ABSTRACT

Title of dissertation: COLLATZ CONJECTURE: GENERALIZING THE ODD PART

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Our aim is to investigate the Collatz conjecture. Because the chaotic mixing from iterating the piecewise Collatz function takes place in the odd case, we restrict attention to the odd integers in the orbits to identify some regularities. The parity sequence is reinterpreted and then used to show that if a counterexample exists then there are infinitely many counterexamples with any given initial behavior. When replacing the subfunction $3x + 1$ in the odd case with other affine functions, our results generalize. We show that the prime factorizations of the coefficients can be used to put a lower bound on the number of weak components in the digraph generated. Furthermore, we identify pairs of functions in this class such that the graph generated by one is isomorphic to a subgraph of the graph generated by the other. In the end, the Collatz conjecture is generalized and several new conjectures are raised.

COLLATZ CONJECTURE: GENERALIZING THE ODD PART

by

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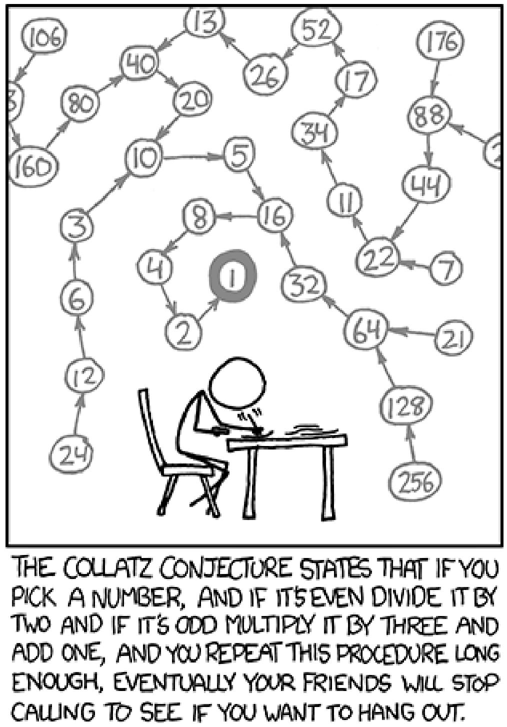

"*I cannot judge my work while I am doing it. I have to do as painters do, stand back and view it from a distance, but not too great a distance. How great? Guess.*"

Blaise Pascal

Dedication

My thesis is dedicated to my family, friends and educators who have supported and taught me. Included with it is a promise to them that I will continue my studies in mathematics until my goal of earning my Ph.D. has been completed. I also submit this thesis with my future children in mind, hoping to inspire them to pursue their areas of passion with heart and determination.

Acknowledgments

First and foremost, I thank my wife and mother for being the two biggest supporters in my life. I used to ask my mother if she knew what it was about the way she raised me that instilled the mentality to never disappoint her, because I want to raise my children the same way. I could never regret the year I took off from grad school, because I took advantage of opportunities to travel for work and met my lovely wife. She has provided incredible support, taking great care of me and helping me have more time to focus on my studies. I look forward to doing the same for her as she returns to grad school.

I am very grateful and have the utmost respect for all of the educators who have taught me mathematics and made this thesis possible. In particular, I thank Dr. Koralov for being my mentor during my research.

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List of Abbreviations

$\mathbb{O}$	The odd integers
$\mathbb{E}$	The even integers
$[r \bmod m]$	$\{x \in \mathbb{Z}^+ \mid x \equiv r \bmod m\}$

Chapter 1: Introduction

1.1 Background

The Collatz conjecture has eluded mathematicians for decades, despite the fact that it can be stated using basic arithmetic, the oldest and most elementary branch of mathematics. Even the most prolific mathematician, Paul Erdös, stated "mathematics is not ready for such a problem" about the Collatz conjecture [3, 330]. It is named after the German mathematician Lothar Collatz, who is credited with posing it back in 1937. Tying together his research interests in number theory and graph theory, he considered iterable functions defined on the positive integers and raised the question of how to determine the graph structure of such functions.

A function is *iterable* if its range is a subset of its domain, thereby guaranteeing we can compose the function with itself. For any iterable function ρ, we shall commonly refer to the digraph with nodes labeled by dom ρ and edges $x \mapsto \rho(x)$ as *the graph of* ρ. For any $k \in \mathbb{N}$, the exponential notation $\rho^k(x)$ is a common shorthand notation for

$$\underbrace{\rho \circ \rho \circ \cdots \circ \rho}_{k \text{ times}}(x). \tag{1.1}$$

The sequence $\{\rho^i(x)\}_{i=0}^{\infty}$ is known as the ρ-*orbit*, or ρ-*trajectory*, of x. Orbits of an iterable function can be classified based on characteristics of their infinite tails. Since orbits are infinite in length, by the pigeonhole principle, a bounded orbit of a function with domain a subset of $\mathbb{Z}$ must contain some integer multiple times. Furthermore, since we can assume that our function is well-defined, a bounded orbit must eventually become cyclic. The function being well-defined also implies that any two orbits are either disjoint or share an infinite tail. In particular, for any $x, y \in \text{dom } \rho$,

$$\rho^k(x) = \rho^\ell(y) \quad \Leftrightarrow \quad \rho^{k+m}(x) = \rho^{\ell+m}(y), \quad m \in \mathbb{N}. \tag{1.2}$$

Although Collatz considered a variety of iterable functions, the main one which bears his name is $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \text{ even}, \\ 3x+1, & x \text{ odd}, \end{cases} \tag{1.3}$$

also commonly referred to as the *3x+1 function*. The Collatz conjecture asserts that every f-orbit reaches 1, in which case the orbit is said to *converge*. This is equivalent to saying that the graph of f, which has been called the Collatz graph, is weakly connected. Throughout the remainder of this paper, we shall come across additional equivalent statements which the reader is encouraged to keep an eye out for.

Remark 1. For the remainder, we shall use the alternate definition of f given by

$$f(x) = \begin{cases} \frac{x}{2}, & x \text{ even}, \\ \frac{3x+1}{2}, & x \text{ odd}, \end{cases} \tag{1.4}$$

because $3x+1$ is obviously even when x is odd. The Collatz conjecture is unaffected and we shall see that many of our formulas will be simplified by using this version.

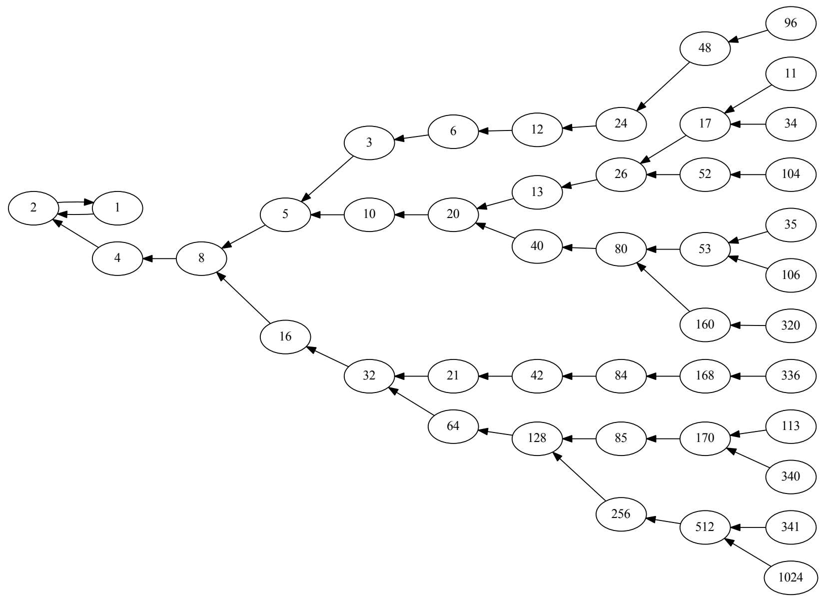

Figure 1.1: The Graph of the Collatz Function

Studied by many, the Collatz conjecture has picked up a lot of names along the way: *Syracuse problem*, *Hasse's algorithm*, *Ulam's problem*, *Kakutani's problem*, and *Thwaites' conjecture*. The University of Michigan's Jeffrey Lagarias is one of those who put time into solving the problem and has become its unofficial historian [5]. The conjecture has spread mainly by word of mouth over the years, because those interested in mathematics have been fascinated by how a function so easily definable could behave so chaotically. So far, the conjecture has been shown to hold for at least the positive integers up to $5 \times 2^{60} \approx 5.764 \times 10^{18}$ [6].

1.2 Characteristics

Beyond answering whether or not the f-orbit of an integer converges to 1, we can measure how long it takes to do so in terms of the number of iterations. For this purpose, the *total stopping time* $\sigma_\infty : \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{\infty\}$ is defined by

$$\sigma_\infty(x) = \begin{cases} \min\left\{k \in \mathbb{Z}^+ \mid f^k(x) = 1\right\}, & f\text{-orbit}(x) \text{ converges}, \\ \infty, & \text{otherwise}. \end{cases} \tag{1.5}$$

A classic example of the chaotic nature of the Collatz function is the f-orbit of 27, which begins

$$27, 41, 62, 31, 47, 71, 107, 161, 242, 121, \ldots$$

and eventually reaches a maximum value of 4616, converging after 70 iterations. In contrast, the f-orbits of 28 and 29 each converge in only 13 iterations. In Figure 1.2 is a plot of the total stopping time for $x \leq 1000$. Some of the curvature noticeable is simply due to the fact that $f(2x) = x$, and thus $\sigma_\infty(2x) = \sigma_\infty(x) + 1$.

The *stopping time* $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{\infty\}$ is defined similarly by

$$\sigma(x) = \begin{cases} \min\left\{k \in \mathbb{Z}^+ \mid f^k(x) < x\right\}, & f\text{-orbit}(x) \text{ decreases}, \\ \infty, & \text{otherwise}. \end{cases} \tag{1.6}$$

While this function definition resembles that of the total stopping time, Figure 1.3 demonstrates a very different distribution. Note that the y-axis is on a log scale, and thus the even integers appear on the x-axis. In the next section, we shall see that the set of integers with a given stopping time can be expressed as a finite union of arithmetic sequences, explaining the periodicity seen on horizontal lines in the stopping time plot.

