ABSTRACT

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The Operational Calculus is a construction used for analyzing the behavior of linear operators that arise in the study of ordinary and partial differential equations. Given a linear operator T and a class of functions $\mathcal{F}$, one rigorously defines a new operator $f(T)$ for each f in $\mathcal{F}$ and establishes properties of the transformation $f \mapsto f(T)$, among which is that, if $\mathcal{F}$ is an algebra of functions, then the transformation induces an algebra homomorphism from $\mathcal{F}$ to the algebra of bounded linear operators on a Banach space. This paper begins with a discussion of an operational calculus for compact symmetric operators. This motivates the construction of the Dunford operational calculus for general bounded linear operators. Next, a treatment for bounded symmetric operators is provided, together with a rigorous presentation of all background material. All this is the basis of an operational calculus for unbounded symmetric operators T on a complex Hilbert space. This latter construction is based on a representation theorem of Riesz and Lorch for unbounded self-adjoint operators: the presentation is simpler and more illuminating than the customary one.

Operational Calculus

by

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Introduction

Throughout, H is a complex Hilbert space equipped with an Hermitian inner-product $\langle \cdot, \cdot \rangle$, and $\mathcal{L}(H)$ denotes the space of bounded linear operators on H, equipped with the operator norm. For an entire function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and an operator T in $\mathcal{L}(H)$, since $\mathcal{L}(H)$ is complete with respect to the operator norm, the operator $f(T) \in \mathcal{L}(H)$ is properly defined by

$$f(T) = \sum_{k=0}^{\infty} a_k T^k. \tag{1}$$

The transformation $f \mapsto f(T)$, which we call an operational calculus, induces an algebra homeomorphism from the algebra of entire functions to the algebra $\mathcal{L}(H)$. If the coefficients a_k are real and the operator T is symmetric, then the operator $f(T)$ also is symmetric.

It was David Hilbert who first extended the operational calculus. In the context of his study of integral equations, he considered compact symmetric operators on a Hilbert space. He and his student Erhart Schmidt proved the Hilbert-Schmidt Theorem, an expression of a compact symmetric operator $T : H \to H$ as an eigenvalue expansion: there is an orthonormal basis for H such that, for all h in H,

$$Th = \sum_{k=1}^{\infty} \lambda_k \langle h, e_k \rangle e_k.$$

An operational calculus follows immediately: for f a bounded real-valued function,

$$f(T)h = \sum_{k=1}^{\infty} f(\lambda_k) \langle h, e_k \rangle e_k.$$

We devote Chapter 1 to the Hilbert-Schmidt theory.

The Hilbert-Schmidt Theory

The Dunford Theory   Bounded Symmetric Operators

Unbounded Self-Adjoint Operators

There are two branches that grow from the Hilbert-Schmidt theory. The first is the operational calculus for general bounded operators. This was created by Nelson Dunford (see [1]), where he defines $f(T)$ for a general bounded linear operator $T : X \to X$ on a complex Banach space X, and an analytic function f defined on a neighborhood of the spectrum of T. Among other things, he proved that this established an algebra homeomorphism from the algebra of analytic functions to the algebra of bounded operators on X. We devote Chapter 2 to this Dunford calculus.

The second branch that extends from the Hilbert-Schmidt theory is the operational calculus for symmetric operators on a complex Hilbert space. This itself has two branches; one for bounded symmetric operators and the other for unbounded self-adjoint operators. Bounded symmetric operators play an important part in the study of integral equations. In Chapter 3 we develop the operational calculus for bounded symmetric operators, based on a thorough exposition and proof of all of the background material, including the three fundamental theorems of spectral theory: the spectral mapping theorem, the spectral radius theorem and the spectral boundary theorem.

In the final chapter, we consider unbounded self-adjoint operators $T\colon D \subseteq H \to H$. John von Neumann based his mathematical presentation of quantum mechanics on such operators, and indeed such operators play a fundamental role in the study of boundary-value problems for partial differential equations (see [3], [4]). We first consider a special construction of Lax [3] for $f(T)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and has finite limits at infinity. We then develop a general operational calculus for these operators based on the Riesz-Lorch theorems [5] for the representation of unbounded self-adjoint operators as appropriate limits of bounded symmetric operators. The proofs here are much shorter than the customary proofs, all details are presented, and the development makes very transparent the manner in which properties of the operational calculus for bounded symmetric operators are inherited by unbounded self-adjoint operators. The novelty of the approach lies in that we pass directly from the calculus for bounded operators to the calculus for unbounded operators, thereby eliminating many technical details that arise when this passage is made by first creating a spectral resolution for unbounded operators.

Our treatment of the Hilbert-Schmidt theory is adapted from Lax [3] and Taylor [8]. The presentation of the background material on spectral theory and the Dunford calculus is adapted from Taylor [8] and Dunford-Schwarz [1]. Out treatment of the operational calculus for bounded symmetric operators has its roots in Reed-Simon [4], Riesz-Nagy [6] and Fitzpatrick [2]. The final extension of the operational calculus for unbounded symmetric operators is an evolution of the presentation in Fitzpatrick [2].

1 The Hilbert-Schmidt Theory

In this section, we will examine an operational calculus for compact symmetric operators that supports a very general class of functions. The construction relies on an extension to linear symmetric operators on an infinite dimensional space of the Principal Axis Theorem of matrix theory.

Definition. *A linear operator $T : D \subseteq H \to H$ is said to be* symmetric *if for every $x, y \in D$ we have $\langle T(x), y\rangle = \langle x, T(y)\rangle$.*

If $T : H \to H$ is a bounded symmetric operator and h belongs to H, by definition $\langle Th, h\rangle = \langle h, Th\rangle$. Since $\langle\cdot,\cdot\rangle$ is a Hermitian inner product, $\langle h, Th\rangle$ is the complex conjugate of $\langle Th, h\rangle$. Therefore $\langle Th, h\rangle$ is real.

The resolvent of T, denoted by $\rho(T)$, comprises the complex numbers λ for which $\lambda - T$ is invertible, and the spectrum of T, denoted by $\sigma(T)$, is the complement in $\mathbb{C}$ of the resolvent. Define the spectral bounds for T by

$$m(T) = \inf_{h\neq 0} \frac{\langle Th, h\rangle}{\langle h, h\rangle} \quad \text{and} \quad M(T) = \sup_{h\neq 0} \frac{\langle Th, h\rangle}{\langle h, h\rangle}.$$

A bounded symmetric operator is said to be positive definite provided $m(T) > 0$, and said to be nonnegative provided $m(T) \geq 0$. The sum of nonnegative symmetric operators is nonnegative. Moreover, for any $S \in \mathcal{L}(H)$, the operator S^*TS is symmetric and nonnegative since for each $h \in H$ we have $\langle S^*TS(h), h\rangle = \langle T(S(h)), S(h)\rangle \geq 0$.

Theorem 1.1. *If $T : H \to H$ is a bounded symmetric operator then*

$$\|T\| = \sup\{|\langle Th, h\rangle| : h \in H \textit{ and } \|h\| = 1\}.$$

Proof. For notational convenience, let $M = \sup\{|\langle Th, h\rangle| : \|h\| = 1\}$. Observe that for any $x, y \in H$ we have

$$\frac{1}{4}\Big(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\Big) = \langle T(x), y\rangle. \tag{2}$$

If $M = 0$ we would have $\langle T(x+y), x+y\rangle = \langle T(x-y), x-y\rangle = 0$ and then (2) would force $T = 0$, so it suffices to consider the case when $M > 0$.

Let h be a member of H with $\|h\| = 1$. The Cauchy-Schwartz inequality tells us that $|\langle T(h), h\rangle| \leq \|T(h)\| \cdot \|h\| \leq \|T\|$, which implies

$$M \leq \|T\|. \tag{3}$$

On the other hand, the symmetric operators $MI + T$ and $MI - T$ are both nonnegative since

$$\langle (MI + T)h, h\rangle = M\|h\|^2 + \langle Th, h\rangle \geq 0$$

and

$$\langle (MI - T)h, h\rangle = M\|h\|^2 - \langle Th, h\rangle \geq 0$$

follow immediately from the definition of M as a supremum. Then both $(MI - T)(MI + T)(MI - T)$ and $(MI + T)(MI - T)(MI + T)$ are nonnegative and their sum $2M(M^2I - T^2)$ is also nonnegative and so we have

$$0 \leq \langle (M^2I - T^2)h, h\rangle = M^2\|h\|^2 - \langle T^2(h), h\rangle$$

so that $\|T(h)\|^2 = \langle T(h), T(h)\rangle = \langle T^2(h), h\rangle \leq M^2\|h\|^2$. Hence,

$$\|T\| \leq M. \tag{4}$$

Together (3) and (4) establish our result.

□

Definition. *A bounded linear operator $T : H \to H$ is* compact *if the image of the closed unit ball under T has compact closure.*

The Hilbert-Schmidt Theorem, which we will use to construct the operational calculus, follows quickly from the following lemma.

