# ABSTRACT

Title of dissertation: NONCOLLISION SINGULARITIES IN A PLANAR TWO-CENTER -TWO-BODY PROBLEM

Jinxin Xue, Doctor of Philosophy, 2013

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In this work, we study a model of simplified four-body problem called planar two-center-two-body problem. In the plane, we have two fixed centers $Q_1 = (-\chi, 0), Q_2 = (0, 0)$ of masses 1, and two moving bodies $Q_3$ and $Q_4$ of masses $\mu \ll 1$. They interact via Newtonian potential. $Q_3$ is captured by $Q_2$, and $Q_4$ travels back and forth between two centers. Based on a model of Gerver, we prove that there is a Cantor set of initial conditions which lead to solutions of the Hamiltonian system whose velocities are accelerated to infinity within finite time avoiding all early collisions. We consider this model as a simplified model for the planar four-body problem case of the Painlevé conjecture.

# NONCOLLISION SINGULARITIES IN A PLANAR TWO-CENTER-TWO-BODY PROBLEM

by

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## 1.1 Introduction

### 1.1.1 Statement of the main result

We study a two-center two-body problem. Consider two fixed centers $Q_1$ and $Q_2$ of masses $m_1 = m_2 = 1$ located at distance $\chi$ from each other and two small particles $Q_3$ and $Q_4$ of masses $m_3 = m_4 = \mu \ll 1$. $Q_i$s interact with each other via Newtonian potential. If we choose coordinates so that $Q_2$ is at $(0,0)$ and $Q_1$ is at $(-\chi, 0)$ then the Hamiltonian of this system can be written as

$$H = \frac{|P_3|^2}{2\mu} + \frac{|P_4|^2}{2\mu} - \frac{\mu}{|Q_3|} - \frac{\mu}{|Q_3 - (-\chi,0)|} - \frac{\mu}{|Q_4|} - \frac{\mu}{|Q_4 - (-\chi,0)|} - \frac{\mu^2}{|Q_3 - Q_4|}. \tag{1.1.1}$$

We assume that the total energy of the system is zero.

We want to study singular solutions of this system, that are solutions which can not be continued for all positive times. We will exhibit a rich variety of singular solutions. Fix $\varepsilon_0 < \chi$. Let $\boldsymbol{\omega} = \{\boldsymbol{\omega}_j\}_{j=1}^{\infty}$ be a sequence of 3s and 4s.

**Definition 1.** We say that $(Q_3(t), Q_4(t))$ is a **singular solution with symbolic sequence** $\boldsymbol{\omega}$ if there exists a positive increasing sequence $\{t_j\}_{j=0}^{\infty}$ such that

- $t^* = \lim_{j\to\infty} t_j < \infty$.
- $|Q_3(t_j) - Q_2| \le \varepsilon_0$, $|Q_4(t_j) - Q_2| \le \varepsilon_0$.
- If $\boldsymbol{\omega}_j = 4$ then for $t \in [t_{j-1}, t_j]$, $|Q_3(t) - Q_2| \le \varepsilon_0$ and $\{Q_4(t)\}_{t\in[t_{j-1},t_j]}$ winds around $Q_1$ exactly once.

  If $\boldsymbol{\omega}_j = 3$ then for $t \in [t_{j-1}, t_j]$, $|Q_4(t) - Q_2| \le \varepsilon_0$ and $\{Q_3(t)\}_{t\in[t_{j-1},t_j]}$ winds around $Q_1$ exactly once.

- $|\dot{Q}_i(t)| \to \infty$ as $t \to t^*$.

During the time interval $[t_{j-1}, t_j]$ we refer to $Q_{\boldsymbol{\omega}_j}$ as the traveling particle and to $Q_{7-\boldsymbol{\omega}_j}$ as the captured particle. Thus $\boldsymbol{\omega}_j$ prescribes which particle is the traveler during the $j$ trip.

We denote by $\Sigma_{\boldsymbol{\omega}}$ the set of initial conditions of singular orbits with symbolic sequence $\boldsymbol{\omega}$. Note that if $\boldsymbol{\omega}$ contains only finitely many 3s then there is a collision of $Q_3$ and $Q_2$ at time $t^*$. If $\boldsymbol{\omega}$ contains only finitely many 4s then there is a collision of $Q_4$ and $Q_2$ at time $t^*$. Otherwise at we have a **collisionless singularity** at $t^*$.