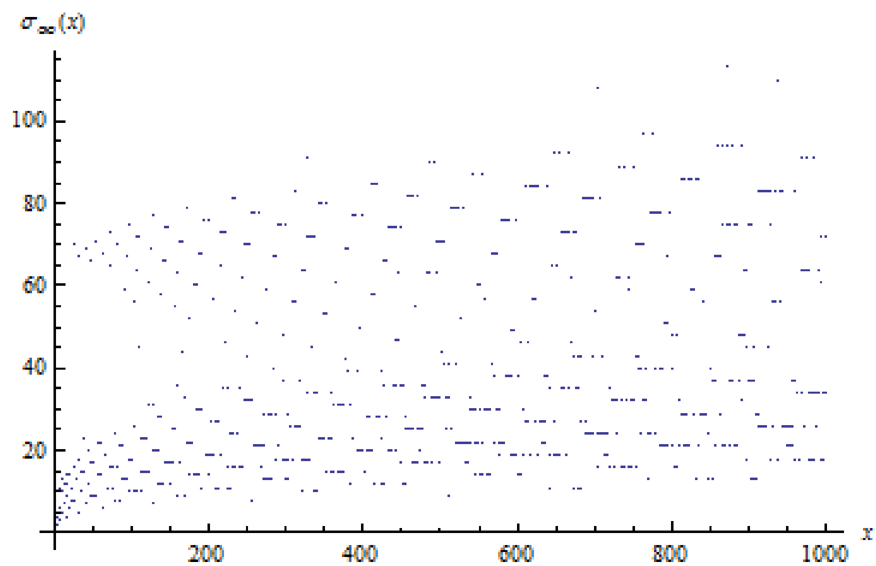

Figure 1.2: Total Stopping Time

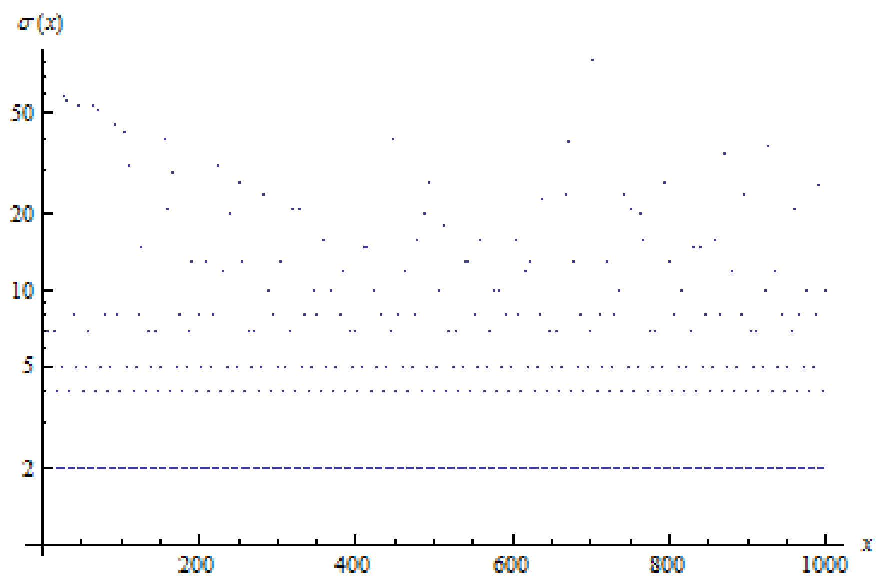

Figure 1.3: Stopping Time

Note that it's easy to show

$$k \leq \sigma\left(2^k - 1\right) \leq \sigma_\infty\left(2^k - 1\right), \quad k \in \mathbb{N},$$

and thus both functions are unbounded. From these functions, we already obtain another way of stating the Collatz conjecture.

Proposition 1.1. *The following statements are equivalent:*

1. *for every* $x > 1$, $\sigma_\infty(x) < \infty$;

2. *for every* $x > 1$, $\sigma(x) < \infty$.

Proof. The proof of (1) $\Rightarrow$ (2) follows immediately from

$$\sigma(x) \leq \sigma_\infty(x), \quad x > 1. \tag{1.7}$$

So suppose (2) holds. Fix $x_0 \in \mathbb{Z}^+$ and define the sequence $\{x_i\}_{i=0}^{\infty}$ recursively by

$$x_{n+1} = \begin{cases} f^{\sigma(x_n)}(x_n), & x_n > 1, \\ 1, & x_n = 1. \end{cases}$$

From this, we obtain a subsequence of the f-orbit of x which is strictly decreasing until reaching 1. Moreover,

$$\sigma_\infty(x_0) = \sum_{i=0}^{\bar{n}-1} \sigma(x_i), \quad \bar{n} = \min\left\{n \in \mathbb{Z}^+ \mid x_n = 1\right\}. \tag{1.8}$$

$\square$

While we will not look at the problem from the perspective of analysis much, one could also investigate upper bounds on orbits. Again applying the pigeonhole principle, any upper bound on an orbit is also an upper bound on how long it takes for the orbit to become cyclic. So we define

$$\tau(x) = \sup\left\{f^k(x) \mid k \in \mathbb{N}\right\} \tag{1.9}$$

as the *top*, or *height*, of an f-orbit. In terms of τ, the *expansion factor* is

$$\omega(x) = \frac{\tau(x)}{x}. \tag{1.10}$$

Both of these functions can be used to measure the growth in the f-orbit.

1.3 The Parity Sequence

Because the Collatz function f is piecewise-defined, split based on the parity of x, the infinite sequence $\{p_i(x)\}_{i=0}^{\infty}$ over $\mathbb{Z}/2\mathbb{Z}$ satisfying

$$p_k(x) \equiv f^k(x) \bmod 2, \quad k \in \mathbb{N}, \tag{1.11}$$

is called the *parity sequence*, or *parity vector*, of x. This sequence identifies which of the subfunctions is used at each iteration in the f-orbit. Therefore, using the initial k terms of the parity sequence, we can compute a direct formula for $f^k(x)$.

Theorem 1.2. *If $\{p_i(x)\}_{i=0}^{\infty}$ is the parity sequence of x then*

$$f^k(x) = \frac{3^{\sum_{i=0}^{k-1} p_i(x)}}{2^k}x + \sum_{i=0}^{k-1} p_i(x)\,\frac{3^{\sum_{j=i+1}^{k-1} p_j(x)}}{2^{k-i}}, \quad k \in \mathbb{N}. \tag{1.12}$$

Proof. For $k = 0$, using the standard convention that the empty sum is 0, we obtain the identity function as desired. So suppose the statement holds for some fixed k.

From our inductive hypothesis,

$$f^{k+1}(x) = f \circ f^{k}(x) \tag{1.13}$$

$$= f\left(\frac{3^{\sum_{i=0}^{k-1} p_i(x)}}{2^k} x + \sum_{i=0}^{k-1} p_i(x) \frac{3^{\sum_{j=i+1}^{k-1} p_j(x)}}{2^{k-i}}\right). \tag{1.14}$$

From the definition of a parity sequence, $p_k(x)$ determines which subfunction to apply next. In both cases, it is easily verified that we obtain the desired equation. □

One immediate benefit of this formula is a finite procedure to prove cycles of a given length don't exist. To do such, each element of $(\mathbb{Z}/2\mathbb{Z})^k$ can be substituted into $f^k(x) = x$ for the parity sequence, leaving x as the only free variable remaining. One can then check whether the unique solution for x is an integer. Note, however, that is algorithm is extremely inefficient.

For any countable sequence $\{\bar{p}_i\}$ over $\mathbb{Z}/2\mathbb{Z}$ of length κ, consider the set

$$\mathfrak{P}(\{\bar{p}_i\}) = \left\{x \in \mathbb{Z}^+ \mid p_i(x) = \bar{p}_i, 0 \leq i < \kappa\right\}, \tag{1.15}$$

which we shall describe as a *level-κ parity set* to signify the number of iterations restricted. From these sets, we obtain a lattice with a binary tree structure. Just consider that for a parity sequence to have initial segment $\{\bar{p}_i\}$, it is necessary that every initial segment of $\{\bar{p}_i\}$ itself be obeyed. That is, if $j \leq k$ then

$$\left\{x \in \mathbb{Z}^+ \mid p_i(x) = \bar{p}_i, 0 \leq i < k\right\} \subseteq \left\{x \in \mathbb{Z}^+ \mid p_i(x) = \bar{p}_i, 0 \leq i < j\right\}. \tag{1.16}$$

Thus we have a complete binary tree with the nodes at level k each representing a unique element of $(\mathbb{Z}/2\mathbb{Z})^k$ and the relation that ancestors represent initial segments. This structure coincides nicely with another complete binary tree from number theory, namely, the partitioning of congruence classes modulo powers of 2.

Theorem 1.3. *For any finite binary sequence* $\{\bar{p}_i\}_{i=0}^{k-1}$*, there exists* $r \in \mathbb{N}$ *such that*

$$\mathfrak{P}\left(\{\bar{p}_i\}_{i=0}^{k-1}\right) = \left[r \text{ } mod \text{ } 2^k\right]. \tag{1.17}$$

Proof. We shall prove the statement by induction on k. The definition of f covers the base case of $k = 1$, so suppose the statement holds for some fixed k. Let $\{\bar{p}_i\}_{i=0}^{k}$ be a fixed binary sequence. From the definition of a parity set,

$$\mathfrak{P}\left(\{\bar{p}_i\}_{i=0}^{k}\right) = \left\{x \in \mathbb{Z}^+ \mid p_0(x) = \bar{p}_0 \text{ and } f(x) \in \mathfrak{P}\left(\{\bar{p}_i\}_{i=1}^{k}\right)\right\}. \tag{1.18}$$

This perspective of $\mathfrak{P}\left(\{\bar{p}_i\}_{i=0}^{k}\right)$ allows us to simply solve for it algebraically. Using our inductive hypothesis, let $r \in \mathbb{N}$ such that

$$\mathfrak{P}\left(\{\bar{p}_i\}_{i=1}^{k}\right) = \left[r \bmod 2^k\right]. \tag{1.19}$$

In the case of $\bar{p}_0 = 0$,

$$f(x) = \frac{x}{2} \in \left[r \bmod 2^k\right] \tag{1.20}$$

implies

$$\mathfrak{P}\left(\{\bar{p}_i\}_{i=0}^{k}\right) = \left[2r \bmod 2^{k+1}\right]. \tag{1.21}$$

In the case of $\bar{p}_0 = 1$,

$$f(x) = \frac{3x+1}{2} \in \left[r \bmod 2^k\right] \tag{1.22}$$

requires partitioning of $\left[r \bmod 2^k\right]$ into

$$\left[r \bmod 2^k \cdot 3\right] \cup \left[r + 2^k \bmod 2^k \cdot 3\right] \cup \left[r + 2^{k+1} \bmod 2^k \cdot 3\right] \tag{1.23}$$

in order to solve. Using the proper set from this partition, we again obtain a congruence class modulo 2^{k+1} as the solution set. $\square$

Corollary 1.4. *If $x, y, k \in \mathbb{Z}^+$ with $x < y < 2^k$ then the parity sequences of x and y differ within the first k terms.*

Proof. Since $x \not\equiv y \bmod 2^k$, the result follows from Theorem 1.3. □

While Theorem 1.3 only considers identifying the positive integers with parity sequences initially obeying a given binary sequence of finite length, since parity sequences are infinite, we can do the same with a given infinite binary sequence. From Corollary 1.4, the set of positive integers with parity sequence matching an infinite sequence has cardinality at most 1. Of course, because of countability, most infinite binary sequences are not the parity sequence of any integer. Recall that orbits are either disjoint or share an infinite tail. In terms of parity sequences, this translates into simply whether or not an infinite tail is shared. In particular, the parity sequence of 1 is the alternating binary sequence, and thus the f-orbit of an integer converges if and only if its parity sequence has the alternating binary sequence as an infinite tail.

Applying Theorem 1.3, we can identify parity sets based on sampling. For example, observation of the convergent orbit

$$7 \quad 11 \quad 17 \quad 26 \quad 13 \quad 20 \quad 10 \quad 5 \quad 8 \quad 4 \quad 2 \quad 1 \quad 2 \quad 1 \quad \ldots$$

with corresponding parity sequence

$$1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad \ldots$$

can be used to infer the following:

$$
\begin{array}{lcl}
x \equiv 7 \bmod 2 & \Rightarrow & \{p_i(x)\}_{i=0}^{0} = (1) \\
x \equiv 7 \bmod 4 & \Rightarrow & \{p_i(x)\}_{i=0}^{1} = (1,1) \\
x \equiv 7 \bmod 8 & \Rightarrow & \{p_i(x)\}_{i=0}^{2} = (1,1,1) \\
\vdots & & \vdots \\
x \equiv 17 \bmod 2 & \Rightarrow & \{p_i(x)\}_{i=0}^{0} = (1) \\
x \equiv 17 \bmod 4 & \Rightarrow & \{p_i(x)\}_{i=0}^{1} = (1,0) \\
x \equiv 17 \bmod 8 & \Rightarrow & \{p_i(x)\}_{i=0}^{2} = (1,0,1) \\
\vdots & & \vdots
\end{array}
$$

The congruence class which is equal to a parity set based on a finite binary sequence is easy to compute in specific cases. Trivial examples include constant sequences. Note that Theorem 1.3 says that if we know $\left[r \bmod 2^k\right]$ is the parity set of $\bar{p} \in (\mathbb{Z}/2\mathbb{Z})^k$ then changing the k^{th} term of $\bar{p}$ must result in obtaining the congruence class $\left[r + 2^{k-1} \bmod 2^k\right]$. Another consequence of Theorem 1.3, when combined with Theorem 1.2, is that the set

$$S_k = \left\{x \in \mathbb{Z}^+ \mid \sigma(x) = k\right\}$$

is actually a finite union of arithmetic sequences, because the expression for f^k is an affine mapping on a congruence class modulo 2^k.

