# ABSTRACT

Title of dissertation: SURFACE TENSION FREE BOUNDARY PROBLEMS: FORMULATION, OPTIMAL CONTROL AND NUMERICS

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The goal of this work is to treat the formulation, optimal control and numerical analysis of free boundary problems with surface tension effects. From a formulation point of view, we introduce a (dimension independent) abstract framework which captures the essential behavior of free boundary problems with surface tension effects. We then apply this framework to two scenarios. The first is where the underlying bulk system is governed by the Laplacian with non-homogeneous essential boundary condition, and the second is modeled by the Stokes equations with slip and no-slip boundary conditions. We do not impose a fixed contact angle between the free surface and any fixed part of the boundary.

Although the formulation and numerics involving the Laplacian was available in the literature, the Stokes free boundary problem in $\mathbb{R}^n$ is novel. To obtain this last result we also had to prove the existence and uniqueness in Sobolev spaces for the pure slip problem for domains of type $C^{1,\epsilon}$. This is a significant improvement over the current best result involving $C^{1,1}$ domains.

The results from the abstract formulation also carry over to the optimal control aspect. We obtain differentiability conditions which guarantee existence and (local) uniqueness of a minimizer to well-behaved cost functions. In the Laplacian case we go beyond the theoretical results and give precise second-order sufficient conditions for the (local) uniqueness of a minimizer for cost functions of the tracking type. The contribution in this area is significant in the sense that sufficient conditions are usually only assumed to be true, while we actually show that it indeed holds for our specific problem.

The last piece of this work is the numerical treatment of the free boundary optimal control problem based on the Laplace equation. We are able to prove optimal convergence results using the finite element method. Moreover, we construct experiments to study the behavior of various metrics associated with the optimization problem.

# SURFACE TENSION FREE BOUNDARY PROBLEMS: FORMULATION, OPTIMAL CONTROL AND NUMERICS

by

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# Foreword

"We can't let the perfect be the enemy of the absolutely necessary."

– (a variation on Voltaire)

Dedication

to $S^4$.

# Table of Contents

# List of Tables

# List of Figures

# List of Abbreviations

| | |
|---|---|
| FBP | Free Boundary Problem |
| MFBP | Model (Laplace) Free Boundary Problem |
| OC-FBP | Optimal Control of a Free Boundary Problem |
| PDE | Partial Differential Equation |
| SFBP | Stokes Free Boundary Problem |

# Part I

# Surface Tension Framework

# Chapter 1

# The Surface Tension Framework

## 1.1 Introduction

Free boundary problems (FBPs) are challenging due to their highly nonlinear nature. Besides the state variables, the domain is also an unknown. FBPs find a wide range of applications from phase separation (Stefan problem, Cahn-Hilliard), shape optimization (minimal surface area), optimal control problems with state constraints, fluid dynamics (flow in porous media), crystal growth, biomembranes, electrowetting on dielectric, and to finance. For many of these problems there is a close interplay between the surface tension and the curvature of the interface [WSN09, WBN10]. There are several methodologies to formulate FBPs depending on the role of the free boundary. We deal with a sharp interface method, i.e. one where the surface of moving interface can be parametrized. Alternative approaches to treat FBPs are the level set method and the diffuse interface method [DDE05, BH11].

In the context of this dissertation a FBP with surface tension effect is a nonlinear coupled system composed of a bulk PDE, e.g. the Laplacian or Stokes, and the free boundary interface governed by the Young-Laplace relation, i.e. the boundary curvature is proportional to some normal trace quantity. In the Laplacian case this quantity is the normal derivative, and in the Stokes case the normal stress. A mathematical presentation is given in §1.2.

Optimal control of partial differential equations (PDEs) consists of enforcing a specific optimization goal subject to a PDE (1.4) and other constraints; this can be highly beneficial in practice (see [Trö10] for more details). For example using the reverse electrowetting i.e., by applying a control to change the shape of fluid droplets, one can generate enough power to charge a cellphone [KT11], by mere stroll in the park. There has been various attempts to solve optimal control problems with a FBP constraint. We refer to [HZ07a, HZ07b] for control of a two phase Stefan problem in graph formulation and [BH11] for the same problem in level set formulation. Paper [RMP11] discusses optimal control of a FBP with Stokes flow.

In this dissertation we provide a general framework for the optimal control free boundary problem OC-FBP in §1.3. In the particular case where the bulk quantity is governed by the Laplace equation (vs. Stokes) we provide complete control theory, i.e. first-order necessary and second-order sufficient conditions, and numerical analysis in Part II. Even though the Laplace based problem is relatively simple, it captures the essential features associated with surface tension effects found in more complex systems. Moreover, a complete second-order analysis, is absent in the existing literature on OC-FBP.

The first-order necessary conditions for the OC-FBP/Stokes falls under the general framework. On the other hand, the second-order sufficient conditions as well as the numerical analysis are left as part of future work.

We use a fixed domain approach to solve OC-FBP. In fact, we transform the physical domain $\Omega$ into a reference domain $\hat{\Omega}$ at the expense of having a governing PDE with unknown rough coefficients. This avoids dealing with shape sensitivity analysis [SZ92, DZ11a]. We refer to [vdZvBAdB10] for a comparison between these approaches applied to a FBP. One of the challenges of an OC-FBP is dealing with possible topological changes of the domain by introducing state constraints. Our analysis provides control constraints which always enforce the state constraints i.e., we can simply treat OC-FBP as a control constrained problem.

We organize this chapter as follows

- In §1.2 we introduce the abstract formulation which captures a FBP with surface tension effects.
- In §1.3 we present a simple optimal control formulation with first- and second-order sufficient conditions.
- In §1.4 we state the numerical convergence result for the optimal control of a free boundary problem.

## 1.2 Formulation

The goal of this section is to provide an abstract formulation which is shared by FBPs governed by surface tension effects using a sharp interface model, i.e. the free boundary is described by a surface, perhaps parametrically.

### 1.2.1 A Model Free Boundary Problem

The problem of interest is to find the pair $(\Omega, u)$ which is locally a unique solution to

$$-\Delta u = f \text{ in } \Omega, \tag{1.1a}$$

where $\Omega$ is an open subset of $\mathbb{R}^n$, together with the essential boundary condition

$$u = 0 \quad \text{on } \Sigma, \tag{1.1b}$$

and the Young-Laplace relation on the free boundary, i.e. the surface tension effect

$$u = 0, \quad \partial_{\boldsymbol{\nu}} u = \alpha\kappa \quad \text{on } \Gamma, \tag{1.1c}$$

where $\Delta u = \sum_{i=1}^{n} \partial_{x_i}^2 u$ is the Laplacian of $u$, $\partial_{\boldsymbol{\nu}} u = \nabla u \cdot \boldsymbol{\nu}$ is the normal derivative of $u$, $\alpha \equiv \mathsf{constant}$ is the surface tension coefficient, and $\kappa$ is the mean curvature. The Young-Laplace relation induces a strong non-linear coupling between the domain $\Omega$ and the unknown $u$.

Next we present a implicit weak formulation of the above problem, i.e. one where the test functions also depend on $\Omega$.

**Formulation 1.2.1** (Implicit weak form)**.** If $(\Omega, u)$ is a sufficiently smooth solution to (1.1a), then

$$\begin{aligned} \left\langle \mathcal{T}^*\mathcal{H}(\Omega) + \mathcal{L}(u;\Omega) - \mathcal{F}(\Omega), v \right\rangle &= 0 \qquad \text{for all } v \in D(\Omega) \\ u &= 0 \qquad \text{on } \Sigma \cup \Gamma = \partial\Omega \end{aligned}$$

where $D(\Omega) := \left\{ v \in C^\infty(\overline{\Omega}) : v = 0 \text{ on } \Sigma \right\}$, and $\langle \cdot, \cdot \rangle$ is the duality pairing. More importantly, the underlying operators are given by

$$\begin{aligned} \mathcal{T}^*\mathcal{H}(\Omega)(v) = \left\langle \mathcal{H}(\Omega), \mathcal{T}v \right\rangle &:= \int_\Gamma \kappa v \,\mathrm{d}s, \\ \mathcal{L}(u;\Omega)(v) &:= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \\ \mathcal{F}(\Omega)(v) &:= \int_\Omega f v \,\mathrm{d}x. \end{aligned}$$

*Derivation.* It suffices to multiply the bulk equation (1.1a) by $v$, integrate parts and use the Young-Laplace equation to obtain the curvature term. □

**Remark 1.2.2** (Simplified FBP)**.** The above problem is a generalization of the model problem posed by Saavedra-Scott[SS91] where $\Omega \subset \mathbb{R}^2$. We will use the formulation in $\mathbb{R}^2$ to develop a full optimal control theory with the MFBP as a constraint along with numerics in Part II.

### 1.2.2 The Stokes Free Boundary Problem

The problem of interest is to find the triple $(\Omega, \boldsymbol{u}, p)$ which is locally a unique solution to

$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = g \quad \text{in } \Omega, \tag{1.2a}$$

together with the no-slip boundary condition

$$\boldsymbol{u} = 0 \quad \text{on } \Sigma, \tag{1.2b}$$

and the free boundary condition with surface tension effects

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = 0, \quad \boldsymbol{\sigma}(\boldsymbol{u}, p)\, \boldsymbol{\nu} = \alpha \boldsymbol{\kappa} \quad \text{on } \Gamma, \tag{1.2c}$$

where $\boldsymbol{\sigma} = 2\eta\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{I}p$ is the stress tensor, $\eta \equiv \mathsf{constant}$ (Newtonian fluid) is the viscosity parameter, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)/2$ is the strain tensor (or symmetric gradient), $\boldsymbol{\nu}$ is the exterior unit normal to $\Gamma$, $\alpha \equiv \mathsf{constant}$ is the surface tension coefficient, $\boldsymbol{\kappa}(\boldsymbol{x}) = \kappa(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})$ with $\kappa(\boldsymbol{x})$ the mean curvature of $\Gamma$ at the point $\boldsymbol{x}$.

Equivalently we may split the boundary condition (1.2c) as

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = 0, \quad \boldsymbol{T}^\top \boldsymbol{\sigma}\left(\boldsymbol{u}, p\right) \boldsymbol{\nu} = \mathbf{0}, \tag{1.2d}$$

$$\boldsymbol{\nu}^\top \boldsymbol{\sigma}(\boldsymbol{u}, p) \boldsymbol{\nu} = \alpha\kappa, \tag{1.2e}$$

where $\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ is the projection operator onto the tangent plane of $\Gamma$. In order to put this problem in a suitable functional framework we will first write the system in its implicit variational form.

**Formulation 1.2.3** (Implicit variational form)**.** If $(\Omega, \boldsymbol{u}, p)$ is a sufficiently smooth solution to (1.2), then

$$\begin{aligned} \left\langle \mathcal{T}^*\mathcal{H}(\Omega) + \mathcal{S}(\boldsymbol{u}, p; \Omega) - \mathcal{F}(\Omega), (\boldsymbol{v}, q) \right\rangle &= 0 \qquad && \text{for all } (\boldsymbol{v}, q) \in D(\Omega) \\ \boldsymbol{u} \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma \\ \boldsymbol{u} &= \mathbf{0} && \text{on } \Sigma \end{aligned}$$

where $D(\Omega) := \left\{ (\boldsymbol{v}, q) \in C^\infty(\overline{\Omega}) \times C_0^\infty(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \Sigma \right\}$, and $\langle \cdot, \cdot \rangle$ is the duality pairing. More importantly, the underlying operators are given by

$$\begin{aligned} \mathcal{T}^*\mathcal{H}(\Omega)(\boldsymbol{v}) &:= \alpha \int_\Gamma \boldsymbol{\kappa} \cdot \boldsymbol{v} \, \mathrm{d}s, \\ \mathcal{S}(\boldsymbol{u}, p; \Omega)(\boldsymbol{v}, q) &:= \int_\Omega \eta \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p \operatorname{div} \boldsymbol{v} + q \operatorname{div} \boldsymbol{u} \, \mathrm{d}x, \\ \mathcal{F}(\Omega)(\boldsymbol{v}, q) &:= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + gq \, \mathrm{d}x. \end{aligned}$$

*Derivation.* It suffices to multiply the first bulk equation (1.2a) by $\boldsymbol{v}$, and integrate by parts, and multiply the second equation of (1.2a) by $q$. Finally, we add both equations together to obtain the desired expression. □

**Remark 1.2.4.** The above formulation is a generalization of some of the problems studied by [JP04] where the contact angle is fixed, [JJ05] where the contact angle is always $\pi/2$, [Sol95], [PS10], [Bae11] where no contact angle is given, [Nit86] where no proofs are provided.

### 1.2.3 The Abstract Framework

As we could see from the two previous examples there is a very clear structure between them. In spirit they both share the same surface operator, $\mathcal{T}^*\mathcal{H}$, and differed only by the choice of the bulk system, $\mathcal{L}$ and $\mathcal{S}$. Given this highly nonlinear structure, our goal will be to present conditions on $\mathcal{F}$ so that the inverse problem may be solved. The main tool behind our framework is the implicit function theorem which we state next together with a basic theorem on dual operators.

**Theorem 1.2.5** (Implicit Function Theorem)**.** *Let $X$ and $Z$ be Banach spaces, $Y$ a complete metric space, $U \times V$ an open subset of $X \times Y$, and $\mathcal{N} : U \times V \to Z$ a continuous map which is Fréchet differentiable with respect to $\boldsymbol{x}$. Additionally, suppose that:*

*(i) the Fréchet derivative $\mathcal{N}_x(\boldsymbol{x}, \boldsymbol{y})$ is continuous in $U \times V$,*

*(ii) $\mathcal{N}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \mathbf{0}_Z$ for some $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ in $U \times V$,*

*(iii) $\mathcal{A} := \mathcal{N}_x(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is an isomorphism from $X$ onto $Z$.*

*Then, there exists an open ball $B_r(\boldsymbol{y}_0) := \left\{\boldsymbol{y} \in Y : \|\boldsymbol{y}_0 - \boldsymbol{y}\|_Y < r\right\}$ and a* unique *continuous map $\mathcal{U} : B_r(\boldsymbol{y}_0) \to X$ such that $\mathcal{U}(\boldsymbol{y}_0) = \boldsymbol{x}_0$, and $\mathcal{N}(\mathcal{U}(\boldsymbol{y}), \boldsymbol{y}) = \mathbf{0}_Z$ for* *every $\boldsymbol{y}$ in $\mathcal{B}_r(\boldsymbol{y}_0)$.*

**Definition 1.2.6** (Identity Operator)**.** Given a Banach space $X$, we denote by $\mathcal{I}_X : X \to X$ the identity on $X$.

**Proposition 1.2.7.** *Let $X$ and $Y$ be Banach Spaces. If $\mathcal{A} : X \to Y^*$ is a linear and continuous operator, then there exists a unique linear and continuous operator $\mathcal{A}^* : Y^{**} \to X^*$ such that*

$$\langle \mathcal{A}x, y \rangle_{Y^*,Y} = \langle x, \mathcal{A}^* y \rangle_{X,X^*}.$$

*We will call $\mathcal{A}^*$ the dual operator to $\mathcal{A}$.*

The first common thread is the surface tension operator which acts on (part of) the boundary of the domain $\Omega$. Leaving aside for the moment which set of domains can be characterized by a Banach space we propose the following.

**Definition 1.2.8** (Surface Space)**.** Let $Y_s^2$ and $Y_{s'}^1$ be Banach spaces. We will call $Y_s^2$ and $(Y_{s'}^1)^*$ the *surface trial* and *surface data* spaces.

**Definition 1.2.9** (Surface Operator)**.** Let $V_H$ be an open subset of the surface trial space $Y_s^2$ and $\mathcal{H} : V_H \to (Y_{s'}^1)^*$ be an operator. We say $\mathcal{H}$ is an *admissible surface operator* if the following holds:

(i) $\mathcal{H}$ is continuously Fréchet differentiable on $V_H$, and $\partial_\omega \mathcal{H}$ denotes its derivative;

(ii) there exists $\omega_0$ in $V_H$ such that $\partial_\omega \mathcal{H}(\omega_0)$ is an isomorphism between the surface trial space $Y_s^2$ and the surface data space $(Y_{s'}^1)^*$. We will call $\omega_0$ a *resting surface.*

**Remark 1.2.10.** Condition (ii) states that there exists at least one surface where we could begin to look for solutions to the free boundary problem.

The next part is the space which characterizes the bulk quantities of interest, e.g. the velocity and pressure in the Stokes Free Boundary Problem (SFBP), or the unknown $u$ in the MFBP. Moreover, we need to define the operator which links both the surface and bulk equations.

**Definition 1.2.11** (Bulk Space)**.** Let $X_s$ and $X_{s'}$ be reflexive Banach spaces. We will call $X_s$ the *bulk trial space* and $X_{s'}^*$ the *bulk data space*. We use $X_{s'}^*$ to represent the dual of $X_{s'}$ and notice that $X_{s'}^{**} = X_{s'}$.

**Definition 1.2.12** (Trace Operator)**.** Let $r = \{s', s\}$ and $\mathcal{T}_r : X_r \to Y_r^1$ be a linear operator. We will say $\mathcal{T}_r$ is an *admissible* trace operator whenever:

(i) $\mathcal{T}_r$ is continuous,

(ii) if $r = s'$ there exists a linear and continuous operator $\mathcal{T}_{s'}^- : Y_{s'}^1 \to X_{s'}$ such that

$$\mathcal{T}_{s'}\mathcal{T}_{s'}^- = \mathcal{I}_{Y_{s'}^1},$$

i.e. $\mathcal{T}_{s'}$ has a right inverse.

**Proposition 1.2.13** (Characterization of Right Inverse)**.** *Let $\mathcal{A}$ be a bounded linear operator defined on a Banach space $X$ and mapping into a Banach space $Y$. $\mathcal{A}$ has a right inverse if and only it is surjective.*

*Proof.* [KPS82, pg.33] □

**Proposition 1.2.14** (Dual Trace Operators)**.** *If $\mathcal{T}_{s'}$ is an* admissible *trace operator, then its dual operator $\mathcal{T}_{s'}^* : (Y_{s'}^1)^* \to X_{s'}^*$ has a linear and continuous left inverse. In particular, the dual operator of $\mathcal{T}_{s'}^-$ is such an inverse.*

*Proof.* If $\ell$ is in $(Y_{s'}^1)^*$ and $\zeta$ is in $Y_{s'}^1$, then it follows

$$\left\langle \mathcal{T}_{s'}^{-*}\mathcal{T}_{s'}^*\ell, \boldsymbol{v}\right\rangle_{(Y_{s'}^1)^*, Y_{s'}^1} = \left\langle \mathcal{T}_{s'}^*\ell, \mathcal{T}_{s'}^-\zeta\right\rangle_{X_{s'}^*, X_{s'}} = \left\langle \ell, \mathcal{T}_{s'}\mathcal{T}_{s'}^-\zeta\right\rangle_{(Y_{s'}^1)^*, Y_{s'}^1} = \langle \ell, \zeta\rangle_{(Y_{s'}^1)^*, Y_{s'}^1},$$

which is the assertion. □

The trace operators are in general not injective, i.e. an arbitrary function in $X_s$ cannot be *uniquely* characterized by the image of its trace. Nevertheless, the null-space of $\mathcal{T}_{\{s',s\}}$ will be of extreme importance to us.

**Lemma 1.2.15** (Restricted Bulk Space)**.** *The null-space of $\mathcal{T}_{\{s',s\}}$ is a closed subspace of $X_{\{s',s\}}$. We denote this space by $\mathring{X}_{\{s',s\}}$, call it the* restricted bulk space*.*

*Proof.* This follows from the continuity of $\mathcal{T}_{\{s',s\}}$. □

**Remark 1.2.16.** In the MFPB example the null-space consists of the functions $u$ which vanishes identically on $\Sigma$ and $\Gamma$. In the SFBP example the null-space consists of the vector fields $\boldsymbol{u}$ which vanish on $\Sigma$ and satisfy the "no-flow" condition on $\Gamma$, i.e. $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$.

**Definition 1.2.17** (Bulk Operator)**.** Let $V_B$ be an open subset of the surface trial space $Y_s^2$. An operator $\mathcal{B} : X_s \times V_B \to X_{s'}^*$ which is

(i) linear in the bulk trial space $X_s$ for every $\omega$ in $V_B$,

(ii) continuously Fréchet differentiable in $V_B$,

(iii) an isomorphism between the restricted bulk space $\mathring{X}_s$ and $\mathring{X}^*_{s'}$,

will be called an *admissible bulk operator*.

**Definition 1.2.18** (Applied Force Space)**.** Let $Z$ be a Banach space. We will call $Z$ the *applied force space.*

**Definition 1.2.19** (Applied Force Operator)**.** Let $V_F$ be an open subset of the surface trial space $Y^2_s$. An operator $\mathcal{F} : Z \times V_F \to X^*_{s'}$ which is

(i) linear in the applied force space $Z$,

(ii) continuously Fréchet differentiable in $V_F$.

will be called an *admissible applied force operator.*

**Lemma 1.2.20** ($\partial_\omega \mathcal{B}(\cdot, \omega)$ and $\partial_\omega \mathcal{F}(\cdot, \omega)$ are bilinear)**.** *Let $\omega_0$ be in $V_B$ (or $V_F$) and $\mathcal{A}(\cdot, \delta\omega) := \partial_\omega \mathcal{B}(\boldsymbol{u}, \omega_0)(\delta\omega)$ (or $:= \partial_\omega \mathcal{F}(\cdot, \omega_0)(\delta\omega)$) denote the Fréchet derivative of $\mathcal{B}$ (or $\mathcal{F}$) at the point $\omega_0$. The operator $\mathcal{A}$ is a bounded* bilinear *map from $X_s \times Y^2_s$ (or $Z \times Y^2_s$) to $X^*_{s'}$.*

**Definition 1.2.21** (Free Boundary Operator)**.** Let $V := V_H \cap V_S \cap V_F$. We define the *free boundary operator* $\mathcal{N} : \mathring{X}_s \times V \times Z \to X^*_{s'}$ as

$$\mathcal{N}(\boldsymbol{u}, \omega, f) := \mathcal{T}^*_{s'} \mathcal{H}(\omega) + \mathcal{B}(\boldsymbol{u}, \omega) - \mathcal{F}(f, \omega),$$

whenever $\mathcal{H}$, $\mathcal{B}$, and $\mathcal{F}$ are admissible.

**Definition 1.2.22** (Initial Configuration)**.** Let $\omega_0$ be a resting surface. We call the point $(\boldsymbol{u}_0, \omega_0)$ an initial configuration whenever $\mathcal{N}(\boldsymbol{u}_0, \omega_0, 0) = 0$ and $\partial_\omega \mathcal{B}(\boldsymbol{u}_0, \omega_0)(\delta\omega) = 0$ for every $\delta\omega$ in the surface trial space.

**Proposition 1.2.23.** *Let* $(\boldsymbol{u}_0, \omega_0)$ *be an initial configuration. There exists an open ball of radius* $r$ *around the point* $0_Z$ *in the static data space, namely* $B(0_Z, r) := \{f \in Z : \|f\|_Z < r\}$*, and a* unique *continuously Fréchet differentiable operator* $\mathcal{U} : B(0_Z, r) \to (\mathring{X}_s^1 \times V)$ *such that* $\mathcal{U}(0_Z) = (\mathbf{0}, \omega_0)$ *and*

$$\mathcal{N}(\mathcal{U}(f), f) = 0 \quad \forall f \in B(0_Z, r).$$

*We call* $\mathcal{U}$ *the* solution operator *to the free boundary problem.*

*Proof.* The first thing to note is that the set $V$ is open because it is the finite intersection of open sets, thus we will use the Implicit Function Theorem. Since $(\boldsymbol{u}_0, \omega_0)$ is already an initial configuration and the underlying maps are all continuously Fréchet differentiable, we only need to verify that the operator $\mathcal{A} : \mathring{X}_s^1 \times Y_s^2 \to (X_{s'}^1)^*$, $\mathcal{A} := \partial_{(u,\omega)}\mathcal{N}(\boldsymbol{u}_0, \omega_0; f_0)$ is an isomorphism.

A direct computation yields that for every $(\delta\boldsymbol{u}, \delta\omega)$ in the domain of $\mathcal{A}$ the following holds,

$$\begin{aligned}\mathcal{A}(\delta\boldsymbol{u}, \delta\omega) &= \mathcal{B}(\delta\boldsymbol{u}, \omega_0) + \partial_\omega\mathcal{B}(\boldsymbol{u}_0, \omega_0)(\delta\omega) + \mathcal{T}_{s'}^*\partial_\omega\mathcal{H}(\omega_0)(\delta\omega) - \partial_\omega\mathcal{F}(f_0, \omega_0)\\ &= \mathcal{B}(\delta\boldsymbol{u}, \omega_0) + \mathcal{T}_{s'}^*\partial_\omega\mathcal{H}(\omega_0)(\delta\omega),\end{aligned}$$

where the last equality follows from the choice $f_0 = 0$ and the choice of $\boldsymbol{u}_0$.

Although it appears that we have two unknowns and only one equation, we can split $\mathcal{A}(\delta u, \delta\omega) = \mathcal{G}$ in $X_{s'}^*$ by multiplying $\mathcal{A}(\delta u, \delta\omega)$ with the left inverse of $\mathcal{T}_{s'}^*$, and by also testing $\mathcal{A}(\delta u, \delta\omega)$ with $\boldsymbol{v}$ in $\mathring{X}_{s'}$. Once these operations are performed we are left with the system,

$$\begin{cases} \partial_\omega H(\omega_0)(\delta\omega) + \mathcal{T}_{s'}^{-*}\mathcal{B}(\delta\boldsymbol{u},\omega_0) = \mathcal{T}_{s'}^{-*}\mathcal{G} & \text{in } (Y_{s'}^1)^* \\ \mathcal{B}(\delta\boldsymbol{u},\omega_0) = \mathcal{G} & \text{in } \mathring{X}_{s'}^* \end{cases}.$$

We conclude that $\mathcal{A}$ is invertible because of its upper triangular structure, and the assumption that $\mathcal{H}$ and $\mathcal{B}$ are admissible surface and bulk operators. $\square$

## 1.3 Optimal Control

**Definition 1.3.1** (Cost Function)**.** A function $\mathcal{J} : \mathring{X}_s \times Y_s \times Z \to \mathbb{R}$ which is $C^2 \cap C^{2,1}_{\text{loc}}$, i.e twice continuously Fréchet differentiable with locally Lipschitz second derivative, and convex in $Z$ is called an *admissible cost function.*

**Formulation 1.3.2** (FBP Constraint)**.** Given an admissible cost function $\mathcal{J}$ we want solve

$$\min_{u,\omega,f} \mathcal{J}(u,\omega,f) \tag{1.3}$$

subject to the free boundary problem

$$\mathcal{N}(u,\omega,f) = 0, \tag{1.4}$$

and control constraints

$$f \in Z_{ad}, \tag{1.5}$$

where $Z_{ad}$ is a non-empty closed and convex subset of $Z$.

**Formulation 1.3.3** (Reduced Control Problem)**.** Given a cost function $\mathcal{J}$ our task is to find a minimizer to

$$\min_{f \in Z_{ad}} \mathcal{J}(\mathcal{U}(f), f) \tag{1.6}$$

where $\mathcal{U}$ is the solution operator given in Proposition 1.2.23.

**Proposition 1.3.4** (Necessary and Sufficient Conditions)**.** *A necessary condition for $\bar{f}$ to be a minimizer of Formulation 1.3.3 is*

$$\left\langle \mathcal{J}'(\bar{f}), f - \bar{f} \right\rangle \geq 0 \quad \forall f \in Z_{ad}.$$

*A sufficient condition for the local uniqueness of $\bar{f}$ is that there exists $\delta > 0$ such that for every $h$ in $Z$*

$$\mathcal{J}''(\bar{f})h^2 \geq \delta \|h\|_Z^2 .$$

*Proof.* See [Trö10, Theorem 4.23]. □

**Remark 1.3.5.** The novelty in this thesis with respect to second-order sufficient conditions is that we actually show that the optimal control of the MFBP possesses a second-order sufficient condition.

## 1.4 Numerics

For numerics this thesis restricts itself to the MFBP. An in-depth discussion is provided in §4.2, but for the moment we just state the main result.

**Theorem 1.4.1** (Optimal control error estimate)**.** *Let $h \leq h_0$ and both $h_0$ and be sufficiently small, and $\bar{f}$ and $\bar{F}$ denote the continunous and discrete optimal controls.*

*The following error estimate holds,*

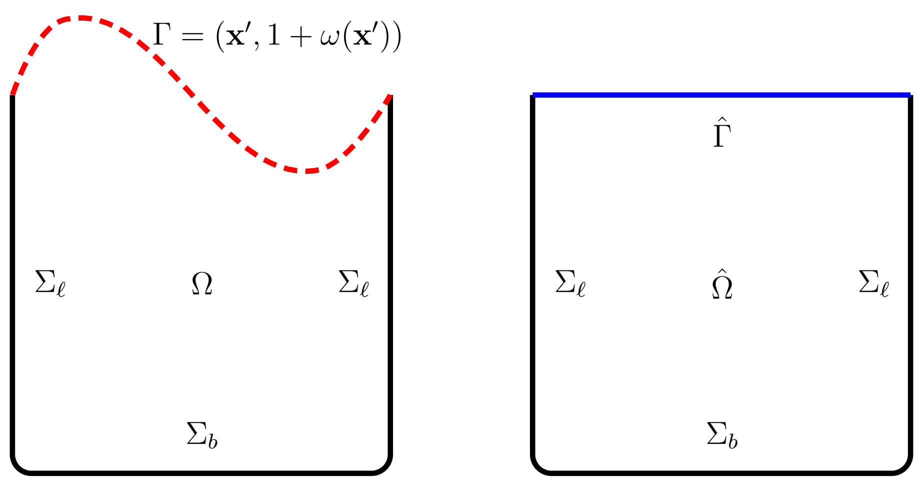
$$\left\|\bar{f} - \bar{F}\right\| \lesssim h. \tag{1.7}$$

# Chapter 2

# Admissible Surfaces

## 2.1 Surface Space: Graph Formulation

The goal of this section is to define the admissible surface space in the context of small perturbations of a resting configuration. In the mathematical context this is usually called the reference domain; c.f. Figure 2.1.




(a) Physical Domain (b) Reference Domain

Figure 2.1: The physical and reference domains.

**Definition 2.1.1** (Reference Domain)**.** Let $\tilde{\Gamma}$ be the unit disc in $\mathbb{R}^{n-1}$ and $\tilde{\Omega} = \tilde{\Gamma} \times (0,1)$. Let $\hat{\Gamma}$ be the top of $\tilde{\Omega}$ and define the *reference domain* $\hat{\Omega}$ by mollifying its bottom edge. Moreover, we set $\Sigma = \Sigma_b \cup \Sigma_\ell$, where $\Sigma_b$ is the bottom of $\hat{\Omega}$ and $\Sigma_\ell$ its lateral surface.

**Remark 2.1.2.** The mollification of the bottom edge can be dropped in the context of the MFBP with $n = 2$. It is enforced here to guarantee the existence of a solution to the Stokes problem with no-slip boundary conditions without having to resort to Besov spaces [MW12].

**Lemma 2.1.3** (Surface Space)**.** *Let $1 < s' < n < s < \infty$. The space $Y_s^2 := W_s^{2-1/s}(\hat{\Gamma}) \cap \mathring{W}_s^1(\hat{\Gamma})$ is an* admissible surface space *with the semi-norm*

$$|\omega|^s_{W_s^{2-1/s}(\hat{\Gamma})} := \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \frac{\left|\nabla'\omega(\boldsymbol{x}') - \nabla'\omega(\boldsymbol{y}')\right|^s}{\left|\boldsymbol{x}' - \boldsymbol{y}'\right|^{s+n-2}} \, \mathrm{d}\boldsymbol{y}' \, \mathrm{d}\boldsymbol{x}'$$

*where $\nabla'$ is the gradient in $\mathbb{R}^{n-1}$. Moreover, the dual of $Y_{s'}^1 := W_{s'}^{1-1/s'}(\hat{\Gamma})$ is an admissible surface data space with its canonical norm.*

*Proof.* They are both Banach spaces when equipped with their canonical norms. The only step left it to show $Y_s^2$ is complete under the semi-norm given above. First, the Poincaré inequality gives for every $\omega$ in $\mathring{W}_s^1(\hat{\Gamma})$

$$\|\omega\|_{L^s(\hat{\Gamma})} \leq C_{\hat{\Gamma},n,s} |\omega|_{W_s^1(\hat{\Gamma})} \, .$$

To obtain an equivalent result for $\nabla'\omega$ we rely on the fact that $\nabla'\omega$ is Hölder continuous, i.e. $\nabla'\omega$ is in $C^{0,1-n/s}(\hat{\Gamma})^{n-1}$, and apply Rolle's Theorem for each component of $\nabla'\omega$. Let $i = 1, \ldots, n-1$ be fixed but arbitrary, since $\omega$ vanishes on $\partial\hat{\Gamma}$, we have that $\omega(-\boldsymbol{e}_i) = \omega(\boldsymbol{e}_i) = 0$, whence there exists $t$ in $(0, 1)$ such that $\partial_{x^i}\omega(t\boldsymbol{e}_i + (t-1)\boldsymbol{e}_i) = 0$. We label that point $\boldsymbol{\xi}_i'$ and estimate as follows,

$$\left|\partial_{x^i}\omega(\boldsymbol{x}')\right| = \left|\partial_{x^i}\omega(\boldsymbol{x}') - \partial_{x^i}\omega(\boldsymbol{\xi}_i')\right| \leq [\partial_{x^i}\omega]_{C^{0,1-n/s}} \left|\boldsymbol{x}' - \boldsymbol{\xi}_i'\right|^{1-n/s} \quad \text{for all } \boldsymbol{x}' \in \hat{\Gamma},$$

where $[\cdot]_{C^{0,1-n/s}}$ indicates the Hölder semi-norm. Using Morrey's inequality [Eva98, Section 5.6, Theorem 4] on $\hat{\Gamma}$, we obtain that

$$\sup_{\boldsymbol{x}' \in \hat{\Gamma}} \left|\partial_{x^i}\omega(\boldsymbol{x}')\right| \leq C_{\hat{\Gamma},n,s}|\partial_{x^i}\omega|_{W_s^{1-1/s}(\hat{\Gamma})} \leq C_{\hat{\Gamma},n,s}|\omega|_{W_s^{2-1/s}(\hat{\Gamma})} .$$

Since $i$ is arbitrary this implies in turn that $|\omega|_{W_s^1(\hat{\Gamma})} \leq C_{\hat{\Gamma}}|\omega|_{W_\infty^1(\hat{\Gamma})} \leq C_{\hat{\Gamma},n,s}|\omega|_{W_s^{2-1/s}(\hat{\Gamma})}$. We thus deduce

$$|\omega|_{W_s^{2-1/s}(\hat{\Gamma})} \leq \|\omega\|_{W_s^{2-1/s}(\hat{\Gamma})} = \|\omega\|_{L^s(\hat{\Gamma})} + |\omega|_{W_s^1(\hat{\Gamma})} + |\omega|_{W_s^{2-1/s}(\hat{\Gamma})} \leq C_{\hat{\Gamma},n,s}|\omega|_{W_s^{2-1/s}(\hat{\Gamma})} .$$

which is the desired equivalence of norms. □

**Lemma 2.1.4** (Compact Extension). *If $\omega$ is in $Y_s^2$, then there exists a linear continuous extension $\mathcal{E} : Y_s^2 \to W_s^2(\hat{\Omega})$ such that $\mathcal{E}\omega\big|_{\hat{\Gamma}} = \omega$ and $\mathcal{E}\omega\big|_{\Sigma_b} = 0$, and*

$$\|\mathcal{E}\omega\|_{W_s^2(\hat{\Omega})} \leq C_{\hat{\Omega},n,s}|\omega|_{W_s^{2-1/s}(\hat{\Gamma})} . \tag{2.1}$$

*Proof.* The proof consists of three steps. First, Stein's total extension [AF03, Section 5] together with operator interpolation theory yields a function $\tilde{\omega}$ in $W_s^{2-1/s}(\mathbb{R}^{n-1})$ such that $\tilde{\omega}(\boldsymbol{x}') = \omega(\boldsymbol{x}')$ for all $\boldsymbol{x}'$ in $\hat{\Gamma}$ and $\|\tilde{\omega}\|_{W_s^{2-1/s}(\mathbb{R}^{n-1})} \leq C_{\hat{\Gamma},n,s}\|\omega\|_{W_s^{2-1/s}(\hat{\Gamma})}$.

Second, the surjectivity of the trace operator [AF03, Theorem 7.19] $\mathcal{T} : W_s^2(\mathbb{R}^n) \to W_s^{2-1/s}(\mathbb{R}^{n-1})$ implies the existence of a right inverse operator $\mathcal{T}^- : W_s^{2-1/s}(\mathbb{R}^{n-1}) \to W_s^2(\mathbb{R}^n)$. Thus $\mathcal{T}^-\tilde{\omega}$ belongs to $W_s^2(\mathbb{R}^{n-1})$, $\mathcal{T}\mathcal{T}^-\tilde{\omega} = \tilde{\omega}$, and $\left\|\mathcal{T}^-\tilde{\omega}\right\|_{W_s^2(\mathbb{R}^n)} \leq C_{n,s}\|\tilde{\omega}\|_{W_s^{2-1/s}(\mathbb{R}^{n-1})}$.

Third, let $\varrho$ be an exponential cutoff function such that $\varrho(x^n = 1) = 1$ and $\varrho(x^n = 0) = 0$.

We define the extension operator $\mathcal{E} : Y_s^2 \to W_s^2(\mathbb{R}^n)$ by $\mathcal{E}\omega := \varrho^2(x^n)\mathcal{T}^-\tilde{\omega}$.

Recalling Lemma 2.1.3 we have that

$$\|\mathcal{E}\omega\|_{W_s^2(\hat{\Omega})} \leq \|\mathcal{E}\omega\|_{W_s^2(\mathbb{R}^n)} \leq C_{\hat{\Omega},n,s}\left\|\mathcal{T}^{-}\tilde{\omega}\right\|_{W_s^2(\mathbb{R}^n)} \leq C_{\hat{\Omega},\hat{\Gamma},n,s}|\omega|_{W_s^{2-1/s}(\hat{\Gamma})},$$

which is what we wanted to show. □

**Proposition 2.1.5** (Graph Diffeomorphism)**.** *Let $\mathcal{E}$ be the compact extension of Lemma 2.1.4, and $\mathcal{D} : Y_s^2 \to W_s^2(\mathbb{R}^n)$ be defined by $\mathcal{D}\omega := \mathcal{I}_{\mathbb{R}^n} + \boldsymbol{e}^n\mathcal{E}\omega$. There exists a constant $L$ such that for every $\omega$ in $V_D := \left\{v \in W_s^{2-1/s}(\hat{\Gamma}) : |v|_{W_s^{2-1/s}(\hat{\Gamma})} < L\right\}$ the vector field $\hat{\Psi} = \mathcal{D}\omega$ is a $W_s^2$-diffeomorphism such that $\Omega = \hat{\Psi}(\hat{\Omega})$.*

*Proof.* This is based on [Sch05]. For every $\hat{\boldsymbol{x}}$ in $\hat{\Omega}$, $\hat{\Psi}(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}} + \boldsymbol{e}^n\mathcal{E}\omega(\hat{\boldsymbol{x}})$. The differentiability of $\hat{\Psi}$ follows from the function $\mathcal{E}\omega$; in particular we have

$$\hat{\nabla}\hat{\Psi} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \nabla'\mathcal{E}\omega^{\top} & 1 + \partial_n\mathcal{E}\omega \end{bmatrix}$$

Clearly $\hat{\Psi}$ is surjective as map from $\hat{\Omega}$ onto its range $\Omega = \hat{\Psi}(\hat{\Omega})$. To verify that $\hat{\Psi}$ is injective it suffices to show that $\hat{\Psi}^n$ is strictly monotone in the $\boldsymbol{e}_n$ direction. This follows by taking $L$ sufficiently small so that

$$\partial_n\hat{\Psi}^n(\hat{\boldsymbol{x}}) = 1 + \partial_n\mathcal{E}\omega(\hat{\boldsymbol{x}}) > 1/2.$$

We can use the inverse function theorem to assert the differentiability of $\hat{\Psi}^{-1}$, thus we conclude that $\hat{\Psi} : \hat{\Omega} \to \Omega$ is a diffeomorphism. □

**Corollary 2.1.6** (Fréchet derivative of $\mathcal{D}$)**.** *The mapping $\mathcal{D}$ is Fréchet differentiable in $V_D$ with the variation in the direction $h$ in $Y_s^2$ equal to $\partial_\omega\mathcal{D}(\omega_0)\langle h\rangle = \boldsymbol{e}_n\mathcal{E}h$.*

*Moreover, the following additional variations hold,*

$$\nabla \partial_\omega \hat{\Psi}\langle h\rangle = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \nabla' \mathcal{E}h^\top & \partial_n \mathcal{E}h \end{bmatrix}, \quad \partial_\omega \left(\frac{1}{\det \nabla \hat{\Psi}}\right)\langle h\rangle = -\frac{\partial_n \mathcal{E}h}{(1+\partial_n \mathcal{E}\omega_0)^2}.$$

*where* $\hat{\Psi} := \mathcal{D}(\mathcal{E}\omega)$*.*

*Proof.* The proof follows from the fact that $\mathcal{E}$ is linear so that $\mathcal{D}$ is affine in $\omega$. The definition of $\hat{\nabla}\Psi$ yields that $\det \hat{\nabla}\hat{\Psi} = 1 + \partial_n \mathcal{E}\omega$ as well as the additional Fréchet derivatives listed above. To conclude that $\partial_\omega(1/\det \hat{\nabla}\hat{\Psi})\langle h\rangle$ is in $W^1_s(\hat{\Omega})$ we use that for $s > n$ the product of two functions in $W^1_s(\hat{\Omega})$ is also in $W^1_s(\hat{\Omega})$, c.f [ST95, Remark 3.3.2] and [AF03, Theorem 4.39]. $\square$

## 2.2 Surface Operator: The Mean Curvature

The goal of this section is to show that the mean curvature operator is an admissible *surface operator*. To this end we must show that it is continuously Fréchet differentiable and invertible, at least at one point. We begin by recalling the mean curvature operator on graphs.

**Definition 2.2.1** (Mean curvature for graphs)**.** The mean curvature of a graph surface $\omega : \hat{\Gamma} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is given by

$$\mathcal{H}(\omega) := -\operatorname{div}' \frac{\nabla' \omega}{\mathcal{Q}(\omega)}, \quad \mathcal{Q}(\omega) := \sqrt{1 + |\nabla' \omega|^2}.$$

where $\nabla'$ indicates the gradient in $\mathbb{R}^{n-1}$.

**Lemma 2.2.2** (Fréchet derivative of $\mathcal{Q}$)**.** *$\mathcal{Q}$ is Fréchet differentiable as a map from $W_s^{2-1/s}(\hat{\Gamma}) \to W_s^{1-1/s}(\hat{\Gamma})$. Moreover, its derivative at $\omega_0$ satisfies*

$$\partial_\omega \mathcal{Q}(\omega_0)\langle h\rangle = \frac{\nabla'\omega_0 \cdot \nabla' h}{\mathcal{Q}(\omega_0)} \quad \forall h \in W_s^{2-1/s}(\hat{\Gamma}), \tag{2.2}$$

*and it is locally Lipschitz continuous as a function of $\omega_0$. In fact, $\mathcal{Q}$ is infinitely Fréchet differentiable in $W_s^{2-1/s}(\hat{\Gamma})$.*

*Proof.* Let $\omega_0$ and $\omega_1$ in $W_s^{2-1/s}(\hat{\Gamma})$ be fixed but arbitrary.

First, we show that $\mathcal{Q}$ is Lipchitz continuous,

$$\begin{aligned}\mathcal{Q}(\omega_1) - \mathcal{Q}(\omega_0) &= \big(\mathcal{Q}(\omega_1) - \mathcal{Q}(\omega_0)\big)\frac{\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)}{\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)} = \frac{|\nabla'\omega_1|^2 - |\nabla'\omega_0|^2}{\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)} \\ &= \frac{\nabla'(\omega_1 + \omega_0)\cdot \nabla'(\omega_1 - \omega_0)}{\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)}\end{aligned}$$

Because $\partial_{x^i}\omega$ is in $W_s^{1-1/s}(\hat{\Gamma})$ for $i = 1, \ldots, n-1$ and $W_s^{1-1/s}(\hat{\Gamma})$ is a multiplication algebra [ST95, Remark 3.3.2], we conclude that

$$\big\|\mathcal{Q}(\omega_1) - \mathcal{Q}(\omega_0)\big\|_{W_s^{1-1/s}(\hat{\Gamma})} \le C_{\hat{\Gamma},n,s}\left\|\frac{\nabla'\omega_1 + \nabla'\omega_0}{\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)}\right\|_{W_s^{1-1/s}(\hat{\Gamma})} \|\omega_1 - \omega_0\|_{W_s^{2-1/s}(\hat{\Gamma})}.$$

Second, we compute the Gateux derivative of $\mathcal{Q}$ at $\omega_0$, namely for $h$ in $W_s^{2-1/s}(\hat{\Gamma})$

$$\mathcal{Q}_\omega(\omega_0)\langle h\rangle = \lim_{t\to 0}\frac{\mathcal{Q}(\omega_0 + th) - \mathcal{Q}(\omega_0)}{t} = \lim_{t\to 0}\frac{\nabla'(2\omega_0 + th)}{\mathcal{Q}(\omega_0 + th) + \mathcal{Q}(\omega_0)}\cdot \nabla' h.$$

Using the continuity of $\mathcal{Q}$ we have formally derived expression (2.2).

Third, we address the Fréchet differentiability of $\mathcal{Q}$. We must show that for every $\omega_0$ in $W_s^{2-1/s}(\hat{\Gamma})$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\omega_1 - \omega_0\|_{W_s^{2-1/s}(\hat{\Gamma})} < \delta$ then

$$\frac{\big\|\mathcal{Q}(\omega_1) - \mathcal{Q}(\omega_0) - \partial_\omega \mathcal{Q}(\omega_0)\langle \omega_1 - \omega_0\rangle\big\|_{W_s^{1-1/s}(\hat{\Gamma})}}{\|\omega_1 - \omega_0\|_{W_s^{2-1/s}(\hat{\Gamma})}} \le \epsilon.$$

Setting $\mathcal{R} := \mathcal{Q}(\omega_1) - \mathcal{Q}(\omega_0) - \mathcal{Q}_\omega(\omega_0)\langle \omega_1 - \omega_0 \rangle$ we obtain

$$\begin{aligned}
\mathcal{R} &= \left( \frac{\nabla'(\omega_1 + \omega_0)}{\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)} - \frac{\nabla'\omega_0}{\mathcal{Q}(\omega_0)} \right) \cdot \nabla'(\omega_1 - \omega_0) \\
&= \frac{\big(\mathcal{Q}(\omega_0) - \mathcal{Q}(\omega_1)\big)\, \nabla'\omega_0 + \mathcal{Q}(\omega_0)\nabla'(\omega_1 - \omega_0)}{\mathcal{Q}(\omega_0)\,\big(\mathcal{Q}(\omega_1) + \mathcal{Q}(\omega_0)\big)} \cdot \nabla'(\omega_1 - \omega_0),
\end{aligned}$$

whence, we obtain $\|\mathcal{R}\|_{W_s^{1-1/s}(\hat{\Gamma})} \lesssim \|\omega_1 - \omega_0\|^2_{W_s^{2-1/s}(\hat{\Gamma})}$ where the constant depends on the Lipschitz constant of $\mathcal{Q}$ in a neighborhood of $\omega_0$.

Finally, to show that $\partial_\omega \mathcal{Q}$ is locally Lipschitz continuous, we take $h$ in $W_s^{2-1/s}(\hat{\Gamma})$ fixed but arbitrary to arrive at

$$\begin{aligned}
\big(\partial_\omega \mathcal{Q}(\omega_1) - \partial_\omega \mathcal{Q}(\omega_0)\big)\, \langle h \rangle = &\frac{(\nabla'(\omega_1 - \omega_0)) \cdot \nabla' h}{\mathcal{Q}(\omega_1)} \\
&+ \frac{\nabla'\omega_0 \cdot \nabla' h}{\mathcal{Q}(\omega_1)\mathcal{Q}(\omega_0)} \big(\mathcal{Q}(\omega_0) - \mathcal{Q}(\omega_1)\big)\,.
\end{aligned}$$

From the multiplication algebra property [ST95, Remark 3.3.2] we obtain

$$\begin{aligned}
&\frac{\left\| \big(\partial_\omega \mathcal{Q}(\omega_1) - \partial_\omega \mathcal{Q}(\omega_0)\big)\, \langle h \rangle \right\|_{W_s^{1-1/s}(\hat{\Gamma})}}{\|h\|_{W_s^{2-1/s}(\hat{\Gamma})}} \\
&\quad \le \left\| \frac{1}{\mathcal{Q}(\omega_1)} \right\|_{W_s^{1-1/s}(\hat{\Gamma})} \|\omega_1 - \omega_2\|_{W_s^{2-1/s}(\hat{\Gamma})} \\
&\quad\quad + \left\| \frac{\nabla'\omega_0}{\mathcal{Q}(\omega_1)\mathcal{Q}(\omega_0)} \right\|_{W_s^{1-1/s}(\hat{\Gamma})} \big\|\mathcal{Q}(\omega_1) - \mathcal{Q}(\omega_0)\big\|_{W_s^{1-1/s}(\hat{\Gamma})}\,.
\end{aligned}$$

The Lipschitz continuity of $\mathcal{Q}$ yields the second part of the result. To conclude that $\mathcal{Q}$ is infinitely Fréchet differentiable it suffices apply the same steps again to $\partial_\omega \mathcal{Q}$ and reuse the multiplication algebra property. $\square$

**Lemma 2.2.3** (Fréchet derivative of $\mathcal{H}$)**.** *Let $\omega_0$ be in $W_s^{2-1/s}(\hat{\Gamma})$, $s > n$. The mean curvature operator $\mathcal{H}$ is Fréchet differentiable as a map from $W_s^{2-1/s}(\hat{\Gamma}) \to$*

$W_s^{-1/s}(\hat{\Gamma})$. *Moreover, its derivative at* $\omega_0$ *satisfies*

$$\partial_\omega \mathcal{H}(\omega_0)\langle h\rangle = -\operatorname{div}'\left[\left(\boldsymbol{I} - \frac{\nabla'\omega_0 \otimes \nabla'\omega_0}{\mathcal{Q}(\omega_0)^2}\right)\frac{\nabla' h}{\mathcal{Q}(\omega_0)}\right] \tag{2.3}$$

*for all* $h \in W_s^{2-1/s}(\hat{\Gamma})$*, and it is Lipschitz continuous as a function of* $\omega_0$*. In fact,* $\mathcal{H}$ *is infinitely Fréchet differentiable in* $W_s^{2-1/s}(\hat{\Gamma})$*.*

*Proof.* We obtain expression (2.3) by formally computing the Gâteux derivative of $\mathcal{H}$. We skip this derivation.

Next, we must show that for $\omega_0$ in $W_s^{2-1/s}(\hat{\Gamma})$ and $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|\omega_1 - \omega_0\|_{W_s^{2-1/s}(\hat{\Gamma})} < \delta$ we have

$$\frac{\left\|\mathcal{H}(\omega_1) - \mathcal{H}(\omega_0) - \partial_\omega \mathcal{H}(\omega_0)\langle \omega_1 - \omega_0\rangle\right\|_{W_s^{-1/s}(\hat{\Gamma})}}{\|\omega_1 - \omega_0\|_{W_s^{2-1/s}(\hat{\Gamma})}} \leq \epsilon.$$

The numerator can be further simplified by using the characterization of $W_s^{-1/s}(\hat{\Gamma})$ as the derivative of functions in $W_s^{1-1/s}(\hat{\Gamma})$, i.e. $\left\|\operatorname{div}' \boldsymbol{f}\right\|_{W_s^{-1/s}(\hat{\Gamma})} \approx \|\boldsymbol{f}\|_{W_s^{1-1/s}(\hat{\Gamma})}$. Thus we are left with the task of showing that $\nabla'\omega/\mathcal{Q}(\omega)$ is Fréchet differentiable.

In order to infer the Fréchet differentiability of $\nabla'\omega/\mathcal{Q}(\omega)$ it suffices to note that the only nonlinear term is $1/\mathcal{Q}(\omega)$. Since this term inherits all its continuity and differentiability properties from $\mathcal{Q}$, the result follows from Lemma 2.2.2. $\square$

**Proposition 2.2.4** (Laplace operator on $W_s^{2-1/s}(\hat{\Gamma})$)**.** *Let* $\hat{\Gamma}$ *be a* $C^{1,1}$ *bounded domain of* $\mathbb{R}^{n-1}$ *and* $s > 2$*. The Laplace operator with Dirichlet boundary conditions* $\Delta_0$ *is an isomorphism between* $W_s^{2-1/s}(\hat{\Gamma}) \cap \mathring{W}_s^1(\hat{\Gamma})$ *and* $W_s^{-1/s}(\hat{\Gamma})$*.*

*Proof.* The result will follow from interpolation theory [Tar07, Lemma 28.1, Lemma 41.3][AF03] . Because the domain is $C^{1,1}$ we have that $\Delta_0 : W_s^2(\hat{\Gamma}) \cap \mathring{W}_s^1(\hat{\Gamma}) \to L^s(\hat{\Gamma})$

and $\Delta_0 : \mathring{W}_s^1(\hat{\Gamma}) \to W_s^{-1}(\hat{\Gamma})$ are isomorphisms [GT01, Theorem 9.17]. Applying the space interpolation theory to $\Delta_0$ we obtain that

$$\Delta_0 : (W_s^2(\hat{\Gamma}), W_s^1(\hat{\Gamma}))_{\theta,p} \cap \mathring{W}_s^1(\hat{\Gamma}) \to (L^s(\hat{\Gamma}), W_s^{-1}(\hat{\Gamma}))_{\theta,p}$$

is an isomorphism for $0 < \theta < 1$ and $1 \leq p \leq \infty$. The notation $(E_0, E_1)_{\theta,p}$ entails the interpolation space between $E_0$ and $E_1$. By choosing $\theta = 1/s$ and $p = s$ we have that

$$(W_s^2(\hat{\Gamma}), W_s^1(\hat{\Gamma}))_{1/s,s} = W_s^{2(1-\theta)+1\theta}(\hat{\Gamma}) = W_s^{2-1/s}(\hat{\Gamma}).$$

To obtain the interpolation space for the dual spaces we use [Tar07, Lemma 41.3] and [LM61, Théorèm 5.1], namely

$$(L^s(\hat{\Gamma}), W_s^{-1}(\hat{\Gamma}))_{1/s,s} = (L^{s'}(\hat{\Gamma}), \mathring{W}_{s'}^1(\hat{\Gamma}))_{1/s,s'}^* = \mathring{W}_{s'}^{1/s}(\hat{\Gamma})^* = W_s^{-1/s}(\hat{\Gamma}).$$

This concludes the proof. □

**Theorem 2.2.5** (Admissible Surface Operator)**.** *There exists an open ball centered at zero $V_H \subset W_s^{2-1/s}(\hat{\Gamma})$ such that for every $\omega_0$ in $V_H$, the Fréchet derivative of $\mathcal{H}$ is an isomorphism between $W_s^{2-1/s}(\hat{\Gamma}) \cap \mathring{W}_s^1(\hat{\Gamma})$ and $W_s^{1-1/s}(\hat{\Gamma})$.*

*Proof.* We start by noticing that $\partial_\omega \mathcal{H}(0)$ coincides with the Laplace operator. Since $\mathcal{H}$ is in fact twice Fréchet differentiable we obtain that

$$\partial_\omega \mathcal{H}(\omega_0) = \partial_\omega \mathcal{H}(0) - \partial_\omega \mathcal{H}(0) + \partial_\omega \mathcal{H}(\omega_0) = \partial_\omega \mathcal{H}(0) + \int_0^1 \partial_\omega^2 \mathcal{H}(t\omega_0)\langle \omega_0 \rangle \, \mathrm{d}t.$$

Applying $\partial_\omega \mathcal{H}(0)^{-1}$ to both sides yields,

$$\partial_\omega \mathcal{H}^{-1}(0) \partial_\omega \mathcal{H}(\omega_0) = \mathcal{I}_{W_s^{2-1/s}(\hat{\Gamma})} + \partial_\omega \mathcal{H}^{-1}(0) \int_0^1 \partial_\omega^2 \mathcal{H}(t\omega_0)\langle \omega_0 \rangle \, \mathrm{d}t.$$

We now use von Neumann's stability result to conclude that the right-hand-side is an isomorphism provided

$$\left| \int_0^1 \partial_\omega \mathcal{H}^{-1}(0) \partial_\omega^2 \mathcal{H}(t\omega_0) \,\mathrm{d}t \langle \omega_0 \rangle \right| < 1.$$

The above inequality follows by taking $\|\omega_0\|_{W_s^{2-1/s}(\hat{\Gamma})}$ sufficiently small. We thus define $V_H$ based on this choice. $\square$

**Remark 2.2.6.** We notice here that the Laplacian is an admissible surface operator with $V_H$ restricted only by the condition that its graph induces a $W_s^2$-diffeomorphism of the reference/resting domain. Although the Laplacian does not model the true surface tension effects, we will use it in the context of studying the model free boundary problem.