**Theorem 1.** *There exists $\mu_* \ll 1$ such that for $\mu < \mu_*$ the set $\Sigma_{\boldsymbol{\omega}} \neq \emptyset$.*

*Moreover there is an open set $U$ in the phase space and a foliation of $U$ by two-dimensional surfaces such that for any leaf $S$ of our foliation $\Sigma_{\boldsymbol{\omega}} \cap S$ is a Cantor set.*

**Remark 1.** By rescaling space and time variables we can assume that $\chi \gg 1$. In the proof we shall make that assumption and set $\varepsilon_0 = 2$.

**Remark 2.** It follows from the proof that the Cantor set described in Theorem 1 can be chosen to depend continuously on $S$. In other words $\Sigma_{\boldsymbol{\omega}}$ contains a set which is local a product of a five dimensional disc and a Cantor set. The fact that on each surface we have a Cantor set follows from the fact that we have a freedom of choosing how many rotations the captured particle makes during $j$-th trip.

**Remark 3.** The construction presented in this paper also works for small nonzero energies. Namely, it is sufficient that the total energy is much smaller than the kinetic energies of the individual particles. The assumption that the total energy is zero is made to simplify notation since then the energies of $Q_3$ and $Q_4$ have the same absolute values.

**Remark 4.** One can ask if Theorem 1 holds for other choices of masses. The fact that the masses of the fixed centers $Q_1$ and $Q_2$ are the same is not essential and is made only for convenience. The assumption that $Q_3$ and $Q_4$ are light is important since it allows us to treat their interaction as a perturbation except during the close encounters of $Q_3$ and $Q_4$. The fact that the masses of $Q_3$ and $Q_4$ are equal allows us to use an explicit periodic solution of a certain limiting map (*Gerver map*) which is found in [G1]. It seems likely that the conclusion of Theorem 1 is valid if $m_3 = \mu, m_4 = c\mu$ where $c$ is a fixed constant close to 1 and $\mu$ is sufficiently small but we do not have a proof of that.

### 1.1.2 Motivations.

#### 1.1.2.1 Non-collision singularity in N-body problem

Our work is motivated by the following fundamental problem in celestial mechanics. *Describe the set of initial conditions of the Newtonian N-body problem leading to global solutions.* The compliment to this set splits into the initial conditions leading to the collision and non-collision singularities.

It is clear that the set of initial conditions leading to collisions is non-empty for all $N > 1$ and it is shown in [Sa1] that it has zero measure. Much less is known about the non-collision singularities. In particular the following basic problems are still open.

**Conjecture 1.** The set of non-collision singularities is non-empty for all $N > 3$.

**Conjecture 2.** The set of non-collision singularities has zero measure for all $N > 3$.

Conjecture 1 probably goes back to Poincaré who was motivated by King Oscar II prize problem about analytic representation of collision less solutions of the $N$ body problem. It was explicitly mentioned in Painlevé's lectures [Pa] where the author proved that for $N = 3$ there are no non-collision singularities. Soon after Painlevé, von Ziepel showed that if the system of $N$ bodies has a non-collision singularity then some particle should fly off to infinity in finite time. Thus non-collision singularities seem quite counterintuitive. However in [MM] Mather and McGehee constructed a system of four bodies on the line where the particles go to infinity in finite time after an infinite number of binary collisions (it was known since the work of Sundman [Su]) that binary collisions can be regularized so that the solutions can be extended beyond the collisions). Since Mather-McGehee example had collisions it did not solve Conjecture 1 but it made it plausible. Conjecture 1 was solved independently by Xia [X] for the spacial five-body problem and by Gerver [G1] for a planar $N$ body problem where $N$ is sufficiently large. The problem still remains open for $N = 4$ and for small $N$ in the planar case. However in [G1] Gerver sketched a scenario which may lead to a non-collision singularity in the planar four-body problem. Gerver has not published the details of his construction due to a large amount of computations involved (it suffices to mention that even technically simpler large $N$ case took 68 pages in [G1]). The goal of this paper is to realize

Gerver's scenario in the simplified setting of two-center-two-body problem.

Conjecture 2 is mentioned by several authors, see e.g. [Sim, Sa3, K]. It is known that the set of initial conditions leading to the collisions has zero measure [Sa1] and that the same is true for non-collisions singularities if $N = 4$. To obtain the complete solution of this conjecture one needs to understand better of the structure of the non-collision singularities and our paper is one step in this direction.