Lemma 1.2. *Let $T : H \to H$ be a non-zero compact symmetric operator. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue of T, and there is a corresponding eigenvector h such that $\|h\| = 1$ and $|\langle Th, h\rangle| = \|T\|$.*

Proof. For notational convenience, we let $\lambda = \|T\|$. Theorem 1.1 tells us that there exists a sequence of unit vectors $\{h_n\}$ in H so that the sequence of real numbers $\{\langle Th_n, h_n\rangle\}$ converges to λ. We see from

$$0 \le \|Th_n - \lambda h_n\|^2 = \|Th_n\|^2 - 2\lambda\langle Th_n, h_n\rangle + \lambda^2\|h_n\|^2 \le \|T\|^2 - 2\lambda\langle Th_n, h_n\rangle + \lambda^2$$

that $Th_n - \lambda h_n \to 0$. Since T is compact, $\{Th_n\}$ contains a convergent subsequence which we denote by $\{Th_{n_k}\}$. Then, since $\lambda \neq 0$ by assumption, the sequence h_{n_k} also converges, and we denote the limit by h. Thus we have $\|h\| = 1$ and $Th = \lambda h$. Furthermore, we observe that

$$|\langle Th, h\rangle| = |\lambda| \cdot \|h\|^2 = \|T\|.$$

□

If a symmetric operator maps a subspace into itself, it maps the orthogonal complement into the orthogonal complement. The preceding lemma therefore leads to a proof of the following theorem. The assertion regarding the accumulation points of the set of eigenvalues follows from compactness.

Theorem 1.3 (Hilbert-Schmidt Theorem)**.** *Let $T : H \to H$ be a compact symmetric operator. Then there is an orthonormal base for H consisting of the eigenvectors $\{e_n \in H : Te_n = \lambda_n e_n\}$ of T. The corresponding eigenvalues are real and their only point of accumulation is zero.*

With the preceding theorem in hand, the construction of the operational calculus is almost immediate.

Definition. *Let $T : H \to H$ be a compact symmetric operator and $\{e_k\}_{k=1}^{\infty}$ an orthonormal basis of H for which*

$$Th = \sum_{k=1}^{\infty} \lambda_k \langle h, e_k\rangle e_k \text{ for all } h \in H.$$

For f a bounded real-valued function on the spectrum of T, define

$$f(T)h = \sum_{k=1}^{\infty} f(\lambda_k)\langle h, e_k\rangle e_k \text{ for all } h \in H. \tag{5}$$

The above operator is properly defined, bounded, and symmetric since f is real-valued. Furthermore, since $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis, the transformation $f \mapsto f(T)$ is an isometric isomorphism from the ring of bounded functions on $\sigma(T)$ into the algebra of bounded maps of H to itself.

Proposition 1.4. *If p is a polynomial, definitions (5) and (1) are the same.*

Proof. Let $T : H \to H$ be a compact symmetric operator and $p(x) = a_n x^n + ... + a_1 x + a_0$ be a polynomial. The Hilbert-Schmidt theorem tells us that there exists an orthonormal base of H consisting of eigenvectors $\{e_n \in H : Te_n = \lambda_n e_n\}$ of T. Let $h = \sum_{k=1}^{\infty} c_k e_k$ belong to H and observe that

$$\begin{aligned}
p(T)h &= \sum_{k=1}^{\infty} p(\lambda_k) c_k e_k \\
&= \sum_{k=1}^{\infty} (a_n \lambda_k^n + ... a_1 \lambda_k + a_0)\, c_k e_k \\
&= a_n \sum_{k=1}^{\infty} c_k \lambda_k^n e_k + ... + \sum_{k=1}^{\infty} c_k \lambda_k e_k + a_0 \sum_{k=1}^{\infty} c_k e_k \\
&= a_n \sum_{k=1}^{\infty} c_k T^n(e_k) + ... + a_1 \sum_{k=1}^{\infty} c_k T(e_k) + a_0 \sum_{k=1}^{\infty} c_k e_k \\
&= a_n T^n \left(\sum_{k=1}^{\infty} c_k e_k \right) + ... + a_1 T \left(\sum_{k=1}^{\infty} c_k e_k \right) + a_0 \sum_{k=1}^{\infty} c_k e_k \\
&= a_n T^n(h) + ... + a_1 T(h) + a_0 h.
\end{aligned}$$

□

Theorem 1.5. *Let $T : H \to H$ be a non-zero compact symmetric operator with $\{e_n\}$ an orthonormal base of the eigenvectors $\{e_n \in H : Te_n = \lambda_n e_n\}$ of T. If $\lambda \neq 0$ and $\lambda \neq \lambda_k$ for each k, then $\lambda I - T$ has a continuous inverse defined on all of H given by*

$$(\lambda I - T)^{-1} x = \frac{1}{\lambda} x + \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda_k \frac{\langle x, e_k\rangle}{\lambda - \lambda_k} e_k \tag{6}$$

Proof. First, we will show that the sum in (6) is convergent. Let $\{s_n\}$ be the sequence of partial sums

$$s_n = \sum_{k=1}^{n} \lambda_k \frac{\langle x, e_k \rangle}{\lambda - \lambda_k} e_k.$$

If $m < n$ then we have

$$\|s_n - s_m\|^2 = \sum_{k=m+1}^{n} \left| \frac{\lambda_k}{\lambda - \lambda_k} \right|^2 \cdot |\langle x, e_k \rangle|^2 \leq \alpha^2 \sum_{k=m+1}^{n} |\langle x, e_k \rangle|^2 \tag{7}$$

where α is the supremum over all natural numbers k of the quantity $\lambda_k/(\lambda - \lambda_k)$. Since $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$ is convergent, we deduce from (7) that $\{s_n\}$ is a Cauchy sequence in H and is therefore convergent. Now, suppose that

$$y = \frac{1}{\lambda} x + \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda_k \frac{\langle x, e_k \rangle}{\lambda - \lambda_k} e_k,$$

and observe that

$$\begin{aligned}
(\lambda I - T)y &= x + \sum_{k=1}^{\infty} \lambda_k \frac{\langle x, e_k \rangle}{\lambda - \lambda_k} e_k - \frac{1}{\lambda} Tx - \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda_k \frac{\langle x, e_k \rangle}{\lambda - \lambda_k} Te_k \\
&= x - \frac{1}{\lambda} Tx + \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k \\
&= x.
\end{aligned}$$

Furthermore, we deduce that

$$\|y\| \leq \frac{1}{|\lambda|} \|x\| + \frac{1}{|\lambda|} \alpha \|x\|.$$

We conclude that $(\lambda I - T)^{-1}$ is continuous and defined on all of H by (6), as required.

□

We deduce from the Cauchy Integral Theorem, together with the expression (6) that

$$T = \frac{1}{2\pi i} \int_{\Gamma} \lambda (\lambda I - T)^{-1} d\lambda$$

where Γ is a closed, rectifiable, positively oriented path in the complex plane surrounding the spectrum of T. This identity motivates the forthcoming Dunford operational calculus.

2 The Dunford Theory

Our goal in this section is to use the above integral representation to present Dunford's construction of an operational calculus for general bounded linear operators $T\colon X \to X$, where X is a general complex Banach space.

2.1 Spectral Theory: General results

Given a bounded linear operator T, it will be helpful to know when the operator $I - T$ is invertible. The following useful proposition tells us that whenever T is a contraction, $I - T$ is invertible.

Proposition 2.1. *If $T : H \to H$ is a bounded linear operator such that $\|T\| < 1$ then the operator $I - T$ is invertible with inverse given by*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n. \qquad \text{(Neumann Series Expansion)}$$

Proof. Since H is complete, the space of bounded linear operators from H, $\mathcal{L}(X)$, equipped with the operator norm, is complete. Since $\|T\| < 1$, the sequence of partial sums for the series on the right hand side is Cauchy in the space of bounded linear operators, and therefore convergent. Letting $S = \sum_{n=0}^{\infty} T^n$, we have

$$TS = T\sum_{n=0}^{\infty} T^n = \sum_{n=1}^{\infty} T^n = S - I = \left(\sum_{n=0}^{\infty} T^n\right) T = ST$$

so that $(I - T)S = I$ and $S(I - T) = I$, establishing that S is the inverse of $I - T$.

□

Lemma 2.2. *Let T be a bounded linear operator and $\mu \in \mathbb{C}$ belong to the resolvent set of T. Then $\lambda I - T$ is invertible for all $\lambda \in \mathbb{C}$ that satisfy*

$$|\lambda - \mu| \cdot \|(\mu I - T)^{-1}\| < 1.$$

Proof. We have $\mu I - T$ invertible by assumption, and since we are given that $|\lambda - \mu| \cdot \|(\mu I - T)^{-1}\| \leq 1$ we know from proposition 2.1 that $I - (\lambda - \mu)(\mu I - T)^{-1}$ is invertible as well. It follows that the product

$$(\mu I - T)(I - (\mu - \lambda)(\mu I - T)^{-1}) = \mu I - T - \mu I + \lambda I = \lambda I - T$$

is also invertible.

