} \bar{G}_{\alpha(s-2)} = \\
& = -\frac{s}{s+1} D^2 G_{\alpha(s-1) \dot{\alpha}(s-2)} \\
& + \frac{i}{(s-1)!} \frac{s(s-1)}{(s+1)^2} \partial_{(\alpha_{s-1})}^{\dot{\alpha}_{s-1}} \bar{G}_{\alpha(s-2) \dot{\alpha}(s-1)} \\
& - 2ic \frac{s^2}{(2s+1)(s+1)} \partial^{\alpha_s \dot{\alpha}_{s-1}} \left\{ D^2 \bar{D}^{\dot{\gamma}} H_{\alpha(s) \dot{\gamma} \dot{\alpha}(s-1)} \right. \\
& \quad \left. + \frac{i(s+1)}{s!} D_{(\alpha_s} \partial^{\gamma \dot{\gamma}} H_{\gamma \alpha(s-1) \dot{\gamma} \dot{\alpha}(s-1)} \right. \\
& \quad \left. + \frac{s+1}{s!(s-1)!} D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}) \dot{\alpha}(s-2))} \chi_{\alpha(s-1)} \right\} \\
& + 2ic \frac{s(s-1)}{(s+1)^2} \frac{1}{(s-1)!} \partial_{(\alpha_{s-1})}^{\dot{\alpha}_{s-1}} \left\{ i D^{\beta} \partial^{\gamma \dot{\gamma}} H_{\beta \gamma \alpha(s-2) \dot{\gamma} \dot{\alpha}(s-1)} \right. \\
& \quad \left. + \frac{1}{(s-1)!} D^{\gamma} \bar{D}_{(\dot{\alpha}_{s-1}) \dot{\alpha}(s-2))} \chi_{\gamma \alpha(s-2)} \right\}
\end{aligned} \tag{3.107}$$

Let's define the following fields

$$\begin{aligned}
& \frac{1}{(s+1)!} \bar{D}^2 D_{(\alpha_{s+1}) \dot{\alpha}(s)} H_{\alpha(s) \dot{\alpha}(s)} \Big| \equiv N_1 \psi_{\alpha(s+1) \dot{\alpha}(s)} \\
& \left\{ D^2 \bar{D}^{\dot{\alpha}_s} H_{\alpha(s) \dot{\alpha}(s)} + \frac{i(s+1)}{s!} D_{(\alpha_s} \partial^{\gamma \dot{\gamma}} H_{\gamma \alpha(s-1) \dot{\gamma} \dot{\alpha}(s-1)} \right. \\
& \quad \left. + \frac{s+1}{s!(s-1)!} D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}) \dot{\alpha}(s-2))} \chi_{\alpha(s-1)} \right\} \Big| \equiv N_2 \psi_{\alpha(s) \dot{\alpha}(s-1)} \\
& \left\{ i \bar{D}^{\dot{\beta}} \partial^{\gamma \dot{\gamma}} H_{\gamma \alpha(s-1) \dot{\beta} \dot{\gamma} \dot{\alpha}(s-2)} \right. \\
& \quad \left. + \frac{1}{(s-1)!} \bar{D}^{\dot{\alpha}_{s-1}} D_{(\alpha_{s-1}) \dot{\alpha}(s-1)} \bar{\chi}_{\alpha(s-2)} \right\} \Big| \equiv N_3 \psi_{\alpha(s-1) \dot{\alpha}(s-2)}
\end{aligned}$$

Putting everything together, the component lagrangian for the fermions takes

the form

$$\begin{aligned}
\mathcal{L}_F = & 2ic|N_1|^2 \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& - 2ic \frac{s^2}{(2s+1)(s+1)} N_1 N_2 \bar{\psi}^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& - 2ic \frac{s^2}{(2s+1)(s+1)^2} |N_2|^2 \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s \dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& - 2ic \frac{s(s-1)}{(2s+1)(s+1)} N_2 N_3 \bar{\psi}^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \quad (3.108) \\
& - 2ic \left(\frac{s-1}{s+1} \right)^2 |N_3|^2 \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1} \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \\
& + \frac{1}{2c} \frac{(2s+1)(s-1)}{s^2(s+1)^2} G^{\alpha(s)\dot{\alpha}(s-1)} \left| \left(D^2 G_{\alpha(s-1)\dot{\alpha}(s-2)} \right. \right. \\
& \quad \left. \left. - \frac{i}{2} \frac{s-1}{s+1} \frac{1}{(s-1)!} \partial_{(\alpha_{s-1} \dot{\alpha}_{s-1}} \bar{G}_{\alpha(s-2))\dot{\alpha}(s-1)} \right) \right| + c.c
\end{aligned}$$

The last term in the lagrangian is the algebraic term for the two auxiliary fields and the rest of the terms are exactly the structure of a theory that describes helicity $\lambda = s + 1/2$ (A.2). To have an exact match we choose coefficients

$$\begin{aligned}
c = 1, \quad N_2 = & -\frac{1}{\sqrt{2}} \frac{2s+1}{s} \\
N_1 = \frac{1}{\sqrt{2}}, \quad N_3 = & \frac{1}{\sqrt{2}} \frac{s+1}{s-1}
\end{aligned}$$

So the fields that appear in the fermionic action are defined as

$$\begin{aligned}
\rho_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv G_{\alpha(s-1)\dot{\alpha}(s-2)}| \\
\beta_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv \left\{ D^2 G_{\alpha(s-1)\dot{\alpha}(s-2)} \right. \\
&\quad \left. - \frac{i}{2} \frac{s-1}{s+1} \frac{1}{(s-1)!} \partial_{(\alpha_{s-1}} \dot{\alpha}_{s-1} \bar{G}_{\alpha(s-2))\dot{\alpha}(s-1)} \right\} | \\
\psi_{\alpha(s+1)\dot{\alpha}(s)} &\equiv \frac{\sqrt{2}}{(s+1)!} \bar{D}^2 D_{(\alpha_{s+1}} H_{\alpha(s))\dot{\alpha}(s)}| \\
\psi_{\alpha(s)\dot{\alpha}(s-1)} &\equiv -\sqrt{2} \frac{s}{2s+1} \left\{ D^2 \bar{D}^{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} \right. \\
&\quad \left. + \frac{i(s+1)}{s!} D_{(\alpha_s} \partial^{\gamma\dot{\gamma}} H_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)} \right. \\
&\quad \left. + \frac{s+1}{s!(s-1)!} D_{(\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}} \chi_{\alpha(s-1))\dot{\alpha}(s-2)} \right\} | \\
\psi_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv \sqrt{2} \frac{s-1}{s+1} \left\{ i \bar{D}^{\dot{\beta}} \partial^{\gamma\dot{\gamma}} H_{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)} \right. \\
&\quad \left. + \frac{1}{(s-1)!} \bar{D}^{\dot{\alpha}_{s-1}} D_{(\alpha_{s-1}} \bar{\chi}_{\alpha(s-2))\dot{\alpha}(s-1)} \right\} |
\end{aligned} \tag{3.109}$$

The lagrangian is

$$\begin{aligned}
\mathcal{L}_F &= \rho^{\alpha(s)\dot{\alpha}(s-1)} \beta_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
&+ i \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}}_{\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
&+ i \left[\frac{s}{s+1} \right] \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
&- i \left[\frac{2s+1}{(s+1)^2} \right] \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s}_{\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
&+ i \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
&- i \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned} \tag{3.110}$$

and the gauge transformations of the fields are

$$\begin{aligned}
\delta_G \rho_{\alpha(s)\dot{\alpha}(s-1)} &= 0, \quad \delta_G \psi_{\alpha(s+1)\dot{\alpha}(s)} = \frac{1}{s!(s+1)!} \partial_{(\alpha_{s+1}(\dot{\alpha}_s \xi_{\alpha(s)\dot{\alpha}(s-1)})} \\
\delta_G \beta_{\alpha(s)\dot{\alpha}(s-1)} &= 0, \quad \delta_G \psi_{\alpha(s)\dot{\alpha}(s-1)} = -\frac{1}{s!} \partial_{(\alpha_s \dot{\alpha}_s \bar{\xi}_{\alpha(s-1)\dot{\alpha}(s)}} \\
\delta_G \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \frac{s-1}{s} \partial^{\alpha_s \dot{\alpha}_{s-1}} \xi_{\alpha(s)\dot{\alpha}(s-1)} \\
\text{with } \xi_{\alpha(s)\dot{\alpha}(s-1)} &= -i\sqrt{2} \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} |
\end{aligned} \tag{3.111}$$

3.4.4.2 Bosonic component lagrangian

We do the same for the bosons. The definition of the fields that will appear in the lagrangian is

$$\begin{aligned}
A_{\alpha(s)\dot{\alpha}(s)} &\equiv T_{\alpha(s)\dot{\alpha}(s)} | \\
U_{\alpha(s)\dot{\alpha}(s-2)} &\equiv \frac{1}{s!} D_{(\alpha_s} G_{\alpha(s-1)\dot{\alpha}(s-2)} | \\
u_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2(s-1)!} \{ \bar{D}_{(\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} - D_{(\alpha_{s-1}} \bar{G}_{\alpha(s-2)\dot{\alpha}(s-1)} \} | \\
v_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv -\frac{i}{2(s-1)!} \{ \bar{D}_{(\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} + D_{(\alpha_{s-1}} \bar{G}_{\alpha(s-2)\dot{\alpha}(s-1)} \} | \\
S_{\alpha(s-2)\dot{\alpha}(s-2)} &\equiv \frac{1}{2} \{ D^{\alpha_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} + \bar{D}^{\dot{\alpha}_{s-1}} \bar{G}_{\alpha(s-2)\dot{\alpha}(s-1)} \} | \\
P_{\alpha(s-2)\dot{\alpha}(s-2)} &\equiv -\frac{i}{2} \{ D^{\alpha_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} - \bar{D}^{\dot{\alpha}_{s-1}} \bar{G}_{\alpha(s-2)\dot{\alpha}(s-1)} \} | \\
h_{\alpha(s+1)\dot{\alpha}(s+1)} &\equiv \frac{1}{2} \frac{1}{(s+1)!^2} [D_{(\alpha_{s+1}}, \bar{D}_{(\dot{\alpha}_{s+1}}] H_{\alpha(s)\dot{\alpha}(s)} | \\
h_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv -\frac{1}{2} \frac{s}{(2s+1)(s+1)^2} [D^{\alpha_s}, \bar{D}^{\dot{\alpha}_s}] H_{\alpha(s)\dot{\alpha}(s)} | \\
&\quad - \frac{s}{(2s+1)(s+1)} \frac{1}{(s-1)!} \left(D_{(\alpha_{s-1}} \bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)} \right. \\
&\quad \quad \left. - \bar{D}_{(\dot{\alpha}_{s-1}} \chi_{\alpha(s-1)\dot{\alpha}(s-2)} \right) |
\end{aligned} \tag{3.112}$$

and their gauge transformations are

$$\begin{aligned}
\delta_G U_{\alpha(s)\dot{\alpha}(s-2)} &= 0, & \delta_G A_{\alpha(s)\dot{\alpha}(s)} &= 0 \\
\delta_G u_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0, & \delta_G S_{\alpha(s-2)\dot{\alpha}(s-2)} &= 0 \\
\delta_G v_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0, & \delta_G P_{\alpha(s-2)\dot{\alpha}(s-2)} &= 0 \\
\delta_G h_{\alpha(s+1)\dot{\alpha}(s+1)} &= \frac{1}{(s+1)!^2} \partial_{(\alpha_{s+1}(\dot{\alpha}_{s+1} \zeta_{\alpha(s)\dot{\alpha}(s)})} \\
\delta_G h_{\alpha(s-1)\dot{\alpha}(s-1)} &= \frac{s}{(s+1)^2} \partial^{\alpha_s \dot{\alpha}_s} \zeta_{\alpha(s)\dot{\alpha}(s)} \\
\text{where } \zeta_{\alpha(s)\dot{\alpha}(s)} &= \frac{i}{2s!} \left(D_{(\alpha_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} + \bar{D}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1)}) \right) |
\end{aligned} \tag{3.113}$$