#### 1.1.2.2 Well-posedness in other systems

Recently the question of global well-posedness in PDE attracted a lot of attention motivated in part by the Clay Prize problem about well-posedness of the Navier-Stokes equation (see e.g. [LS]). One approach to constructing a blowup solutions for PDEs is to find a fixed point of a suitable renormalization scheme and to prove the convergence towards this fixed point. The same scheme is also used to analyze two-center-two-body problem and so we hope that the techniques developed in this paper can be useful in constructing singular solutions in more complicated systems.

#### 1.1.2.3 Poincaré's second species solution.

In his book [Po] , Poincaré claimed the existence of the so-called second species solution in three-body problem, which are periodic orbits converging to collision chains as $\mu \to 0$. The concept of second species solution was generalized to the non-periodic case. In recent years significant progress was made in understanding second species solutions of both restricted [BM, FNS] and full [BN] three-body problem. However the understanding of general second species solutions generated by infinite aperiodic collision chains is still incomplete. Our result can be considered as a generalized version of second species solution. All masses are positive and there are infinitely many close encounters. Therefore the techniques developed in this paper can be useful in the study of the second species solutions.

### 1.1.3 A glimpse of the 4-body problem

Consider the same setting as in our main result but suppose that $Q_1$ and $Q_2$ are also free (not fixed). Then we can expect that during each encounter light particle transfers a fixed proportion of their energy and momentum to the heavy particle . The exponential growth of energy and momentum would cause $Q_1$ and $Q_2$ to go to infinity in finite time leading to a non-collision singularity.

A proof of this would however involve a significant amount of additional computation due to higher dimensionality of the full four-body problem. Indeed planar four-body problem has 16 dimensions since each particle has two position and two momentum coordinates. Removing the translation invariance we are left with 12 dimensions. Taking into account the rotation invariance leaves us with 10 dimensions. Energy conservation and taking a Poincaré section kills two more dimensions so we obtain a eight dimensional Poincaré map. We expect however that similarly to the problem at hand the Poincaré of the full four-body problem will have only two strongly expanding directions while other directions will be dominated by the most expanding ones. This would allow our strategy to extend to the full four-body problem leading to the complete solution of the Painlevé conjecture.

### 1.1.4 Plan of the paper.

The paper is organized as follows. Section 2.2 and 3.3 constitute the main framework of the proof. We give a proof of the main Theorem 1 based on a careful study of the hyperbolicity of the Poincaré map. In Section 3.3, we summarize all later calculations and we prove the hyperbolicity results of Section 3.3. All the later sections provide calculations needed in Section 3.3. We define the local map to study the local interaction between $Q_3$ and $Q_4$ and global map to cover the time interval when $Q_4$ is traveling between $Q_1$ and $Q_2$. The Section 4.1, 4.3, 4.4 and 4.5 are devoted to the calculations of the derivative of the global map, while Section 5.1,5.2, and 5.3 computes the derivative of the local map. Finally, we have two appendices. In Appendix A.1, we include an introduction to the Delaunay coordinates for Kepler motion, which is used extensively in our calculation. In Appendix B.2, we summarize the main information concerning Gerver's model in [G1].

## 2.2 Proof of the main theorem

### 2.2.1 Idea of the proof.

The proof of the Theorem 1 is based on studying the hyperbolicity of the Poincaré map. Our system has four degrees of freedom. We pick the zero energy surface and then consider a Poincaré section. The resulting Poincaré map is six dimensional. In turns out that for orbits of interest (that is, the orbits where the captured particle rotates around $Q_2$ and the traveler moves back and forth between $Q_1$ and $Q_2$) there is an invariant cone which consists of vectors close to a certain two dimensional subspace such that all vectors in the cone are strongly expanding. This expansion comes from the combination of shearing (there are long stretches then the motion of the light particles is well approximated by the Kepler motion and so the derivatives are almost upper triangular) and twisting caused by the close encounters between $Q_4$ and $Q_3$ and between $Q_4$ and $Q_1$. We restrict our attention to a two dimensional surface whose tangent space belong to the invariant cone and construct on such a surface a Cantor set of singular orbits as follows. The two parameters coming from the two dimensionality of the surface will be used to control the phase of the close encounter between the particles and their relative distance. The strong expansion will be used to ensure that the choices made at the next step will have a little effect on the parameters at the previous steps. This Cantor set construction based on the instability of near colliding orbits is also among the key ingredients of the singular orbit constructions in [MM] and [X].