# Part II

# The Model Free Boundary Problem

# Chapter 3

## Laplace Free Boundary Problem: Optimal Control

The MFBP is formulated based on the work by [SS91], it is posed in the unit square of $\mathbb{R}^2$ with the free surface modeled by a Lipschitz curve Figure 3.1. With the current literature it is not clear how to extend the results to arbitrary dimensions, the bottleneck being the invertibility of the Laplacian for the curve in $\mathring{W}^1_\infty(\hat{\Gamma})$, Proposition 3.3.1. Nevertheless, in §3.3.1.2 we return to the framework by showing that the curve is in $W^{2-1/s}_s(\hat{\Gamma})$, $s > n$.

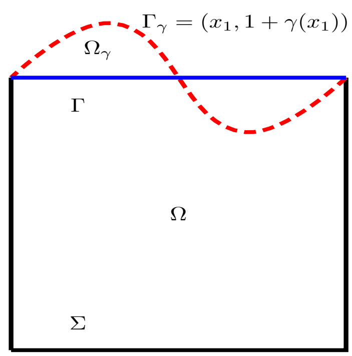



Figure 3.1: $\Omega_\gamma$ denotes a physical domain with boundary $\partial\Omega_\gamma = \Sigma \cup \Gamma_\gamma$. Here $\Sigma$ includes the lateral and the bottom boundary and is assumed to be fixed. Furthermore, the top boundary $\Gamma_\gamma$ (dotted line) is "free" and is assumed to be a graph of the form $(x_1, 1+\gamma(x_1))$, where $\gamma \in \mathring{W}^1_\infty(0,1)$ denotes a parametrization. $\Gamma_\gamma$ is further mapped to a fixed boundary $\Gamma = (0,1)$ and in turn the physical domain $\Omega_\gamma$ is mapped to a reference domain $\Omega = (0,1)^2$, where all computations are carried out.

Of particular interest to us is the control of a model FBP previously studied by P. Saavedra and L. R. Scott in [SS91] and formulated in graph form; see Figure 3.1 where the free boundary $\Gamma_\gamma$ is the dotted line. The state equations (3.2b) involve a Laplace equation in the bulk and a Young-Laplace equation on the free boundary to account for surface tension. This amounts to solving a second-order system both in the bulk and on the interface. Below we give a detailed description of the problem.

Let $\gamma \in \mathring{W}^1_\infty(0,1)$ denote a parametrization of the top boundary (see Figure 3.1) of the physical domain $\Omega_\gamma \subset \Omega^* \subset \mathbb{R}^2$ with boundary $\partial\Omega_\gamma := \Gamma_\gamma \cup \Sigma$, defined as

$$
\begin{aligned}
\Omega^* &= (0,1) \times (0,2), \\
\Omega_\gamma &= \left\{(x_1,x_2)\,:\,0 < x_1 < 1,\ 0 < x_2 < 1 + \gamma(x_1)\right\}, \\
\Gamma_\gamma &= \left\{(x_1,x_2)\,:\,0 < x_1 < 1,\ x_2 = 1 + \gamma(x_1)\right\}, \\
\Sigma &= \partial\Omega_\gamma \setminus \Gamma_\gamma, \\
\Gamma &= \left\{(x_1,x_2)\,:\,0 < x_1 < 1,\ x_2 = 1\right\}.
\end{aligned}
$$

Here, $\Omega^*$ and $\Sigma$ are fixed while $\Omega_\gamma$ and $\Gamma_\gamma$ deform according to $\gamma$. The Sobolev space $\mathring{W}^1_\infty(0,1)$ consists of the Lipschitz continuous function on the unit interval $(0,1)$ which vanish at 0 and 1.

We want to find an optimal control $u \in \mathcal{U}_{ad} \subset L^2(0,1)$ so that the solution pair $(\gamma, y)$ of the FBP approximates a given boundary $\gamma_d : (0,1) \to \mathbb{R}$ and potential $y_d : \Omega^* \to \mathbb{R}$. This amounts to solving the problem: minimize

$$
\mathcal{J}(\gamma, y, u) := \frac{1}{2}\|\gamma - \gamma_d\|^2_{L^2(0,1)} + \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_\gamma)} + \frac{\lambda}{2}\|u\|^2_{L^2(0,1)}, \tag{3.2a}
$$

subject to the state equations

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega_\gamma \\ y = v & \text{on } \partial\Omega_\gamma \\ -\kappa\mathcal{H}\left[\gamma(x_1)\right] + \partial_\nu y\left(x_1, 1+\gamma(x_1)\right) = u(x_1) & x_1 \in (0,1) \\ \gamma(0) = \gamma(1) = 0, & \end{cases}, \tag{3.2b}$$

the state constraints

$$\left|\mathrm{d}_{x_1}\gamma(x_1)\right| \leq 1 \quad \text{a.e. } x_1 \in (0,1)\,, \tag{3.2c}$$

with $\mathrm{d}_{x_1}$ being the total derivative with respect to $x_1$, and the control constraint

$$u \in \mathcal{U}_{ad} \tag{3.2d}$$

dictated by $\mathcal{U}_{ad}$, a closed ball in $L^2(0,1)$, to be specified later in Definition 3.3.5. Here $\lambda > 0$ is the stabilization parameter; $v$ is given which in principle could act as a Dirichlet boundary control;

$$\mathcal{H}[\gamma] := \mathrm{d}_{x_1}\left(\frac{\mathrm{d}_{x_1}\gamma}{\sqrt{1+\left|\mathrm{d}_{x_1}\gamma\right|^2}}\right)$$

is the *curvature* of $\gamma$; and $\kappa > 0$ plays the role of surface tension coefficient.

Depending on the role of the free boundary there are several methodologies to formulate a FBPs. We choose the sharp interface method written in graph form (see Figure 3.1). The (free) interface $\Gamma_\gamma$ is governed by the explicit nonlinear PDE

$$-\kappa\mathcal{H}[\gamma] + \partial_\nu y = u.$$

A similar approach was used in [HZ07a, HZ07b] for the optimal control of a Stefan problem, but without the full accompanying theory developed herein. Alternative approaches to treat FBPs are the level set method and the diffuse interface method [DDE05, BH11].

We have organized this chapter as follows. A detailed problem description on a fixed domain is given in section 3.1. We introduce the Lagrangian functional to formally derive the first-order necessary optimality conditions in section 3.2. We present a rigorous justification of the Lagrangian results in the remaining sections. To this end, we introduce a control-to-state operator in section 3.3 and show that for a particular set of admissible controls it is twice Fréchet differentiable. Finally, we write the optimal control problem in its reduced form and show the existence of a control under slightly higher regularity together with second-order sufficient conditions in section 3.4.

## 3.1 OC-FBP on Reference Domain

For simplicity we consider the FBP (3.2b) with linearized curvature. To analyze the minimization problem (3.2), we map the physical domain $\Omega_\gamma$ onto the fixed reference domain $\Omega = (0,1)^2$. This results in an optimal control problem subject to PDE constraints with nonlinear coefficients depending on $\gamma$ but without an explicit interface. The idea is to map the unknown domain $\Omega_\gamma$ onto the fixed domain $\Omega = (0,1)^2$ using the inverse of the Lipschitz map, $\Psi : \Omega \to \Omega_\gamma$, defined as

$$\Psi(x_1, x_2) = \Big(x_1, \big(1 + \gamma\,(x_1)\big)\, x_2\Big) \in \Omega_\gamma, \quad \text{for } \ (x_1, x_2) \in \Omega. \tag{3.3}$$

Since $\gamma$ is Lipschitz continuous according to the state constraint (3.5c) with constant 1, we deduce that $|\gamma| \leq 1/2$ which in turn implies that $\Psi$ is invertible. Furthermore, the inverse of $\Psi$ is also Lipschitz. Moreover, it becomes routine to check that the Laplace equation $\Delta y = 0$ in $\Omega_\gamma$ and $\partial_\nu y$ on $\Gamma_\gamma$ can be written as

$$\operatorname{div}(A[\gamma]\,\nabla y) = 0 \quad \text{in } \Omega, \qquad A[\gamma]\,\nabla y \cdot \nu \left(1 + |\mathrm{d}_{x_1}\gamma|^2\right)^{-1/2} \quad \text{on } (0,1)\,,$$

where $\nu = [0,1]^T$, and $A : \mathring{W}^1_\infty(0,1) \to L^\infty(\Omega)^{2\times 2}$ is the Nemytskii operator [Trö10, Chapter 4] defined by

$$A[\gamma] = \begin{bmatrix} 1+\gamma(x_1) & -\mathrm{d}_{x_1}\gamma(x_1)x_2 \\ -\mathrm{d}_{x_1}\gamma(x_1)x_2 & \frac{1+\left(\mathrm{d}_{x_1}\gamma(x_1)x_2\right)^2}{1+\gamma(x_1)} \end{bmatrix}. \tag{3.4}$$

It is convenient to write $A_{2,2}[\gamma]$ as $\Phi[\gamma] := \varphi\left(\gamma(x_1), \mathrm{d}_{x_1}\gamma(x_1)x_2\right)$, where $\varphi(a,b) := \left(1+b^2\right)/(1+a)$.

To simplify the exposition we make the following assumptions:

($\mathbf{A_1}$) Linearized curvature: $\mathcal{H}_{lin}[\gamma] = \mathrm{d}^2_{x_1}\gamma\left(1 + |\mathrm{d}_{x_1}\gamma|^2\right)^{-1/2}$.

($\mathbf{A_2}$) Scaled control: $u = u\left(1 + |\mathrm{d}_{x_1}\gamma|^2\right)^{-1/2}$.

These assumptions are not crucial. In case of assumption ($\mathbf{A_1}$), if $\gamma$ is sufficiently smooth the nonlinear curvature $\mathcal{H}[\gamma]$ is given by

$$\mathcal{H}[\gamma] = \mathrm{d}_{x_1}\left(\frac{\mathrm{d}_{x_1}\gamma}{\sqrt{1 + |\mathrm{d}_{x_1}\gamma|^2}}\right) = \frac{\mathrm{d}^2_{x_1}\gamma}{\left(1 + |\mathrm{d}_{x_1}\gamma|^2\right)^{3/2}},$$

which is similar to the linearized curvature $\mathcal{H}_{lin}$ except for the $L^\infty(0,1)$ factor $1/(1+|\mathrm{d}_{x_1}\gamma|^2)$.

As far as assumption ($\mathbf{A_2}$), the scaling of the control avoids unnecessarily complicating the right-hand-side of(3.5b) below, which would contain $u\left(1 + |\mathrm{d}_{x_1}\gamma|^2\right)^{1/2}$

instead of simply $u$. This can be justified as we are only interested in small perturbations of the free boundary, i.e. $\mathrm{d}_{x_1}\gamma$ is small.

Dirichlet boundary condition: we identify $v$ on the boundary with the trace of a function $v \in W_p^1(\Omega)$, $p > 2$, see [SS91, Lemma 2].

Under these assumptions and the application of the map $\Psi$, the optimal control problem (3.2) becomes: minimize

$$\mathcal{J}(\gamma, y, u) := \frac{1}{2}\|\gamma - \gamma_d\|^2_{L^2(0,1)} + \frac{1}{2}\left\|(y + v - y_d)\sqrt{1+\gamma}\right\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(0,1)} \tag{3.5a}$$

subject to the state equations $(\gamma, y) \in \mathring{W}^1_\infty(0,1) \times \mathring{W}^1_p(\Omega)$

$$\begin{cases} -\operatorname{div}(A[\gamma]\nabla(y+v)) = 0 \quad \text{in } \Omega \\ -\kappa\,\mathrm{d}^2_{x_1}\gamma + A[\gamma]\nabla(y+v)\cdot\nu = u \quad \text{in } (0,1) \end{cases} \tag{3.5b}$$

the state constraints

$$\left|\mathrm{d}_{x_1}\gamma(x_1)\right| \leq 1 \quad \text{a.e. } x_1 \in (0,1), \tag{3.5c}$$

with $\mathrm{d}_{x_1}$ being the total derivative with respect to $x_1$, and the control constraint

$$u \in \mathcal{U}_{ad} \tag{3.5d}$$

dictated by $\mathcal{U}_{ad}$, a closed ball in $L^2(0,1)$, to be specified later in Definition 3.3.5.

In order to derive the first- and second-order optimality conditions in later sections, we need to compute the first- and second-order directional derivatives of $A$, which in turn requires computing the directional derivative of the Nemytskii operator $\Phi$ defined above. To simplify notation, we drop the evaluation of $\gamma$ and $\mathrm{d}_{x_1}\gamma$ at $x_1$. The derivative of $\Phi$ in the direction $h$ at $(\gamma, x_2\,\mathrm{d}_{x_1}\gamma)$ is given by

$$\mathrm{D}\Phi[\gamma]\langle h\rangle = \partial_a\varphi(\gamma, x_2\,\mathrm{d}_{x_1}\gamma)\,h + \partial_b\varphi(\gamma, x_2\,\mathrm{d}_{x_1}\gamma)\,x_2\,\mathrm{d}_{x_1}h$$

where

$$\nabla\varphi(a,b) := \begin{bmatrix} \partial_a\varphi(a,b) & \partial_b\varphi(a,b) \end{bmatrix} = \begin{bmatrix} -\frac{1+b^2}{(1+a)^2} & \frac{2b}{1+a} \end{bmatrix}.$$

Furthermore, we obtain the following representation for $\mathrm{D}A$ in terms of $h$ and $\mathrm{d}_{x_1} h$

$$\begin{aligned} \mathrm{D}A\,[\gamma]\,\langle h\rangle &:= A_1\,[\gamma]\,h + A_2\,[\gamma]\,\mathrm{d}_{x_1} h \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \partial_a\varphi\left(\gamma, x_2\,\mathrm{d}_{x_1}\gamma\right) \end{bmatrix} h + \begin{bmatrix} 0 & -x_2 \\ -x_2 & x_2\partial_b\varphi\left(\gamma, x_2\,\mathrm{d}_{x_1}\gamma\right) \end{bmatrix} \mathrm{d}_{x_1} h, \end{aligned} \tag{3.6}$$

whence the remainder $\mathrm{R}_A\,[\gamma, h]$ at $\gamma$ in the direction $h$ reads

$$\mathrm{R}_A\,[\gamma, h] := A\,[\gamma + h] - A\,[\gamma] - \mathrm{D}A\,[\gamma]\,\langle h\rangle \tag{3.7a}$$

and

$$\lim_{|h|_{W^1_\infty(0,1)}\to 0} \frac{\left\|\mathrm{R}_A\,[\gamma, h]\right\|_{L^\infty(\Omega)}}{|h|_{W^1_\infty(0,1)}} = 0, \tag{3.7b}$$

where (3.7b) follows directly from the structure of $A$, and the proof is therefore omitted. The Hessian of $\varphi$ is

$$\nabla^2\varphi(a,b) = \begin{bmatrix} \partial_a^2\varphi(a,b) & \partial_{ab}\varphi(a,b) \\ \partial_{ba}\varphi(a,b) & \partial_b^2\varphi(a,b) \end{bmatrix} = 2\begin{bmatrix} \frac{1+b^2}{(1+a)^3} & \frac{-b}{(1+a)^2} \\ \frac{-b}{(1+a)^2} & \frac{1}{1+a} \end{bmatrix}.$$

The second-order derivative of $\Phi$ in the direction $h_1$ followed by $h_2$ evaluated at $(\gamma, x_2\,\mathrm{d}_{x_1}\gamma)$ is

$$\begin{aligned} \mathrm{D}^2\,\Phi\,[\gamma]\,\langle h_2, h_1\rangle &= \partial_a^2\varphi\left(\gamma, x_2\,\mathrm{d}_{x_1}\gamma\right) h_2 h_1 + \partial_{ab}\varphi\left(\gamma, x_2\,\mathrm{d}_{x_1}\gamma\right) x_2 h_2\,\mathrm{d}_{x_1} h_1 \\ &\quad + \partial_{ab}\varphi\left(\gamma, x_2\,\mathrm{d}_{x_1}\gamma\right) x_2\,\mathrm{d}_{x_1} h_2 h_1 + \partial_b^2\varphi\left(\gamma, x_2\,\mathrm{d}_{x_1}\gamma\right) x_2^2\,\mathrm{d}_{x_1} h_2\,\mathrm{d}_{x_1} h_1. \end{aligned}$$

Finally, we obtain the following representation for $\mathrm{D}^2 A$ in terms of $h_1$ and $h_2$

$$\mathrm{D}^2 A\left[\gamma\right]\langle h_2, h_1\rangle = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{D}^2\,\Phi\left[\gamma\right]\langle h_2, h_1\rangle \end{bmatrix}, \tag{3.8}$$

whence the remainder $\mathrm{R}_{\mathrm{D}A}\left[\gamma, h_1, h_2\right]$ at $\gamma$ reads

$$\mathrm{R}_{\mathrm{D}A}\left[\gamma, h_1, h_2\right] := \mathrm{D}A\left[\gamma + h_2\right]\langle h_1\rangle - \mathrm{D}A\left[\gamma\right]\langle h_1\rangle - \mathrm{D}^2 A\left[\gamma\right]\langle h_1, h_2\rangle, \tag{3.9a}$$

and

$$\lim_{\substack{|h_1|_{W^1_\infty(0,1)}\to 0 \\ |h_2|_{W^1_\infty(0,1)}\to 0}} \frac{\left\|\mathrm{R}_{\mathrm{D}A}\left[\gamma, h_1, h_2\right]\right\|_{L^\infty(\Omega)}}{|h_1|_{W^1_\infty(0,1)}|h_2|_{W^1_\infty(0,1)}} = 0. \tag{3.9b}$$

**Proposition 3.1.1** (bounds on $A$)**.** *Applying the state constraint* (3.5c) *to* (3.4), (3.6) *and* (3.8) *implies the existence of a positive constant* $C_A < \infty$ *such that*

$$\begin{aligned} \left\|A\left[\gamma\right]\right\|_{L^\infty(\Omega)^{2\times 2}} &+ \sup_{|h|_{W^1_\infty(0,1)}=1} \left\|\mathrm{D}A\left[\gamma\right]\langle h\rangle\right\|_{L^\infty(\Omega)^{2\times 2}} \\ &+ \sup_{\substack{|h_1|_{W^1_\infty(0,1)}=1 \\ |h_2|_{W^1_\infty(0,1)}=1}} \left\|\mathrm{D}^2 A\left[\gamma\right]\langle h_1, h_2\rangle\right\|_{L^\infty(\Omega)^{2\times 2}} \le C_A. \end{aligned} \tag{3.10}$$

## 3.2 Lagrangian Multipliers Formulation

In this section we formally derive, the first-order necessary optimality conditions using the Lagrangian approach described in [Trö10]. We will assume that the state constraints (3.5c) are always satisfied upon an appropriate choice of admissible control set $\mathcal{U}_{ad}$; see §3.3 for details. For a rigorous analysis of the existence of Lagrange multipliers in Banach spaces we refer to [ZK79].

It is well known that for a convex optimal control problem with linear constraints, the first-order necessary optimality conditions are also sufficient conditions

[Trö10, Lemma 2.21]. However, despite linearizing the curvature via assumption ($\mathbf{A_1}$), the state equations (3.5b) are still highly nonlinear, and the optimization nonconvex, whence the first-order optimality conditions are in general not sufficient. We will derive the second-order sufficient optimality conditions in section 3.4.

Let $s, r$ denote the adjoint variables corresponding to states $\gamma, y$ respectively, and $p, q$ be Hölder conjugate indices i.e., $1/p + 1/q = 1$ with $p > 2$. Then the Lagrangian functional is given by

$$\begin{aligned}\mathcal{L}(\gamma, y, u, r, s) := \mathcal{J}(\gamma, y, u) + \int_\Omega \operatorname{div}(A[\gamma] \nabla (y+v)) r \, \mathrm{d}x \\ + \int_0^1 \left(\kappa \, \mathrm{d}^2_{x_1} \gamma - A[\gamma] \nabla (y+v) \cdot \nu + u\right) s \, \mathrm{d}\sigma,\end{aligned} \tag{3.11}$$

where the integrals are understood in duality sense, similarly other equations written in the strong form with an integral, in this section are understood as duality pairings as well. Additionally, if $(\bar{\gamma}, \bar{y}, \bar{u}, \bar{r}, \bar{s})$ is a critical point for the Lagrangian $\mathcal{L}$, then the first-order necessary optimality conditions are

$$\left\langle \mathrm{D}_{\{\gamma,y,r,s\}} \mathcal{L}(\bar{\gamma}, \bar{y}, \bar{u}, \bar{r}, \bar{s}), h \right\rangle = 0 \quad \forall h \in \left\{ \mathring{W}^1_\infty(0,1), \mathring{W}^1_p(\Omega), \mathring{W}^1_q(\Omega), \mathring{W}^1_1(0,1) \right\}, \tag{3.12a}$$

$$\left\langle \mathrm{D}_u \mathcal{L}(\bar{\gamma}, \bar{y}, \bar{u}, \bar{r}, \bar{s}), u - \bar{u} \right\rangle \geq 0 \quad \forall u \in \mathcal{U}_{ad}, \tag{3.12b}$$

where $\{\}$ denotes a list, e.g. $\left\langle \mathrm{D}_s \mathcal{L}(\bar{\gamma}, \bar{y}, \bar{u}, \bar{r}, \bar{s}), h \right\rangle = 0$ for all $h \in \mathring{W}^1_1(0,1)$, with $\langle \cdot, \cdot \rangle$ denoting the duality pairing between $\mathring{W}^1_1(0,1)^* = W^{-1}_\infty(0,1)$ and $\mathring{W}^1_1(0,1)$, and $^*$ indicates the dual of $\mathring{W}^1_1(0,1)$. Therefore, computing $(\bar{\gamma}, \bar{y}, \bar{u}, \bar{r}, \bar{s})$ requires solving the nonlinear system (3.12). In practice this can be realized using techniques described in [AHL07, Kel99, Trö10]. To solve variational inequalities of first (3.12b) and second kind we refer to [Glo08] for relaxation and augmented Lagrangian techniques and to [DLRH11] for semi-smooth Newton methods, and references therein. The remainder of this section is devoted to the derivation of the equations satisfied by $(\bar\gamma, \bar y, \bar u, \bar r, \bar s)$ using the nonlinear system above.

Since $\left\langle \mathrm{D}_{\{s,r\}}\, \mathcal{L}(\bar\gamma, \bar y, \bar u, \bar r, \bar s), h \right\rangle = 0$ implies that $(\bar\gamma, \bar y)$ solves the state equations (3.5b), we focus on the adjoint equations $\left\langle \mathrm{D}_{\{\gamma,y\}}\, \mathcal{L}(\bar\gamma, \bar y, \bar u, \bar r, \bar s), h \right\rangle = 0$. Using Green's theorem and assuming smoothness, the Lagrangian $\mathcal{L}$ can be rewritten as:

$$\begin{aligned} \mathcal{L}(y,\gamma,u,r,s) &= \mathcal{J}(\gamma,y,u) + \int_\Omega \operatorname{div}(A[\gamma]\,\nabla r)\,(y+v)\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \big(rA[\gamma]\,\nabla(y+v) - (y+v)\,A[\gamma]\,\nabla r\big)\cdot\nu\,\mathrm{d}\sigma \\ &\quad + \int_0^1 \big(\kappa\gamma\,\mathrm{d}_{x_1}^2 s - sA[\gamma]\,\nabla(y+v)\cdot\nu + us\big)\,\mathrm{d}\sigma + \kappa\left(\mathrm{d}_{x_1}\gamma s - \gamma\,\mathrm{d}_{x_1} s\right)\Big|_0^1. \end{aligned} \tag{3.13}$$

Imposing $\left\langle \mathrm{D}_y\, \mathcal{L}(\bar\gamma, \bar y, \bar u, \bar r, \bar s), h \right\rangle = 0$ to (3.13) implies that for every $h \in \mathring{W}_p^1(\Omega)$

$$\begin{aligned} -\int_\Omega \operatorname{div}(A[\bar\gamma]\,\nabla\bar r)h\,\mathrm{d}x &= \int_\Omega (\bar y + v - y_d)\,(1+\bar\gamma)\,h\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \bar r A[\bar\gamma]\,\nabla h\cdot\nu\,\mathrm{d}\sigma - \int_0^1 \bar s A[\bar\gamma]\,\nabla h\cdot\nu\,\mathrm{d}\sigma. \end{aligned} \tag{3.14a}$$

Next, without loss of generality ($C_0^\infty(\Omega)$ is dense in $\mathring{W}_p^1(\Omega)$), we obtain

$$-\int_\Omega \operatorname{div}(A[\bar\gamma]\,\nabla\bar r)h\,\mathrm{d}x = \int_\Omega (\bar y + v - y_d)\,(1+\bar\gamma)\,h\,\mathrm{d}x \qquad \forall h\in C_0^\infty(\Omega), \tag{3.14b}$$

whereas, using that $A[\gamma]\,\nabla h\cdot\nu$ can be chosen arbitrarily on $\partial\Omega$ we deduce from (3.14a) and (3.14b) that

$$\bar r - \bar s|_\Gamma = 0, \qquad \bar r|_\Sigma = 0. \tag{3.14c}$$

In view of (3.14b-c), the strong form of the boundary value problem for $\bar r$ is: seek

$\bar{r} \in W_q^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(A[\bar{\gamma}]\nabla\bar{r}) = (\bar{y}+v-y_d)(1+\bar{\gamma}) & \text{in } \Omega \\ \bar{r} = \bar{s} & \text{on } \Gamma \\ \bar{r} = 0 & \text{on } \Sigma. \end{cases} \tag{3.15}$$

Next we employ the same technique to obtain the equations for the second adjoint variable $\bar{s}$: we impose $\langle \mathrm{D}_\gamma \mathcal{L}(\bar{\gamma},\bar{y},\bar{u},\bar{r},\bar{s}), h\rangle = 0$ to (3.13) and make use of the boundary conditions in (3.15) to obtain for every $h \in \mathring{W}_\infty^1(0,1)$

$$\begin{aligned} -\int_0^1 \kappa\, \mathrm{d}_{x_1}^2 sh \,\mathrm{d}\sigma = \int_0^1 (\bar{\gamma}-\gamma_d)\, h\,\mathrm{d}\sigma \\ + \frac{1}{2}\int_\Omega |\bar{y}+v-y_d|^2\, h\,\mathrm{d}x - \int_\Omega \mathrm{D}A[\bar{\gamma}]\langle h\rangle \nabla(\bar{y}+v)\cdot\nabla\bar{r}\,\mathrm{d}x. \end{aligned}$$

Therefore, the strong form of the boundary value problem for $\bar{s}$ is: seek $\bar{s} \in \mathring{W}_1^1(0,1)$

$$\begin{cases} \begin{aligned} -\kappa\, \mathrm{d}_{x_1}^2 \bar{s} = (\bar{\gamma}-\gamma_d) + \frac{1}{2}\int_0^1 |\bar{y}+v-y_d|^2\,\mathrm{d}x_2 - \int_0^1 A_1[\bar{\gamma}]\nabla(\bar{y}+v)\cdot\nabla\bar{r}\,\mathrm{d}x_2 \\ + \mathrm{d}_{x_1}\left(\int_0^1 A_2[\bar{\gamma}]\nabla(\bar{y}+v)\cdot\nabla\bar{r}\,\mathrm{d}x_2\right) \quad \text{in } (0,1) \end{aligned} \\ \bar{s}(0) = \bar{s}(1) = 0, \end{cases} \tag{3.16}$$

where $A_1, A_2$ denote the representation of $DA$ given in (3.6). We note that the integrals on the right hand side of (3.16) correspond to integration in $x_2$ (vertical) direction.

Finally, (3.12b) implies

$$\langle \lambda\bar{u}+\bar{s}, u-\bar{u}\rangle_{L^2(0,1),L^2(0,1)} \geq 0, \qquad \forall u \in \mathcal{U}_{ad}. \tag{3.17}$$

To summarize, the solution $(\bar{\gamma},\bar{y},\bar{u},\bar{r},\bar{s})$ to the first-order optimality system (3.12) satisfies (3.5b), (3.15), (3.16) and (3.17). We stress that the formal approach presented in this section is very systematic and highly useful even though it is not clear at the moment how to show the existence and (local) uniqueness of the optimal control $\bar{u}$. A rigorous analysis will be developed in the next two sections.

## 3.3 The Control-to-state Map $G_v$

Let $G_v$ denote the nonlinear map

$$\begin{array}{rcccc} G_v & : & \mathcal{U} & \longrightarrow & \mathbb{W}^1 \\ & & u & \longmapsto & (\gamma, y) \end{array}, \tag{3.18}$$

where $\mathbb{W}^1 := \mathring{W}^1_\infty(0,1) \times \mathring{W}^1_p(\Omega)$, $(\gamma, y)$ solves (3.5b), and the subscript on $G_v$ denotes dependence on a fixed and non-trivial $v \in W^1_p(\Omega)$. Furthermore, $\mathcal{U} \subset L^2(0,1)$ is open, such that

$$\mathcal{U}_{ad} \subset \mathcal{U} \subset L^2(0,1),$$

which will be precisely specified in Definition 3.3.4. Our goal is to show the existence of a control, derive the first-order necessary and second-order sufficient optimality conditions within the realm of a rigorous mathematical framework. The first-order optimality conditions requires to show that $G_v$ is Fréchet differentiable (subsection 3.3.3) and the second order conditions require $G_v$ to be twice Fréchet differentiable (subsection 3.3.4).

The steps described above are standard for PDE-constrained optimization in fixed domains [Trö10], but our analysis for the linearized curvature OC-FBP is novel. The novelty resides in the highly nonlinear structure of the underlying FBP, which is posed in a pair of Banach spaces one being non-reflexive, and yet we deal with *minimal regularity*. A number of other control problems for FBPs fall under a similar functional framework [MEl07, MEl10], but their theory is not as complete and conclusive as ours. This appears to be an area of intense current research.

The first step in this voyage is to show that there exists a unique weak solution to (3.5b), which implies that $G_v$ is a well defined one-to-one nonlinear operator. In fact, it is known [SS91] that for $u = 0$ and $v$ small, a fixed point argument asserts the existence and uniqueness of a weak solution $(\gamma, y)$ in $\mathbb{W}^1$ to (3.5b). We further extend this analysis to the case where $u \neq 0$. This gives us an open ball $\mathcal{U} \subset L^2(0,1)$ where we can show the existence of solution to (3.5b).

### 3.3.1 Well-posedness of the State System (3.5b)

The weak form of the system (3.5b) is: find $(\gamma, y) \in \mathbb{W}^1$ such that

$$\begin{cases} \mathcal{B}_\Omega\left[y+v, z; A[\gamma]\right] = 0 & \forall z \in \mathring{W}^1_q(\Omega) \\ \mathcal{B}_\Gamma[\gamma,\zeta] + \mathcal{B}_\Omega\left[y+v, E\zeta; A[\gamma]\right] = \langle u, \zeta\rangle_{W^{-1}_\infty(0,1), \mathring{W}^1_1(0,1)} & \forall \zeta \in \mathring{W}^1_1(0,1), \end{cases} \tag{3.19}$$

where $\mathcal{B}_\Gamma : \mathring{W}^1_\infty(0,1) \times \mathring{W}^1_1(0,1) \to \mathbb{R}$, $\mathcal{B}_\Omega : \mathring{W}^1_p(\Omega) \times \mathring{W}^1_q(\Omega) \to \mathbb{R}$ are defined by

$$\begin{aligned} \mathcal{B}_\Gamma[\gamma,\zeta] &:= \kappa \int_0^1 \mathrm{d}_{x_1}\gamma(x_1)\mathrm{d}_{x_1}\zeta(x_1)\,\mathrm{d}x_1, \\ \mathcal{B}_\Omega\left[y, z; A[\gamma]\right] &:= \int_\Omega A[\gamma]\,\nabla y \cdot \nabla z\,\mathrm{d}x. \end{aligned} \tag{3.20}$$

Furthermore, $E : \mathring{W}^1_1(0,1) \to W^1_q(\Omega)$, $q < 2$ denotes a continuous extension such that $E\zeta|_\Gamma = \zeta$, $E\zeta|_\Sigma = 0$ (cf. [SS91, Lemma 2]). In particular, this implies that there exists a constant $C_E \geq 1$ such that

$$|E\zeta|_{W^1_q(\Omega)} \leq C_E |\zeta|_{W^1_1(0,1)}, \quad \forall \zeta \in \mathring{W}^1_1(0,1). \tag{3.21}$$

Moreover, when $u \in \mathcal{U} \subset L^2(0,1)$ and the test function $\zeta \in \mathring{W}_1^1(0,1)$, then $\zeta \in L^2(0,1)$ and we may write

$$\langle u, \zeta \rangle_{W_\infty^{-1}(0,1), \mathring{W}_1^1(0,1)} = \int_0^1 u\zeta, \tag{3.22}$$

where $W_\infty^{-1}(0,1)$ is the dual space of $\mathring{W}_1^1(0,1)$, we refer to [AF03]. This also enables us to deduce that for $u \in L^2(0,1)$

$$\|u\|_{W_\infty^{-1}(0,1)} \leq \|u\|_{L^2(0,1)}. \tag{3.23}$$

We will make use of these two facts repeatedly throughout the rest of the chapter.

**Proposition 3.3.1** (inf-sup conditions)**.** *The following conditions hold for the bilinear forms* $\mathcal{B}_\Gamma[\cdot,\cdot]$ *and* $\mathcal{B}_\Omega\left[\cdot,\cdot;A[\gamma]\right]$ *defined in* (3.20) *:*

*(i)* $\mathcal{B}_\Gamma[\cdot,\cdot]$ *is continuous and there exists a constant* $\alpha > 0$ *such that for every* $\gamma \in \mathring{W}_\infty^1(0,1)$ *and* $s \in \mathring{W}_1^1(0,1)$

$$|\gamma|_{W_\infty^1(0,1)} \leq \alpha \sup_{0 \neq \zeta \in \mathring{W}_1^1(0,1)} \frac{\mathcal{B}_\Gamma[\gamma,\zeta]}{|\zeta|_{W_1^1(0,1)}}, \tag{3.24a}$$

$$|s|_{W_1^1(0,1)} \leq \alpha \sup_{0 \neq \zeta \in \mathring{W}_\infty^1(0,1)} \frac{\mathcal{B}_\Gamma[\zeta,s]}{|\zeta|_{W_\infty^1(0,1)}}. \tag{3.24b}$$

*(ii) If* $\gamma \in \mathring{W}_\infty^1(0,1)$ *satisfies* (3.5c)*, then* $\mathcal{B}_\Omega\left[\cdot,\cdot;A[\gamma]\right]$ *is continuous and there exist constants* $P, Q$ *with* $Q < 2 < P$ *and* $\beta > 0$*, such that for* $p \in (Q,P)$ *and for all* $y \in \mathring{W}_p^1(\Omega)$

$$|y|_{W_p^1(\Omega)} \leq \beta \sup_{0 \neq z \in \mathring{W}_q^1(\Omega)} \frac{\mathcal{B}_\Omega\left[y,z;A[\gamma]\right]}{|z|_{W_q^1(\Omega)}}. \tag{3.25}$$

*Proof.* For (3.24a) and (3.25) we refer to [SS91, Proposition 2.2-2.3] for a proof. For (3.24b) we proceed as follows: applying the definition of the $L^1$-norm and the homogeneous Dirichlet values of $s$, we obtain

$$|s|_{W_1^1(0,1)} = \int_0^1 |\mathrm{d}_{x_1} s| = \int_0^1 \operatorname{sgn}(\mathrm{d}_{x_1} s)\, \mathrm{d}_{x_1} s$$

Using the fact that $s \in \mathring{W}_1^1(0,1)$, for any constant $c > 0$ we get, $c\int_0^1 \mathrm{d}_{x_1} s = 0$, whence

$$|s|_{W_1^1(0,1)} = \int_0^1 \underbrace{\left(\operatorname{sgn}(\mathrm{d}_{x_1} s) - \int_0^1 \operatorname{sgn}(\mathrm{d}_{x_1} s)\right)}_{=\mathrm{d}_{x_1}\zeta} \mathrm{d}_{x_1} s = \frac{1}{\kappa}\mathcal{B}_\Gamma\left[\zeta, s\right],$$

where $\zeta(x_1) = \int_0^{x_1}\left(\operatorname{sgn}(\mathrm{d}_{x_1} s) - \int_0^1 \operatorname{sgn}(\mathrm{d}_{x_1} s)\right) \in \mathring{W}_\infty^1(0,1)$. Estimate (3.24b) follows by noting that $|\zeta|_{W_\infty^1(0,1)} \leq 2$, and taking the sup over every $\zeta \in \mathring{W}_\infty^1(0,1)$. □

The following lemma demonstrates how one can improve the integrability index of a solution to a PDE obtained by standard methods.

**Lemma 3.3.2** (improved integrability)**.** *Let $\Omega$ be an open Lipschitz bounded domain of $\mathbb{R}^d$ and $\mathcal{B} : \mathring{W}_\infty^1(\Omega) \times \mathring{W}_1^1(\Omega) \to \mathbb{R}$ be a continuous bilinear form. Furthermore, suppose that*

*(i) there exists $\alpha > 0$ such that*

$$|\chi|_{W_\infty^1(\Omega)} \leq \alpha \sup_{0\neq\psi\in\mathring{W}_1^1(\Omega)} \frac{\mathcal{B}\left[\chi,\psi\right]}{|\psi|_{W_1^1(\Omega)}} \qquad \forall \chi \in \mathring{W}_\infty^1(\Omega), \tag{3.26}$$

*(ii) and $\mathcal{B}$ is continuous and coercive in $\mathring{W}_2^1(\Omega)$.*

*Then for every $F \in \mathring{W}_1^1(\Omega)^*$, there exists a unique $\chi \in \mathring{W}_\infty^1(\Omega)$ such that*

$$\mathcal{B}\left[\chi,\psi\right] = F(\psi) \text{ for all } \psi \in \mathring{W}_1^1(\Omega) \text{ and } |\chi|_{W_\infty^1(\Omega)} \leq \alpha \|F\|_{W_1^1(\Omega)^*}. \tag{3.27}$$

*Proof.* Since $\mathring{W}^1_2(\Omega) \subset \mathring{W}^1_1(\Omega)$, it follows that $F \in \mathring{W}^1_1(\Omega)^* \subset \mathring{W}^1_2(\Omega)^*$, where $^*$ denotes the dual space. The Lax-Milgram lemma guarantees the existence and uniqueness of $\chi \in \mathring{W}^1_2(\Omega)$ such that $\mathcal{B}[\chi, \psi] = F(\psi)$ for all $\psi \in \mathring{W}^1_2(\Omega)$.

Next, we extend $\mathcal{B}[\chi, \cdot]$ as a linear functional on $\mathring{W}^1_1(\Omega)$. To this end, let $\{\psi_n\} \subset \mathring{W}^1_2(\Omega)$ be a Cauchy sequence in the $\mathring{W}^1_1(\Omega)$-norm. It immediately follows that $\{\mathcal{B}[\chi, \psi_n]\}$ is also Cauchy in $\mathbb{R}$, i.e.

$$\left|\mathcal{B}[\chi, \psi_n - \psi_m]\right| = \left|F(\psi_n - \psi_m)\right| \leq \|F\|_{\mathring{W}^1_1(\Omega)^*} |\psi_n - \psi_m|_{W^1_1(\Omega)} .$$

Finally, by the density of $\mathring{W}^1_2(\Omega)$ in $\mathring{W}^1_1(\Omega)$, not only do we obtain $\psi_n \to \psi \in \mathring{W}^1_1(\Omega)$, but also

$$\mathcal{B}[\chi, \psi] := \lim_{n\to\infty} \mathcal{B}[\chi, \psi_n] = \lim_{n\to\infty} F(\psi_n) = F(\psi).$$

The estimate for $|\chi|_{W^1_\infty(0,1)}$ follows from (3.26). $\square$

#### 3.3.1.1 First-order regularity

Now we are ready to prove that there exists a unique solution to (3.19) with first-order regularity. Since the system (3.19) is nonlinear we will obtain this result by applying the Banach fixed point theorem combined with a smallness assumption on a non-trivial $v$. To this end, we let $2 < p < P$ and equip the space $\mathbb{W}^1 = \mathring{W}^1_\infty(0,1) \times \mathring{W}^1_p(\Omega)$ with the equivalent norm

$$\left\|(\gamma, y)\right\|_{\mathbb{W}^1} := (1 + \beta C_A)|v|_{W^1_p(\Omega)} |\gamma|_{W^1_\infty(0,1)} + |y|_{W^1_p(\Omega)} , \tag{3.28}$$

where $C_A$ and $\beta$ are given in (3.10) and (3.25), and define the closed (convex) ball

$$\mathbb{B}_v := \left\{(\gamma, y) \in \mathbb{W}^1 : |y|_{W^1_p(\Omega)} \leq \beta C_A |v|_{W^1_p(\Omega)} , |\gamma|_{W^1_\infty(0,1)} \leq 1\right\}. \tag{3.29}$$

Furthermore, consider the operator $T : \mathbb{B}_v \to \mathbb{W}^1$ defined as

$$T(\gamma, y) := \big(T_1(\gamma, y), T_2(\gamma, y)\big) = (\widetilde{\gamma}, \widetilde{y}) \qquad \forall\, (\gamma, y) \in \mathbb{B}_v, \tag{3.30}$$

where $\widetilde{\gamma} = T_1(\gamma, y) \in \mathring{W}^1_\infty(0,1)$ satisfies for every $\zeta \in \mathring{W}^1_1(0,1)$

$$\mathcal{B}_\Gamma\left[\widetilde{\gamma}, \zeta\right] = -\mathcal{B}_\Omega\left[y + v, E\zeta; A\left[\gamma\right]\right] + \langle u, \zeta\rangle_{W_\infty^{-1}(0,1), \mathring{W}^1_1(0,1)}, \tag{3.31}$$

and $\widetilde{y} = T_2(\gamma, y) \in \mathring{W}^1_p(\Omega)$ satisfies for every $z \in \mathring{W}^1_q(\Omega)$

$$\mathcal{B}_\Omega\Big[\widetilde{y} + v, z; A\left[T_1\left(\gamma, y\right)\right]\Big] = 0. \tag{3.32}$$

With these definitions at hand we proceed to find conditions under which $T$ not only maps $\mathbb{B}_v$ into itself but is in fact a contraction in $\mathbb{B}_v$.

**Lemma 3.3.3** (range of $T$)**.** *Let $T_1$ and $T_2$ be the operators defined in* (3.31) *and* (3.32)*, and $C_A$ and $C_E$ be the constants defined in* (3.10) *and* (3.21)*. Furthermore, suppose there exists $\theta_1 \in \big(\beta C_A/(1+\beta C_A), 1\big)$ such that*

$$|v|_{W^1_p(\Omega)} \leq (1-\theta_1)\left(\alpha C_E C_A\left(1+\beta C_A\right)\right)^{-1}. \tag{3.33}$$

*If $u \in L^2(0,1)$ with $\|u\|_{L^2(0,1)} \leq \theta_1/\alpha$, then the range of $T$ is contained in $\mathbb{B}_v$.*

*Proof.* Let $(\gamma, y) \in \mathbb{B}_v$ be fixed but arbitrary. First we rely on Lemma 3.3.2 to show the well-posedness of $T_1$. Since it is straight-forward to check that $\mathcal{B}_\Gamma$ is continuous and coercive in $\mathring{W}^1_2(0,1)$, we only need to show the regularity of the forcing term in (3.31). If we define $F(\zeta) := -\mathcal{B}_\Omega\left[y + v, E\zeta; A\left[\gamma\right]\right] + \langle u, \zeta\rangle$ and use (3.10), (3.21) and (3.29) we find that

$$\begin{aligned}\big|F(\zeta)\big| &\leq C_A\Big(|y|_{W^1_p(\Omega)} + |v|_{W^1_p(\Omega)}\Big)|E\zeta|_{W^1_q(\Omega)} + \|u\|_{L^2(0,1)}|\zeta|_{W^1_1(0,1)} \\ &\leq \Big(C_E C_A\left(1+\beta C_A\right)|v|_{W^1_p(\Omega)} + \|u\|_{L^2(0,1)}\Big)|\zeta|_{W^1_1(0,1)}\,,\end{aligned} \tag{3.34}$$

whence $F \in \mathring{W}_1^1(0,1)^*$ and we conclude from (3.27) that

$$|\widetilde{\gamma}|_{W^1_\infty(0,1)} \leq \alpha \left( C_E C_A \left(1 + \beta C_A\right) |v|_{W^1_p(\Omega)} + \|u\|_{L^2(0,1)} \right) \leq (1 - \theta_1) + \theta_1 = 1.$$

The well-posedness of $T_2$ follows by Proposition 3.3.1 and the Banach-Nečas theorem for reflexive Banach spaces [EG04, Theorem 2.6]. Applying (3.25) we obtain

$$|\widetilde{y}|_{W^1_p(\Omega)} \leq \beta C_A |v|_{W^1_p(\Omega)} .$$

Since $(\gamma, y)$ is arbitrary, we conclude that the range of $T$ is contained in $\mathbb{B}_v$. □

**Definition 3.3.4** (an open set $\mathcal{U}$)**.** Let $\theta_1$ be as defined in Lemma 3.3.3. We can define a nontrivial open ball $\mathcal{U} \subset L^2(0,1)$ as

$$\mathcal{U} := \left\{ u \in L^2(0,1) \ : \ \|u\|_{L^2(0,1)} < \theta_1/\alpha \right\}. \tag{3.35}$$

To this end we are ready to define the closed, convex set of admissible controls.

**Definition 3.3.5** (admissible control set)**.** Let $\theta_1$ be as defined in Lemma 3.3.3. The admissible set of controls $\mathcal{U}_{ad}$ is the (nontrivial) closed ball

$$\mathcal{U}_{ad} := \left\{ u \in \mathcal{U} \ : \ \|u\|_{L^2(0,1)} \leq \theta_1/2\alpha \right\}. \tag{3.36}$$

We remark that we have chosen a closed subset of an open set $\mathcal{U}$, as our admissible set of controls. We need $\mathcal{U}$ to open because we will discuss the Fréchet differentiability of $G_v$ in §3.3.3.2. In the next theorem we will show that the state equations are solvable for any $u \in \mathcal{U}$.

For practical implementation a precise bound $\theta_1/2\alpha$ is needed in the Definition of admissible control set 3.3.5. This bound depends on $\alpha$, $\beta$ and $C_A$ via the definition of $\theta_1$. We remark that $\alpha$ can be precisely estimated from the proof of [SS91, Proposition 2.1] and is given by $2/\kappa$, where $\kappa$ is the surface tension coefficient. $\beta$ depends on the smallest eigenvalue of the operator $A$ (3.4). Finally, $C_A > 0$ is an upper bound on the $L^\infty$ norm of $A$. Using the fact that $|\mathrm{d}_{x_1}\gamma| \leq 1$, it might be possible to estimate for $\beta$ and $C_A$. In our numerical experiments, we have found that the constraint in (3.36) is not so restrictive, this discussion is part of a forthcoming paper.

**Theorem 3.3.6** (T is a contraction)**.** *Let the assumptions of Lemma 3.3.3 hold and suppose further that there exists a* $\theta_2 \in (0,1)$ *such that*

$$|v|_{W^1_p(\Omega)} \leq (1-\theta_2)\left(\alpha C_E C_A \left(1+\beta C_A\right)^2\right)^{-1}. \tag{3.37}$$

*Then, the map* $T$ *defined in* (3.30) *is a contraction in* $\mathbb{B}_v$ *with constant* $1-\theta_2$.

*Proof.* Consider $(\gamma_1, y_1), (\gamma_2, y_2) \in \mathbb{B}_v$ such that $(\gamma_1, y_1) \neq (\gamma_2, y_2)$. Using (3.30) we have that $T(\gamma_i, y_i) = (\widetilde{\gamma}_i, \widetilde{y}_i)$ solves (3.31) and (3.32) for $i = 1, 2$. Therefore, combining Proposition 3.3.1 (i) and Lemma 3.3.3 with (3.37) implies

$$\begin{aligned}
|\widetilde{\gamma}_1 - \widetilde{\gamma}_2|_{W^1_\infty(0,1)} &\leq \alpha \sup_{|\zeta|_{W^1_1(0,1)}=1} \mathcal{B}_\Gamma\left[\widetilde{\gamma}_1 - \widetilde{\gamma}_2, \zeta\right] \\
&= \alpha \sup_{|\zeta|_{W^1_1(0,1)}=1} \mathcal{B}_\Omega\left[y_2 - y_1, E\zeta; A\left[\gamma_1\right]\right] + \mathcal{B}_\Omega\left[y_2 + v, E\zeta; A\left[\gamma_2\right] - A\left[\gamma_1\right]\right] \\
&\leq \alpha C_E C_A \left(|y_1 - y_2|_{W^1_p(\Omega)} + (1+\beta C_A)|v|_{W^1_p(\Omega)}|\gamma_1 - \gamma_2|_{W^1_\infty(0,1)}\right) \\
&= \alpha C_E C_A \left\|(\gamma_1 - \gamma_2, y_1 - y_2)\right\|_{\mathbb{W}^1}. 
\end{aligned} \tag{3.38}$$

Similarly, Proposition 3.3.1 (ii) in conjunction with (3.32) leads to

$$\begin{aligned}
|\widetilde{y}_1 - \widetilde{y}_2|_{W^1_p(\Omega)} &\leq \beta \sup_{|z|_{W^1_q(\Omega)}=1} \mathcal{B}_\Omega\left[\widetilde{y}_1 - \widetilde{y}_2, z; A\left[\widetilde{\gamma}_1\right]\right] \\
&= \beta \sup_{|z|_{W^1_q(\Omega)}=1} \mathcal{B}_\Omega\left[\widetilde{y}_2 + v, z; A\left[\widetilde{\gamma}_2\right] - A\left[\widetilde{\gamma}_1\right]\right] \qquad (3.39) \\
&\leq \beta C_A |\widetilde{y}_2 + v|_{W^1_p(\Omega)} |\widetilde{\gamma}_1 - \widetilde{\gamma}_2|_{W^1_\infty(0,1)} \\
&\leq \beta C_A\left(1 + \beta C_A\right) |v|_{W^1_p(\Omega)} |\widetilde{\gamma}_1 - \widetilde{\gamma}_2|_{W^1_\infty(0,1)} .
\end{aligned}$$

Finally, (3.38) and (3.39) yield

$$\begin{aligned}
\left\|(\widetilde{\gamma}_1, \widetilde{y}_1) - (\widetilde{\gamma}_2, \widetilde{y}_2)\right\|_{\mathbb{W}^1} &\leq \left(1 + \beta C_A\right)^2 |v|_{W^1_p(\Omega)} |\widetilde{\gamma}_1 - \widetilde{\gamma}_2|_{W^1_\infty(0,1)} \\
&\leq \alpha C_E C_A \left(1 + \beta C_A\right)^2 |v|_{W^1_p(\Omega)} \left\|(\gamma_1, y_1) - (\gamma_2, y_2)\right\|_{\mathbb{W}^1} \\
&\leq (1 - \theta_2) \left\|(\gamma_1, y_1) - (\gamma_2, y_2)\right\|_{\mathbb{W}^1} ,
\end{aligned}$$

where the last inequality follows from (3.37). Since $\theta_2 \in (0, 1)$, $T$ is a contraction with constant $1 - \theta_2$ as asserted. □

We remark that the state constraint (3.5c) is used in the proof of Theorem 3.3.6 at two instances. The first is when we estimate $A[\gamma_2] - A[\gamma_1]$. The second use is when we invoke the inf-sup constant $\beta$ for $y$; see (3.25). For details on how $\beta$ depends on the state constraint we refer to [SS91, Proposition 2.3].

**Corollary 3.3.7** (well-posedness of state system)**.** *For every $u \in \mathcal{U}$, the open ball of Definition 3.3.4, and $v$ satisfying* (3.33) *and* (3.37)*, there exists a unique solution* $(\gamma, y) \in \mathbb{W}^1$ *to the state equations* (3.19)*. This further implies that $G_v$ in* (3.18) *is a well defined, one-to-one, nonlinear operator.*

*Proof.* Let $u \in \mathcal{U}$ be fixed but arbitrary. It now follows that $T$ is a contraction in the closed convex set $\mathbb{B}_v$ (cf. Theorem 3.3.6) and applying the Banach fixed point theorem we obtain a unique $(\gamma, y) \in \mathbb{B}_v$ such that $T(\gamma, y) = (\gamma, y)$. In view of (3.31) and (3.32), this is equivalent to saying that $(\gamma, y)$ is the weak solution to the FBP (3.19), i.e. $G_v(u) = (\gamma, y)$. □

#### 3.3.1.2 Enhanced Regularity of $\gamma$

Corollary 3.3.7 implies the existence and uniqueness of a solution $(\gamma, y)$ to (3.5b) with first-order regularity, provided $u \in \mathcal{U}$ and $v$ satisfies (3.33) and (3.37). That is, we only have one weak derivative for $\gamma$ and $y$. In the sequel we will show that the solution $(\gamma, y) = G_v(u)$ is slightly more regular *without* any extra restrictions on $u$ or $v$. More specifically, we will show that

$$\gamma \in W_p^{1+1/q}(0,1) \cap \mathring{W}_\infty^1(0,1). \tag{3.40}$$

The importance of this result will be evident in subsection 3.4.1 where the existence of an optimal control is proven. Despite its importance, the proof is rather simple.