The bosonic lagrangian is

$$\begin{aligned}
\mathcal{L}_B &= -\frac{1}{4} \left[\frac{(2s+1)(s-1)}{s^2(s+1)} \right] U^{\alpha(s)\dot{\alpha}(s-2)} U_{\alpha(s)\dot{\alpha}(s-2)} + c.c. \\
&+ \frac{1}{8} \left[\frac{2s+1}{s+1} \right] A^{\alpha(s)\dot{\alpha}(s)} A_{\alpha(s)\dot{\alpha}(s)} \\
&- \frac{1}{2} \left[\frac{2s+1}{s^2} \right] u^{\alpha(s-1)\dot{\alpha}(s-1)} u_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \frac{1}{2} \left[\frac{2s+1}{s^2} \right] v^{\alpha(s-1)\dot{\alpha}(s-1)} v_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \frac{1}{2} \left[\frac{(2s+1)(s-1)^2}{s^3} \right] S^{\alpha(s-2)\dot{\alpha}(s-2)} S_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&+ \frac{1}{2} \left[\frac{(s-1)^2}{s^3} \right] P^{\alpha(s-2)\dot{\alpha}(s-2)} P_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&+ h^{\alpha(s+1)\dot{\alpha}(s+1)} \square h_{\alpha(s+1)\dot{\alpha}(s+1)} \\
&- \left[\frac{s+1}{2} \right] h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha_{s+1}\dot{\alpha}_{s+1}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s)} \\
&+ [s(s+1)] h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha_{s+1}\dot{\alpha}_{s+1}} \partial_{\alpha_s \dot{\alpha}_s} h_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- [(s+1)(2s+1)] h^{\alpha(s-1)\dot{\alpha}(s-1)} \square h_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&- \left[\frac{(s+1)(s-1)^2}{2} \right] h^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned} \tag{3.114}$$

and gives rise to the theory of helicity $\lambda = s + 1$ (A.1) as expected.

3.4.5 Off-shell degrees of freedom

The counting of the bosonic degrees of freedom is

<i>fields</i>	<i>d.o.f</i>	<i>redundancy</i>	<i>net</i>
$h_{\alpha(s+1)\dot{\alpha}(s+1)}$	$(s+2)^2$	$(s+1)^2$	$s^2 + 2s + 3$
$h_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2		
$u_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
$v_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
$A_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$U_{\alpha(s)\dot{\alpha}(s-2)}$	$2(s+1)(s-1)$	0	$2(s+1)(s-1)$
$S_{\alpha(s-2)\dot{\alpha}(s-2)}$	$(s-1)^2$	0	$(s-1)^2$
$P_{\alpha(s-2)\dot{\alpha}(s-2)}$	$(s-1)^2$	0	$(s-1)^2$
		<i>Total</i>	$8s^2 + 4$

Table 3.5: Off-shell bosonic degrees of freedom for a half-integer(II) superhelicity

and for the fermionic degrees of freedom we get

<i>fields</i>	<i>d.o.f</i>	<i>redundancy</i>	<i>net</i>
$\psi_{\alpha(s+1)\dot{\alpha}(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2 + 4s + 4$
$\psi_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot{\alpha}(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s-1)\dot{\alpha}(s-2)}$	$2(s-1)s$	0	$2(s-1)s$
$\beta_{\alpha(s-1)\dot{\alpha}(s-2)}$	$2(s-1)s$	0	$2(s-1)s$
		<i>Total</i>	$8s^2 + 4$

Table 3.6: Off-shell fermionic degrees of freedom for a half-integer(II) superhelicity

From this we conclude that this theory of half-integer superhelicity is an $8s^2+4|8s^2+4$ system. Therefore it has different field spectrum and number of degrees of freedom than case (I). So although on-shell they describe the same physics, the off-shell structure is different and from that point of view the two theories are not equivalent to each other. That means that we can not find a transformation that will map, the one theory to the other in a one to one fashion.

3.4.6 Supersymmetric transformation laws for the components

Here we calculate the supersymmetric transformation laws for the components in exactly the same way as in the previous two cases. For this reason we give directly the results.

3.4.6.1 Transformation laws for fermions

$$\begin{aligned}
\delta_S \rho_{\alpha(s-1)\dot{\alpha}(s-2)} &= -\epsilon^{\alpha s} U_{\alpha(s)\dot{\alpha}(s-2)} \\
&+ \left[\frac{s-1}{s} \right] \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} [S_{\alpha(s-2))\dot{\alpha}(s-2)} + iP_{\alpha(s-2))\dot{\alpha}(s-2)}] \\
&- \bar{\epsilon}^{\dot{\alpha} s-1} [u_{\alpha(s-1)\dot{\alpha}(s-1)} + iv_{\alpha(s-1)\dot{\alpha}(s-1)}]
\end{aligned} \tag{3.115}$$

$$\begin{aligned}
\delta_S \beta_{\alpha(s-1)\dot{\alpha}(s-2)} &= \\
&= \frac{i}{2} \frac{s^2}{s+1} \bar{\epsilon}^{\dot{\alpha} s-1} \partial^{\alpha s \dot{\alpha} s} A_{\alpha(s)\dot{\alpha}(s)} \\
&+ \frac{s^2}{2s+1} \bar{\epsilon}^{\dot{\alpha} s-1} \partial^{\alpha s+1 \dot{\alpha} s+1} \partial^{\alpha s \dot{\alpha} s} h_{\alpha(s+1)\dot{\alpha}(s+1)} \\
&- 2s \bar{\epsilon}^{\dot{\alpha} s-1} \square h_{\alpha(s-1)\dot{\alpha}(s-2)} \\
&- \frac{s(s-1)^2}{2s+1} \frac{1}{(s-1)!^2} \bar{\epsilon}^{\dot{\alpha} s-1} \partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1} \partial^{\beta \dot{\beta}} h_{\beta\alpha(s-2))\dot{\beta}\dot{\alpha}(s-2))} \\
&- \frac{i}{(s-1)!} \bar{\epsilon}^{\dot{\alpha} s-1} \partial^{\alpha s}_{(\dot{\alpha}_{s-1}} U_{\alpha(s)\dot{\alpha}(s-2)} \\
&+ \frac{s-2}{s-1} \frac{i}{(s-2)!} \bar{\epsilon}_{(\dot{\alpha}_{s-2}} \partial^{\beta \dot{\beta}} U_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-3)} \\
&+ \frac{1}{2} \frac{s-1}{s+1} \frac{i}{(s-1)!} \bar{\epsilon}^{\dot{\alpha} s-1} \partial_{(\alpha_{s-1}}^{\dot{\alpha} s} \bar{U}_{\alpha(s-2))\dot{\alpha}(s)} \\
&+ \frac{(s-1)(2s^2+2s+1)}{2s(s+1)} \frac{i}{(s-1)!^2} \bar{\epsilon}^{\dot{\alpha} s-1} \partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1} S_{\alpha(s-2))\dot{\alpha}(s-2)} \\
&- \frac{(s-1)(2s^2+4s+3)}{2s(s+1)(2s+1)} \frac{1}{(s-1)!^2} \bar{\epsilon}^{\dot{\alpha} s-1} \partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1} P_{\alpha(s-2))\dot{\alpha}(s-2)} \\
&- \frac{(s-2)(3s+2)}{2s(s+1)} \frac{i}{(s-2)!(s-1)!} \bar{\epsilon}_{(\dot{\alpha}_{s-2}} \partial_{(\alpha_{s-1}}^{\dot{\beta}} S_{\alpha(s-2))\dot{\beta}\dot{\alpha}(s-3)} \\
&+ \frac{(s-2)(s+2)}{2s(s+1)} \frac{1}{(s-2)!(s-1)!} \bar{\epsilon}_{(\dot{\alpha}_{s-2}} \partial_{(\alpha_{s-1}}^{\dot{\beta}} P_{\alpha(s-2))\dot{\beta}\dot{\alpha}(s-3)} \\
&- \frac{1}{2} \frac{s-1}{s+1} \frac{i}{(s-1)!} \epsilon^{\beta} \partial_{(\alpha_{s-1}}^{\dot{\alpha} s-1} u_{\beta\alpha(s-2))\dot{\alpha}(s-1)} \\
&- \frac{1}{2} \frac{s-1}{s+1} \frac{1}{(s-1)!} \epsilon^{\beta} \partial_{(\alpha_{s-1}}^{\dot{\alpha} s-1} v_{\beta\alpha(s-2))\dot{\alpha}(s-1)}
\end{aligned} \tag{3.116}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s+1)\dot{\alpha}(s)} &= \frac{\sqrt{2}i}{(s+2)!} \epsilon^{\alpha_{s+2}} \partial_{(\alpha_{s+2}} \dot{\alpha}_{s+1} h_{\alpha(s+1))\dot{\alpha}(s+1)} \\
&\quad - \frac{1}{\sqrt{2}} \frac{s}{s+2} \frac{i}{(s+1)!} \epsilon_{(\alpha_{s+1}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s))\dot{\gamma}\dot{\alpha}(s)} \\
&\quad + \frac{1}{2\sqrt{2}} \frac{2s+1}{s+1} \frac{1}{(s+1)!} \epsilon_{(\alpha_{s+1}} A_{\alpha(s))\dot{\alpha}(s)}
\end{aligned} \tag{3.117}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s)\dot{\alpha}(s-1)} &= \frac{1}{\sqrt{2}} \frac{s+1}{s} \frac{1}{s!} \epsilon_{(\alpha_s} \left[-u_{\alpha(s-1))\dot{\alpha}(s-1)} + i v_{\alpha(s-1))\dot{\alpha}(s-1)} \right] \\
&\quad + \frac{1}{\sqrt{2}} \frac{s-1}{s} \frac{1}{(s-1)!} \bar{\epsilon}_{(\dot{\alpha}_{s-1}} U_{\alpha(s)\dot{\alpha}(s-2))} \\
&\quad - \frac{1}{2\sqrt{2}} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_s} A_{\alpha(s)\dot{\alpha}(s)} \\
&\quad - \frac{is}{\sqrt{2}} \bar{\epsilon}^{\dot{\alpha}_s} \partial^{\alpha_{s+1}\dot{\alpha}_{s+1}} h_{\alpha(s+1)\dot{\alpha}(s+1)} \\
&\quad + \frac{is(s+2)}{\sqrt{2}s!s!} \bar{\epsilon}^{\dot{\alpha}_s} \partial_{(\alpha_s(\dot{\alpha}_s} h_{\alpha(s-1))\dot{\alpha}(s-1))}
\end{aligned} \tag{3.118}$$