### 2.2.2 Main ingredients.

In this section we present the main steps in proving Theorem 1. In Subsection 2.2.3 we describe a simplified model for constructing singular solutions given by Gerver [G1]. This model is based on the following simplifying assumptions:

- $\mu = 0, \ \chi = \infty$.
- The particles do not interact except during a close encounter.
- Velocity exchange during close encounters can be modeled by an elastic collision.
- The action of $Q_1$ on light particles can be ignored except that during the close encounters of the traveler particle with $Q_1$ the angular momentum of the traveler with respect to $Q_2$ can be changed arbitrarily.

The main conclusion of [G1] is that the energy of the captured particle can be increased by a fixed factor while keeping the shape of its orbit unchanged. Gerver designs a procedure with two steps of collisions having the following properties:

- The incoming and outgoing asymptotes of the traveler are horizontal.
- The major axis of the captured particle remains vertical.
- After two steps of collisions, the elliptic orbit of the captured particle has the same eccentricity but smaller semimajor compared with the elliptic orbit before the first collision (see Fig 1 and 2).

For quantitative information, see the Appendix B.2.

Since the shape is unchanged after the two trips described above the procedure can be repeated. Then the kinetic energies of the particles grow exponentially and so the time needed for $j$-th trip is exponentially small. Thus the particles can make infinitely many trips in finite time leading to a singularity. Our goal therefore is to get rid of the above mentioned simplifying assumptions.

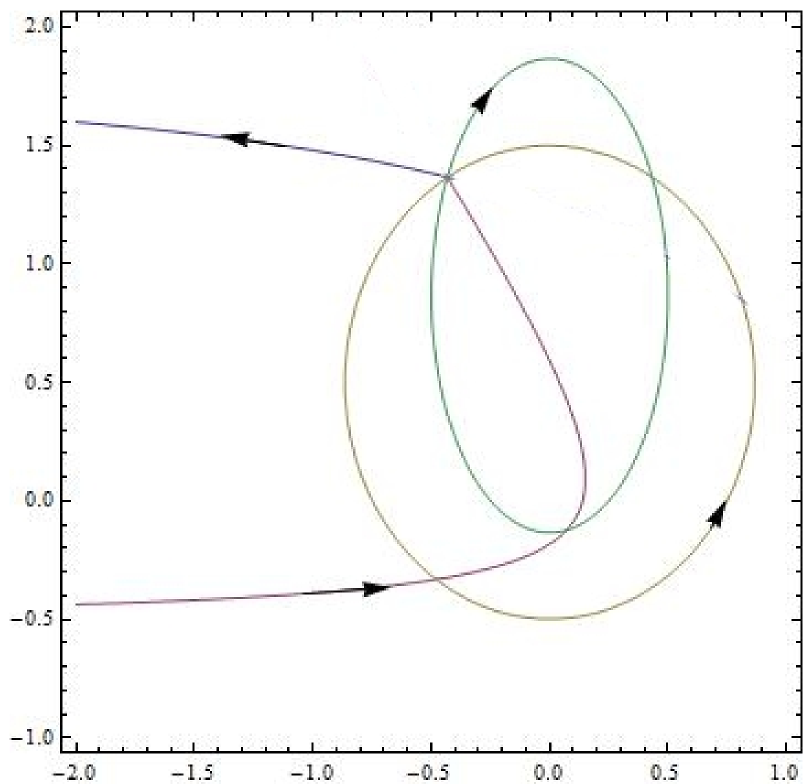


Figure 1: Angular momentum transfer collision

In Subsection 2.2.4 we study near collision of the light particles. This assumption that velocity exchange can be modeled by elastic collision is not very restrictive since both energy and momentum are conserved during the exchange and any change of velocities conserving energy and momentum amounts to rotating the relative ve-