Let $(\gamma, y) \in \mathbb{W}^1$ be a weak solution to (3.19). Function $\gamma$ satisfies

$$-\kappa\, \mathrm{d}_{x_1}^2 \gamma = -A[\gamma] \nabla (y + v) \cdot \nu + u =: f,$$

in the sense of distributions. If we assume, for the moment, that $f \in W_p^{-1/p}(0,1)$, then $\mathrm{d}_{x_1}^2 \gamma \in W_p^{-1/p}(0,1)$. This directly implies $\gamma \in W_p^{-1/p+2}(0,1)$, i.e. $\gamma \in W_p^{1+1/q}(0,1)$ as desired. Thus, it remains to show that $f$ is in $W_p^{-1/p}(0,1)$ as suggested. Since $u \in L^2(0,1)$, we just need to deal with the first term.

Using the Lions-Magenes [LM61, Théorèm 3.1] we have that $\mathring{W}_q^{1/p}(0,1) = W_q^{1/p}(0,1)$ for $q < 2$. Therefore every $\phi \in \mathring{W}_q^{1/p}(0,1)$ can be seen as the restriction of the trace of a function $E\phi \in W_q^1(\Omega)$, in particular $E\phi|_\Gamma = \phi$, $E\phi|_\Sigma = 0$, and $\|E\phi\|_{W_q^1(\Omega)} = \|\phi\|_{W_q^{1/p}(0,1)}$. With this in mind,

$$\left\langle A\left[\gamma\right]\nabla\left(y+v\right)\cdot\nu,\phi\right\rangle_{W_p^{-1/p}(\Gamma),\mathring{W}_q^{1/p}(\Gamma)} = \int_\Omega A\left[\gamma\right]\nabla\left(y+v\right)\cdot\nabla E\phi,$$

whence

$$\left\|A\left[\gamma\right]\nabla\left(y+v\right)\cdot\nu\right\|_{W_p^{-1/p}(\Gamma)} \le C_E C_A\left(1+\beta C_A\right)|v|_{W_p^1(\Omega)}.$$

We collect this result in the next theorem.

**Theorem 3.3.8** (enhanced regularity). *If $G_v(u) = (\gamma, y) \in \mathbb{W}^1$ is the solution in Corollary 3.3.7, then $\gamma \in W_p^{1+1/q}(0,1) \cap \mathring{W}_\infty^1(0,1)$ without any further assumptions on $u$ or $v$.*

### 3.3.2 $G_v$ is Lipschitz Continuous

The first step to show that $G_v$ is twice Fréchet differentiable is to demonstrate that it is Lipschitz continuous.

In the interest of saving some space we will rewrite the variational system (3.19) in the following form: find $(\gamma, y) \in \mathbb{W}^1$ such that for every $(\zeta, z) \in \mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$

$$\mathcal{B}_\Gamma\left[\gamma,\zeta\right] + \mathcal{B}_\Omega\left[y+v, z+E\zeta; A\left[\gamma\right]\right] = \langle u,\zeta\rangle_{W_\infty^{-1}(0,1),\mathring{W}_1^1(0,1)}. \tag{3.41}$$

With this new notation in place we are ready to study the Lipschitz continuity of $G_v$.

**Theorem 3.3.9** (Lipschitz continuity of $G_v$). *If $v$ fulfills the conditions of Corollary 3.3.7, then $G_v$ satisfies*

$$\left\|G_v(u_1) - G_v(u_2)\right\|_{\mathbb{W}^1} \le L_G\|u_1-u_2\|_{L^2(0,1)} \qquad \forall u_1,u_2 \in \mathcal{U}, \tag{3.42}$$

*with constant* $L_G = \frac{\alpha}{\theta_2}\left(1+\beta C_A\right)^2 |v|_{W_p^1(\Omega)}$.

*Proof.* Given $u_1, u_2 \in \mathcal{U}$, set $(\gamma_1, y_1) - (\gamma_2, y_2) = G_v(u_1) - G_v(u_2)$. Using (3.41), we have for every $(\zeta, z) \in \mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$

$$\begin{aligned}\mathcal{B}_\Gamma\left[\gamma_1-\gamma_2,\zeta\right] &+ \mathcal{B}_\Omega\left[y_1+v, z+E\zeta; A\left[\gamma_1\right]\right] \\ &- \mathcal{B}_\Omega\left[y_2+v, z+E\zeta; A\left[\gamma_2\right]\right] = \langle u_1-u_2,\zeta\rangle_{L^2(0,1),L^2(0,1)}.\end{aligned}$$

Subtracting $\mathcal{B}_\Omega\left[y_2+v, z+E\zeta; A\left[\gamma_1\right]\right]$ from both sides and rearranging terms yields

$$\begin{aligned}\mathcal{B}_\Gamma\left[\gamma_1-\gamma_2,\zeta\right] &+ \mathcal{B}_\Omega\left[y_1-y_2, z+E\zeta; A\left[\gamma_1\right]\right] \\ &= \mathcal{B}_\Omega\left[y_2+v, z+E\zeta; A\left[\gamma_2\right]-A\left[\gamma_1\right]\right] + \langle u_1-u_2,\zeta\rangle_{L^2(0,1),L^2(0,1)}.\end{aligned}$$

The inf-sup estimates from Proposition 3.3.1, together with $(\gamma_i, y_i) \in \mathbb{B}_v$ for $i = 1, 2$, imply for $\zeta = 0$

$$\begin{aligned}|y_1-y_2|_{W_p^1(\Omega)} &\le \beta C_A |y_2+v|_{W_p^1(\Omega)} |\gamma_1-\gamma_2|_{W_\infty^1(0,1)} \\ &\le \beta C_A\left(1+\beta C_A\right)|v|_{W_p^1(\Omega)} |\gamma_1-\gamma_2|_{W_\infty^1(0,1)}, \end{aligned} \tag{3.43}$$

and for $z = 0$

$$\begin{aligned}|\gamma_1-\gamma_2|_{W_\infty^1(0,1)} &\le \alpha C_E C_A\left(|y_1-y_2|_{W_p^1(\Omega)} + |y_2+v|_{W_p^1(\Omega)}|\gamma_1-\gamma_2|_{W_\infty^1(0,1)}\right) \\ &\quad + \alpha\|u_1-u_2\|_{L^2(0,1)} \\ &\le \alpha C_E C_A\left(1+\beta C_A\right)^2 |v|_{W_p^1(\Omega)} |\gamma_1-\gamma_2|_{W_\infty^1(0,1)} + \alpha\|u_1-u_2\|_{L^2(0,1)}.\end{aligned}$$

Finally, in view of (3.37), we infer that

$$|\gamma_1-\gamma_2|_{W_\infty^1(0,1)} \le \frac{\alpha}{\theta_2}\|u_1-u_2\|_{L^2(0,1)}. \tag{3.44}$$

The asserted estimate follows immediately from the definition of $\|\cdot\|_{\mathbb{W}^1}$ in (3.28). $\square$

### 3.3.3 $G_v$ is Fréchet Differentiable

The next step towards showing the twice Fréchet differentiability of $G_v$ entails analyzing the well-posedness of the *linear* variational system: find $(\gamma, y) \in \mathbb{W}^1$ such that for every $(\zeta, z) \in \mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$

$$\mathcal{B}_\Gamma[\gamma, \zeta] + \mathcal{D}_\Omega\left[(\gamma, y), z + E\zeta; \bar{\gamma}, \bar{y}\right] = F_\Omega(z + E\zeta) + F_\Gamma(\zeta) \tag{3.45}$$

where

$$\mathcal{D}_\Omega\left[(\gamma, y), \cdot; \bar{\gamma}, \bar{y}\right] := \mathcal{B}_\Omega\left[y, \cdot; A[\bar{\gamma}]\right] + \mathcal{B}_\Omega\left[\bar{y} + v, \cdot; \mathrm{D}A[\bar{\gamma}]\langle\gamma\rangle\right],$$

$(\bar{\gamma}, \bar{y}) = G_v(\bar{u}) \in \mathbb{B}_v$ for a fixed $\bar{u}$ in $\mathcal{U}$, $\mathrm{D}A[\bar{\gamma}]\langle\gamma\rangle$ is given in (3.6), and $F_\Omega \in W_q^1(\Omega)^*$ and $F_\Gamma \in \mathring{W}_1^1(0,1)^*$ are fixed but arbitrary.

#### 3.3.3.1 Preliminary Estimates

Given that the coupled system (3.45) is linear, one would be inclined to use the standard Banach-Nečas theorem to prove its well-posedness directly. We deviate from this approach and resort to the machinery already put in place.

Consider the operator $T : \mathbb{W}^1 \to \mathbb{W}^1$ given by

$$T(\gamma, y) := \big(T_1(\gamma, y), T_2(\gamma, y)\big) = (\widehat{\gamma}, \widehat{y}) \quad \forall (\gamma, y) \in \mathbb{W}^1, \tag{3.46}$$

where $\widehat{\gamma} = T_1(\gamma, y) \in \mathring{W}_\infty^1(0,1)$ satisfies for every $\zeta \in \mathring{W}_1^1(0,1)$

$$\mathcal{B}_\Gamma[\widehat{\gamma}, \zeta] = -\mathcal{D}_\Omega\left[(\gamma, y), E\zeta; \bar{\gamma}, \bar{y}\right] + F_\Omega(E\zeta) + F_\Gamma(\zeta), \tag{3.47}$$

and $\widehat{y} = T_2(\gamma, y) \in \mathring{W}_p^1(\Omega)$ satisfies for every $z \in \mathring{W}_q^1(\Omega)$

$$\mathcal{B}_\Omega\left[\widehat{y}, z; A[\bar{\gamma}]\right] = -\mathcal{B}_\Omega\left[\bar{y} + v, z; \mathrm{D}A[\bar{\gamma}]\langle T_1(\gamma, y)\rangle\right] + F_\Omega(z). \tag{3.48}$$

We point out that any fixed point of $T$ is also a solution to (3.45). To infer the existence of a fixed point we exploit the linear structure of (3.45). Therefore, it suffices to show the well-posedness of the intermediate operators $T_1$ and $T_2$, and to show that $T$ is a contraction in $\mathbb{W}^1$.

**Lemma 3.3.10** (well-posedness of $T_1$ and $T_2$)**.** *Let $T_1$, $T_2$ be the operators defined in* (3.47) *and* (3.48) *with $(\bar{\gamma}, \bar{y}) \in \mathbb{B}_v$. The following holds*

*(i) for every $(\gamma, y) \in \mathbb{W}^1$, there exists a unique $\widehat{\gamma} = T_1(\gamma, y)$ satisfying* (3.47) *and*

$$|\widehat{\gamma}|_{W^1_\infty(0,1)} \leq \alpha \left( C_E C_A \big\|(\gamma, y)\big\|_{\mathbb{W}^1} + C_E \|F_\Omega\|_{W^1_q(\Omega)^*} + \|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*} \right),$$

*(ii) for every $(\gamma, y) \in \mathbb{W}^1$, there exists a unique $\widehat{y} = T_2(\gamma, y)$ satisfying* (3.48) *and*

$$|\widehat{y}|_{W^1_p(\Omega)} \leq \beta C_A \,(1 + \beta C_A) |v|_{W^1_p(\Omega)} |\widehat{\gamma}|_{W^1_\infty(0,1)} + \beta \|F_\Omega\|_{W^1_q(\Omega)^*}\,.$$

*Proof.* To prove (i) we proceed as in Lemma 3.3.3. It suffices to check that the right-hand-side RHS$(\zeta)$ of (3.47) is in $\mathring{W}^1_1(0,1)^*$, namely

$$\big|\mathrm{RHS}(\zeta)\big| \leq \Bigg( C_E C_A \Big( |y|_{W^1_p(\Omega)} + (1 + \beta C_A) |v|_{W^1_p(\Omega)} |\gamma|_{W^1_\infty(0,1)} \Big) + C_E \|F_\Omega\|_{W^1_q(\Omega)^*} + \|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*} \Bigg) |\zeta|_{W^1_1(0,1)}\,.$$

The desired estimate follows from Lemma 3.3.2 with the coercivity of $\mathcal{B}_\Gamma$ in $\mathring{W}^1_2(0,1)$, and the definition of $\|\cdot\|_{\mathbb{W}^1}$ in (3.28).

Estimate (ii) is a straightforward application of the Banach-Nečas theorem [EG04]. $\square$

**Theorem 3.3.11** (T is a contraction)**.** *Let* (3.37) *hold for some $\theta_2 \in (0,1)$. The operator $T$ defined in* (3.46) *is a contraction in $\mathbb{W}^1$ with constant $1 - \theta_2$.*

*Proof.* We proceed in a similar fashion to Theorem 3.3.6. Consider not identical $(\gamma_1, y_1)$ and $(\gamma_2, y_2)$ in $\mathbb{W}^1$, and use (3.46) to write $(\widehat{\gamma}_i, \widehat{y}_i) = T(\gamma_i, y_i)$ for $i = 1, 2$. Applying Lemma 3.3.10 (i), we obtain

$$|\widehat{\gamma}_1 - \widehat{\gamma}_2|_{W^1_\infty(0,1)} \leq \alpha C_E C_A \big\|(\gamma_1 - \gamma_2, y_1 - y_2)\big\|_{\mathbb{W}^1}.$$

Similarly, Lemma 3.3.10 (ii) implies

$$|\widehat{y}_1 - \widehat{y}_2|_{W^1_p(\Omega)} \leq \beta C_A (1 + \beta C_A)|v|_{W^1_p(\Omega)}|\widehat{\gamma}_1 - \widehat{\gamma}_2|_{W^1_\infty(0,1)}.$$

Lastly, the upper bound (3.37) on $v$ yields

$$\begin{aligned}\big\|(\widehat{\gamma}_1 - \widehat{\gamma}_2, \widehat{y}_1 - \widehat{y}_2)\big\|_{\mathbb{W}^1} &\leq (1 + \beta C_A)^2|v|_{W^1_p(\Omega)}|\widehat{\gamma}_1 - \widehat{\gamma}_2|_{W^1_\infty(0,1)} \\ &\leq (1 - \theta_2)\big\|(\gamma_1 - \gamma_2, y_1 - y_2)\big\|_{\mathbb{W}^1}.\end{aligned}$$

Hence, $T$ is a contraction with constant $1 - \theta_2$, as asserted. □

**Corollary 3.3.12** (well-posedness of the linear system (3.45))**.** *Under the assumptions of Theorem 3.3.11, there exists a unique solution* $(\gamma, y) \in \mathbb{W}^1$ *to the variational equation* (3.45) *and the following estimates hold*

$$|\gamma|_{W^1_\infty(0,1)} \leq \frac{\alpha}{\theta_2}\Big(C_E (1 + \beta C_A)\|F_\Omega\|_{W^1_q(\Omega)^*} + \|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*}\Big) \tag{3.49}$$

$$|y|_{W^1_p(\Omega)} \leq \frac{\beta}{\theta_2}\Big(\|F_\Omega\|_{W^1_q(\Omega)^*} + \alpha C_A (1 + \beta C_A)|v|_{W^1_p(\Omega)}\|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*}\Big). \tag{3.50}$$

*Therefore*

$$\begin{aligned}\big\|(\gamma, y)\big\|_{\mathbb{W}^1} \leq{}& \frac{1}{\theta_2}\Big(\alpha C_E (1 + \beta C_A)^2|v|_{W^1_p(\Omega)} + \beta\Big)\|F_\Omega\|_{W^1_q(\Omega)^*} \\ &+ \frac{\alpha}{\theta_2}(1 + \beta C_A)^2|v|_{W^1_p(\Omega)}\|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*}.\end{aligned}$$

*Proof.* Existence and uniqueness follows from Theorem 3.3.11. As far as the estimates, we will only derive (3.49) since the other two are a mere a consequence.

To this end we apply Lemma 3.3.10 and the upper bound (3.37) for $v$ to get

$$\begin{aligned}
|\gamma|_{W^1_\infty(0,1)} &\leq \alpha C_E C_A \left((1+\beta C_A)|v|_{W^1_p(\Omega)}|\gamma|_{W^1_\infty(0,1)} + |y|_{W^1_p(\Omega)}\right) \\
&\quad + \alpha\left(C_E\|F_\Omega\|_{W^1_q(\Omega)^*} + \|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*}\right) \\
&\leq \alpha C_E C_A \left(1+\beta C_A\right)^2 |v|_{W^1_p(\Omega)}|\gamma|_{W^1_\infty(0,1)} \\
&\quad + \alpha C_E\left(1+\beta C_A\right)\|F_\Omega\|_{W^1_q(\Omega)^*} + \alpha\|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*}\,. \\
&\leq (1-\theta_2)|\gamma|_{W^1_\infty(0,1)} + \alpha C_E\left(1+\beta C_A\right)\|F_\Omega\|_{W^1_q(\Omega)^*} + \alpha\|F_\Gamma\|_{\mathring{W}^1_1(0,1)^*}\,.
\end{aligned}$$

The estimate (3.49) follows immediately. □

#### 3.3.3.2 The first-order Fréchet Derivative

In this section we will prove the first-order differentiability of the control-to-state map $G_v$.

**Theorem 3.3.13** (the Fréchet derivative of $G_v$)**.** *The control-to-state map $G_v : \mathcal{U} \to \mathbb{W}^1$ is Fréchet differentiable. The first Fréchet derivative $G_v' : \mathcal{U} \to \mathcal{L}\left(L^2(0,1), \mathbb{W}^1\right)$. Then for every $\bar{u} \in \mathcal{U}$ and every $h \in L^2(0,1)$, $(\gamma, y) := G_v'(\bar{u})\, h \in \mathbb{W}^1$ satisfies the linear variational system* (3.45) *with $F_\Omega = 0$ and $F_\Gamma(\zeta) = \int_0^1 h\zeta$, namely*

$$\mathcal{B}_\Gamma\left[\gamma, \zeta\right] + \mathcal{D}_\Omega\left[\left(\gamma, y\right), z + E\zeta; \bar{\gamma}, \bar{y}\right] = \int_0^1 h\zeta, \qquad \forall\left(\zeta, z\right) \in \mathring{W}^1_1(0,1) \times \mathring{W}^1_q(\Omega). \tag{3.51}$$

*Moreover, the following estimate holds*

$$\left\|(\gamma, y)\right\|_{\mathbb{W}^1} \leq \frac{\alpha}{\theta_2}\left(1+\beta C_A\right)^2 |v|_{W^1_p(\Omega)}\|h\|_{L^2(0,1)}\,. \tag{3.52}$$

*Proof.* The derivation of (3.51) is tedious but straightforward, so we skip it. The estimate (3.52) follows from Corollary 3.3.12.

We turn our focus to proving that $G'_v$ is the Fréchet derivative of $G_v$. To this end, we must show that the (possibility unbounded) remainder operator $\mathrm{R}_{G_v} : \mathcal{U} \times L^2(0,1) \to \mathbb{W}^1$, defined as

$$\mathrm{R}_{G_v}[\bar{u}, h] := G_v(\bar{u} + h) - G_v(\bar{u}) - G'_v(\bar{u})\,h, \tag{3.53}$$

satisfies for all $\bar{u} \in \mathcal{U}$

$$\lim_{\|h\|_{L^2(0,1)} \to 0} \frac{\left\|\mathrm{R}_{G_v}[\bar{u}, h]\right\|_{\mathbb{W}^1}}{\|h\|_{L^2(0,1)}} = 0.$$

Since we do not have direct access to $\left\|\mathrm{R}_{G_v}[\bar{u}, h]\right\|_{\mathbb{W}^1}$, the strategy of the proof is to first show that $\mathrm{R}_{G_v}[\bar{u}, h]$ satisfies (3.45) for some $F_\Omega \in W^1_q(\Omega)^*$ and $F_\Gamma = 0$, assuming $h$ is small enough such that $\bar{u} + h \in \mathcal{U}$. Next, owing to the estimates in Corollary 3.3.12, it suffices to check that

$$\lim_{\|h\|_{L^2(0,1)} \to 0} \frac{\|F_\Omega\|_{W^1_q(\Omega)^*}}{\|h\|_{L^2(0,1)}} = 0.$$

To avoid any ambiguity we adopt the following notation in this proof,

$$\big(\gamma(\bar{u}), y(\bar{u})\big) := G_v(\bar{u}), \qquad \big(\gamma(\bar{u} + h), y(\bar{u} + h)\big) := G_v(\bar{u} + h)$$

$$\big(\gamma_u(\bar{u})h, y_u(\bar{u})h\big) := G'_v(\bar{u})h, \qquad (\delta\gamma, \delta y) = \big(\mathrm{R}_\gamma[\bar{u}, h], \mathrm{R}_y[\bar{u}, h]\big) := \mathrm{R}_{G_v}(\bar{u}, h),$$

whence

$$\delta\gamma = \gamma(\bar{u} + h) - \gamma(\bar{u}) - \gamma_u(\bar{u})\,h \qquad \delta y = y(\bar{u} + h) - y(\bar{u}) - y_u(\bar{u})\,h.$$

According to the remainder definition in (3.53) we start by combining (3.19) for $G_v(\bar{u}+h)$ and $G_v(\bar{u})$ with (3.51) to obtain for every $(\zeta, z)$ in $\mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$

$$\begin{aligned} 0 = \mathcal{B}_\Gamma\left[\gamma(\bar{u}+h) - \gamma(\bar{u}) - \gamma_u(\bar{u})h, \zeta\right] - \mathcal{D}_\Omega\left[\left(\gamma_u(\bar{u})h, y_u(\bar{u})h\right), z + E\zeta; \gamma(\bar{u}), y(\bar{u})\right] \\ + \mathcal{B}_\Omega\left[y(\bar{u}+h) + v, z + E\zeta; A\left[\gamma(\bar{u}+h)\right]\right] - \mathcal{B}_\Omega\left[y(\bar{u}) + v, z + E\zeta; A\left[\gamma(\bar{u})\right]\right]. \end{aligned}$$

Adding and subtracting $\mathcal{D}_\Omega\left[\left(\gamma(\bar{u}+h) - \gamma(\bar{u}), y(\bar{u}+h) - y(\bar{u})\right), z + E\zeta; \gamma(\bar{u}), y(\bar{u})\right]$ to the previous equation and utilizing the definition of $\delta\gamma$ and $\delta y$ above, yields for every $(\zeta, z)$ in $\mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$

$$\mathcal{B}_\Gamma\left[\delta\gamma, \zeta\right] + \mathcal{D}_\Omega\left[(\delta\gamma, \delta y), z + E\zeta; \gamma(\bar{u}), y(\bar{u})\right] = F_\Omega(z + E\zeta),$$

where

$$\begin{aligned} F_\Omega(\cdot) = \mathcal{B}_\Omega\left[y(\bar{u}+h) + v, \cdot; A\left[\gamma(\bar{u})\right] - A\left[\gamma(\bar{u}+h)\right]\right] \\ + \mathcal{B}_\Omega\left[y(\bar{u}) + v, \cdot; \mathrm{D}A\left[\gamma(\bar{u})\right]\langle\gamma(\bar{u}+h) - \gamma(\bar{u})\rangle\right]. \end{aligned}$$

The fact that $F_\Omega$ is in $W_q^1(\Omega)^*$ follows from the continuity of $\mathcal{B}_\Omega[w, \cdot; V]$ with $|w|_{W_p^1(\Omega)}$ and $\|V\|_{L^\infty(\Omega)}$ bounded uniformly (c.f. (3.10)). Our last step is to add and subtract $\mathcal{B}_\Omega\left[y(\bar{u}+h) + v, \cdot; \mathrm{D}A\left[\gamma(\bar{u})\right]\langle\gamma(\bar{u}+h) - \gamma(\bar{u})\rangle\right]$ to $F_\Omega$, employ the definition of the remainder $\mathrm{R}_A$ (3.7) and the Lipschitz estimates (3.43) and (3.44) to obtain

$$\lim_{\|h\|_{L^2(0,1)} \to 0} \frac{\left\|\mathrm{R}_A\left[\gamma(\bar{u}), \gamma(\bar{u}+h) - \gamma(\bar{u})\right]\right\|_{L^\infty(\Omega)}}{\|h\|_{L^2(0,1)}} = 0,$$

as well as

$$\begin{aligned} \|F_\Omega\|_{W_q^1(\Omega)^*} &\le (1 + \beta C_A)|v|_{W_p^1(\Omega)}\left\|\mathrm{R}_A\left[\gamma(\bar{u}), \gamma(\bar{u}+h) - \gamma(\bar{u})\right]\right\|_{L^\infty(\Omega)} \\ &\quad + C_A\left|y(\bar{u}+h) - y(\bar{u})\right|_{W_p^1(\Omega)}\left|\gamma(\bar{u}+h) - \gamma(\bar{u})\right|_{W_\infty^1(0,1)} \\ &= o\left(\|h\|_{L^2(0,1)}\right). \end{aligned}$$

This concludes the proof. □

### 3.3.4 The second-order Fréchet Derivative

The main result of this subsection is to show that $G_v(u)$ is twice Fréchet differentiable with respect to $u$. We adopt a direct approach in line with sections 3.3.2 and 3.3.3 in favor of the technique based on the implicit function theorem described in [Trö10, Pg. 239-240]. The reason being that the modifications to the latter are not straight forward in view of the high nonlinearity of our state equations.

Proceeding as in Theorem 3.3.9 we get the following.

**Proposition 3.3.14** (Lipschitz continuity of $G_v'$)**.** *There exists a constant $L_{G'} > 0$, such that for every $u_1, u_2 \in \mathcal{U}$*

$$\sup_{0 \neq h \in L^2(0,1)} \frac{\left\| G_v'(u_1)h - G_v'(u_2)h \right\|_{\mathbb{W}^1}}{\|h\|_{L^2(0,1)}} \leq L_{G'} \|u_1 - u_2\|_{L^2(0,1)} . \tag{3.54}$$

**Theorem 3.3.15** (the Fréchet derivative of $G_v'(\bar{u})\, h$)**.** *The control-to-state map $G_v : \mathcal{U} \to \mathbb{W}^1$ is twice Fréchet differentiable. The second Fréchet derivative $G_v'' : \mathcal{U} \to \mathcal{L}\left(L^2(0,1), \mathcal{L}\left(L^2(0,1), \mathbb{W}^1\right)\right)$. Then for every $\bar{u} \in \mathcal{U}$ and every $(h_1, h_2) \in L^2(0,1) \times L^2(0,1)$, $(\gamma, y) := G_v''(\bar{u})\, h_1 h_2 \in \mathbb{W}^1$ satisfies the linear variational system* (3.45)*, namely for every $(\zeta, z)$ in $\mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$*

$$\mathcal{B}_\Gamma[\gamma, \zeta] + \mathcal{D}_\Omega\left[(\gamma, y), z + E\zeta; \bar{\gamma}, \bar{y}\right] = F_\Omega(z + E\zeta), \tag{3.55}$$

*with $F_\Omega \in W_q^1(\Omega)^*$ given by*

$$\begin{aligned} F_\Omega(\cdot) := &-\mathcal{B}_\Omega\left[y_1, \cdot; \mathrm{D}A[\bar{\gamma}]\langle\gamma_2\rangle\right] \\ &- \mathcal{B}_\Omega\left[y_2, \cdot; \mathrm{D}A[\bar{\gamma}]\langle\gamma_1\rangle\right] - \mathcal{B}_\Omega\left[\bar{y} + v, \cdot; \mathrm{D}^2A[\bar{\gamma}]\langle\gamma_1, \gamma_2\rangle\right], \end{aligned} \tag{3.56}$$

*and* $(\gamma_i, y_i) := G_v'(\bar{u})\, h_i$, *for* $i = 1, 2$. *Moreover, the following estimates hold*

$$|\gamma|_{W^1_\infty(0,1)} \le \frac{\alpha^3}{\theta_2^3} C_E C_A (1 + 2\beta C_A)(1 + \beta C_A)^2 |v|_{W^1_p(\Omega)} \|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)}, \quad (3.57)$$

$$|y|_{W^1_p(\Omega)} \le \frac{\alpha^2}{\theta_2^3} \beta C_A (1 + 2\beta C_A)(1 + \beta C_A) |v|_{W^1_p(\Omega)} \|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)}. \quad (3.58)$$

*Proof.* We skip the derivation of (3.55) because it is tedious but straightforward. The estimates for $|\gamma|_{W^1_\infty(0,1)}$ and $|y|_{W^1_p(\Omega)}$ are a consequence of Corollary 3.3.12 with $F_\Gamma = 0$ after estimating (3.56), namely

$$\begin{aligned}
\|F_\Omega\|_{W^1_q(\Omega)^*} &\le C_A \left( |y_1|_{W^1_p(\Omega)} |\gamma_2|_{W^1_\infty(0,1)} + |y_2|_{W^1_p(\Omega)} |\gamma_1|_{W^1_\infty(0,1)} \right) \\
&\quad + C_A (1 + \beta C_A) |v|_{W^1_p(\Omega)} |\gamma_1|_{W^1_\infty(0,1)} |\gamma_2|_{W^1_\infty(0,1)} \\
&\le 2 \left( \frac{\alpha}{\theta_2} \right)^2 \beta C_A^2 (1 + \beta C_A) |v|_{W^1_p(\Omega)} \|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)} \\
&\quad + \left( \frac{\alpha}{\theta_2} \right)^2 C_A (1 + \beta C_A) |v|_{W^1_p(\Omega)} \|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)} \\
&= \left( \frac{\alpha}{\theta_2} \right)^2 C_A (1 + 2\beta C_A)(1 + \beta C_A) |v|_{W^1_p(\Omega)} \|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)},
\end{aligned}$$

where we have used (3.49)-(3.50) with $F_\Omega = 0$ for $(\gamma_i, y_i)$ along with (3.23).

The strategy for showing second-order Fréchet differentiability of $G_v$ is the same as in Theorem 3.3.13: we first show that the remainder

$$(\delta\gamma, \delta y) := G_v'(\bar{u} + h_2)\, h_1 - G_v'(\bar{u})\, h_1 - G_v''(\bar{u})\, h_1 h_2, \quad (3.59)$$

satisfies the linear variational system in (3.45) for a suitable right-hand side $\delta F_\Omega \in W^1_q(\Omega)^*$, and prove that

$$\sup_{0 \ne h_1 \in L^2(0,1)} \frac{\|\delta F_\Omega\|_{W^1_q(\Omega)^*}}{\|h_1\|_{L^2(0,1)}} = o\left( \|h_2\|_{L^2(0,1)} \right), \quad (3.60)$$

where $h_1, h_2 \in L^2(0,1)$ are arbitrary but small enough such that if $\bar{u} \in \mathcal{U}$, then $\bar{u} + h_1, \bar{u} + h_2 \in \mathcal{U}$.

As a tradeoff between clarity and space we denote $u_i = \bar{u} + h_i$, and

$$\begin{aligned}
\big(\gamma(u_i), y(u_i)\big) &:= G_v(u_i), \\
\big(\gamma_u\,(u_i)\,h_j, y_u\,(u_i)\,h_j\big) &:= G_v'\,(u_i)\,h_j \\
\big(\gamma_{uu}\,(\bar{u})\,h_1 h_2, y_{uu}\,(\bar{u})\,h_1 h_2\big) &:= G_v''\,(\bar{u})\,h_1 h_2,
\end{aligned}$$

for $i, j = 1, 2$, whence

$$\begin{aligned}
\delta\gamma &= \gamma_u\,(u_2)\,h_1 - \gamma_u\,(\bar{u})\,h_1 - \gamma_{uu}\,(\bar{u})\,h_1 h_2, \\
\delta y &= y_u\,(u_2)\,h_1 - y_u\,(\bar{u})\,h_1 - y_{uu}\,(\bar{u})\,h_1 h_2.
\end{aligned}$$

According to the remainder definition in (3.59) we start by combining (3.51) for $G_v'\,(u_2)\,h_1$ and $G_v'\,(\bar{u})\,h_1$ with (3.55) to obtain for every $(\zeta, z)$ in $\mathring{W}_1^1(0,1) \times \mathring{W}_q^1(\Omega)$

$$\begin{aligned}
-F_\Omega\,(z + E\zeta) = \mathcal{B}_\Gamma\,[\delta\gamma, \zeta] &+ \mathcal{D}_\Omega\left[G_v'(u_2)h_1, z + E\zeta; G_v(u_2)\right] \\
&- \mathcal{D}_\Omega\left[G_v'(\bar{u})h_1 + G_v''(\bar{u})h_1 h_2, z + E\zeta; G_v(\bar{u})\right],
\end{aligned}$$

where $-F_\Omega(\cdot) = \sum_{i=1}^{3} F_i(\cdot)$ is defined in (3.56). Further manipulation, based on adding to both sides the following two additional terms,

$$\begin{aligned}
F_4\,(z + E\zeta) &= \mathcal{D}_\Omega\left[G_v'(u_2)h_1, z + E\zeta; G_v(\bar{u})\right], \\
F_5\,(z + E\zeta) &= -\mathcal{D}_\Omega\left[G_v'(u_2)h_1, z + E\zeta; G_v(u_2)\right]
\end{aligned}$$

leads to

$$\mathcal{B}_\Gamma\,[\delta\gamma, \zeta] + \mathcal{D}_\Omega\left[(\delta\gamma, \delta y)\,, z + E\zeta; \gamma(\bar{u}), y(\bar{u})\right] = -\delta F_\Omega\,(z + E\zeta)\,,$$

where $\delta F_\Omega = \sum_{i=1}^5 F_i$ is clearly in $W_q^1(\Omega)^*$. To create additional cancellations we further decompose $\delta F_\Omega = \sum_{i=1}^9 T_i$ as follows:

$$T_1 = \mathcal{B}_\Omega \Big[ y_u(\bar{u})h_1, \cdot; \mathrm{D}A \left[\gamma(\bar{u})\right] \langle \gamma(u_2) - \gamma(\bar{u}) - \gamma_u(\bar{u})h_2 \rangle \Big],$$

$$T_2 = \mathcal{B}_\Omega \Big[ y_u(u_2)h_1 - y_u(\bar{u})h_1, \cdot; \mathrm{D}A \left[\gamma(\bar{u})\right] \langle \gamma(u_2) - \gamma(\bar{u}) \rangle \Big],$$

$$T_3 = \mathcal{B}_\Omega \Big[ y_u(u_2)h_1, \cdot; A\left[\gamma(u_2)\right] - A\left[\gamma(\bar{u})\right] - \mathrm{D}A \left[\gamma(\bar{u})\right] \langle \gamma(u_2) - \gamma(\bar{u}) \rangle \Big],$$

$$T_4 = \mathcal{B}_\Omega \Big[ y_u(\bar{u})h_2, \cdot; \mathrm{D}A \left[\gamma(\bar{u})\right] \langle \gamma_u(u_2)h_1 - \gamma_u(\bar{u})h_1 \rangle \Big],$$

$$T_5 = \mathcal{B}_\Omega \Big[ y(u_2) - y(\bar{u}) - y_u(\bar{u})h_2, \cdot; \mathrm{D}A \left[\gamma(\bar{u})\right] \langle \gamma_u(u_2)h_1 \rangle \Big],$$

$$\begin{aligned} T_6 = \mathcal{B}_\Omega \Big[ y(u_2) + v, \cdot; \Big( &\mathrm{D}A \left[\gamma(u_2)\right] - \mathrm{D}A \left[\gamma(\bar{u})\right] \Big) \langle \gamma_u(u_2)h_1 \rangle \\ &- \mathrm{D}^2 A \left[\gamma(\bar{u})\right] \langle \gamma_u(u_2)h_1, \gamma(u_2) - \gamma(\bar{u}) \rangle \Big], \end{aligned}$$

$$T_7 = \mathcal{B}_\Omega \left[ y(u_2) + v, \cdot; \mathrm{D}^2 A \left[\gamma(\bar{u})\right] \langle \gamma_u(u_2)h_1, \gamma(u_2) - \gamma(\bar{u}) - \gamma_u(\bar{u})h_2 \rangle \right],$$

$$T_8 = \mathcal{B}_\Omega \Big[ y(u_2) + v, \cdot; \mathrm{D}^2 A \left[\gamma(\bar{u})\right] \langle \gamma_u(u_2)h_1 - \gamma_u(\bar{u})h_1, \gamma_u(\bar{u})h_2 \rangle \Big],$$

$$T_9 = \mathcal{B}_\Omega \Big[ y(u_2) - y(\bar{u}), \cdot; \mathrm{D}^2 A \left[\gamma(\bar{u})\right] \langle \gamma_u(\bar{u})h_1, \gamma_u(\bar{u})h_2 \rangle \Big].$$

We now estimate each of these terms separately and show

$$\sup_{0 \neq h_1 \in L^2(0,1)} \frac{\|T_i\|_{W_q^1(\Omega)^*}}{\|h_1\|_{L^2(0,1)}} = o\left( \|h_2\|_{L^2(0,1)} \right). \tag{3.61}$$

which obviously imply (3.60).

- Term $T_1$: Since

$$\|T_1\|_{W_q^1(\Omega)^*} \le C_A \left| y_u(\bar{u})\, h_1 \right|_{W_p^1(\Omega)} \left| \gamma(u_2) - \gamma(\bar{u}) - \gamma_u(\bar{u})\, h_2 \right|_{W_\infty^1(0,1)},$$

  the estimate (3.52), together with

$$\left| \gamma(u_2) - \gamma(\bar{u}) - \gamma_u(\bar{u})\, h_2 \right|_{W_\infty^1(0,1)} = o\left( \|h_2\|_{L^2(0,1)} \right),$$

implies (3.61).

- Term $T_2$: Since

$$\|T_2\|_{W^1_q(\Omega)^*} \leq C_A \big| y_u(u_2)\, h_1 - y_u(\bar{u})\, h_1 \big|_{W^1_p(\Omega)} \big| \gamma(u_2) - \gamma(\bar{u}) \big|_{W^1_\infty(0,1)},$$

it suffices to recall the Lipschitz properties (3.42) of $G_v$, and (3.54) of $G_v'$ to deduce (3.61).

- Term $T_3$: Invoking the Fréchet differentiability (3.7) of $A$, and the Lipschitz property (3.44) of $\gamma(\bar{u})$ we infer that

$$\Big\| A\big[\gamma(u_2)\big] - A\big[\gamma(\bar{u})\big] - \mathrm{D}A\big[\gamma(\bar{u})\big]\langle \gamma(u_2) - \gamma(\bar{u})\rangle \Big\|_{L^\infty(\Omega)}$$
$$= o\Big( \big|\gamma(u_2) - \gamma(\bar{u})\big|_{W^1_\infty(0,1)} \Big) = o\Big( \|h_2\|_{L^2(0,1)} \Big).$$

This, in conjuction with $\big|y_u(u_2)\, h_1\big|_{W^1_p(\Omega)} \lesssim \|h_1\|_{L^2(0,1)}$, yields (3.61).

- Term $T_4$: In view of the Lipschitz property (3.54) of $G_v'$

$$\big|\gamma_u(u_2)\, h_1 - \gamma_u(\bar{u})\, h_1\big|_{W^1_\infty(0,1)} \lesssim \|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)},$$

property (3.61) follows from $\big|y_u(\bar{u})\, h_2\big|_{W^1_p(\Omega)} \lesssim \|h_2\|_{L^2(0,1)}$.

- Term $T_5$: Since $y(u)$ is Fréchet differentiable according to Theorem 3.3.13, namely

$$\big|y(u_2) - y(\bar{u}) - y_u(\bar{u})\, h_2\big|_{W^1_p(\Omega)} = o\Big( \|h_2\|_{L^2(0,1)} \Big),$$

the bound (3.61) is a consequence of $\big|\gamma_u(u_2)\, h_1\big|_{W^1_\infty(0,1)} \lesssim \|h_1\|_{L^2(0,1)}$.

- Term $T_6$: We recall the second-order Fréchet differentiability of the matrix $A$ with respect to $\gamma$, (c.f.) (3.9), and the Lipschitz continuity (3.42) of $G_v$, to write

$$\begin{aligned}\Bigg\| &\mathrm{D}A\left[\gamma\left(u_2\right)\right]\left\langle\gamma_u(u_2)h_1\right\rangle-\mathrm{D}A\left[\gamma\left(\bar{u}\right)\right]\left\langle\gamma_u(u_2)h_1\right\rangle\\ &\qquad-\mathrm{D}^2A\left[\gamma\left(\bar{u}\right)\right]\left\langle\gamma_u\left(u_2\right)h_1,\gamma\left(u_2\right)-\gamma\left(\bar{u}\right)\right\rangle\Bigg\|_{L^\infty(0,1)}\\ &=\|h_1\|_{L^2(0,1)}\,o\left(\left|\gamma\left(u_2\right)-\gamma\left(\bar{u}\right)\right|_{W^1_\infty(0,1)}\right)=\|h_1\|_{L^2(0,1)}\,o\left(\|h_2\|_{L^2(0,1)}\right).\end{aligned}$$

  Since $\left|y(u_2)+v\right|_{W^1_p(\Omega)}\lesssim|v|_{W^1_p(\Omega)}$, this implies (3.61).

- Term $T_7$: We proceed as with $T_6$, now appealing to (3.10) and the Fréchet differentiability of $\gamma$ at $\bar{u}$ (Theorem 3.3.13), to obtain

$$\left|\gamma_u(u_2)h_1\right|_{W^1_\infty(0,1)}\left|\gamma(u_2)-\gamma(\bar{u})-\gamma_u(\bar{u})h_2\right|_{W^1_\infty(0,1)}=\|h_1\|_{L^2(0,1)}\,o\left(\|h_2\|_{L^2(0,1)}\right),$$

  whence (3.61).

- Term $T_8$: We employ the Lipschitz property (3.54) of $G'_v$ in to write

$$\left|\gamma_u(u_2)h_1-\gamma_u(\bar{u})h_1\right|_{W^1_\infty(0,1)}\lesssim\|h_1\|_{L^2(0,1)}\|h_2\|_{L^2(0,1)}\,.$$

  The desired bound (3.61) follows from $\left|\gamma_u\left(\bar{u}\right)h_2\right|_{W^1_\infty(0,1)}\lesssim\|h_2\|_{L^2(0,1)}$.

- Term $T_9$: We use the Lipschitz property (3.42) of $G_v$,

$$\left|y(u_2)-y(\bar{u})\right|_{W^1_p(\Omega)}\lesssim\|h_2\|_{L^2(0,1)}\,,$$

  together with $\left|\gamma_u\left(\bar{u}\right)h_j\right|_{W^1_\infty(0,1)}\lesssim\left\|h_j\right\|_{L^2(0,1)}$ to deduce (3.61).

Altogether, this concludes the proof. $\square$

Next we state that the second derivative of the control-to-state map is Lipschitz continuous, the proof is based on Theorem 3.3.9 and is omitted here.

**Proposition 3.3.16** (Lipschitz continuity of $G_v''$)**.** *There exists a constant $L_{G''} > 0$, such that for every $u_1, u_2 \in \mathcal{U}$*

$$\sup_{0 \neq h_1, h_2 \in L^2(0,1)} \frac{\left\| G_v''(u_1)h_1 h_2 - G_v''(u_2)h_1 h_2 \right\|_{\mathbb{W}^1}}{\|h_1\|_{L^2(0,1)} \|h_2\|_{L^2(0,1)}} \leq L_{G''} \|u_1 - u_2\|_{L^2(0,1)} . \tag{3.62}$$

## 3.4 Optimal Control

Let us summarize what we have accomplished so far. We have formally derived the first-order necessary optimality conditions in section 3.2. If $G_v$ denotes the control-to-state map, we have proved in section 3.3 that $G_v$ is well posed, i.e., there exists a unique weak solution to the state equations (3.5b) for every $u \in \mathcal{U}$ in (3.35), and $v$ satisfying (3.33) and (3.37). As a crucial step forward we have shown that $G_v$ is twice Fréchet differentiable on $\mathcal{U}$.

This background work puts us in the position to show the existence and (local) uniqueness of the optimal control $u$ solving the OC-FBP in (3.5a)-(3.5b). We will achieve this result in three stages. We first show the existence of $u$ in Theorem 3.4.1 of subsection 3.4.1. We next derive the first-order necessary optimality conditions and show the existence and uniqueness of the solution to the adjoint equations in subsection 3.4.2. Finally in subsection 3.4.3 we end this voyage by proving the second-order sufficient conditions for the control $u$.

### 3.4.1 Existence of Optimal Control

In order to show the existence of a solution to our optimal control problem we first rewrite the cost functional $\mathcal{J} : \mathbb{W}^1 \times \mathcal{U}_{ad} \to \mathbb{R}$ from (3.5a) in its reduced form. This is accomplished by utilizing the control-to-state map $G_v$ from Section 3.3 as follows:

$$\mathcal{J}(u) := \mathcal{J}(G_v(u), u) = \mathcal{J}(\gamma, y, u) = \mathcal{J}_1(G_v(u)) + \mathcal{J}_2(u),$$

with

$$\mathcal{J}_1(G_v(u)) := \frac{1}{2}\|\gamma - \gamma_d\|^2_{L^2(0,1)} + \frac{1}{2}\|y + v - y_d\|^2_{L^2(\Omega)}, \qquad \mathcal{J}_2(u) := \frac{\lambda}{2}\|u\|^2_{L^2(0,1)}.$$

Thus, after recalling that $\mathcal{U}_{ad}$ is a closed subset of $\mathcal{U}$, we obtain that

$$\min_{u \in \mathcal{U}_{ad}} \mathcal{J}(u) \tag{3.63}$$

is an equivalent minimization problem to (3.5a).

**Theorem 3.4.1.** *For every $v$ satisfying* (3.33), (3.37), *there exists an optimal control $\bar{u} \in \mathcal{U}_{ad}$ minimizing the cost functional* (3.5a) *with optimal state $(\bar{\gamma}, \bar{y}) \in \left(W_p^{1+1/q}(0,1) \cap \mathring{W}^1_\infty(0,1)\right) \times \mathring{W}^1_p(\Omega)$ which solves the free boundary problem* (3.5b) *and satisfies the state constraint* (3.5c).

*Proof.* In order to show the existence of an optimal control we use the direct method of the calculus of variations. We first note that the cost functional $\mathcal{J}$ in (3.63) is bounded below by zero, whence $j = \inf_{u \in \mathcal{U}_{ad}} \mathcal{J}(u)$ is finite. We thus construct a minimizing sequence $\{u_n\}_{n \in \mathbb{N}}$ such that

$$j = \lim_{n \to \infty} \mathcal{J}(u_n).$$

By Definition 3.3.5, $\mathcal{U}_{ad}$ is nonempty, closed, bounded and convex in $L^2(0,1)$, thus weakly sequentially compact. Consequently, we can extract a weakly convergent subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset L^2(0,1)$, i.e.

$$u_{n_k} \rightharpoonup \bar{u} \quad \text{in } L^2(0,1), \qquad \bar{u} \in \mathcal{U}_{ad}.$$

Here $\bar{u}$ is our optimal control candidate.

Henceforth, we drop the subindex $k$ when extracting subsequences. According to Corollary 3.3.7 and (3.40), we let $G_v(u_n) = (\gamma_n, y_n) \in \left(W_p^{1+1/q}(0,1) \cap \mathring{W}_\infty^1(0,1)\right) \times \mathring{W}_p^1(\Omega)$ denote the unique state corresponding to $u_n$, thereby solving the free boundary problem (3.5b) and satisfying the state constraint (3.5c). Since $W_p^{1+1/q}(0,1) \cap \mathring{W}_\infty^1(0,1)$ is compactly embedded into $\mathring{W}_\infty^1(0,1)$ the Rellich-Kondrachov theorem yields a strongly convergent subsequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathring{W}_\infty^1(0,1)$, i.e.

$$\gamma_n \to \bar{\gamma} \quad \text{in } \mathring{W}_\infty^1(0,1), \qquad \text{and} \qquad y_n \rightharpoonup \bar{y} \quad \text{in } \mathring{W}_p^1(\Omega).$$

Note that the limit pair $(\bar{\gamma}, \bar{y})$ is the state corresponding to the control $\bar{u}$. This results from replacing $(\gamma, y)$ with $(\gamma_n, y_n)$ in the variational equation (3.19) taking the limit, and making use of the embedding $L^2(0,1) \subset W_\infty^{-1}(0,1)$.

Finally, using the fact that $\mathcal{J}_2(u)$ is continuous in $L^2$ and convex, together with the the strong convergence $(\gamma_n, y_n) \to (\bar{\gamma}, \bar{y})$ in $L^\infty(0,1) \times L^\infty(\Omega)$, again due to Rellich-Kondrachov theorem, it follows that $\mathcal{J}$ is weakly lower semicontinuous, whence

$$\inf_{u\in\mathcal{U}_{ad}} \mathcal{J}(u) = \liminf_{n\to\infty} \big(\mathcal{J}_1(G_v(u_n)) + \mathcal{J}_2(u_n)\big) \geq \mathcal{J}_1(G_v(\bar{u})) + \mathcal{J}_2(\bar{u}) = \mathcal{J}(\bar{u}).$$

This concludes the proof. $\square$

### 3.4.2 First-order Necessary Condition

We start with a classical result [Trö10].

**Lemma 3.4.2** (variational inequality)**.** *If $\bar{u} \in \mathcal{U}_{ad}$ denotes an optimal control, given by Theorem 3.4.1, then the first order necessary optimality condition satisfied by $\bar{u}$ is*

$$\left\langle \mathcal{J}'(\bar{u}), u - \bar{u} \right\rangle_{L^2(0,1),L^2(0,1)} \geq 0 \quad \forall u \in \mathcal{U}_{ad}. \tag{3.64}$$

We will show that the variational inequality (3.64) is the same as (3.17) as well as prove that (3.15) and (3.16) are the correct adjoint equations. This furnishes a rigorous derivation of the formal results of Section 3.2.

To this end we recall that the set $\mathcal{U}_{ad}$ defined in (3.36) is not open, thus we need to define a proper set of admissible directions.

**Definition 3.4.3.** (admissible directions) Given $u \in \mathcal{U}_{ad}$, the convex set $\mathcal{C}(u)$ comprises of all directions $h \in L^2(0,1)$ such that $u + h \in \mathcal{U}_{ad}$, i.e.,

$$\mathcal{C}(u) := \left\{ h \in L^2(0,1) : \ u + h \in \mathcal{U}_{ad} \right\}.$$

**Theorem 3.4.4** (First-order conditions)**.** *If $\bar{u} \in \mathcal{U}_{ad}$ denotes an optimal control of OC-FBP, then the first-order necessary optimality conditions are given by* (3.15), (3.16) *and* (3.17)*.*

*Proof.* We can infer that $\mathcal{J}$ is Fréchet differentiable by recalling from Theorems 3.3.13 and 3.3.15 that $G_v$ is twice differentiable and that $\mathcal{J}_1$ is quadratic. In fact, the Fréchet derivative of $\mathcal{J}$ in (3.63) at $\bar{u}$ in a direction $h \in \mathcal{C}(\bar{u})$ is

$$\mathcal{J}'(\bar{u})h = \mathcal{J}_1'(G_v(\bar{u}))G_v'(\bar{u})h + \mathcal{J}_2'(\bar{u})h = \mathcal{J}_1'(G_v(\bar{u}))(\gamma_u(\bar{u})h, y_u(\bar{u})h) + \mathcal{J}_2'(\bar{u})h,$$

where $(\gamma_u(\bar{u})h, y_u(\bar{u})h) = G_v'(\bar{u})h$ satisfies (3.51) and

$$\begin{aligned}\mathcal{J}'(\bar{u})h = &\left\langle (\bar{y} + v - y_d)\,(1 + \bar{\gamma})\,, y_u(\bar{u})h \right\rangle_{L^2(\Omega),L^2(\Omega)} \\ &+ \left\langle \bar{\gamma} - \gamma_d + \frac{1}{2}\int_0^1 |\bar{y} + v - y_d|^2 \,\mathrm{d}x_2, \gamma_u(\bar{u})h \right\rangle_{L^2(0,1)\times L^2(0,1)} \\ &+ \lambda\langle \bar{u}, h\rangle_{L^2(0,1),L^2(0,1)}.\end{aligned} \tag{3.65}$$

Introducing the adjoint states $(\bar{r}, \bar{s}) \in W_q^1(\Omega) \times \mathring{W}_1^1(0,1)$, which satisfy the system (3.15)-(3.16) in weak form, and noting that $h \in L^2(0,1)$, we obtain

$$\mathcal{J}'(\bar{u})h = \mathcal{B}_\Gamma\left[\gamma_u(\bar{u})h, \bar{s}\right] + \mathcal{D}_\Omega\left[\left(\gamma_u(\bar{u})h, y_u(\bar{u})h\right), \bar{r}; \bar{\gamma}, \bar{y}\right] + \lambda\langle \bar{u}, h\rangle_{L^2(0,1),L^2(0,1)}.$$

Utilizing (3.51) with $\zeta = \bar{s}$ and $z = \bar{r}$, we arrive at

$$\mathcal{J}'(\bar{u})h = \langle \bar{s} + \lambda\bar{u}, h\rangle_{L^2(0,1),L^2(0,1)} + \mathcal{D}_\Omega\left[\left(\gamma_u(\bar{u})h, y_u(\bar{u})h\right), \bar{r} - E\bar{s}; \bar{\gamma}, \bar{y}\right].$$

Since the Dirichlet condition $\bar{r}|_\Gamma = \bar{s}$ implies $\bar{r} - E\bar{s} \in \mathring{W}_q^1(\Omega)$, (3.51) with $\zeta = 0$ and $z \in \mathring{W}_q^1(\Omega)$ yields $\mathcal{D}_\Omega\left[\left(\gamma_u(\bar{u})h, y_u(\bar{u})h\right), z; \bar{\gamma}, \bar{y}\right] = 0$, whence

$$\mathcal{J}'(\bar{u})h = \langle \bar{s} + \lambda\bar{u}, h\rangle_{L^2(0,1),L^2(0,1)}.$$

In view of (3.64), this coincides with (3.17) for $h = u - \bar{u}$ admissible. □

#### 3.4.2.1 Well-posedness of the Adjoint System

Before we dwell upon the second-order sufficient optimality conditions we put together the last piece of the puzzle: the well-posedness of the adjoint system (3.15) and (3.16). This will be done using a contraction argument in Banach spaces, assuming that we have a solution $(\bar{\gamma}, \bar{y}) \in \mathbb{B}_v$ to the state equations in (3.5b) satisfying Proposition 3.3.1 and Lemma 3.3.3.