□

Theorem 2.3. *The spectrum of a bounded linear operator is a closed, bounded subset of $\mathbb{C}$.*

Proof. As a consequence of the preceding lemma, it is clear that the resolvent set of a bounded linear operator is open. Since the spectrum is the complement of the resolvent set, we deduce that the spectrum of a bounded linear operator is closed.

To show that the spectrum is bounded, suppose $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$. Then $\|\lambda^{-1}T\| \leq |\lambda^{-1}| \cdot \|T\| < 1$ so that $I - \lambda^{-1}T$ is invertible which implies that $\lambda(I - \lambda^{-1}T) = \lambda - T$ is invertible and so $\lambda \in \rho(T)$, by definition. It follows that any λ that lies in the spectrum of T must satisfy $|\lambda| \leq \|T\|$, and so $\sigma(T)$ is bounded.

Definition. *Given a bounded linear operator $T : H \to H$, the operator-valued function $R_\lambda = (\lambda I - T)^{-1}$ defined on $\rho(T)$ is called the* resolvent operator *of T.*

By an analytic function of a complex variable whose values lie in a Banach space over $\mathbb{C}$ is meant a function that is locally expressed as a power series. Since $\mathcal{L}(H)$ is a Banach space over $\mathbb{C}$, we may speak of analytic functions whose values lie in $\mathcal{L}(H)$. The product of such analytic functions is analytic, and the standard theory of analytic functions, e.g. the Cauchy integral formula, power series, Laurent series, etc., are meaningful and valid for functions that take their values in $\mathcal{L}(H)$.

Proposition 2.4. *The resolvent operator R_λ of a bounded linear operator T is an analytic function of λ on $\rho(T)$.*

Proof. Let λ_0 belong to the resolvent set of T and let λ be a complex scalar such that
$$|\lambda_0 - \lambda| < \frac{1}{\|(\lambda_0 I - T)^{-1}\|}.$$
Then we know from lemma 2.2 that $\lambda \in \rho(T)$ and we may apply proposition 2.1 to the operator $(\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}$ for

$$(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1})^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - T)^{-n}. \tag{8}$$

Observe that

$$\lambda I - T = (\lambda_0 I - T) - (\lambda_0 I - \lambda I) = (\lambda_0 I - T)(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}).$$

Since we have already established that $\lambda I - T$ is invertible, we may rearrange the terms so that

$$(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1})^{-1} = (\lambda_0 I - T)(\lambda I - T)^{-1}$$

which can be substituted into the left hand side of (8) yielding

$$R_\lambda = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}^{n+1} \text{ for } |\lambda_0 - \lambda| < r$$

where $r = 1/\|(\lambda_0 I - T)^{-1}\|$. This shows that the resolvent operator R_λ of $T : H \to H$ is analytic when viewed as a function of a complex variable λ whose values lie in the Banach space H.

$\square$

Theorem 2.5. *The spectrum of a bounded linear operator on a complex Banach space is a non-empty subset of the complex plane.*

Proof. Recall that when $|\lambda| > \|T\|$ lemma 2.2 gives an explicit representation for the inverse of $\lambda I - T$ as

$$(\lambda I - T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^{-k} T^k.$$

Substituting R_λ for $(\lambda I - T)^{-1}$, applying the norm to both sides, and summing the geometric series gives

$$\|R_\lambda\| \leq \frac{1}{|\lambda|} \sum_{k=0}^{\infty} \left(\frac{\|T\|}{|\lambda|} \right)^k = \frac{1}{|\lambda| - \|T\|} \tag{9}$$

which implies that $\|R_\lambda\| \to 0$ as $|\lambda| \to \infty$. In light of proposition 2.4, for any $x, y \in H$ the function $f(\lambda) = \langle R_\lambda x, y \rangle$ is analytic. If $\sigma(T)$ were empty then the resolvent set would be all of H and f would be a bounded entire function, so that Liouville's theorem would force $f = 0$ and we would have $R_\lambda = 0$. We conclude that $\sigma(T)$ is non-empty.

$\square$

2.2 An Operational Calculus for Bounded Linear Operators

In this section, we present an operational calculus for bounded linear operators that supports the following collection of functions.

Definition. *Given a bounded linear operator $T : H \to H$ and an open set $\mathcal{O}$ containing $\sigma(T)$, define $\mathcal{A}(\mathcal{O})$ to be the collection of analytic functions $f : \mathcal{O} \to \mathbb{C}$.*

Lemma 2.6 (The Resolvent Identity)**.** *Let T be a bounded linear operator with $\lambda, \mu \in \rho(T)$. Then*

$$R_\lambda R_\mu = (\lambda - \mu)^{-1}(R_\mu - R_\lambda)$$

Proof. Observe that

$$(\lambda I - T) - (\mu I - T) = (\lambda - \mu) I$$

Multiplying both sides by $(\lambda I - T)^{-1}(\mu I - T)^{-1}(\lambda - \mu)^{-1}$ and canceling appropriate terms leaves

$$(\mu I - T)^{-1}(\lambda - \mu)^{-1} - (\lambda I - T)^{-1}(\lambda - \mu)^{-1} = (\lambda I - T)^{-1}(\mu I - T)^{-1}$$

which can be written more concisely as

$$(R_\mu - R_\lambda)(\lambda - \mu)^{-1} = R_\lambda R_\mu$$

□

Given a bounded linear operator T, an open set $\mathcal{O} \subset \mathbb{C}$ containing $\sigma(T)$, a function $f \in \mathcal{A}(\mathcal{O})$, and a contour Γ in $\mathcal{O} \cap \rho(T)$ that winds once around each point in $\sigma(T)$ and winds zero times around any point in the complement of $\mathcal{O}$, define

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R_\lambda d\lambda. \tag{10}$$

The integral on the right-hand side is properly defined by Riemann sums and is a bounded linear operator on H.

Theorem 2.7. *The transformation $f \mapsto f(T)$ defined above is a homomorphism from the algebra of functions analytic on $\mathcal{O}$ into the Banach algebra of bounded linear operators from H to itself.*

Proof. The linearity of the mapping follows immediately from the linearity of the integral. It remains to verify that for $f, g \in \mathcal{A}(\mathcal{O})$ we have

$$(f \cdot g)(T) = f(T) \circ g(T).$$

Let C_1 and C_2 be two circular closed contours contained in $\mathcal{O} \cap \rho(T)$ with C_2 inside of C_1 so that C_1 winds once around each point in C_2, and C_2 winds zero times around each point in C_1. By definition we have

$$f(T)g(T) = \frac{1}{2\pi i} \int_{C_1} f(\lambda)(\lambda I - T)^{-1} \left(\frac{1}{2\pi i} \int_{C_2} g(\mu)(\mu I - T)^{-1} d\mu \right) d\lambda$$

which the resolvent identity tells us is equal to

$$\left(\frac{1}{2\pi i}\right)^2 \int\limits_{C_1} \left(\int\limits_{C_2} f(\lambda) g(\mu)(\lambda - \mu)^{-1}(R_\mu - R_\lambda) d\mu \right) d\lambda.$$

We deduce from (9) and Fubini's theorem that the preceding integral is equal to

$$\frac{1}{2\pi i} \int\limits_{C_2} \left(\frac{1}{2\pi i} \int\limits_{C_1} \frac{f(\lambda) d\lambda}{\lambda - \mu} \right) g(\mu) R_\mu d\mu - \frac{1}{2\pi i} \int\limits_{C_1} \left(\frac{1}{2\pi i} \int\limits_{C_2} \frac{g(\mu) d\mu}{\lambda - \mu} \right) f(\lambda) R_\lambda d\lambda.$$

Since each point μ on C_2 lies inside the closed contour C_1, Cauchy's integral formula tells us that

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\lambda)}{\lambda-\mu}d\lambda = f(\mu).$$

On the other hand, each point λ on C_1 lies outside of the closed contour C_2 and so

$$\frac{1}{2\pi i}\int_{C_2}\frac{g(\mu)}{\mu-\lambda}d\mu = 0.$$

Then our integral simplifies to

$$f(T)g(T) = \frac{1}{2\pi i}\int_{C_2} f(\mu)g(\mu)R_\mu d\mu = (f\cdot g)(T).$$

□

It is easy to verify that $f(T) = I$ when $f = 1$ and that $f(T) = T$ when f is the identity. The following corollary further highlights the relationship between f and $f(T)$.

Corollary 2.8. *If T is a bounded linear operator, $\mathcal{O}$ is a subset of the complex plane that contains the spectrum of T, and f belongs to $\mathcal{A}(\mathcal{O})$ with the property that $f \neq 0$ on $\mathcal{O}$, then $f^{-1}(T) = f(T)^{-1}$.*

Proof. The result follows immediately from the observation that

$$f(T)\circ f^{-1}(T) = (f\cdot f^{-1})(T) = I = (f^{-1}\cdot f)(T) = f^{-1}(T)\circ f(T)$$

□

The following commutativity property follows as an easy consequence of the analyticity of the resolvent operator on $\rho(T)$.