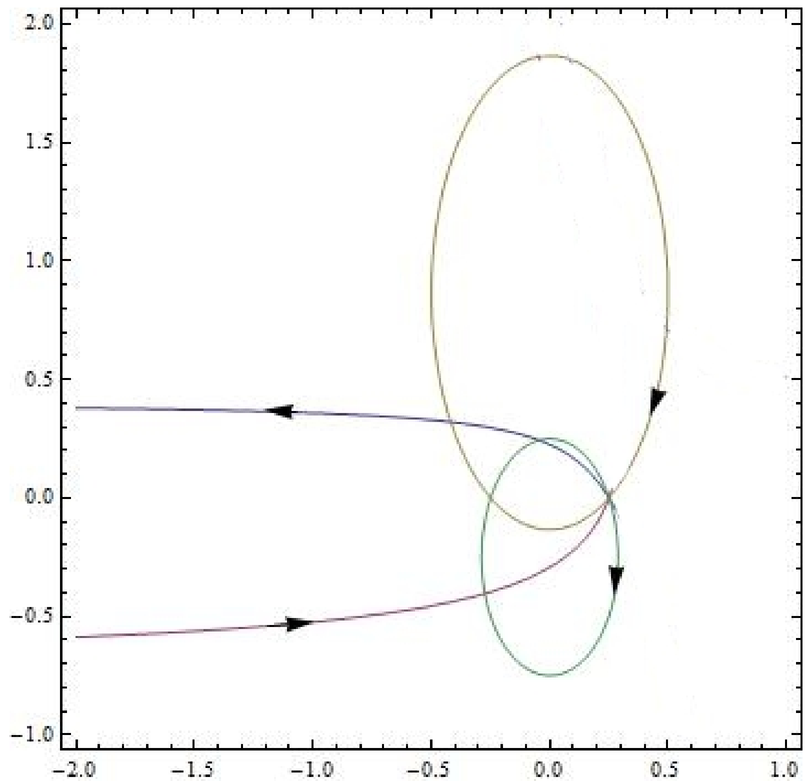


Figure 2: Energy transfer collision

locity by some angle and so it can be effected by an elastic collision. We recall a formula relating the angle of rotation to the minimal distance between the particles. In Subsection 2.2.5 we state a result saying that away from the close encounters we can disregard interaction between the light particles and the action of $Q_1$ to the particle which is captured by $Q_2$ can indeed be disregarded. In Subsection 2.2.6 we study the Poincaré map corresponding to one trip of one light particle around $Q_1$. After some technical preparations we present the main result of that section Lemma 2.2.6 which says that after this trip the angular momentum of the traveler particle indeed can change in an arbitrary way. Finally in Subsection 2.2.7 we show how to combine the above ingredients to construct a Cantor set of singular orbits.

### 2.2.3 Gerver map.

Following [G1], we discuss in this section the limit case $\mu = 0, \chi = \infty$. We assume that $Q_3$ has elliptic motion and $Q_4$ has hyperbolic motion with respect to the focus $Q_2$. Since $\mu = 0$, $Q_3$ and $Q_4$ do not interact unless they have exact collision. Since we assume that $Q_4$ just comes from the interaction from $Q_1$ located at $(-\infty, 0)$ and the new traveler particle is going to interact with $Q_1$ in the future, the slope of incoming asymptote $\theta_4^-$ of $Q_4$ and that of the outgoing asymptote $\bar{\theta}^+$ of the traveler particle should satisfy $\theta^- = 0,\ \bar{\theta}^+ = \pi$.

The Kepler motions of $Q_3$ and $Q_4$ has three first integrals $E_i, G_i$ and $g_i$ where$E_i$ denotes the energy, $G_i$ denotes the angular momentum and $g_i$ denotes the argument of periapsis. Since the total energy of the system is zero we have $E_4 = -E_3$. It turns out convenient to use eccentricities $e_i = \sqrt{1 + 2G_i^2 E_i}$ instead of $G_i$ since the proof of Theorem 1 involves a renormalization transformation and $e_i$ are scaling invariant. The Gerver map describes the parameters of the elliptic orbit change during the interaction of $Q_3$ and $Q_4$. The orbits of $Q_3$ and $Q_4$ intersect in two points. We pick one of them. We use a discrete parameter $j \in \{1, 2\}$ to describe if the points meet at the first or at the second intersection (the intersection points will be numbered chronologically along the orbit of $Q_4$).

Since $Q_3$ and $Q_4$ only interact when they are at the same point the only effect of the interaction is to change their velocities. Any such change which satisfies energy and momentum conservation can be described by an elastic collision. That is, velocities before and after the collision are related by

$$v_3^+ = \frac{v_3^- + v_4^-}{2} + \left|\frac{v_3^- - v_4^-}{2}\right| n(\alpha), \quad v_4^+ = \frac{v_3^- + v_4^-}{2} - \left|\frac{v_3^- - v_4^-}{2}\right| n(\alpha), \tag{2.2.1}$$

where $n(\alpha)$ is a unit vector making angle $\alpha$ with $v_3^- - v_4^-$.