Let $\mathbb{V} := \left\{ r \in W_q^1(\Omega) \,:\, r|_\Gamma \in \mathring{W}_1^1(0,1),\, r|_\Sigma = 0 \right\}$, and the operator $T_1 : \mathbb{V} \to \mathring{W}_1^1(0,1)$ be defined as $\widetilde{s} = T_1(r)$ where $\widetilde{s}$ satisfies for every $\zeta \in \mathring{W}_\infty^1(0,1)$

$$\mathcal{B}_\Gamma \left[\zeta, \widetilde{s}\right] = \left\langle \zeta, \mathbf{f}\left[\bar{\gamma}, \bar{y}, r\right] \right\rangle_{\mathring{W}_\infty^1(0,1), W_1^{-1}(0,1)}, \tag{3.66}$$

with

$$\begin{aligned} &\langle \zeta, \mathbf{f} \rangle_{\mathring{W}_\infty^1(0,1), W_1^{-1}(0,1)} \\ &\qquad := \left\langle \zeta, f_0\left[x_1; \bar{\gamma}, \bar{y}, r\right] \right\rangle_{L^\infty(0,1), L^1(0,1)} + \left\langle \mathrm{d}_{x_1}\zeta, f_1\left[x_1; \bar{\gamma}, \bar{y}, r\right] \right\rangle_{L^\infty(0,1), L^1(0,1)} \end{aligned}$$

and

$$\begin{aligned} f_0[\cdot; \bar{\gamma}, \bar{y}, r] &:= \bar{\gamma} - \gamma_d + \frac{1}{2}\int_0^1 |\bar{y} + v - y_d|^2 \, \mathrm{d}x_2 - \int_0^1 A_1\left[\bar{\gamma}\right] \nabla\left(\bar{y} + v\right) \cdot \nabla r \, \mathrm{d}x_2, \\ f_1[\cdot; \bar{\gamma}, \bar{y}, r] &:= -\int_0^1 A_2\left[\bar{\gamma}\right] \nabla\left(\bar{y} + v\right) \cdot \nabla r \, \mathrm{d}x_2. \end{aligned} \tag{3.67}$$

Given $\widetilde{s} \in \mathring{W}_1^1(0,1)$, let $T_2 : \mathring{W}_1^1(0,1) \to \mathbb{V}$ be the operator defined as $\widetilde{r} = T_2(\widetilde{s}) = \ell + E\widetilde{s}$ with $\ell \in \mathring{W}_q^1(\Omega)$ satisfying

$$\mathcal{B}_\Omega\left[z, \widetilde{r}; A\left[\bar{\gamma}\right]\right] = \left\langle z, \left(\bar{y} + v - y_d\right)\left(1 + \bar{\gamma}\right) \right\rangle_{L^p(\Omega), L^q(\Omega)} \qquad \forall z \in \mathring{W}_p^1(\Omega). \tag{3.68}$$

**Lemma 3.4.5** (ranges of $T_1$ and $T_2$)**.** *Let $T_1$, $T_2$ be defined in* (3.66) *and* (3.68)*. If $(\bar{\gamma}, \bar{y}) \in \mathbb{B}_v$, defined in* (3.29)*, then for every*

*(i) $r \in \mathbb{V}$, the solution $\widetilde{s} = T_1(r)$ to* (3.66) *satisfies*

$$\begin{aligned} |\widetilde{s}|_{W_1^1(0,1)} \leq \alpha\Big( &\|\bar{\gamma} - \gamma_d\|_{L^1(0,1)} + \frac{1}{2}\|\bar{y} + v - y_d\|_{L^2(\Omega)}^2 \\ &+ C_A\left(1 + \beta C_A\right)|v|_{W_p^1(\Omega)}|r|_{W_q^1(\Omega)}\Big); \end{aligned}$$

*(ii) $\widetilde{s} \in \mathring{W}_1^1(0,1)$, the solution $\widetilde{r} = T_2\left(\widetilde{s}\right)$ to* (3.68) *satisfies*

$$|\widetilde{r}|_{W_q^1(\Omega)} \leq \beta\left(2\|\bar{y} + v - y_d\|_{L^q(\Omega)} + C_E C_A |\widetilde{s}|_{W_1^1(0,1)}\right).$$

*Proof.* Using (3.25) of Proposition 3.3.1, and applying Banach-Nečas theorem [EG04], there exists a unique solution $\widetilde{r}$ to (3.68). Estimate (ii) follows from (3.25) and (3.10), as well as the Poincaré inequality $\|z\|_{L^p(\Omega)} \leq |z|_{W^1_p(\Omega)}$ for the unit square.

In order to show the existence of solution to (3.66) we note that we are looking for an absolutely continuous function on the interval $(0,1)$ with zero Dirichlet values. Therefore, by the characterization of such functions in $\mathbb{R}$ [Roy88, Theorem 5.14], there exists a $g \in L^1(0,1)$ such that

$$\widetilde{s}(x_1) = \int_0^{x_1} \left( g(t) - \int_0^1 g(\tau)\,\mathrm{d}\tau \right) \mathrm{d}t \qquad \forall x_1 \in (0,1)$$

because $s_1(0) = s_1(1) = 0$. The variational equation (3.66) is satisfied by $\widetilde{s}$ with

$$g(t) = -\frac{1}{\kappa} \left( f_1[t;\bar{\gamma},\bar{y},r] + \int_0^t f_0[\tau;\bar{\gamma},\bar{y},r]\,\mathrm{d}\tau \right)$$

and $f_0, f_1$ defined in (3.67).

It remains to check that $\int_0^t f_0 d\tau$ and $f_1$ are in $L^1(0,1)$. This follows by applying Fubini's theorem and Hölder's inequality. We consider first $f_0$:

$$\begin{aligned}
&\int_0^1 \left| f_0[x_1;\cdot] \right| \mathrm{d}x_1 \\
&\qquad \leq \|\bar{\gamma} - \gamma_d\|_{L^1(0,1)} + \frac{1}{2}\int_\Omega |\bar{y} + v - y_d|^2\,\mathrm{d}x + \int_\Omega \left| A_1[\bar{\gamma}]\,\nabla(\bar{y}+v)\cdot\nabla r \right| \mathrm{d}x \\
&\qquad \leq \|\bar{\gamma} - \gamma_d\|_{L^1(0,1)} + \frac{1}{2}\|\bar{y} + v - y_d\|^2_{L^2(\Omega)} + C_A(1+\beta C_A)|v|_{W^1_p(\Omega)}|r|_{W^1_q(\Omega)},
\end{aligned}$$

because $|\bar{y}+v|_{W^1_p(\Omega)} \leq (1+\beta C_A)|v|_{W^1_p(\Omega)}$. Similarly, we obtain an $L^1$ estimate for $f_1$

$$\int_0^1 \left| f_1(x_1) \right| \mathrm{d}x_1 \leq \int_\Omega \left| A_2[\bar{\gamma}]\,\nabla(\bar{y}+v)\cdot\nabla r \right| \mathrm{d}x \leq C_A(1+\beta C_A)|v|_{W^1_p(\Omega)}|r|_{W^1_q(\Omega)}.$$

This implies (i). Finally, the uniqueness of $\widetilde{s}$ follows from the estimate in (3.24b). $\square$

**Theorem 3.4.6** (existence of (3.15) and (3.16))**.** *Under the assumptions of Lemma 3.4.5, the operator $T = T_2 \circ T_1 : \mathbb{V} \to \mathbb{V}$ is a contraction with constant $1 - \theta_2$.*

*Proof.* The proof is similar to the one in Theorem 3.3.6, therefore we will be brief. Consider $r_1$, $r_2 \in \mathbb{V}$ such that $r_1 \neq r_2$ and let $\widetilde{s}_i = T_1(r_i), \widetilde{r}_i = T_2(\widetilde{s}_i)$, where $\widetilde{s}_i, \widetilde{r}_i$ solve (3.66) and (3.68) for $i = 1, 2$. Then Proposition 3.3.1 (i) and Lemma 3.4.5 (i) imply

$$
\begin{aligned}
|\widetilde{s}_1 - \widetilde{s}_2|_{W_1^1(0,1)} &\leq \alpha \sup_{|\zeta|_{W_\infty^1(0,1)}=1} \mathcal{B}_\Gamma\left[\zeta, \widetilde{s}_1 - \widetilde{s}_2\right] \\
&= \alpha \sup_{|\zeta|_{W_\infty^1(0,1)}=1} \left\langle \zeta, \mathbf{f}\left[\bar{\gamma}, \bar{y}, r_1\right] - \mathbf{f}\left[\bar{\gamma}, \bar{y}, r_2\right] \right\rangle \\
&\leq \alpha C_A \left(1 + \beta C_A\right) |v|_{W_p^1(\Omega)} |r_1 - r_2|_{W_q^1(\Omega)} .
\end{aligned}
$$

In addition, since

$$
|\widetilde{r}_1 - \widetilde{r}_2|_{W_q^1(\Omega)} = \left|\tilde{\ell}_1 + E\widetilde{s}_1 - \tilde{\ell}_2 - E\widetilde{s}_2\right|_{W_q^1(\Omega)} \leq C_E |\widetilde{s}_1 - \widetilde{s}_2|_{W_1^1(0,1)} + \left|\tilde{\ell}_1 - \tilde{\ell}_2\right|_{W_q^1(\Omega)} ,
$$

and Proposition 3.3.1 (ii) implies

$$
\begin{aligned}
\left|\tilde{\ell}_1 - \tilde{\ell}_2\right|_{W_q^1(\Omega)} &\leq \beta \sup_{|z|_{W_p^1(\Omega)}=1} \mathcal{B}_\Omega\left[z, \tilde{\ell}_1 - \tilde{\ell}_2; A\left[\bar{\gamma}\right]\right] \\
&= \beta \sup_{|z|_{W_p^1(\Omega)}=1} \mathcal{B}_\Omega\left[z, E\widetilde{s}_2 - E\widetilde{s}_1; A\left[\bar{\gamma}\right]\right] ,
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
|\widetilde{r}_1 - \widetilde{r}_2|_{W_q^1(\Omega)} &\leq C_E \left(1 + \beta C_A\right) |\widetilde{s}_1 - \widetilde{s}_2|_{W_1^1(0,1)} \\
&\leq \alpha C_E C_A \left(1 + \beta C_A\right)^2 |v|_{W_p^1(\Omega)} |r_1 - r_2|_{W_q^1(\Omega)} .
\end{aligned}
$$

Invoking (3.37) we obtain

$$|\widetilde{r}_1 - \widetilde{r}_2|_{W^1_q(\Omega)} \leq (1-\theta_2)|r_1 - r_2|_{W^1_q(\Omega)}$$

Therefore $T = T_2 \circ T_1 : \mathbb{V} \to \mathbb{V}$ is a contraction in $\mathbb{V}$. $\square$

### 3.4.3 second-order sufficient condition

The final step is to prove the second-order sufficient condition for the optimal control $\bar{u}$ found earlier, which in turn guarantees that $\bar{u}$ is locally unique. This imposes an additional condition on $|v|_{W^1_p(\Omega)}$ depending on the parameter $\theta_3$ given by

$$\theta_3 = \frac{\theta_2^2}{\alpha^2}\left(1+\beta C_A\right)^{-1}\left[\frac{1}{\theta_2}C_A\left(1+2\beta C_A\right)\left\{\alpha C_E\left(1+\beta C_A\right)\left(1+\|\gamma_d\|_{L^2(0,1)}+\frac{1}{2}\omega^2\right)+2\beta\omega\right\}+2\left(1+\beta C_A\right)^2\omega\right]^{-1}, \tag{3.69}$$

where $\omega = \frac{1-\theta_1}{\alpha C_E C_A}$.

**Theorem 3.4.7** (second-order sufficient conditions)**.** *If $\theta_1$, $\theta_2$ satisfy* (3.33), (3.37), *and in addition*

$$|v|_{W^1_p(\Omega)} \leq \frac{\theta_3 \lambda}{2}, \tag{3.70}$$

*then*

$$\mathcal{J}''(\bar{u})h^2 \geq \frac{\lambda}{2}\|h\|^2_{L^2(0,1)} \qquad \forall h \in \mathcal{C}\left(\bar{u}\right). \tag{3.71}$$

*Proof.* Since $G_v$ is twice Fréchet differentiable, according to Theorem 3.3.15, we can write the second-order Fréchet derivative of $\mathcal{J}$ from (3.63) at $\bar{u}$ in the direction $h^2$

as

$$\mathcal{J}''(\bar{u})h^2 = \mathcal{J}_1''(G_v(\bar{u}))(G_v'(\bar{u})h)^2 + \mathcal{J}_1'(G_v(\bar{u}))G_v''(\bar{u})h^2 + \mathcal{J}_2''(\bar{u})h^2. \tag{3.72}$$

By recalling that $(\gamma_u(\bar{u})h, y_u(\bar{u})h) = G_v'(\bar{u})h$ solves (3.51) and $(\gamma_{uu}(\bar{u})h^2, y_{uu}(\bar{u})h^2) = G_v''(\bar{u})h^2$ solves (3.55), we can write

$$\mathcal{J}''(\bar{u})h^2 = \mathcal{J}_1''(G_v(\bar{u}))(\gamma_u(\bar{u})h, y_u(\bar{u})h)^2 + \mathcal{J}_1'(G_v(\bar{u}))(\gamma_{uu}(\bar{u})h^2, y_{uu}(\bar{u})h^2) + \mathcal{J}_2''(\bar{u})h^2,$$

where

$$\begin{aligned}
&\mathcal{J}_1''(G_v(\bar{u}))(\gamma_u(\bar{u})h, y_u(\bar{u})h)^2 \\
&\qquad = \left\|\gamma_u(\bar{u})h\right\|_{L^2(0,1)}^2 + \int_0^1 \left(\int_0^1 (\bar{y} + v - y_d)\left(y_u(\bar{u})h\right) \mathrm{d}x_2\right)\left(\gamma_u(\bar{u})h\right) \mathrm{d}x_1 \\
&\qquad\quad + \int_\Omega \left(y_u(\bar{u})h\right)^2 (1+\bar{\gamma})\, \mathrm{d}x + \int_\Omega (\bar{y} + v - y_d)\left(\gamma_u(\bar{u})h\right)\left(y_u(\bar{u})h\right) \mathrm{d}x, \\
&\mathcal{J}_1'(G_v(\bar{u}))(\gamma_{uu}(\bar{u})h^2, y_{uu}(\bar{u})h^2) \\
&\qquad = \int_0^1 (\bar{\gamma} - \gamma_d)\left(\gamma_{uu}(\bar{u})h^2\right) \mathrm{d}x_1 + \int_0^1 \left(\frac{1}{2}\int_0^1 \left|\bar{y} + v - y_d\right|^2 \mathrm{d}x_2\right)\left(\gamma_{uu}(\bar{u})h^2\right) \mathrm{d}x_1 \\
&\qquad\quad + \int_\Omega (\bar{y} + v - y_d)(1+\bar{\gamma})\left(y_{uu}(\bar{u})h^2\right) \mathrm{d}x, \\
&\mathcal{J}_2''(\bar{u})h^2 = \lambda\|h\|_{L^2(0,1)}^2 .
\end{aligned}$$

This yields

$$\begin{aligned}
\mathcal{J}''(\bar{u})h^2 \geq{}& \lambda\|h\|_{L^2(0,1)}^2 + \left\|\gamma_u(\bar{u})h\right\|_{L^2(0,1)}^2 + \left\|y_u(\bar{u})h\sqrt{1+\bar{\gamma}}\right\|_{L^2(\Omega)}^2 \\
&- \left(\|\bar{\gamma} - \gamma_d\|_{L^2(0,1)} + \frac{1}{2}\|\bar{y} + v - y_d\|_{L^2(\Omega)}^2\right)\left|\gamma_{uu}(\bar{u})h^2\right|_{W^1_\infty(0,1)} \\
&- 2\|\bar{y} + v - y_d\|_{L^2(\Omega)}\left|y_{uu}(\bar{u})h^2\right|_{W^1_p(\Omega)} \\
&- 2\|\bar{y} + v - y_d\|_{L^2(\Omega)}\left|y_u(\bar{u})h\right|_{W^1_p(\Omega)}\left|\gamma_u(\bar{u})h\right|_{W^1_\infty(0,1)},
\end{aligned}$$

because of the Poincaré inequalities $\|z\|_{L^2(\Omega)} \le |z|_{W_2^1(\Omega)} \le |z|_{W_p^1(\Omega)}$ for the unit square $\Omega$ and $\|\zeta\|_{L^2(0,1)} \le |\zeta|_{W_2^1(0,1)} \le |\zeta|_{W_p^1(0,1)}$ for the unit interval $(0,1)$.

Estimates (3.57) and (3.58) in Theorem 3.3.15 and estimate (3.52) in Theorem 3.3.13 imply

$$\begin{aligned}
&\mathcal{J}''(\bar{u})h^2 \ge \\
&\quad \lambda\|h\|^2_{L^2(0,1)} - \frac{\alpha^2}{\theta_2^2}\left(1+\beta C_A\right)|v|_{W_p^1(\Omega)}\Bigg[\frac{1}{\theta_2}C_A\left(1+2\beta C_A\right) \\
&\qquad \left\{\alpha C_E\left(1+\beta C_A\right)\left(\|\bar{\gamma}-\gamma_d\|_{L^2(0,1)} + \frac{1}{2}\|\bar{y}+v-y_d\|^2_{L^2(\Omega)}\right) + 2\beta\|\bar{y}+v-y_d\|_{L^2(\Omega)}\right\} \\
&\qquad + 2\left(1+\beta C_A\right)^2\|\bar{y}+v-y_d\|_{L^2(\Omega)}\Bigg]\|h\|^2_{L^2(0,1)}\,.
\end{aligned}$$

Furthermore, from $(\bar{\gamma},\bar{y}) \in \mathbb{B}_v$ we obtain

$$\|\bar{\gamma}-\gamma_d\|_{L^2(0,1)} \le \|\bar{\gamma}\|_{L^2(0,1)} + \|\gamma_d\|_{L^2(0,1)} \le 1 + \|\gamma_d\|_{L^2(\Omega)}\,,$$

and

$$\|\bar{y}+v-y_d\|_{L^2(\Omega)} \le |\bar{y}+v|_{W_p^1(\Omega)} + \|y_d\|_{L^2(\Omega)}$$

with

$$|\bar{y}+v|_{W_p^1(\Omega)} \le (1+\beta C_A)|v|_{W_p^1(\Omega)} \le \frac{1-\theta_1}{\alpha C_E C_A}.$$

In view of the definition (3.69) of $\theta_3$, this in turn implies

$$\mathcal{J}''(\bar{u})h^2 \ge \left(\lambda - \frac{|v|_{W_p^1(\Omega)}}{\theta_3}\right)\|h\|^2_{L^2(\Omega)}\,.$$

Therefore, the smallness condition (3.70) on $v$ yields (3.71). $\square$

**Corollary 3.4.8** (quadratic growth)**.** *Under the assumptions of Theorem 3.4.7 combined with Proposition 3.3.16 there exist* $\theta > 0$ *such that for all* $h \in \mathcal{C}(\bar{u})$ *with* $\|h\|_{L^2(0,1)} \leq \theta$ *we have*

$$\mathcal{J}(\bar{u} + h) \geq \mathcal{J}(\bar{u}) + \frac{\lambda}{8}\|h\|^2_{L^2(0,1)}. \tag{3.73}$$

*Proof.* We refer to [Trö10, Theorem 4.23]. $\square$

Corollary 3.4.8 implies that there exists a unique local minimum $\bar{u}$ solution to our OC-FBP. Moreover, (3.73) is equivalent to

$$\left\langle \mathcal{J}'(u) - \mathcal{J}'(\bar{u}), u - \bar{u} \right\rangle_{L^2(0,1),L^2(0,1)} \geq \frac{\lambda}{4}\|u - \bar{u}\|^2_{L^2(0,1)} \quad \forall u \in \bar{u} + \mathcal{C}(\bar{u}). \tag{3.74}$$

**Remark 3.4.9.** The results of this chapter are contained in [ANS12].

# Chapter 4

# Laplace Free Boundary Problem: Numerics

## 4.1 Strong Solutions

The goal of this section is to prove the existence of strong solutions to the state and adjoint equations. It is important to show such solutions exist because they are the basis for the a-priori error estimates of §4.2.2. A presentation of these two results follows next.

### 4.1.1 State Equations

The technique is the fixed point argument used in [SS91, Section 2] and §3.3.1 . It consists of three steps: defining a convex set which can act as the domain of the fixed point iterator, linearizing the free boundary problem, and identifying conditions to guarantee a contraction on the convex set.

We are looking for solutions with second-order regularity. Therefore, the least restrictive Banach space is $\mathbb{W}^2 := \Big(W^2_\infty(0,1) \times W^2_p(\Omega)\Big) \cap \Big(\mathring{W}^1_\infty(0,1) \times \mathring{W}^1_p(\Omega)\Big)$, endowed with the norm,

$$\big\|(\gamma, y)\big\|_{\mathbb{W}^2} := (1 + C_{GT})\|v\|_{W^2_p(\Omega)}\|\gamma\|_{W^2_\infty(0,1)} + \|y\|_{W^2_p(\Omega)}\,.$$

We also want to guarantee that the assumptions for the first-order regularity results in §3.3.1 satisfied by iterating on the set (3.29)

$$\mathbb{B}_1 := \Big\{(\gamma, y) \in \mathbb{W}^{1,1}_{\infty,p} : |y|_{W^1_p(\Omega)} \le \beta C_A |v|_{W^1_p(\Omega)} \,, |\gamma|_{W^1_\infty(0,1)} \le 1\Big\} \,.$$

For the purposes of finding a classical solution, we further restrict $\mathbb{B}_1$ as follows,

$$\mathbb{B}_2 := \Big\{(\gamma, y) \in \mathbb{B}_1 \cap \mathbb{W}^2 : \|y\|_{W^2_p(\Omega)} \le C_{GT} \|v\|_{W^2_p(\Omega)} \,, |\gamma|_{W^2_\infty(0,1)} \le 1\Big\} \,,$$

where $C_{GT}$ can be found in [GT01, Lemma 9.17] and is related to the well-posedness of uniformly elliptic operators on $W^2_p(\Omega) \cap \mathring{W}^1_p(\Omega)$.

We linearize the free boundary problem by considering the following operator $T : \mathbb{B}_2 \to \mathbb{W}^2$ defined as

$$T(\gamma, y) := \big(T_1(\gamma, y), T_2(\gamma, y)\big) = (\widetilde{\gamma}, \widetilde{y}) \qquad \forall\, (\gamma, y) \in \mathbb{B}_2, \tag{4.1}$$

where $\widetilde{\gamma} = T_1(\gamma, y) \in W^2_\infty(0,1) \cap \mathring{W}^1_\infty(0,1)$ is the unique solution to

$$-\,\kappa\, \mathrm{d}^2_{x_1} \widetilde{\gamma} = A\,[\gamma]\, \nabla y \cdot \nu + u \quad \text{in } \ (0,1)\,, \tag{4.2}$$

and $\widetilde{y} = T_2(\gamma, y) \in W^2_p(\Omega) \cap \mathring{W}^1_p(\Omega)$ is the unique solution to

$$-\,A\,\big[T_1(\gamma, y)\big] : \mathrm{D}^2\, \widetilde{y} = A\,\big[T_1(\gamma, y)\big] : \mathrm{D}^2\, v + \operatorname{div} A\,\big[T_1(\gamma, y)\big] \cdot \nabla (y + v) \quad \text{in } \Omega, \tag{4.3}$$

where $\operatorname{div} A\,\big[T_1(\gamma, y)\big]$ is computed row-wise. The operators $T_1$ and $T_2$ are well-defined. We refer to §3.4.2.1 for $T_1$, and to [GT01, Lemma 9.17] for $T_2$.

**Theorem 4.1.1.** *If the following two conditions holds*

$$\begin{aligned} \Big(2C_{W^2_p(\Omega)\hookrightarrow W^1_\infty(\Omega)}\,(1 + C_{GT})\|v\|_{W^2_p(\Omega)} + \|u\|_{L^\infty(0,1)}\Big) &\le \kappa, \\ 5C_{A,W^2_p(\Omega)\hookrightarrow W^1_\infty(\Omega)}\,(1 + C_{GT})\|v\|_{W^2_p(\Omega)} &< 1. \end{aligned} \tag{4.4}$$

*then $T$ is a contraction on $\mathbb{B}_2$.*

*Proof.* The contraction argument is similar to Theorem 3.3.11 and relies of two estimates. The first estimate in (4.4) guarantees that $T$ maps $\mathbb{B}_2$ back into $\mathbb{B}_2$, and the second estimate enforces the contraction. $\square$

**Remark 4.1.2.** We point out that, within the context of the optimal control problem, the $L^\infty$-estimate requirement on $\bar{u}$ can be satisfied. The reason is that the variational inequality (3.17) implies the optimal control $\bar{u}$ is proportional to the adjoint function $\bar{s}$ which in turn is absolutely continuous, i.e $\bar{s} \in \mathring{W}_1^1(0,1) \subset L^\infty(0,1)$.

### 4.1.2 Adjoint Equations

We begin by recalling the *adjoint equations* in non-divergence form assuming that $(\bar{\gamma}, \bar{y})$ belongs to $\mathbb{B}_2$,

$$
\begin{aligned}
-A\left[\bar{\gamma}\right] : \mathrm{D}^2\, \bar{r} - \operatorname{div} A\left[\bar{\gamma}\right] \cdot \nabla \bar{r} = \left(\bar{y} + v - y_d\right)\left(1 + \bar{\gamma}\right) \quad \text{in } \Omega& \\
-\, \kappa\, \mathrm{d}_{x_1}^2 \bar{s} = \left(\bar{\gamma} - \gamma_d\right) + \frac{1}{2}\int_0^1 \left|\bar{y} + v - y_d\right|^2 \mathrm{d}x_2 - \int_0^1 A_1\left[\bar{\gamma}\right] \nabla\left(\bar{y} + v\right) \cdot \nabla \bar{r}\, \mathrm{d}x_2& \\
+\, \mathrm{d}_{x_1}\left(\int_0^1 A_2\left[\bar{\gamma}\right] \nabla\left(\bar{y} + v\right) \cdot \nabla \bar{r}\, \mathrm{d}x_2\right) \quad \text{in } \left(0,1\right),&
\end{aligned}
\tag{4.5}
$$

together with the boundary conditions $\bar{r} = 0$ on $\Sigma$, $\bar{r} = \bar{s}$ on $\Gamma$, and $\bar{s}(0) = \bar{s}(1) = 0$. Moreoever, we recall that the matrices $A_1$ and $A_2$ are a decomposition of the derivative of $A$ with respect to $\gamma$, more specifically $\mathrm{D}A\left[\bar{\gamma}\right]\langle h\rangle = A_1\left[\bar{\gamma}\right] h + A_2\left[\bar{\gamma}\right] \mathrm{d}_{x_1} h$.

**Theorem 4.1.3.** *The solution* $(\bar{s}, \bar{r} + E\bar{s})$ *to the adjoint equations* (4.5) *belongs to* $\left(\mathring{W}_1^1(0,1) \cap W_1^2(0,1)\right) \times \left(\mathring{W}_q^1(\Omega) \cap W_q^2(\Omega)\right)$ *and the following a-priori error esti-*

*mates are true*

$$\|\bar{s}\|_{W_1^2(0,1)} \lesssim \|\bar{\gamma} - \gamma_d\|_{L^1(0,1)} + \|\bar{y} + v - \gamma_d\|^2_{L^2(\Omega)} + \|v\|_{W_p^2(\Omega)} \|\bar{y} + v - \gamma_d\|_{L^q(\Omega)}, \quad (4.6)$$

$$\|\bar{r}\|_{W_q^2(\Omega)} \lesssim \|\bar{\gamma} - \gamma_d\|_{L^1(\Gamma)} + \|\bar{y} + v - \gamma_d\|^2_{L^2(\Omega)} + \Big(1 + \|v\|_{W_p^2(\Omega)}\Big) \|\bar{y} + v - \gamma_d\|_{L^q(\Omega)}. \quad (4.7)$$

*Proof.* Since $\Omega$ and $(0,1)$ are convex, the existence of a strong solution follows from [GT01, Lemma 9.17]. Furthermore, using (3.5c) we readily obtain a preliminary estimate for $\bar{r}$

$$\begin{aligned} \|\bar{r}\|_{W_q^2(\Omega)} &\lesssim \|\bar{s}\|_{W_q^{2-1/q}(0,1)} + \|\bar{y} + v - y_d\|_{L^q(\Omega)} \\ &\lesssim \|\bar{s}\|_{W_1^2(0,1)} + \|\bar{y} + v - y_d\|_{L^q(\Omega)} \end{aligned}$$

and similarly for $\bar{s}$,

$$\begin{aligned} \|\bar{s}\|_{W_1^2(0,1)} &\lesssim \|\bar{\gamma} - \gamma_d\|_{L^1(0,1)} + \|\bar{y} + v - y_d\|^2_{L^2(\Omega)} \\ &\quad + |\bar{y} + v|_{W_p^1(\Omega)} \Big(|\bar{r}|_{W_q^1(\Omega)} + |\bar{r}|_{W_q^2(\Omega)}\Big) + |\bar{y} + v|_{W_p^2(\Omega)} |\bar{r}|_{W_q^1(\Omega)} \\ &\lesssim \|\bar{\gamma} - \gamma_d\|_{L^1(0,1)} + \|\bar{y} + v - y_d\|^2_{L^2(\Omega)} + \|v\|_{W_p^2(\Omega)} \|\bar{r}\|_{W_q^2(\Omega)}. \end{aligned}$$

Once again, we invoke the smallness condition on $v$ to obtain the final a-priori estimate for $(\bar{s}, \bar{r})$. $\square$

This accomplishes the goal of showing estimates for strong solutions to the optimal control problem.

## 4.2 Numerics

The goal of this section is to introduce the discrete version of the optimization problem (3.2) and show an a-priori error estimate relating it to its continuous counterpart. The discretization uses the finite element method and is classical. The error estimate relies on the second-order regularity results from §4.1.

### 4.2.1 Discrete Optimal Control Problem

Let $\mathcal{T}$ denote a geometrically conforming rectangular quasi-uniform triangulation of the fixed domain $\Omega$ such that $\overline{\Omega} = \cup_{K\in\mathcal{T}} K$ and $h \approx h_K$ be the meshsize of $\mathcal{T}$. Additionally, suppose $[0,1] = \cup_{i=0}^{M} [\zeta_i, \zeta_{i+1}]$ with $0 = \zeta_0 < \zeta_1 < \ldots < \zeta_{M+1} = 1$ and $\zeta_i$ is compatible with $\mathcal{T}$. Consider the following finite dimensional spaces, where the capital letters stand for discrete objects

$$\mathbb{V}_h := \left\{ Y \in C^0(\bar{\Omega}) : Y|_K \in \mathcal{P}^1(K), K \in \mathcal{T} \right\}, \tag{4.8a}$$

$$\mathring{\mathbb{V}}_h := \mathbb{V}_h \cap \mathring{W}^1_p(\Omega), \tag{4.8b}$$

$$\mathbb{S}_h := \left\{ G \in C^0([0,1]) : G|_{[\zeta_i,\zeta_{i+1}]} \in \mathcal{P}^1([\zeta_i, \zeta_{i+1}]),\ 0 \le i \le M \right\}, \tag{4.8c}$$

$$\mathring{\mathbb{S}}_h := \mathbb{S}_h \cap \mathring{W}^1_\infty(0,1), \tag{4.8d}$$

$$\mathbb{U}_{ad} := \mathbb{S}_h \cap \mathcal{U}_{ad}. \tag{4.8e}$$

The spaces $\mathring{\mathbb{V}}_h, \mathring{\mathbb{S}}_h$ and $\mathbb{U}_{ad}$ in (4.8) will be used to approximate the continuous solution of (3.5b). The spaces are based on the finite dimensional space $\mathcal{P}^1(D)$ which are the bi-linear polynomials on the domain $D$, where $D$ is either a rectangle in $\mathcal{T}$ or an interval $[\zeta_i, \zeta_{i+1}]$. This discretization is classical and is detailed in [BS08, Chapter 3]. We remark that in our numerical implementation the $L^2$ constraints in $\mathbb{U}_{ad}$ are enforced by scaling the functions with their $L^2$-norm; for more details we refer to Section 4.4.

Next we present a discrete analog of the continuous extension (3.21), namely

$$E_h(G) := (\mathcal{S}_h \circ E)(G), \quad \forall G \in \mathring{\mathbb{S}}_h.$$

The caveat is that functions in $W^1_q(\Omega)$ are not necessarily continuous. This issue is addressed by utilizing the Scott-Zhang interpolant $\mathcal{S}_h : W^1_q(\Omega) \to \mathbb{V}_h$. This operator satisfies the optimal estimate [BS08],

$$|w - \mathcal{S}_h w|_{W^1_q(\Omega)} \lesssim h|w|_{W^2_q(\Omega)}, \, \forall w \in W^2_q(\Omega), \, 1 \leq q \leq \infty. \tag{4.9}$$

For functions in $W^1_p(\Omega)$ with $p > 2$, $W^1_\infty(0,1)$ and $W^1_1(0,1)$ we will use the standard Lagrange interpolant $\mathcal{I}_h$. This is justified by the Sobolev embedding theorems, i.e. we can identify functions in those spaces with their continuous equivalents. Moreover, the following optimal interpolation estimates hold,

$$|y - \mathcal{I}_h y|_{W^1_p(\Omega)} \lesssim h|y|_{W^2_p(\Omega)}, \, \forall y \in W^2_p(\Omega), \, 2 < p, \tag{4.10a}$$

$$|\gamma - \mathcal{I}_h \gamma|_{W^1_p(0,1)} \lesssim h|\gamma|_{W^2_p(0,1)}, \, \forall \gamma \in W^2_p(0,1), \, 1 \leq p \leq \infty. \tag{4.10b}$$

Next we state the discrete version of the optimal control problem (3.5a) in its variational form: minimize

$$\mathcal{J}_h(G,Y,U) := \frac{1}{2}\|G - \gamma_d\|^2_{L^2(0,1)} + \frac{1}{2}\left\|(Y + v - y_d)\sqrt{1+G}\right\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|U\|^2_{L^2(0,1)}, \tag{4.11a}$$

subject to the discrete *state equation* $(G,Y) \in \mathring{\mathbb{S}}_h \times \mathring{\mathbb{V}}_h$

$$\mathcal{B}_\Gamma[G,\Xi] + \mathcal{B}_\Omega\left[Y + v, Z + E_h\Xi; A[G]\right] = \int_0^1 U\Xi \quad \forall(\Xi, Z) \in \mathring{\mathbb{S}}_h \times \mathring{\mathbb{V}}_h, \tag{4.11b}$$

the state constraints

$$\left|G'\right| \leq 1 \qquad \text{on every } \; (\zeta_i, \zeta_{i+1}), \; i = 0, ..., M-1, \tag{4.11c}$$

and the control constraints

$$u \in \mathbb{U}_{ad}.$$

We remark that in (4.11b) $Y|_{\partial\Omega} = 0$. This is not the standard approach in finite element literature because it requires knowing an extension of $v$ to $\Omega$; we adopt this approach to simplify the exposition. We must include the following regularity assumption on the given data in order to obtain an order of convergence,

($\mathbf{A_3}$) The given data $v$, $\gamma_d$ and $y_d$ belong to $W_p^2(\Omega)$, $L^2(0,1)$, and $L^2(\Omega^*)$ respectively.

Now let $\bar{U}$ denote the optimal control to (4.11a) and $\left(\bar{G}, \bar{Y}\right)$ be the optimal state, which satisfy discrete *state equations* in variational form (4.11b). The existence of $\bar{U}$ will be shown in Theorem 4.2.2. We approximate the solutions $(\bar{S}, \bar{R})$ to the adjoint equations using the finite dimensional spaces $\mathring{\mathbb{S}}_h \times \mathring{\mathbb{V}}_h$. The discrete *adjoint equations* in variational form read: Find $(\bar{S}, \bar{R}) \in \mathring{\mathbb{S}}_h \times \mathring{\mathbb{V}}_h$ such that for every $(\Xi, Z) \in \mathring{\mathbb{S}}_h \times \mathring{\mathbb{V}}_h$,

$$\mathcal{B}_\Gamma\left[\Xi, \bar{S}\right] + \mathcal{D}_\Omega\left[\Xi, Z, \bar{R} + E_h\bar{S}; \bar{G}, \bar{Y}\right] = \left\langle \Xi, \bar{G} - \gamma_d + \frac{1}{2}\int_0^1 \left|\bar{Y} + v - y_d\right|^2 \mathrm{d}x_2 \right\rangle \tag{4.12}$$

$$+ \left\langle Z, \left(\bar{Y} + v - y_d\right)\left(1 + \bar{G}\right)\right\rangle.$$

Finally, the control satisfies the variational inequality

$$\left\langle \mathcal{J}_h'(\bar{U}), U - \bar{U}\right\rangle_{L^2(0,1),L^2(0,1)} \geq 0, \quad \forall U \in \mathbb{U}_{ad} \tag{4.13}$$

where $\mathcal{J}_h'(\bar{U}) = \bar{S} + \lambda\bar{U}$, therefore (4.13) reads

$$\left\langle \bar{S} + \lambda\bar{U}, U - \bar{U} \right\rangle_{L^2(0,1),L^2(0,1)} \geq 0, \quad \forall U \in \mathbb{U}_{ad}. \tag{4.14}$$

The following discrete estimates are analogous to the continuous inf-sup in Proposition 3.3.1

**Proposition 4.2.1** (Discrete inf-sup). *There exists constants $0 < \alpha, \beta < +\infty$ independent of $h$ such that*

*(i)*

$$|G|_{W^1_\infty(0,1)} \leq \alpha \sup_{0 \neq \Xi \in \mathring{\mathbb{S}}_h} \frac{\mathcal{B}_\Gamma[G, \Xi]}{|\Xi|_{W^1_1(0,1)}}, \tag{4.15a}$$

$$|S|_{W^1_1(0,1)} \leq \alpha \sup_{0 \neq \Xi \in \mathring{\mathbb{S}}_h} \frac{\mathcal{B}_\Gamma[\Xi, S]}{|\Xi|_{W^1_\infty(0,1)}}. \tag{4.15b}$$

*(ii) There exist constants $Q < 2 < P$, $h_0 > 0$, such that for $p \in [Q, P]$ and $0 < h \leq h_0$*

$$|Y|_{W^1_p(\Omega)} \leq \beta \sup_{0 \neq Z \in \mathring{\mathbb{V}}_h} \frac{\mathcal{B}_\Omega\left[Y, Z; A[G]\right]}{|Z|_{W^1_q(\Omega)}}. \tag{4.16}$$

*Proof.* We refer to [SS91, Proposition 3.2] for obtaining estimate (4.15a). The technique explained there can also be extended to (4.15b).

We deal with (4.16) in two stages. The estimate for $Q \geq 2$ was derived in detail in [SS91, Proposition 3.3] and relied on the fact that the Ritz projection $\mathcal{R}$ on a convex domain is continuous only for $P > 2$; for a proof we refer to [RS82].

To obtain the estimate for $Q < 2$, we use the following standard duality argument:

$$\|\nabla\mathcal{R}y\|_{L^p(\Omega)} = \sup_{\boldsymbol{z} \in L^q(\Omega)^2} \frac{(\nabla\mathcal{R}y, \boldsymbol{z})}{\|\boldsymbol{z}\|_{L^q(\Omega)}}, \quad p \in (1, Q]. \tag{4.17}$$

Using Green's theorem we have $(\nabla \mathcal{R} y, \boldsymbol{z}) = -(\mathcal{R} y, \operatorname{div} \boldsymbol{z})$, then setting $\operatorname{div} \boldsymbol{z} = \Delta w$, which is understood in a weak sense, whence $\|\nabla w\|_{L^q(\Omega)} \lesssim \|z\|_{L^q(\Omega)}$. We have

$$(\nabla \mathcal{R} y, z) = (\nabla \mathcal{R} y, \nabla w) = (\nabla y, \nabla \mathcal{R} w) \,. \tag{4.18}$$

Finally, (4.17) and (4.18) imply

$$\|\nabla \mathcal{R} y\|_{L^p(\Omega)} \leq \sup_{w \in \mathring{W}^1_q(\Omega)} \frac{(\nabla y, \nabla \mathcal{R} w)}{\|\nabla w\|_{L^q(\Omega)}} \lesssim \|\nabla y\|_{L^p(\Omega)} \sup_{w \in \mathring{W}^1_q(\Omega)} \frac{\|\nabla \mathcal{R} w\|_{L^q(\Omega)}}{\|\nabla w\|_{L^q(\Omega)}} \lesssim \|\nabla y\|_{L^p(\Omega)} \,,$$

where the last inequality follows from the fact that the Ritz projection is continuous for $q > 2$. We can now repeat the proof in [SS91, Proposition 3.3] for $Q < 2$ to conclude our result. □

The existence and uniqueness of the solution to *state* and *adjoint equations* can be shown similarly to the continuous case Corollary 3.3.7 and Theorem 3.4.6 under the assumption that $U$ belongs to $\mathbb{U}_{ad}$ and $|v|_{W^1_p(\Omega)}$ is small. Next we will prove the existence of an optimal control $\bar{U}$ solving (4.11a).

**Theorem 4.2.2.** *There exists an optimal control $\bar{U}$ in $\mathbb{U}_{ad}$ which solves* (4.11a)*.*

*Proof.* The proof follows by using a minimizing sequence argument similar to the continuous proof. The main difference is that in finite dimensional spaces the weak convergence of the minimizing sequence $U_n$ is also yields strong convergence. Therefore, using the Lipschitz continuity of the discrete control-to-state map, we also obtain strong convergence of the associated state sequence $(G_n, Y_n)$. □

### 4.2.2 A-priori Error Estimates: State and Adjoint Variables

The goal of this section is to derive a-priori error estimates between the continuous and discrete solutions of the *state* and *adjoint* equations for given functions $u \in \mathcal{U}_{ad}$ and $U \in \mathbb{U}_{ad}$. This is the content of Lemmas 4.2.3 through 4.2.8. These estimates are the stepping stone for the optimal control $L^2$ estimate in Theorem 4.3.1.

First we rewrite the adjoint equations in a compact but equivalent form, $(\bar{s}, \bar{r})$ in $\mathring{W}^1_1(0,1) \times \mathring{W}^1_q(\Omega)$ satisfies the *adjoint equation* in variational form

$$\mathcal{B}_\Gamma[\xi, \bar{s}] + \mathcal{D}_\Omega[\xi, z, \bar{r} + E\bar{s}; \bar{\gamma}, \bar{y}] = \langle \xi, \bar{\gamma} - \gamma_d \rangle + \langle z, \bar{y} + v - y_d \rangle, \tag{4.19}$$

for all $(\xi, z)$ in $W^1_\infty(0,1) \times W^1_p(\Omega)$ with $(\bar{\gamma}, \bar{y})$ set to $\big(\gamma(\bar{u}), y(\bar{u})\big)$. The parametrized form

$$\mathcal{D}_\Omega[\xi, z, r; \bar{\gamma}, \bar{y}] := \mathcal{B}_\Omega\big[z, r; A[\bar{\gamma}]\big] + \mathcal{B}_\Omega\big[\bar{y} + v, r; \mathrm{D}A[\bar{\gamma}]\langle \xi \rangle\big]. \tag{4.20}$$

$\mathcal{D}_\Omega$ was introduced while analysing the Fréchet differentiability of the control-to-state map §3.3.3. Moreover, the duality pairings on the right-hand-side of (4.19) are reduced to standard integrals due to the $L^2$-regularity imposed by the cost functional.

**Lemma 4.2.3** (Preliminary $G$-error estimate)**.** *Let* $(\gamma, y)$ *and* $(G, Y)$ *solve* (3.5b) *and* (4.11b) *respectively for* $u \in \mathcal{U}_{ad}$*,* $v \in W^2_p(\Omega)$ *with* $|v|_{W^1_p(\Omega)}$ *small. Then the following error estimate for* $\gamma - G$ *holds*

$$|\gamma - G|_{W^1_\infty(0,1)} \lesssim h|\gamma|_{W^2_\infty(0,1)} + |y - Y|_{W^1_p(\Omega)} + \|u - U\|_{L^2(0,1)}.$$

*Proof.* The goal is to use the *discrete* inf-sup (4.15a). By the triangle inequality

$$\begin{aligned}|\gamma - G|_{W^1_\infty(0,1)} &\le |\gamma - \mathcal{I}_h\gamma|_{W^1_\infty(0,1)} + |\mathcal{I}_h\gamma - G|_{W^1_\infty(0,1)} \\ &\lesssim h|\gamma|_{W^2_\infty(0,1)} + \sup_{0\neq\Xi\in\mathring{\mathbb{S}}_h} \frac{\mathcal{B}_\Gamma\left[\mathcal{I}_h\gamma - G, \Xi\right]}{|\Xi|_{W^1_1(0,1)}}.\end{aligned}$$

Next, we write $\mathcal{B}_\Gamma\left[\mathcal{I}_h\gamma - G, \Xi\right] = \mathcal{B}_\Gamma\left[\mathcal{I}_h\gamma - \gamma, \Xi\right] + \mathcal{B}_\Gamma\left[\gamma - G, \Xi\right]$. We estimate the first term using Hölder's inequality. For the second term we derive the estimates below. By setting $w = y + v$ and $W = Y + v$, using that $\gamma$ and $G$ satisfy (3.5b) and (4.11b) respectively, and the fact that $|y|_{W^1_p(\Omega)} \lesssim |v|_{W^1_p(\Omega)}$ we obtain,

$$\begin{aligned}\mathcal{B}_\Gamma\left[\gamma - G, \Xi\right] &= -\mathcal{B}_\Omega\left[w, E_h\Xi; A\left[\gamma\right]\right] + \mathcal{B}_\Omega\left[W, E_h\Xi; A\left[G\right]\right] + \langle u - U, \Xi\rangle_{L^2(0,1),L^2(0,1)} \\ &= \mathcal{B}_\Omega\left[w, E_h\Xi; -A\left[\gamma\right] + A\left[G\right]\right] + \mathcal{B}_\Omega\left[W - w, E_h\Xi; A\left[G\right]\right] \\ &\quad + \langle u - U, \Xi\rangle_{L^2(0,1),L^2(0,1)} \\ &\lesssim \left(|\gamma - G|_{W^1_\infty(0,1)}|v|_{W^1_p(\Omega)} + |y - Y|_{W^1_p(\Omega)} + \|u - U\|_{L^2(0,1)}\right)|\Xi|_{W^1_1(0,1)}.\end{aligned}$$

Combining the above two estimates we have

$$|\gamma - G|_{W^1_\infty(0,1)} \lesssim h|\gamma|_{W^2_\infty(0,1)} + |\gamma - G|_{W^1_\infty(0,1)}|v|_{W^1_p(\Omega)} + |y - Y|_{W^1_p(\Omega)} + \|u - U\|_{L^2(0,1)}.$$

Using that $|v|_{W^1_p(\Omega)}$ is small yields the desired result. □

**Lemma 4.2.4** (Y-error estimate)**.** *Let* $(\gamma, y)$ *and* $(G, Y)$ *solve* (3.5b) *and* (4.11b) *respectively for* $u \in \mathcal{U}_{ad}$, $|v|_{W^1_p(\Omega)}$ *small. Then the following estimate for* $y - Y$ *holds*

$$|y - Y|_{W^1_p(\Omega)} \lesssim h\left(|\gamma|_{W^2_\infty(0,1)} + |y|_{W^2_p(\Omega)}\right) + |v|_{W^1_p(\Omega)}\|u - U\|_{L^2(0,1)}.$$

*Proof.* The idea is the same as in the previous lemma. We use the triangle inequality and the *discrete* inf-sup followed by an interpolation estimate together with the state constraint $|G|_{W^1_\infty(0,1)} < 1$ to obtain

$$
\begin{aligned}
|y - Y|_{W^1_p(\Omega)} &\leq |y - \mathcal{I}_h y|_{W^1_p(\Omega)} + |\mathcal{I}_h y - Y|_{W^1_p(\Omega)} \\
&\lesssim h|y|_{W^2_p(\Omega)} + \sup_{0 \neq Z \in \mathring{\mathbb{V}}_h} \frac{\mathcal{B}_\Omega \left[\mathcal{I}_h y - Y, Z; A\left[G\right]\right]}{|Z|_{W^1_q(\Omega)}} \\
&\lesssim h|y|_{W^2_p(\Omega)} + \sup_{0 \neq Z \in \mathring{\mathbb{V}}_h} \frac{\mathcal{B}_\Omega \left[y - Y, Z; A\left[G\right]\right]}{|Z|_{W^1_q(\Omega)}}.
\end{aligned}
$$

We handle the last term by using that $y$ and $Y$ are solutions to (3.5b) and (4.11b), i.e.

$$
\begin{aligned}
\mathcal{B}_\Omega \left[y - Y, Z; A\left[G\right]\right] &= \mathcal{B}_\Omega \left[y + v, Z; A\left[G\right]\right] - \mathcal{B}_\Omega \left[Y + v, Z; A\left[G\right]\right] \\
&= \mathcal{B}_\Omega \left[y + v, Z; A\left[\gamma\right]\right] + \mathcal{B}_\Omega \left[y + v, Z; A\left[G\right] - A\left[\gamma\right]\right] \\
&= \mathcal{B}_\Omega \left[y + v, Z; A\left[G\right] - A\left[\gamma\right]\right],
\end{aligned}
$$

followed by the bound $|y|_{W^1_p(\Omega)} \lesssim |v|_{W^1_p(\Omega)}$ in the definition of $\mathbb{B}_1$ to yield

$$
\mathcal{B}_\Omega \left[y - Y, Z; A\left[G\right]\right] \lesssim |\gamma - G|_{W^1_\infty(0,1)} |v|_{W^1_p(\Omega)} |Z|_{W^1_q(\Omega)}.
$$

Combining the above estimates with Lemma 4.2.3, we obtain

$$
\begin{aligned}
|y - Y|_{W^1_p(\Omega)} &\lesssim h|y|_{W^2_p(\Omega)} + |\gamma - G|_{W^1_\infty(0,1)} |v|_{W^1_p(\Omega)} \\
&\lesssim h\left(|\gamma|_{W^2_\infty(0,1)} |v|_{W^1_p(\Omega)} + |y|_{W^2_p(\Omega)}\right) + \|u - U\|_{L^2(0,1)} |v|_{W^1_p(\Omega)} \\
&\quad + |y - Y|_{W^1_p(\Omega)} |v|_{W^1_p(\Omega)}
\end{aligned}
$$

The final result is obtained by resorting to the smallness of $|v|_{W^1_p(\Omega)}$. □

**Lemma 4.2.5** ($G$-error estimate)**.** *Let* $(\gamma, y)$ *and* $(G, Y)$ *solve* (3.5b) *and* (4.11b) *respectively for* $u \in \mathcal{U}_{ad}$, $U \in \mathbb{U}_{ad}$, $|v|_{W^1_p(\Omega)}$ *small. Then the following error estimate* *for $\gamma - G$ holds*

$$|\gamma - G|_{W^1_\infty(0,1)} \lesssim h\left(|\gamma|_{W^2_\infty(0,1)} + |y|_{W^2_p(\Omega)}\right) + \|u - U\|_{L^2(0,1)}.$$

*Proof.* The estimate follows by combining Lemma 4.2.4 with Lemma 4.2.3. $\square$

**Lemma 4.2.6** (Preliminary $S$-error estimate)**.** *Let $(s, r + Es) \in \mathring{W}^1_1(0,1) \times \mathring{W}^1_q(\Omega)$ satisfy the continuous adjoint system* (4.19)*, and $(S, R)$ satisfy the discrete counterpart* (4.12)*. Then the following error estimate for $s - S$ is valid*

$$\begin{aligned}
&|s - S|_{W^1_1(0,1)} \\
&\quad\lesssim h|s|_{W^2_1(0,1)} + \left(1 + |r|_{W^1_q(\Omega)}|v|_{W^1_p(\Omega)}\right)|\gamma - G|_{W^1_\infty(0,1)} \\
&\qquad + \left(\|y + 2v - 2y_d\|_{L^2(\Omega)} + |y - Y|_{W^1_p(\Omega)} + \|y\|_{L^2(\Omega)} + |r|_{W^1_q(\Omega)}\right)|y - Y|_{W^1_p(\Omega)} \\
&\qquad + |r - R|_{W^1_q(\Omega)}|v|_{W^1_p(\Omega)}.
\end{aligned}$$

*Proof.* Again, the goal is to use the *discrete* inf-sup (4.15b), now taking the form

$$\begin{aligned}
|s - S|_{W^1_1(0,1)} &\lesssim |s - \mathcal{I}_h s|_{W^1_1(0,1)} + \sup_{0 \neq \Xi \in \mathring{\mathbb{S}}_h} \frac{\mathcal{B}_\Gamma\left[\Xi, \mathcal{I}_h s - S\right]}{|\Xi|_{W^1_\infty(0,1)}} \\
&\lesssim h|s|_{W^2_1(0,1)} + \sup_{0 \neq \Xi \in \mathring{\mathbb{S}}_h} \frac{\mathcal{B}_\Gamma\left[\Xi, s - S\right]}{|\Xi|_{W^1_\infty(0,1)}},
\end{aligned}$$

where the last inequality follows by adding and subtracting $s$, the continuity of $\mathcal{B}_\Gamma$, and an interpolation estimate for $s - \mathcal{I}_h s$. It remains to control the last term. We use that $s$ and $S$ satisfy equations (4.19) and (4.12) to obtain

$$\begin{aligned}
\mathcal{B}_\Gamma\left[\Xi, s-S\right] &= \langle \Xi, \gamma-\gamma_d\rangle - \langle \Xi, G-\gamma_d\rangle \\
&\quad + \left\langle \Xi, \frac{1}{2}\int_0^1 |y+v-y_d|^2 \,\mathrm{d}x_2 \right\rangle - \left\langle \Xi, \frac{1}{2}\int_0^1 |Y+v-y_d|^2 \,\mathrm{d}x_2 \right\rangle \\
&\quad + \mathcal{B}_\Omega\left[y+v, r; DA\left[\gamma\right]\langle\Xi\rangle\right] - \mathcal{B}_\Omega\left[Y+v, R; DA\left[G\right]\langle\Xi\rangle\right] \\
&= \langle \Xi, \gamma - G\rangle + \left\langle \Xi, \frac{1}{2}\int_0^1 \left(y-Y\right)\left(y+Y+2v-2y_d\right)\mathrm{d}x_2 \right\rangle \\
&\quad + \mathcal{B}_\Omega\left[y-Y, r; DA\left[\gamma\right]\langle\Xi\rangle\right] + \mathcal{B}_\Omega\left[Y+v, r; \left(DA\left[\gamma\right] - DA\left[G\right]\right)\langle\Xi\rangle\right] \\
&\quad + \mathcal{B}_\Omega\left[Y+v, r-R; DA\left[G\right]\langle\Xi\rangle\right],
\end{aligned}$$

whence after normalization ($|\Xi|_{W^1_\infty(0,1)} = 1$),

$$\begin{aligned}
\left|\mathcal{B}_\Gamma\left[\Xi, s-S\right]\right| \lesssim & \|\gamma - G\|_{L^1(0,1)} \\
&+ \|y-Y\|_{L^2(\Omega)}\left(\|y+2v-2y_d\| + \|Y-y\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}\right) \\
&+ |y-Y|_{W^1_p(\Omega)} |r|_{W^1_q(\Omega)} \\
&+ |v|_{W^1_p(\Omega)}\left(|r|_{W^1_q(\Omega)} |\gamma - G|_{W^1_\infty(0,1)} + |r-R|_{W^1_q(\Omega)}\right).
\end{aligned}$$