$$\begin{aligned}
\delta_S \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= -\frac{1}{\sqrt{2}} \frac{(2s+1)(s-1)}{s^2(s+1)} \bar{\epsilon}^{\dot{\alpha}_{s-1}} u_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&\quad - \frac{i}{\sqrt{2}} \frac{(2s+1)(s-1)}{s^2(s+1)} \bar{\epsilon}^{\dot{\alpha}_{s-1}} v_{\alpha(s-1)\dot{\alpha}(s-1)} \\
&\quad - \frac{1}{\sqrt{2}} \frac{(s-1)^2(2s+1)}{s^2(s+1)} \frac{1}{(s-1)!} \epsilon_{\alpha_{s-1}} S_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad + \frac{i}{\sqrt{2}} \frac{(s-1)^2}{s^2(s+1)} \frac{1}{(s-1)!} \epsilon_{\alpha_{s-1}} P_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad + i\sqrt{2} \frac{(s-1)^2(s+1)}{s} \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-2))\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned} \tag{3.119}$$

3.4.6.2 Transformation laws for bosons

The susy-transformation laws for the bosonic fields are

$$\begin{aligned}
\delta_S A_{\alpha(s)\dot{\alpha}(s)} = & -\frac{i\sqrt{2}}{(s+1)!} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial^{\alpha_{s+1}}_{(\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)})} + c.c. \\
& + \frac{i\sqrt{2}}{s!} \frac{s^2}{(s+1)(2s+1)} \bar{\epsilon}_{(\dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} \psi_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)})} + c.c. \\
& + \frac{i\sqrt{2}}{(s+1)!s!} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s+1}} \partial_{(\alpha_s(\dot{\alpha}_{s+1}} \bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)})} + c.c. \\
& + \frac{i\sqrt{2}}{s!s!} \frac{s}{(s+1)^2} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s} \dot{\gamma} \bar{\psi}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)})} + c.c. \\
& - \frac{i\sqrt{2}}{s!s!} \frac{s}{2s+1} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1))\dot{\alpha}(s-2)})} + c.c. \\
& - \frac{i}{s!s!} \frac{s-1}{s+1} \bar{\epsilon}_{(\dot{\alpha}_s} \partial_{(\alpha_s \dot{\alpha}_{s-1}} \rho_{\alpha(s-1))\dot{\alpha}(s-2)})} + c.c.
\end{aligned} \tag{3.120}$$

$$\begin{aligned}
\delta_S U_{\alpha(s)\dot{\alpha}(s-2)} = & \frac{1}{s!} \epsilon_{(\alpha_s} \beta_{\alpha(s-1))\dot{\alpha}(s-2)} \\
& - \frac{i}{s!(s-1)!} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial_{(\alpha_s(\dot{\alpha}_{s-1}} \rho_{\alpha(s-1))\dot{\alpha}(s-2)})} \\
& + \frac{i}{s!(s-2)!} \frac{s-2}{s-1} \bar{\epsilon}_{(\dot{\alpha}_{s-2}} \partial_{(\alpha_s} \dot{\gamma} \rho_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-3)})} \\
& + \frac{i}{s!} \frac{s-1}{2(s+1)} \epsilon_{(\alpha_s} \partial_{\alpha_{s-1}} \dot{\alpha}_{s-1} \bar{\rho}_{\alpha(s-2))\dot{\alpha}(s-1)} \\
& - i\sqrt{2} \frac{s^2}{2s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial^{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& - \frac{i\sqrt{2}}{s!} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial_{(\alpha_s} \dot{\alpha}_s \bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)} \\
& + \frac{i\sqrt{2}}{s!(s-1)!} \frac{s(s+1)}{2s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial_{(\alpha_s(\dot{\alpha}_{s-1}} \psi_{\alpha(s-1))\dot{\alpha}(s-2)})}
\end{aligned} \tag{3.121}$$

$$\begin{aligned}
& \delta_S (u_{\alpha(s-1)\dot{\alpha}(s-1)} + iv_{\alpha(s-1)\dot{\alpha}(s-1)}) = \\
& = i\sqrt{2} \frac{s^2}{2s+1} \epsilon^{\alpha_s} \partial^{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& + i\sqrt{2} \frac{s}{s+1} \frac{1}{s!} \epsilon^{\alpha_s} \partial_{(\alpha_s}^{\dot{\alpha}_s} \bar{\psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - i\sqrt{2} \frac{s(s+1)}{2s+1} \frac{1}{s!(s-1)!} \epsilon^{\alpha_s} \partial_{(\alpha_s(\dot{\alpha}_{s-1} \psi_{\alpha(s-1)\dot{\alpha}(s-2)})} \\
& - \frac{s^2}{(s+1)(s-1)} \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} \bar{\beta}_{\alpha(s-2)\dot{\alpha}(s-1)} \\
& - i\sqrt{2} \frac{s^2}{(s+1)(s-1)} \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} \partial^{\gamma\dot{\alpha}_s} \bar{\psi}_{\gamma\alpha(s-2)\dot{\alpha}(s)} \\
& + i\sqrt{2} \frac{s^2}{(s+1)(s-1)} \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} \partial^{\gamma} (\dot{\alpha}_{s-1} \psi_{\gamma\alpha(s-2)\dot{\alpha}(s-2)})
\end{aligned} \tag{3.122}$$

$$\begin{aligned}
& \delta_S (S_{\alpha(s-2)\dot{\alpha}(s-2)} + iP_{\alpha(s-2)\dot{\alpha}(s-2)}) = \\
& = \epsilon^{\alpha_{s-1}} \beta_{\alpha(s-1)\dot{\alpha}(s-2)} \\
& + \frac{i}{2} \frac{s-1}{s+1} \frac{1}{(s-1)!} \epsilon^{\alpha_{s-1}} \partial_{(\alpha_{s-1}}^{\dot{\alpha}_{s-1}} \bar{\rho}_{\alpha(s-2)\dot{\alpha}_{s-1}} \\
& - \frac{i}{(s-1)!} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial^{\alpha_{s-1}} (\dot{\alpha}_{s-1} \rho_{\alpha(s-1)\dot{\alpha}(s-2)}) \\
& + \frac{i}{(s-2)!} \frac{s-2}{s-1} \bar{\epsilon}_{(\dot{\alpha}_{s-2}} \partial^{\gamma\dot{\gamma}} \rho_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-3)}) \\
& - \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \bar{\beta}_{\alpha(s-2)\dot{\alpha}(s-1)} \\
& - i\sqrt{2} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial^{\alpha_{s-1}\dot{\alpha}_s} \bar{\psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \frac{i\sqrt{2}}{(s-1)!} \frac{s}{s+1} \bar{\epsilon}^{\dot{\alpha}_{s-1}} \partial^{\alpha_{s-1}} (\dot{\alpha}_{s-1} \psi_{\alpha(s-1)\dot{\alpha}(s-2)})
\end{aligned} \tag{3.123}$$

$$\delta_S h_{\alpha(s+1)\dot{\alpha}(s+1)} = \frac{1}{\sqrt{2}(s+1)!} \epsilon_{(\alpha_{s+1}} \bar{\psi}_{\alpha(s)\dot{\alpha}(s+1)} + c.c. \tag{3.124}$$

$$\begin{aligned}
\delta_S h_{\alpha(s-1)\dot{\alpha}(s-1)} &= \frac{1}{\sqrt{2}} \frac{1}{(s+1)^2} \epsilon^{\alpha_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
&+ \frac{1}{\sqrt{2}} \frac{1}{(s+1)} \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} \bar{\psi}_{\alpha(s-2)\dot{\alpha}(s-1)} + c.c. \\
&- \frac{1}{2} \frac{s-1}{s(s+1)^2} \frac{1}{(s-1)!} \epsilon_{(\alpha_{s-1}} \bar{\rho}_{\alpha(s-2)\dot{\alpha}(s-1)} + c.c.
\end{aligned} \tag{3.125}$$

3.5 Map of superhelicity theories

To summarize the results, the landscape of the massless irreducible representations that describe the highest superhelicity supermultiplets looks as following

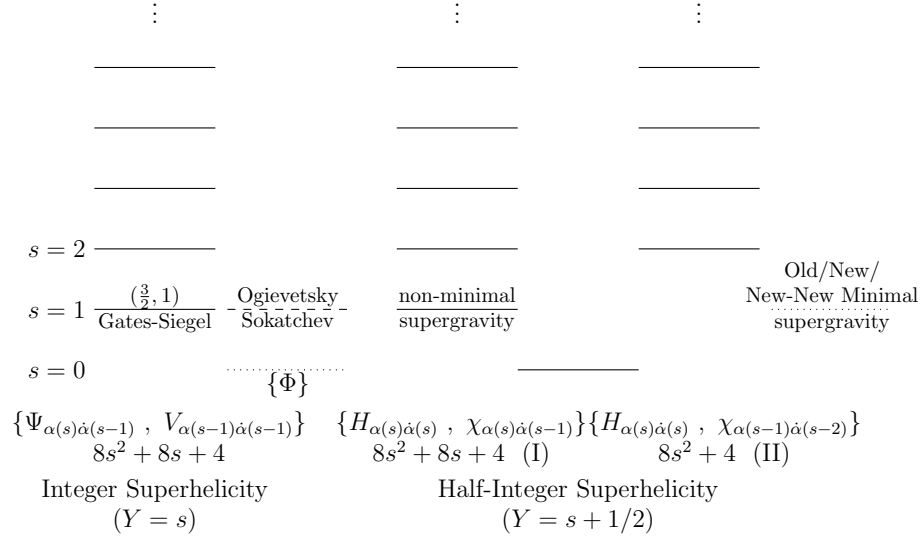


Figure 3.1: Map of massless representations for the highest possible superhelicity