With this in mind we proceed to define the Gerver map $\mathbf{G}_{e_4,j,\omega}(E_3, e_3, g_3)$. This map depends on two discrete parameters $j \in \{1, 2\}$ and $\omega \in \{3, 4\}$. The role of $j$ has been explained above, and $\omega$ will tell us which particle will be the traveler after the collision.

To define $\mathbf{G}$ we assume that $Q_4$ moves along the hyperbolic orbit with parameters $(-E_3, e_4, g_4)$ where $g_4$ is fixed by requiring that the incoming asymptote of $Q_4$ is horizontal. We assume that $Q_3$ and $Q_4$ arrive to the $j$-th intersection point of their orbit simultaneously. At this point their velocities are changed by (2.2.1). After that the particle proceed to move independently. Thus $Q_3$ moves on an orbit with parameters $(\bar{E}_3, \bar{e}_3, \bar{g}_3)$, and $Q_4$ moves on an orbit with parameters $(\bar{E}_4, \bar{e}_4, \bar{g}_4)$.

If $\omega = 4$, we choose $\alpha$ so that after the exchange $Q_4$ moves on hyperbolic orbit and $\bar{\theta}_4^+ = \pi$ and let

$$\mathbf{G}_{e_4,j,4}(E_3, e_3, g_3) = (\bar{E}_3, \bar{e}_3, \bar{g}_3).$$

If $\omega = 3$ we choose $\alpha$ so that after the exchange $Q_3$ moves on hyperbolic orbit and $\bar{\theta}_3^+ = \pi$ and let

$$\mathbf{G}_{e_4,j,3}(E_3, e_3, g_3) = (\bar{E}_4, \bar{e}_4, \bar{g}_4).$$

*In the following, to fix our notation, we always call the captured particle $Q_3$ and the traveler $Q_4$.*

We will denote the ideal orbit parameters in Gerver's paper [G1] of $Q_3$ and $Q_4$ before the first (respectively second) collision with * (respectively **). Thus, for example, $G_4^{**}$ will denote the angular momentum of $Q_4$ before the second collision. Moreover, the realistic values after the first (respectively, after the second) collisions are denoted with a *bar* or *double bar*.

Note $\mathbf{G}$ has a skew product form

$$\bar{e}_3 = f_e(e_3, g_3, e_4), \quad \bar{g}_3 = f_g(e_3, g_3, e_4), \quad \bar{E}_3 = E_3 f_E(e_3, g_3, e_4).$$

This skew product structure will be crucial in the proof of Theorem 1 since it will allow us to iterate $\mathbf{G}$ so that $E_3$ grows exponentially while $e_3$ and $g_3$ remains almost unchanged.

The following fact plays a key role in constructing singular solutions.

**Lemma 2.2.1.** *([G1]) There exist $(e_3^*, g_3^*)$, such that for sufficiently small $\bar{\delta} > 0$ given $\omega', \omega'' \in \{3, 4\}$, there exist $\lambda_0 > 1$ and functions $e_4'(e_3, g_3)$, $e_4''(e_3, g_3)$, defined in a small (depending on $\bar{\delta}$) neighborhood of $(e_3^*, g_3^*)$, such that*

*(a) for $e_4^*, e_4^{**}$ given by $e_4'(e_3^*, g_3^*) = e_4^*$ and $e_4''(e_3^*, g_3^*) = e_4^{**}$, we have*

$$(e_3, g_3, E_3)^{**} = \mathbf{G}_{e_4^*,1,\omega'}\left(e_3, g_3, E_3\right)^*, \quad (e_3, -g_3, \lambda_0 E_3)^* = \mathbf{G}_{e_4^{**},2,\omega''}\left(e_3, g_3, E_3\right)^{**},$$

*(b) If $(e_3, g_3)$ lie in a $\bar{\delta}$ neighborhood of $(e_3^*, g_3^*)$, we have*

$$(\bar{e}_3, \bar{g}_3, \bar{E}_3) = \mathbf{G}_{e_4'(e_3,g_3),1,\omega'}\left(e_3, g_3, E_3\right), \quad (\bar{\bar{e}}_3, -\bar{\bar{g}}_3, \bar{\bar{E}}_3) = \mathbf{G}_{e_4''(e_3,g_3),2,\omega''}\left(\bar{e}_3, \bar{g}_3, \bar{E}_3\right),$$

*and*

$$\bar{\bar{e}}_3 = e_3^*, \quad \bar{\bar{g}}_3 = g_3^*, \quad \bar{\bar{E}}_3 = \lambda(e_3, g_3) E_3$$

*where $\lambda_0 - \bar{\delta} < \lambda < \lambda_0 + \bar{\delta}$.*

Part (a) is the main result of the above lemma. It allows us to increase energy after two collisions without changing the shape of the orbit in the limit case $\mu = 0, \chi = \infty$. Part (b) is of the mrs technical nature, which allows us to fight against the effect of perturbations coming from the fact that $\mu > 0$ and $\chi < \infty$.