Proposition 2.9. *Every bounded linear operator $S : H \to H$ which commutes with T also commutes with $f(T)$.*

Proof. Observe that if $|\lambda| > \|T\|$, then, using a Neumann series expansion, we have

$$R_\lambda = (\lambda I - T)^{-1} = \frac{1}{\lambda}(I-\lambda^{-1}T)^{-1} = \sum_{n=0}^{\infty}\lambda^{-(n+1)}T^n.$$

Let r be a real number such that $r > \|T\|$ and define the closed contour $\Gamma = \{\lambda : |\lambda| = r\}$. Let $\mathcal{O}$ be an open subset of $\mathbb{C}$ such that $\sigma(T)\subset\mathcal{O}$ and Γ is contained in $\mathcal{O}$. Then, recalling (10), we may integrate term by term for

$$f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)R_\lambda d\lambda = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_\Gamma \frac{f(\lambda)}{\lambda^{n+1}}T^n d\lambda.$$

Since S commutes with T, it also commutes with each term in the above series. Thus S commutes with $f(T)$, as required.

□

Proposition 2.10. *For a polynomial p, definitions (10) and (1) are the same.*

Proof. Let $p(x) = a_n x^n + ... + a_1 x + a_0$ be a polynomial and let $T : H \to H$ be a bounded linear operator. Let r be a real number such that $r > \|T\|$ and define the closed contour $\Gamma = \{\lambda : |\lambda| > r\}$. Let $\mathcal{O}$ be an open subset of $\mathbb{C}$ that contains both Γ and the spectrum of T. Let λ be a complex scalar such that $\lambda > \|T\|$. Then

$$R_\lambda = (\lambda I - T)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty} \lambda^{-k} T^k.$$

Observe that

$$\begin{aligned}
2\pi i \cdot p(T) &= \int_\Gamma p(\lambda) R_\lambda d\lambda \\
&= a_n \int_\Gamma \lambda^n R_\lambda d\lambda + ... + a_1 \int_\Gamma \lambda R_\lambda d\lambda + a_0 \int_\Gamma R_\lambda d\lambda \\
&= a_n \int_\Gamma \frac{1}{\lambda}\left(\sum_{k=0}^{\infty} \lambda^{n-k} T^k\right) d\lambda + ... + a_0 \int_\Gamma \frac{1}{\lambda}\left(\sum_{k=0}^{\infty} \lambda^{-k} T^k\right) d\lambda \\
&= \left(a_n T^k + ... + a_1 T + a_0 I\right) 2\pi i.
\end{aligned}$$

□

3 Bounded Symmetric Operators

In this section we consider an operational calculus for bounded symmetric operators on a complex Hilbert space. We establish three basic theorems regarding the spectrum of a bounded symmetric operator: the spectral boundary theorem, spectral radius theorem, and the spectral mapping theorem. With these in hand, we use the Riesz-Markov Representation Theorem for the bounded linear functionals on $C(K, \mathbb{R})$) where K is compact, to prove theorem 3.8 which is the foundation on which we construct the operational calculus for bounded symmetric operators.

3.1 Spectral Theory for Bounded Symmetric Operators

We begin by showing that the spectrum of a bounded symmetric operator is necessarily real.

Lemma 3.1. *If $T : H \to H$ is a bounded symmetric operator with closed image, then*

$$H = T(H) \oplus \ker T.$$

Proof. Since the image of T is closed, we may decompose H into the direct sum $T(H) \oplus T(H)^\perp$, and so all that remains is to show that $T(H)^\perp = \ker T$. To this end, let z belong to the kernel of T and let y belong to the image of T so that $T(z) = 0$ and there exists $x \in H$ with $T(x) = y$. Since T is symmetric, the inner product

$$\langle y, z\rangle = \langle T(x), z\rangle = \langle x, T(z)\rangle = \langle x, 0\rangle = 0$$

which establishes $T(H)^{\perp} = \ker T$ as required.

□

Proposition 3.2. *Each non-real $\lambda \in \mathbb{C}$ lies in the resolvent set of T.*

Proof. Let $\lambda = \alpha + i\beta$, with α and β real and $\beta \neq 0$. We claim that $\lambda I - T$ is invertible. Indeed, since

$$\langle \lambda h - Th, h\rangle = \langle \lambda h, h\rangle - \langle Th, h\rangle = \alpha\|h\|^2 + i\beta\|h\|^2 - \langle Th, h\rangle$$

and, by the symmetry if T, $\langle Th, h\rangle$ is real, while, by choice, α and β are real, we deduce from the Cauchy-Schwartz Inequality that

$$|\beta| \cdot \|h\| \leq \|\lambda h - Th\| \text{ for all } h \text{ in } H.$$

We deduce that $\lambda I - T$ has trivial kernel, so that it is one-to-one, and has closed range. Then lemma 3.1 establishes that $\lambda I - T$ is onto. We conclude that $\lambda \in \rho(T)$

□

Theorem 3.3. *The spectrum of a bounded symmetric operator acting on a complex Hilbert space is a compact subset of the real line.*

Proof. Let T be a bounded symmetric operator. The preceding lemma establishes that the spectrum of T lies on the real line. Theorem 2.5 says the spectrum is closed and bounded. Together, these imply that $\sigma(T) \subset \mathbb{R}$ is compact.

□

Definition. *The* spectral radius *of a bounded linear operator T, denoted by* $\text{rad}_\sigma(T)$*, is defined as*

$$\text{rad}_\sigma(T) = \max\{|\lambda| \in \mathbb{C} | \lambda \in \sigma(T)\}.$$

Lemma 3.4. *A positive definite bounded symmetric operator is invertible.*

Proof. For notational convenience, we define $m = M(T)$. Then for every $h \in H$ the following inequality holds

$$m\langle h, h\rangle \leq \langle Th, h\rangle$$

which we can rewrite using the Cauchy Schwartz inequality as

$$m\|h\| \leq \|T(h)\|.$$

In particular, this inequality holds for $h = u - v$. Recalling that $m > 0$, we write

$$0 \leq m\|u - v\| \leq \|T(u) - T(v)\| \tag{11}$$

from which it is clear that the kernel of T is trivial so that T is one-to-one.

Next we will show that the image of T is closed. Let $\{T(x_n)\}$ be a sequence in the image of T that converges to $y \in H$. Since $\{T(x_n)\}$ is convergent it is also Cauchy, and so (11) forces $\{x_n\}$ Cauchy as well. But H is complete and so $\{x_n\}$ must converge to some $x \in H$. Then the continuity of T forces $T(x) = y$ which establishes that the image of T is closed.

Since the kernel of T is trivial and the image of T is closed, we may conclude from lemma 3.1 that the image of T is onto.

□

For a nonnegative symmetric operator T, there is the following useful inequality:

$$\|Th\|^4 \leq (\|T\|^3\|h\|^2)\langle Th, h\rangle \text{ for all } h \text{ in } H. \tag{12}$$

To see this, for $u, v \in H$, observe that, since $T \geq 0$, $g(t) = \langle T(u+tv), u+tv\rangle \geq 0$ for all real numbers t. Therefore the discriminant of the quadratic polynomial g is non-positive, that is,

$$\langle Tu, v\rangle^2 \leq \langle Tu, u\rangle\langle Tv, v\rangle.$$

Substitute Tu for v in this inequality and use the Cauchy-Schwartz Inequality to deduce (12).

Theorem 3.5 (The Spectral Boundary Theorem)**.** *Let T be a bounded symmetric operator on H and let $m = m(T)$ and $M = M(T)$ denote the spectral bounds of T. Then $\sigma(T) \subseteq [m, M]$ and both m and M belong to $\sigma(T)$.*

Proof. To verify the inclusion $\sigma(T) \subseteq [m, M]$, first consider $\lambda > M$. Then $\lambda I - T$ is positive definite so that lemma 3.4 implies $\lambda I - T$ is invertible and so $\lambda \in \rho(T)$. Similarly, if $\lambda < m$ then $T - \lambda I$ is positive definite we have $\lambda \in \rho(T)$. We conclude that $\sigma(T) \subseteq [m, M]$.

To show that M belongs to $\sigma(T)$, observe that since $MI - T$ is a nonnegative symmetric operator, according to (12),

$$\|(MI - T)h\|^4 \leq \|MI - T\|^3\|h\|^2\langle(MI - T)h, h\rangle \text{ for all } h \text{ in } H.$$

By the definition of M, there is a sequence $\{h_n\}$ of unit vectors such that $\{\langle(MI - T)h_n, h_n\rangle\} \to 0$. The above inequality tells us that $\{(MI - T)h_n\} \to 0$. Therefore $MI - T$ cannot be invertible, since otherwise, by the Open Mapping Theorem, its inverse would be continuous. Hence M belongs to $\sigma(T)$. Replacing $MI - T$ by $T - mI$, the same argument shows that m also belongs to $\sigma(T)$.