There are three infinite towers of theories, one for the integer superhelicity and two for the half-integer superhelicity. A solid line represents the corresponding theory for that value of s . The corresponding theory for integer and half-integer (I) is a two parameter family of actions, but for half-integer (II) it is a unique action. At the bottom, the superfield structure of the action and the number of degrees of freedom involved are being displayed. For the $s = 0$ case of the integer

tower, there is a gap. The reason is that for $s = 0$ there is no superfield Ψ and the tower starts from $s = 1$. The $Y = s = 0$ theory is being generated by a chiral superfield Φ . Similarly the for the $s = 1$ case of the half-integer (II) theories, where its place takes a triplet of theories, the old minimal, the new minimal and the new-new minimal. The dash line represent theories that don't fall in the pattern. These are low helicity 'accidents' that don't generalize to arbitrary s . It is very easy to understand the reason why. When $s = 1$ the corresponding superfields (Ψ_α) do not have rich enough index structure to generate many terms. Therefore when we are calculating the deformation of the action under the gauge symmetry some terms can be simplified. Thus there are alternative approaches that will lead to different formulations of the theory.

For example when $s = 1$ equation (3.5) can take the form

(the gauge parameter $\Lambda_{\alpha(s)\dot{\alpha}(s-2)}$, in order to survive, in the $s = 1$ case must be modified to $\bar{D}^{\dot{\alpha}s-1}\Lambda_{\alpha(s)\dot{\alpha}(s-1)}$)

$$\begin{aligned} \delta S = \int d^8z & (-2a_1 D_\alpha \Psi^\alpha + a_4 \bar{D}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}}) D^\beta \bar{D}^2 \Lambda_\beta + c.c. \\ & + 2a_2 \Psi^\alpha \bar{D}^2 D_\alpha K + c.c. \\ & - a_3 \bar{\Psi}^{\dot{\alpha}} D^2 \bar{D}_{\dot{\alpha}} K + c.c. \\ & + (2a_4 - a_3) \bar{\Psi}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^2 K + c.c. \end{aligned}$$

Then it's obvious that besides the approach we followed there is another way: If

$$-2a_1 = a_4 \quad , \quad 2a_2 = a_3 \quad , \quad 2a_4 = a_3$$

$$K = \bar{K}, \quad \Lambda_\alpha = iD_\alpha U, \quad U = \bar{U}$$

the change of the action vanishes and we have a gauge invariant theory. This configuration will lead to the Ogievetsky / Sokatchev theory [21, 22].

3.6 Hints for $\mathcal{N} = 2$

The massless irreducible representations of $4D$, $\mathcal{N} = 2$ Super-Poincaré group for superhelicity Y describe helicities $\lambda = Y + 1$, $\lambda = Y + 1/2$, $\lambda = Y + 1/2$, $\lambda = Y$. That looks like the direct sum of two $\mathcal{N} = 1$ massless irreducible representations, one describing superhelicity $Y + 1/2$ and the other one describing superhelicity Y . Therefore one will be tempted to try to combine the theory of integer superhelicity $Y = s$ with one of the theories of half-integer superhelicity $Y = s + 1/2$ in order to construct an $\mathcal{N} = 2$ representation. The question is which pair [integer, half-integer(I)] or [integer, half-integer(II)] will be the one to give the $\mathcal{N} = 2$ representation. In an attempt to find the answer the authors of [23], by trial and error concluded that the answer was [integer, half-integer(I)].

The counting of the degrees of freedom argument provides a very simple explanation why this is the case. The integer theory has exactly the same degrees of freedom as the half-integer (I) theory. This is a sign that if we add together the two theories then in principle we can have a second direction of supersymmetry that will map the bosons (fermions) of one theory to the fermions (bosons) of the other theory. This can only happen if the number of bosons and fermions match exactly, as they do. Therefore we can construct an irreducible representation of $4D$, $\mathcal{N} = 2$ Super-Poincaré group. Also in the same manner we can understand why a possible

pair of integer theory with half-integer (II) theory can never work.

Chapter 4: Lagrangians for massive representations

We have constructed massless representations. Now we want to do the same for massive ones. That means to construct actions that dynamically generate all the required constraints in order to describe a massive irreducible representation. Again we will focus on the highest superspin theories, the theories where the superfield, given its index structure, describes the highest possible superspin multiplet. Unlike the massless case where there was a gauge symmetry to guide us, the massive case has no guidelines. So the strategy is to work case by case from low superspin theories to higher ones, trying to understand the pattern of the set of auxiliary fields needed and the mechanisms involved. Then generalize it to the arbitrary superspin case. The one clue that we have, is that the massless theories are by construction the massless limit of the massive theories. Therefore we should start with the massless theories that we know and then add terms proportional to mass and other (auxiliary) superfields that in the massless limit decouple. Now it is obvious that if we want to understand the pattern of massive theories for arbitrary superspin, it would be smart to use massless theories that are formulated for arbitrary superhelicities. That means a massive extension of the Ogievetsky-Sokatchev or old-minimal or new-minimal or new-new-minimal theories will not help us to see the big picture.

4.1 Massive superspin $Y = 0$ (Wess-Zumino action)

That is the lowest superspin we can start. The action is formulated in terms of a chiral scalar superfield Φ

$$S = \int d^8z \Phi \bar{\Phi} + \int d^6z a m \Phi \Phi + \int d^6\bar{z} a m \bar{\Phi} \bar{\Phi} \quad (4.1)$$

The equation of motion is:

$$\mathcal{E}^{(\Phi)} = -\bar{D}^2 \bar{\Phi} + 2a m \Phi \rightsquigarrow D^2 \mathcal{E} + 2am \bar{\mathcal{E}} = -\square \bar{\Phi} + 2a^2 m^2 \bar{\Phi} = 0 \quad (4.2)$$

so for $a = \pm \frac{1}{\sqrt{2}}$ we get $\square \Phi = m^2 \Phi$ and because it is chiral, it describes the $Y = 0$ representation.

4.2 Massive superspin $Y = \frac{1}{2}$

We start with the $Y = 1/2$ massless theory and we add all possible mass terms

$$S = \int d^8z \left\{ HD^\gamma \bar{D}^2 D_\gamma H + a_1 m H (D^2 H + \bar{D}^2 H) + a_2 m^2 H H \right\} \quad (4.3)$$

The equation of motion is

$$\mathcal{E}^{(H)} = 2D^\gamma \bar{D}^2 D_\gamma H + 2a_1 m (D^2 H + \bar{D}^2 H) + 2a_2 m^2 H$$

To describe $Y = \frac{1}{2}$, H must satisfy $D^2 H = 0$ and $\square H = m^2 H$.

$$D^2 \mathcal{E}^{(H)} = 2a_1 m D^2 \bar{D}^2 H + 2a_2 m^2 D^2 H$$

so by choosing $a_1 = 0, a_2 \neq 0$ we get $D^2H = 0 \rightsquigarrow \bar{D}^2H = 0$ (reality) and if we plug it back in to $\mathcal{E}^{(H)}$ we get $\square H = a_2 m^2 H$ which fixes $a_2 = 1$ for compatibility with the Klein-Gordon equation. The final action is:

$$S = \int d^8z \left\{ HD^\gamma \bar{D}^2 D_\gamma H + m^2 HH \right\} \quad (4.4)$$

4.3 Massive superspin $Y = \frac{1}{2}$ as the direct sum of massless superhelicities $Y = \frac{1}{2}$ and $Y = 0$

There is also another way to get that result. The observation is that at least on-shell the massive superspin $\frac{1}{2}$ can be seen as the result of combining the massless superhelicity $\frac{1}{2}$ with the massless superhelicity 0. That is directly related to the fact that massive spin states can be seen as the direct sum of massless spin states. So we would like to see if that idea can be transferred to superspin as well. We start with the actions for superhelicity $\frac{1}{2}$ and 0 and we introduce terms proportional to m and m^2 . The action takes the form

$$\begin{aligned} S = \int d^8z \left\{ \right. & HD^\gamma D^2 \bar{D}_\gamma H \\ & + a_1 m H (D^2 H + \bar{D}^2 H) \\ & + a_2 m^2 H^2 \\ & + \gamma m H (\Phi + \bar{\Phi}) \\ & \left. + b_1 \Phi \bar{\Phi} \right\} \\ & + \int d^6z b_2 m \Phi \Phi + c.c. \end{aligned} \quad (4.5)$$

The equations of motion are

$$\begin{aligned}\mathcal{E}^{(H)} &= 2D^\gamma \bar{D}^2 D_\gamma H + 2a_1 m (D^2 H + \bar{D}^2 H) + \gamma m (\Phi + \bar{\Phi}) + 2a_2 m^2 H \\ \mathcal{E}^{(\Phi)} &= -b_1 \bar{D}^2 \bar{\Phi} - \gamma m \bar{D}^2 H + 2b_2 m \Phi\end{aligned}\tag{4.6}$$

If we manage to show that on-shell $\Phi = 0$ then $\mathcal{E}^{(\Phi)} = 0 \rightsquigarrow D^2 H = 0$. That takes us to the previous case and we can show that if $a_2 = 1$, then $\square H = m^2 H$. With that goal in mind we try, by eliminating H , to form an equation that depends only on Φ . Then by choosing coefficients in clever way we will show that $\Phi = 0$. So consider the following:

$$\begin{aligned}I &= \bar{D}^2 \mathcal{E}^{(H)} + m \mathcal{E}^{(\Phi)} \\ &= (\gamma - b_1) m \bar{D}^2 \bar{\Phi} \\ &\quad + (2a_2 - \gamma) m^2 \bar{D}^2 H \\ &\quad + 2a_1 \bar{D}^2 D^2 H \\ &\quad + 2b_2 m^2 \Phi\end{aligned}\tag{4.7}$$

If we choose $\gamma - 2a_2 = 0$, $a_1 = 0$ we remove the H -dependence and then if $\gamma - b_1 = 0$ we get $I = 2b_2 m^2 \Phi$.