We obtain the desired result after combining the above estimate with $\|\gamma - G\|_{L^1(0,1)} \lesssim |\gamma - G|_{W^1_\infty(0,1)}$ and $\|y-Y\|_{L^2(\Omega)} \lesssim |y-Y|_{W^1_p(\Omega)}$. □

**Lemma 4.2.7** ($R$-error estimate)**.** *Let* $(s, r+Es) \in \mathring{W}^1_1(0,1) \times \mathring{W}^1_q(\Omega)$ *satisfy the continuous adjoint equations, and* $(S,R)$ *satisfy the discrete version. The following* *a-priori error estimate for $r - R$ holds*

$$
\begin{aligned}
&|r-R|_{W_q^1(\Omega)} \\
&\quad \lesssim h\left(|s|_{W_1^2(0,1)} + |r|_{W_q^2(\Omega)}\right) \\
&\qquad + \left(1 + \left(1 + |v|_{W_p^1(\Omega)}\right)|r|_{W_q^1(\Omega)} + \|y\|_{L^q(\Omega)} + \|v - y_d\|_{L^q(\Omega)}\right)|\gamma - G|_{W_\infty^1(0,1)} \\
&\qquad + \left(1 + \|y + 2v - 2y_d\|_{L^2(\Omega)} + |y - Y|_{W_p^1(\Omega)} + \|y\|_{L^2(\Omega)} + |r|_{W_q^1(\Omega)}\right)|y - Y|_{W_p^1(\Omega)}.
\end{aligned}
$$

*Proof.* Again, the goal is to use the *discrete* inf-sup (4.16). Since it is only applicable for functions in $\mathring{\mathbb{V}}_h$, we write $r = r_0 + Es$, and $R = R_0 + E_h S$, where $r_0$ and $R_0$ are in $\mathring{W}_q^1(\Omega)$ and $\mathring{\mathbb{V}}_h$, to obtain

$$
\begin{aligned}
|r-R|_{W_q^1(\Omega)} &\leq |r_0 - \mathcal{S}_h r_0|_{W_q^1(\Omega)} + |\mathcal{S}_h r_0 - R_0|_{W_q^1(\Omega)} + |Es - E_h S|_{W_q^1(\Omega)} \\
&\lesssim h|r_0|_{W_q^2(\Omega)} + |Es - E_h S|_{W_q^1(\Omega)} + \sup_{0 \neq Z \in \mathring{\mathbb{V}}_h} \frac{\mathcal{B}_\Omega\left[Z, r_0 - R_0; A\left[G\right]\right]}{|Z|_{W_p^1(\Omega)}},
\end{aligned}
$$

where we have added and subtracted $r_0$ to get the last term from (4.16). Moreover, we handle this term as before, i.e.

$$
\begin{aligned}
\mathcal{B}_\Omega\left[Z, r_0 - R_0; A\left[G\right]\right] &= \mathcal{B}_\Omega\left[Z, r_0 + Es; A\left[\gamma\right]\right] - \mathcal{B}_\Omega\left[Z, R_0 + E_h S; A\left[G\right]\right] \\
&\quad + \mathcal{B}_\Omega\left[Z, r_0 + Es; A\left[G\right] - A\left[\gamma\right]\right] + \mathcal{B}_\Omega\left[Z, E_h S - Es; A\left[G\right]\right].
\end{aligned}
$$

Invoking the *adjoint equations* (4.19) and (4.12), we see that

$$
\begin{aligned}
&\mathcal{B}_\Omega\left[Z, r_0 + Es; A\left[\gamma\right]\right] - \mathcal{B}_\Omega\left[Z, R_0 + E_h S; A\left[G\right]\right] \\
&\qquad = \left\langle (y + v - y_d)(1 + \gamma), Z\right\rangle - \left\langle (Y + v - y_d)(1 + G), Z\right\rangle \\
&\qquad = \langle y - Y, Z\rangle + \langle y\gamma - YG, Z\rangle + \left\langle (v - y_d)(\gamma - G), Z\right\rangle.
\end{aligned}
$$

Since $y\gamma - YG = y(\gamma - G) - (Y - y)G$, therefore

$$\mathcal{B}_\Omega\left[Z, r_0 + Es; A[\gamma]\right] - \mathcal{B}_\Omega\left[Z, R_0 + E_h S; A[G]\right]$$
$$= \langle y - Y, Z\rangle + \left\langle y(\gamma - G) - (Y - y)G, Z\right\rangle + \left\langle (v - y_d)(\gamma - G), Z\right\rangle.$$

After normalization ($|Z|_{W^1_p(\Omega)} = 1$) and using (4.11c), we obtain the estimate

$$\begin{aligned}\left|\mathcal{B}_\Omega\left[Z, r_0 - R_0; A[G]\right]\right| \lesssim{}& \|y - Y\|_{L^q(\Omega)} \\ &+ |\gamma - G|_{W^1_\infty(0,1)}\left(|r|_{W^1_q(\Omega)} + \|y\|_{L^q(\Omega)} + \|v - y_d\|_{L^q(\Omega)}\right) \\ &+ |Es - E_h S|_{W^1_q(\Omega)}.\end{aligned}$$

Combining this together with $|Es - E_h S|_{W^1_q(\Omega)} \lesssim h|s|_{W^2_1(0,1)}$, and $|Es - E_h S|_{W^1_q(\Omega)} \lesssim |s - S|_{W^1_1(0,1)}$, we end up with

$$\begin{aligned}|r - R|_{W^1_q(\Omega)} \lesssim{}& h\left(|s|_{W^2_1(0,1)} + |r|_{W^2_q(\Omega)}\right) \\ &+ \|y - Y\|_{L^q(\Omega)} + |\gamma - G|_{W^1_\infty(0,1)}\left(|r|_{W^1_q(\Omega)} + \|y\|_{L^q(\Omega)} + \|v - y_d\|_{L^q(\Omega)}\right) \\ &+ |s - S|_{W^1_1(0,1)}.\end{aligned}$$

Finally, under the smallness assumption on $|v|_{W^1_p(\Omega)}$ and $\|y - Y\|_{L^q(\Omega)} \lesssim |y - Y|_{W^1_p(\Omega)}$, Lemma 4.2.6 yields the desired result. □

**Lemma 4.2.8** ($S$-error estimate)**.** *The following a-priori estimate for $s - S$ holds*

$$\begin{aligned}|s - S|_{W^1_1(0,1)} \lesssim{}& h\Bigg(\left(1 + |r|_{W^1_q(\Omega)} + \|y_d\|_{L^2(\Omega)} + \|u - U\|_{L^2(0,1)}\right)\left(|\gamma|_{W^2_\infty(0,1)} + |y|_{W^2_p(\Omega)}\right) \\ &+ |s|_{W^2_1(0,1)} + |r|_{W^2_q(\Omega)}\Bigg) \\ &+ \left(1 + |r|_{W^1_q(\Omega)} + \|y_d\|_{L^2(\Omega)} + \|u - U\|_{L^2(0,1)}\right)\|u - U\|_{L^2(0,1)}.\end{aligned}$$

*Proof.* We use lemmas 4.2.6 and 4.2.7, to obtain

$$
\begin{aligned}
|s-S|_{W_1^1(0,1)} \lesssim h & \Big(|s|_{W_1^2(0,1)} + |v|_{W_p^1(\Omega)} \Big(|s|_{W_1^2(0,1)} + |r|_{W_q^2(\Omega)}\Big)\Big) \\
& + \Big(c_1 + |v|_{W_p^1(\Omega)} (c_1 + c_3)\Big) |\gamma - G|_{W_\infty^1(0,1)} \\
& + \Big(c_2 + |v|_{W_p^1(\Omega)} (1 + c_2)\Big) |y - Y|_{W_p^1(\Omega)} ,
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= 1 + |r|_{W_q^1(\Omega)} |v|_{W_p^1(\Omega)} , \\
c_2 &= \|y + 2v - 2y_d\|_{L^2(\Omega)} + |y - Y|_{W_p^1(\Omega)} + \|y\|_{L^2(\Omega)} + |r|_{W_q^1(\Omega)} , \\
c_3 &= |r|_{W_q^1(\Omega)} + \|y\|_{L^q(\Omega)} + \|v - y_d\|_{L^q(\Omega)} .
\end{aligned}
$$

The assertion follows by applying Lemmas 4.2.4 and 4.2.5, together with $|v|_{W_q^1(\Omega)} \leq 1$. □

## 4.3 A-priori Error Estimates: Optimal Control

Next we derive the a-priori error estimate between $\bar{u}$ and $\bar{U}$.

**Theorem 4.3.1** ($\bar{U}$-error estimate)**.** *Let $h \leq h_0$ and both $h_0$ and $|v|_{W_p^1(\Omega)}$ be sufficiently small, then*

$$
\left\|\bar{u} - \bar{U}\right\|_{L^2(0,1)} \leq \frac{4}{\lambda} \left\|s(\bar{U}) - S(\bar{U})\right\|_{L^2(0,1)} , \tag{4.21}
$$

*where $s(\bar{U})$ is the solution of the continuous adjoint equation* (4.19) *with $\big(\gamma(\bar{U}), y(\bar{U})\big)$ solutions of the state equation* (3.5b) *with control $\bar{U}$, and $S(\bar{U})$ solution of the discrete adjoint equation* (4.12)*.*

*Proof.* The proof relies primarily on the continuous quadratic growth condition (3.74) and on the continuous and discrete first-order optimality conditions (3.17) and (4.13). Since $\bar{U} \in \mathbb{U}_{ad}$ is admissible, replacing $u$ by $\bar{U}$ in (3.74) we get

$$\frac{\lambda}{4}\|\bar{U}-\bar{u}\|^2_{L^2(0,1)} \leq \langle \mathcal{J}'(\bar{U}) - \mathcal{J}'(\bar{u}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)}.$$

Adding and subtracting $\mathcal{J}_h'(\bar{U})$ gives

$$\begin{aligned}\frac{\lambda}{4}\|\bar{U}-\bar{u}\|^2_{L^2(0,1)} \leq{}& \langle \mathcal{J}'(\bar{U}) - \mathcal{J}_h'(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)} \\ &+ \langle \mathcal{J}_h'(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)} + \langle \mathcal{J}'(\bar{u}), \bar{u}-\bar{U}\rangle_{L^2(0,1),L^2(0,1)}.\end{aligned}$$

Since $\langle \mathcal{J}'(\bar{u}), \bar{u}-\bar{U}\rangle_{L^2(0,1),L^2(0,1)} \leq 0$, according to (3.17), we deduce

$$\begin{aligned}\frac{\lambda}{4}\|\bar{U}-\bar{u}\|^2_{L^2(0,1)} \leq{}& \langle \mathcal{J}'(\bar{U}) - \mathcal{J}_h'(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)} \\ &+ \langle \mathcal{J}_h'(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)}.\end{aligned}$$

Add and subtract $\mathcal{P}_h\bar{u}$, the $L^2$ orthogonal projection of $\bar{u}$ onto $\mathbb{U}_{ad}$, to get

$$\begin{aligned}\frac{\lambda}{4}\|\bar{U}-\bar{u}\|^2_{L^2(0,1)} \leq{}& \langle \mathcal{J}'(\bar{U}) - \mathcal{J}_h'(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)} \\ &+ \langle \mathcal{J}_h'(\bar{U}), \mathcal{P}_h\bar{u}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)} + \langle \mathcal{J}_h'(\bar{U}), \bar{U}-\mathcal{P}_h\bar{u}\rangle_{L^2(0,1),L^2(0,1)}.\end{aligned}$$

Since $\mathcal{J}_h'(\bar{U}) \in \mathbb{S}_h$ the middle term vanishes. In view of (4.13) and the fact that $\mathcal{P}_h\bar{u} \in \mathbb{U}_{ad}$, we deduce $\langle \mathcal{J}_h'(\bar{U}), \bar{U}-\mathcal{P}_h\bar{u}\rangle_{L^2(0,1),L^2(0,1)} \leq 0$ and

$$\frac{\lambda}{4}\|\bar{U}-\bar{u}\|^2_{L^2(0,1)} \leq \langle \mathcal{J}'(\bar{U}) - \mathcal{J}_h'(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)},$$

The explicit expressions $\mathcal{J}'(\bar{U}) = \lambda\bar{U} + s(\bar{U})$ and $\mathcal{J}_h'(\bar{U}) = \lambda\bar{U} + S(\bar{U})$ yield

$$\frac{\lambda}{4}\|\bar{U}-\bar{u}\|^2_{L^2(0,1)} \leq \langle s(\bar{U}) - S(\bar{U}), \bar{U}-\bar{u}\rangle_{L^2(0,1),L^2(0,1)},$$

which imply the desired estimate (4.21). □

**Corollary 4.3.2** (rate of convergence)**.** *Let $h \le h_0$ and both $h_0$ and $|v|_{W^1_p(\Omega)}$ be sufficiently small. Furthermore, let $(s(\bar{U}), r(\bar{U}))$ be the solutions of the continuous adjoint equation* (4.19) *with $\big(\gamma(\bar{U}), y(\bar{U})\big)$ solutions for the continuous state equation* (3.5b) *with control $\bar{U}$. Let $\big(S(\bar{U}), R(\bar{U})\big)$ solve the discrete adjoint equation* (4.12) *with $\big(G(\bar{U}, Y(\bar{U}))\big)$ solutions for the discrete state equation* (4.11b) *with control $\bar{U}$. Then, there is a constant $C_0 \ge 1$, depending on $\|\gamma\|_{W^2_\infty(0,1)}$, $\|y\|_{W^2_p(\Omega)}$, $\|s\|_{W^2_1(0,1)}$, $\|r\|_{W^2_q(\Omega)}$, $\gamma_d, y_d$, such that*

$$
\begin{aligned}
&\big|\gamma(\bar{U}) - G(\bar{U})\big|_{W^1_\infty(0,1)} + \big|y(\bar{U}) - Y(\bar{U})\big|_{W^1_p(\Omega)} \\
&\quad + \big|s(\bar{U}) - S(\bar{U})\big|_{W^1_1(0,1)} + \big|r(\bar{U}) - R(\bar{U})\big|_{W^1_q(\Omega)} + \big\|\bar{u} - \bar{U}\big\|_{L^2(0,1)} \le C_0 h.
\end{aligned} \tag{4.22}
$$

*Proof.* In view of the continuity estimate

$$
\big\|s(\bar{U}) - S(\bar{U})\big\|_{L^2(0,1)} \le \big|s(\bar{U}) - S(\bar{U})\big|_{W^1_1(0,1)},
$$

and using (4.21) we get the error estimate for $\big\|\bar{u} - \bar{U}\big\|_{L^2(0,1)}$ in (4.22). For the remaining estimates in (4.22) set $u = \bar{U}$, and $U = \bar{U}$ in Lemmas (4.2.4), (4.2.5), (4.2.7) and (4.2.8) to complete the proof. □

**Remark 4.3.3** (linear rate)**.** The first-order convergence rate of (4.22) is optimal for a piecewise-linear finite element discretization of $(\gamma, y, s, r)$. For a control $u$ in $L^2$, one might expect an increased rate of convergence. For example, it would be possible to use the standard Aubin-Nitsche duality result if we were in a traditional linear setting to obtain

$$
\big\|s(\bar{U}) - S(\bar{U})\big\|_{L^2(0,1)} \le h^{1/2}\big|s(\bar{U}) - S(\bar{U})\big|_{W^1_1(0,1)},
$$

which in turn would yield an optimal rate of convergence $h^{3/2}$ for $\bar{u} - \bar{U}$ in the proof of Corollary 4.3.2. Unfortunately, the duality method fails in our setting because the right-hand-side is discretized as well. Thus we are left with the merely the Sobolev embedding result obtaining

$$\left\|s(\bar{U}) - S(\bar{U})\right\|_{L^2(0,1)} \lesssim \left|s(\bar{U}) - S(\bar{U})\right|_{W_1^1(0,1)}.$$

In turn, this yields the linear rate of convergence for $\bar{u} - \bar{U}$.

## 4.4 Simulations

The goal is to compute an approximation to the optimization problem presented in §3.2 with the cost functional set to

$$\mathcal{J}(\gamma, y, u) := \frac{1}{2}\|\gamma - \gamma_d\|^2_{L^2(0,1)} + \frac{\lambda}{2}\|u\|^2_{L^2(0,1)},$$

the Dirichlet data $v = x_2(1 - x_2)(1 - 2x_1)$ applied to the entire boundary of $\Omega$, and the desired configuration $\gamma_d$ set to an inverted hat function (see Figure 4.1). Moreover, we recall that $\gamma$ satisfies the state equations (3.2b), and we remark that the curvature is not linearized as was done for the analysis of (3.5b) and (3.20).

In view of the control constraint $u \in \mathcal{U}_{ad}$ (3.36), we need $\|u\|_{L^2(0,1)} \leq \theta_1/2\alpha$. Since $\alpha \sim 1/\kappa$ and $\theta_1 < 1$, we have $\|u\|_{L^2(0,1)} \sim \kappa$. In our computations we have $\kappa \leq 1$, this motivates us to consider the following set for the admissible controls.

$$\mathcal{U}_{ad} = \left\{u \in L^2(0,1) : \|u\|_{L^2(0,1)} \leq 3\right\}.$$

We discretize the state $(\gamma, y)$, the adjoint $(s, r)$ and the control $u$ using piecewise linear finite elements on rectangular meshes. To solve the state equations we use an affine invariant Newton strategy from [Deu04, NLEQ-ERR, Pg. 148-149].

Figure 4.1: The desired state $\gamma_d$.

We choose Newton's method instead of a Picard iteration because it is a second order method, and the transpose of the Jacobian is equivalent to the adjoint equations. The adjoint equations further involve the coupling between the 2d bulk and 1d interface, precisely speaking the bilinear form $\mathcal{D}_\Omega$ in (4.19). In view of the aforementioned equivalence, the seemingly complicated coupling $\mathcal{D}_\Omega$ can be assembled with ease. We work on the platform provided by the deal.II finite element library [BHK07] and use a direct (built-in) solver to invert the Jacobian at every Newton iteration, as well as the linear adjoint algebraic system. Consequently we can compute the derivative $\mathcal{J}'$ of the cost functional.

We use a gradient based minimization algorithm to solve the minimization problem in Matlab. In particular, we use the built-in Matlab functions *fmincon* (constrained case), and *fminunc* (unconstrained case). *Stopping criterion*: the optimization algorithm stops when the gradient of the cost function is less then or equal to $\lambda \cdot$ 1e-4, or if the difference between two consecutive values of the cost function are less than or equal to $\lambda \cdot$ 1e-4.

We are interested in three examples which computationally justifies our theoretical a priori estimate for the control in Corollary 4.3.2. In particular, the first two study the behavior of the solution as the regularization parameter $\lambda$ goes to zero; they differ on whether or not the control is a constrained quantity. The third example studies the behavior of the solution for a fixed $\lambda$ but with the surface tension coefficient $\kappa$ going to zero.

For each of these examples we collect the following metrics

- The cost function value $\mathcal{J}(\bar{u})$.
- The smallest eigenvalue of $\mathcal{J}''(\bar{u})$, representing the constant $\delta$ in 2nd order sufficient condition $\mathcal{J}''(\bar{u})h^2 \geq \delta\|h\|^2_{L^2(0,1)}$. This metric is obtained in Matlab through the approximated Hessian provided by the *fmincon* or *fminunc* functions.
- The discrete $L^2$ norm of the optimal control $\bar{u}$ is equal to $\left(\bar{U}^T M \bar{U}\right)^{1/2}$, where $M$ denotes the mass matrix corresponding to 1d problem in the interval $(0,1)$.
- The "self-convergence" rate of the optimal control as we uniformly refine the finite element mesh. We first solve the problem on a very fine mesh, 8 uniform refinement cycles, and use it in place of a closed form solution. Deriving a closed form solution to a nonlinear optimization problem is rather complicated and thus impractical.

### 4.4.1 Example 1: Unconstrained control

We begin with the nominal case $u \in L^2(0,1)$ and $\kappa = 1$, i.e. the control is unconstrained and the surface tension coefficient is fixed. We are interested in the metrics $\mathcal{J}(\bar{u})$, $\mathcal{J}''(\bar{u})$, $\|\bar{u}\|_{L^2(\Gamma)}$ and convergence rate as the control regularization parameter $\lambda$ approaches zero; see Table 4.1.

Recall that we used a fixed point argument to prove the existence and uniqueness of a solution for the state equations which required $\bar{u} \in \mathcal{U}_{ad}$. For $\lambda =$ 1e-6, we have $\|\bar{u}\|_{L^2(0,1)} = 8$, i.e. $\bar{u} \notin \mathcal{U}_{ad}$. Nevertheless, we can still solve the state equations. This indicates that our choice of $\mathcal{U}_{ad}$ is not sharp and we can solve the state equations even for larger $\bar{u}$.

The smallest eigenvalue of the approximated hessian $\mathcal{J}''(\bar{u})$ for $\lambda =$ 1e-6 is 5.5e-4 i.e. the control $\bar{u}$ is also locally unique. The last row in Table 4.1 justifies the theoretical findings in Corollary 4.3.2.

| $\lambda$ | $\infty$ | 1 | 1e-1 | 1e-2 | 1e-3 | 1e-4 | 1e-5 | 1e-6 |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{J}(\bar{u})$ | 7.46e-2 | 7.32e-2 | 6.27e-2 | 2.77e-2 | 7.80e-3 | 1.60e-3 | 2.52e-4 | 4.20e-5 |
| $\mathcal{J}''(\bar{u})$ | - | 7.56e-2 | 8.10e-3 | 1.30e-3 | 6.24e-4 | 5.57e-4 | 5.50e-4 | 5.49e-4 |
| $\|\bar{u}\|_{L^2(\Gamma)}$ | 0 | 0.05 | 0.45 | 1.71 | 3.00 | 5.00 | 6.30 | 8.06 |
| rate | - | 1.1610 | 2.0202 | 1.1224 | 1.8402 | 1.70 | 1.5019 | 1.2117 |

Table 4.1: Example 1 (Unconstrained case): the values of the cost function $\mathcal{J}(\bar{u})$, the smallest eigenvalue of $\mathcal{J}''(\bar{u})$, the $L^2$-norm of $\bar{u}$ and the convergence rate of optimal control as $\lambda$ varies from $\infty$ to 1e-6.

The second column in Table 4.2-4.4 shows the optimal state $(\bar{\gamma}, \bar{y})$ as $\lambda$ approaches zero. The third column shows the control applied (solid blue); for reference we also plot the previous control (dotted red). The last column indicates the largest and smallest values of the control. For $\lambda = 1$ to $\lambda =$ 1e-2 one can see that the control acts at the center and tries to move $\gamma$ towards $\gamma_d$. For $\lambda =$ 1e-3 the control needs to push $\gamma$ in the right-half up, and in the left-half down and therefore it adjusts accordingly. For $\lambda =$ 1e-6 the control again mostly acts at the center. Moreover $\gamma$ matches $\gamma_d$ almost perfectly.

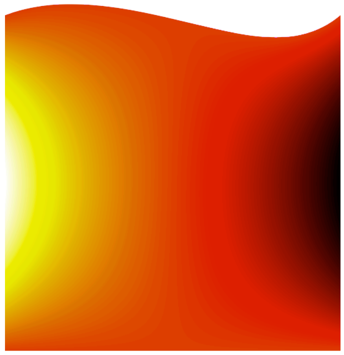
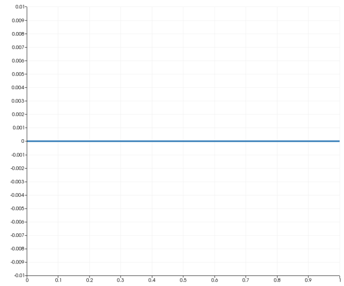
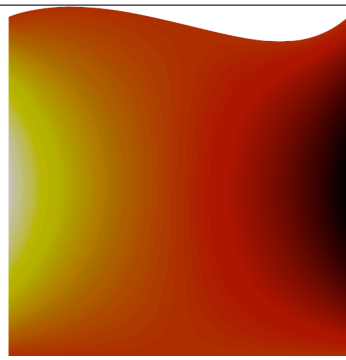
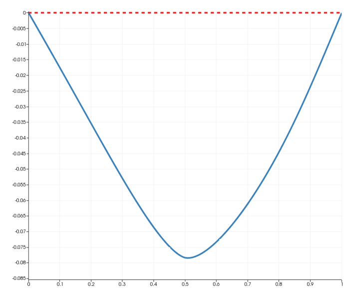
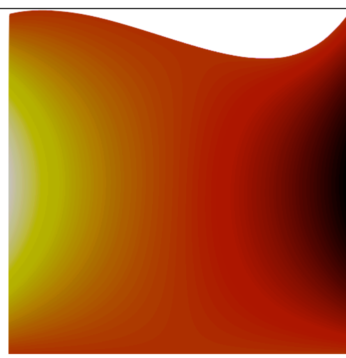
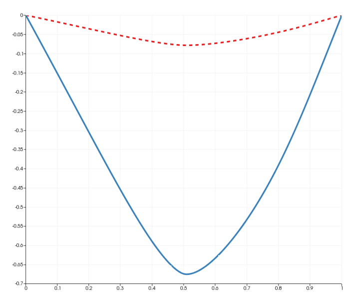
| $\lambda$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ Range |
|---|---|---|---|
| $\infty$ |  |  | $[0, 0]$ |
| 1 |  |  | $(-0.0784971, 0]$ |
| 1e-1 |  |  | $(-0.675277, 0]$ |

Table 4.2: Example 1 (Unconstrained case): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest value of the control for $\lambda = \infty$ to $\lambda =$ 1e-1.

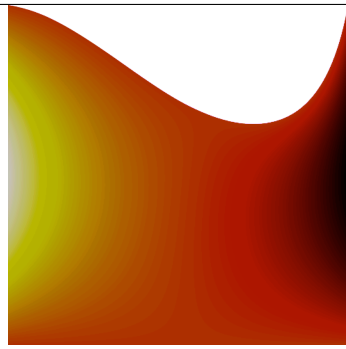
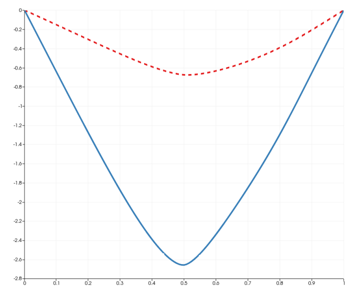
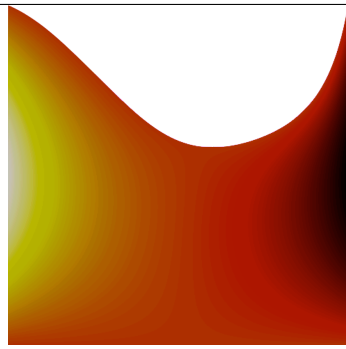
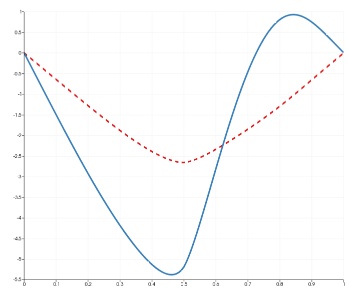
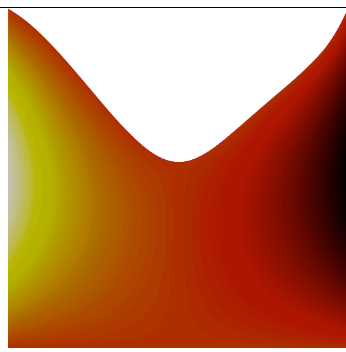
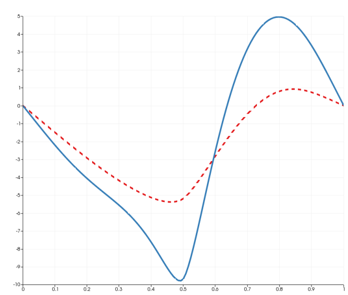
| $\lambda$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ Range |
|---|---|---|---|
| 1e-2 |  |  | $(-2.65675, 0]$ |
| 1e-3 |  |  | $(-5.38, 0.93)$ |
| 1e-4 |  |  | $(-9.78, 4.96)$ |

Table 4.3: Example 1 (Unconstrained case): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest values of control for $\lambda$ = 1e-2 to $\lambda$ = 1e-4.

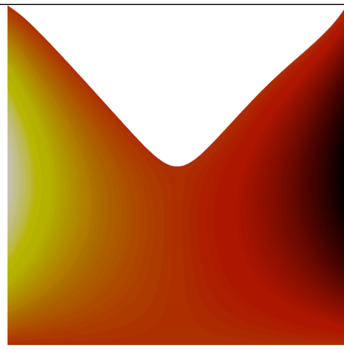
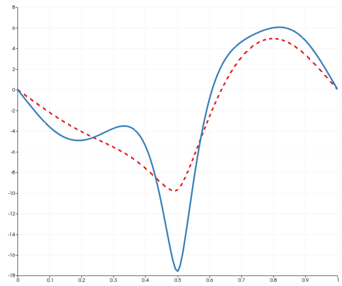
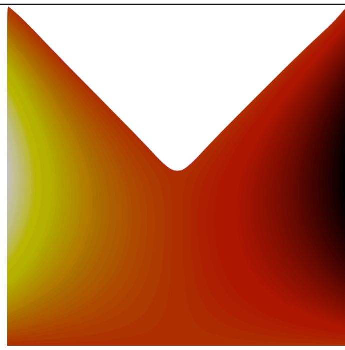
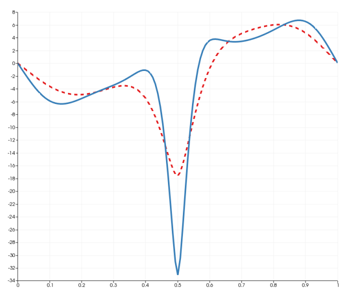
| $\lambda$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ Range |
|---|---|---|---|
| 1e-5 |  |  | $(-17.63, 6.07)$ |
| 1e-6 |  |  | $(-33.1146, 6.73)$ |

Table 4.4: Example 1 (Unconstrained case): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest value of control for $\lambda$ = 1e-5 to $\lambda$ = 1e-6.

### 4.4.2 Example 2: Constrained Control

This example differs from Example 1 only due to the fact that now we impose $u \in \mathcal{U}_{ad}$. The metrics are shown in Table 4.5. We first remark that similar to previous example the control is locally unique and the control convergence rate is linear.

When $\lambda$ = 1e-4 the control constraints become active and as a result the reduction in the cost function is severely impacted. This becomes clear after comparing the constrained and unconstrained cases for $\lambda$ set to 1e-5.

Table 4.6-4.7 shows the optimal state $(\bar{\gamma}, \bar{y})$ and the two consecutive optimal controls (blue: current, red: previous). For $\lambda$ = 1e-4 and 1e-5 the applied control lie on top of the previous control because the constraints are active. We also remark that we can not get as close to the desired configuration $\gamma_d$ as in the unconstrained case.

| $\lambda$ | $\infty$ | 1 | 1e-1 | 1e-2 | 1e-3 | 1e-4 | 1e-5 |
|---|---|---|---|---|---|---|---|
| $\mathcal{J}(\bar{u})$ | 0.07462 | 0.07317 | 0.06276 | 0.02773 | 0.00780 | 0.00375 | 0.00334 |
| $\mathcal{J}''(\bar{u})$ | - | 0.8571 | 0.5143 | 0.8571 | 0.1429 | 8.49e-5 | 9.74e-5 |
| $\Vert\bar{u}\Vert_{L^2(\Gamma)}$ | 0 | 0.0516 | 0.4415 | 1.7092 | 2.9970 | 3 | 3 |
| rate | - | 1.4353 | 2.7840 | 1.2716 | 1.5117 | 1.2134 | 1.1942 |

Table 4.5: Constrained case: the values of the cost function $\mathcal{J}(\bar{u})$, the smallest eigenvalue of $\mathcal{J}''(\bar{u})$, the $L^2$-norm of $\bar{u}$ and the convergence rate of the optimal control as $\lambda$ approaches 0. Notice that the constraint is active for $\lambda$ at 1e-4 and 1e-5.

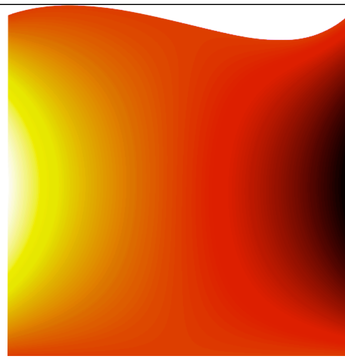
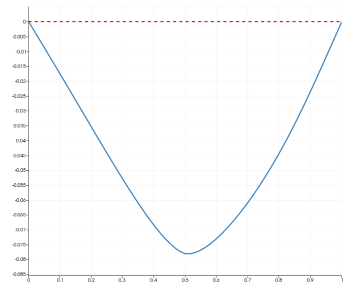
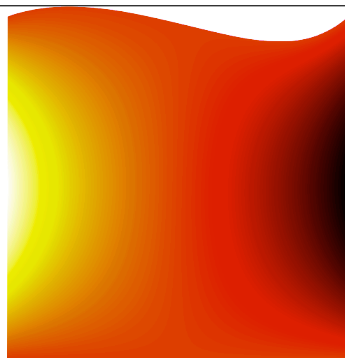
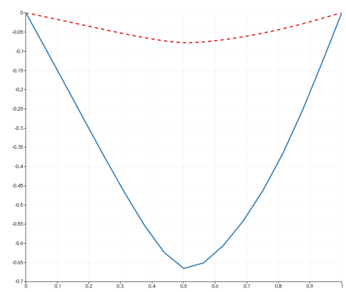
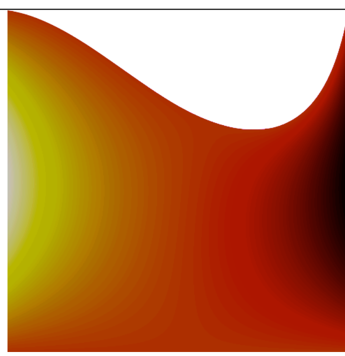
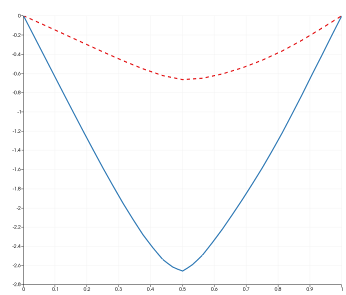
| $\lambda$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ range |
|---|---|---|---|
| 1 |  |  | $(-.0780707, 0]$ |
| 1e-1 |  |  | $(-0.665452, 0]$ |
| 1e-2 |  |  | $(-2.65713, 0]$ |

Table 4.6: Example 2 (Constrained case): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest value of control for $\lambda = 1$ to $\lambda =$ 1e-2.

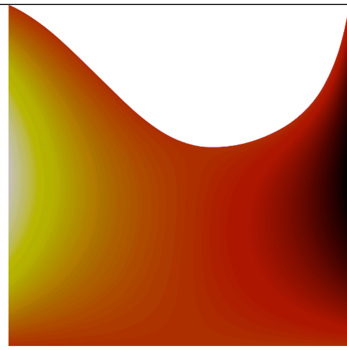
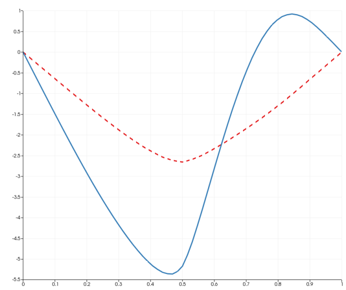
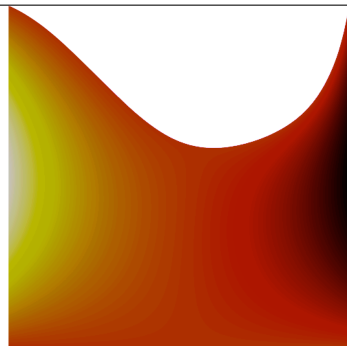
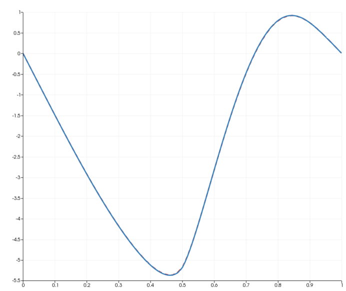
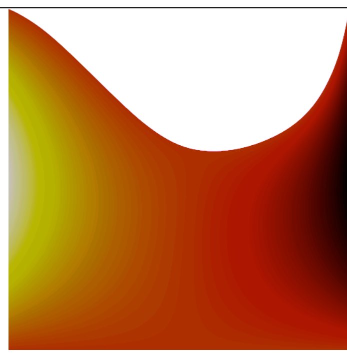
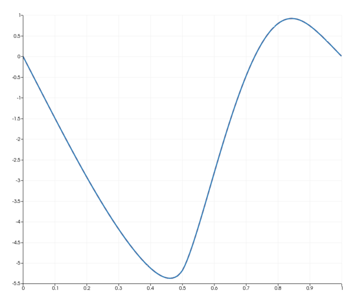
| $\lambda$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ range |
|---|---|---|---|
| 1e-3 |  |  | $(-5.37, 0.93)$ |
| 1e-4 |  |  | $(-5.37, 0.93)$ |
| 1e-5 |  |  | $(-5.37, 0.93)$ |

Table 4.7: Example 2 (Constrained case): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest value of control for $\lambda$ = 1e-3 to $\lambda$ = 1e-5. Notice that is no visual difference between the optimal control for $\lambda$ at 1e-3, 1e-4 and 1e-5, this is because the control constraints are active.

### 4.4.3 Example 3: Surface Tension Effects

We are interested in a study when the surface tension coefficient $\kappa$ approaches zero for a fixed control regularization, namely $\lambda$ = 1e-4. We limit ourselves to the unconstrained case since an extension to the constrained case is straightforward. The metrics are shown in Table 4.8.

Table 4.9-4.10 show the optimal state and corresponding two consecutive controls. We only show the results for $\kappa$ from 1 to 0.7513, as the Newton's solver for the state equations fails to converge for $\kappa$ smaller than 0.7513. This is not surprising, as the existence and uniqueness of the state equations requires us to have a small data $v$ and small $u$.

| $\kappa$ | 1 | .9091 | .8264 | .7513 |
|---|---|---|---|---|
| $\mathcal{J}(\bar{u})$ | 0.0016 | 0.0015 | 0.0014 | 0.0013 |
| $\mathcal{J}''(\bar{u})$ | 5.57e-4 | 6.72e-4 | 8.11e-4 | 9.80e-4 |
| $\\|\bar{u}\\|_{L^2(\Gamma)}$ | 5.00 | 4.90 | 4.81 | 4.72 |
| rate | 1.70 | 1.78 | 1.77 | 1.77 |

Table 4.8: Surface tension effect: for a fixed $\lambda$ = 1e-4, the values of the cost function $\mathcal{J}(\bar{u})$, the smallest eigenvalue of $\mathcal{J}''(\bar{u})$, the $L^2$-norm of $\bar{u}$ and the convergence rate of the optimal control as $\kappa$ range from 1 to 0.7513. We remark that for $\kappa$ smaller than 0.7513, we can not solve the state equations.

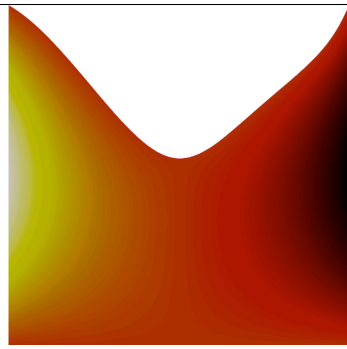
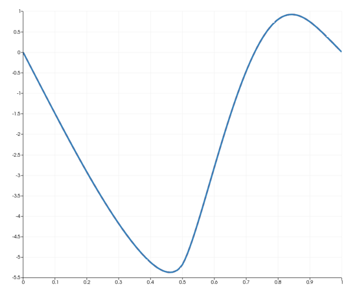
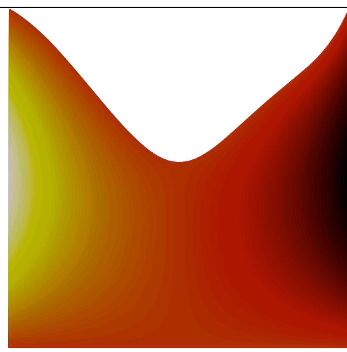
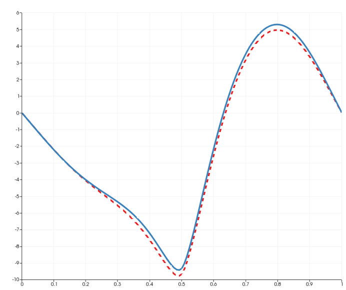
| $\kappa$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ range |
|---|---|---|---|
| 1 |  |  | $(-9.76, 4.958)$ |
| .9091 |  |  | $(-9.42, 5.29)$ |

Table 4.9: Example 3 (Surface tension effect): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest value of control for $\kappa = 1$ and 0.9091.

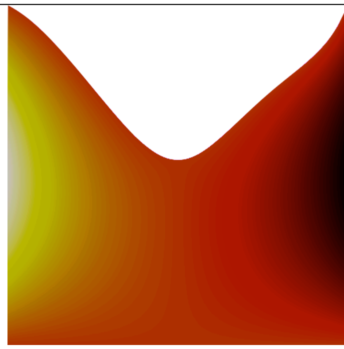
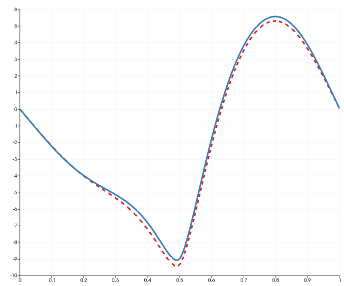
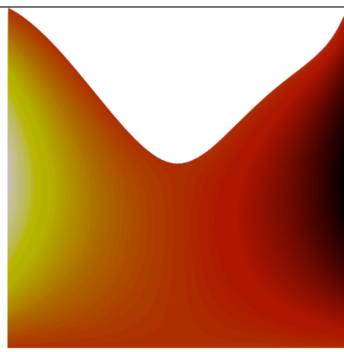
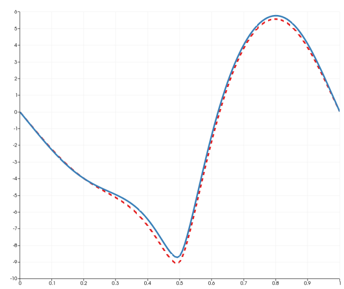
| $\kappa$ | $(\bar{\gamma}, \bar{y})$ | $\bar{u}$ | $\bar{u}(x)$ range |
|---|---|---|---|
| .8264 |  |  | $(-9.07, 5.56)$ |
| .7513 |  |  | $(-8.71, 5.76)$ |

Table 4.10: Example 3 (Surface tension effect): The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The final column gives the smallest and largest value of control for $\kappa = 0.8264$ and $\kappa = 0.7513$. We are not able to solve the state equations for $\kappa$ smaller than 0.7513.

# Part III

# The Stokes Free Boundary Problem

# Chapter 5

# Stokes Problem with Navier Slip Boundary Condition

## 5.1 Introduction

A bounded connected domain $\Omega$ in $\mathbb{R}^n$ $(n \geq 2)$ is said to be of fractional Sobolev class $W_s^{2-1/s}$, $(n < s < \infty)$ whenever its boundary $\partial\Omega$ is locally the graph of a function $\omega$ in $W_{s,\mathrm{loc}}^{2-1/s}(\mathbb{R}^{n-1})$. An equivalent definition for Sobolev domains was introduced by Delfour-Zolésio through an oriented distance function [DZ98, DZ11b]. We immediately remark that $W_s^{2-1/s}$-domains are a strict subset of Hölder domains of class $C^{1,1-n/s}$.

The Stokes problem of interest is to find a unique solution to

$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = g \quad \text{in } \Omega, \tag{5.1a}$$

together with the Navier slip boundary condition,

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = \phi, \quad \beta \boldsymbol{T}\boldsymbol{u} + \boldsymbol{T}^\top \boldsymbol{\sigma}(\boldsymbol{u}, p)\, \boldsymbol{\nu} = \boldsymbol{\psi} \quad \text{on } \partial\Omega, \tag{5.1b}$$

where $\boldsymbol{\sigma} = 2\eta\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{I}p$ is the stress tensor, $\eta \equiv \mathsf{constant}$ (Newtonian fluid) is the viscosity parameter, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)/2$ is the strain tensor (or symmetric gradient), $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$, $\beta(\boldsymbol{x}) \geq 0$ is the friction coefficient, and $\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ is the projection operator onto the tangent plane of $\partial\Omega$.

The slip boundary condition in (5.1b) arises in many applications related to free boundary problems. We refer to [AFV11] and the references therein for a polymer extrusion process. In [ANS13] we study a Stokes free boundary problem with surface tension effects, where (5.1b) appears naturally when the balance of forces along the free boundary is taken into account. Further references can be found in [AS11, PS10].

The pioneering work in this direction was done by Solonnikov-Ščadilov [SŠ73] This was extended by Beirão da Veiga [BdV04], who showed the existence of weak and strong solutions to a generalized Stokes system with $C^{1,1}$-domains in Hilbert space setting. While still working with $C^{1,1}$-domains, Amrouche-Seloula [AS11] recently obtained existence and uniqueness of weak, ultra-weak, and strong solutions to (5.1) in reflexive Sobolev spaces. We also refer to Mitrea-Monniaux [MM09] for time-dependent Navier-Stokes on $C^2$ domains. A comprehensive survey for this problem is given by Berselli [Ber10].

Our goal in this chapter is to prove the existence-uniqueness of weak solution to (5.1), in reflexive Sobolev spaces under the assumption that $\Omega$ is of class $W_s^{2-1/s}$, $s > n$. As a by-product, we significantly improve the best known result by Amrouche-Seloula in $C^{1,1}$ domains [AS11]. They consider an equivalent form of (5.1b) with $\beta = 0$, which for homogeneous boundary data can be written as $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ and $\operatorname{curl} \boldsymbol{u} \times \boldsymbol{\nu} = \boldsymbol{0}$. No such reformulation is carried out in our proofs. Our approach is inspired by Galdi, Simader and Sohr [GSS94], who study (5.1a) with pure no-slip ($\boldsymbol{u} = \boldsymbol{0}$) boundary condition. We consider a standard localization technique and rely only upon the existence and uniqueness of solutions to the Stokes problem in the whole space $\mathbb{R}^n$ for compactly supported data.

While studying the localized problems near the boundary it is crucial to preserve the normal vector once the boundary is flattened, which can be guaranteed by the Piola transform [BF91]. This is the second key difference, besides the boundary conditions, between our approach and [GSS94]. Due to the regularity required by the Piola transform (5.22c) it seems that $W_s^{2-1/s}$ is the *(nearly) optimal* domain regularity needed to study (5.1). Moreover, the proof in [GSS94] is based on *inf-sup* conditions [EG04, Corollary A.45], but our proof is based on an equivalent but higher-level *index-theory* [Lax02, Chapter 27].

For the moment, we will make two simplifications. The first is to treat the frictionless problem, i.e. $\beta = 0$. We will add it back in §5.7. The second concerns the non-trivial essential boundary condition $\phi$ which can be addressed by a standard lifting argument §5.6. To the best of our knowledge, this lifting is continuous only when the domain is of class $W_s^{2-1/s}$ with $s > n$, see §5.6 for details. This is the second place where we find $W_s^{2-1/s}$ to be *(nearly) optimal*.

Returning to the Stokes system (5.1), it is routine to check that the pressure $p$ can only be defined up to a constant. Less apparent is that the velocity field kernel is non-trivial *if and only if* $\Omega$ is axisymmetric. More importantly, when this kernel is not the empty-set, it is characterized by a small subspace of the rigid body motions,

$$Z(\Omega) := \left\{ \boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} : \boldsymbol{x} \in \Omega,\ \boldsymbol{A} = -\boldsymbol{A}^\top \in \mathbb{R}^{n\times n},\ \boldsymbol{b} \in \mathbb{R}^n,\ \boldsymbol{z}\cdot\boldsymbol{\nu}|_{\partial\Omega} = 0 \right\} \tag{5.2}$$

See [LM11, Appendix A] and [BdV04, Appendix I]. for more details.

The standard variational framework requires us to work with two spaces. The first space is that of trial functions $X_r(\Omega)$, which we define as

$$X_r(\Omega) := V_r(\Omega) \times L_0^r(\Omega), \text{ with } s' \leq r \leq s, \tag{5.3a}$$

with $V_r(\Omega) := \left\{ \boldsymbol{v} \in W_r^1(\Omega)/Z(\Omega) : \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \right\}$, $L_0^r(\Omega) := L^r(\Omega)/\mathbb{R}$ and $1/s + 1/s' = 1$.

It follows from its product definition that $X_r(\Omega)$ is complete under the norm

$$\left\| (\boldsymbol{v}, p) \right\|_{X_r(\Omega)} := \|\boldsymbol{v}\|_{W_r^1(\Omega)} + \|p\|_{L^r(\Omega)} . \tag{5.3b}$$

The second space is that of prescribed data, which we take to be $X_{r'}(\Omega)^*$, the topological dual of $X_{r'}(\Omega)$ where $1/r + 1/r' = 1$. Moreover, $X_{r'}(\Omega)^*$ is complete under the operator norm

$$\|\mathcal{F}\|_{X_{r'}(\Omega)^*} = \sup_{\left\|(\boldsymbol{v},q)\right\|_{X_{r'}(\Omega)}=1} \left| \mathcal{F}(\boldsymbol{v}, q) \right| . \tag{5.3c}$$

With the functional setting in place, we state our main result.

**Theorem 5.1.1.** *Let $\Omega$ be a $W_s^{2-1/s}$-domain with $s > n$, and $s' \leq r \leq s$. For every $\mathcal{F}$ in $X_{r'}(\Omega)^*$ there exists a unique $(\boldsymbol{u}, p)$ in $X_r(\Omega)$ such that*

$$\mathcal{S}_\Omega(\boldsymbol{u}, p)(\boldsymbol{v}, q) = \mathcal{F}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in X_{r'}(\Omega), \tag{5.4a}$$

$$\left\| (\boldsymbol{u}, p) \right\|_{X_r(\Omega)} \leq C_{\Omega,\eta,n,r} \|\mathcal{F}\|_{X_{r'}(\Omega)^*} , \tag{5.4b}$$

*where the Stokes bilinear form reads*

$$\mathcal{S}_\Omega(\boldsymbol{u}, p)(\boldsymbol{v}, q) := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p \operatorname{div} \boldsymbol{v} + q \operatorname{div} \boldsymbol{u}. \tag{5.4c}$$

**Definition 5.1.2** (Well-posedness)**.** We will say $\mathcal{S}_\Omega$ is well-posed (in the sense of Hadamard) over the spaces $X_r(\Omega) \times X_{r'}(\Omega)^*$ whenever (5.4a)-(5.4b) is satisfied.

The weak formulation (5.4) is consistent with the strong equations (5.1). This is verified by noting that any smooth solution to (5.1) satisfies the variational system (5.4) with

$$\mathcal{F}(\boldsymbol{v}, q) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\partial\Omega} \boldsymbol{\psi} \cdot \gamma_0 \boldsymbol{v} + \int_\Omega gq,$$

as long as $g$ is in $L^r(\Omega)$, $\boldsymbol{f}$ belongs to $V_{r'}(\Omega)^*$, $\boldsymbol{\psi}$ lies in the trace space $\big(\gamma_0(V_{r'}(\Omega))\big)^*$ and all three satisfy the compatibility conditions

$$\int_\Omega gc = 0 \quad \forall c \in \mathbb{R}, \text{ and } \int_\Omega \boldsymbol{f} \cdot \boldsymbol{z} + \int_{\partial\Omega} \boldsymbol{\psi} \cdot \boldsymbol{z} \ = 0 \quad \forall \boldsymbol{z} \in Z(\Omega). \tag{5.5}$$

Moreover, by taking the integrals involving $\boldsymbol{f}$ and $\boldsymbol{\psi}$ as duality pairings we conclude

$$\|\mathcal{F}\|_{X_{r'}(\Omega)^*} \leq \|\boldsymbol{f}\|_{V_{r'}(\Omega)^*} + \|\boldsymbol{\psi}\|_{\left(\gamma_0(V_{r'}(\Omega))\right)^*} + \|g\|_{L^r(\Omega)} \,.$$

Proof of well-posedness generally relies on one of the following three equivalent techniques: the Banach-Nečas' inf-sup conditions, Brezzi's saddle-point characterization, and the isomorphism of the induced operator $\mathcal{S}_\Omega : X_r(\Omega) \to X_{r'}(\Omega)^*$.

To obtain well-posedness of $\mathcal{S}_\Omega$ in $X_r(\Omega) \times X_{r'}(\Omega)^*$ we will split our analysis in six sections:

§5.2 gives a short proof for the Hilbert space case ($r = 2$). The importance of this result is the direct implication of the uniqueness for the solutions to (5.4a) when $r \geq 2$ and $\Omega$ is bounded.

§5.3 presents a fundamental result on well-posedness of Stokes in $\mathbb{R}^n$, from where we also derive the well-posedness of Stokes in the half-space $\mathbb{R}^n_-$. These two building blocks will be instrumental in constructing a solution of (5.4a) for $s' \leq r \leq r$.

§5.4 develops a sufficiently smooth diffeomorphism which locally flattens $\partial\Omega$, and analyzes a transformation which preserves the essential boundary condition $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$.

§5.5 uses a localization procedure and index theory to prove the well-posedness of $\mathcal{S}_\Omega$ in $X_r(\Omega) \times X_{r'}(\Omega)^*$ for $s' \leq r \leq s$.

§5.6 deals with the inhomogeneous essential boundary conditions.

§5.7 extends the theory to the full Navier boundary condition, i.e. $\beta \neq 0$.

## 5.2 The Hilbert Space Case

In this section we prove the well-posedness of $\mathcal{S}_\Omega$ in $X_2(\Omega) \times X_2(\Omega)^*$. Results in this direction are known for a generalized Stokes system on $C^{1,1}$ domains [BdV04]. Our proof relies on Korn's inequality, Brezzi's result for saddle-point problems, and Nečas' result on the right inverse of the divergence operator. We collect these results in the sequel.