Something the reader may have noticed by now is that it is often beneficial to think of the positive integers in their binary representation in order to understand what is going on. We have seen that, for any countable κ, the last κ digits of the binary representation of a positive integer determine the first κ terms of its parity sequence.

x	base 2	parity seq.	x	base 2	parity seq.
0	000000	0,0,0,0,0,0	32	100000	0,0,0,0,0,1
1	000001	1,0,1,0,1,0	33	100001	1,0,1,0,1,1
2	000010	0,1,0,1,0,1	34	100010	0,1,0,1,0,0
3	000011	1,1,0,0,0,1	35	100011	1,1,0,0,0,0
4	000100	0,0,1,0,1,0	36	100100	0,0,1,0,1,1
5	000101	1,0,0,0,1,0	37	100101	1,0,0,0,1,1
6	000110	0,1,1,0,0,0	38	100110	0,1,1,0,0,1
7	000111	1,1,1,0,1,0	39	100111	1,1,1,0,1,1
8	001000	0,0,0,1,0,1	40	101000	0,0,0,1,0,0
9	001001	1,0,1,1,1,0	41	101001	1,0,1,1,1,1
10	001010	0,1,0,0,0,1	42	101010	0,1,0,0,0,0
11	001011	1,1,0,1,0,0	43	101011	1,1,0,1,0,1
12	001100	0,0,1,1,0,0	44	101100	0,0,1,1,0,1
13	001101	1,0,0,1,0,0	45	101101	1,0,0,1,0,1
14	001110	0,1,1,1,0,1	46	101110	0,1,1,1,0,0
15	001111	1,1,1,1,0,0	47	101111	1,1,1,1,0,1
16	010000	0,0,0,0,1,0	48	110000	0,0,0,0,1,1
17	010001	1,0,1,0,0,1	49	110001	1,0,1,0,0,0
18	010010	0,1,0,1,1,1	50	110010	0,1,0,1,1,0
19	010011	1,1,0,0,1,1	51	110011	1,1,0,0,1,0
20	010100	0,0,1,0,0,0	52	110100	0,0,1,0,0,1
21	010101	1,0,0,0,0,0	53	110101	1,0,0,0,0,1
22	010110	0,1,1,0,1,0	54	110110	0,1,1,0,1,1
23	010111	1,1,1,0,0,0	55	110111	1,1,1,0,0,1
24	011000	0,0,0,1,1,0	56	111000	0,0,0,1,1,1
25	011001	1,0,1,1,0,0	57	111001	1,0,1,1,0,1
26	011010	0,1,0,0,1,0	58	111010	0,1,0,0,1,1
27	011011	1,1,0,1,1,1	59	111011	1,1,0,1,1,0
28	011100	0,0,1,1,1,0	60	111100	0,0,1,1,1,1
29	011101	1,0,0,1,1,0	61	111101	1,0,0,1,1,1
30	011110	0,1,1,1,1,0	62	111110	0,1,1,1,1,1
31	011111	1,1,1,1,1,0	63	111111	1,1,1,1,1,1

Table 1.1: Level-6 Parity Sets

Chapter 2: The Odd Behavior

2.1 The Odd Part

When investigating the Collatz function and seeking to narrow focus, the set of odd integers is a reasonable start since it is where the “mixing” occurs. With $f(2x) = x$, obviously the Collatz conjecture holds if and only if the orbits starting from odd integers all converge. However, domain restriction alone is uninteresting, because the only even integers removed from the range are those divisible by 3. The function $\delta : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$\delta(x) = \min\left\{k \in \mathbb{Z}^+ \mid f^k(x) \in \mathbb{O}^+\right\} \tag{2.1}$$

will be the tool which helps us remove all even integers by specifying precisely how many must be skipped. From the fundamental theorem of arithmetic, δ is certainly well-defined.

The function $g : \mathbb{O}^+ \to \mathbb{O}^+$ defined by

$$g(x) = f^{\delta(x)}(x) \tag{2.2}$$

is called the *Syracuse function*. So now we have the g-orbit of x which consists of just the odd terms in the f-orbit of x. For any $x \in \mathbb{O}^+$, regardless of whether the f-orbit of x converges, diverges, or reaches a non-trivial cycle, the g-orbit of x

will do the same. Although minor, one advantage of the g-orbit over the f-orbit is that convergence of a g-orbit agrees with the standard definition for convergence of a sequence, because 1 is the unique fixed point of g.

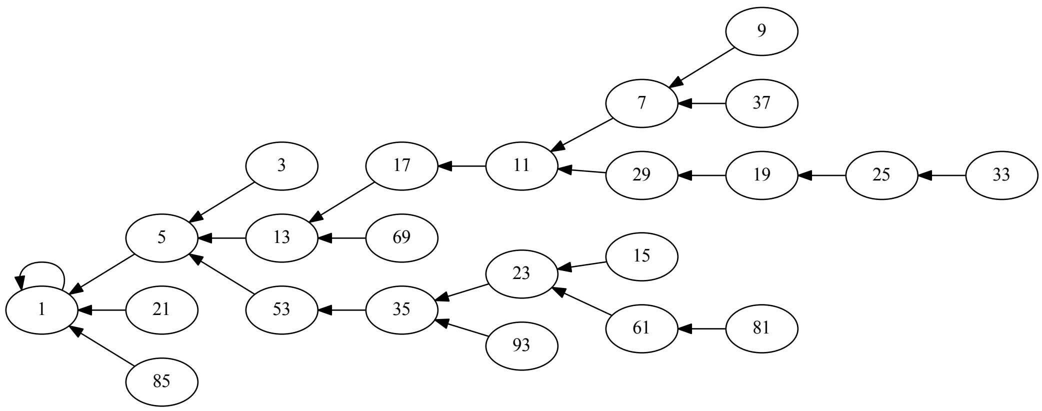

Figure 2.1: The Graph of the Syracuse Function

While there is no such thing as truly choosing a positive integer at random, considering Theorem 1.3, one could intuitively say $P(\delta(x) = k) = \frac{1}{2^k}$. From this, the expected value after one iteration of g is estimated by

$$E(g(x)) \approx \sum_{i=1}^{\infty} \left(\frac{3x}{2^i} \cdot \frac{1}{2^i} \right) = 3x \sum_{i=1}^{\infty} \frac{1}{4^i} = x. \tag{2.3}$$

However, if we consider an entire orbit and imagine that the δ values appearing in the g-orbit are independent, we obtain an expected scale factor of

$$\prod_{i=1}^{\infty} \left(\frac{3}{2^i} \right)^{1/2^i} = \frac{3}{4} < 1. \tag{2.4}$$

This leads one to believe that g-orbits will decrease over time.

Next, we begin examining the regularities which are more easily identified in the g-orbit than the f-orbit.

Theorem 2.1. *Define* $\phi : \mathbb{O}^+ \to \mathbb{O}^+$ *by* $\phi(x) = 4x + 1$*. Then*

1. *for any* $x \in \mathbb{O}^+$ *and* $k \in \mathbb{N}$, $\phi^k(x) = 4^k x + \frac{4^k - 1}{3}$;

2. *for any* $x, y \in \mathbb{O}^+$ *with* $x \leq y$, $g(x) = g(y)$ *if and only if* $y = \phi^k(x)$ *for some* $k \in \mathbb{N}$, *in which case* $\delta(y) = \delta(x) + 2k$;

3. *for any* $y \in ran\ g$, $g^{-1}(y) = \left\{\phi^k(w) \mid k \in \mathbb{N}\right\}$ *for a unique* $w \in \mathbb{O}^+$.

We shall defer the formal proof of Theorem 2.1 for now, because two generalizations will be proven later. This theorem is another statement about the Collatz conjecture that is perhaps more clear when working in binary. Multiplying by 4 and adding 1 is essentially just adding the digits 01 to the end of the binary representation. We write $3\,\phi(x) + 1$ as $2\,\phi(x) + \phi(x) + 1$ and shall mask the digits in the binary representation of x.

$$
\begin{array}{cccccccc:cc}
 & & & & & & & {\scriptstyle 1} & & \\
 & X & X & \cdots & X & X & & 0 & 1 & 0 \\
 & & X & X & \cdots & X & & X & 0 & 1 \\
+ & & & & & & & & & 1 \\
\hline
 & & & & & & & & 0 & 0
\end{array}
$$

The work above shows that we have $g(\phi(x)) = g(x)$ and we divide by 2 precisely two additional times.

Having identified the sets on which g has a given value as ϕ-orbits, we next determine the range of g and show how to completely calculate $g^{-1}(y)$ for any

$y \in \operatorname{ran} g$.

Theorem 2.2. *The range of g is $\{x \in \mathbb{O}^+ \mid x \not\equiv 0 \, mod \, 3\}$. In fact, we have*

$$\begin{aligned} g\left(\{x \in \mathbb{O}^+ \mid \delta\left(x\right) \in \mathbb{O}^+\}\right) &= [5 \, mod \, 6] \,, \\ \textit{and} \quad g\left(\{x \in \mathbb{O}^+ \mid \delta\left(x\right) \in \mathbb{E}^+\}\right) &= [1 \, mod \, 6] \,. \end{aligned} \tag{2.5}$$

Moreover, if $y \in ran \; g$ then $min \; g^{-1}\left(y\right)$ is given by

$$\gamma\left(y\right) = \begin{cases} \frac{2y-1}{3}, & y \equiv 5 \, mod \, 6, \\ \\ \frac{4y-1}{3}, & y \equiv 1 \, mod \, 6. \end{cases} \tag{2.6}$$

Proof. Fix $x \in \mathbb{O}^+$. Let q and r be the unique integers such that $g\left(x\right) = \frac{3x+1}{2^{\delta(x)}} = 6q+r$ and $0 \leq r < 6$. Then $3x + 1 = 3 \cdot 2^{\delta(x)+1}q + 2^{\delta(x)}r$, which implies $2^{\delta(x)}r \equiv 1 \bmod 3$. So clearly $r \neq 3$. Because $2^{\delta(x)} \equiv \left(-1\right)^{\delta(x)} \bmod 3$ and r is the multiplicative inverse of $2^{\delta(x)} \bmod 3$, $r \equiv \left(-1\right)^{\delta(x)} \bmod 3$ as well. Hence, since $g\left(x\right) \in \mathbb{O}^+$, $g\left(x\right) \equiv 5 \bmod 6$ iff $\delta\left(x\right) \in \mathbb{O}^+$ and $g\left(x\right) \equiv 1 \bmod 6$ iff $\delta\left(x\right) \in \mathbb{E}^+$.