Now we can follow two possible routes:

1. $b_2 \neq 0$: b_2 can be anything besides zero. In that case on-shell $I = 0 \rightsquigarrow \Phi = 0$ and so we can generate all the desired constraints for H ($D^2 H = 0$, $\square H = m^2 H$) and the action is

$$\begin{aligned}
S_{m \neq 0}^{Y=\frac{1}{2}}[H, \Phi] = \int d^8 z \{ & HD^\gamma D^2 \bar{D}_\gamma H \\
& + m^2 H^2 \\
& + 2m H (\Phi + \bar{\Phi}) \\
& + 2 \Phi \bar{\Phi} \} \\
& + \int d^6 z b_2 m \Phi \Phi
\end{aligned} \tag{4.8}$$

2. $b_2 = 0$: If we set b_2 to zero, then I vanish identically. That means the equation $\bar{D}^2 \mathcal{E}^{(H)} + m \mathcal{E}^{(\Phi)} = 0$ can be treated as a Bianchi identity and the corresponding action is invariant under a symmetry. The symmetry of the action that generates the above Bianchi identity is

$$\begin{aligned}
\delta_G H &\sim \bar{D}^2 L + D^2 \bar{L} \\
\delta_G \Phi &\sim m \bar{D}^2 L
\end{aligned}$$

Because of that symmetry the chiral superfield Φ can be gauged away completely and therefore its equation of motion will give the desired constraint $D^2 H = 0$. The action for this case is

$$\begin{aligned}
S = \int d^8 z \{ & HD^\gamma D^2 \bar{D}_\gamma H \\
& + m^2 H^2 \\
& + 2m H (\Phi + \bar{\Phi}) \\
& + 2 \Phi \bar{\Phi} \}
\end{aligned} \tag{4.9}$$

and the gauge fixed action ($\Phi = 0$) is identical with the action (4.4).

In both cases the result is, that the massless limit of massive superspin $Y = \frac{1}{2}$ will give the sum of the actions that describe the massless superhelicities $Y = \frac{1}{2}$ and $Y = 0$

$$\lim_{m \rightarrow 0} S_{m \neq 0}^{Y=\frac{1}{2}}[H, \Phi] = S_{m=0}^{Y=\frac{1}{2}} + S_{m=0}^{Y=0} \quad (4.10)$$

Also we should point out that the $b_2 = 0$ story is the exact superspace analogue of the Stüeckelberg construction of massive spin 1

4.4 Massive superspin $Y = 1$

We start with the massless superhelicity $Y = 1$ theory and add mass corrections. The nice thing here is that the massless theory will provide the first auxiliary field, which played the role of the compensator in the massless case. So the starting action is:

$$\begin{aligned} S = \int d^8z \left\{ -\frac{1}{2} \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. + a_1 m V (D^2 V + \bar{D}^2 V) \right. \\ \left. + \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + a_2 m V (D^a \Psi_\alpha + \bar{D}^{\dot{a}} \bar{\Psi}_{\dot{a}}) \right. \\ \left. - V D^\alpha \bar{D}^2 \Psi_\alpha + c.c. + a_3 m \Psi^\alpha \Psi_\alpha + c.c. \right. \\ \left. + \frac{1}{2} V D^\gamma \bar{D}^2 D_\gamma V + a_4 m^2 V^2 \right\} \end{aligned} \quad (4.11)$$

and the equations of motion are

$$\mathcal{E}_\alpha^{(\Psi)} = -\bar{D}^2 \Psi_\alpha + \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + \bar{D}^2 D_\alpha V - a_2 m D_\alpha V + 2a_3 m \Psi_\alpha \quad (4.12)$$

$$\begin{aligned} \mathcal{E}^{(V)} = D^\gamma \bar{D}^2 D_\gamma V - (D^\alpha \bar{D}^2 \Psi_\alpha + \bar{D}^{\dot{\alpha}} D^2 \bar{\Psi}_{\dot{\alpha}}) + 2a_1 m (D^2 V + \bar{D}^2 V) \\ + a_2 m (D^a \Psi_\alpha + \bar{D}^{\dot{a}} \bar{\Psi}_{\dot{a}}) + 2a_4 m^2 V \end{aligned}$$

The strategy is to use the equation of motion of Ψ in order to eliminate any Ψ -term from the equation of V . In this way we get an equation that depends only on V . For that reason we calculate the following linear combination:

$$\begin{aligned}
I &= A(D^\alpha \bar{D}^2 \mathcal{E}_\alpha^\Psi + c.c.) + B(D^2 \mathcal{E}^V + c.c.) + m\Gamma(D^\alpha \mathcal{E}_\alpha^\Psi + c.c.) + m\mathcal{E}^V \\
&= [2Aa_3 - 2\Gamma - 1] mD^\alpha \bar{D}^2 \Psi_\alpha + c.c. + [-2Aa_2 + 2\Gamma + 1] mD^\gamma \bar{D}^2 D_\gamma V \\
&\quad + [Ba_2 - \Gamma] m\bar{D}^2 D^\alpha \Psi_\alpha + c.c. + [2Ba_1] mD^2 \bar{D}^2 V + c.c. \quad (4.13) \\
&\quad + [2\Gamma a_3 + a_2] m^2 D^\alpha \Psi_\alpha + c.c. + [-2\Gamma a_2 + 2Ba_4 + 2a_1] m^2 D^2 V + c.c. \\
&\quad \quad \quad + [2a_4] m^3 V
\end{aligned}$$

In order to eliminate Ψ we choose

$$(\Sigma_1) : \quad 2Aa_3 - 2\Gamma - 1 = 0 \quad , \quad Ba_2 - \Gamma = 0 \quad , \quad 2\Gamma a_3 + a_2 = 0 \quad (4.14)$$

so I becomes an equation that depends on V only

$$\begin{aligned}
I &= +[-2Aa_2 + 2\Gamma + 1] mD^\gamma \bar{D}^2 D_\gamma V \\
&\quad + [2Ba_1] mD^2 \bar{D}^2 V + c.c. \quad (4.15) \\
&\quad + [-2\Gamma a_2 + 2Ba_4 + 2a_1] m^2 D^2 V + c.c. \\
&\quad + [2a_4] m^3 V
\end{aligned}$$

Now we check whether we can choose coefficients in order to make V vanish on-shell.

That means setting

$$(\Sigma_2) : \quad -2Aa_2 + 2\Gamma + 1 = 0 \quad , \quad 2Ba_1 = 0 \quad , \quad -2\Gamma a_2 + 2Ba_4 + 2a_1 = 0 \quad (4.16)$$

$$a_4 \neq 0$$

If the systems of equations (Σ_1) and (Σ_2) have non-trivial solutions then we can make V vanish on-shell. A solution exists and it is

$$a_1 = 0 \quad , \quad a_2 = a_3 = -\frac{1}{2B} \quad , \quad a_4 = \frac{1}{4B^2} \quad , \quad A = 0 \quad , \quad \Gamma = -\frac{1}{2} \quad , \quad B \neq 0$$

Since $V = 0$ on-shell, the equations of motion become

$$\mathcal{E}_\alpha^\Psi = -\bar{D}^2\Psi_\alpha + \bar{D}^\alpha D_\alpha \bar{\Psi}_{\dot{\alpha}} + 2a_3 m \Psi_\alpha \quad (4.17)$$

$$\mathcal{E}^V = -(D^\alpha \bar{D}^2\Psi_\alpha + \bar{D}^{\dot{\alpha}} D^2 \bar{\Psi}_{\dot{\alpha}}) + a_3 m (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}})$$

and we get the following constraints

$$\bar{D}^2 \mathcal{E}_\alpha^\Psi = 2a_3 m \bar{D}^2 \Psi_\alpha \quad \rightsquigarrow \quad \bar{D}^2 \Psi_\alpha = 0 \quad (4.18)$$

$$\mathcal{E}^V = a_3 m (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) \quad \rightsquigarrow \quad D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} = 0 \quad \rightsquigarrow \quad D^2 \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} = 0$$

The above constraints will help us show that

$$D^\alpha \mathcal{E}_\alpha^\Psi = 2a_3 m D^\alpha \Psi_\alpha \quad \rightsquigarrow \quad D^\alpha \Psi_\alpha = 0 \quad (4.19)$$

$$\mathcal{E}_\alpha^\Psi = \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + 2a_3 m \Psi_\alpha = i \partial_\alpha^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} + 2a_3 m \Psi_\alpha \quad \rightsquigarrow \quad i \partial_\alpha^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} + m \Psi_\alpha = 0$$

$$\text{with } 2a_3 = 1$$

We managed to dynamically generate all the constraints required and in the

process we fixed all the coefficients. The final action is

$$\begin{aligned}
S_{m \neq 0}^{Y=1}[\Psi_\alpha, V] = \int d^8z \left\{ & -\frac{1}{2} \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \right. \\
& + \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} \\
& - V D^\alpha \bar{D}^2 \Psi_\alpha + c.c. \\
& + \frac{1}{2} V D^\gamma \bar{D}^2 D_\gamma V \\
& + \frac{1}{2} m V (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) \\
& + \frac{1}{4} m^2 V^2 \\
& \left. + \frac{1}{2} m \Psi^\alpha \Psi_\alpha + c.c. \right\}
\end{aligned} \tag{4.20}$$

There is another approach to this problem presented in [24]. They are using a chiral superfield Φ instead of the real scalar V we used in order to generate the constraint $\bar{D}^2 D^\alpha \Psi_\alpha = 0$. In the massless limit the two approaches are dual to each other through a non-local transformation $\Psi_\alpha \rightarrow \Psi_\alpha + \frac{c_1}{\square} D_\alpha (\bar{D}^2 \bar{\Phi} + i c_2 D^\gamma \bar{D}^2 D_\gamma V)$. But in the full massive case this is no longer true.