**Lemma 5.2.1** (Korn's inequality)**.** *Let $1 < r < \infty$ and $\Theta$ be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists constants $C_1$ and $C_2$ depending only on $\Theta, n$ and $r$ such that for every $\boldsymbol{v}$ in $W^1_r(\Theta)$*

$$\|\boldsymbol{v}\|_{W^1_r(\Theta)} \leq C_1 \left( \|\boldsymbol{v}\|_{L^r(\Theta)} + \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_{L^r(\Theta)} \right) \leq C_2 \|\boldsymbol{v}\|_{W^1_r(\Theta)} . \tag{5.6a}$$

*Moreover, for every $\boldsymbol{v} \in W^1_r(\Theta)$ there exists a skew symmetric matrix $\boldsymbol{A}$ in $\mathbb{R}^{n\times n}$, and $\boldsymbol{b} \in \mathbb{R}^n$ such that*

$$\left\|\boldsymbol{v} - (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})\right\|_{W^1_r(\Theta)} \leq C_{\Theta,n,r} \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_{L^r(\Theta)} . \tag{5.6b}$$

*In case* $\Theta$ *is a Lipschitz cone in* $\mathbb{R}^n$ *the same holds for the seminorm,*

$$\left|\boldsymbol{v} - (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})\right|_{W_r^1(\Theta)} \leq C_{\Theta,n,r} \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_{L^r(\Theta)}. \tag{5.6c}$$

*Proof.* See [LM11, Theorem A.1] for $r = 2$, [DM04, Section 2] for $1 < r < \infty$ in bounded domains, and [KO88, Section 3, Theorem 2] for the unbounded case. □

**Proposition 5.2.2** (Equivalence of norms)**.** *Let* $1 < r < \infty$ *and* $\Theta$ *be a bounded Lipschitz domain. For every* $\boldsymbol{v}$ *in* $V_r(\Theta)$ *the following holds*

$$\|\boldsymbol{v}\|_{W_r^1(\Theta)} \leq C_{\Theta,n,r} \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_{L^r(\Theta)}.$$

*Proof.* We argue by contradiction. Suppose the above inequality was not true, then we have a sequence $\{\boldsymbol{v}_\ell\}_{\ell=0}^{\infty}$ in $V_r(\Theta)$ such that for every $\ell \geq 0$

$$\|\boldsymbol{v}_\ell\|_{W_r^1(\Theta)} = 1, \text{ while } \lim_{\ell\to\infty} \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_{L^r(\Theta)} = 0.$$

We immediately have that $\boldsymbol{v}_\ell$ converges weakly to some $\boldsymbol{v}$ in $V_r(\Theta)$ from where we obtain $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$. By the Rellich-Kondrachov theorem a subsequence converges strongly in the $L^r(\Theta)$ topology which together with Korn's first inequality (5.6a) yields $1 \leq C_1 \|\boldsymbol{v}\|_{L^r(\Theta)}$, i.e. $\boldsymbol{v} \neq \boldsymbol{0}$. On the other hand, Korn's second inequality (5.6b) yields that $\boldsymbol{v}_\ell$ converges strongly to some $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ in the $W_r^1(\Theta)$ topology, whence $\boldsymbol{v}$ is in $Z(\Theta)$ and equivalent to $\boldsymbol{0}$, which is a contradiction. □

The above argument is fairly standard it was based on the case $r = 2$ from [LM11, Theorem A.2].

**Remark 5.2.3.** The difference between Proposition 5.2.2 and Lemma 5.2.1 is that the vector-fields $\boldsymbol{z}$ in $Z(\Theta)$ satisfy $\boldsymbol{z} \cdot \boldsymbol{\nu} = 0$ while the ones from Korn's inequality do not have this requirement.

Next we state Brezzi's characterization [BF91, Section II.1, Theorem 1.1] by rewriting (5.4) in its saddle-point form. Namely, find a unique $(\boldsymbol{u}, p)$ in $V_r(\Omega) \times L_0^r(\Omega)$ such that

$$\begin{aligned} \eta \left\langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_\Omega - \langle p, \operatorname{div} \boldsymbol{v} \rangle_\Omega &= \mathcal{F}(\boldsymbol{v}, 0) \quad \forall \boldsymbol{v} \in V_{r'}(\Omega), \\ \langle \operatorname{div} \boldsymbol{u}, q \rangle_\Omega &= \mathcal{F}(\boldsymbol{0}, q) \quad \forall q \in L_0^{r'}(\Omega). \end{aligned} \tag{5.7}$$

**Lemma 5.2.4** (Brezzi)**.** *The saddle point problem* (5.7) *is well-posed in* $(V_r(\Omega) \times L_0^r(\Omega)) \times (V_{r'}(\Omega)^* \times L_0^{r'}(\Omega)^*)$ *if and only if there exist constants* $\alpha, \beta > 0$ *such that*

$$\inf_{\boldsymbol{w} \in \mathring{V}_r} \sup_{\boldsymbol{v} \in \mathring{V}_{r'}} \frac{\left\langle \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle}{\|\boldsymbol{w}\|_{V_r} \|\boldsymbol{v}\|_{V_{r'}}} = \inf_{\boldsymbol{v} \in \mathring{V}_{r'}} \sup_{\boldsymbol{w} \in \mathring{V}_r} \frac{\left\langle \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle}{\|\boldsymbol{w}\|_{V_r} \|\boldsymbol{v}\|_{V_{r'}}} = \alpha > 0, \tag{5.8a}$$

$$\inf_{q \in L_0^{r'}} \sup_{\boldsymbol{w} \in V_r} \frac{\langle \operatorname{div} \boldsymbol{w}, q \rangle}{\|\boldsymbol{w}\|_{V_r} \|q\|_{L^{r'}}} = \beta > 0, \tag{5.8b}$$

*where* $\mathring{V}_r := \left\{ \boldsymbol{w} \in V_r(\Omega) : \langle \operatorname{div} \boldsymbol{w}, q \rangle = 0, \forall q \in L_0^{r'}(\Omega) \right\}$. *In addition, there exists* $\gamma = \gamma(\alpha, \beta, \eta)$ *such that the solution* $(\boldsymbol{u}, p)$ *is bounded by*

$$\left\| (\boldsymbol{u}, p) \right\|_{X_r(\Omega)} \leq \gamma \|\mathcal{F}\|_{X_{r'}(\Omega)^*}.$$

*Proof.* See [BF91, Section II.1, Theorem 1.1]. □

**Theorem 5.2.5** (Well-posedness for $r = 2$)**.** *Let* $\Omega$ *be a bounded* $W_s^{2-1/s}$*-domain. The Stokes problem* (5.4) *is well-posed in* $X_2(\Omega) \times X_2(\Omega)^*$.

*Proof.* It suffices to check Brezzi's conditions (5.8).

Using Proposition 5.2.2 we obtain for every $\boldsymbol{w} \in \mathring{V}_2(\Omega) \subset V_2(\Omega)$,

$$\|\boldsymbol{w}\|_{V_2}^2 \leq \eta C_{\Omega,n} \left\| \boldsymbol{\varepsilon}(\boldsymbol{w}) \right\|_{L^2(\Omega)}^2 = C_{\Omega,n} \left\langle \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{w}) \right\rangle.$$

which yields (5.8a) with constant $\alpha = C_{\Omega,n}$.

Following the work by Nečas [BF91, Section II.1, Proposition 1.2, Equation 1.16], for every $q$ in $L^2_0(\Omega)$ there exists $\boldsymbol{w}$ in $W^1_2(\Omega)$ such that $\operatorname{div}\boldsymbol{w} = q$ in $\Omega$, $\boldsymbol{w} = \mathbf{0}$ on $\partial\Omega$, and $\|\boldsymbol{w}\|_{W^1_2(\Omega)} \leq C_{\Omega,n}\|q\|_{L^2(\Omega)}$. It is immediate that $\boldsymbol{w} \in V_2(\Omega)$ because $\boldsymbol{w} = \mathbf{0}$ on $\partial\Omega$ implies $\boldsymbol{w}\cdot\boldsymbol{\nu} = 0$ as well. Moreover,

$$\|q\|_{L^2(\Omega)} = \frac{1}{\|q\|_{L^2(\Omega)}}\int_\Omega q^2 = \frac{\langle \operatorname{div}\boldsymbol{w}, q\rangle}{\|q\|_{L^2(\Omega)}} \leq C_{\Omega,n}\frac{\langle \operatorname{div}\boldsymbol{w}, q\rangle}{\|\boldsymbol{w}\|_{W^1_2(\Omega)}},$$

which yields (5.8b) with constant $\beta = 1/C_{\Omega,n}$. □

## 5.3 Stokes Problem on Unbounded Domains ($\mathbb{R}^n$ and $\mathbb{R}^n_-$)

The purpose of this section is to prove the existence, uniqueness and *local* regularity of the Stokes problem (5.4a) in the whole space $\mathbb{R}^n$ and the half-space $\mathbb{R}^n_-$ for data with *compact* support. These two problems are the essential building blocks for the localization procedure in §5.5. With exception of the symmetric gradient $\boldsymbol{\varepsilon}(\cdot)$, this problem has been extensively studied under different functional frameworks; we refer to [AA99, Introduction] for an overview.

Weighted Sobolev spaces are an extremely general framework for it provides a wealth of predictable behaviors at $\infty$ when considering different weight functions. A different framework is the one of Homogeneous Sobolev spaces, its main disadvantage being the lack of control on the $L^r$-norm of the function. Fortunately, these two frameworks are interchangeable as long as the data in question has compact support and one is not interested in the behavior at $\infty$ of the functions being analyzed [AA99, Proposition 4.8].

With this equivalence in hand, and the fact that our work was originally inspired by that of Galdi-Simader-Sohr [GSS94], we choose to work with the Galdi-Simader's characterization for homogeneous Sobolev spaces [GS90]. The remainder of this section is split into three parts. In §5.3.1 we recall this essential characterization and define the equivalent $X_r(\cdot)$ spaces for unbounded domains. In §5.3.2 we prove the well-posedness of the Stokes problem in its symmetric gradient form in $\mathbb{R}^n$. Finally, in §5.3.3 we extend the result to the half-space $\mathbb{R}^n_-$.

### 5.3.1 Homogeneous Sobolev Spaces

The solution space $X_r$ is too small to prove an existence and uniqueness result for unbounded domains [GSS94, Section 2]. In these cases we are led to consider the homogeneous Sobolev spaces

$$\begin{aligned} G^1_r(\mathbb{R}^n) = \mathring{G}^1_r(\mathbb{R}^n) &:= \overline{C^\infty_c(\mathbb{R}^n)^n}^{|\cdot|_{W^1_r(\mathbb{R}^n)}}, \\ G^1_r(\mathbb{R}^n_-) &:= \overline{C^\infty_c(\overline{\mathbb{R}^n_-})^n}^{|\cdot|_{W^1_r(\mathbb{R}^n_-)}}, \\ \mathring{G}^1_r(\mathbb{R}^n_-) &:= \overline{C^\infty_c(\overline{\mathbb{R}^n_-})^{n-1} \times C^\infty_c(\mathbb{R}^n_-)}^{|\cdot|_{W^1_r(\mathbb{R}^n_-)}}, \end{aligned} \tag{5.9}$$

where $C^\infty_c(\cdot)$ are smooth functions with compact support, the half-space $\mathbb{R}^n_-$ is given by $\{\boldsymbol{x} = (\boldsymbol{x}', x^n) : \boldsymbol{x}' \in \mathbb{R}^{n-1},\, x^n < 0\}$, and $\overline{\mathbb{R}^n_-} = \mathbb{R}^n_- \cup \partial\mathbb{R}^n_-$ with $\boldsymbol{x}$ in $\partial\mathbb{R}^n_-$ if and only if $x^n = 0$. The statement $\boldsymbol{v} = (\boldsymbol{v}', v^n) \in C^\infty_c(\overline{\mathbb{R}^n_-})^{n-1} \times C^\infty_c(\mathbb{R}^n_-)$ implies $\boldsymbol{v}|_{\partial\mathbb{R}^n_-} = (\boldsymbol{v}'|_{\partial\mathbb{R}^n_-}, 0)$ for $\boldsymbol{v}'$ in $C^\infty_c(\overline{\mathbb{R}^n_-})^{n-1}$. For a detailed presentation on these spaces, their duals and trace spaces see [Gal11, Chapter II], in particular [Gal11, Theorem II.10.2] for the trace space results.

Next we recall a result by Galdi-Simader on the characterization of $G^1_r(\Theta)$ and $\mathring{G}^1_r(\Theta)$ with $\Theta$ equal to $\mathbb{R}^n$ or $\mathbb{R}^n_-$ [KS92, Lemma 2.2],[BdV05, Section 1].

**Proposition 5.3.1** (Galdi-Simader)**.** *Let* $1 < r < \infty$ *and* $G_r^1(\Theta)$ *and* $\mathring{G}_r^1(\Theta)$ *be the spaces defined in* (5.9)*. The following characterization holds,*

$$\begin{aligned} G_r^1(\mathbb{R}^n) &= \left\{[\boldsymbol{v}]_1 \in L^r_{\text{loc}}(\mathbb{R}^n)^n : \nabla \boldsymbol{v} \in L^r(\mathbb{R}^n)^{n\times n}\right\}, \\ \mathring{G}_r^1(\mathbb{R}^n_-) &= \left\{\boldsymbol{v} = ([\boldsymbol{v}']_1, v^n) \in L^r_{\text{loc}}(\overline{\mathbb{R}^n_-})^n : \nabla \boldsymbol{v} \in L^r(\mathbb{R}^n_-)^{n\times n},\, v^n|_{\partial\mathbb{R}^n_-} = 0\right\}, \end{aligned} \tag{5.10}$$

*where* $[\boldsymbol{v}]_1$ *is the equivalence class of functions in* $L^r_{\text{loc}}(\Theta)^n$ *which differ by a constant vector. Furthermore, if* $1 < r < n$ *then additionally*

$$\begin{aligned} G_r^1(\Theta) &= \left\{\boldsymbol{v} \in L^{r^*}_{\text{loc}}(\Theta)^n : \nabla \boldsymbol{v} \in L^r(\Theta)^{n\times n}\right\}, \quad \Theta = \mathbb{R}^n \text{ or } \mathbb{R}^n_- \\ \mathring{G}_r^1(\mathbb{R}^n_-) &= \left\{\boldsymbol{v} \in (L^{r^*}_{\text{loc}}(\mathbb{R}^n_-)^{n-1} \times L^{r^*}(\mathbb{R}^n_-)) \cap G_r^1(\mathbb{R}^n_-) : v^n|_{\partial\mathbb{R}^n_-} = 0\right\}, \end{aligned} \tag{5.11}$$

*where* $r^*$ *is the Sobolev conjugate of* $r$ *and is given by* $1/r^* = 1/r - 1/n$.

*Proof.* See [KS92, Lemma 2.2], [GS90], and [GSS94]. □

We conclude by introducing the functional space $X_r(\Theta)$ when $\Theta$ is $\mathbb{R}^n$ or $\mathbb{R}^n_-$. The distinction from (5.3a) is that we use the homogeneous Sobolev spaces defined above, and the pressure space is simply $L^r(\Theta)$, i.e.

$$X_r(\Theta) := V_r(\Theta) \times L^r(\Theta) \quad 1 < r < \infty, \tag{5.12a}$$

with $V_r(\mathbb{R}^n) = G_r^1(\mathbb{R}^n)$, and $V_r(\mathbb{R}^n_-) = \mathring{G}_r^1(\mathbb{R}^n_-)$.

It follows from the product definition of $X_r(\Theta)$ that it is complete under the semi-norm

$$\left|(\boldsymbol{v}, p)\right|_{X_r(\Theta)} := |\boldsymbol{v}|_{W_r^1(\Theta)} + \|p\|_{L^r(\Theta)}\,. \tag{5.12b}$$

Second, the space for the prescribed data, which we take to be $X_{r'}(\Theta)^*$, the topological dual of $X_{r'}(\Theta)$ where $1/r + 1/r' = 1$. Moreover, $X_{r'}(\Theta)^*$ is complete under the operator norm

$$\|\mathcal{F}\|_{X_{r'}(\Theta)^*} = \sup_{\left|(\boldsymbol{v},q)\right|_{X_{r'}(\Theta)}=1} \left|\mathcal{F}(\boldsymbol{v},q)\right|. \tag{5.12c}$$

### 5.3.2 Stokes Problem in $\mathbb{R}^n$

In this section we will demonstrate the well-posedness of the Stokes problem (5.4) from $X_r(\mathbb{R}^n)$ to $X_{r'}(\mathbb{R}^n)^*$. Our proof relies on the fundamental solution in $\mathbb{R}^n$ of the Stokes problem without the symmetric gradient and on its extension to homogeneous Sobolev spaces.

**Lemma 5.3.2.** *Let $1 < r < \infty$, $n \geq 2$. For each $\boldsymbol{f} \in G_r^{-1}(\mathbb{R}^n) = G_{r'}^1(\mathbb{R}^n)^*$ and $g \in L^r(\mathbb{R}^n) = L^{r'}(\mathbb{R}^n)^*$, there exists a unique pair $(\boldsymbol{u}, p) \in G_r^1(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ satisfying*

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = g \text{ in } \mathbb{R}^n$$

*in the sense of distributions, and that depends continuously on the data, i.e.*

$$|\boldsymbol{u}|_{W_r^1(\mathbb{R}^n)} + \|p\|_{L^r(\mathbb{R}^n)} \leq C_{n,r}\left(\|\boldsymbol{f}\|_{G_r^{-1}(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)}\right).$$

*Additionally, if $1 < t < \infty$, $\boldsymbol{f} \in G_t^{-1}(\mathbb{R}^n)$ and $g \in L^t(\mathbb{R}^n)$, then $(\boldsymbol{u}, p)$ is in $G_r^1(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \cap G_t^1(\mathbb{R}^n) \times L^t(\mathbb{R}^n)$.*

**Remark 5.3.3.** Because the functions are defined over $\mathbb{R}^n$, the second part of the above lemma is not trivial consequence of the first, e.g. $g \in L^\infty(\mathbb{R}^n)$ does not imply $g \in L^r(\mathbb{R}^n)$ for any $r \neq \infty$.

*Proof.* See [Gal11, Section IV.2] and [GSS94, Section 3], with the caveat of a typo on latter. Just as importantly, we remark that the case of compactly supported data is equivalent to [AA99, Proposition 4.8]. □

Next we will provide a proof of Theorem 5.3.4 by showing that any solution to $\mathcal{S}_{\mathbb{R}^n}(\boldsymbol{u}, p) = \mathcal{F}$, i.e. with the symmetric gradient, is a solution of Lemma 5.3.2 for a particular choice of $\boldsymbol{f}$ and $g$. Formally, this result follows by the identity $\operatorname{div} \nabla \boldsymbol{u}^\top = \nabla \operatorname{div} \boldsymbol{u}$.

**Theorem 5.3.4** (Well-posedness of $\mathcal{S}_{\mathbb{R}^n}$)**.** *Let* $1 < r < \infty$*,* $n \geq 2$*. The Stokes problem* (5.4) *is well-posed from* $X_r(\mathbb{R}^n)$ *to* $X_{r'}(\mathbb{R}^n)^*$*. Additionally, suppose* $1 < t < \infty$ *and* $\mathcal{F}$ *is in* $X_{t'}(\mathbb{R}^n)^*$*, then* $(\boldsymbol{u}, p)$ *is in* $X_t(\mathbb{R}^n)$

*Proof.* Let $\mathcal{F}$ in $X_{r'}(\mathbb{R}^n)^*$ be fixed but arbitrary and consider

$$\boldsymbol{f}(\boldsymbol{v}) := \mathcal{F}(\boldsymbol{v}, -\operatorname{div} \boldsymbol{v}), \text{ and } g(q) := \mathcal{F}(\boldsymbol{0}, q).$$

Note that $\boldsymbol{f}$ and $g$ are bounded linear functionals on $G^1_{r'}(\mathbb{R}^n)$ and $L^{r'}(\mathbb{R}^n)$,

$$\begin{aligned} \left|\boldsymbol{f}(\boldsymbol{v})\right| &\leq \|\mathcal{F}\|_{X_{r'}(\mathbb{R}^n)^*} \left\|(\boldsymbol{v}, \operatorname{div} \boldsymbol{v})\right\|_{X_r(\mathbb{R}^n)} \leq 2\|\mathcal{F}\|_{X_{r'}(\mathbb{R}^n)^*} |\boldsymbol{v}|_{W^1_{r'}(\mathbb{R}^n)}, \\ \left|g(q)\right| &\leq \|\mathcal{F}\|_{X_{r'}(\mathbb{R}^n)^*} \|q\|_{L^{r'}(\mathbb{R}^n)}. \end{aligned}$$

Additionally, the same estimate shows that if $\mathcal{F}$ is in $X_{t'}(\mathbb{R}^n)$ then $\boldsymbol{f}$ and $g$ are in $G^{-1}_t(\mathbb{R}^n)$ and $L^{t'}_0(\mathbb{R}^n)^*$ respectively.

We can apply Lemma 5.3.2 to obtain the existence and uniqueness of a solution $(\boldsymbol{w}, \pi)$ to the data $(\boldsymbol{f}, g)$, where for every $(\boldsymbol{v}, q) \in C^\infty_c(\mathbb{R}^n)^n \times L^{r'}(\mathbb{R}^n)$,

$$\langle \nabla \boldsymbol{w}, \nabla \boldsymbol{v} \rangle_{\mathbb{R}^n} - \langle \pi, \operatorname{div} \boldsymbol{v} \rangle_{\mathbb{R}^n} + \langle \operatorname{div} \boldsymbol{w}, q \rangle_{\mathbb{R}^n} = \mathcal{F}(\boldsymbol{v}, q - \operatorname{div} \boldsymbol{v}), \tag{5.13}$$

and $\left\|(\boldsymbol{w}, \pi)\right\|_{X_r(\mathbb{R}^n)} \leq C_{n,r} \|\mathcal{F}\|_{X_{r'}(\mathbb{R}^n)^*}$. Furthermore, $(\boldsymbol{w}, \pi)$ is in $X_t(\mathbb{R}^n)$ in case $\mathcal{F}$ belongs to $X_{t'}(\mathbb{R}^n)^*$.

By setting $\boldsymbol{v} = \boldsymbol{0}$ in (5.13) we obtain $\langle \operatorname{div} \boldsymbol{w}, q \rangle_{\mathbb{R}^n} = \mathcal{F}(\boldsymbol{0}, q)$ for every $q$ in $L^{r'}(\mathbb{R}^n)$. Since $\operatorname{div} \boldsymbol{v}$ is in $L^{r'}(\mathbb{R}^n)$ as well, we obtain $\mathcal{F}(\boldsymbol{0}, \operatorname{div} \boldsymbol{v}) = \langle \operatorname{div} \boldsymbol{w}, \operatorname{div} \boldsymbol{v} \rangle_{\mathbb{R}^n}$.

In turn, applying Green's theorem twice

$$\begin{aligned}\langle \operatorname{div} \boldsymbol{w}, \operatorname{div} \boldsymbol{v}\rangle_{\mathbb{R}^n} &= \int_\Omega \sum_{i,j=1}^n \partial_i w^i \partial_j v^j = -\int_\Omega \sum_{i,j=1}^n w^i \partial_{ij} v^j = \int_\Omega \sum_{i,j=1}^n \partial_j w^i \partial_i v^j \\ &= \int_\Omega \operatorname{Tr}(\nabla \boldsymbol{v}^\top \nabla \boldsymbol{w}^\top) = \int_\Omega \nabla \boldsymbol{w}^\top : \nabla \boldsymbol{v}. \\ &= \left\langle \nabla \boldsymbol{w}^\top, \nabla \boldsymbol{v} \right\rangle_{\mathbb{R}^n},\end{aligned}$$

from where, after noting that $2\left\langle \boldsymbol{\varepsilon}(\boldsymbol{w}), \nabla \boldsymbol{v}\right\rangle = \left\langle \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v})\right\rangle$, we obtain the well-posedness of (5.4) with $\eta = 1$. For an arbitrary $\eta > 0$ it suffices to take $(\boldsymbol{u}, p) = (\eta \boldsymbol{w}, \pi)$ where $(\boldsymbol{w}, \pi)$ solves (5.4) with $\eta = 1$ for the data

$$\mathcal{F}(\boldsymbol{v}, 0) + \eta \mathcal{F}(\mathbf{0}, q).$$

The result now follows by density of $C_c^\infty(\mathbb{R}^n)^n$ in $V_r(\mathbb{R}^n)$. $\square$

### 5.3.3 Stokes Problem in $\mathbb{R}^n_-$

In this section we demonstrate the well-posedness of the Stokes problem (5.4) from $X_r(\mathbb{R}^n_-)$ to $X_{r'}(\mathbb{R}^n_-)^*$. Although the reflection technique employed is well-known, the construction sets the stage for the localization section. A very general result in this direction is the work by Beirão da Veiga-Crispo-Grisanti [BdVCG11].

**Theorem 5.3.5** (Well-posedness of $\mathcal{S}_{\mathbb{R}^n_-}$)**.** *Let $1 < r < \infty$, $n \geq 2$. The Stokes problem* (5.4) *is well-posed from $X_r(\mathbb{R}^n_-)$ to $X_{r'}(\mathbb{R}^n_-)^*$. Additionally, if $1 < t < \infty$ and $\mathcal{F}$ is in $X_{t'}(\mathbb{R}^n_-)^*$, then $(\boldsymbol{u}, p)$ is in $X_t(\mathbb{R}^n_-)$*

*Proof.* In view of the uniqueness results in [Far94, Theorem 3.1], it suffices to construct a solution to the Stokes problem in the half-space which depends continuously on the data.

Define for each function $\hat{\varphi} : \mathbb{R}^n \to \mathbb{R}$ its upper and lower parts as

$$\varphi_+(\boldsymbol{x}) := \hat{\varphi}(\boldsymbol{x}', -x^n), \text{ and } \varphi_-(\boldsymbol{x}) := \hat{\varphi}(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^n_-.$$

Take $(\hat{\boldsymbol{v}}, \hat{q})$ in $X_{r'}(\mathbb{R}^n)$ and define their pullbacks into $\mathbb{R}^n_-$ as follows,

$$\boldsymbol{v} = \frac{1}{2}\left(\hat{\boldsymbol{v}}'_- + \hat{\boldsymbol{v}}'_+, \hat{v}^n_- - \hat{v}^n_+\right)^\top, \text{ and } q = \frac{1}{2}\left(\hat{q}_- + \hat{q}_+\right).$$

It is simple to show that $(\boldsymbol{v}, q)$ is in $X_{r'}(\mathbb{R}^n_-)$ and

$$\left|(\boldsymbol{v}, q)\right|_{X_{r'}(\mathbb{R}^n_-)} \leq \left|(\hat{\boldsymbol{v}}, \hat{q})\right|_{X_{r'}(\mathbb{R}^n)}.$$

Let $\mathcal{F}$ in $X_{r'}(\mathbb{R}^n_-)^*$ be fixed but arbitrary and define $\hat{\mathcal{F}}(\hat{\boldsymbol{v}}, \hat{q}) := \mathcal{F}(\boldsymbol{v}, q)$. It follows immediately from our current results that $\hat{\mathcal{F}}$ is a linear functional on $X_{r'}(\mathbb{R}^n)$ and $\left\|\hat{\mathcal{F}}\right\|_{X_{r'}(\mathbb{R}^n)^*} \leq \|\mathcal{F}\|_{X_{r'}(\mathbb{R}^n_-)^*}$. Therefore, Theorem 5.3.4 for $\mathbb{R}^n$ asserts the existence and uniqueness of a solution $(\boldsymbol{w}, \pi)$ in $X_r(\mathbb{R}^n)$ to (5.4) with the forcing function $\hat{\mathcal{F}}$, i.e.

$$\mathcal{S}_{\mathbb{R}^n}(\boldsymbol{w}, \pi)(\hat{\boldsymbol{v}}, \hat{q}) = \hat{\mathcal{F}}(\hat{\boldsymbol{v}}, \hat{q}) \quad \forall (\hat{\boldsymbol{v}}, \hat{q}) \in X_{r'}(\mathbb{R}^n), \tag{5.14}$$

and $\left\|(\boldsymbol{w}, \pi)\right\|_{X_r(\mathbb{R}^n)} \leq C_{n,r}\|\mathcal{F}\|_{X_{r'}(\mathbb{R}^n_-)^*}$.

Since the test functions are arbitrary, we take $(\boldsymbol{v}, q)$ in $X_{r'}(\mathbb{R}^n_-)$ and test (5.14) with $(\hat{\boldsymbol{v}}, \hat{q})$ defined as even reflections for $q$ and $\boldsymbol{v}'$, and an odd reflection for $v^n$, i.e.

$$\hat{q}_- = \hat{q}_+ = q, \quad \hat{\boldsymbol{v}}'_- = \hat{\boldsymbol{v}}'_+ = \boldsymbol{v}', \quad \hat{v}^n_- = -\hat{v}^n_+ = v^n.$$

We can immediately verify that $\hat{\mathcal{F}}(\hat{\boldsymbol{v}}, \hat{q}) = \mathcal{F}(\boldsymbol{v}, q)$ still holds, and after some technical computations,

$$\mathcal{S}_{\mathbb{R}^n}(\boldsymbol{w}, \pi)(\hat{\boldsymbol{v}}, \hat{q}) = 2\mathcal{S}_{\mathbb{R}^n_-}(\boldsymbol{u}, p)(\boldsymbol{v}, q),$$

where $p(\boldsymbol{x}) = \frac{1}{2}\left(\pi(\boldsymbol{x}) + \pi(\boldsymbol{x}', -x^n)\right)$, and $\boldsymbol{u} = (\boldsymbol{u}', u^n)$ is given by

$$\boldsymbol{u}'(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{w}'(\boldsymbol{x}) + \boldsymbol{w}'(\boldsymbol{x}', -x^n)\right), \text{ and } u^n(\boldsymbol{x}) = \frac{1}{2}\left(w^n(\boldsymbol{x}) - w^n(\boldsymbol{x}', -x^n)\right).$$

Finally, $\boldsymbol{u} \cdot \boldsymbol{\nu}(\boldsymbol{x}', 0) = u^n(\boldsymbol{x}', 0) = 0$ and $(\boldsymbol{u}, p)$ also satisfies estimate (5.4b). This concludes the proof. □

## 5.4 Sobolev Domains and the Piola Transform

### 5.4.1 Sobolev Domains

The goal of this section is to start with a graph definition of a Sobolev domain and obtain a characterization based on local $W^2_s$-diffeomorphisms which are a compact perturbations of the identity with arbitrarily small Lipschitz norms.

**Definition 5.4.1** ($W^{2-1/s}_s$-domain)**.** An open and connected set $\Omega$ in $\mathbb{R}^n$ is called a $W^{2-1/s}_s$-domain, $s > n$, if at each point $\boldsymbol{x}_\lambda$ in $\partial\Omega$ there exists $\delta_\lambda > 0$ and a function $\omega$ in $W^{2-1/s}_{s,\mathrm{loc}}(\mathbb{R}^{n-1})$ such that, after a possible relabelling and reorientation of the coordinate axis, $\Omega \cap B(\boldsymbol{x}_\lambda, \delta_\lambda) = \Omega(\boldsymbol{x}_\lambda, \delta_\lambda, \omega)$ where the latter is defined by

$$\Omega(\boldsymbol{x}_\lambda, \delta_\lambda, \omega) := \left\{ \left(\boldsymbol{x}', x^n\right) = \boldsymbol{x} \in B(\boldsymbol{x}_\lambda, \delta_\lambda) : x^n < \omega(\boldsymbol{x}') \right\}.$$

A $W^{2-1/s}_s$-domain where $\delta_\lambda$ can be chosen independently of $\boldsymbol{x}_\lambda$ is said to be a *uniform* $W^{2-1/s}_s$-domain. It is easy to verify that every bounded $W^{2-1/s}_s$-domain is a *uniform* $W^{2-1/s}_s$-domain.

**Lemma 5.4.2** (Rotation and translation)**.** *Let $\Omega$ be a $W^{2-1/s}_s$-domain. For every $\boldsymbol{x}_\lambda$ in $\partial\Omega$, we may choose $\omega$ in Definition 5.4.1 to additionally satisfy*

$$\nabla'\omega(\boldsymbol{x}'_\lambda) = \mathbf{0} \tag{5.15a}$$

*and for some* $0 < \delta \leq \delta_\lambda/2$,

$$\fint_{D(\boldsymbol{x}'_\lambda, 2\delta)} \omega = 0. \tag{5.15b}$$

*where* $D(\boldsymbol{x}'_\lambda, 2\delta)$ *is the open disc in* $\mathbb{R}^{n-1}$ *centered at* $\boldsymbol{x}'_\lambda$ *of radius* $2\delta$.

*Proof.* The gradient condition can be obtained by rotating $\Omega$ about $\boldsymbol{x}_\lambda$ so that the outward unit normal at $\boldsymbol{x}_\lambda$ coincides with the canonical vector $\boldsymbol{e}_n$. The integral condition can be obtained by translating $\Omega$ in the $\boldsymbol{e}_n$ direction. $\square$

**Definition 5.4.3** (Smooth characteristic function)**.** Let $\boldsymbol{x} \in \mathbb{R}^n$ and $B(\boldsymbol{x}, \delta)$ be the open ball in $\mathbb{R}^n$ centered at $\boldsymbol{x}$ of radius $\delta$. A function $\varrho$ in $C_0^\infty(\mathbb{R}^n)$ such that $\varrho = 1$ in $B(\boldsymbol{x}, \delta)$, $0 \leq \varrho \leq 1$ in $B(\boldsymbol{x}, 2\delta) \setminus B(\boldsymbol{x}, \delta)$ and $\varrho = 0$ in $\mathbb{R}^n \setminus B(\boldsymbol{x}, 2\delta)$ will be called a *smooth characteristic function* of $B(\boldsymbol{x}, \delta)$.

**Lemma 5.4.4** (Compactly supported graph)**.** *Let* $\Omega$ *be a* $W_s^{2-1/s}$*-domain and* $\boldsymbol{x}_\lambda$ *be in* $\partial\Omega$ *with* $\omega$ *its associated (local) graph. If* $0 < \delta \leq \delta_\lambda/2$, *there exists a function* $\mathcal{C}\omega$ *in* $W_s^{2-1/s}(D(\boldsymbol{x}'_\lambda, 2\delta))$ *such that its extension by zero is in* $W_s^{2-1/s}(\mathbb{R}^n)$, $\mathcal{C}\omega(\boldsymbol{x}') = \omega(\boldsymbol{x}')$ *for* $\boldsymbol{x}' \in D(\boldsymbol{x}'_\lambda, \delta)$ *and*

$$\|\mathcal{C}\omega\|_{W_s^{2-1/s}(D(\boldsymbol{x}'_\lambda, 2\delta))} \leq C\|\omega\|_{W_s^{2-1/s}(D(\boldsymbol{x}'_\lambda, 2\delta))}. \tag{5.16}$$

*where the constant is independent of* $\delta$ *and* $\omega$. *Moreover,* $\Omega \cap B(\boldsymbol{x}_\lambda, \delta) = \Omega(\boldsymbol{x}_\lambda, \delta, \mathcal{C}\omega)$.

*Proof.* Without loss of generality assume that $\boldsymbol{x}'_\lambda = \boldsymbol{0}'$ and write $D_\delta = D(\boldsymbol{0}', \delta)$. Let $\varrho$ be the smooth characteristic function of $D_\delta$ generated by the exponential cut-off and define $\mathcal{C}\omega = \varrho^2\omega$. It is clear that $\mathcal{C}\omega = \omega$ in $D_\delta$. We will derive the estimate (5.16) in three stages.

First, we estimate $\mathcal{C}\omega$ in $W_s^1(D_{2\delta})$-norm,

$$\|\mathcal{C}\omega\|_{W_s^1(D_{2\delta})} = \left\|\varrho^2\omega\right\|_{L^s(D_{2\delta})} + \left\|2\varrho\nabla'\varrho\omega + \varrho^2\nabla'\omega\right\|_{L^s(D_{2\delta})}$$
$$\leq \|\omega\|_{W_s^1(D_{2\delta})} + 2\|\omega\|_{L^s(D_{2\delta})}\left\|\nabla'\varrho\right\|_{L^\infty(D_{2\delta})}$$

where we used that $\left\|\varrho^2\right\|_{L^\infty(D_{2\delta})} \leq 1$, and a direct computation shows that $\|\nabla'\varrho\|_{L^\infty(D_{2\delta})}$ is proportional to $\delta^{-1}$. In view of the integral condition (5.15b), the Poincaré-Friedrichs' inequality yields

$$\|\omega\|_{L^s(D_{2\delta})} \leq C_{n,s}\delta|\omega|_{W_s^1(D_{2\delta})}\,,$$

whence $\|\mathcal{C}\omega\|_{W_s^1(D_{2\delta})} \leq C|\omega|_{W_s^1(D_{2\delta})}$ with constant $C$ independent of $\delta$ and $\omega$.

Next, we obtain an intermediate estimate for the semi-norm,

$$\left|\nabla'\mathcal{C}\omega\right|_{W_s^{1-1/s}(D_{2\delta})} = \left(\int_{D_{2\delta}}\int_{D_{2\delta}} \frac{\left|\nabla'\mathcal{C}\omega(\boldsymbol{x}') - \nabla'\mathcal{C}\omega(\boldsymbol{y}')\right|^s}{\left|\boldsymbol{x}'-\boldsymbol{y}'\right|^{s+n-2}}\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{y}'\right)^{1/s}$$

by splitting it as follows,

$$\begin{aligned}\left|\nabla'\mathcal{C}\omega\right|_{W_s^{1-1/s}(D_{2\delta})} \leq C_{n,s}\Big(&\|\omega\|_{L^\infty(D_{2\delta})}\left|\nabla'\varrho^2\right|_{W_s^{1-1/s}(D_{2\delta})}\\ &+ 2\left\|\nabla'\varrho\right\|_{L^\infty(D_{2\delta})}|\omega|_{W_s^{1-1/s}(D_{2\delta})}\\ &+ \left\|\nabla'\omega\right\|_{L^\infty(D_{2\delta})}\left|\varrho^2\right|_{W_s^{1-1/s}(D_{2\delta})} + \left|\nabla'\omega\right|_{W_s^{1-1/s}(D_{2\delta})}\Big).\end{aligned}$$

Upon scaling the domain we obtain

$$\left|\varrho^2\right|_{W_s^{1-1/s}(D_{2\delta})} = (2\delta)^{\frac{n}{s}-1}\left|\hat{\varrho}^2\right|_{W_s^{1-1/s}(D_1)}\,,$$
$$\left|\nabla'\varrho^2\right|_{W_s^{1-1/s}(D_{2\delta})} = (2\delta)^{\frac{n}{s}-2}\left|\hat{\nabla}'\hat{\varrho}^2\right|_{W_s^{1-1/s}(D_1)}\,,$$

where $\hat{\boldsymbol{x}}' = \boldsymbol{x}'/\delta$ and $\hat{\varrho}(\hat{\boldsymbol{x}}') = \varrho(\boldsymbol{x}')$. Moreover, using the fact that $\omega$ is Lipschitz continuous we have,

$$|\omega|_{W_s^{1-1/s}(D_{2\delta})} \leq \left\|\nabla'\omega\right\|_{L^\infty(D_{2\delta})}\left|\boldsymbol{x}'\right|_{W_s^{1-1/s}(D_{2\delta})} \leq C_{n,s}(2\delta)^{n/s}\left\|\nabla'\omega\right\|_{L^\infty(D_{2\delta})}.$$

Combining these estimates and the Poincaré-Friedrichs' inequality for $\|\omega\|_{L^\infty(D_{2\delta})}$ we obtain the intermediate result,

$$\left|\nabla'\mathcal{C}\omega\right|_{W_s^{1-1/s}(D_{2\delta})} \leq C\delta^{\frac{n}{s}-1}\left\|\nabla'\omega\right\|_{L^\infty(D_{2\delta})} + C_{n,s}\left|\nabla'\omega\right|_{W_s^{1-1/s}(D_{2\delta})},$$

where $C$ depends on $n$, $s$ and $\hat{\varrho}$.

Lastly, using that $\partial_{x^i}\omega$ is in the trace space of $W_s^1(\mathbb{R}^n)$, which continuously embeds into $C^{0,1-n/s}(\mathbb{R}^n)$, together with the gradient condition (5.15a), we conclude

$$\begin{aligned}\left\|\nabla'\omega\right\|_{L^\infty(D_{2\delta})} = \sup_{\boldsymbol{x}'\in D_{2\delta}}\left|\nabla'\omega(\boldsymbol{x}') - \nabla'\omega(\boldsymbol{0}')\right| &\leq \left[\nabla'\omega\right]_{C^{0,1-n/s}} \sup_{\boldsymbol{x}'\in D_{2\delta}}\left|\boldsymbol{x}'\right|^{1-n/s} \\ &\leq C\,(2\delta)^{1-\frac{n}{s}}\left|\nabla'\omega\right|_{W_s^{1-1/s}(D_{2\delta})},\end{aligned}$$

where the last result is Morrey's inequality on $D_{2\delta}$, [Eva98, Section 5.6, Theorem 4]. This concludes the proof. □

**Definition 5.4.5** (Bubble domain)**.** An open set $U(\boldsymbol{0},1)$ is called a *bubble domain* whenever its $C^\infty$ boundary is obtained by smoothing the "corners" of the lower half-ball $B_-(\boldsymbol{0},3/2) = B\left(\boldsymbol{0},3/2\right)\cap\mathbb{R}^n_-$ and

$$\overline{B_-(\boldsymbol{0},1)} \subsetneq \overline{U(\boldsymbol{0},1)} \subsetneq \overline{B_-(\boldsymbol{0},3/2)}.$$

This domain is represented in Figure 5.1.

**Definition 5.4.6** (Local diffeomorphism)**.** We say a diffeomorphism $\hat{\Psi}:\mathbb{R}^n\to\mathbb{R}^n$ is local whenever the set $U = \operatorname{supp}(\mathcal{I}-\hat{\Psi})$ is compactly embedded in $\mathbb{R}^n$, i.e. $\hat{\Psi}$ is a *compact* perturbation of the identity.

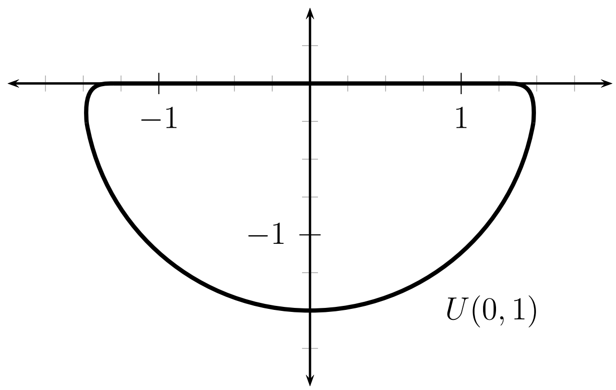



Figure 5.1: The bubble domain in $\mathbb{R}^2$.

**Proposition 5.4.7** ($W^2_s$-diffeomorphism)**.** *Let $\Omega$ be a $W^{2-1/s}_s$-domain and $\mathcal{I}$ the identity on $\mathbb{R}^n$. For each point $\boldsymbol{x}_\lambda$ in $\partial\Omega$ there exists a $\delta > 0$ and a local diffeomorphism $\hat{\Psi}$ in $W^2_s(\mathbb{R}^n_-)$ whose inverse $\Psi^{-1}$ flattens the boundary of $\Omega$ near $\boldsymbol{x}_\lambda$ and maps $\Omega(\boldsymbol{x}_\lambda, \delta, \omega)$ into the bubble domain $U(\mathbf{0}, 2\delta)$. Moreover, modulus a rotation and translation, the diffeomorphism satisfies*

$$\left\|1 - \det \hat{\nabla}\hat{\Psi}\right\|_{L^\infty(U(\mathbf{0},2\delta))} \leq C_{n,s}\delta^{1-n/s}|\omega|_{W^{2-1/s}_s(D(\boldsymbol{x}'_\lambda,2\delta))}\,, \tag{5.17a}$$

$$\left|\mathcal{I} - \hat{\Psi}\right|_{W^1_\infty(U(\mathbf{0},2\delta))} \leq C_{n,s}\delta^{1-n/s}|\omega|_{W^{2-1/s}_s(D(\boldsymbol{x}'_\lambda,2\delta))}\,, \tag{5.17b}$$

$$\left\|\mathcal{I} - \hat{\Psi}\right\|_{W^2_s(U(\mathbf{0},2\delta))} \leq C_{n,s}\|\omega\|_{W^{2-1/s}_s(D(\boldsymbol{x}'_\lambda,2\delta))}\,. \tag{5.17c}$$

*Proof.* Without any loss of generality we assume that $\boldsymbol{x}'_\lambda = \mathbf{0}'$ and define a mapping $\hat{\Psi}(\hat{\boldsymbol{x}}) := \hat{\boldsymbol{x}} + \tilde{\mathcal{E}}\omega(\hat{\boldsymbol{x}})\boldsymbol{e}_n$, where $\tilde{\mathcal{E}}\omega$ is the harmonic lifting of $\omega$ which we shall describe in the sequel.

Let $U = U(\mathbf{0}, 2\delta)$ be a bubble domain and $D = D(\mathbf{0}', 2\delta)$ the disc in $\mathbb{R}^{n-1}$. By Definition 5.4.5 it immediately follows that $\partial U \cap D = D$. A sufficient condition for $\hat{\Psi}(U)$ to be an open domain of $\mathbb{R}^n$ is $\|\omega\|_{L^\infty(D)} \leq \delta$. This is possible when $\delta$ is small enough because $\omega$ is proportional to $\delta^{2-n/s}$, i.e.

$$\|\omega\|_{L^\infty(D)} < C_{n,s}\delta^{2-n/s}|\omega|_{W_s^{2-1/s}(D)}, \tag{5.18}$$

where this was verified in Lemma 5.4.4 through the Poincaré-Friedrichs' inequality.

Additionally, if $\tilde{\omega}$ is the extension by zero of $\mathcal{C}\omega \in W_s^{2-1/s}(D)$, then $\tilde{\omega}$ is also a function in $W_s^{2-1/s}(\partial U)$ in view of Lemma 5.4.4. Let $\mathcal{E}\omega$ be the harmonic lifting of $\tilde{\omega}$ into $U$, i.e. $\Delta\mathcal{E}\omega = 0$ in $U$ and $\mathcal{E}\omega = \tilde{\omega}$ on $\partial U$. Since $\partial U$ is smooth and $\tilde{\omega}$ has compact support, we may extend $\mathcal{E}\omega$ to $\mathbb{R}^n_-$ and obtain

$$\left\|\tilde{\mathcal{E}}\omega\right\|_{W_s^2(\mathbb{R}^n_-)} \leq C_{n,s}\|\mathcal{E}\omega\|_{W_s^2(U)} \leq C_{n,s}\|\omega\|_{W_s^{2-1/s}(D)}.$$

This proves estimate (5.17c).

Next we obtain estimate (5.17b). Let $B := B(-3\delta/4, \delta/2)$ and $\overline{\nabla\mathcal{E}\omega} := \fint_B \nabla\mathcal{E}\omega$, i.e. the average of each gradient component of $\mathcal{E}\omega$ over the ball $B$, see Figure 5.2. The triangle and Poincaré inequalities (or equivalently the theory on averaged Taylor polynomials [BS08, 4.1.3, 4.3.2, 4.3.4]) yields

$$\begin{aligned}\|\nabla\mathcal{E}\omega\|_{L^\infty(U)} &\leq \left\|\nabla\mathcal{E}\omega - \overline{\nabla\mathcal{E}\omega}\right\|_{L^\infty(U)} + \fint_B |\nabla\mathcal{E}\omega| \\ &\leq C_{n,s}\delta^{1-n/s}|\nabla\mathcal{E}\omega|_{W_s^1(U)} + \|\nabla\mathcal{E}\omega\|_{L^\infty(B)}.\end{aligned}$$

To estimate the last term we invoke the interior estimate for derivatives of a harmonic function [GT01, Theorem 2.10] and obtain

$$\|\nabla\mathcal{E}\omega\|_{L^\infty(B)} \leq n\delta^{-1}\|\mathcal{E}\omega\|_{L^\infty(U)} \leq n\delta^{-1}\|\omega\|_{L^\infty(D)},$$

where the last inequality followed by the maximum principle and the support of $\tilde{\omega}$. To conclude estimate (5.17b) it suffices to utilize (5.18) one more time.

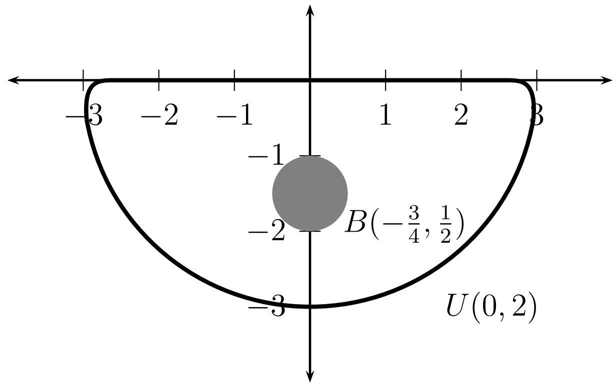



Figure 5.2: The bubble domain $U$ and averaging ball $B$.

Estimate (5.17a) is a direct consequence of the definition of $\hat{\Psi}$. It remains to show that $\hat{\Psi}$ is invertible. In view of the inverse function theorem, a sufficient condition is for $\hat{\nabla}\hat{\Psi}$ to be invertible. This is verifiable from (5.17a) by taking $\delta$ small enough to obtain $\left|1 - \det \hat{\nabla}\hat{\Psi}\right| < 1/2$, whence $\left|\det \hat{\nabla}\hat{\Psi}\right| \geq 1/2$. $\square$

### 5.4.2 Piola Transform

The purpose of this section is to analyze the Piola transform, a mapping which preserves the essential boundary condition $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ after the boundary of $\Omega$ has been flattened. We will restrict the presentation to a *local* $W_s^2$*-diffeomorphism* $\hat{\Psi}$, $s > n$ which maps the reference domain $\hat{\Theta}$ (bounded or unbounded) one-to-one and onto a physical domain $\Theta$, i.e.

$$\begin{array}{rccccl} \hat{\Psi} & : & \hat{\Theta} & \longrightarrow & \Theta & \\ & & \hat{\boldsymbol{x}} & \longmapsto & \boldsymbol{x} & , \end{array} \tag{5.19}$$

with $\hat{U} = \operatorname{supp}(\mathcal{I} - \hat{\Psi})$, and $U = \hat{\Psi}(\hat{U})$.

**Definition 5.4.8** (Piola transform)**.** We say two vector fields $\hat{\boldsymbol{v}}$ and $\boldsymbol{v}$ are the Piola transforms of each other if and only if

$$\hat{\boldsymbol{v}} = \boldsymbol{P}^{-1}\boldsymbol{v} \circ \hat{\Psi}, \text{ and } \boldsymbol{v} = \hat{\boldsymbol{P}}\hat{\boldsymbol{v}} \circ \Psi^{-1}, \tag{5.20}$$

where $\boldsymbol{P}^{-1} := \nabla\Psi^{-1}/\det(\nabla\Psi^{-1})$, $\hat{\boldsymbol{P}} = \hat{\nabla}\hat{\Psi}/\det\hat{\nabla}\hat{\Psi}$. In view of the inverse function theorem we also have $\boldsymbol{P}^{-1} \circ \hat{\Psi} = \hat{\boldsymbol{P}}^{-1} = \det\hat{\nabla}\hat{\Psi}\left(\hat{\nabla}\hat{\Psi}\right)^{-1}$.

**Lemma 5.4.9.** *Let $1 < r < \infty$ and $\hat{\boldsymbol{v}}$ be the Piola transform of $\boldsymbol{v}$. There exists constants $C'$ and $C''$ which depend only on the Lipschitz semi-norms of $\hat{\Psi}$ and $\Psi^{-1}$ such that*

$$C'\|\hat{\boldsymbol{v}}\|_{L^r(\hat{\Theta})} \le \|\boldsymbol{v}\|_{L^r(\Theta)} \le C''\|\hat{\boldsymbol{v}}\|_{L^r(\hat{\Theta})}. \tag{5.21}$$

*Proof.* The proof follows directly from the definition, i.e.

$$\left\|\hat{\boldsymbol{P}}^{-1}(\boldsymbol{v}\circ\hat{\Psi})\right\|_{L^r(\hat{\Theta})} \le \left\|\hat{\boldsymbol{P}}^{-1}\right\|_{L^\infty(\hat{\Theta})}\left\|\det(\nabla\Psi^{-1})\right\|_{L^\infty(\Theta)}\|\boldsymbol{v}\|_{L^r(\Theta)}.$$

Conversely,

$$\left\|\hat{\boldsymbol{P}}\hat{\boldsymbol{v}}\circ\Psi^{-1}\right\|_{L^r(\Theta)} \le \left\|\det\hat{\nabla}\hat{\Psi}\right\|_{L^\infty(\hat{\Theta})}\left\|\hat{\boldsymbol{P}}\right\|_{L^\infty(\hat{\Theta})}\|\hat{\boldsymbol{v}}\|_{L^r(\hat{\Theta})}.$$

which concludes the proof. □

**Lemma 5.4.10** (Piola gradient)**.** *Let $\hat{\boldsymbol{v}}$ and $\boldsymbol{v}$ be Piola transforms of each other. The gradient admits the following decomposition*

$$\hat{\nabla}\hat{\boldsymbol{v}} = \det\hat{\nabla}\hat{\Psi}[\![\nabla\boldsymbol{v}\circ\hat{\Psi}]\!]_{\hat{\boldsymbol{P}}^{-1}} + \hat{\nabla}\hat{\boldsymbol{P}}^{-\top}\otimes\boldsymbol{v}\circ\hat{\Psi}, \tag{5.22a}$$

$$\nabla\boldsymbol{v}\circ\hat{\Psi} = \frac{1}{\det\hat{\nabla}\hat{\Psi}}\left([\![\hat{\nabla}\hat{\boldsymbol{v}}]\!]_{\hat{\boldsymbol{P}}} - [\![\hat{\nabla}\hat{\boldsymbol{P}}^{-\top}\otimes\hat{\boldsymbol{P}}\hat{\boldsymbol{v}}]\!]_{\hat{\boldsymbol{P}}}\right), \tag{5.22b}$$

*where $[\![\boldsymbol{M}]\!]_{\hat{\boldsymbol{P}}} := \hat{\boldsymbol{P}}\boldsymbol{M}\hat{\boldsymbol{P}}^{-1}$ is a similarity transformation, and $(\hat{\nabla}\boldsymbol{M}\otimes\boldsymbol{w})^{i,j} := (\partial_{x^j}\boldsymbol{M})^i\boldsymbol{w}$, where $\hat{\nabla}\boldsymbol{M}$ is the tensor $(\partial_{x^1}\boldsymbol{M},\ldots,\partial_{x^n}\boldsymbol{M})^\top$. Moreoever, if $\hat{\Psi}$ is a* *local $W^2_s$-diffeomorphism, $s > n$, then for $s' \le t^\bullet \le s$ and $1/t^\bullet = 1/s + 1/t^\circ$*

$$\begin{aligned} |\hat{\boldsymbol{v}}|_{W^1_{t^\bullet}(\hat{\Theta})} &\le C' \left( |\boldsymbol{v}|_{W^1_{t^\bullet}(\Theta)} + \|\boldsymbol{v}\|_{L^{t^\circ}(U)} \right) \\ |\boldsymbol{v}|_{W^1_{t^\bullet}(\Theta)} &\le C'' \left( |\hat{\boldsymbol{v}}|_{W^1_{t^\bullet}(\hat{\Theta})} + \|\hat{\boldsymbol{v}}\|_{L^{t^\circ}(\hat{U})} \right) \end{aligned} \tag{5.22c}$$

*where the constants $C'$ and $C''$ depend only on $n, r, s$, the Lipschitz and $W^2_s$ semi-norms of $\hat{\Psi}$ and $\Psi^{-1}$, and on the sets $\hat{U} = \operatorname{supp}(\mathcal{I} - \hat{\Psi})$ and $U = \hat{\Psi}(\hat{U})$.*

*Proof.* To obtain expression (5.22a) it suffices to differentiate the Piola transform given in the definition (5.20). To derive (5.22b) we apply the similarity transformation $[\![\cdot]\!]_{\hat{\boldsymbol{P}}}$ to both sides of (5.22a) and reorder terms to obtain

$$[\![\hat{\nabla}\hat{\boldsymbol{v}}]\!]_{\hat{\boldsymbol{P}}} - [\![\hat{\nabla}\hat{\boldsymbol{P}}^{-\top} \otimes \boldsymbol{v} \circ \hat{\Psi}]\!]_{\hat{\boldsymbol{P}}} = [\![\det \hat{\nabla}\hat{\Psi} [\![\nabla \boldsymbol{v} \circ \hat{\Psi}]\!]_{\hat{\boldsymbol{P}}^{-1}}]\!]_{\hat{\boldsymbol{P}}} = \left( \det \hat{\nabla}\hat{\Psi} \right) \nabla \boldsymbol{v} \circ \hat{\Psi},$$

whence, after replacing $\boldsymbol{v} \circ \hat{\Psi}$ by $\hat{\boldsymbol{P}}\hat{\boldsymbol{v}}$, we achieve the desired expression.