Lemma 2.2.1 is a slight restatement of the main result of [G1]. Namely part (a) is proven in Sections 3 and 4 of [G1] and part (b) is stated in Section 5 of [G1] (see equations (5-10)–(5-13)). The proof of part (b) proceeds by a routine numerical computation. For the reader's convenience we review the proof of Lemma 2.2.1 in Appendix B.2 through explaining how the numerics is done.

**Remark 5.** 1. In fact Gerver produces a one parameter family of the periodic solution. Namely one can take $e_3^*$ to be any number between 0 and $\frac{\sqrt{2}}{2}$ and $g_3^* = 0$. In the course of the proof of Theorem 1 we need to check several non-degeneracy conditions. This will be done numerically for $e_3^* = \frac{1}{2}$.

2. We try to minimize the use of numerics in our work. The use of numerics is always preceded by mathematical derivations. Readers can see that the numerics in this paper can also be done without using computer, but we do not expect interesting mathematics there.

### 2.2.4 Asymptotic analysis. Local map.

We assume that the two centers are at distance $\chi \gg 1$ and that $Q_3, Q_4$ have positive masses $0 < \mu \ll 1$. We also assume that $Q_3$ and $Q_4$ have initial orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ in the section $\{x_4(0) = -2,\ \dot{x}_4(0) > 0\}$ (Here $\ell_3$

stands for the mean anomaly of $Q_3$, see Appendix A.1). We let particles move until one of the particles reach the surface $\{x_4 = -2,\ \dot{x}_4 < 0\}$ moving on hyperbolic orbit. We measure the final orbit parameters $(\bar{E}_3, \bar{l}_3, \bar{e}_3, \bar{g}_3, , \bar{e}_4, \bar{g}_4)$. We call the mapping moving initial positions of the particles to their final positions the **local map** $\mathbb{L}$. In Fig 3, the local map is to the right of the section $\{x = -2\}$.

**Lemma 2.2.2.** *Suppose that the initial orbit parameters* $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ *are such that the traveler particle(s) satisfy* $\theta^- = O(\mu)$ *and* $\bar{\theta}^+ = \pi + O(\mu)$ *then the following asymptotics holds uniformly*

$$(\bar{E}_3, \bar{e}_3, \bar{g}_3) = \mathbf{G}_{e_4}(E_3, e_3, g_3) + o(1),$$

*as* $\mu \to 0, \chi \to \infty$.

The lemma tells us Gerver map is a good approximation of the local map $\mathbb{L}$ for the real case $0 < \mu \ll 1 \ll \chi < \infty$ for the orbits of interest. Lemma 2.2.2 will be proven in Section 5.3 where we also present some additional information about the local map (see Lemma 5.2.2).

### 2.2.5 Asymptotic analysis. Global map.

As before we assume that the two centers are at distance $\chi \gg 1$. We assume that initially $Q_3$ moves on an elliptic orbit, $Q_4$ moves on hyperbolic orbit and $\{x_4(0) = -2,\ \dot{x}_4(0) < 0\}$. We assume that $|y_4(0)| < C$ and after moving around $Q_1$ it hits the surface $\{x_4 = -2,\ \dot{x}_4 > 0\}$ so that $|y_4| < C$. We call the mapping moving initial positions of the particles to their final positions the **global map** $\mathbb{G}$. In Fig

3, the global map is to the left of the section $\{x = -2\}$. We let $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ denote the initial orbit parameters measured in the section $\{x_4 = -2,\ \dot{x}_4 < 0\}$ and $(\bar{E}_3, \bar{l}_3, \bar{e}_3, \bar{g}_3, \bar{e}_4, \bar{g}_4)$ denote the final orbit parameters measured in the section $\{x_4 = -2,\ \dot{x}_4 > 0\}$.