4.5 Massive superspin $Y = 1$ as the direct sum of massless

superhelicities $Y = 1$, $Y = \frac{1}{2}$ and $Y = 0$

We want to test whether the idea of writing the massive action as the sum of all massless actions with superhelicities less or equal to the superspin value holds for the case of superspin 1. We start by taking a linear combination of the massless actions for superhelicities 1, $\frac{1}{2}$ and 0 and adding all possible interaction terms proportional

to m and m^2

$$S_{m \neq 0}^{Y=1} = S_{m=0}^{Y=1} + c S_{m=0}^{Y=\frac{1}{2}} + d S_{m=0}^{Y=0} + m(\dots) + m^2(\dots)$$

In this way we make sure that in the massless limit all we get is the sum of the three superhelicities. So the starting action has the form

$$\begin{aligned}
S = \int d^8 z \left\{ & -\frac{1}{2} \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \quad + a_1 m V (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) + f_1 m \Psi^\alpha \Psi_\alpha + c.c. \\
& + \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} \quad + a_2 m H (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) + f_2 m^2 V^2 \\
& - V D^\alpha \bar{D}^2 \Psi_\alpha + c.c. \quad + a_3 m V (\Phi + \bar{\Phi}) \quad + f_3 m^2 V H \quad (4.21) \\
& + \frac{1}{2} V D^\gamma \bar{D}^2 D_\gamma V \quad + a_4 m H (\Phi + \bar{\Phi}) \quad + f_4 m^2 H^2 \\
& + c H D^\gamma \bar{D}^2 \bar{D}_\gamma H \quad + b_1 m V (D^2 V + \bar{D}^2 V) \\
& + d \Phi \bar{\Phi} \quad + b_2 m V (D^2 H + \bar{D}^2 H) \\
& \quad + b_3 m H (D^2 H + \bar{D}^2 H) \quad \left. \vphantom{\int} \right\} \\
& + \int d^6 z f_5 m \Phi \Phi
\end{aligned}$$

It will be useful to keep in mind that some of these coefficients can be changed by rescaling some of the superfields. For example, the relative scales of Ψ and V are fixed by the $S_{m=0}^{Y=1}$ action. The overall scale of the entire action is also fixed by choosing the coefficient of $S_{m=0}^{Y=1}$ to be 1. But we have the freedom of rescaling H

and Φ . So we can choose $a_3 = 1$, $f_3 = 1$. The equations of motion are

$$\begin{aligned}
\mathcal{E}_\alpha^{(\Psi)} &= -\bar{D}^2\Psi_\alpha + \bar{D}^{\dot{\alpha}}D_\alpha\bar{\Psi}_{\dot{\alpha}} + \bar{D}^2D_\alpha V - a_1mD_\alpha V - a_2mD_\alpha H + 2f_1m\Psi_\alpha \\
\mathcal{E}^{(V)} &= D^\gamma\bar{D}^2D_\gamma V - (D^\alpha\bar{D}^2\Psi_\alpha + \bar{D}^{\dot{\alpha}}D^2\bar{\Psi}_{\dot{\alpha}}) + a_1m(D^\alpha\Psi_\alpha + \bar{D}^{\dot{\alpha}}\bar{\Psi}_{\dot{\alpha}}) \\
&\quad + a_3m(\Phi + \bar{\Phi}) + 2b_1m(D^2V + \bar{D}^2V) + b_2m(D^2H + \bar{D}^2H) \\
&\quad + 2f_2m^2V + f_3m^2H \\
\mathcal{E}^{(H)} &= 2cD^\gamma\bar{D}^2D_\gamma H + a_2m(D^\alpha\Psi_\alpha + \bar{D}^{\dot{\alpha}}\bar{\Psi}_{\dot{\alpha}}) + a_4m(\Phi + \bar{\Phi}) + b_2m(D^2V + \bar{D}^2V) \\
&\quad + 2b_3m(D^2H + \bar{D}^2H) + f_3m^2V + 2f_4m^2H \\
\mathcal{E}^{(\Phi)} &= -d\bar{D}^2\bar{\Phi} - a_3m\bar{D}^2V - a_4m\bar{D}^2H + 2f_5m\Phi
\end{aligned} \tag{4.22}$$

We want to choose coefficients so one by one the auxiliary superfields vanish and we generate the constraints for Ψ to describe a superspin 1 system. First we try to eliminate Φ . For that reason let's consider the following combination of the equations of motion:

$$\begin{aligned}
I &= A\bar{D}^2\mathcal{E}^H + m\mathcal{E}^\Phi \\
&= [Aa_2] m\bar{D}^2D^\alpha\Psi_\alpha + [Af_3 - a_3] m^2\bar{D}^2V \\
&\quad [Aa_4 - d] m\bar{D}^2\bar{\Phi} + [2Af_4 - a_4] m^2\bar{D}^2H \\
&\quad [Ab_2] m\bar{D}^2D^2V + (2f_5) m^2\Phi \\
&\quad [2Ab_3] m\bar{D}^2D^2H
\end{aligned}$$

If we choose coefficients such that

$$\begin{aligned}
& a_2 = 0 \quad , \quad b_3 = 0 \\
(\Sigma_1) \quad & Aa_4 - d = 0 \quad , \quad Af_3 - a_3 = 0 \\
& b_2 = 0 \quad , \quad 2Af_4 - a_4 = 0
\end{aligned} \tag{4.23}$$

then we get $I = (2f_5) m^2 \Phi$. Notice that f_5 is not constrained in anyway, so if $f_5 \neq 0$ then on-shell ($I = 0$) we get that $\Phi = 0$. On the other hand if $f_5 = 0$ then off-shell, I vanishes identically, thus it can be interpreted as a Bianchi identity and that means that the action must have a symmetry. The symmetry that can generate such a Bianchi identity would have to be similar as in the superspin $\frac{1}{2}$ case

$$\delta_G H \sim \bar{D}^2 L + D^2 \bar{L}, \quad \delta_G \Phi \sim m \bar{D}^2 L$$

No matter what the case is, the conclusion is the same: Φ will vanish (or gauged away) on-shell. Then the updated equations of motion are

$$\begin{aligned}
\mathcal{E}_\alpha^{(\Psi)} &= -\bar{D}^2 \Psi_\alpha + \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + \bar{D}^2 D_\alpha V - a_1 m D_\alpha V + 2f_1 m \Psi_\alpha \\
\mathcal{E}^{(V)} &= D^\gamma \bar{D}^2 D_\gamma V - (D^\alpha \bar{D}^2 \Psi_\alpha + \bar{D}^{\dot{\alpha}} D^2 \bar{\Psi}_{\dot{\alpha}}) + a_1 m (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) \\
&\quad + 2b_1 m (D^2 V + \bar{D}^2 V) + 2f_2 m^2 V + f_3 m^2 H \\
\mathcal{E}^{(H)} &= 2c D^\gamma \bar{D}^2 D_\gamma H + f_3 m^2 V + 2f_4 m^2 H \\
\mathcal{E}^{(\Phi)} &= -a_3 m \bar{D}^2 V - a_4 m \bar{D}^2 H
\end{aligned} \tag{4.24}$$

Now we try to eliminate H . Consider the following combination of equations

of motion

$$\begin{aligned}
J &= B(D^\alpha \bar{D}^2 \mathcal{E}_\alpha^\Psi + c.c.) + m\Gamma(D^\alpha \mathcal{E}_\alpha^\Psi + c.c.) + \Delta(D^2 \mathcal{E}^V + c.c.) + mZ(\mathcal{E}^\Phi + c.c.) + m\mathcal{E}^V \\
&= [2Bf_1 - 2\Gamma - 1] mD^\alpha \bar{D}^2 \Psi_\alpha + c.c. \quad + [-2\Gamma a_1 + 2\Delta f_2 - Za_3 + 2b_1] m^2 D^2 V \\
&\quad + [-\Gamma + \Delta a_1] mD^2 \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} + c.c. \quad + [2\Delta f_3 - Za_4] m^2 \bar{D}^2 H \\
&\quad + [2\Gamma f_1 + a_1] m^2 D^\alpha \Psi_\alpha + c.c. \quad + [2f_2] m^3 V \\
&\quad + [-2Ba_1 + 2\Gamma + 1] mD^\gamma \bar{D}^2 D_\gamma V \quad + (f_3) m^3 H \\
&\quad + [2\Delta b_1] mD^2 \bar{D}^2 V + c.c.
\end{aligned}$$

By choosing coefficients such that

$$\begin{aligned}
&2Bf_1 - 2\Gamma - 1 = 0 \quad , \quad b_1 = 0 \\
(\Sigma_2) \quad &-\Gamma + \Delta a_1 = 0 \quad , \quad -2\Gamma a_1 + 2\Delta f_2 - Za_3 + 2b_1 = 0 \\
&2\Gamma f_1 + a_1 = 0 \quad , \quad 2\Delta f_3 - Za_4 = 0 \quad (4.25) \\
&-2Ba_1 + 2\Gamma + 1 = 0 \quad , \quad f_2 = 0
\end{aligned}$$

we get that if $f_3 \neq 0$, then on-shell $H = 0$. Vanishing of H will immediately mean (through $\mathcal{E}^{(H)}$) that V must vanish as well. We update the equations of motion again

$$\begin{aligned}
\mathcal{E}_\alpha^{(\Psi)} &= -\bar{D}^2 \Psi_\alpha + \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + 2f_1 m \Psi_\alpha \\
\mathcal{E}^{(V)} &= -(D^\alpha \bar{D}^2 \Psi_\alpha + \bar{D}^{\dot{\alpha}} D^2 \bar{\Psi}_{\dot{\alpha}}) + a_1 m (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) \quad (4.26) \\
\mathcal{E}^{(H)} &= 0 \\
\mathcal{E}^{(\Phi)} &= 0
\end{aligned}$$

and we are back to equations (4.17). So for $a_1 \neq 0$, $2f_1 = 1$ we will describe

a massive superspin $Y = 1$ system. The last thing to do is check if equations $\Sigma_1, \Sigma_2, f_3 \neq 0, a_1 \neq 0, 2f_1 = 1$ are consistent with each other and have a non trivial solution. The answer is that they are consistent and the solution is

$$\begin{aligned}
a_1 &= \frac{1}{2} \quad , \quad b_1 = 0 \quad , \quad f_1 = \frac{1}{2} \quad , \quad d = -4 \quad , \quad A = 1 \\
a_2 &= 0 \quad , \quad b_2 = 0 \quad , \quad f_2 = 0 \quad , \quad c = \text{free} \quad , \quad B = 0 \\
a_3 &= 1 \quad , \quad b_3 = 0 \quad , \quad f_3 = 1 \quad , \quad \Gamma = -\frac{1}{2} \\
a_4 &= -4 \quad , \quad f_4 = -2 \quad , \quad \Delta = -1 \\
& \quad \quad \quad f_5 = \text{free} \quad , \quad Z = \frac{1}{2}
\end{aligned}$$

The final action is:

$$\begin{aligned}
S_{m \neq 0}^{Y=1}[\Psi_\alpha, V, H, \Phi] &= \\
&= \int d^8z \left\{ -\frac{1}{2} \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \quad + \frac{1}{2} m V (D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}) + \frac{1}{2} m \Psi^\alpha \Psi_\alpha + c.c. \right. \\
&\quad + \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} \quad + m V (\Phi + \bar{\Phi}) \quad + m^2 V H \\
&\quad - V D^\alpha \bar{D}^2 \Psi_\alpha + c.c. \quad - 4m H (\Phi + \bar{\Phi}) \quad - 2m^2 H^2 \\
&\quad \left. + \frac{1}{2} V D^\gamma \bar{D}^2 D_\gamma V \quad + c H D^\gamma \bar{D}^2 \bar{D}_\gamma H \quad - 4\Phi \bar{\Phi} \right\} \\
&+ \int d^6z f_5 m \Phi \Phi
\end{aligned} \tag{4.27}$$

We discussed about the freedom of f_5 , but what about c ?