With γ as defined, clearly g is a left inverse of γ. Next, assume by contradiction that $\gamma\left(y\right) \neq \min g^{-1}\left(y\right)$. Then, by Theorem 2.1, there exists $w \in \mathbb{O}^+$ such that $\gamma\left(y\right) = \phi\left(w\right) = 4w + 1$. But, for either subfunction, this implies $y \in \mathbb{E}^+$ and we obtain a contradiction. □

Corollary 2.3. *The ranges of γ and ϕ partition $\mathbb{O}^+$.*

Proof. Directly from the definitions and domains,

$$\operatorname{ran} \gamma = [3 \bmod 4] \cup [1 \bmod 8] \tag{2.7}$$

and

$$\operatorname{ran} \phi = [5 \bmod 8] \tag{2.8}$$

are easily computed. $\square$

Viewing $g^{-1}(y)$ as the ϕ - orbit of $\gamma(y)$, the conjecture can now be approached by traveling away from 1 rather than supposedly towards.

Corollary 2.4. *The g - orbit of x converges if and only if there exists a finite sequence $\{j_i\}_{i=0}^{k-1}$ of natural numbers such that $x = \phi^{j_0} \circ \gamma \circ \cdots \circ \phi^{j_{k-1}} \circ \gamma(1)$.*

Proof. Applying Theorem 2.1 and Theorem 2.2 iteratively we obtain

$$g^k(x) = y \Leftrightarrow x = \phi^{\lfloor \frac{p_0(x)-1}{2} \rfloor} \circ \gamma \circ \cdots \circ \phi^{\lfloor \frac{p_{k-1}(x)-1}{2} \rfloor} \circ \gamma(y) \tag{2.9}$$

and apply it $y = 1$. $\square$

The following theorem is just an interesting side track which can be used to show specific orbits merge.

Theorem 2.5. *Let $\{t_i\}_{i=0}^{\infty}$ be defined by $t_0 = 1$, $t_1 = 11$, and $t_{k+2} = 4t_k + 3$. Then, for every $k \in \mathbb{N}$, $x \equiv t_k \, mod \, 2^{k+3}$ implies $g^{k+2}(x) = g^{k+2}(2x+1)$.*

Proof. For $k = 0$, suppose $x \equiv 1 \bmod 8$. Say $x = 8i + 1$ for some $i \in \mathbb{N}$. Then $2x + 1 = 16i + 3$ and we easily compute

$$g(x) = \frac{3(8i+1)+1}{4} = 6i + 1 \tag{2.10}$$

and

$$g(2x+1) = \frac{3(16i+3)+1}{2} = 24i + 5. \tag{2.11}$$

So we have $g(2x+1) = \phi(g(x))$, and thus $g^2(x) = g^2(2x+1)$, by Theorem 2.1.

For $k = 1$, suppose $x \equiv 11 \bmod 16$. Say $x = 16i + 11$ for some $i \in \mathbb{N}$. Then $2x + 1 = 32i + 23$. Again, we easily compute

$$g(x) = \frac{3(16i + 11) + 1}{2} = 24i + 17$$

and

$$g(2x + 1) = \frac{3(32i + 23) + 1}{2} = 48i + 35.$$

From these equations we see $g(x) \equiv 1 \bmod 8$ and $g(2x + 1) = 2g(x) + 1$. So we can apply the case of $k = 0$ to $g(x)$ and conclude $g^3(x) = g^3(2x + 1)$.

For our inductive step, suppose our statement holds for a given $k \in \mathbb{N}$ and let $x \equiv t_{k+2} \equiv 4t_k + 3 \bmod 2^{k+5}$. Say $x = 2^{k+5}i + 4t_k + 3$ for some $i \in \mathbb{N}$. Then

$$g(x) = \frac{3\left(2^{k+5}i + 4t_k + 3\right) + 1}{2} \tag{2.12}$$

$$= 3\left(2^{k+4}i + 2t_k\right) + 5 \tag{2.13}$$

and

$$g^2(x) = \frac{3\left(3\left(2^{k+4}i + 2t_k\right) + 5\right) + 1}{2} \tag{2.14}$$

$$= 3^2\left(2^{k+3}i + t_k\right) + 8 \tag{2.15}$$

$$= 3^2 2^{k+3}i + 3^2 t_k + 8 \tag{2.16}$$

$$= 3^2 2^{k+3}i + 8(t_k + 1) + t_k. \tag{2.17}$$

From the recursive definition of the sequence $\{t_i\}_{i=0}^{\infty}$, it's easily shown by induction that $t_k \equiv -1 \bmod 2^{k+1}$. Therefore, the last line in our computation of $g^2(x)$ gives us $g^2(x) \equiv t_k \bmod 2^{k+3}$. Finally,

$$g(2x+1) = \frac{3\left(2\left(2^{k+5}i + 4t_k + 3\right) + 1\right) + 1}{2} \tag{2.18}$$

$$= 3\left(2^{k+5}i + 4t_k\right) + 11 \tag{2.19}$$

and

$$g^2(2x+1) = \frac{3\left(3\left(2^5 i + 4t_k\right) + 11\right) + 1}{2} \tag{2.20}$$

$$= 3^2\left(2^{k+4} + 2t_k\right) + 17 \tag{2.21}$$

again shows $g^2(2x+1) = 2g^2(x) + 1$. Applying the inductive hypothesis to $g^2(x)$ completes the proof. $\square$

It is worthwhile to start looking at the structure of the set of counterexamples to Collatz conjecture. Without loss of generality, suppose x is a counterexample which belongs to ran g. Since the g - orbit of x doesn't converge, we can apply ϕ or γ any finite number of times and every integer obtained is also a counterexample to the conjecture. Since dom $\phi = \mathbb{O}^+$ and the ϕ - orbit is strictly increasing, we obtain infinitely many distinct counterexamples, regardless of whether the g - orbit reaches a non-trivial cycle or diverges. We shall see in the next section that the existence of a counterexample implies we can identify infinitely many counterexamples exhibiting any finite orbital behavior.

Lemma 2.6. *For every $x \in \mathbb{O}^+$ and $k \in \mathbb{N}$, any 3^k consecutive terms in the ϕ - orbit of x form a complete set of residues module 3^k.*

Proof. The cases of $k = 0$ and $k = 1$ are trivial, so suppose the statement holds for some fixed k. By inductive hypothesis, the particular sets

$$\left\{x, \phi(x), \ldots, \phi^{3^k-1}(x)\right\} \text{ and } \left\{\phi(x), \phi^2(x), \ldots, \phi^{3^k}(x)\right\}$$

are each complete sets of residues modulo 3^k. With a difference of only one element, this implies $x \equiv \phi^{3^k}(x) \mod 3^k$. Applying this logic to the entire ϕ-orbit of x, it suffices to show that $\phi^{\ell}(x)$, $\phi^{\ell+3^k}(x)$ and $\phi^{\ell+2\cdot 3^k}(x)$ are distinct modulo 3^{k+1}. From Theorem 2.1, $\phi^{3^k}(x) = 4^{3^k}x + \frac{4^{3^k}-1}{3}$. Easily shown by simple induction, $4^{3^k} \equiv 1 \mod 3^{k+1}$ and $\frac{4^{3^k}-1}{3} \equiv 3^k \mod 3^{k+1}$. Using these two congruences, we obtain the distinct residues modulo 3^{k+1}. □

Theorem 2.7. *If there exists a counterexample to the Collatz conjecture then every congruence class modulo* 3^k *contains infinitely many.*

Proof. Let $x \in \mathbb{O}^+$ be a counterexample. Since g is constant when restricted to a ϕ-orbit, the ϕ-orbit of x is an infinite set of counterexamples. By Lemma 2.6, we obtain the desired distribution. □

In Figure 2.1, the directed graph with edges determined by ϕ and γ is a way to transform the graph of g. In the graph of g, nodes had either no parents or infinitely many, depending on whether the integer at the node was divisible by 3. An advantage in this new graph is that every node has precisely one parent and either one or two children. This means that the graph can more naturally be constructed iteratively.

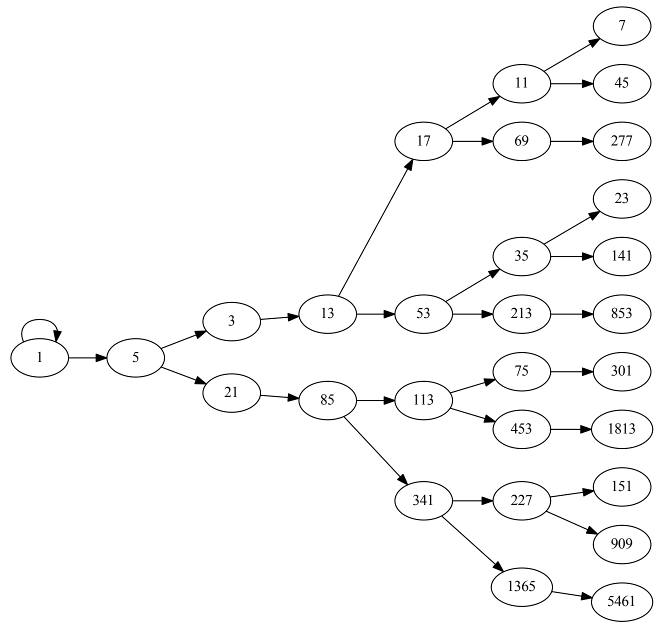

Figure 2.2: The Graph of ϕ and γ

2.2 Behavior

Throughout this section, $x \in \mathbb{O}^+$ unless otherwise specified. The sequence

$$\left\{\delta\left(g^i\left(x\right)\right)\right\}_{i=0}^{\infty} \tag{2.22}$$

is called the *behavioral sequence*, or more simply the *behavior*, of x. We shall use the shorthand notation $\Delta_i\left(x\right)$ to represent $\delta\left(g^i\left(x\right)\right)$. Since ran $\delta = \mathbb{Z}^+$, we shall commonly refer to sequences consisting of only positive integers as behaviors.

Notice that the parity sequence and behavioral sequence are decipherable from one another. Parsing the parity sequence by splitting on the left-hand side of each 1, where we are at an odd term in the orbit, we can identify behavior by the lengths of the chunks. For example, recall from Chapter 1 the parity sequence of 7

$$\underbrace{1}_{1}\quad\underbrace{1}_{1}\quad\underbrace{1\quad 0}_{2}\quad\underbrace{1\quad 0\quad 0}_{3}\quad\underbrace{1\quad 0\quad 0\quad 0}_{4}\quad\underbrace{1\quad 0}_{2}\quad\underbrace{1\quad 0}_{2}\quad 1\quad\ldots \tag{2.23}$$

and observe how the behavior is easily recognized. Although either sequence would suffice, behavior will help us express some structure in the Collatz graph more elegantly.

Behavior holds a wealth of information. Similar to Equation 1.12, the formula

$$g^k(x) = \frac{3^k x + \sum_{i=0}^{k-1} 3^{k-i-1} 2^{\sum_{j=0}^{i-1} \Delta_j(x)}}{2^{\sum_{i=0}^{k-1} \Delta_i(x)}} \tag{2.24}$$

can be obtained by translating a parity sequence into a behavioral sequence. The tail of $\{\Delta_i(x)\}_{i=0}^{\infty}$ indicates whether the orbit is convergent, divergent, or cyclic. Furthermore, the conditions can be stated more than one way. An orbit converges if and only if its behavior becomes a constant sequence since 1 is the only fixed point. Since our domain is bounded below, we could also say that it is necessary and sufficient for the behavior to have only finitely many 1's in it. Divergent and cyclic orbits can be distinguished by whether behavior itself becomes cyclic.

Similar to the definition of a parity set, for any countable behavior $\{b_i\}$ of length κ, the *behavioral set based on* $\{b_i\}$ is the set

$$\mathfrak{B}(\{b_i\}) = \left\{x \in \mathbb{O}^+ \mid \Delta_i(x) = b_i, i < \kappa\right\}, \tag{2.25}$$

i.e., the odd positive integers with initial behavior obeying $\{b_i\}$. Equivalently, we could say that for any $k < \kappa$, $j \leq k$ implies $\Delta_j(x) = b_j$ and $g^k(x) \in \mathfrak{B}(\{b_i\}_{i=k}^{\kappa})$. The latter description of behavioral sets will be useful for proving statements by induction on the length of behaviors. Note that the behavioral set based on the empty sequence is $\mathbb{O}^+$, because there are no restrictions placed on behavior. As parity sets in disguise, the reader is encouraged to look back at the theorems proven about the parity sequence and parity sets and translate them into the equivalent for behaviors and behavioral sets, respectively.