To show estimate (5.22c) we write

$$\begin{aligned} |\hat{\boldsymbol{v}}|_{W^1_{t^\bullet}(\hat{\Theta})} &= |\hat{\boldsymbol{v}}|_{W^1_{t^\bullet}(\hat{\Theta}\setminus\hat{U})} + |\hat{\boldsymbol{v}}|_{W^1_{t^\bullet}(\hat{U})} = |\boldsymbol{v}|_{W^1_{t^\bullet}(\Theta\setminus U)} + |\hat{\boldsymbol{v}}|_{W^1_{t^\bullet}(\hat{U})} \\ &\le |\boldsymbol{v}|_{W^1_{t^\bullet}(\Theta\setminus U)} + \left\| \det \hat{\nabla}\hat{\Psi} [\![\nabla \boldsymbol{v} \circ \hat{\Psi}]\!]_{\hat{\boldsymbol{P}}^{-1}} \right\|_{L^{t^\bullet}(\hat{U})} + \left\| \hat{\nabla}\hat{\boldsymbol{P}}^{-\top} \otimes \boldsymbol{v} \circ \hat{\Psi} \right\|_{L^{t^\bullet}(\hat{U})} \\ &\le |\boldsymbol{v}|_{W^1_{t^\bullet}(\Theta\setminus U)} + \left\| \hat{\boldsymbol{P}} \right\|_{L^\infty(\hat{U})} \left\| \hat{\boldsymbol{P}}^{-1} \right\|_{L^\infty(\hat{U})} \left\| \det \nabla \Psi^{-1} \right\|_{L^\infty(U)} |\boldsymbol{v}|_{W^1_{t^\bullet}(U)} \\ &\quad + \left\| \hat{\nabla}\hat{\boldsymbol{P}}^{-\top} \otimes \boldsymbol{v} \circ \hat{\Psi} \right\|_{L^{t^\bullet}(\hat{U})} \end{aligned}$$

where we have used the assumption that $\hat{\Psi}$ is a local diffeomorphism, the triangle inequality and a change of variables. To estimate the last term above it suffices to apply Hölder's inequality with $1/t^\bullet = 1/s + 1/t^\circ$. It is easy to check that, in case $t^\bullet = s$ then $t^\circ = \infty$, and if $t^\bullet = s'$ then $1/t^\circ = 1 - 2/s$.

Combining these results yields the first half of (5.22c). To obtain the second half, it suffices to follow the same steps above starting with (5.22b). □

**Proposition 5.4.11** (Piola symmetric gradient)**.** *Let $\hat{\boldsymbol{w}}$ be the Piola transform of $\boldsymbol{w}$. The symmetric gradient admits the following decomposition*

$$\boldsymbol{\varepsilon}(\boldsymbol{w}) \circ \hat{\Psi} = \frac{1}{\det \hat{\nabla}\hat{\Psi}} \left( \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}) - \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}) \right), \tag{5.23a}$$

*with*

$$\begin{aligned} \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}) &:= \frac{1}{2} \left( [\![\hat{\nabla}\hat{\boldsymbol{w}}]\!]_{\hat{\boldsymbol{P}}} + [\![\hat{\nabla}\hat{\boldsymbol{w}}]\!]_{\hat{\boldsymbol{P}}}^{\top} \right), \\ \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}) &:= \frac{1}{2} \left( [\![\hat{\nabla}\hat{\boldsymbol{P}}^{-\top} \otimes \hat{\boldsymbol{P}}\hat{\boldsymbol{w}}]\!]_{\hat{\boldsymbol{P}}} + [\![\hat{\nabla}\hat{\boldsymbol{P}}^{-\top} \otimes \hat{\boldsymbol{P}}\hat{\boldsymbol{w}}]\!]_{\hat{\boldsymbol{P}}}^{\top} \right). \end{aligned}$$

*Moreover, if $\hat{\Psi}$ is a local $W^2_s$-diffeomorphism, $s > n$, then for $s' \leq t^\circ \leq \infty$, and $1/t^\bullet = 1/s + 1/t^\circ$*

$$\begin{aligned} \left\| \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}) \right\|_{L^{t^\circ}(\hat{\Theta})} &\leq C' |\hat{\boldsymbol{w}}|_{W^1_{t^\circ}(\hat{\Theta})} \\ \left\| \boldsymbol{\vartheta}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}) \right\|_{L^{t^\bullet}(\hat{\Theta})} &\leq C'' \|\hat{\boldsymbol{w}}\|_{L^{t^\circ}(\hat{U})}, \end{aligned} \tag{5.23b}$$

*with constants $C'$ and $C''$ depend only on $n, r, s, t^\circ$, the Lipschitz and $W^2_s$ semi-norms of $\Psi$ and $\Psi^{-1}$, and on the sets $\hat{U} = \operatorname{supp}(\mathcal{I} - \hat{\Psi})$ and $U = \hat{\Psi}(\hat{U})$.*

*Proof.* The decomposition follows directly by the definition of the symmetric gradient and (5.22b). The bounds for $\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}$ and $\boldsymbol{\vartheta}_{\hat{\boldsymbol{P}}}$ follow from the bounds in Lemma 5.22b for $[\![\hat{\nabla}\hat{\boldsymbol{v}}]\!]_{\hat{\boldsymbol{P}}}$ and $[\![\hat{\nabla}\hat{\boldsymbol{P}}^{-\top} \otimes \hat{\boldsymbol{P}}\hat{\boldsymbol{v}}]\!]_{\hat{\boldsymbol{P}}}$ respectively. □

**Lemma 5.4.12** (Piola identity)**.** *If $\hat{\boldsymbol{v}}$ is the Piola transform of $\boldsymbol{v}$, then the following are equivalent*

$$\widehat{\operatorname{div}}\, \hat{\boldsymbol{v}} = \left( \det \hat{\nabla}\hat{\Psi} \right) \operatorname{div} \boldsymbol{v} \circ \hat{\Psi}, \tag{5.24a}$$

$$\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\nu}}\, \mathrm{d}\hat{s} = \boldsymbol{v} \cdot \boldsymbol{\nu}\, \mathrm{d}s \tag{5.24b}$$

$$\widehat{\operatorname{div}}\, \hat{\boldsymbol{P}}^{-\top} = \boldsymbol{0}. \tag{5.24c}$$

*where* $\mathrm{d}\hat{s}$ *and* $\mathrm{d}s$ *denote the surface measures of* $\partial\hat{\Theta}$ *and* $\partial\Theta$*. Moreover, as long as the right-hand-side is meaningful, we have*

$$\int_{\Theta} \phi \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_{\hat{\Theta}} \hat{\phi} \, \widehat{\operatorname{div}} \, \hat{\boldsymbol{v}} \, \mathrm{d}\hat{\boldsymbol{x}}, \tag{5.24d}$$

$$\int_{\Theta} \nabla\phi \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_{\hat{\Theta}} \hat{\nabla}\hat{\phi} \cdot \hat{\boldsymbol{v}} \, \mathrm{d}\hat{\boldsymbol{x}}, \tag{5.24e}$$

$$\int_{\partial\Theta} \phi \boldsymbol{v} \cdot \boldsymbol{\nu} \, \mathrm{d}s = \int_{\partial\hat{\Theta}} \hat{\phi} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}\hat{s} \tag{5.24f}$$

*Proof.* See [Cia88, Theorem 1.7-1]. □

**Theorem 5.4.13** (Space isomorphism)**.** *Let* $s' \leq r \leq s$*, and* $\hat{\Psi}$ *a local* $W^2_s$*-diffeormorphism between* $\hat{\Theta}$ *and* $\Theta$*. The operator*

$$\begin{array}{rccc} \widehat{\mathcal{P}} & : & X_r(\hat{\Theta}) & \longrightarrow & X_r(\Theta) \\ & & (\hat{\boldsymbol{v}}, \hat{q}) & \longmapsto & \left(\hat{\boldsymbol{P}}\hat{\boldsymbol{v}}, \hat{q}\right) \circ \Psi^{-1} = (\boldsymbol{v}, q) \end{array}$$

*is bounded and invertible. Therefore,* $X_r(\hat{\Theta})$ *and* $X_r(\Theta)$ *are isomorphic.*

*Proof.* Let $\hat{q}$ be in $L^r_0(\hat{\Theta})$, then

$$\|q\|_{L^r(\Theta)} = \left\|\hat{q} \circ \Psi^{-1}\right\|_{L^r(\Theta)} \leq C \left\|\det \hat{\nabla}\hat{\Psi}\right\|_{L^\infty(\hat{\Theta})} \|\hat{q}\|_{L^r(\hat{\Theta})} \,.$$

The same estimate holds by starting with $q$ in $L^r_0(\Theta)$.

If $\hat{\boldsymbol{v}}$ is in $V_r(\hat{\Theta})$, then $\boldsymbol{v}$ will be in $V_r(\hat{\Theta})$ as long as $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$ and $|\boldsymbol{v}|_{W^1_r(\Theta)}$ is bounded. In view of the Piola identity (5.24b) we have $\boldsymbol{v} \cdot \boldsymbol{\nu} \, \mathrm{d}s = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}\hat{s} = 0$. According to estimate(5.22c), $|\boldsymbol{v}|_{W^1_r(\Theta)}$ will be bounded as long as we can control $\|\hat{\boldsymbol{v}}\|_{L^{t^\circ}(\hat{U})}$. This splits into three cases:

- Case $s' \leq r < n$: $\hat{\boldsymbol{v}}$ belongs to $W^1_r(\hat{\Theta}) \hookrightarrow L^{r^*}(\hat{U})$ and $t^\circ \leq r^*$;
- Case $r = n$: $\hat{\boldsymbol{v}}$ belongs to $W^1_r(\hat{\Theta}) \hookrightarrow L^t(\hat{U})$ for $1 \leq t < \infty$ and $t^\circ < t$;

- Case $n < r \leq s$: $\hat{\boldsymbol{v}}$ is in $W_r^1(\hat{\Theta}) \hookrightarrow L^\infty(\hat{U})$ and $t^\circ \leq \infty$.

Thus in all three cases we have $\|\hat{\boldsymbol{v}}\|_{L^{t^\circ}(\hat{U})} \leq C_{n,r,s,\hat{U}} \|\hat{\boldsymbol{v}}\|_{W_r^1(\hat{U})} \leq C|\hat{\boldsymbol{v}}|_{W_r^1(\Theta)}$, where the last inequality follows from either

(i) Proposition 5.2.2 on equivalence of norms if $\hat{\Theta}$ is bounded; or

(ii) Proposition 5.3.1 and the Poincaré inequality when $\hat{\Theta}$ is unbounded, i.e. use that $\hat{\boldsymbol{v}}$ is in $[\cdot]_1$ hence only defined up to a constant.

The same process yields that $\widehat{\mathcal{P}}^{-1}$ is bounded and we conclude $X_r(\hat{\Theta})$ and $X_r(\Theta)$ are isomorphic. $\square$

**Remark 5.4.14.** As we can see above, it is absolutely necessary for $\hat{\Psi}$ to have two derivatives for the Piola transform to make sense as an isomorphism between $X_r(\hat{\Theta})$ and $X_r(\Theta)$. This differs from the canonical use of the Piola transform for $H(\mathrm{div})$ spaces.

**Corollary 5.4.15** (Fréchet derivatives of $\hat{\boldsymbol{P}}$ and $\hat{\boldsymbol{P}}^{-1}$)**.** *Suppose* $\hat{\Psi} = \mathcal{I} + \boldsymbol{e}^n \mathcal{E}\omega$ *is a local* $W_s^2$*-diffeomorphism and* $\mathcal{E} : W_s^{2-1/s}(\partial\hat{U}) \to W_s^2(\mathbb{R}^n)$ *is a bounded linear operator. The Piola matrices* $\hat{\boldsymbol{P}}$ *and* $\hat{\boldsymbol{P}}^{-1}$ *are Fréchet differentiable with respect to* $\omega$ *as a map from* $W_s^{2-1/s}(\partial\hat{U})$ *to* $W_s^1(\mathbb{R}^n)$*. Moreover, their variations are given by*

$$\partial_\omega \hat{\boldsymbol{P}}^{-1}(\omega_0)\langle h\rangle = \begin{bmatrix} \boldsymbol{I}\partial_n \mathcal{E}h & 0 \\ -\nabla' \mathcal{E}h^\top & 0 \end{bmatrix}, \tag{5.25}$$

$$\partial_\omega \hat{\boldsymbol{P}}(\omega_0)\langle h\rangle = \partial_\omega \left(\frac{1}{\det \hat{\nabla}\hat{\Psi}}\right)\langle h\rangle \hat{\nabla}\hat{\Psi} + \frac{1}{\det \hat{\nabla}\hat{\Psi}} \partial_\omega \hat{\nabla}\hat{\Psi}\langle h\rangle. \tag{5.26}$$

*Proof.* The Fréchet derivatives related to $\hat{\nabla}\hat{\Psi}$ can be found in Corollary 2.1.6. $\square$

## 5.5 The Sobolev Space Case

Let $X$, $Y$ and $Z$ be arbitrary Banach spaces with $X^*$, $Y^*$ and $Z^*$ their duals.

**Definition 5.5.1** (Index)**.** A (bounded) linear operator $\mathcal{A} : X \to Y$ is said to have *finite index* if it has the following properties:

(i) The nullspace $N_{\mathcal{A}}$ of $\mathcal{A}$ is a finite dimensional subspace of $X$.

(ii) The quotient space $Y/R_{\mathcal{A}}$ is finite dimensional, with $R_{\mathcal{A}}$ the range of $\mathcal{A}$.

For such an operator we define the *index* as

$$\operatorname{ind}\mathcal{A} := \dim N_{\mathcal{A}} - \dim Y/R_{\mathcal{A}}.$$

**Definition 5.5.2** (Pseudoinverse)**.** Two bounded linear operators $\mathcal{A} : X \to Y$ and $\mathcal{A}^\dagger : Y \to X$ are called *pseudoinverses* of each other if

$$\mathcal{A}\mathcal{A}^\dagger = \mathcal{I}_Y + \mathcal{K}, \quad \mathcal{A}^\dagger\mathcal{A} = \mathcal{I}_X + \mathcal{C},$$

where $\mathcal{K}$ and $\mathcal{C}$ are *compact* operators of $Y$, respectively $X$, into themselves.

**Definition 5.5.3** (Dual operator)**.** Every bounded linear operator $\mathcal{A} : X \to Y$ has a dual (operator) $\mathcal{A}^* : Y^* \to X^*$ given by the relation,

$$\langle \mathcal{A}^*\ell, x\rangle_{X^*,X} := \langle \ell, \mathcal{A}x\rangle_{Y^*,Y}, \quad x \in X, \, \ell \in Y^*.$$

**Theorem 5.5.4.** *A bounded linear operator $\mathcal{A} : X \to Y$ has finite index if and only if $\mathcal{A}$ has a pseudoinverse. Moreover,*

$$\operatorname{ind} A = -\operatorname{ind} A^\dagger.$$

*Proof.* See [Lax02, Chapter 27: Theorems 1,2]. □

**Lemma 5.5.5.** *If $\mathcal{B} : Y \to Z$ is a bounded linear map and $\mathcal{C} : X \to Y$ is compact, then $\mathcal{BC} : X \to Z$ is compact. Moreover, the same result holds if $\mathcal{B}$ is a compact linear map and $\mathcal{C}$ is only bounded.*

*Proof.* See [Lax02, Chapter 21: Theorem 1]. □

**Lemma 5.5.6.** *Suppose that $\mathcal{A} : X \to Y$ has finite index, and $\mathcal{K} : X \to Y$ is a compact linear map. Then $\mathcal{A} + \mathcal{K}$ has finite index and*

$$\operatorname{ind}(\mathcal{A} + \mathcal{K}) = \operatorname{ind} \mathcal{A}.$$

*Proof.* [Lax02, Chapter 21: Theorem 3] □

**Theorem 5.5.7.** *Let $\mathcal{A} : X \to Y$ be a bounded linear operator. If $\mathcal{A}$ has finite index, then so does its dual $\mathcal{A}^*$. Moreover,*

$$\operatorname{ind} \mathcal{A}^* = -\operatorname{ind} \mathcal{A}.$$

*Proof.* See [Lax02, Chapter 27: Theorem 4]. □

**Corollary 5.5.8.** *Let $\mathcal{A} : X \to Y$ be a bounded operator with a pseudoinverse. If $\mathcal{A}$ and $\mathcal{A}^*$ are injective then they are bijective.*

*Proof.* From Theorem 5.5.4 we have that $\mathcal{A}$ has finite index. Since $\mathcal{A}$ and $\mathcal{A}^*$ are injective, $\dim N_{\mathcal{A}} = \dim N_{\mathcal{A}^*} = 0$. According to Theorem 5.5.7 we have, $-\dim X^*/R_{\mathcal{A}^*} = \dim Y/R_{\mathcal{A}}$. Obviously the dimension of a space is not negative so that

$$\dim X^*/R_{\mathcal{A}^*} = \dim Y/R_{\mathcal{A}} = 0,$$

i.e. $\mathcal{A}$ and $\mathcal{A}^*$ are surjective which concludes our proof. $\square$

The above result can be easily reinterpreted in terms of the more common Banach-Nečas' inf-sup theorem.

**Corollary 5.5.9.** *Let $\mathcal{A} : X \to Y$ be a bounded linear operator. $\mathcal{A}$ is bijective if and only if there exists a constant $\alpha > 0$ such that*

$$\inf_{x \in X} \sup_{\ell \in Y^*} \frac{\langle \ell, \mathcal{A}x \rangle_{Y^*,Y}}{\|\ell\|_{Y^*} \|x\|_X} \geq \alpha, \tag{5.27a}$$

$$\forall \ell \in Y^*, \quad (\langle \ell, \mathcal{A}x \rangle_{Y^*,Y} = 0, \forall x \in X) \implies (\ell = 0). \tag{5.27b}$$

*Proof.* See [EG04, Corollary A.45]. $\square$

Our strategy is to use Corollary 5.5.8 to infer the invertibility of the Stokes operator $\mathcal{S}_\Omega : X_r(\Omega) \to X_{r'}(\Omega)^*$. First, we will decompose $\mathcal{S}_\Omega$ into its interior and boundary parts. Second, we will use the boundedness of the domain $\Omega$ to construct a pseudoinverse of $\mathcal{S}_\Omega$, hence showing that it has a finite index. Third, we will show that $\mathcal{S}_\Omega$ and $\mathcal{S}_\Omega^*$ are injective.

### 5.5.1 Localized Equations

The goal of this section is to localize the Stokes equations. The technique's essence is to test the Stokes variational system with a smooth cutoff version of a velocity-pressure pair $(\hat{\boldsymbol{v}}, \hat{p})$ defined over an unbounded domain. This exposes the local behavior, in operator terms, of the Stokes linear map. In particular, we will see that it splits (locally) into bounded operators including $\mathcal{S}_{\mathbb{R}^n}$ or $\mathcal{S}_{\mathbb{R}^n_-}$ plus a compact part.

**Definition 5.5.10** (Localization operator)**.** Let $\Omega$ be a $W_s^{2-1/s}$-domain and $\boldsymbol{x}_\lambda$ be a point in $\overline{\Omega}$. The operators defined for every $s' \le r \le s$ by

$$\begin{array}{rcrcl} \mathcal{R}_{\lambda,\zeta} & : & X_r(\Omega) & \longrightarrow & X_r(\hat{\Theta}_\lambda) \\ & & (\boldsymbol{u}, p) & \longmapsto & \widehat{\mathcal{P}}_\lambda^{-1}(\zeta\boldsymbol{u}, \zeta p) \\ \widehat{\mathcal{R}}_{\lambda,\zeta} & : & X_r(\hat{\Theta}_\lambda) & \longrightarrow & X_r(\Omega) \\ & & (\hat{\boldsymbol{v}}, \hat{q}) & \longmapsto & \zeta\widehat{\mathcal{P}}_\lambda(\hat{\boldsymbol{v}}, \hat{q}) \end{array}$$

will be called *localization operators* whenever $\zeta$ is in $C_c^\infty(B(\boldsymbol{x}_\lambda, \delta_\lambda))$ and

(i) if $\boldsymbol{x}_\lambda$ is in $\Omega$, then $\delta_\lambda = \mathrm{dist}(x_\lambda, \partial\Omega)$, $\hat{\Psi} = \Psi^{-1} = \mathcal{I}$, so $\hat{\Theta}_\lambda = \Theta_\lambda = \mathbb{R}^n$;

(ii) if $\boldsymbol{x}_\lambda$ is in $\partial\Omega$, then $\delta_\lambda \le \delta$ is given by the local $W_s^2$-diffeomorphism $\hat{\Psi}_\lambda$ from Proposition 5.4.7, $\hat{\Theta}_\lambda = \mathbb{R}^n_-$, $\Theta_\lambda = \hat{\Psi}(\hat{\Theta}_\lambda)$; and

(iii) in both cases $\widehat{\mathcal{P}}_\lambda$ is given by Theorem 5.4.13.

We remark that $\mathcal{R}_{\lambda,\zeta}$ and $\widehat{\mathcal{R}}_{\lambda,\zeta}$ are *not* invertible, but we will address this issue in §5.5.2.

**Lemma 5.5.11.** *The localization operators $\mathcal{R}_{\lambda,\zeta}$ and $\widehat{\mathcal{R}}_{\lambda,\zeta}$ are continuous. As a result, their corresponding duals $\mathcal{R}^*_{\lambda,\zeta}$ and $\widehat{\mathcal{R}}^*_{\lambda,\zeta}$ exist and are continuous.*

*Proof.* If $\boldsymbol{x}_\lambda$ is in $\Omega$ the proof is direct since we are simply multiplying $(\boldsymbol{u}, p)$ or $(\hat{\boldsymbol{v}}, \hat{q})$ by a smooth function whose support lies in the interior of $\Omega$. If $\boldsymbol{x}_\lambda$ is in $\partial\Omega$ then the difficulty lies in analyzing the continuity of the Piola transform, but this was already treated in Theorem 5.4.13. $\square$

The next two propositions gives the equations satisfied by the localized Stokes operator. Each localization is performed in three steps: a composition of $\widehat{\mathcal{R}}^*_{\lambda,\zeta}$

with $\mathcal{S}_\Omega$ resulting in range localization; a composition of $S_\Omega$ with $\widehat{\mathcal{R}}_{\lambda,\zeta}$ resulting in domain localization; and follow-up results on the topological properties of the ensuing decomposition. This process is applied both to the interior of $\Omega$ and to its boundary.

**Proposition 5.5.12** (Interior localization)**.** *Let $\boldsymbol{x}_\lambda$ be in $\Omega$. The interior Stokes operators $\widehat{\mathcal{R}}^*_{\lambda,\zeta}\mathcal{S}_\Omega : X_r(\Omega) \to X_{r'}(\hat{\Theta}_\lambda)^*$, and $S_\Omega\widehat{\mathcal{R}}_{\lambda,\zeta} : X_r(\hat{\Theta}_\lambda) \to X_{r'}(\Omega)^*$ are continuous and equivalent to,*

$$\begin{aligned} \widehat{\mathcal{R}}^*_{\lambda,\zeta}\mathcal{S}_\Omega &= \tilde{\mathcal{S}}_\lambda\mathcal{R}_{\lambda,\zeta} + \widehat{\mathcal{P}}^*_\lambda\mathcal{K}_{\lambda,\zeta} \\ S_\Omega\widehat{\mathcal{R}}_{\lambda,\zeta} &= \mathcal{R}^*_{\lambda,\zeta}\tilde{\mathcal{S}}_\lambda + \mathcal{K}_{\lambda,\zeta}\widehat{\mathcal{P}}_\lambda \end{aligned} \tag{5.28a}$$

*where $\tilde{\mathcal{S}}_\lambda := \mathcal{S}_{\mathbb{R}^n}$ is the Stokes operator in $\mathbb{R}^n$, $\mathcal{K}_{\lambda,\zeta} : X_r(\Theta_\lambda) \to X_{r'}(\Theta_\lambda)^*$*

$$\begin{aligned} \mathcal{K}_{\lambda,\zeta}(\boldsymbol{u},p)(\boldsymbol{v},q) := &- \langle p, \nabla\zeta\cdot\boldsymbol{v}\rangle_{\Omega_\lambda} - \langle \nabla\zeta\cdot\boldsymbol{u}, q\rangle_{\Omega_\lambda} \\ &- \eta\left\langle \boldsymbol{\vartheta}_\zeta(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\right\rangle_{\Omega_\lambda} + \eta\left\langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\vartheta}_\zeta(\boldsymbol{v})\right\rangle_{\Omega_\lambda}, \end{aligned} \tag{5.28b}$$

*where $\Omega_\lambda := \Omega \cap B(\boldsymbol{x}_\lambda, \delta_\lambda)$. Furthermore,*

$$\begin{aligned} \boldsymbol{\vartheta}_\zeta(\boldsymbol{w}) &:= \frac{1}{2}(\nabla\zeta\otimes\boldsymbol{w} + \boldsymbol{w}\otimes\nabla\zeta), \\ \left\|\boldsymbol{\vartheta}_\zeta(\boldsymbol{w})\right\|_{L^t(\mathbb{R}^n)} &\le C_\lambda\|\boldsymbol{w}\|_{L^t(\Omega_\lambda)}, \end{aligned} \tag{5.28c}$$

*where $1 \le t \le \infty$.*

*Proof.* Continuity of the decompositions follows from the fact that a composition of two continuous linear operators is continuous and linear. Derivation of (5.28a) relies mostly on the fact that for any vector valued function $\boldsymbol{w}$, $\nabla(\zeta\boldsymbol{w}) = \nabla\zeta\otimes\boldsymbol{w} + \zeta\nabla\boldsymbol{w}$ so that $\boldsymbol{\varepsilon}(\zeta\boldsymbol{w}) = \zeta\boldsymbol{\varepsilon}(\boldsymbol{w}) + \boldsymbol{\vartheta}_\zeta(\boldsymbol{w})$, and $\operatorname{div}\zeta\boldsymbol{w} = \zeta\operatorname{div}\boldsymbol{w} + \nabla\zeta\cdot\boldsymbol{w}$.

If $(\boldsymbol{u},p) \in X_r(\Omega)$ and $(\hat{\boldsymbol{v}},\hat{q}) \in X_{r'}(\hat{\Theta}_\lambda)$ are fixed, then it follows that

$$\begin{aligned}\left\langle \widehat{\mathcal{R}}^*_{\lambda,\zeta}\mathcal{S}_\Omega(\boldsymbol{u},p),(\hat{\boldsymbol{v}},\hat{q})\right\rangle_{X_{r'}(\hat{\Theta}_\lambda)^*,X_{r'}(\hat{\Theta}_\lambda)} &= \mathcal{S}_\Omega(\boldsymbol{u},p)\widehat{\mathcal{R}}_{\lambda,\zeta}(\hat{\boldsymbol{v}},\hat{q}) \\ &= \tilde{\mathcal{S}}_\lambda\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p)(\hat{\boldsymbol{v}},\hat{q}) \\ &\quad + \mathcal{K}_{\lambda,\zeta}(\boldsymbol{u},p)\widehat{\mathcal{P}}_\lambda(\hat{\boldsymbol{v}},\hat{q}),\end{aligned}$$

which gives the first equation of (5.28a). The second equation follows from an identical approach.

To obtain the boundedness of $\boldsymbol{\vartheta}_\zeta$ it suffices to note that $\nabla\zeta$ is smooth with compact support, and use Hölder's inequality. □

**Lemma 5.5.13.** *Consider two Banach spaces $X$ and $Y$, together with their duals $X^*$ and $Y^*$. Moreover let $Y$ be reflexive.*

*(i) Let $\mathcal{A} : X \to Y$ be a bounded linear map. If $\{x_k\}_{k=1}^\infty$ converges weakly to $x$ in $X$, then $\mathcal{A}x_k$ converges weakly to $\mathcal{A}x$.*

*(ii) If $\{x_k\}_{k=1}^\infty$ converges weakly to $x$ in $X$, and $\{\ell_k\}_{k=1}^\infty$ converges strongly to $\ell$ in $X^*$, then $\langle \ell_k, x_k\rangle_{X^*,X}$ converges strongly to $\langle \ell, x\rangle_{X^*,X}$.*

*Proof.* See [Lax02, Chapter 15: Exercise 2] for the first result.

To prove the second result let $\epsilon > 0$ be given. By the principle of uniform boundedness [Lax02, Chapter 10: Theorem 4'] there exists $M > 0$ such that $\|x_k\| \le M$ for $k = 1,\dots,\infty$. Moreover, by strong convergence there exists $K_1$ large enough such that $\|\ell_k - \ell\|_{X^*} \le \epsilon/(2M)$ for every $k > K$. Furthermore, by the weak convergence there exists $K_2$ large enough such that $\left|\langle \ell, x_k - x\rangle_{X^*,X}\right| < \epsilon/2$ for all

$k > K_2$. Hence for all $k > \max K_1, K_2$ we have that

$$\begin{aligned}\left|\langle \ell_k, x_k\rangle_{X^*,X} - \langle \ell, x\rangle_{X^*,X}\right| &\le \left|\langle \ell_k - \ell, x_k\rangle_{X^*,X}\right| + \left|\langle \ell, x_k - x\rangle_{X^*,X}\right| \\ &\le \|\ell_k - \ell\|_{X^*} \|x_k\|_X + \epsilon/2 \\ &\le \epsilon.\end{aligned}$$

This concludes the proof. □

**Lemma 5.5.14** ($\mathcal{K}_{\lambda,\zeta}$ is compact)**.** *Let $s' \le r \le s$ and $\boldsymbol{x}_\lambda$ be in $\overline{\Omega}$. The operator $\mathcal{K}_{\lambda,\zeta} : X_r(\Theta_\lambda) \to X_{r'}(\Theta_\lambda)^*$ is compact.*

*Proof.* To show $\mathcal{K}_{\lambda,\zeta}$ is compact it suffices to prove compactness of the bilinear functional $\mathcal{A} : X_r(\Theta_\lambda) \times X_{r'}(\Theta_\lambda) \to \mathbb{R}$,

$$\mathcal{A}\left[\boldsymbol{u}, p, \boldsymbol{v}, q\right] := \mathcal{K}_\lambda(\boldsymbol{u}, p)(\boldsymbol{v}, q).$$

Consider a bounded sequence $\left\{(\boldsymbol{u}_\ell, p_\ell, \boldsymbol{v}_\ell, q_\ell)\right\}_{\ell=1}^\infty$ in $X_r(\Theta_\lambda) \times X_{r'}(\Theta_\lambda)$. By weak compactness and the Rellich-Kondrachov theorem there exists a subsequence $\left\{(\boldsymbol{u}_{\ell_k}, p_{\ell_k}, \boldsymbol{v}_{\ell_k}, q_{\ell_k})\right\}_{k=1}^\infty$ which converges weakly to $(\boldsymbol{u}, p, \boldsymbol{v}, q)$, and $\left\{(\boldsymbol{u}_{\ell_k}, \boldsymbol{v}_{\ell_k})\right\}_{k=1}^\infty$ converges strongly in $L^r_{\text{loc}}(\Theta_\lambda) \times L^{r'}_{\text{loc}}(\Theta_\lambda)$ [AF03, Theorem 6.2].

Additionally, in view of bound (5.28c), the sequence $\left\{\boldsymbol{\vartheta}_\zeta(\boldsymbol{u}_{\ell_k}), \boldsymbol{\vartheta}_\zeta(\boldsymbol{v}_{\ell_k})\right\}_{k=1}^\infty$ is Cauchy in $L^r(\mathbb{R}^n)^{n\times n} \times L^{r'}(\mathbb{R}^n)^{n\times n}$, whence it converges in the strong sense as well. Finally, because $\boldsymbol{\varepsilon}(\cdot)$ is continuous, and $1 < r', r < \infty$, the sequence $\left\{\boldsymbol{\varepsilon}(\boldsymbol{u}_{\ell_k}), \boldsymbol{\varepsilon}(\boldsymbol{v}_{\ell_k})\right\}_{k=1}^\infty$ converges weakly in $L^r(\Theta_\lambda)^{n\times n} \times L^{r'}(\Theta_\lambda)^{n\times n}$.

Apply Lemma 5.5.13 to $\left\{A\left[\boldsymbol{u}_{\ell_k}, p_{\ell_k}, \boldsymbol{v}_\ell, q_{\ell_k}\right]\right\}_{k=1}^\infty$ with $X = L^r(\Theta_\lambda)^{n\times n}$ to conclude the proof. □

**Proposition 5.5.15** (Boundary localization)**.** *Let $\boldsymbol{x}_\lambda$ be in $\partial\Omega$. The boundary Stokes operators $\widehat{\mathcal{R}}^*_{\lambda,\zeta}\mathcal{S}_\Omega : X_r(\Omega) \to X_{r'}(\hat{\Theta}_\lambda)^*$, and $S_\Omega\widehat{\mathcal{R}}_{\lambda,\zeta} : X_r(\hat{\Theta}_\lambda) \to X_{r'}(\Omega)^*$ are continuous and equivalent to,*

$$\begin{aligned}\widehat{\mathcal{R}}^*_{\lambda,\zeta}\mathcal{S}_\Omega &= \left(\tilde{\mathcal{S}}_\lambda + \mathcal{C}_\lambda\right)\mathcal{R}_{\lambda,\zeta} + \widehat{\mathcal{P}}^*_\lambda\mathcal{K}_{\lambda,\zeta} \\ S_\Omega\widehat{\mathcal{R}}_{\lambda,\zeta} &= \mathcal{R}^*_{\lambda,\zeta}\left(\tilde{\mathcal{S}}_\lambda + \mathcal{C}_\lambda\right) + \mathcal{K}_{\lambda,\zeta}\widehat{\mathcal{P}}_\lambda\end{aligned} \tag{5.29a}$$

*where $\tilde{\mathcal{S}}_\lambda := \mathcal{S}_{\mathbb{R}^n_-} + \mathcal{B}_\lambda$ is a perturbation of the Stokes operator in $\mathbb{R}^n_-$,*

$$\mathcal{B}_\lambda(\hat{\boldsymbol{w}},\hat{\pi})(\hat{\boldsymbol{v}},\hat{q}) := \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{\mathbb{R}^n_-,\xi} - \left\langle \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}})\right\rangle_{\mathbb{R}^n_-}, \tag{5.29b}$$

*and*

$$\begin{aligned}\mathcal{C}_\lambda(\hat{\boldsymbol{w}},\hat{\pi})(\hat{\boldsymbol{v}},\hat{q}) := &\left\langle \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{\hat{U},\xi} \\ &- \left\langle \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{\hat{U},\xi} - \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{\hat{U},\xi},\end{aligned} \tag{5.29c}$$

$\xi = 1/\det\hat{\nabla}\hat{\Psi}$ *is a weight for the integral pairing,* $\hat{U} = \operatorname{supp}(\mathcal{I} - \hat{\Psi}_\lambda)$*, and $K_{\lambda,\zeta}$ is defined in* (5.28b)*.*

*Proof.* Using the same technique for the interior decomposition we obtain for every $(\boldsymbol{u},p) \in X_r(\Omega)$ and $(\hat{\boldsymbol{v}},\hat{q}) \in X_{r'}(\hat{\Theta}_\lambda)$

$$\mathcal{S}_\Omega(\boldsymbol{u},p)\widehat{\mathcal{R}}_{\lambda,\zeta}(\hat{\boldsymbol{v}},\hat{q}) = \left(\mathcal{S}_{\Omega_\lambda}(\zeta\boldsymbol{u},\zeta p) + \mathcal{K}_{\lambda,\zeta}(\boldsymbol{u},\boldsymbol{p})\right)\widehat{\mathcal{P}}_\lambda(\hat{\boldsymbol{v}},\hat{q}).$$

Let $\widehat{\zeta\boldsymbol{u}}$ be the Piola transform of $\zeta\boldsymbol{u}$. The divergence terms in $\mathcal{S}_{\Omega_\lambda}$ can be immediately simplified after using the Piola identity (5.24d), e.g.

$$\int_\Omega \zeta p \operatorname{div} v = \int_{\mathbb{R}^n_-} (\zeta p \operatorname{div} v) \circ \hat{\Psi} \det\hat{\nabla}\hat{\Psi} \tag{5.30}$$

$$= \int_{\mathbb{R}^n_-} \widehat{\zeta p}\, \widehat{\operatorname{div}}\, \hat{v}. \tag{5.31}$$

where $\widehat{\zeta p} := \zeta p \circ \hat{\Psi}$. Moreoever, by adding and subtracting $\left\langle \hat{\boldsymbol{\varepsilon}}(\widehat{\boldsymbol{\zeta u}}), \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-}$ we arrive at

$$\mathcal{S}_{\Omega_\lambda}(\zeta \boldsymbol{u}, \zeta p)(\boldsymbol{v}_\lambda, q_\lambda) = \mathcal{S}_{\mathbb{R}^n_-}\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u}, p)(\hat{\boldsymbol{v}}, \hat{q}) - \left\langle \hat{\boldsymbol{\varepsilon}}(\widehat{\zeta \boldsymbol{u}}), \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-}$$
$$+ \left\langle \boldsymbol{\varepsilon}(\zeta \boldsymbol{u}), \boldsymbol{\varepsilon}(\hat{\boldsymbol{P}}_\lambda \hat{\boldsymbol{v}}) \circ \Psi^{-1} \right\rangle_{\Omega_\lambda}.$$

To obtain the expressions for $\mathcal{B}_\lambda$ and $\mathcal{C}_\lambda$ it suffices to change variables, i.e mapping $\langle \cdot, \cdot \rangle_{\Omega_\lambda}$ into $\langle \cdot, \cdot \rangle_{\mathbb{R}^n_-}$, writing $\hat{\boldsymbol{w}} = \widehat{\zeta \boldsymbol{u}}$, $\zeta \boldsymbol{u}$ as $\hat{\boldsymbol{P}}_\lambda \hat{\boldsymbol{w}}$, and using the *symmetric gradient decomposition* of Proposition 5.4.11. The expression for $\mathcal{S}_\Omega \widehat{\mathcal{R}}_{\lambda,\zeta}$ follows analogously. $\square$

We delay the discussion about $\tilde{\mathcal{S}}_\lambda$ and $\mathcal{B}_\lambda$ to Theorem 5.5.18, where we show that $\tilde{\mathcal{S}}_\lambda$ has a bounded inverse. We now deal with $\mathcal{C}_\lambda$.

**Lemma 5.5.16** ($\mathcal{C}_\lambda$ is compact)**.** *Let $s' \le r \le s$ and $\boldsymbol{x}_\lambda$ be in $\partial\Omega$. The operator $\mathcal{C}_\lambda : X_r(\hat{\Theta}_\lambda) \to X_{r'}(\hat{\Theta}_\lambda)^*$ defined at* (5.29c) *is compact.*

*Proof.* The technique is the same as that employed for $\mathcal{K}_\lambda$ in Lemma 5.5.14. Let $\left\{(\hat{\boldsymbol{w}}_\ell, \hat{\boldsymbol{v}}_\ell)\right\}_{\ell=1}^\infty$ be a bounded sequence in $\widehat{V}_r(\hat{\Theta}_\lambda) \times \widehat{V}_{r'}(\hat{\Theta}_\lambda)$. By weak compactness and the Rellich-Kondrachov theorem there exists a subsequence $\left\{(\hat{\boldsymbol{w}}_{\ell_k}, \hat{\boldsymbol{v}}_{\ell_k})\right\}_{k=1}^\infty$ which converges weakly to $(\hat{\boldsymbol{w}}, \hat{\boldsymbol{v}})$, and strongly in $L^t_{\text{loc}}(\Theta_\lambda) \times L^{t'}_{\text{loc}}(\Theta_\lambda)$ for $1 \le t < r^*$ and $1 \le t' < (r')^*$ [AF03, Theorem 6.2]. If $r$ or $r'$ are greater than $n$, we respectively set $t$ or $t'$ to $\infty$.

Additionally, for $t^\bullet = r$ in the symmetric gradient bounds (5.23b) we have,

$$\left\| \boldsymbol{\vartheta}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}_{\ell_k}) \right\|_{L^r(\hat{\Theta}_\lambda)} \le C \left\| \hat{\boldsymbol{w}}_{\ell_k} \right\|_{L^{r^\circ}(\hat{U})}, \quad \left\| \boldsymbol{\vartheta}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{v}}_{\ell_k}) \right\|_{L^{r'}(\hat{\Theta}_\lambda)} \le C \left\| \hat{\boldsymbol{v}}_{\ell_k} \right\|_{L^{(r')^\circ}(\hat{U})},$$

where $1/r^\circ = 1/r - 1/s$ and $1/(r')^\circ = 1/r' - 1/s$. A simple calculation verifies that either $r^\circ < r^*$ or $r = s$ and $r^\circ = \infty$, whence $W^1_r(\Theta_\lambda)$ compactly embedds into $L^{r^\circ}_{\text{loc}}(\Theta_\lambda)$.

Since a similar computation holds for $(r')^\circ$ and $\hat{U} \subset\subset \hat{\Theta}_\lambda$, we conclude that the sequence $\left\{(\boldsymbol{\vartheta}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}_{\ell_k}), \boldsymbol{\vartheta}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{v}}_{\ell_k}))\right\}_{k=1}^{\infty}$ converges strongly in $L^r(\hat{\Theta}_\lambda) \times L^{r'}(\hat{\Theta}_\lambda)$. Finally, because $\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\cdot)$ is continuous and $1 < r', r < \infty$, the sequence $\left\{\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}_{\ell_k}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{w}}_{\ell_k})\right\}$ converges weakly in $L^r(\hat{U})^{n\times n} \times L^{r'}(\hat{U})^{n\times n}$.

Finally, apply Lemma 5.5.13 with $X = L^r(\hat{U})^{n\times n}$ to conclude the proof. $\square$

Now that we have obtained a local decomposition of $S_\Omega$ it is important to show that $\tilde{\mathcal{S}}_\lambda$ is invertible and enjoys the same smoothing property as $\mathcal{S}_{\mathbb{R}^n}$ and $\mathcal{S}_{\mathbb{R}^n_-}$. The strategy is to use an argument known as *von Neumann's perturbation of the identity*. We restate this result in a form that suits our needs.

**Lemma 5.5.17** (von Neumann)**.** *Consider two Banach spaces $X$ and $Y$, and two bounded linear operators $\mathcal{A}$ and $\mathcal{B}$ from $X$ to $Y$. Suppose $\mathcal{A}$ has a bounded inverse from $Y$ to $X$ and that*

$$\|\mathcal{B}x\|_Y \leq C\|\mathcal{A}x\|_Y \quad \forall x \in X,$$

*with a constant $0 < C < 1$. Then $\mathcal{A} + \mathcal{B} : X \to Y$ is bijective with a bounded inverse.*

*Proof.* See [Kat66, Chapter 4: Theorem 1.16], [GSS94, Lemma 3.1]. $\square$

**Theorem 5.5.18** (Well-posedness of $\tilde{\mathcal{S}}_\lambda$)**.** *Let $s' \leq r \leq s$, and $\boldsymbol{x}_\lambda$ be in $\overline{\Omega}$. If $\delta_\lambda < C(n, r, s, \partial\Omega)$, the (perturbed) Stokes problem*

$$\tilde{\mathcal{S}}_\lambda(\hat{\boldsymbol{w}}, \hat{\pi}) = \hat{\mathcal{F}}$$

*is well-posed from $X_r(\hat{\Theta}_\lambda)$ to $X_{r'}(\hat{\Theta}_\lambda)^*$, i.e. $\hat{\mathcal{F}}$ belongs to $X_{r'}(\hat{\Theta}_\lambda)^*$. Additionally, if $r < t \leq s$ and $\hat{\mathcal{F}}$ is in $X_{t'}(\hat{\Theta}_\lambda)^*$, then $(\hat{\boldsymbol{w}}, \hat{\pi})$ is in $X_t(\hat{\Theta}_\lambda)$.*

*Proof.* If $\boldsymbol{x}_\lambda$ is in $\Omega$ then $\tilde{\mathcal{S}}_{\Theta_\lambda} = \mathcal{S}_{\mathbb{R}^n}$, we can keep $\delta_\lambda = \operatorname{dist}(\boldsymbol{x}_\lambda, \partial\Omega)$ and apply Theorem 5.3.4. If $\boldsymbol{x}_\lambda$ is in $\partial\Omega$ then we must show that $\|\mathcal{B}_\lambda\|$ is proportional to $(\delta_\lambda)^\alpha$ for some $\alpha > 0$. To this end, we add and subtract $\left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-} + \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-}$ to obtain,

$$\begin{aligned} \mathcal{B}_\lambda(\hat{\boldsymbol{w}}, \hat{\pi})(\hat{\boldsymbol{v}}, \hat{q}) = & \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-, \xi-1} \\ & + \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}}) - \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-} \\ & + \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}) - \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-}, \end{aligned} \tag{5.32}$$

where $\xi = 1/\det \hat{\nabla}\hat{\Psi}$. We assume for the remainder of this proof that $|\hat{\boldsymbol{v}}|_{W^1_r(\mathbb{R}^n_-)} = 1$.

We can readily estimate the first term as,

$$\begin{aligned} \left| \left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-, \xi-1} \right| & \leq (1 + C) \left\| 1 - \det \hat{\nabla}\hat{\Psi} \right\|_{L^\infty(\hat{U})} |\hat{\boldsymbol{w}}|_{W^1_r(\hat{U})} \\ & \leq (1 + C')\delta_\lambda^{1-n/s} |\hat{\boldsymbol{w}}|_{W^1_r(\hat{U})} . \end{aligned}$$

where the last inequality followed from (5.17a) and the constant $C'$ depends on $n$, $r$, $s$, $U$, and $\partial\Omega$.

Next note that for any matrix $\boldsymbol{M}$,

$$\begin{aligned} [\![\boldsymbol{M}]\!]_{\hat{\boldsymbol{P}}} - \boldsymbol{M} = \hat{\boldsymbol{P}}\boldsymbol{M}\hat{\boldsymbol{P}}^{-1} - \boldsymbol{M} &= \hat{\nabla}\hat{\Psi}\boldsymbol{M}\hat{\nabla}\hat{\Psi}^{-1} - \boldsymbol{M} \\ &= \left(\hat{\nabla}\hat{\Psi} - \boldsymbol{I}\right)\boldsymbol{M}\hat{\nabla}\hat{\Psi}^{-1} + \boldsymbol{M}\left(\hat{\nabla}\hat{\Psi}^{-1} - \boldsymbol{I}\right) \\ &= \left(\hat{\nabla}\hat{\Psi} - \boldsymbol{I}\right)\boldsymbol{M}\hat{\nabla}\hat{\Psi}^{-1} + \boldsymbol{M}\hat{\nabla}\hat{\Psi}\left(\boldsymbol{I} - \hat{\nabla}\hat{\Psi}\right). \end{aligned}$$

Applying the above identity, in conjunction with (5.17b), the two remaining terms of (5.32) yields

$$\left|\left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{w}}), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}}) - \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \right\rangle_{\mathbb{R}^n_-}\right| \leq (1+C)\left|\mathcal{I} - \hat{\Psi}\right|_{W^1_\infty(\hat{U})} |\hat{\boldsymbol{w}}|_{W^1_r(\hat{U})}$$
$$\leq \left(1 + C'\right) \delta_\lambda^{1-n/s} |\hat{\boldsymbol{w}}|_{W^1_r(\hat{U})} .$$

Finally, collecting the estimates and using that $\mathcal{S}_{\mathbb{R}^n_-}$ is invertible, we obtain

$$\begin{aligned}
\left\|\mathcal{B}_\lambda(\hat{\boldsymbol{w}}, \hat{p})\right\|_{X_{r'}(\mathbb{R}^n_-)^*} &\leq C\delta_\lambda^{1-n/s} \left\|(\hat{\boldsymbol{w}}, \hat{p})\right\|_{X_r(\mathbb{R}^n_-)} \\
&= C\delta_\lambda^{1-n/s} \left\|\mathcal{S}^{-1}_{\mathbb{R}^n_-}\mathcal{S}_{\mathbb{R}^n_-}(\hat{\boldsymbol{w}}, \hat{p})\right\|_{X_r(\mathbb{R}^n_-)} \\
&\leq \delta_\lambda^{1-n/s} C \left\|\mathcal{S}^{-1}_{\mathbb{R}^n_-}\right\|_{\mathcal{L}(X_{r'}(\mathbb{R}^n_-)^*, X_r(\mathbb{R}^n_-))} \left\|\mathcal{S}_{\mathbb{R}^n_-}(\hat{\boldsymbol{w}}, \hat{p})\right\|_{X_{r'}(\mathbb{R}^n_-)^*} ,
\end{aligned}$$

where $C$ depends on $n$, $r$, $s$, $U$ and $\partial\Omega$. By choosing $\delta_\lambda$ small enough we satisfy the assumption of the stability Lemma 5.5.17 and conclude the first part of our theorem.

To prove the second part we follow Galdi-Simader-Sohr [GSS94, pg. 159]. $\square$

### 5.5.2 $\mathcal{S}_\Omega$ has finite index

We borrow ideas from the Domain Decomposition community in the finite element literature.

**Lemma 5.5.19** (Domain decomposition)**.** *Let $\Omega$ be a bounded $W^{2-1/s}_s$-domain in $\mathbb{R}^n$. There exists a finite open cover of $\overline{\Omega}$*

$$\overline{\Omega} \subseteq \bigcup_{i=1}^{k} B(\boldsymbol{x}_i, \delta_i/2), \tag{5.33}$$

*such that for each $\boldsymbol{x}_i$*

*(i) if $\boldsymbol{x}_i$ is in $\Omega$ then $B(\boldsymbol{x}_i, \delta_i/2) \cap \partial\Omega = \emptyset$.*

*(ii) if $\boldsymbol{x}_i$ is in $\partial\Omega$ then the associated local $W^2_s$ diffeomorphism $\Psi_i^{-1}$ maps $\Omega \cap B(\boldsymbol{x}, \delta_i)$ one-to-one and onto a neighborhood of $\mathbb{R}^n_-$, with $\partial\Omega \cap B(\boldsymbol{x}, \delta_i)$ mapped to $D(\mathbf{0}', \delta_i)$, i.e. it flattens the boundary of $\Omega$ near $\boldsymbol{x}_i$.*

*(iii) the (perturbed) Stokes operator $\tilde{\mathcal{S}}_i$ is invertible.*

*Furthermore, we associate with the covering*

*(a) a smooth partition of unity $\{\varphi_i\}_{i=1}^k$ of $\overline{\Omega}$, i.e. $\varphi_i$ in $C_c^\infty(B(\boldsymbol{x}_i, \delta_i/2))$ with $0 \leq \varphi_i \leq 1$, $\sum_{i=1}^k \varphi_i(\boldsymbol{x}) = 1$ for every $\boldsymbol{x} \in \overline{\Omega}$.*

*(b) smooth characteristic functions $\{\varrho_i\}_{i=1}^k$ of $B(\boldsymbol{x}_i, \delta_i/2)$ with support on $B(\boldsymbol{x}_i, \delta_i)$, i.e. $\varrho_i$ is in $C_c^\infty(B(\boldsymbol{x}_i, \delta_i))$ with $\varrho_i = 1$ on $B(\boldsymbol{x}_i, \delta_i/2)$.*

*Proof.* $\overline{\Omega}$ is compact so the trivial covering generated by the $\{B(\boldsymbol{x}_\lambda, \delta_\lambda/2)\}$, where $\delta_\lambda$ is computed in Theorem 5.5.18 has a finite sub-covering. Results (i)-(iii) follow immediately while (a)-(b) are standard fare. □

**Lemma 5.5.20** (Space decomposition of $X_r(\Omega)$)**.** *Let $s' \leq r \leq s$, $\mathcal{R}_{\varphi_i} := \mathcal{R}_{i,\varphi_i}$ and $\widehat{\mathcal{R}}_{\varrho_i} := \widehat{\mathcal{R}}_{i,\varrho_i}$ be localization operators. The following identities hold,*

$$\mathcal{I}_{X_r(\Omega)} = \sum_{i=1}^k \widehat{\mathcal{R}}_{\varphi_i}\mathcal{R}_{\varrho_i}, \quad \mathcal{I}_{X_r(\Omega)^*} = \sum_{i=1}^k \mathcal{R}^*_{\varphi_i}\widehat{\mathcal{R}}^*_{\varrho_i}.$$

*As a result we may decompose $X_r(\Omega)$ and $X_r(\Omega)^*$ as*

$$X_r(\Omega) = \sum_{i=1}^k \widehat{\mathcal{R}}_{\varphi_i} X_r(\hat{\Theta}_i), \quad X_r(\Omega)^* = \sum_{i=1}^k \mathcal{R}^*_{\varphi_i} X_r(\hat{\Theta}_i)^*.$$

*Proof.* Let $(\boldsymbol{u}, p)$ in $X_r(\Omega)$ be fixed but arbitrary. In view of $\mathcal{R}_{\varrho_i}$'s continuity it follows that the "vector" $(\mathcal{R}_{\varrho_1}(\boldsymbol{u}, p), \ldots, \mathcal{R}_{\varrho_k}(\boldsymbol{u}, p))$ belongs to the product space

$X_r(\Theta_1) \times \ldots \times X_r(\Theta_k)$. Moreover,

$$\sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varphi_i} \mathcal{R}_{\varrho_i}(\boldsymbol{u}, p) = \sum_{i=1}^{k} \varphi_i \widehat{\mathcal{P}}_\lambda \widehat{\mathcal{P}}_\lambda^{-1}(\varrho_i \boldsymbol{u}, \varrho_i q) = \sum_{i=1}^{k} \varphi_i \varrho_i(\boldsymbol{u}, q) = (\boldsymbol{u}, q).$$

This proves the identity relation $\mathcal{I}_{X_r(\Omega)}$ and that every element of $X_r(\Omega)$ has at least one decomposition in terms of $X_r(\Theta_i)$.