- If $c = 0$ then the H superfield will not have any kinetic energy terms and it's equation of motion can be solved to express it in terms of V and Φ , thus eliminating it completely from the action. The massless limit of this theory will be

$$\lim_{m \rightarrow 0} S_{m \neq 0}^{Y=1}[\Psi_\alpha, V, H, \Phi] = S_{m=0}^{Y=1} - 4 S_{m=0}^{Y=0}$$

- If $c \neq 0$ then H can not be eliminated from the action and the massless limit of the theory will be

$$\lim_{m \rightarrow 0} S_{m \neq 0}^{Y=1}[\Psi_\alpha, V, H, \Phi] = S_{m=0}^{Y=1} + c S_{m=0}^{Y=\frac{1}{2}} - 4 S_{m=0}^{Y=0}$$

- if $c = -2$ then the action can be written

$$S_{m \neq 0}^{Y=1}[\Psi_\alpha, V, H, \Phi] = S_{m \neq 0}^{Y=1}[\Psi_\alpha, V] - 2S_{m \neq 0}^{Y=\frac{1}{2}}[H, \Phi] - \frac{1}{4}m^2V^2$$

4.6 Massive superspin $Y = \frac{3}{2}$ (non-minimal supergravity)

There are in the literature constructions for massive supergravity [25] but most of them have as a massless limit old-minimal or new-minimal or new-new minimal supergravities. These theories can not be generalized to arbitrary superhelicity so they are not very helpful. What we need is a massive extension of non-minimal supergravity. There is a construction like that [26] but it uses a lagrange multiplier to impose constraints that can not be generated dynamically. We will show that there is another formulation where all the constraints are generated in a dynamic way.

We start with the non-minimal formulation of superhelicity $\frac{3}{2}$ (3.64) and add all possible mass corrections. The starting action is

$$\begin{aligned}
S = \int d^8z \left\{ & H^{\alpha\dot{\alpha}} D^\gamma \bar{D}^2 D_\gamma H_{\alpha\dot{\alpha}} & + a_1 m H^{\alpha\dot{\alpha}} (\bar{D}_{\dot{\alpha}} \chi_\alpha - D_\alpha \bar{\chi}_{\dot{\alpha}}) \\
& - 2 H^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^2 \chi_\alpha + c.c. & + a_2 m H^{\alpha\dot{\alpha}} (D^2 H_{\alpha\dot{\alpha}} + \bar{D}^2 H_{\alpha\dot{\alpha}}) \\
& - 2 \chi^\alpha D^2 \chi_\alpha + c.c. & + a_3 m \chi^\alpha \chi_\alpha + c.c. \\
& + 2 \chi^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} & + a_4 m^2 H^{\alpha\dot{\alpha}} H_{\alpha\dot{\alpha}} \left. \right\} \quad (4.28)
\end{aligned}$$

impose a constraint on χ_α when it vanishes. That constraint will be used to simplify the above expression for I_α and set χ_α to zero. But a more careful examination of I_α will convince us that there is no unique constraint on χ_α that will make all terms (except the last one) vanish. The inescapable conclusion is that we have to treat $\chi_\alpha = 0$ as the desired constraint. This suggests that we must introduce a spinorial superfield u_α that couples with χ_α through only a mass term $\sim mu^\alpha\chi_\alpha$. Hence when $u_\alpha = 0$ then immediately we will get $\chi_\alpha = 0$.

We must update the action with the introduction of a few new terms: the interaction term $mu^\alpha\chi_\alpha$ and the kinetic energy terms for u_α (the most general quadratic action). The new action is

$$\begin{aligned}
S = \int d^8z \left\{ & H^{\alpha\dot{\alpha}}D^\gamma\bar{D}^2D_\gamma H_{\alpha\dot{\alpha}} & +\gamma mu^\alpha\chi_\alpha & (4.33) \\
& -2 H^{\alpha\dot{\alpha}}\bar{D}_{\dot{\alpha}}D^2\chi_\alpha + c.c. & +a_2mH^{\alpha\dot{\alpha}}D^2H_{\alpha\dot{\alpha}} + c.c. & +b_1u^\alpha D^2u_\alpha + c.c. \\
& -2 \chi^\alpha D^2\chi_\alpha + c.c. & +a_3m\chi^\alpha\chi_\alpha + c.c. & +b_2u^\alpha\bar{D}^2u_\alpha + c.c. \\
& +2 \chi^\alpha D_\alpha\bar{D}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} & +a_4m^2H^{\alpha\dot{\alpha}}H_{\alpha\dot{\alpha}} & +b_3u^\alpha\bar{D}^{\dot{\alpha}}D_\alpha\bar{u}_{\dot{\alpha}} \\
& & & +b_4u^\alpha D_\alpha\bar{D}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}} \\
& & & +b_5mu^\alpha u_\alpha \left. \right\}
\end{aligned}$$

and the updated equations of motion are

$$\begin{aligned}
\mathcal{E}_{\alpha\dot{\alpha}}^{(H)} = & 2D^\gamma\bar{D}^2D_\gamma H_{\alpha\dot{\alpha}} + 2(D_\alpha\bar{D}^2\bar{\chi}_{\dot{\alpha}} - \bar{D}_{\dot{\alpha}}D^2\chi_\alpha) + 2a_2m(D^2H_{\alpha\dot{\alpha}} + \bar{D}^2H_{\alpha\dot{\alpha}}) \\
& +2a_4m^2H_{\alpha\dot{\alpha}} & (4.34)
\end{aligned}$$

$$\mathcal{E}_\alpha^{(\chi)} = -4D^2\chi_\alpha + 2D_\alpha\bar{D}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} - 2D^2\bar{D}^{\dot{\alpha}}H_{\alpha\dot{\alpha}} + 2a_3m\chi_\alpha + \gamma mu_\alpha$$

$$\mathcal{E}_\alpha^{(u)} = 2b_1D^2u_\alpha + 2b_2\bar{D}^2u_\alpha + b_3\bar{D}^{\dot{\alpha}}D_\alpha\bar{u}_{\dot{\alpha}} + b_4D_\alpha\bar{D}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}} + 2b_5mu_\alpha + \gamma m\chi_\alpha$$

Now we repeat the process of eliminating $H_{\alpha\dot{\alpha}}$, but because u_α doesn't couple with $H_{\alpha\dot{\alpha}}$ nothing will be changed regarding the $H_{\alpha\dot{\alpha}}$ depending terms. The same choice of coefficients as in (Σ_1) must be made to remove $H_{\alpha\dot{\alpha}}$. So the updated expression for I_α is

$$\begin{aligned}
I_\alpha = & 2Ba_3mD^2\bar{D}^2\chi_\alpha - 4m^2D^2\chi_\alpha & (4.35) \\
& + B\gamma mD^2\bar{D}^2u_\alpha + 2m^2D_\alpha\bar{D}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \\
& + \gamma m^3u_\alpha + 2a_3m^3\chi_\alpha
\end{aligned}$$

Now we want to use the equation of motion of u_α to remove any dependences on χ_α in order to get an equation of u_α . For that we calculate

$$\begin{aligned}
J_\alpha = & I_\alpha + mKD^2\mathcal{E}_\alpha^{(u)} + m\Lambda D_\alpha\bar{D}^{\dot{\alpha}}\bar{\mathcal{E}}_{\dot{\alpha}}^{(u)} \\
= & [2Ba_3]D^2\bar{D}^2\chi_\alpha + [B\gamma + 2Kb_2 + \Lambda b_3]mD^2\bar{D}^{\dot{\alpha}}u_\alpha \\
& + [-4 + K\gamma]m^2D^2\chi_\alpha + [Kb_3 + 2\Lambda b_2]mD^2\bar{D}^{\dot{\alpha}}D_\alpha\bar{u}_{\dot{\alpha}} & (4.36) \\
& + [2 + \Lambda\gamma]m^2D_\alpha\bar{D}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} + [\Lambda(2b_4 - b_3)]D_\alpha\bar{D}^2D^\beta u_\beta \\
& + [2a_3]m^3\chi_\alpha + \gamma m^3u_\alpha \\
& + [Kb_5]m^2D^2u_\alpha + [\Lambda b_5]m^2D_\alpha\bar{D}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}}
\end{aligned}$$

If we choose

$$(\Sigma_2) : \quad a_3 = 0 \quad , \quad -4 + K\gamma = 0 \quad , \quad 2 + \Lambda\gamma = 0$$

we get an equation for u_α

$$\begin{aligned}
J_\alpha = & [B\gamma + 2Kb_2 + \Lambda b_3]mD^2\bar{D}^\alpha u_\alpha + [Kb_5]m^2D^2u_\alpha \\
& + [Kb_3 + 2\Lambda b_2]mD^2\bar{D}^\alpha D_\alpha \bar{u}_{\dot{\alpha}} + [\Lambda b_5]m^2D_\alpha \bar{D}^\alpha \bar{u}_{\dot{\alpha}} \\
& + [\Lambda(2b_4 - b_3)]D_\alpha \bar{D}^2 D^\beta u_\beta \\
& + \gamma m^3 u_\alpha
\end{aligned} \tag{4.37}$$

Now we are in position to choose coefficients so as to make u_α vanish on-shell. Select

$$(\Sigma_3) : B\gamma + 2Kb_2 + \Lambda b_3 = 0, \quad Kb_3 + 2\Lambda b_2 = 0, \quad 2b_4 - b_3 = 0, \quad b_5 = 0, \quad \gamma \neq 0$$