Similar to how parity sets partition $\mathbb{Z}^+$, behavioral sets generate a natural partitioning of the positive odd integers, based on just the fact that every odd integer has an infinite behavior. For any fixed finite behavior $\{\bar{b}_i\}$ and countable cardinal κ,

$$\mathfrak{B}\left(\{\bar{b}_i\}\right) = \bigcup_{\{b'_j\}\in X} \mathfrak{B}\left(\{\bar{b}_i\}^\frown \{b'_j\}\right), \quad X = \underbrace{\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+}_{\kappa}, \tag{2.26}$$

where $\{\bar{b}_i\}^\frown \{b'_j\}$ denotes concatenation of the two behaviors. For calculating which integers belong to behavioral sets, while adding terms to the end of a sequence is the more natural direction for extending, there is less guesswork when constructing the behavior in reverse order. That is, the procedure of

$$\begin{array}{cccc} \text{constructing} & \mathfrak{B}\left(b_{k-1}\right) & \text{from} & \mathbb{O}^+ \\ \text{constructing} & \mathfrak{B}\left(b_{k-2}, b_{k-1}\right) & \text{from} & \mathfrak{B}\left(b_{k-1}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \text{constructing} & \mathfrak{B}\left(b_0, \ldots, b_{k-1}\right) & \text{from} & \mathfrak{B}\left(b_1, \ldots, b_{k-1}\right) \end{array} \tag{2.27}$$

is both more efficient and will be useful in proofs.

Theorem 2.8 (Head-On Construction). *Suppose $\{b_i\}_{i=0}^{k-1}$ is a finite behavior and $\mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right) = [r \bmod 2^m]$ for some $r \in \mathbb{O}^+$ and $m \in \mathbb{Z}^+$. Then*

1. *letting* $r', r'' \in \{r, r+2^m, r+2^{m+1}\}$ *such that* $r' \equiv 5 \, mod \, 6$ *and* $r'' \equiv 1 \, mod \, 6$,

$$\mathfrak{B}(1, b_0, \ldots, b_{k-1}) = \left[\gamma(r') \; mod \; 2^{m+1}\right] \tag{2.28}$$

$$\mathfrak{B}(2, b_0, \ldots, b_{k-1}) = \left[\gamma(r'') \; mod \; 2^{m+2}\right] \tag{2.29}$$

2. *for every* $j \in \mathbb{N}$,

$$\mathfrak{B}(b_0 + 2j, \ldots, b_{k-1}) = \left[\phi^j(r) \; mod \; 2^{m+2j}\right]. \tag{2.30}$$

Proof.

1. Since $\gcd(2^m, 3) = 1$ and $r \in \mathbb{O}^+$, we are guaranteed $\{r, r+2^m, r+2^{m+1}\}$ contains unique elements which satisfy the desired residues for r' and r''. Let $i \in \mathbb{N}$ be such that $r' = 6i + 5$. As pointed out in the proof of Theorem 2.2,

$$g(\gamma(6i+5)) = g(4i+3) = f(\gamma(r')) = r', \tag{2.31}$$

and thus $\delta(\gamma(r')) = 1$. Since $r' \in [r \bmod 2^m] = \mathfrak{B}(b_0, \ldots, b_{k-1})$, we conclude from the definition of behavioral sets that $\gamma(r') \in \mathfrak{B}(1, b_0, \ldots, b_{k-1})$. A similar argument, *mutatis mutandis*, shows that $\gamma(r'') \in \mathfrak{B}(2, b_0, \ldots, b_{k-1})$.

2. From Theorem 2.1,

$$x \in \mathfrak{B}(b_0, \ldots, b_{k-1}) \Leftrightarrow \phi(x) \in \mathfrak{B}(b_0 + 2, \ldots, b_{k-1}), \quad x \in \mathbb{O}^+. \tag{2.32}$$

Given $\mathfrak{B}(b_0, \ldots, b_{k-1}) = [r \bmod 2^m]$, $\mathfrak{B}(b_0 + 2, \ldots, b_{k-1}) = [\phi(r) \bmod 2^{m+2}]$ is easily computed. Since b_0 was arbitrary, our inductive step was implicit.

□

The following theorem is describing the lattice of parity sets with binary tree structure in terms of behavioral sets.

Theorem 2.9. *Let $r \in \mathbb{O}$ and $m \in \mathbb{Z}^+$. Then*

1. *if $[r \text{ mod } 2^m] = \mathfrak{B}(b_0, \ldots, b_{k-1})$ then one of the child nodes, $[r \text{ mod } 2^{m+1}]$ and $[r + 2^m \text{ mod } 2^{m+1}]$, is the behavioral set $\mathfrak{B}(b_0, \ldots, b_{k-1}, 1)$ and the other is $\bigcup_{b'=2}^{\infty} \mathfrak{B}(b_0, \ldots, b_{k-1}, b')$;*

2. *if $[r \text{ mod } 2^m]$ is not a behavioral set and $\mathfrak{B}(b_0, \ldots, b_{k-1}) = \left[r' \text{ mod } 2^{m'}\right]$ is the highest level behavioral set such that $\mathfrak{B}(b_0, \ldots, b_{k-1}) \supset [r \text{ mod } 2^m]$ then one of the child nodes, $[r \text{ mod } 2^{m+1}]$ and $[r + 2^m \text{ mod } 2^{m+1}]$, is the behavioral set $\mathfrak{B}(b_0, \ldots, b_{k-1}, m - m')$ and the other is $\bigcup_{b'=m-m'+1}^{\infty} \mathfrak{B}(b_0, \ldots, b_{k-1}, b')$;*

Proof. This follows immediately from Theorem 1.3. □

Theorem 2.10. *For every finite behavior $\{b_i\}_{i=0}^{k-1}$*

1. *$\mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right) = [r \text{ mod } 2^m]$, for some $r \in \mathbb{O}^+$ and $m = 1 + \sum_{i=0}^{k-1} b_i$;*

2. *for any $h, j \in \mathbb{N}$ with $j \leq k$,*

$$g^j(2^m h + r) = 2^{m-n} \cdot 3^j h + g^j(r), \quad n = \sum_{i=0}^{j-1} b_i, \tag{2.33}$$

   i.e. g^j is a one-to-one affine mapping of the behavioral set $\mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right)$ onto $[g^j(r) \text{ mod } 2^{m-n} \cdot 3^j]$;

3. *$g^{k+1}\left(\mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right)\right) = ran\ g$.*

Proof.

1. We shall use induction on the length of behaviors. Recall that the behavioral set based on the empty sequence is $\mathbb{O}^+$. Using the standard convention that the sum of the empty sequence is 0, this agrees with our claim. Now suppose the statement holds for all behaviors of length k and consider the behavior $\{b_i\}_{i=0}^k$ of length $k+1$. By our inductive hypothesis, $\mathfrak{B}\left(\{b_i\}_{i=1}^k\right) = [r \bmod 2^m]$ for some $r \in \mathbb{O}^+$ and $m = 1 + \sum_{i=1}^k b_i$. The inductive step follows from Theorem 2.8, when we construct $\mathfrak{B}\left(\{b_i\}_{i=0}^k\right)$ from $\mathfrak{B}\left(\{b_i\}_{i=1}^k\right)$.

2. Proof follows by simple induction on j up to $k-1$ via the observation that

$$g^0 = (2^m h + r) = 2^m h + g^0(r) \tag{2.34}$$

and when $0 \le j < k$,

$$\begin{aligned}
g^{j+1}(2^m h + r) &= g\left(g^j(2^m h + r)\right) && (2.35)\\
&= g\left(2^{m-n} \cdot 3^j h + g^j(r)\right), \quad n = \sum_{i=0}^{j-1} b_i && (2.36)\\
&= \frac{3\left(2^{m-n} \cdot 3^j h + g^j(r)\right) + 1}{2^{b_j}}, \quad n = \sum_{i=0}^{j-1} b_i && (2.37)\\
&= 2^{m-n} \cdot 3^{j+1} h + \frac{3\, g^j(r) + 1}{2^{b_j}}, \quad n = \sum_{i=0}^{j} b_i && (2.38)\\
&= 2^{m-n} \cdot 3^{j+1} h + g^{j+1}(r), \quad n = \sum_{i=0}^{j} b_i. && (2.39)
\end{aligned}$$

3. Suppose $y \in \operatorname{ran} g$. By Lemma 2.6, the set $\left\{\gamma(y), \phi(\gamma(y)), \ldots, \phi^{3^k-1}(\gamma(y))\right\}$ is a complete set of residues modulo 3^k. So we can choose $\ell \in \mathbb{N}$ to satisfy $\phi^\ell(\gamma(y)) \equiv r \bmod 3^k$. Then, by part (2), there exists $x \in \mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right)$ such that $g^k(x) = \phi^\ell(\gamma(y))$, and hence $g^{k+1}(x) = g\left(\phi^\ell(\gamma(y))\right) = y$.

$\square$

Corollary 2.11. *Let $x \in \mathbb{O}^+$ and $\{b_i(x)\}_{i=0}^{\infty}$ be the behavior of x. Then*

1. $\mathfrak{B}\left(\{b_i(x)\}_{i=0}^{k-1}\right) = \left[x \text{ } mod \text{ } 2^{1+\sum_{i=0}^{k-1} b_i(x)}\right]$;

2. g^k *is an affine mapping from* $\mathfrak{B}\left(\{b_i(x)\}_{i=0}^{k-1}\right)$ *onto* $\left[\, g^k(x) \text{ } mod \text{ } 2 \cdot 3^k \,\right]$.

Theorem 2.10 highlights some benefits of studying the Syracuse function and behavior over the Collatz function and the parity sequence. As seen in part (2), the number of iterations now matching how many times we have multiplied by 3 gives us some regularity with respect to the image. For example,

$$f(\mathfrak{P}(0)) = \mathbb{Z}^+ \qquad \text{and} \qquad f(\mathfrak{P}(1)) = [\,2 \bmod 3\,], \tag{2.40}$$

whereas $g^k\left(\mathfrak{B}\left(\{\bar{b}_i\}_{i=0}^{k-1}\right)\right)$ is a congruence class modulo $2 \cdot 3^k$ for any behavior of length k. So we can go further than Corollary 2.11 with sampling, identifying not only level-k behavioral sets from $\{g^i(x)\}_{i=0}^{k-1}$, but also their images under g^k.