Conversely, let $((\hat{\boldsymbol{v}}_1, \hat{q}_1), \ldots, (\hat{\boldsymbol{v}}_k, \hat{q}_k))$ be in the product space. Because $\widehat{\mathcal{R}}_{\varphi_i}$ is continuous, $k$ is finite and $X_r(\Omega)$ is a normed vector space it follows that $(\boldsymbol{v}, q) := \sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varphi_i}(\hat{\boldsymbol{v}}_i, \hat{q}_i)$ belongs to $X_r(\Omega)$.

Let $\mathcal{F}$ in $X_r(\Omega)^*$ be fixed but arbitrary. The "vector" $(\widehat{\mathcal{R}}^*_{\varrho_1}\mathcal{F}, \ldots, \widehat{\mathcal{R}}^*_{\varrho_k}\mathcal{F})$ belongs to the product space $X_r(\Theta_1)^* \times \ldots \times X_r(\Theta_k)^*$ because $\widehat{\mathcal{R}}^*_{\varrho_i}$ is continuous. Moreover, for any $(\boldsymbol{u}, p)$ in $X_r(\Omega)$

$$\begin{aligned}\left\langle \sum_{i=1}^{k} \mathcal{R}^*_{\varphi_i} \widehat{\mathcal{R}}^*_{\varrho_i} \mathcal{F}, (\boldsymbol{u}, p) \right\rangle &= \sum_{i=1}^{k} \left\langle \mathcal{F}, \widehat{\mathcal{R}}_{\varrho_i} \mathcal{R}_{\varphi_i}(\boldsymbol{u}, p) \right\rangle = \sum_{i=1}^{k} \left\langle \mathcal{F}, \varrho_i \varphi_i(\boldsymbol{u}, p) \right\rangle \\ &= \left\langle \mathcal{F}, (\boldsymbol{u}, p) \right\rangle.\end{aligned}$$

This proves the identity relation $\mathcal{I}_{X_r(\Omega)^*}$ and that every element of $X_r(\Omega)^*$ has at least one decomposition in terms of $X_r(\Theta_i)^*$.

Conversely, let $(\hat{\mathcal{F}}_1, \ldots, \hat{\mathcal{F}}_k)$ be in the product space. Because $\mathcal{R}^*_{\varphi_i}$ is continuous, $k$ is finite and $X_r(\Omega)^*$ is a normed vector space it follows that $\mathcal{F} := \sum_{i=1}^{k} \mathcal{R}^*_{\varphi_i} \hat{\mathcal{F}}_i$ belongs to $X_r^*(\Omega)$. $\square$

**Theorem 5.5.21** (Pseudoinverse of $\mathcal{S}_\Omega$)**.** *Let $s' \leq r \leq s$, $n \geq 2$ and $\mathcal{S}_\Omega$ the Stokes operator defined in* (5.4a)*. The operator $\mathcal{S}_\Omega^\dagger : X_{r'}(\Omega)^* \to X_r(\Omega)$*

$$\mathcal{S}_\Omega^\dagger := \sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varrho_i} \tilde{\mathcal{S}}_i^{-1} \widehat{\mathcal{R}}^*_{\varphi_i}$$

*is a pseudoinverse of $\mathcal{S}_\Omega$.*

*Proof.* To simplify the notation take $\mathcal{C}_i = 0$ whenever $\boldsymbol{x}_i$ is not in $\partial\Omega$. Then

$$\begin{aligned}
\mathcal{S}_\Omega^\dagger \mathcal{S}_\Omega &= \sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varrho_i} \tilde{\mathcal{S}}_i^{-1} (\tilde{S}_i \mathcal{R}_{\varphi_i} + \mathcal{C}_i \mathcal{R}_{\varphi_i} + \widehat{\mathcal{P}}_i^* \mathcal{K}_i) \\
&= \sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varrho_i} \mathcal{R}_{\varphi_i} + \sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varrho_i} \tilde{\mathcal{S}}_i^{-1} (\mathcal{C}_i \mathcal{R}_{\varphi_i} + \widehat{\mathcal{P}}_i^* \mathcal{K}_i) \\
&= \mathcal{I}_{X_r(\Omega)} + \sum_{i=1}^{k} \widehat{\mathcal{R}}_{\varrho_i} \tilde{\mathcal{S}}_i^{-1} (\mathcal{C}_i \mathcal{R}_{\varphi_i} + \widehat{\mathcal{P}}_i^* \mathcal{K}_i).
\end{aligned}$$

Apply Lemma 5.5.5 to conclude that $\mathcal{S}_\Omega^\dagger \mathcal{S}_\Omega - \mathcal{I}_{X_r(\Omega)}$ is compact. Conversely,

$$\begin{aligned}
\mathcal{S}_\Omega \mathcal{S}_\Omega^\dagger &= \sum_{i=1}^{k} (\mathcal{R}_{\varrho_i}^* \tilde{\mathcal{S}}_i + \mathcal{R}_{\varrho_i}^* \mathcal{C}_i + \mathcal{K}_{\varrho_i} \widehat{\mathcal{P}}_i) \tilde{\mathcal{S}}_i^{-1} \widehat{\mathcal{R}}_{\varphi_i}^* \\
&= \sum_{i=1}^{k} \mathcal{R}_{\varrho_i}^* \mathcal{R}_{\varphi_i}^* + \sum_{i=1}^{k} (\mathcal{R}_{\varrho_i}^* \mathcal{C}_i + \mathcal{K}_{\varrho_i} \widehat{\mathcal{P}}_i) \tilde{\mathcal{S}}_i^{-1} \widehat{\mathcal{R}}_{\varphi_i}^* \\
&= \mathcal{I}_{X_r^*(\Omega)} + \sum_{i=1}^{k} (\mathcal{R}_{\varrho_i}^* \mathcal{C}_i + \mathcal{K}_{\varrho_i} \widehat{\mathcal{P}}_i) \tilde{\mathcal{S}}_i^{-1} \widehat{\mathcal{R}}_{\varphi_i}^*.
\end{aligned}$$

We conclude the proof by applying Lemma 5.5.5 once again. □

### 5.5.3 $\mathcal{S}_\Omega$ and $\mathcal{S}_\Omega^*$ are injective

**Proposition 5.5.22.** *Let $2 \le r \le s$. The Stokes operator $\mathcal{S}_\Omega : X_r(\Omega) \to X_{r'}^*(\Omega)$ is injective.*

*Proof.* We already know from Theorem 5.2.5 that the only solution to the homogeneous problem in $X_2(\Omega)$ is the zero solution. To obtain the result for $r > 2$ we use that $\Omega$ is bounded so that $X_r(\Omega) \hookrightarrow X_2(\Omega)$, and as a result any homogeneous solution in $X_r(\Omega)$ is necessarily a homogeneous solution in $X_2(\Omega)$. We conclude that $\mathcal{S}_\Omega$ is injective for $2 \le r \le s$. □

To prove that $\mathcal{S}_\Omega$ is injective for $s' \le r < 2$ we follow a modified version of the induction argument by Galdi-Simader-Sohr [GSS94, pg. 159]. Leveraging the boundedness of $\Omega$ it is sufficient to show uniqueness for the case $r = s'$. We will show in a finite number of steps that a homogeneous solution in $X_{s'}(\Omega)$ is in fact in $X_t(\Omega)$ for some $t \geq 2$.

**Remark 5.5.23.** It appears that the induction argument below is rooted on the work introduced by Moser in the context of elliptic differential equations [Mos60][GT01, Section 8.5].

**Definition 5.5.24** (Smoothing sequence)**.** Let $s > n$ and $t_{-1} = r = s'$. The smoothing sequence conformal to $\mathcal{S}_\Omega$ is given by

$$
\begin{aligned}
\frac{1}{t_0} &:= 1 - \frac{2}{s+n} \\
\frac{1}{t_m} &:= \frac{1}{t_{m-1}} + \frac{1}{s} - \frac{1}{n} \text{ for } m = 1, \ldots, M
\end{aligned}
$$

where $M = \left\lceil \left(\frac{1}{n} - \frac{1}{s}\right)^{-1} \left(\frac{1}{2} - \frac{2}{s+n}\right) \right\rceil$ guarantees that $t_M \geq 2$.

We ask the reader to check that $t_m$ is monotone increasing, and that for $m \geq 1$ we have $t'_m < n$, where $t'_m$ is the Hölder conjugate of $t_m$.

**Lemma 5.5.25** (Sobolev embedding)**.** *Let $\Theta$ be a bounded Lipschitz domain of $\mathbb{R}^n$. The following holds,*

$$
W^1_{t_{m-1}}(\Theta) \hookrightarrow L^{t_m}(\Theta), \quad W^1_{t'_m}(\Theta) \hookrightarrow L^{t'_{m-1}}(\Theta)
$$

*or equivalently, for every $u$ in $W^1_{t_{m-1}}(\Theta)$ and $v$ in $W^1_{t'_m}(\Theta)$,*

$$
\begin{aligned}
\|u\|_{L^{t_m}(\Theta)} &\leq C_{m,n,s,\Theta} \|u\|_{W^1_{t_{m-1}}(\Theta)} \\
\|v\|_{L^{t'_{m-1}}(\Theta)} &\leq C_{m,n,s,\Theta} \|v\|_{W^1_{t_m}(\Theta)} .
\end{aligned} \tag{5.34a}
$$

*Proof.* We split the proof into two cases:

- Case $m = 0$: Since $t_{-1} = s' < 2 \leq n$, $W^1_{t_{-1}}(\Theta)$ is continuously embedded in $L^{t_0}(\Theta)$ if $t_0 \leq t^*_{-1}$, i.e $1/t_0 - 1/t^*_{-1} \geq 0$

$$1 - \frac{2}{s+n} - \left(\frac{1}{t_{-1}} - \frac{1}{n}\right) = \frac{1}{s} + \frac{1}{n} - \frac{2}{s+n} = \frac{n^2+s^2}{sn(s+n)} > 0.$$

  Since $t'_{-1} = s$, and $t'_0 = (s+n)/2 > n$, we have $W^1_{t'_0}(\Theta) \hookrightarrow L^s(\Theta)$.

- Case $m \geq 1$: Since $t_{m-1} < 2 \leq n$, it suffices to verify $t_m \leq t^*_{m-1}$,

$$\frac{1}{t_m} - \frac{1}{t^*_{m-1}} = \frac{1}{t_{m-1}} + \frac{1}{s} - \frac{1}{n} - \left(\frac{1}{t_{m-1}} - \frac{1}{n}\right) = \frac{1}{s} \geq 0.$$

  A similar computation shows that $t'_{m-1} \leq (t'_m)^*$.

This completes the proof. □

**Lemma 5.5.26** (Interior regularity)**.** *Let $\boldsymbol{x}_\lambda$ be in $\Omega$. If $(\boldsymbol{u}, p)$ in $X_{t_{m-1}}(\Omega)$ is a homogeneous solution of* (5.4a)*, then $\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u}, p)$ belongs to $X_{t_m}(\hat{\Theta}_\lambda)$.*

*Proof.* Because $(\boldsymbol{u}, p)$ is a homogeneous solution we have that

$$\tilde{\mathcal{S}}_\lambda \mathcal{R}_{\lambda,\zeta}(\boldsymbol{u}, p) = -\widehat{\mathcal{P}}^*_\lambda \mathcal{K}_{\lambda,\zeta}(\boldsymbol{u}, p).$$

The strategy is to show that the right-hand-side is in $X_{t'_m}(\hat{\Theta}_\lambda)^*$, and use the smoothing property of the Stokes operator in Theorem 5.5.18. Since $\widehat{\mathcal{P}}$ is an isomorphism when $s' \leq t \leq s$, it suffices to check the smoothness of $\mathcal{K}_{\lambda,\zeta}(\boldsymbol{u}, p)$.

Let $(\boldsymbol{v}, q)$ be in $X_{t'_m}(\Theta_\lambda) = \widehat{\mathcal{P}}_\lambda(X_{t'_m}(\hat{\Theta}_\lambda))$. Invoking Hölder's inequality and

(5.28c) one obtains,

$$\begin{aligned}
\left|\langle p, \nabla\zeta\cdot\boldsymbol{v}\rangle_{\Omega_\lambda}\right| &\le C\|p\|_{L^{t_{m-1}}(\Omega_\lambda)}\|\boldsymbol{v}\|_{L^{t'_{m-1}}(\Omega_\lambda)}\,,\\
\left|\langle \nabla\zeta\cdot\boldsymbol{u}, \hat{q}\rangle_{\Omega_\lambda}\right| &\le C\|\boldsymbol{u}\|_{L^{t_m}(\Omega_\lambda)}\|\hat{q}\|_{L^{t'_m}(\Omega_\lambda)}\,,\\
\left|\left\langle \boldsymbol{\vartheta}_\zeta(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\right\rangle_{\Omega_\lambda}\right| &\le C\|\boldsymbol{u}\|_{L^{t_m}(\Omega_\lambda)}\left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_{L^{t'_m}(\Omega_\lambda)}\,,\\
\left|\left\langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\vartheta}_\zeta(\boldsymbol{v})\right\rangle_{\Omega_\lambda}\right| &\le C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t_{m-1}}(\Omega_\lambda)}\|\boldsymbol{v}\|_{L^{t'_{m-1}}(\Omega_\lambda)}\,.
\end{aligned}$$

Using Lemma 5.5.25 and $\boldsymbol{v}$ in $[\cdot]_1$, we have $\|\boldsymbol{v}\|_{L^{t'_{m-1}}(\Omega_\lambda)} \le C|\boldsymbol{v}|_{W^1_{t_m}(\Omega_\lambda)}$. Similarly, using Lemma 5.5.25, $\boldsymbol{u}$ in $V_{t_{m-1}}(\Omega)$, together with Korn's inequality gives $\|\boldsymbol{u}\|_{L^{t_m}(\Omega_\lambda)} \le C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t_{m-1}}(\Omega)}$. Therefore, $\mathcal{K}_{\lambda,\zeta}(\boldsymbol{u},p)$ belongs to $X_{t'_m}(\Theta_\lambda)^*$. □

**Lemma 5.5.27** (Boundary regularity)**.** *Let $\boldsymbol{x}_\lambda$ be in $\partial\Omega$. If $(\boldsymbol{u},p)$ in $X_{t_{m-1}}(\Omega)$ is a homogeneous solution of* (5.4a) *then $\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p)$ belongs to $X_{t_m}(\hat{\Theta}_\lambda)$.*

*Proof.* Because $(\boldsymbol{u},p)$ is a homogeneous solution we have

$$\tilde{\mathcal{S}}_\lambda\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p) = -\mathcal{C}_\lambda\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p) - \widehat{\mathcal{P}}^*_\lambda\mathcal{K}_{\lambda,\zeta}(\boldsymbol{u},p).$$

The strategy is the same as before: we show the right-hand-side is in $X_{t'_m}(\hat{\Theta}_\lambda)^*$ and use the smoothing property of $\tilde{\mathcal{S}}_\lambda$ from Theorem 5.5.18. In particular, the regularity for $K_{\lambda,\zeta}(\boldsymbol{u},p)$ follows from the exact same argument used in the interior regularity lemma. To show the additional regularity for $\mathcal{C}_\lambda\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p)$ we rely mostly on Proposition 5.4.11.

Let $(\hat{\boldsymbol{v}},\hat{q})$ be in $X_{t'_m}(\hat{\Theta}_\lambda)$ and set $(\hat{\boldsymbol{u}}_\lambda,\hat{p}_\lambda) = \mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p)$.

- Case $m \ge 1$: Since $t_{m-1} < 2 \le n$ we have $\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)$ in $L^{t^\bullet_{m-1}}(U)$ with $1/t^\bullet_{m-1} = 1/s + 1/t^*_{m-1} = 1/t_{m-1} + 1/s - 1/n = 1/t_m$, while $\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})$ belongs to $L^{(t'_m)^\bullet}(U)$

with $(t'_m)^\bullet = t'_{m-1} > t'_m$; recall that $\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}$ is defined in (5.23). Therefore, Hölder's inequality with $1/t_m + 1/t'_m = 1$ yields

$$\begin{aligned}\left|\left\langle \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{U,\xi}\right| &\le C\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)\right\|_{L^{t_m}(U)}\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\|_{L^{t'_m}(U)} \\ &\le C\|\hat{\boldsymbol{u}}_\lambda\|_{W^1_{t_{m-1}}(U)}\|\hat{\boldsymbol{v}}\|_{W^1_{t'_m}(U)} \\ &\le C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t_{m-1}}(\Omega)}|\hat{\boldsymbol{v}}|_{W^1_{t'_m}(\hat{\Theta})},\end{aligned}$$

where the last inequality followed by a change of variables, Korn's inequality 5.2.1, and that $\hat{\boldsymbol{v}}$ is in $[\cdot]_1$.

For the second term in $\mathcal{C}_\lambda$ note that $\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})$ is in $L^{t'_m}(\hat{\Theta}_\lambda)^{n\times n}$ and follow the same steps above to obtain

$$\begin{aligned}\left|\left\langle \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{U,\xi}\right| &\le C\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)\right\|_{L^{t_m}(U)}\left\|\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\|_{L^{t'_m}(U)} \\ &\le C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t_{m-1}}(\Omega)}|\hat{\boldsymbol{v}}|_{W^1_{t'_m}(\hat{\Theta})}.\end{aligned}$$

Finally, using that $\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)$ is in $L^{t_{m-1}}(\hat{\Theta}_\lambda)$ gives

$$\begin{aligned}\left|\left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{U,\xi}\right| &\le C\left\|\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)\right\|_{L^{t_{m-1}}(U)}\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\|_{L^{t'_{m-1}}(U)} \\ &\le C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t_{m-1}}(\Omega)}|\hat{\boldsymbol{v}}|_{W^1_{t'_m}(\hat{\Theta}_\lambda)},\end{aligned}$$

which concludes this case.

- Case $m = 0$: Since $t_{-1} = s' < n$ we have $\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)$ is in $L^{t^\bullet_{-1}}(U)$ with $1/t^\bullet_{-1} = 1/n'$, while $\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})$ is in $L^s(U)$, $s > n$. Therefore, Hölder's inequality with

$1/n + 1/n' = 1$ yields

$$\left|\left\langle \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{U,\xi}\right| \leq C\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)\right\|_{L^{n'}(U)}\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\|_{L^{n}(U)}$$
$$\leq C\|\hat{\boldsymbol{u}}_\lambda\|_{W^1_{t_{-1}}(U)}\|\hat{\boldsymbol{v}}\|_{W^1_{t_0'}(U)}$$
$$\leq C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t-1}(\Omega)}|\hat{\boldsymbol{v}}|_{W^1_{t_0'}(\hat{\Theta}_\lambda)}\,.$$

To estimate the second term in the definition 5.29c of $\mathcal{C}_\lambda$, note that $\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})$ is in $L^{t_0'}(\Theta_\lambda)^{n\times n}$ where $t_0 = (s+n)/2 > n$ and follow the same steps above to obtain

$$\left|\left\langle \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda), \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{U,\xi}\right| \leq C\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)\right\|_{L^{n'}(U)}\left\|\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\|_{L^{n}(U)}$$
$$\leq C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t-1}(\Omega)}|\hat{\boldsymbol{v}}|_{W^1_{t_0'}(\hat{\Theta}_\lambda)}\,.$$

Finally, using that $\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)$ is in $L^{s'}(\hat{\Theta}_\lambda)$ gives

$$\left|\left\langle \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\rangle_{U,\xi}\right| \leq C\left\|\hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{u}}_\lambda)\right\|_{L^{s'}(U)}\left\|\hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}_\lambda}(\hat{\boldsymbol{v}})\right\|_{L^{s}(U)}$$
$$\leq C\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})\right\|_{L^{t-1}(\Omega)}|\hat{\boldsymbol{v}}|_{W^1_{t_0'}(\hat{\Theta}_\lambda)}\,.$$

which concludes the base case.

Therefore, $\mathcal{C}_\lambda\mathcal{R}_{\lambda,\zeta}(\boldsymbol{u},p)$ belongs to $X_{t_m'}(\hat{\Theta}_\lambda)^*$ which concludes our proof. □

**Lemma 5.5.28** (Global regularity)**.** *If* $(\boldsymbol{u},p)$ *in* $X_{t_{m-1}}(\Omega)$ *is a homogeneous solution of* (5.4a) *then* $(\boldsymbol{u},p)$ *belongs to* $X_{t_m}(\Omega)$.

*Proof.* In view of the interior and boundary regularity results we have that the "vector" $(\mathcal{R}_{\varrho_1}(\boldsymbol{u},p),\ldots,\mathcal{R}_{\varrho_k}(\boldsymbol{u},p))$ belongs to $X_{t_m}(\Theta_1)\times\ldots\times X_{t_m}(\Theta_k)$. The space decomposition property gives $(\boldsymbol{u},p) = \sum_{i=0}^{k}\widehat{\mathcal{R}}_{\varphi_i}\mathcal{R}_{\varrho_i}(\boldsymbol{u},p) \in X_{t_m}(\Omega)$, as asserted. □

**Proposition 5.5.29** (Injectivity of $\mathcal{S}_{\Omega,r}$)**.** *Let $s' \le r < 2$. The Stokes operator $\mathcal{S}_{\Omega,r} : X_r(\Omega) \to X_{r'}(\Omega)$ is injective.*

*Proof.* As before, assume $r = s'$ and suppose $(\boldsymbol{u}, p)$ is a homogeneous solution in $X_{s'}(\Omega)$. By applying the global regularity result $M$ times to obtain that

$$(\boldsymbol{u}, p) \in X_{t_{-1}=s'}(\Omega) \cap \ldots \cap X_{t_M}(\Omega),$$

where $t_M \ge 2$. Since $\mathcal{S}_\Omega$ is injective for any $t \ge 2$ we conclude from Proposition 5.5.22 that $(\boldsymbol{u}, p)$ is the zero solution. □

**Proposition 5.5.30** (Injectivity of $\mathcal{S}^*_{\Omega,r}$)**.** *Let $s' \le r \le s$. The dual Stokes operator $\mathcal{S}^*_{\Omega,r}$ is injective.*

*Proof.* Since $X_{r'}$ is reflexive we have that $\mathcal{S}^*_{\Omega,r}$ maps $X_{r'}(\Omega)$ into $X_r(\Omega)^*$. If $(\boldsymbol{v}, q)$ in $X_{r'}(\Omega)$ and $\mathcal{S}^*_{\Omega,r}(\boldsymbol{v}, q) = 0$, then using the definition of the dual operator we have for every $(\boldsymbol{u}, p)$ in $X_r(\Omega)$

$$\begin{aligned} 0 &= \Big\langle (\boldsymbol{u}, p), \mathcal{S}^*_{\Omega,r}(\boldsymbol{v}, q) \Big\rangle_{X_r(\Omega), X_r(\Omega)^*} = \mathcal{S}_{\Omega,r}(\boldsymbol{u}, p)(\boldsymbol{v}, q) \\ &= \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p \operatorname{div} \boldsymbol{v} + q \operatorname{div} \boldsymbol{u} \\ &= \big\langle \mathcal{S}_{\Omega,r'}(\boldsymbol{v}, -q), (\boldsymbol{u}, -p) \big\rangle_{X_r(\Omega)^*, X_r(\Omega)} . \end{aligned}$$

Since $(\boldsymbol{u}, p)$ was arbitrary and $\mathcal{S}_{\Omega,r'}$ was already shown to be injective, we have that $(\boldsymbol{v}, -q) = (\boldsymbol{0}, 0)$, i.e. $\mathcal{S}^*_{\Omega,r}$ is injective. □

**Remark 5.5.31** (Reflexivity)**.** The above proposition is the first place where we used the reflexivity of our function spaces.

## 5.6 The non-homogeneous case $\boldsymbol{u}\cdot\boldsymbol{\nu}\neq\phi$

We present a framework on how to treat the nonhomogeneous essential boundary condition (5.1b). It relies on the standard practice of lifting the data inside the domain. By the principle of superposition it suffices to study the case when $\phi$ is the only non-trivial data. Note that the compatibility condition (5.5) becomes $\int_\Omega g - \int_{\partial\Omega}\phi = 0$, whence $\int_{\partial\Omega}\phi = 0$.

Given $\phi \in W_r^{1/r'}(\partial\Omega)$ and $\boldsymbol{\nu}\in W_s^{1/s'}(\partial\Omega)$, there exists a vector valued function $\boldsymbol{\varphi}\in W_r^1(\Omega)$ which coincides, in the trace sense, with $\phi\boldsymbol{\nu}$ on the boundary [Tar07, Lemma 13.3], and

$$\|\boldsymbol{\varphi}\|_{W_r^1(\Omega)} \leq C_{\Omega,n,r,s}\|\phi\|_{W_r^{1/r'}(\partial\Omega)}\|\boldsymbol{\nu}\|_{W_s^{1/s'}(\partial\Omega)}\,; \tag{5.35}$$

see [GR86, Corollary 1.1][AF03, Theorem 7.39].

**Corollary 5.6.1.** *The pair* $(\boldsymbol{u},p)$ *is a solution to* (5.1a), (5.1b)*, with only* $\phi$ *non-trivial, if and only if* $(\boldsymbol{w},p)=(\boldsymbol{u}-\boldsymbol{\varphi},p)$ *is a solution to* (5.1a), (5.1b)*, with*

$$\boldsymbol{f}=\operatorname{div}2\eta\boldsymbol{\varepsilon}(\boldsymbol{\varphi}),\quad g=-\operatorname{div}\boldsymbol{\varphi},\quad \boldsymbol{\psi}=-\boldsymbol{T}^\top 2\eta\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\boldsymbol{\nu},\quad \boldsymbol{w}\cdot\boldsymbol{\nu}=0.$$

*In particular,* $(\boldsymbol{w},p)$ *in* $X_r(\Omega)$ *satisfies*

$$\mathcal{S}_\Omega(\boldsymbol{w},p)(\boldsymbol{v},q)=\eta\left\langle\boldsymbol{\varepsilon}(\boldsymbol{\varphi}),\boldsymbol{\varepsilon}(\boldsymbol{v})\right\rangle_\Omega-\langle\operatorname{div}\boldsymbol{\varphi},q\rangle_\Omega\quad\forall(\boldsymbol{v},q)\in X_{r'}(\Omega),$$

*and its norm is controlled by the data* $\phi$*, namely*

$$\big\|(\boldsymbol{w},p)\big\|_{X_r(\Omega)}\leq C_{\Omega,n,r,\eta}\|\phi\|_{W_r^{1-1/r}(\partial\Omega)}\,.$$

*Proof.* The expressions for $\boldsymbol{f}$, $g$, $\boldsymbol{\psi}$, and $\boldsymbol{w}\cdot\boldsymbol{\nu}$ are straightforward to obtain, so we skip their derivation. The variational form follows after integrating by parts and recalling that the test functions $\boldsymbol{v}$ are tangential on the boundary. This causes the boundary integral to vanish. Finally, the continuity estimate is a direct application of the results when $\boldsymbol{w} \cdot \boldsymbol{\nu} = 0$ as well as (5.35). $\square$

**Remark 5.6.2.** Another way to lift the data $\phi$ in $W_r^{1-1/r}(\partial\Omega)$ is by solving the Neumann problem

$$-\Delta\varphi + \varphi = 0 \quad \text{in } \Omega$$

$$\partial_{\boldsymbol{\nu}}\varphi = \phi \quad \text{on } \partial\Omega$$

in $W_r^2(\Omega)$. Then, the pair $(\boldsymbol{u}, p)$ is a solution (5.1a), (5.1b), with only $\phi$ non-trivial, if an only if $(\boldsymbol{w}, p) = (\boldsymbol{u} - \nabla\varphi, p)$ is a solution to (5.1a), (5.1b), with

$$\boldsymbol{f} = \operatorname{div} 2\eta\boldsymbol{\varepsilon}(\nabla\varphi), \quad g = -\operatorname{div}\nabla\varphi, \quad \boldsymbol{\psi} = -\boldsymbol{T}^\top 2\eta\boldsymbol{\varepsilon}(\nabla\varphi)\boldsymbol{\nu}, \quad \boldsymbol{w}\cdot\boldsymbol{\nu} = 0.$$

and solving the Stokes problem satisfied by $(\boldsymbol{w}, p) = (\boldsymbol{u} - \nabla\varphi, p)$.

The existence of a strong solution to the inhomogeneous Neumann problem is controversial when the domain $\Omega$ is Lipschitz continuous [Zan00] and in some cases even if it is $C^1$ [JK89]. On the other hand, it is well-known that for $C^{1,1}$ domains a strong solution always exists [AS11, Theorem 4.4: Step 2]. In this aspect, we see our Sobolev domain choice is (nearly) optimal.

## 5.7 The Navier boundary condition

The goal of this section is to consider the Stokes problem (5.1) with the Navier boundary condition, i.e.

$$\boldsymbol{u}\cdot\boldsymbol{\nu} = 0, \quad \beta\boldsymbol{T}\boldsymbol{u} + \boldsymbol{T}^\top\boldsymbol{\sigma}(\boldsymbol{u}, p)\,\boldsymbol{\nu} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{5.36}$$

with $\beta > 0$. The strategy is to notice that the term $\boldsymbol{T}\boldsymbol{u}$ is a compact perturbation of the pure-slip problem, and in view of Lemma 5.5.6 we have that the index of this new problem is zero. Therefore, its well-posedness will be governed only by its finite dimensional null-space. We structure the rest of this section as follows: first, we state mild integrability assumptions on the parameter $\beta$ which still guarantee compactness of the added term; second, we show that the perturbed problem is injective by constructing a smoothing sequence as was done in §5.5.3; finally, we state the main result as another consequence of Corrollary 5.5.8.

**Lemma 5.7.1** ($\mathcal{T}_{\partial\Omega}$ is compact)**.** *Let $\mathcal{T}_{\partial\Omega}(\boldsymbol{u})(\boldsymbol{v}) := \int_{\partial\Omega} \beta \boldsymbol{T}\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{v}$. If $\beta$ belongs to $L^q(\partial\Omega)$ with $q > n-1$, then for every $s' \leq r \leq s$, the operator $\mathcal{T} : W^1_r(\Omega)^n \to \left(W^1_{r'}(\Omega)^n\right)^*$ is compact. Moreover, if $s' \leq r < n$ then $\mathcal{T}$ is continuous from $W^1_r(\Omega)^n \to \left(W^1_{t'}(\Omega)^n\right)^*$ where $\frac{1}{t'} = 1 - \frac{1}{r} + \frac{1}{n}\left(1 - (n-1)\frac{1}{q}\right)$, that is*

$$\left|\mathcal{T}(\boldsymbol{u})(\boldsymbol{v})\right| \leq C\|\beta\|_{L^q(\partial\Omega)}\|\boldsymbol{u}\|_{W^1_r(\Omega)}\|\boldsymbol{v}\|_{W^1_{t'}(\Omega)}\,, \tag{5.37}$$

*where $C$ depends on $n, r, s$ and $\Omega$.*

*Proof.* To prove that $\mathcal{T}$ is compact it suffices to use that $\boldsymbol{T}$ is in $L^\infty$ for $s > n$, and then show that the product $L^q(\partial\Omega) \cdot W^1_r(\Omega) \cdot W^1_{r'}(\Omega)$ compactly embeds in $L^1(\partial\Omega)$. The case $q = \infty$ is trivial, so we skip it. Otherwise, the embedding can be checked through Sobolev numbers, i.e.

$$\mathrm{sob}(L^q(\partial\Omega)) + \mathrm{sob}(W^1_r(\Omega)) + \mathrm{sob}(W^1_{r'}(\Omega)) - \mathrm{sob}(L^1(\partial\Omega)) > 0,$$

which yields the condition,

$$\left(-(n-1)\frac{1}{q}\right) + \left(1 - \frac{n}{r}\right) + \left(1 - \frac{n}{r'}\right) - \left(-(n-1)\frac{1}{1}\right) > 0,$$

as long as $q > n-1$.

By replacing $r'$ with $t'$ and solving for equality, i.e.

$$\mathrm{sob}(W^1_{t'}(\Omega)) = \mathrm{sob}(L^1(\partial\Omega)) - \mathrm{sob}(L^q(\partial\Omega)) - \mathrm{sob}(W^1_r(\Omega))$$

yields the relation

$$\begin{aligned}\left(1 - \frac{n}{t'}\right) &= -(n-1) + (n-1)\frac{1}{q} - 1 + \frac{n}{r} \\ &= -n + \frac{n}{r} + (n-1)\frac{1}{q}.\end{aligned}$$

The expression for $1/t'$ follows by algebraic manipulation. The precise expression for $1/t'$ is indirectly used to define the smoothing sequence below 5.7.2.

An alternate proof follows by the boundary trace embedding and Rellich-Kondrachov theorems [AF03, Theorems 5.36 and 6.3]. They yield the same results.

$\square$

**Definition 5.7.2** (Smoothing sequence)**.** Let $s > n$, $q > n-1$ and $t_0 = r = s'$. The smoothing sequence conformal to $\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}$ is given by

$$\frac{1}{t_m} := \frac{1}{t_{m-1}} - \frac{1}{n}\left(1 - (n-1)\frac{1}{q}\right) \text{ for } m = 1, \ldots, M$$

where $M \geq n\left(1-(n-1)1/q\right)^{-1}\left\lceil \frac{1}{t_0} - \frac{1}{2}\right\rceil$ guarantees that $t_M \geq 2$.

**Lemma 5.7.3** ($\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}$ is injective)**.** *Let $s' \leq r \leq s$ and suppose $\beta$ is strictly positive in a set $\Gamma \subset \partial\Omega$ of positive measure. If $(\boldsymbol{u}, p) \in X_r$ is a homogeneous solution to the Stokes problem with Navier slip boundary conditions, then $(\boldsymbol{u}, p) = (\mathbf{0}, 0)$.*

*Proof.* Since $\Omega$ is bounded, the case $2 < r \leq s$ follows from the embedding $X_r(\Omega) \hookrightarrow$

$X_2(\Omega)$. The Hilbert space case $r = 2$ follows from the coercivity estimate

$$0 = \mathcal{S}(\boldsymbol{u}, p)(\boldsymbol{u}, p) + \mathcal{T}_{\partial\Omega}(\boldsymbol{u})(\boldsymbol{u}) = \eta \big\| \boldsymbol{\varepsilon}(\boldsymbol{u}) \big\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \beta |\boldsymbol{T}\boldsymbol{u}|^2$$
$$\geq \eta \big\| \boldsymbol{\varepsilon}(\boldsymbol{u}) \big\|^2_{L^2(\Omega)} + \beta_0 \| \boldsymbol{T}\boldsymbol{u} \|^2_{L^2(\Gamma)} .$$

Since $\big\| \boldsymbol{\varepsilon}(\boldsymbol{u}) \big\|_{L^2(\Omega)} = 0$ we conclude that $\boldsymbol{u}$ is an element of $Z$, i.e. it is an affine vector field of the form $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$. By using that $\boldsymbol{T}\boldsymbol{u} = 0$ a.e. on $\Gamma$, we conclude that $\boldsymbol{u} = \boldsymbol{0}$. The uniqueness of $p$, up to a constant, follows as in §5.2.

To obtain injectivity for $s' \leq r \leq 2$ we suppose that $(\boldsymbol{u}, p) \in X_{t_0}(\Omega)$ is a homogeneous solution to $\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}$ with $t_0 = r = s'$. We then use the smoothing property of the Stokes operator to obtain in $M$ steps that

$$(\boldsymbol{u}, p) \in X_{t_0}(\Omega) \cap \ldots \cap \big( X_{t_M}(\Omega) \subset X_2(\Omega) \big) .$$

where $t_m$ is the sequence from Definition 5.7.2, whence $(\boldsymbol{u}, p) = (\boldsymbol{0}, 0)$ as desired. $\square$

**Remark 5.7.4.** The above induction argument is the same one used in Proposition 5.5.29.

**Remark 5.7.5.** If the set $\Gamma = \partial\Omega$ then we may take the set $Z = \emptyset$, i.e. the velocity field $\boldsymbol{u}$ is unique and the pressure is unique up to a constant, [BdV04].

**Theorem 5.7.6** (Slip with friction)**.** *Let $\Omega$ be a bounded $W^{2-1/s}_s$-domain and $\beta$ satisfy the assumptions of Lemmas 5.7.1 and 5.7.3. For every $\mathcal{F}$ in $X_{r'}(\Omega)^*$ there exists a unique $(\boldsymbol{u}, p)$ in $X_r(\Omega)$ such that*

$$\mathcal{S}_\Omega(\boldsymbol{u}, p)(\boldsymbol{v}, q) + \mathcal{T}_{\partial\Omega}(\boldsymbol{u})(\boldsymbol{v}) = \mathcal{F}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in X_{r'}(\Omega),$$

*and*

$$\left\|(\boldsymbol{u}, p)\right\|_{X_r(\Omega)} \leq C_{\Omega,\eta,n,r} \|\mathcal{F}\|_{X_{r'}(\Omega)^*}.$$

*Moreover, if $r < t \leq s$ and $\mathcal{F}$ belongs to $X_{t'}(\Omega)^*$, then $(\boldsymbol{u}, p)$ is in $X_t(\Omega)$.*

*Proof.* The proof relies only the boundedness of $\Omega$ and the compactness of $\mathcal{T}_{\partial\Omega}$. We start by noting that $\mathcal{S}_\Omega^{-1}$ is a pseudo-inverse of $\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}$, i.e.

$$\mathcal{S}_\Omega^{-1}(\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}) = \mathcal{I}_{X_r} + \mathcal{S}_\Omega^{-1}\mathcal{T}_{\partial\Omega}, \quad (\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega})\,\mathcal{S}_\Omega^{-1} = \mathcal{I}_{X_{r'}^*} + \mathcal{T}_{\partial\Omega}\mathcal{S}_\Omega^{-1}.$$

This follows from Lemma 5.5.5, i.e. the product of a bounded operator and a compact one is compact, and the definition of the pseudo-inverse. Moreover, in view of Theorem 5.5.4 we have that

$$\operatorname{ind}(\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}) = -\operatorname{ind}\mathcal{S}_\Omega^{-1} = 0.$$

Using Lemma 5.7.3 and the definition of the index we have that $\operatorname{codim} R_{\mathcal{S}_\Omega+\mathcal{T}_{\partial\Omega}} = \dim N_{\mathcal{S}_\Omega+\mathcal{T}_{\partial\Omega}} = 0$, i.e. $\mathcal{S}_\Omega + \mathcal{T}_{\partial\Omega}$ is bijective. The same process can be applied to the dual operator $\mathcal{S}_\Omega^* + \mathcal{T}_{\partial\Omega}^*$. $\square$

# Chapter 6

# Stokes Free Boundary Problem

## 6.1 Bulk Space and Trace Operator

In this chapter $\hat{\Omega}$ is the n-dimensional mollified cylinder we introduced in §2.1. We use the mollified version in order to use the well-posedness results by Galdi-Simader-Sohr [GSS94] for the Stokes operator with no-slip boundary conditions in $\mathbb{R}^n$.

**Lemma 6.1.1** (Bulk Space)**.** *Let $1 < s' < n < s < \infty$. The space*

$$X_s(\hat{\Omega}) := \left\{ W^1_s(\hat{\Omega})^n \times L^s(\hat{\Omega}) : \hat{\boldsymbol{u}} = \boldsymbol{0} \ \text{on } \Sigma \right\} \tag{6.1}$$

*together with the semi-norm*

$$\left\| (\hat{\boldsymbol{u}}, \hat{p}) \right\|_{X_s(\hat{\Omega})} := \left\| \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{u}}) \right\|_{L^s(\hat{\Omega})} + \| \hat{p} \|_{L^s(\hat{\Omega})} \tag{6.2}$$

*is an admissible trial space and $X_{s'}(\hat{\Omega})^*$ is an admissible bulk data space.*

*Proof.* The vector space $X_s$ is defined as the product of two reflexive Banach spaces and together with the canonical norm it is complete. To show completeness with the semi-norm it suffices for to check that $\|\hat{\boldsymbol{u}}\|_{W^1_s(\hat{\Omega})} \leq C_{\hat{\Omega},n,s} \left\| \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{u}}) \right\|_{L^s(\hat{\Omega})}$.

Following the contraction argument of Proposition 5.2.2 we would obtain a sequence $\{\boldsymbol{v}_\ell\}$ which converges strongly to $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ in $X_s(\hat{\Omega})$, where $\boldsymbol{A} = -\boldsymbol{A}^\top$ and $\boldsymbol{v}$ is supposed to be non-trivial. The contradiction comes from geometry of $\hat{\Omega}$ and the definition of $\Sigma$, i.e. we have an affine function which vanishes on the lateral and bottom boundary of the n-dimensional cylinder $\hat{\Omega}$, whence $\boldsymbol{v} = \boldsymbol{0}$.

See [Cia88, Theorem 6.3-4] for a similar result. $\square$

**Lemma 6.1.2** (Trace Operator)**.** *Let $s' \le r \le s$ and $s > n$. The operator $\mathcal{T}_r : X_r(\hat{\Omega}) \to W_r^{1-1/r}(\hat{\Gamma})$ defined by $\mathcal{T}_r \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\nu}}|_{\hat{\Gamma}}$ is an admissible trace operator.*

*Proof.* We need to show that $\mathcal{T}_r$ is linear, continuous and when $r = s'$ it also needs to be surjective. Linearity is trivial. Continuity follows from the standard trace operator $\gamma_0 : W_r^1(\hat{\Omega}) \to W_r^{1-1/r}(\partial\hat{\Omega})$, i.e.

$$\|\mathcal{T}\hat{\boldsymbol{v}}\|_{W_r^{1-1/r}(\hat{\Gamma})} = \|\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\nu}}\|_{W_r^{1-1/r}(\hat{\Gamma})} = \|\gamma_0 v^n\|_{W_r^{1-1/r}(\hat{\Gamma})} \le \|\gamma_0 v^n\|_{W_r^{1-1/r}(\partial\hat{\Omega})} \le \|v^n\|_{W_r^1(\hat{\Omega})}$$
$$\le \|\boldsymbol{v}\|_{W_r^1(\hat{\Omega})}.$$

In the graph case $\boldsymbol{\nu} = (\hat{\mathbf{0}}', 1)$ and it makes the above estimate simple. Nevertheless, the same estimate holds true if $\hat{\boldsymbol{\nu}}$ were only in $W_s^{1-1/s}(\hat{\Gamma})$ with $s > n$, recall the multiplication algebra property [ST95, Remark 3.3.2].

To proof of surjectivity only holds for $r < 2$, i.e. the bulk test space. Let $\phi$ be in $W_r^{1-1/r}(\hat{\Gamma})$ and $\tilde{\phi}$ be its extension by zero to all of $\partial\hat{\Omega}$. It follows from Lions-Magenes [LM61, Théorèm 3.1] that the extension by zero is a linear and continuous operation only for $r < 2$. Invoking the surjectivity of $\gamma_0$, i.e. it has a right inverse $\gamma_0^-$, we define $\mathcal{T}_r^- : W_r^{1-1/r}(\hat{\Gamma}) \to X_r(\hat{\Omega})$ as the map $\phi \mapsto ((\mathbf{0}', \gamma_0^- \tilde{\phi}), 0)$. The fact that $\mathcal{T}_r^-$ is a right inverse of $\mathcal{T}$ follows next

$$\mathcal{T}\mathcal{T}_r^- \phi = \mathcal{T}\left((\mathbf{0}', \gamma_0^- \tilde{\phi}), 0\right) = \gamma_0(\mathbf{0}', \gamma_0^- \tilde{\phi}) \cdot (\mathbf{0}', 1) = \gamma_0 \gamma_0^- \tilde{\phi} = \tilde{\phi} = \phi.$$

Thus, we have shown that $\mathcal{T}_r$ is surjective for $r < 2$. $\square$

## 6.2 Bulk Operator: Stokes

The goal of this section is to show that the Stokes operator is an admissible bulk operator. To this end we must first convert operator from the physical domain to the reference one. This is accomplished by using the Piola transform for the vector field $\boldsymbol{u}$ and a change of variables for the pressure $p$. We call this resulting operator $\tilde{\mathcal{S}}$ and its derivation is analogous to Proposition 5.5.15.

**Lemma 6.2.1** ($\tilde{\mathcal{S}}$ is admissible)**.** *There exists an open set $V_S \subset Y_s^2$ such that the operator $\tilde{\mathcal{S}} : X_s(\hat{\Omega}) \times V_S \to X_{s'}(\hat{\Omega})^*$ defined by*

$$\begin{aligned}
\tilde{\mathcal{S}}(\hat{\boldsymbol{u}}, \hat{p}; \omega)(\hat{\boldsymbol{v}}, \hat{q}) &:= \mathcal{S}(\hat{\boldsymbol{u}}, \hat{p})(\hat{\boldsymbol{v}}, \hat{q}) + (\mathcal{B} + \mathcal{C})(\hat{\boldsymbol{u}}; \omega)(\hat{\boldsymbol{v}}) \\
\mathcal{S}(\hat{\boldsymbol{u}}, \hat{p})(\hat{\boldsymbol{v}}, \hat{q}) &:= \int_{\hat{\Omega}} \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) - \hat{p}\, \widehat{\operatorname{div}}\, \hat{\boldsymbol{v}} + \hat{q}\, \widehat{\operatorname{div}}\, \hat{\boldsymbol{u}} \\
\mathcal{B}(\hat{\boldsymbol{u}})(\hat{\boldsymbol{v}}) &:= \int_{\hat{\Omega}} \left( \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{v}}) \right) \xi - \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) \\
\mathcal{C}(\hat{\boldsymbol{u}})(\hat{\boldsymbol{v}}) &:= \int_{\hat{\Omega}} \Big( \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{v}}) \Big) \xi \\
&\qquad - \int_{\hat{\Omega}} \Big( \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{v}}) + \hat{\boldsymbol{\varepsilon}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{u}}), \hat{\boldsymbol{\vartheta}}_{\hat{\boldsymbol{P}}}(\hat{\boldsymbol{v}}) \Big) \xi
\end{aligned}$$

*with $\xi = 1/\det \hat{\nabla}\hat{\Psi}$, is an admissible bulk operator. In other other words, it is*

*(i) linear in the bulk trial space $X_s(\hat{\Omega})$ for every $\omega$ in $V_S$,*

*(ii) continuously Fréchet differentiable in $V_S$,*

*(iii) an isomorphism between the restricted bulk space $\mathring{X}_s(\hat{\Omega})$ and $\mathring{X}_{s'}(\hat{\Omega})^*$.*

*Proof.* Linearity and Fréchet differentiability follows immediately from the linearity and differentiability of of symmetric gradient decomposition, Proposition 5.4.11. In particular, the differentiability follows from that of the Piola matrix itself, Corollary 5.4.15. To show that it is an isomorphism we need to reuse the work from §5.5.

The first road-block is showing that $\mathcal{S}$, i.e. the Stokes operator on $\mathring{X}_s(\hat{\Omega})$ is an isomorphism. This follows by the reflection principle, i.e. solving $\mathcal{S}$ on $\hat{\Omega}$ with a slip condition at the top $\hat{\Gamma}$ amounts to solving the Stokes problem with no-slip conditions on the domain $\Omega_R := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in \hat{\Omega} \cup \hat{\Gamma}, \text{ or } (\boldsymbol{x}', 2 - x^n) \in \hat{\Omega} \right\}$; c.f. §5.3.3 for the technique and Figure 6.1 for a picture of $\Omega_R$. Because of the mollification the domain $\Omega_R$ is at least $C^1$, thus we can use Galdi-Simader-Sohr [GSS94] to obtain that $\mathcal{S}$ is an isomorphism between $\mathring{X}_s(\hat{\Omega})$ and $\mathring{X}_{s'}(\hat{\Omega})^*$ for $1 < s < \infty$.

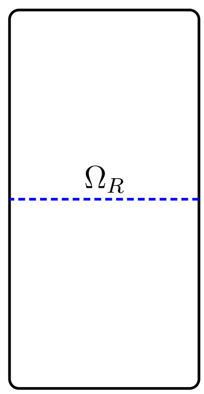



Figure 6.1: The reflected domain $\Omega_R$ and the reflected boundary in dashed blue.

Next we choose $L$ small enough such that $\mathcal{S} + \mathcal{B}$ is an isomorphism for every $\omega$ in the set $V_S := \left\{ \omega \in Y_s^2 : |\omega|_{W^1_\infty(\hat{\Gamma})} < L \right\}$. This follows from the von Neumann perturbation/stability result Lemma 5.5.17.

The last step is to apply Corollary 5.5.8. Since $\tilde{\mathcal{S}}$ is a compact perturbation of an invertible map $\mathcal{S} + \mathcal{B}$ it has finite index, Lemma 5.5.6. To conclude that $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}^*$ are injective it suffices to reapply the results from §5.5.3. $\square$

## 6.3 Applied Force Space and Operator

At this point we need to make a compromise, we will only study the case where the applied force is of the "dead-load" type, [Cia88, Section 2.7]. This implies that we will take $Z = X_{s'}(\hat{\Omega})^*$ and $\mathcal{F}(\mathcal{G};\omega) := \mathcal{G}$ for every $\mathcal{G} \in Z$. This choice is always an admissible applied force. For completeness we collect this result in the next lemma.

**Lemma 6.3.1** (Applied Force Operator)**.** *Let $Z := X_{s'}(\hat{\Omega})^*$ and $\mathcal{F} : Z \times Y_s^2 \to X_{s'}(\hat{\Omega})^*$ be the operator defined by*

$$\mathcal{F}(\mathcal{G},\omega)(\hat{\boldsymbol{v}},\hat{q}) := \mathcal{G}(\hat{\boldsymbol{v}},\hat{q}) \quad \textit{for all } (\hat{\boldsymbol{v}},\hat{q}) \in X_{s'}(\hat{\Omega}).$$

*Then $\mathcal{F}$ is an admissible applied force operator.*

## 6.4 Applying the Abstract Framework

The goal of this section is to apply the abstract surface tension framework to conclude the existence and local uniqueness to the Stokes free boundary problem. To this end we must obtain an explicit variational form from Formulation 1.2.3. This result holds because of the isomorphism between the spaces $\mathring{X}_r(\hat{\Omega})$ and $\mathring{X}_r(\Omega)$ for $s' \le r \le s$, where $\hat{\Omega}$ is the n-dimensional cylinder and $\Omega$ is physical domain, i.e. the top boundary is the graph of a function in surface trial space $Y_s^2$.

**Formulation 6.4.1.** Let $\Omega$ be an admissible domain. The triple $(\Omega, \boldsymbol{u}, p)$ is a weak solution to 1.2 if and only if there exists $(\hat{\boldsymbol{u}}, \hat{p})$ in $\mathring{X}_s(\hat{\Omega})$ such that $(\boldsymbol{u}, p) = \mathcal{P}(\hat{\boldsymbol{u}}, \hat{p})$, with $\mathcal{P}$ the Piola transform associated with the $W_s^2$-graph-diffeomorphism $\hat{\Psi}$ generated by the function $\omega$ in $Y_s^2$, and $\Omega = \hat{\Psi}(\hat{\Omega})$. Furthermore, the triple

$(\omega, \hat{\boldsymbol{u}}, \hat{p})$ satisfies the variational system

$$\left\langle \mathcal{T}_{s'}^{*}\mathcal{H}(\omega) + \tilde{\mathcal{S}}(\hat{\boldsymbol{u}}, \hat{p}; \omega) - \mathcal{G}, (\hat{\boldsymbol{v}}, \hat{q}) \right\rangle \quad \text{for all } (\hat{\boldsymbol{v}}, \hat{q}) \in X_{s'}(\hat{\Omega}), \tag{6.3}$$

where $\langle \cdot, \cdot \rangle$ indicates the duality pairing between $X_{s'}(\hat{\Omega})^{*}$ and $X_{s'}(\hat{\Omega})$ and the underlying operators are defined by

$$\mathcal{T}_{s'}^{*}\mathcal{H}(\omega)(\hat{\boldsymbol{v}}, \hat{q}) := \alpha \int_{\hat{\Gamma}} -\operatorname{div}' \left( \frac{\nabla' \omega}{\mathcal{Q}(\omega)} \right) v^{n} \, \mathrm{d}s$$

with $\xi = 1/\det \hat{\nabla}\hat{\Psi}$.

*Derivation.* The proof consists of reapplying the boundary operator decomposition from Proposition 5.5.15 to obtain $\tilde{\mathcal{S}}(\hat{\boldsymbol{u}}, \hat{p}; \omega)$ from $\mathcal{S}(\boldsymbol{u}, p; \Omega)$. To obtain the converse result it suffices to use that the Piola transform is an isomorphism between $\mathring{X}_r(\hat{\Omega})$ and $\mathring{X}_r(\Omega)$ for $s' \leq r \leq s$, c.f. Theorem 5.4.13. □

**Lemma 6.4.2** (Resting Configuration)**.** *The point $\omega_0 = 0$, $\hat{\boldsymbol{u}}_0 = \mathbf{0}$ and $p_0 = 0$ is a resting configuration to the Stokes free boundary problem.*

*Proof.* The proof follows directly from the definition of the operators $\mathcal{H}$ and $\tilde{\mathcal{S}}$. □

**Corollary 6.4.3** (Stokes FBP)**.** *Let $(\omega_0, \hat{\boldsymbol{u}}_0, \hat{p}_0)$ be the resting configuration given above, and $V = V_S \cap V_H$ There exists an open ball of radius $r$ around the point $0_Z$ in the applied force space, namely $B(0_Z, r) := \left\{ \mathcal{G} \in Z : \|\mathcal{G}\|_Z < r \right\}$, and a* unique *continuously Fréchet differentiable operator $\mathcal{U} : B(0_Z, r) \to (V \times \mathring{X}_s^1)$ such that $\mathcal{U}(0_Z) = (\omega_0, \hat{\boldsymbol{u}}_0, p_0)$ and $(\omega, \hat{\boldsymbol{u}}, \hat{p}) = \mathcal{U}(\mathcal{G})$ satisfies*

$$\mathcal{T}_{s'}^{*}\mathcal{H}(\omega) + \tilde{\mathcal{S}}(\hat{\boldsymbol{u}}, \hat{p}) = \mathcal{G}$$

*for every $\mathcal{G}$ in $B(0_Z, r)$.*

*Proof.* We have shown in Lemma 6.1.2, Lemma 6.2.1 and Lemma 6.3.1 that the Trace, Stokes in reference domain and the dead-load applied forces are admissible in the sense required by the abstract surface tension framework. Moreover, the curvature operator $\mathcal{H}$ was also an admissible surface operator. We invoke Proposition 1.2.23 to conclude the result. $\square$

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