□

Theorem 3.6 (Spectral Radius Theorem)**.** *The spectral radius of a bounded symmetric operator is equal to its norm.*

Proof. Let $m = m(T)$ and $M = M(T)$ be the spectral bounds of T. Then appealing to theorem 1.1 we have

$$\|T\| = \sup\{|\langle T(h), h\rangle| : \|h\| = 1\} = \max\{|m|, |M|\}.$$

On the other hand, the Spectral Boundary Theorem says that $\sigma(T) \subseteq [m, M]$ with $m, M \in \sigma(T)$ so that

$$\mathrm{rad}_\sigma(T) = \max\{|m|, |M|\}.$$

We conclude that $\|T\| = \mathrm{rad}_\sigma(T)$.

□

The following theorem relates the spectrum of an operator with the spectrum of a polynomial in the same operator.

Theorem 3.7 (Spectral Mapping Theorem)**.** *Let $T : H \to H$ be a bounded linear operator and let p be a polynomial. Then*

$$\sigma(p(T)) = p(\sigma(T)).$$

Proof. First suppose that $\lambda_0 \in \sigma(T)$. Then λ_0 is trivially a root of the polynomial $p(\lambda_0) - p(\lambda)$. Therefore there is a polynomial q for which $p(\lambda_0) - p(\lambda) = (\lambda_0 - \lambda)q(\lambda)$. Consequently

$$p(\lambda_0)I - p(T) = (\lambda_0 I - T)q(T).$$

Since $\lambda I - T$ either fails to be one-to-one or fails to be onto, the operator $p(\lambda_0)I - p(T)$ has the same property. Thus $p(\lambda_0)$ belongs to $\sigma(p(T))$.

Now assume μ belongs to $\sigma(p(T))$. We can factor $\mu - p(x) = c\prod_{k=1}^{n}(r_k - x)$ so that

$$\mu I - p(T) = c\prod_{k=1}^{n}(r_k I - T).$$

If all of the r_k were members of $\rho(T)$ we would have μ in the resolvent set of $p(T)$. But μ belongs to $\sigma(p(T))$ by assumption and so at least one of the r_k must belong to $\sigma(T)$. But observe that $\mu = p(r_k)$ so that $\mu \in p(\sigma(T))$.

□

We now establish the cornerstone on which our operational calculus for bounded symmetric operators will be constructed.

Theorem 3.8. *Let $T : H \to H$ be a bounded symmetric operator and let h belong to H. Then there exists a unique real finite measure, μ_h, defined on the Borel algebra of the spectrum of T with the property that for any polynomial p with real coefficients we have*

$$\langle p(T)h, h\rangle = \int_{\sigma(T)} p(\lambda)\,d\mu_h.$$

Proof. For p a polynomial with real coefficients, define $\psi(p) = \langle p(T)h, h\rangle$. Observe that $p(T)$ is symmetric since p has real coefficients and T is symmetric. As a consequence, ψ is real-valued. There is the following estimate of $|\psi(p)|$:

$$\begin{aligned} |\psi(p)| &\leq \|p(T)\| \cdot \|h\|^2 \\ &= \mathrm{rad}_\sigma(p(T)) \cdot \|h\|^2 \\ &= \sup_{\lambda \in \sigma(T)} \{|p(\lambda)|\} \cdot \|h\|^2 \\ &= \|p\|_{\max} \cdot \|h\|^2 \end{aligned}$$

where the first inequality follows from Cauchy-Schwartz inequality, the second from the Spectral Radius Theorem, and the third from the Spectral Mapping Theorem. This estimate tells us that if we equip the linear space $C(\sigma(T), \mathbb{R})$ with the maximum norm and let $\mathcal{P}$ be the subspace of restrictions to $\sigma(T)$ of polynomials with real coefficients, then $\psi : \mathcal{P} \to \mathbb{R}$ is continuous. According to the theorem 3.3, $\sigma(T)$ is compact, and so we may appeal to the Weierstrass Approximation Theorem to deduce that $\mathcal{P}$ is a dense subspace of $C(\sigma(T), \mathbb{R})$. We may therefore extend ψ to a continuous linear functional $\psi : C(\sigma(T), \mathbb{R}) \to \mathbb{R}$. We claim that this functional is positive, in the sense that if $f \geq 0$ on $\sigma(T)$, then $\psi(f) \geq 0$. Indeed, first let $f = p$ be a polynomial with real coefficients which is nonnegative on $\sigma(T)$. Then, by the Spectral Mapping Theorem, the spectrum of $p(T)$ is nonnegative and therefore, by the Spectral Boundary Theorem, its lower spectral bound is nonnegative, that is, $\langle p(T)h, h\rangle \geq 0$ for all $h \in H$. From this we deduce, by a perturbation argument, the continuity of ψ with respect to the maximum norm and the Weierstrass Approximation Theorem, that the functional ψ is positive.

At this point, we may apply the Riesz-Markov Representation Theorem ([7, p. 458]) which tells us that there is a unique finite real Borel measure μ_h defined on the Borel algebra of the spectrum of T such that for every $f \in C(\sigma(T), \mathbb{R})$ we have

$$\psi(f) = \int_{\sigma(T)} f(\lambda) d\mu_h.$$

□

The collection of measures μ_h constructed in the previous theorem are referred to as the family of spectral measures for T. In the next section we shall make frequent use of the following corollary of theorem 3.8.

Corollary 3.9. *Let $T : H \to H$ be a bounded symmetric operator, let h belong to H, and let p be a polynomial with real coefficients. Then*

$$\|p(T)h\|^2 = \int_{\sigma(T)} p(\lambda)^2 d\mu_h \text{ and } \mu_h(\sigma(T)) = \|h\|^2.$$

Proof. Observe that since T is symmetric and p has real coefficients, then $p(T)$

is symmetric so that

$$\|p(T)h\|^2 = \langle p(T)h, p(T)h\rangle = \langle p^2(T)h, h\rangle = \int_{\sigma(T)} p^2(\lambda) d\mu_h.$$

We deduce from the preceding equality, taking $p = 1$, that $\mu_h(\sigma(T)) = \|h\|^2$. □

3.2 An Operational Calculus for Bounded Symmetric Operators

The operational calculus defined earlier for bounded linear operators on a complex Banach space X transformed analytic functions to bounded linear operators, using contour integrals and the Cauchy Integral Formula. For symmetric operators $T\colon H \to H$, we employ an entirely different approach to the construction of an operational calculus. We construct an operational calculus for $f(T)$, where f belongs to a very general family of real-valued functions $f\colon \sigma(T) \to \mathbb{R}$.

Definition. *For a bounded symmetric operator T, define $\mathcal{F}(T)$ to be the collection of real-valued functions on $\sigma(T)$ which are the pointwise limit of a bounded sequence of continuous functions defined on the spectrum of T.*

The space of functions $\mathcal{F}(T)$ has a straightforward classification that follows immediately from theorem 3.3 and the Weierstrass Approximation Theorem.

Proposition 3.10. *Let T be a bounded symmetric operator. A real-valued function $f : \sigma(T) \to \mathbb{R}$ belongs to $\mathcal{F}(T)$ if and only if there exists a sequence of polynomials that is uniformly bounded on the spectrum of T which converges point-wise on $\sigma(T)$ to f.*

Polynomials, with real coefficients, of a bounded symmetric operator are properly defined bounded symmetric operators. The strategy in the forthcoming construction is to approximate a function $f \in \mathcal{F}(T)$ by polynomials $\{p_n\}$, and then build $f(T)$ using the sequence $\{p_n(T)\}$.

Lemma 3.11. *Let T be a bounded symmetric operator on H. For each f in $\mathcal{F}(T)$, there is a bounded symmetric operator $f(T)$ on H with the property that if $\{p_n\}$ is a sequence of polynomials with real coefficients which is bounded on $\sigma(T)$ and converges pointwise to f on $\sigma(T)$, then*

$$\lim_{n\to\infty} p_n(T)h = f(T)h \text{ for all } h \text{ in } H. \tag{13}$$

Proof. Let h belong to H, let f be a member of $\mathcal{F}(T)$, and let $\{p_n\}$ be a sequence of polynomials with real coefficients that is uniformly bounded on $\sigma(T)$ and converges to f. Theorem 3.8 tells us that there exists a real finite measure μ_h defined on the Borel algebra of the spectrum of T. It follows from the Lebesgue

Bounded Convergence Theorem that

$$\lim_{n\to\infty}\int_{\sigma(T)}|p_n(\lambda)-f(\lambda)|^2 d\mu_h = 0$$

which means that $\{p_n\}$ converges in $L^2(\sigma(T),\mu_h)$ to f and, in particular, $\{p_n\}$ is Cauchy in $L^2(\sigma(T),\mu_h)$. For natural numbers n and m, we apply the corollary 3.9 to the polynomial $p_n - p_m$ to obtain

$$\|p_n(T)h - p_m(T)h\|^2 = \int_{\sigma(T)}|p_n(\lambda)-p_m(\lambda)|^2 d\mu_h.$$

Since $\{p_n\}$ is Cauchy in $L^2(\sigma(T),\mu_h)$, $\{p_n(T)h\}$ is Cauchy in H. But H is complete, and so $\{p_n(T)h\}$ converges to a vector $f(T)h$. Since each p_k is symmetric, so is $f(T)$. We deduce from corollary 3.9 that $\{p_k(T)\}$ is a bounded sequence so that $f(T)$ is bounded.