$g^2(\mathfrak{B}(1,1)) = [\,17 \bmod 18\,]$	$g^2(\mathfrak{B}(2,1)) = [\,11 \bmod 18\,]$
$g^2(\mathfrak{B}(1,2)) = [\,13 \bmod 18\,]$	$g^2(\mathfrak{B}(2,2)) = [\,1 \bmod 18\,]$
$g^2(\mathfrak{B}(1,3)) = [\,11 \bmod 18\,]$	$g^2(\mathfrak{B}(2,3)) = [\,5 \bmod 18\,]$
$g^2(\mathfrak{B}(1,4)) = [\,1 \bmod 18\,]$	$g^2(\mathfrak{B}(2,4)) = [\,7 \bmod 18\,]$
$g^2(\mathfrak{B}(1,5)) = [\,5 \bmod 18\,]$	$g^2(\mathfrak{B}(2,5)) = [\,17 \bmod 18\,]$
$g^2(\mathfrak{B}(1,6)) = [\,7 \bmod 18\,]$	$g^2(\mathfrak{B}(2,6)) = [\,13 \bmod 18\,]$
$g^2(\mathfrak{B}(1,7)) = [\,17 \bmod 18\,]$	$g^2(\mathfrak{B}(2,7)) = [\,11 \bmod 18\,]$

Table 2.1: Level-2 Behavioral Set Images

When one computes which congruence classes represent level-k behavioral sets and their images under g^k, the residues of the images are noticeably cyclic as the last term in the behavior is repeatedly incremented, as seen in Table 2.1. Observe from Equation 2.24 that having the first $k-1$ terms of the behavior fixed and incrementing the last term simply means incrementing the power of 2 in the denominator. So, from Theorem 2.10, knowledge of expected modulii for our images allows us to explain such cycles. Over $\mathbb{Z}/3^k\mathbb{Z}$, the fact that the multiplicative inverse of 2 is $\frac{3^k+1}{2}$ is trivial. Since the Syracuse function is defined on $\mathbb{O}^+$, if

$$\mathfrak{B}\left(\bar{b}_0,\ldots,\bar{b}_{k-1}\right) = \left[r \bmod 2^{1+\sum_{i=0}^{k-1}\bar{b}_i}\right] \tag{2.41}$$

then this trivial solution may be translated as

$$g^k\left(\mathfrak{B}\left(\bar{b}_0,\ldots,\bar{b}_{k-1}+1\right)\right) = \left[\left(\frac{9^{\lceil\frac{k}{2}\rceil}+1}{2}\right) g^k\left(r\right) \bmod 2\cdot 3^k\right]. \tag{2.42}$$

Since direct and recursive formulas each have their benefits, note that instead of using using $s_k = \frac{9^{\lceil\frac{k}{2}\rceil}+1}{2}$, the scalar could be defined using the recursive definition

$$s_1 = s_2 = 5, \quad s_{k+2} = 9s_k - 4. \tag{2.43}$$

Since a behavior is a compacted representation of a parity sequence, we have a special case of Corollary 1.4.

Corollary 2.12. *If $x, y, k \in \mathbb{O}^+$ with $x < y < 2^k$ then the behaviors of x and y must differ within k iterations of g.*

Knowing that the orbits of integers in one behavioral set cover ran g, we can now say that the existence of even one counterexample to Collatz conjecture implies infinitely many exist with any given finite initial behavior.

Corollary 2.13. *If a counterexample to the Collatz conjecture exists then every finite-level behavioral set contains infinitely many.*

Proof. Since a level - k behavioral consists of infinitely many level - $(k+1)$ behavioral sets, the result follows from part (3) of Theorem 2.10. $\square$

Theorem 2.14. *For any finite behavior $\{b_i\}_{i=0}^{k-1}$, define the function*

$$\Phi_{\{b\}_{i=0}^{k-1}} : \mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right) \to \mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right)$$

recursively by

$$\Phi(x) = 4x + 1; \tag{2.44}$$

$$\Phi_{\{b\}_{i=j}^{k-1}}(x) = \Phi^{\lfloor \frac{b_j - 1}{2} \rfloor}\left(\gamma\left(\Phi^3_{\{b\}_{i=j+1}^{k-1}}(g(x))\right)\right), \quad 0 \le j < k. \tag{2.45}$$

Then

1. $\Phi^{\ell}_{\{b\}_{i=0}^{k-1}}(x) = 4^{3^k \ell} x + \left(\frac{4^{3^k \ell} - 1}{3^{k+1}}\right) \sum_{i=0}^{k} \left(2^{\sum_{j=0}^{k-i-1} b_j} 3^i\right);$

2. *for any $x, y \in \mathfrak{B}\left(\{b\}_{i=0}^{k-1}\right)$ with $x \le y$, $g^{k+1}(x) = g^{k+1}(y)$ iff $y = \Phi^{\ell}_{\{b\}_{i=0}^{k-1}}(x)$ for some $\ell \in \mathbb{N}$, in which case $\Delta_k(x) = \Delta_k(x) + 2 \cdot 3^k \ell$;*

3. *for every $y \in$ ran g,*

$$\left\{x \in \mathfrak{B}\left(\{b\}_{i=0}^{k-1}\right) \mid g^{k+1}(x) = y\right\} \tag{2.46}$$

is of the form $\left\{\Phi^{\ell}_{\{b\}_{i=0}^{k-1}}(w) \mid \ell \in \mathbb{N}\right\}$ for a unique $w \in \mathfrak{B}\left(\{b\}_{i=0}^{k-1}\right)$.

Proof.

1. We shall prove our claim by induction on k and ℓ. For $k = 0$, the reader can verify that our formula simplifies to agree with Theorem 2.1. For any

finite behavior, if $\ell = 0$ then we obtain the identity function as desired. Next, fix $\ell = 1$ and suppose the statement holds for all behaviors of length k. To simplify our inductive step, let

$$\eta_{m,n} = \frac{4^m - 1}{3^n} \tag{2.47}$$

and

$$\zeta_{\{b_i\}_{i=0}^{k-1}} = \sum_{i=0}^{k} 2^{\sum_{j=0}^{k-i-1} b_j} 3^i. \tag{2.48}$$

In terms of these functions, for our inductive step we need to show

$$\Phi_{\{b\}_{i=0}^{k}}(x) = 4^{3^{k+1}} x + \eta_{3^{k+1},k+2} \cdot \zeta_{\{b_i\}_{i=0}^{k}}. \tag{2.49}$$

$$\begin{aligned}
&\Phi_{\{b\}_{i=0}^{k}}(x) && (2.50)\\
&= \Phi^{\lfloor \frac{b_0-1}{2} \rfloor}\left(\gamma\left(\Phi^3_{\{b_i\}_{i=1}^{k}}(g(x))\right)\right) && (2.51)\\
&= \Phi^{\lfloor \frac{b_0-1}{2} \rfloor}\left(\gamma\left(4^{3^{k+1}} g(x) + \left(4^{2\cdot 3^k} + 4^{3^k} + 1\right) \eta_{3^k,k+1} \cdot \zeta_{\{b_i\}_{i=1}^{k}}\right)\right) && (2.52)\\
&= \Phi^{\lfloor \frac{b_0-1}{2} \rfloor}\left(\frac{2^a\left(4^{3^{k+1}} g(x) + \eta_{3^{k+1},k+1} \cdot \zeta_{\{b_i\}_{i=1}^{k}}\right) - 1}{3}\right), \quad a = 1,2 && (2.53)\\
&= 4^{\lfloor \frac{b_0-1}{2} \rfloor}\left(\frac{2^a\left(4^{3^{k+1}} g(x) + \eta_{3^{k+1},k+1} \cdot \zeta_{\{b_i\}_{i=1}^{k}}\right) - 1}{3}\right) + \frac{4^{\lfloor \frac{b_0-1}{2} \rfloor} - 1}{3} && (2.54)\\
&= \frac{2^{b_0} 4^{3^{k+1}} g(x) - 1}{3} + 2^{b_0} \eta_{3^{k+1},k+2} \cdot \zeta_{\{b_i\}_{i=1}^{k}} && (2.55)\\
&= 4^{3^{k+1}} x + \frac{4^{3^{k+1}} - 1}{3} + 2^{b_0} \eta_{3^{k+1},k+2} \cdot \zeta_{\{b_i\}_{i=1}^{k}} && (2.56)\\
&= 4^{3^{k+1}} x + \eta_{k+1,k+2}\left(\frac{4^{3^{k+1}} - 1}{3\eta_{3^{k+1},k+2}} + 2^{b_0} \cdot \zeta_{\{b_i\}_{i=1}^{k}}\right) && (2.57)\\
&= 4^{3^{k+1}} x + \eta_{3^{k+1},k+2}\left(3^{k+1} + 2^{b_0} \cdot \zeta_{\{b_i\}_{i=1}^{k}}\right) && (2.58)\\
&= 4^{3^{k+1}} x + \eta_{3^{k+1},k+2} \cdot \zeta_{\{b_i\}_{i=0}^{k}}. && (2.59)
\end{aligned}$$

Finally,

$$\Phi^{\ell+1}_{\{b\}_{i=0}^{k-1}}(x) = \Phi_{\{b\}_{i=0}^{k-1}} \circ \Phi^{\ell}_{\{b\}_{i=0}^{k-1}}(x) \tag{2.60}$$

$$= \Phi_{\{b\}_{i=0}^{k-1}}\left(4^{3^k\ell}x + \eta_{3^k\ell,k+1} \cdot \zeta_{\{b_i\}_{i=0}^{k-1}}\right) \tag{2.61}$$

$$= 4^{3^k}\left(4^{3^k\ell}x + \eta_{3^k\ell,k+1} \cdot \zeta_{\{b_i\}_{i=0}^{k-1}}\right) + \eta_{3^k,k+1} \cdot \zeta_{\{b_i\}_{i=0}^{k-1}} \tag{2.62}$$

$$= 4^{3^k(\ell+1)}x + \left(4^{3^k}\eta_{3^k\ell,k+1} + \eta_{3^k,k+1}\right)\zeta_{\{b_i\}_{i=0}^{k-1}} \tag{2.63}$$

$$= 4^{3^k(\ell+1)}x + \eta_{3^k(\ell+1),k+1} \cdot \zeta_{\{b_i\}_{i=0}^{k-1}} \tag{2.64}$$

completes our proof by induction.

□

The proofs of parts 2.14.2 and 2.14.3 are left to the reader. The recursive definition of $\Phi_{\{b\}_{i=0}^{k-1}}(x)$ can be used to compute how the behavior is affected by looking at the sequence of functions in the composition one at a time. A key to seeing that

$$\left\{\Phi^{\ell}_{\{b\}_{i=0}^{k-1}}(x)\right\}_{\ell=0}^{\infty} = \left\{y \in \mathfrak{B}\left(\{b_i(x)\}_{i=0}^{k-1}\right) \mid g^{k+1}(x) = g^{k+1}(y), x \leq y\right\}, \tag{2.65}$$

is using Lemma 2.6. Recall that it says

$$\Phi^{\ell}\left(g^k(x)\right) \equiv g^k(x) \mod 3^k \Leftrightarrow 3^k \mid \ell. \tag{2.66}$$

This congruence is what allows us to apply the same composition of γ and ϕ which we would use to obtain x from $g^k(x)$.

Corollary 2.15. *For every $r \in$ ran g and $k \in \mathbb{N}$, the congruence class $\left[r \mod 2 \cdot 3^k\right]$ is the image under g^k of infinitely many level-k behavioral sets.*

Proof. From Theorem 2.10, for any finite behavior $\{b_k\}_{i=0}^{k-1}$, we have

$$g^k\left(\mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right)\right) = \left[r \bmod 2 \cdot 3^k\right], \quad r \in \operatorname{ran} g \tag{2.67}$$

and

$$g^{k+1}\left(\mathfrak{B}\left(\{b_i\}_{i=0}^{k-1}\right)\right) = \operatorname{ran} g. \tag{2.68}$$

$\square$

Chapter 3: Generalizations

3.1 Alternate Domains

The simplest domain expansion for f is to all of $\mathbb{Z}$. Soon after one starts in this direction, it takes hardly any time at all to discover the non-trivial cycles shown in Figure 3.1. It has been conjectured that these two cycles, the fixed point of 0 and the cycle containing 1 are the only cycles which exist for this extension. Another domain extension considered by several authors is to the 2-adic integers [4].