Since $u_\alpha = 0$ on-shell, now we can reverse the arguments. Its equation of motion will give $\chi_\alpha = 0$ and that will put constraints on $H_{\alpha\dot{\alpha}}$: $D^2\bar{D}^\alpha H_{\alpha\dot{\alpha}} = 0$

$$\begin{aligned}
\mathcal{E}_{\alpha\dot{\alpha}}^{(H)} = & 2D^\gamma \bar{D}^2 D_\gamma H_{\alpha\dot{\alpha}} + 2a_2 m (D^2 H_{\alpha\dot{\alpha}} + \bar{D}^2 H_{\alpha\dot{\alpha}}) + 2a_4 m^2 H_{\alpha\dot{\alpha}} \\
\mathcal{E}_\alpha^{(\chi)} = & -2D^2 \bar{D}^\alpha H_{\alpha\dot{\alpha}}
\end{aligned} \tag{4.38}$$

Finally because of $D^2\bar{D}^\alpha H_{\alpha\dot{\alpha}} = 0$ we get that

$$D^\alpha \mathcal{E}_{\alpha\dot{\alpha}}^{(H)} = 2a_2 m D^\alpha \bar{D}^2 H_{\alpha\dot{\alpha}} + 2a_4 m^2 D^\alpha H_{\alpha\dot{\alpha}}. \text{ For } a_2 = 0, \ a_4 \neq 0 \text{ this gives } D^\alpha H_{\alpha\dot{\alpha}} = 0.$$

Thus the equation of motion for $H_{\alpha\dot{\alpha}}$ becomes the Klein-Gordon equation with

$$2a_4 = 2$$

$$\square H_{\alpha\dot{\alpha}} = m^2 H_{\alpha\dot{\alpha}} \tag{4.39}$$

To complete the analysis we look for the consistency and non-trivial solution of the

systems of equations $\Sigma_1, \Sigma_2, \Sigma_3, a_2 = 0, 2a_4 = 2$. A solution exists and it is

$$\begin{aligned}
a_1 = 0 \quad , \quad b_1 = \text{free, can be set to zero} \quad , \quad \gamma = 1 \quad , \quad \Lambda = -2 \\
a_2 = 0 \quad , \quad b_2 = \frac{1}{6} \quad , \quad A = 1 \\
a_3 = 0 \quad , \quad b_3 = \frac{1}{6} \quad , \quad B = -1 \\
a_4 = 1 \quad , \quad b_4 = \frac{1}{12} \quad , \quad K = 4 \\
b_5 = 0
\end{aligned}$$

The final action takes the form

$$\begin{aligned}
S = \int d^8z \left\{ H^{\alpha\dot{\alpha}} D^\gamma \bar{D}^2 D_\gamma H_{\alpha\dot{\alpha}} \quad + mu^\alpha \chi_\alpha \right. \\
\quad -2 H^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^2 \chi_\alpha + c.c. \quad + \frac{1}{6} u^\alpha \bar{D}^2 u_\alpha + c.c. \\
\quad -2 \chi^\alpha D^2 \chi_\alpha + c.c. \quad + \frac{1}{6} u^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{u}_{\dot{\alpha}} \\
\quad +2 \chi^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} \quad + \frac{1}{12} u^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{u}_{\dot{\alpha}} \\
\quad \left. + m^2 H^{\alpha\dot{\alpha}} H_{\alpha\dot{\alpha}} \right\} \tag{4.40}
\end{aligned}$$

4.7 Conclusion and future directions

The pattern for the massive representations looks like this

$$Y = 0: \quad \Phi$$

$$Y = \frac{1}{2}: \quad H$$

$$Y = 1: \quad \Psi_\alpha - V$$

$$Y = \frac{3}{2}: \quad H_{\alpha\dot{\alpha}} - \chi_\alpha \sim u_\alpha$$

where straight lines represent interactions and wave lines represent interactions proportional to m (these lines will break in the massless limit). We would like to understand how to complete this picture for the higher superspin representations. We are expecting an increasing number of auxiliary superfields but what type of superfields and how many of them, is still unknown. We have some preliminary results for the next case of $Y = 2$ but it is a work in progress.

Another direction for future investigations is to study the component structure of these massive representations. The techniques developed for the massless case can be applied in the massive case and give us the off-shell component structure of the theory, along with the component action and the supersymmetric transformation laws. Furthermore the knowledge of the off-shell degrees of freedom of each theory will make it much easier to identify connections among the various theories. That will be a tremendous help for the construction of higher \mathcal{N} and higher D theories.

One of the motivations that started the entire investigation was the higher spin states of superstring theory. These states they live in some Fock Space and are created by the action of a series of creation operators to a vacuum state. That motivates as to study higher superspin constructions in a Fock space language. An effective description will be to consider the superfields, used in all the above theories, as coefficients in the expansion of a state in some Fock space in the basis of the creation operators. For example if we consider the following commuting set of creation and annihilation operators $c^\alpha, a_\beta | \bar{c}^\alpha, \bar{a}_\beta$

$$[a^\alpha, c_\beta] = \delta^\alpha_\beta, [\bar{a}^{\dot{\alpha}}, \bar{c}_{\dot{\beta}}] = \delta^{\dot{\alpha}}_{\dot{\beta}}$$

and define for each one of these sets a vacuum state $|0\rangle$ and $|\bar{0}\rangle$ in the following way

$$a^\alpha|0\rangle = 0, \bar{a}^{\dot{\alpha}}|\bar{0}\rangle = 0$$

then in this vector space there are states like

$$|\Psi\rangle = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k!(k-1)!}} \Psi^{\alpha(k)\dot{\alpha}(k-1)} c_{\alpha(k)} \bar{c}_{\dot{\alpha}(k-1)} |\Omega\rangle$$

$$|H\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} H^{\alpha(k)\dot{\alpha}(k)} c_{\alpha(k)} \bar{c}_{\dot{\alpha}(k)} |\Omega\rangle$$

We can then attempt to view all the above results from the prism of Fock space and bring us a bit closer to understanding some of the complexities of superstring theory.

Chapter A: Massless representations of the Poincaré group

For the component discussion of the massless irreducible representations of the Super-Poincaré group we must have expressions for the actions that describe the integer and half-integer helicities. This chapter will provide these expressions so we can reference them when needed. These actions are well known in the vector index notation. However we will be using the left-right index notation which is the natural language for a $4D$, $\mathcal{N} = 1$ theory. Therefore a translation of these theories to the left-right index notation is in order.

A.1 Integer helicity, $\lambda = s$

The integer helicity theory is being described in [3, 5]. The theory is formulated in terms of two fields, a rank s symmetric traceless tensor $h_{\mu(s)}$ and a rank $s - 2$ symmetric traceless tensor $h_{\mu(s-2)}$. Also the theory is invariant under a gauge transformation transformations $\delta_G h_{\mu(s)} \sim \partial_{(\mu_s} \zeta_{\mu(s-1))}$, $\delta_G h_{\mu(s-2)} \sim \partial^{\mu_{s-1}} \zeta_{\mu(s-1)}$. In the left-right index notation these fields will be replaced by a field of type $(s, s) : h_{\alpha(s)\dot{\alpha}(s)}$ and a field of type $(s - 2, s - 2) : h_{\alpha(s-2)\dot{\alpha}(s-2)}$ and they are both independently symmetrized in both the undotted and dotted indices. The theory will be invariant under the change $\delta_G h_{\alpha(s)\dot{\alpha}(s)} \sim \partial_{(\alpha_s(\dot{\alpha}_s \zeta_{\alpha(s-1)\dot{\alpha}(s-1)})}$, $\delta_G h_{\alpha(s-2)\dot{\alpha}(s-2)} \sim$

$\partial^{\alpha_{s-1}\dot{\alpha}_{s-1}}\zeta_{\alpha(s-1)\dot{\alpha}(s-1)}$. The exact action is

$$\begin{aligned}
S = \int d^4x \left\{ & h^{\alpha(s)\dot{\alpha}(s)} \square h_{\alpha(s)\dot{\alpha}(s)} \right. \\
& - \frac{s}{2} h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \\
& + s(s-1) h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
& - s(2s-1) h^{\alpha(s-2)\dot{\alpha}(s-2)} \square h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
& \left. - \left[\frac{s(s-2)^2}{2} \right] h^{\alpha(s-2)\dot{\alpha}(s-2)} \partial_{\alpha_{s-2}\dot{\alpha}_{s-2}} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-3)\dot{\gamma}\dot{\alpha}(s-3)} \right\} \tag{A.1}
\end{aligned}$$

and it is invariant under

$$\begin{aligned}
\delta_G h_{\alpha(s)\dot{\alpha}(s)} &= \frac{1}{s!s!} \partial_{(\alpha_s(\dot{\alpha}_s \zeta_{\alpha(s-1)\dot{\alpha}(s-1)}) \\
\delta_G h_{\alpha(s-2)\dot{\alpha}(s-2)} &= \frac{s-1}{s^2} \partial^{\alpha_{s-1}\dot{\alpha}_{s-1}} \zeta_{\alpha(s-1)\dot{\alpha}(s-1)}
\end{aligned}$$

A.2 Half-integer helicity, $\lambda = s + \frac{1}{2}$

The theory for half-integer helicity [4, 6] is formulated in the vector index notation in terms of three fields $\psi_{\mu(s)}$, $\psi_{\mu(s-1)}$, $\psi_{\mu(s-2)}$. All of them are spinors and they are symmetric, traceless and γ -traceless. They also have appropriate gauge transformations. In the left-right index notation they will be replaced by three fields $\psi_{\alpha(s+1)\dot{\alpha}(s)}$, $\psi_{\alpha(s)\dot{\alpha}(s-1)}$, $\psi_{\alpha(s-1)\dot{\alpha}(s-2)}$. They are all symmetric independently in

both left and right indices. The exact action for half-integer helicity is

$$\begin{aligned}
S = \int d^4x \left\{ & +i \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1} \dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \right. \\
& +i \left[\frac{s}{s+1} \right] \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -i \left[\frac{2s+1}{(s+1)^2} \right] \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s \dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& +i \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
& \left. -i \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1} \dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \right\} \tag{A.2}
\end{aligned}$$

and it is invariant under

$$\begin{aligned}
\delta_G \psi_{\alpha(s+1)\dot{\alpha}(s)} &= \frac{1}{s!(s+1)!} \partial_{(\alpha_{s+1}(\dot{\alpha}_s \xi_{\alpha(s)\dot{\alpha}(s-1)})} \\
\delta_G \psi_{\alpha(s)\dot{\alpha}(s-1)} &= -\frac{1}{s!} \partial_{(\alpha_s \dot{\alpha}_s \bar{\xi}_{\alpha(s-1)\dot{\alpha}(s)}} \\
\delta_G \psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \frac{s-1}{s} \partial^{\alpha_s \dot{\alpha}_{s-1}} \xi_{\alpha(s)\dot{\alpha}(s-1)}
\end{aligned}$$

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