□

Theorem 3.12 (Operational Calculus)**.** *Let T be a bounded symmetric operator on H and for a function f in $\mathcal{F}(T)$, let $f(T)$ be the bounded symmetric operator on H defined by (13). Then, for all $h \in H$,*

$$\langle f(T)h,h\rangle = \int_{\sigma(T)} f(\lambda)d\mu_h \text{ and } \|f(T)h\|^2 = \int_{\sigma(T)} f(\lambda)^2 d\mu_h \tag{14}$$

where $\{\mu_h\}_{h\in H}$ is the family of spectral measures for T. The transformation $f \mapsto f(T)$ possesses the following properties: for f and g in $\mathcal{F}(T)$ and real numbers α and β,

- *(i) Linearity: $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$,*
- *(ii) The Product Property: $(f\cdot g)(T) = f(T)\circ g(T)$,*
- *(iii) Commutativity: Every bounded linear operator S which commutes with T also commutes with $f(T)$, and*
- *(iv) Monotonicity: If $f(\lambda)\geq 0$ for all $\lambda\in\sigma(T)$, then $\langle f(T)h,h\rangle \geq 0$ for all $h\in H$.*

Proof. There exist sequences of polynomials with real coefficients $\{f_n\}$ and $\{g_n\}$ which are bounded on $\sigma(T)$ and converge pointwise on $\sigma(T)$ to f and g, respectively. We deduce (14) from (13) together with the Lebesgue Bounded Convergence Theorem. For each n the linearity property holds for the polynomials

$$\alpha f_n(T)h + \beta g_n(T)h = (\alpha f_n + \beta g_n)(T)h.$$

Taking the limit as $n\to\infty$ establishes the linearity property.

Let h belong to H. Define $v = g(T)h$ and, for each n, define $v_n = g_n(T)h$. Note that $\{v_n\}$ converges to v. Observe that, for each n,

$$(f_n \cdot g_n)(T)h = (f_n(T) \circ g_n(T))h = f_n(T)v + f_n(T)(v_n - v). \tag{15}$$

Since $\lim_{n\to\infty} f_n(T)v = f(T)v = (f(T) \circ g(T))h$ and $\lim_{n\to\infty}(f_n \cdot g_n)(T)h = (f \cdot g)(T)h$, we may deduce the product property from (15) provided we verify that

$$\lim_{n\to\infty} f_n(T)(v_n - v) = 0 \tag{16}$$

To do so, let C be a uniform bound for $\{|f_n|\}$ on $\sigma(T)$. For each n, set $u_n = v_n - v$. We deduce from corollary 3.9 that

$$\|f_n(T)(v_n - v)\|^2 = \left| \int_\sigma (T) p_n(\lambda)^2 d\mu_{u_n} \right| \leq C^2 \int_{\sigma(T)} d\mu_{u_n} = C^2 \|v_n - v\|^2.$$

Since $\{v_n\}$ converges to v, (16) holds and therefore the product property is established.

If an operator commutes with T, then it commutes with polynomials in T and therefore, by (13), with $f(T)$, where f belongs to $\mathcal{F}(T)$. Thus the commutativity property is established.

To verify the monotonicity property, let f belong to $\mathcal{F}(T)$ with the property that $f(\lambda) \geq 0$ for all $\lambda \in \sigma(T)$. There is a bounded sequence $\{f_n\}$ of nonnegative continuous functions on $\sigma(T)$ which converges pointwise to f on $\sigma(T)$. Then $\{\sqrt{f_n}\}$ is also a bounded sequence of continuous nonnegative functions. Since $\{\sqrt{f_n}\} \to \sqrt{f}$, we have $\sqrt{f} \in \mathcal{F}(T)$, and we appeal to the product property of the operational calculus to deduce that for each $h \in H$,

$$\langle f(T)h, h\rangle = \langle(\sqrt{f} \cdot \sqrt{f})(T)h, h\rangle = \langle\sqrt{f}(T)h, \sqrt{f}(T)h\rangle = \|\sqrt{f}(T)h\|^2 \geq 0.$$

□

4 Unbounded Self-Adjoint Operators

Let D be a dense subspace of a complex Hilbert space H. For a linear operator $T\colon D \subseteq H \to H$, the adjoint T^* of T is the operator whose domain D^* consists of the vectors v in H for which there is a vector T^*v in H such that, for all u in D,

$$\langle Tu, v\rangle = \langle u, T^*v\rangle.$$

An operator $T\colon D \subseteq H \to H$ is said to be symmetric provided $\langle Tu, v\rangle = \langle u, Tv\rangle$, for all u, v in D. A symmetric operator is said to be self-adjoint provided $T = T^*$, which is equivalent to the assertion that T has no proper symmetric extensions.[1]

[1]Von Neumann attributes the notion of self-adjointness to Erhart Schmidt. Frequently it is exceedingly difficult to establish that a partial differential operator is self-adjoint. For example, given a real valued function $q\colon R^3 \to \mathbb{R}$, for what domain $D \subseteq L^2(\mathbb{R}^3)$ is the operator

The resolvent set of an unbounded operator $T : D \subset H \to H$, $\rho(T)$, is defined to be the set of complex numbers λ for which $\lambda I - T : D \to H$ is one-to-one and onto. The spectrum of T is the complement in the complex numbers of the resolvent of T.

Theorem 4.1 (Hellinger-Toeplitz)**.** *An everywhere-defined symmetric operator is bounded.*

Proof. Let $T : H \to H$ be symmetric. We will show that T is closed, so that the Closed Graph Theorem implies that T is bounded. To this end, let $\{x_n\} \to x$ a convergent sequence in H with the property that the sequence $\{T(x_n)\}$ also converges to some $y \in H$. We must show that $T(x) = y$. Let h belong to H and observe that

$$\langle T(x_n), h\rangle = \langle x_n, T(h)\rangle.$$

Taking the limit as $n \to \infty$ yields

$$\langle y, h\rangle = \langle x, T(h)\rangle = \langle T(x), h\rangle$$

which establishes T closed.

□

We have already proven that the spectrum of a bounded symmetric operator is a subset of the real line. It turns out that the result also holds for unbounded symmetric operators, and we will make use of this fact in constructing our operational calculus.

Lemma 4.2. *If T is a self-adjoint operator acting on a complex Hilbert space H, then all non-real complex numbers belong to the resolvent set of T.*

Proof. Let $T : D \subseteq H$ be a self-adjoint operator, and let $\lambda = \alpha + i\beta$ with α and β real and $\beta \neq 0$. We claim that $\lambda I - T$ is a one-to-one map of D onto H. Let $x \in D$. Indeed, since

$$\langle \lambda h - Th, h\rangle = \langle \lambda h, h\rangle - \langle Th, h\rangle = \alpha\|h\|^2 + i\beta\|h\|^2 - \langle Th, h\rangle$$

and, by the symmetry if T, $\langle Th, h\rangle$ is real, while, by choice, α and β are real, we deduce from the Cauchy-Schwartz Inequality that

$$|\beta| \cdot \|h\| \leq \|\lambda h - Th\| \text{ for all } h \text{ in } H. \tag{17}$$

We deduce that $\lambda I - T$ has trivial kernel, so that it is one-to-one.

Next we will show that $\lambda I - T$ has closed range and maps D onto H. To this end, let $y_n = T(x_n)$ we a sequence in the range of T that converges to some $y \in H$. Then (17) implies that $\{x_n\}$ is Cauchy, and therefore converges to some $x \in H$. Since T is self-adjoint, for any $h \in H$ we have $\langle T(x_n), h\rangle = \langle x_n, T(h)\rangle$.

$u \mapsto \Delta u + qu$ self-adjoint: see [3].

Taking the limit as $n \to \infty$ and again using that T is self-adjoint we have

$$\left\langle \lim_{n\to\infty} T(x_n), h \right\rangle = \langle x, T(h) \rangle = \langle T(x), h \rangle$$

which implies $\lim_{n\to\infty} T(x_n) = T(x)$. We write

$$\lambda x_n - T(x_n) = y_n.$$

Taking the limit as $n \to \infty$ yields

$$\lambda x - Tx = y$$

which shows that y is in the range of $\lambda I - T$ and establishes that the range of $\lambda I - T$ is closed.

Now, if the range of $\lambda I - T$ were not all of H, then there would exist some non-zero orthogonal vector $h^* \in H$ so that $0 = \langle (\lambda I - T)h, h^* \rangle$ for each h in the range of $\lambda I - T$. Expanding, rearranging, and appealing to the symmetry of T, we have

$$\langle h, T(h^*) \rangle = \langle h, \overline{\lambda} h^* \rangle$$

which implies $T(h^*) = \overline{\lambda} h^*$ so that $\langle h^*, T(h^*) \rangle = \lambda \langle h^*, h^* \rangle$. But T is self-adjoint so $\langle T(h), h \rangle$ is real, which contradicts our choice of $\lambda = \alpha + \beta i$ with $\beta \neq 0$.