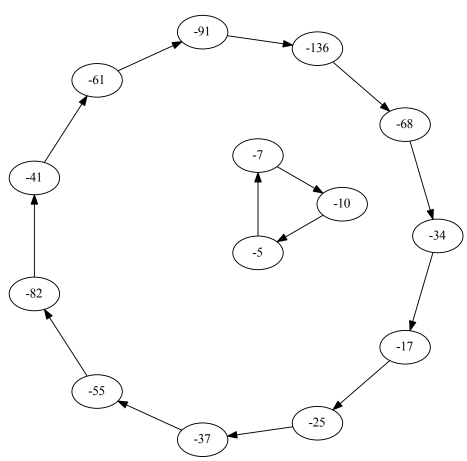

Figure 3.1: Cycles of the Collatz Function Extended to $\mathbb{Z}$

3.2 Alternate Linear Subfunctions

Our focus is on the generalization of the Collatz function which frees the coefficients in the odd case to vary over the positive integers:

$$f_{m,b}(x) = \begin{cases} \frac{x}{2}, & x \text{ even}, \\ \frac{mx+b}{2}, & x \text{ odd}. \end{cases} \tag{3.1}$$

Note that we can restrict our attention to when $m, b \in \mathbb{O}^+$. Since we are dividing by 2 in the odd case, if one is odd and the other is even then the range is no longer a subset of the integers and we can't iterate. Even if we were to use the original $mx + b$ subfunction, the odd case would be inescapable. Thus every orbit would become strictly increasing, which is quite uninteresting. If both terms are even then we are simply being wasteful, because the common power of 2 will immediately be removed in subsequent iterations. Since we have left the even case alone, the functions $\delta_{m,b}$ and $g_{m,b}$ are defined to coincide with $f_{m,b}$ similarly to how they were defined in Chapter 2 for the original Collatz function. Recall the heuristic argument from Chapter 2 which showed that g - orbits are expected to decrease. Observe that this argument generalized for any $m \in \mathbb{O}^+$ gives us an expected value of $\frac{m}{4}$, and thus $m = 3$ is the only case in which we expect the orbit to decrease over time.

Most of our previous theorems about the Syracuse function and behavioral sets generalize nicely. Recall Theorem 2.1 which allowed us to identify the inverse image of an element as a single ϕ - orbit. This statement still holds for $g_{m,b}$ if we generalize ϕ.

Theorem 3.1. *Let $m, b \in \mathbb{O}^+$. Define m' and b' by dividing m and b, respectively,* *by* $gcd(m,b)$*. Then for*

$$\phi_{m,b}(x) = 2^k x + \left(\frac{2^k - 1}{m'}\right) b', \quad k = min\left\{i \in \mathbb{Z}^+ \mid 2^i \equiv 1 \, mod \, m'\right\} \tag{3.2}$$

we have the following:

1. *ran* $\phi_{m,b} = \left[2^k + \left(\frac{2^k-1}{m'}\right) b' \;\; mod \; 2^{k+1}\right]$*;*

2. *for every* $\ell \in \mathbb{N}$*,* $\phi^{\ell}_{m,b}(x) = 2^{k\ell}x + \left(\frac{2^{k\ell}-1}{m'}\right) b'$*;*

3. $g_{m,b}(x) = g_{m,b}(y)$ *with* $x \leq y$ *if and only if there exists* $\ell \in \mathbb{N}$ *such that* $y = \phi^{\ell}_{m,b}(x)$*;*

4. *for any* $y \in ran \; g_{m,b}$*,* $g^{-1}_{m,b}(y) = \left\{\phi^k(w) \mid k \in \mathbb{N}\right\}$ *for a unique* $w \in \mathbb{O}^+$*.*

Proof.

1. Proof is trivial.

2. For $\ell = 0$, we obtain the identity function as desired. Supposing the statement holds for some fixed ℓ, we have

$$\begin{aligned}
\phi^{\ell+1}_{m,b}(x) &= \phi_{m,b} \circ \phi^{\ell}_{m,b}(x) & (3.3)\\
&= \phi_{m,b}\left(2^{k\ell}x + \left(\frac{2^{k\ell}-1}{m'}\right) b'\right) & (3.4)\\
&= 2^k\left(2^{k\ell}x + \left(\frac{2^{k\ell}-1}{m'}\right) b'\right) + \left(\frac{2^k-1}{m'}\right) b' & (3.5)\\
&= 2^{k\ell+k}x + \frac{\left(2^{k\ell+k} - 2^k\right) b'}{m'} + \frac{\left(2^k-1\right) b'}{m'} & (3.6)\\
&= 2^{k(\ell+1)}x + \left(\frac{2^{k(\ell+1)}-1}{m'}\right) b', & (3.7)
\end{aligned}$$

completing our proof by induction.

3. The statement is trivial for $x = y$, so suppose $g_{m,b}(x) = g_{m,b}(y)$ with $x < y$.

That is, for some $k < \ell$,

$$\frac{mx+b}{2^k} = \frac{my+b}{2^\ell}. \tag{3.8}$$

Solving for y, we obtain

$$\begin{aligned} y &= 2^{\ell-k}x + \left(\frac{2^{\ell-k}-1}{m}\right) b && (3.9) \\ &= 2^{\ell-k}x + \left(\frac{2^{\ell-k}-1}{m'}\right) b'. && (3.10) \end{aligned}$$

Since $y \in \mathbb{O}^+$, we must have

$$2^{\ell-k} \equiv 1 \bmod m'. \tag{3.11}$$

Since $m' \in \mathbb{O}^+$, positive integer multiples of the order of the cyclic subgroup of $(\mathbb{Z}/m'\mathbb{Z})^\times$ generated by 2 will be the values of $\ell - k$ satisfying this congruence. Applying part (1), this says y belongs to the $\phi_{m,b}$-orbit of x.

Now suppose $y = \phi_{m,b}(x)$. Then

$$\begin{aligned} f_{m,b}(y) &= f_{m,b}(\phi_{m,b}) && (3.12) \\ &= f_{m,b}\left(2^k x + \left(\frac{2^k-1}{m}\right) b\right) && (3.13) \\ &= \frac{m}{2}\left(2^k x + \left(\frac{2^k-1}{m}\right) b\right) + \frac{b}{2} && (3.14) \\ &= 2^{k-1}mx + \left(2^k - 1\right)\frac{b}{2} + \frac{b}{2} && (3.15) \\ &= 2^{k-1}(mx+b). && (3.16) \end{aligned}$$

From this, we can see that

$$f_{m,b}(x) = f_{m,b}^{k+1}(\phi_{m,b}(x)) \tag{3.17}$$

and no odd terms were passed in the $f_{m,b}$-orbit of $\phi_{m,b}(x)$. Hence,

$$g_{m,b}(x) = g_{m,b}(\phi_{m,b}(x)). \tag{3.18}$$

Since x was arbitrary, this argument can be applied iteratively ℓ times to complete the proof.

4. Choose $w = \min g_{m,b}^{-1}(y)$.

$\square$

The prime factorizations of m and b play a key role in the connectivity of $f_{m,b}$. If there exists any prime p such that $p \mid b$ and $p \nmid m$ then the graph of $f_{m,b}$ is not weakly connected. Furthermore, we can use these prime factorizations to put a lower bound on the number of weak components.

Theorem 3.2. *Let $b, d, m \in \mathbb{O}^+$. Then*

1. *the graph of $f_{m,bd}\,|_{d\mathbb{Z}^+}$ is isomorphic to the graph of $f_{m,b}$;*

2. *the graph of $g_{m,bd}\,|_{d\mathbb{Z}^+}$ is isomorphic to the graph of $g_{m,b}$;*

3. *if $d \mid m$ then the graphs of $f_{m,b}$ and $f_{m,bd}$ (equivalently $g_{m,b}$ and $g_{m,bd}$) have the same number of weak components;*

4. *if $d = 3^{\alpha_1} 5^{\alpha_2} \cdots p_n^{\alpha_n}$ divides b but is coprime to m then the graph of $f_{m,b}$ (equivalently $g_{m,b}$) has at least*

$$\prod_{i=1}^{n} (\alpha_i + 1) \tag{3.19}$$

weak components.

Proof.

1. This holds simply because

$$\frac{dx}{2} = d\left(\frac{x}{2}\right) \tag{3.20}$$

and

$$m\left(dx\right) + bd = d\left(mx + b\right), \tag{3.21}$$

showing

$$f_{m,bd}\left(dx\right) = df_{m,b}\left(x\right). \tag{3.22}$$

This implies that $x \mapsto dx$ gives us an isomorphism from the graph of $f_{m,b}$ to the graph of $f_{m,bd}\mid_{d\mathbb{Z}^+}$.

2. This follows from part (1).

3. Applying part (1), it suffices to show that every $f_{m,b}$-orbit contains an integer divisible by d. By the fundamental theorem of arithmetic, we are guaranteed to apply the subfunction of the odd case, from which we clearly have d dividing $mx + bd$.

4. For any such d, the reader can check that

$$d \mid x \Leftrightarrow d \mid f_{m,b}\left(x\right). \tag{3.23}$$

Therefore, for every $x \in \mathbb{Z}^+$, we have

$$\gcd\left(d, f^k_{m,b}\left(x\right)\right) = \gcd\left(d, f^\ell_{m,b}\left(x\right)\right), \quad \forall k, \ell \in \mathbb{N}. \tag{3.24}$$

Hence, the orbits of distinct divisors of d are disjoint.