□

4.1 Lax's Calculus for Unbounded Self-Adjoint Operators

When we constructed an operational calculus for bounded symmetric operators we explicitly defined the operator $f(T) : H \to H$ by expressing, for each h in H, the value of the inner product $\langle f(T)h, h \rangle$ as an integral with respect to a finite measure which was dependent on the choice of h. There are a number of methods to establish an operational calculus for unbounded self-adjoint operators. In this section, we follow an approach of Lax in [3] which works for a particular class of functions. The general calculus will be described in the following section.

Definition. *The* extended spectrum *of a self-adjoint operator $T : D \subseteq H \to H$ is its spectrum compactified by adjoining ∞.*

We define $\mathcal{R}$ to be the subset of the continuous functions defined on the extended spectrum of T consisting of the following collection of rational functions,

$$\mathcal{R} = \{q(\lambda)(\lambda^2 + 1)^{-n} \mid \deg(q) \leq 2n\}.$$

Lemma 4.3. *The collection $\mathcal{R}$ separates points on the extended spectrum of a self-adjoint operator T, in the sense that given any pair of distinct points p and q, there exists a function $f \in \mathcal{R}$ with the property that $f(p) \neq f(q)$.*

Proof. Let $T : D \subseteq H \to H$ be a self-adjoint operator and let p and q be points on the extended spectrum of T with $p \neq q$. We will consider two cases. First, if p and q have the same sign (or one of them is infinity) then the function $(\lambda^2 + 1)^{-1} \in \mathcal{R}$ separates them. Otherwise, p and q have opposite sign and the function $\lambda(\lambda^2 + 1)^{-1} \in \mathcal{R}$ separates them.

□

Proposition 4.4. *The collection $\mathcal{R}$ is dense in the space of continuous functions on the real line compactified by adjoining ∞, normed by the maximum norm.*

Proof. It is clear that the collection $\mathcal{R}$ forms an algebra over the reals and contain the constant function. The preceding proposition establishes that it separates points. We appeal to the Stone-Weierstrass Theorem to establish our result.

□

Theorem 4.5. *Let $T : D \subseteq H \to H$ be a self-adjoint operator and $r(\lambda) \in \mathcal{R}$ a rational function. Then $r(T)$ is a bounded symmetric operator on H.*

The preceding theorem is proven in [3]. Its proof is rather technical and depends essentially on von Neumann's proof of the special case where $n = 1$ and $r(\lambda) = \lambda$. We omit the proof of the general result and instead prove the special case, using von Neumann's original proof.

Proposition 4.6 (Von Neumann)**.** *Let $T : D \subseteq H \to H$ be self-adjoint. Then $I + T^2$ is a one-to-one and onto mapping from the domain of T^2 to H, and the operator $T(I + T^2)^{-1}$ is bounded and symmetric.*

Proof. To show that $I + T^2$ maps the domain of T^2 onto H, we examine the graph of T,

$$G(T) = \{(u, T(u)) \mid u \in D\} \subseteq H \oplus H.$$

Since T is self-adjoint, $G(T)$ is a closed subspace of $H \oplus H$, considered as a Hilbert space with the natural Hermitian form making the decomposition orthogonal. Therefore, there is the following orthogonal decomposition of $H \oplus H$:

$$H \oplus H = G(T) \oplus G(T)^{\perp}. \tag{18}$$

We deduce from the self-adjointness of T that

$$G(T)^{\perp} = \{(-Tu, u) \mid u \in D\}.$$

Let h belong to H. According to (18), there are vectors u, v in D for which

$$(h, 0) = (u, Tu) + (-Tv, v),$$

that is $h = u - Tv$ and $v = -Tu$. Hence $h = u + T^2(u)$ and u is in the domain of T^2. Thus $I + T^2$ maps the domain of T^2 onto H. Since T is symmetric

$$\langle (I + T^2)u, u\rangle = \langle u, u\rangle + \langle Tu, Tu\rangle \geq \langle u, u\rangle \tag{19}$$

holds for each u in the domain of T^2. The preceding inequality implies the kernel of $I+T^2$ is trivial so that $I+T^2$ is one-to-one. Furthermore, $(I+T^2)^{-1}$ is symmetric since it is the inverse of a symmetric operator.

To verify that $T(I+T^2)^{-1}$ is bounded, let h belong to H. Then

$$T(I+T^2)^{-1}h = Tv \text{ where } (I+T^2)v = h.$$

From $\langle v, v\rangle + \langle Tv, Tv\rangle = \langle h, v\rangle$ we first deduce that $\|v\| \leq \|h\|$ and then that $\|Tv\| \leq \|h\|$. Therefore

$$\|T(I+T^2)^{-1}h\| = \|Tv\| \leq \|h\| \text{ for all } h \text{ in } H$$

□

The final component that we require to construct Lax's operational calculus for unbounded self-adjoint operators is a version of the spectral mapping theorem for the collection $\mathcal{R}$.

Theorem 4.7. *Let T be a self-adjoint operator and let r belong to $\mathcal{R}$. Then*

$$\sigma(r(T)) = r(\sigma(T)).$$

Proof. Let $r(\lambda) = q(\lambda)(\lambda^2+1)^{-n}$ belong to $\mathcal{R}$ and let μ be a real scalar. The polynomial $q(\lambda)-\mu(\lambda^2+1)^n$ has $2n$ zeros. We denote the real zeros by $x_1, ..., x_k$ and the imaginary zeros by $z_1, ..., z_{2n-k}$ so that the polynomial factors as

$$q(\lambda) - \mu(\lambda^2+1)^n = \prod_{i=1}^{k}(\lambda - x_i) \prod_{i=1}^{2n-k}(\lambda - z_i)$$

and $r(x_i) = \mu$ for each x_i. Observe that

$$r(\lambda) - \mu = \frac{q(\lambda) - \mu(\lambda^2+1)^n}{(\lambda^2+1)^n} = (\lambda^2+1)^{-n} \prod_{i=1}^{k}(\lambda - x_i) \prod_{i=1}^{2n-k}(\lambda - z_i)$$

with $(\lambda^2+1)^{-n}$ a member of $\mathcal{R}$. We deduce from theorem 4.5 that

$$r(T) - \mu I = (T^2+I)^{-n} \prod_{i=1}^{k}(T - x_i I) \prod_{i=1}^{2n-k}(T - z_i I).$$

Since every non-real complex number belongs to the resolvent set of T, the product $(\lambda^2+1)^{-n}\prod_{i=1}^{2n-k}(\lambda - z_i)$ is a one-to-one and onto map from H to itself. Then the invertibility of $r(T)-\mu I$ depends entirely on the invertibility of the $T - x_i I$. This establishes the result, because if $\mu = r(\lambda)$ and $\lambda I - T$ is not invertible so that $\lambda \in \sigma(T)$ then $\mu I - r(T)$ is not invertible so that $\mu \in \sigma(r(T))$, and vice-versa.

□

If T is an unbounded self-adjoint operator, then according to proposition 4.5

$r(T)$ is a bounded symmetric operator for each $r \in \mathcal{R}$. Then it follows from the Spectral Radius Theorem for bounded symmetric operators and the preceding spectral mapping theorem that

$$\|r(T)\| = \max\{|r(\lambda)| \ : \ \lambda \in \sigma(T)\} \tag{20}$$

On the other hand, proposition 4.4 tells us that every function f that is continuous on the extended spectrum of T can be approximated uniformly by a sequence $\{r_k\}$ of functions from $\mathcal{R}$ so that $r_k \to f$ in the maximum norm. It follows that $\{r_k\}$ is Cauchy with respect to the maximum norm. Together with (20) this implies

$$\lim_{m,n\to\infty} \|r_m(T) - r_n(T)\| = 0$$

so that the sequence of bounded symmetric operators $\{r_k(T)\}$ is Cauchy in $\mathcal{L}(H)$. But $\mathcal{L}(H)$ is complete, so the sequence $\{r_k(T)\}$ is convergent. We define

$$f(T) = \lim_{n\to\infty} r_n(T) \tag{21}$$

to be the norm limit of the sequence of bounded symmetric operators $\{r_k(T)\}$.

Theorem 4.8. *Let $T : D \subseteq H \to H$ be a self-adjoint operator and let f and g be functions that are continuous on the extended spectrum of T. The bounded symmetric operator $f(T)$ defined by (21) possesses the following properties*

(i) $(f+g)(T) = f(T) + g(T)$, *and* $(f \cdot g)(T) = f(T) \circ g(T)$.

(ii) $\|f(T)\| = \max\{|f(\lambda)| : \lambda \in \sigma(T)\}$.

Proof. Since $(f+g)(T) = f(T) + g(T)$ and $(f \cdot g)(T) = f(T) \circ g(T)$ are both true when f and g are polynomials, they are also true for the uniform limit of polynomials.

Since $f(\lambda)$ is the uniform limit of a sequence of functions $r_n(\lambda)$ defined on the extended spectrum of T we have

$$\max\{|f(\lambda)| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} \max\{|r_n(\lambda)| : \lambda \in \sigma(T)\}.$$

Since $f(T)$ is the uniform limit of $r_k(T)$, we have $\|f(T)\| = \lim_{n\to\infty} \|r_k(T)\|$. From the two preceding inequalities, together with (20), we deduce that

$$\|f(T)\| = \max\{|f(\lambda)| : \lambda \in \sigma(T)\}$$

$\square$

While the construction is both clever and creative, ultimately it does not allow us to work with functions defined on the spectrum, only the extended spectrum. We will construct an operational calculus for self-adjoint operators that is not subject to this constraint in the final section of the paper.

4.2 A General Calculus for Unbounded Self-Adjoint Operators

We call $\{H_k, P_k\}_{k=1}^{\infty}$ an orthogonal decomposition of H provided $\{H_k\}_{k=1}^{\infty}$ is a pairwise orthogonal collection of closed subspaces of H, each P_k is the orthogonal projection of H onto H_k, and $\sum_{k=1}^{\infty} P_k h = h$ for all $h \in H$. Let $T : D \subseteq H \to H$ be a self-adjoint operator. An orthogonal decomposition $\{H_k, P_k\}_{k=1}^{\infty}$ of H is said to reduce T provided that, for each k, $H_k \subseteq D$ and $T(H_k) \subseteq H_k$. The following two theorems are due to Riesz and Lorch [5].

Theorem 4.9. *Let $T : D \subseteq H \to H$ be self-adjoint. Then T is reduced by an orthogonal decomposition $\{H_k, P_k\}_{k=1}^{\infty}$ of H which has the further property that if a bounded linear operator $S : H \to H$ commutes with T, in the sense that $S \circ T = T \circ S$ on D, then, for all k, $S(H_k) \subseteq H_k$.*

Theorem 4.10. *Let $\{H_k, P_k\}_{k=1}^{\infty}$ be an orthogonal decomposition of H, and, for each k, let the linear operator $T_k : H_k \to H_k$ be symmetric. There is one and only one self-adjoint operator $T : D \subseteq H \to H$ such that, for each k, $H_k \subseteq D$ and the restriction of T to H_k is T_k. It is the operator defined as follows:*

$$D = \left\{ h \in H \;\Big|\; \sum_{k=1}^{\infty} \|T_k P_k h\|^2 < \infty \right\} \text{ and } T(h) = \sum_{k=1}^{\infty} T_k P_k h \text{ for } h \text{ in } H.$$

Definition. *Define $\mathcal{F}_B(T)$ to be the collection of real-valued Borel functions $f : \sigma(T) \to \mathbb{R}$ that are the pointwise limit of a sequence of continuous functions that are bounded on bounded sets.*

If $T : D \subset H \to H$ is a self-adjoint operator then we know from theorem 4.9 that there exists an orthogonal decomposition $\{H_k, P_k\}$ that reduces T, and it follows from theorem 4.1 that the restriction $T_k : H_k \to H_k$ of T is a bounded symmetric operator so that $\sigma(T_k)$ is closed and bounded.

Then for each $f \in \mathcal{F}_B(T)$ and every k, the restriction of f to the spectrum of T_k is the point-wise limit on $\sigma(T_k)$ of a bounded sequence of continuous functions on $\sigma(T_k)$ so that $f \in \mathcal{F}(T_k)$. Thus, the bounded linear operator $f(T_k) : H_k \to H_k$ is defined by virtue of the operational calculus established earlier for bounded symmetric operators. Define

$$D(f(T)) = \left\{ h \in H \Big| \sum_{k=1}^{\infty} \|f(T_k) \circ P_k(h)\|^2 < \infty \right\} \text{ and } f(T) = \sum_{k=1}^{\infty} f(T_k) \circ P_k. \tag{22}$$

Theorem 4.10 tells us that the operator $f(T) : D(f(T)) \subseteq H \to H$ is self-adjoint.

Proposition 4.11. *Let $T : D \subseteq H \to H$ be a self-adjoint operator and let f and g belong to $\mathcal{F}$. The self-adjoint operator defined by (22) has the following properties:*

(i) (Linearity) $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$ on $D(f(T)) \cap D(g(T))$;

(ii) *(Product Property)* $(f \cdot g)(T)h = f(T) \circ g(T)h$ *on* $D(f \cdot g)(T))$;

(iii) *(Commutativity) The operator* $f(T)$ *commutes with* T*, and also commutes with any bounded linear operator on* H *which commutes with* T.

(iv) *(Monotonicity) If* $f \geq 0$ *on* $\sigma(T)$*, then* $f(T) \geq 0$ *in the sense that* $\langle f(T)h, h\rangle \geq 0$ *for each* $h \in D(f(T))$.

Proof. Since each $T_k : H_k \to H_k$ is a bounded symmetric operator, for each k we have

$$(\alpha f + \beta g)T_k = \alpha f(T_k) + \beta g(T_k)$$

by virtue of the operational calculus for bounded symmetric operators. Then for each h in $D(f(T)) \cap D(g(T))$ we have

$$\begin{aligned}
(\alpha f + \beta g)(T)h &= \sum_{k=1}^{\infty}(\alpha f + \beta g)(T_k)P_k h \\
&= \sum_{k=1}^{\infty}(\alpha f(T_k) + \beta g(T_k))P_k h \\
&= \alpha \sum_{k=1}^{\infty} f(T_k)P_k h + \beta \sum_{k=1}^{\infty} g(T_k)P_k h \\
&= \alpha f(T_k)h + \beta g(T_k)h.
\end{aligned}$$

Since each P_j is an orthogonal projection, and $H_j \perp H_k$ whenever $j \neq k$, we have $P_j(g(T_k) \circ P_k)$ equal to $g(T_k)$ when $j = k$ and zero otherwise, so that

$$\begin{aligned}
f(T) \circ g(T)h &= f(T) \circ \sum_{k=1}^{\infty} g(T_k) \circ P_k h = \sum_{j=1}^{\infty} f(T_j) \circ P_j \circ \sum_{k=1}^{\infty} g(T_k) \circ P_k h \\
&= \sum_{j=1}^{\infty} f(T_j) \circ g(T_j) \circ P_j h = \sum_{j=1}^{\infty}(f \cdot g)(T_j) \circ P_j h = (f \cdot g)(T)h,
\end{aligned}$$

and

$$\begin{aligned}
T \circ f(T)(h) &= T\left(\sum_{k=1}^{\infty} f(T_k) \circ P_k(h)\right) = \sum_{k=1}^{\infty} T_k \circ f(T_k) \circ P_k(h) \\
&= \sum_{k=1}^{\infty} f(T_k) \circ T_k \circ P_k(h) = \sum_{k=1}^{\infty} f(T_k) \circ P_k \left(\sum_{j=1}^{\infty} T_j \circ P_j(h)\right) \\
&= \sum_{k=1}^{\infty} f(T_k) \circ P_k \circ T\left(\sum_{j=1}^{\infty} P_j(h)\right) = \sum_{k=1}^{\infty} f(T_k) \circ P_k \circ T(h) \\
&= f(T) \circ T(h).
\end{aligned}$$

Suppose $S : H \to H$ is a bounded linear operator that commutes with T in the sense that $S \circ T = T \circ S$ on D. Theorem 4.9 tells us that $S(H_k) \subseteq H_k$ for each k and so we may define the restriction $S_k : H_k \to H_k$ which is bounded, symmetric, and necessarily commutes with T_k on H_k. Also, since P_k is an orthogonal projection, and H_k is orthogonal to H_n whenever $n \neq k$, we have $P_k \circ S \circ P_n = 0$ unless $n = k$ so that

$$P_k \circ S = P_k \circ S \circ \sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} P_k \circ S \circ P_n = P_k \circ S \circ P_k = P_k \circ S_k = S_k \circ P_k.$$

Then for every $h \in D$ that satisfies $S(h) \in D$ we have

$$\begin{aligned}
S \circ f(T)(h) &= S\left(\sum_{k=1}^{\infty} f(T_k) \circ P_k(h)\right) \\
&= \sum_{k=1}^{\infty} S_k \circ f(T_k) \circ P_k(h) \\
&= \sum_{k=1}^{\infty} f(T_k) \circ S_k \circ P_k(h) \\
&= \sum_{k=1}^{\infty} f(T_k) \circ P_k \circ S \circ P_k(h) \\
&= \sum_{k=1}^{\infty} f(T_k) \circ P_k \circ S(h) \\
&= f(T) \circ S(h)
\end{aligned}$$

Finally, if $f \geq 0$ on $\sigma(T)$ then we have $f \geq 0$ on $\sigma(T_k)$ and we may apply the monotonicity property proven for bounded symmetric operators so that $f(T_k) \geq 0$. Then for $h \in D(f(T))$ we have

$$\langle f(T)h, h \rangle = \sum_{k=1}^{\infty} \langle f(T_k) \circ P_k h, h \rangle \geq 0.$$

□

The operational calculus we have just constructed possesses a rich collection of properties, and it is defined for a far wider range of functions than the operation calculus due to Lax that we examined in the previous section. Furthermore, our calculus supports functions which are defined directly on the spectrum of T, so that we may dispense with the concept of the extended spectrum entirely.

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