□

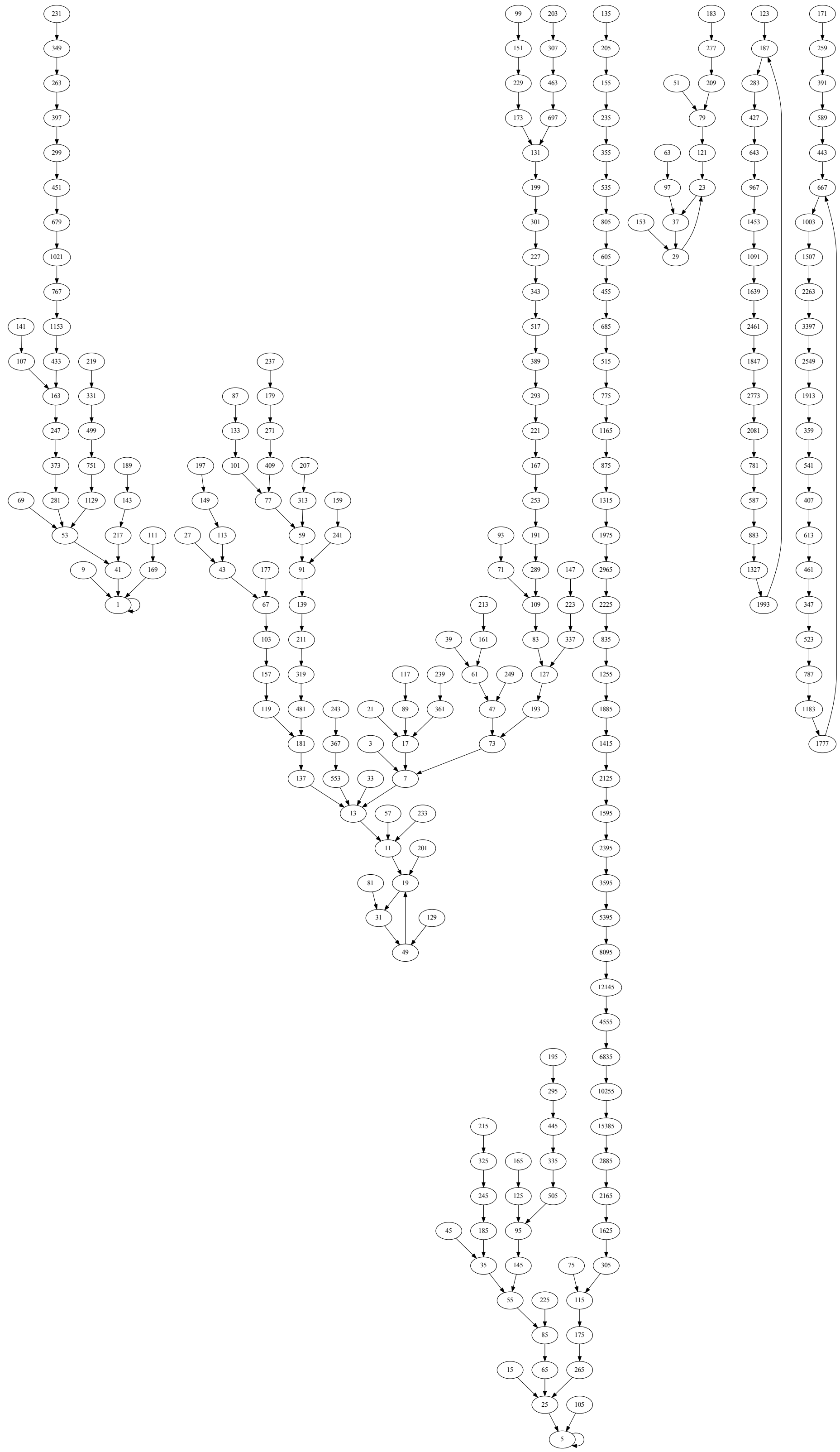

Figure 3.2: The Graph of $f_{3,5}$

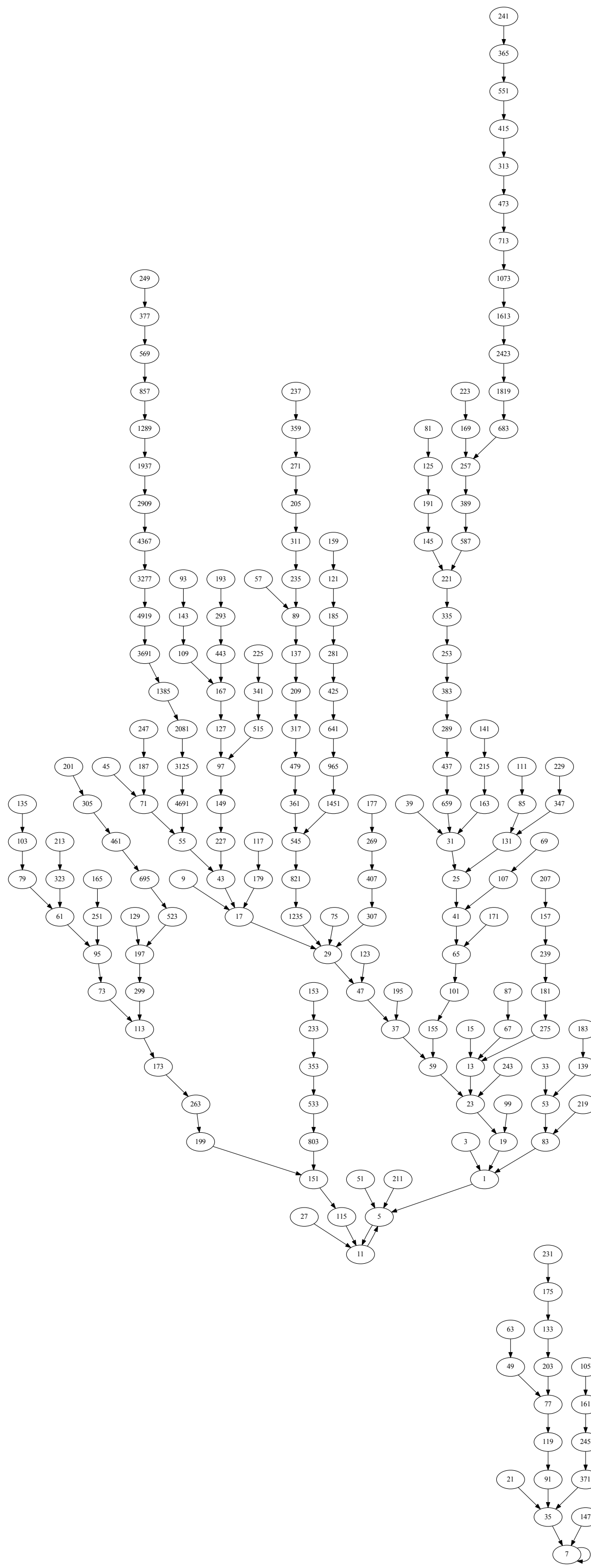

Figure 3.3: The Graph of $f_{3,7}$

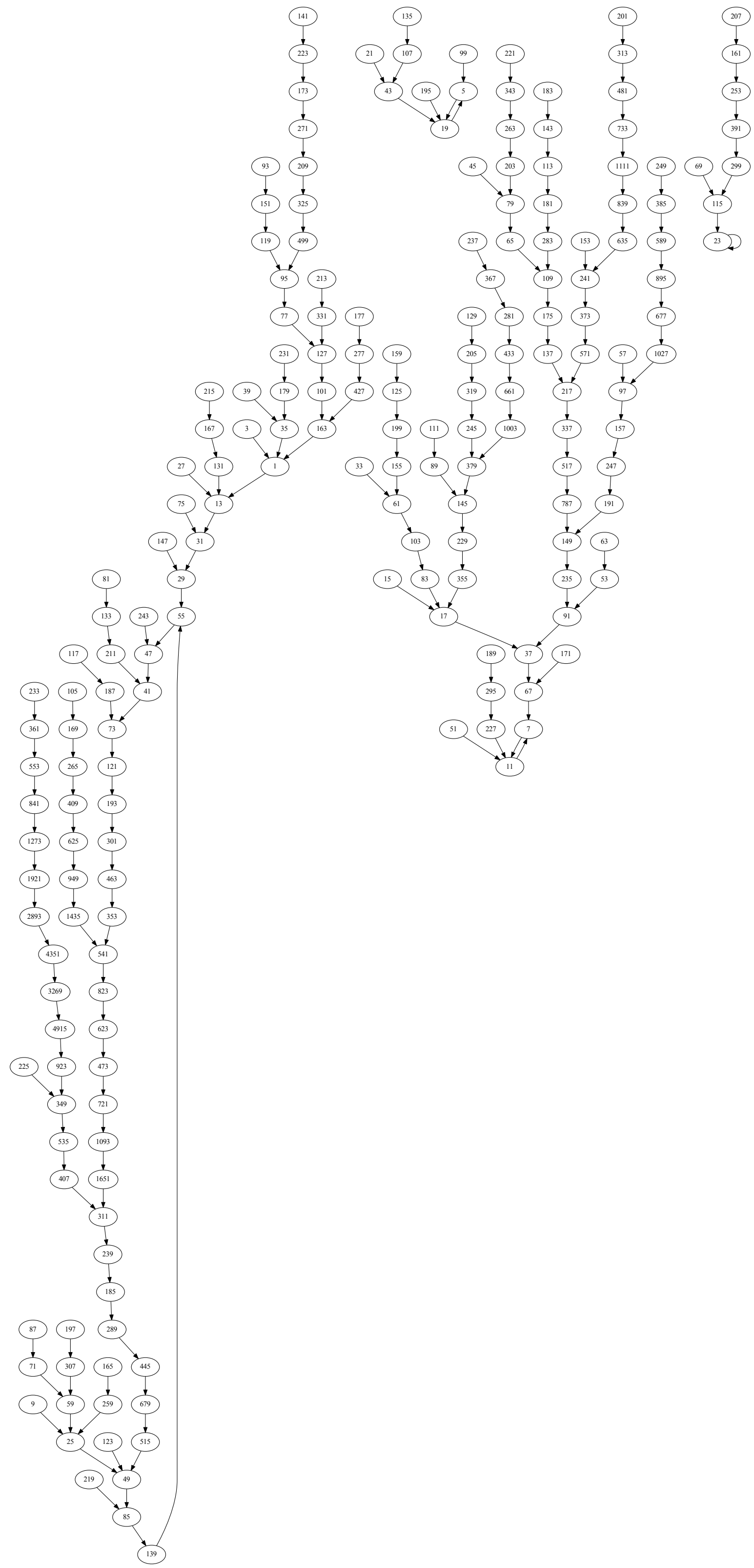

Figure 3.4: The Graph of $f_{3,23}$

Chapter 4: Conclusion

To no surprise, the regularities uncovered are matched with new observations which appear to be just as chaotic as the original Collatz function orbits. An example not touched upon yet is the number of iterations it takes for a γ - orbit to terminate. Since dom $\gamma = [1 \text{ mod } 6] \cup [5 \text{ mod } 6]$, if a γ - orbit reaches an integer divisible by 3 then to continue traversing in the opposite direction of the Collatz function we must apply ϕ. Since ϕ gives us an increase by a factor of 4, the following conjecture is similar to Proposition 1.1.

Conjecture 1. *The Collatz conjecture holds if and only if the γ - orbit terminates for every integer $x > 1$.*

Behavioral sets provide us with valuable insight to the "global" character of the Syracuse function. Regularities such as an expected modulus for the image under g^k of a level - k behavioral set give us reason to focus on the Syracuse function over the Collatz function. Combining Lemma 2.6 and Theorem 2.10, for any counterexample x to the Collatz conjecture and any congruence class of the form $[r \text{ mod } 2^m \cdot 3^n]$, we can find an integer $y \equiv r \text{ mod } 2^m \cdot 3^n$ such that the g - orbit of y contains x. The contrapositive of this statement is that the existence of an arithmetic sequence with step size $2^m \cdot 3^n$ for which all of the f - orbits converge implies the Collatz conjecture holds.

From Chapter 3, the generalization of the Collatz function and theorems about connectivity suggest broadening the Collatz conjecture.

Conjecture 2 (Extended Collatz)**.** *The graph of $f_{m,b}$ is weakly connected if and only if $m = 3$ and $b = 3^k$ for some $k \in \mathbb{N}$.*

We have already shown that the graph of $f_{3,b}$ is either never weakly connected or it is weakly connected precisely when b is a power of 3, determined by the answer to the Collatz conjecture.

While lower bounds were established for the number of weak components in the graph of the function $f_{m,b}$ whenever b has a factor which doesn't divide m, the question of precisely how many there are is still open. If we ignore efficiency, do there exist any finite procedures which allow us to determine the exact number of weak components? If there are only finitely many, determining bounds on the integers contained in cycles would suffice.

Conjecture 3. *For every $b \in \mathbb{O}^+$, the graph of $f_{3,b}$ contains only finitely many weak components. Moreover, all weak components contain cycles.*

The probabilistic argument using Equation 2.4 to suggest that orbits of the Collatz function are expected to decrease over time can be applied to all functions in this class. Because our domain is bounded below, perhaps this "collapsing" of orbits means we can determine a bound on the set of integers belonging to cycles. Because this has not yet been established, counts for the number of cycles were not included in this paper. However, computer trials did tend to show that cycles were revealed early and then no further cycles were seen as far as exhausted.

We can also slightly modify Equation 2.4 to conjecture that there are divergent orbits whenever $m \geq 5$. In this case, since the domain is not bounded above, we are afforded more freedom for orbits to grow without colliding.

Conjecture 4. *For every odd* $m > 3$*, the graph generated by* $f_{m,b}$ *contains infinitely many weak components based on divergent orbits.*

Unfortunately, there is still no way known to even conclude that an orbit will diverge. This is one future focus, along with more research into our function $g_{m,b}$. It is easy to check that behavioral sets are easily adjusted if we view congruence classes as arithmetic sequences. The step sizes for these new behavioral sets and their images will remain the same as our modulii computed in Theorem 2.10.

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