**Lemma 2.2.3.** *Assume that $|y_4| < C$ holds both at initial and final moments. Then uniformly in $\chi, \mu$ we have the following estimates*

$$(a)\ \bar{E}_3 - E_3 = O(\mu), \quad \bar{G}_3 - G_3 = O(\mu), \quad \bar{g}_3 - g_3 = O(\mu).$$

$$(b)\ \theta_4^+ = \pi + O(\mu), \quad \bar{\theta}_4^- = O(\mu).$$

The proof of this lemma is given in Section 4.1.

### 2.2.6 Admissible surfaces.

Given a sequence $\boldsymbol{\omega}$ we need to construct orbits having singularity with symbolic sequence $\boldsymbol{\omega}$.

We will study the Poincaré map $\mathcal{P} = \mathbb{G} \circ \mathbb{L}$ to the surface $\{x_4 = -2,\, \dot{x_4} > 0\}$. It is a composition of the local and global maps defined in the previous sections.

Given $\delta$ consider open sets in the phase space defined by

$$U_1(\delta) = \left\{ \left| E_3 - \left( -\frac{1}{2} \right) \right|,\ |e_3 - e_3^*|,\ |g_3 - g_3^*|,\ |\theta_4^-| < \delta, \quad |e_4 - e_4^*| < \sqrt{\delta} \right\},$$

$$U_2(\delta) = \left\{ |E_3 - E_3^{**}|,\ |e_3 - e_3^{**}|,\ |g_3 - g_3^{**}|,\ |\theta_4^-| < \delta, \quad |e_4 - e_4^{**}| < \sqrt{\delta} \right\}.$$

We will also need the renormalization map $\mathcal{R}$ defined as follows. Partition our section into cubes of size $1/\sqrt{\chi}$ and on each cube we rescale the space and time so that

- in the center of the cube $Q_3$ has elliptic orbit with energy $-\frac{1}{2}$.

- the potential of the fixed centers is still $1/|Q - Q_j|$.

In addition we reflect the coordinates with respect to $x$ axis.

After the rescaling we apply the dynamics until $x_4$ becomes equal to $-2$ again. Note that the rescaling changes (for the orbits of interest, increases) the distance between the fixed centers by sending $\chi$ to $\lambda\chi$. Observe that at each step we have the freedom of choosing the centers of the cubes. We describe how this choice is made in the next section. In the following we give a proof of the main theorem based on the three lemmas, whose proofs are in the next section.

**Lemma 2.2.4.** *There are cone families $\mathcal{K}_1$ on $T_x(T^*\mathbb{T}^3)$, $x \in U_1(\delta)$ and $\mathcal{K}_2$ on $T_x(T^*\mathbb{T}^3)$, $x \in U_2(\delta)$, each of which contains a two dimensional plane, such that*

*(a) $d\mathcal{P}(\mathcal{K}_1) \subset \mathcal{K}_2$, $d(\mathcal{R} \circ \mathcal{P})(\mathcal{K}_2) \subset \mathcal{K}_1$.*

*(b) If $v \in \mathcal{K}_1$, then $||d\mathcal{P}(v)|| \geq c\chi||v||$. If $v \in \mathcal{K}_2$, then $||d(\mathcal{R} \circ \mathcal{P})(v)|| \geq c\chi||v||$ for some $c > 0$ independent of $\chi$.*

We call a $C^1$ surface $S_1 \subset U_1(\delta)$ (respectively $S_2 \subset U_2(\delta)$ **admissible** if $TS_1 \subset \mathcal{K}_1$ (respectively $TS_2 \subset \mathcal{K}_2$).

**Lemma 2.2.5.** *(a) The vector $\tilde{w} = \frac{\partial}{\partial \ell_3}$ is in $\mathcal{K}_i$.*

*(b) Any plane $\Pi$ in $\mathcal{K}_i$ the map projection map $\pi_{e_4,\ell_3} = (de_4, d\ell_3) : \Pi \to \mathbb{R}^2$ is one-to-one. In other words $(e_4, \ell_3)$ can be used as coordinates on admissible surfaces.*

We call an admissible surface **essential** if $\pi_{e_4,\ell_3}$ is an $I \times \mathbb{T}^1$ for some interval $I$. In other words given $e_4 \in I$ we can prescribe $\ell_3$ arbitrarily.

**Lemma 2.2.6.** *(a) Given an essential admissible surface $S_1 \in U_1(\delta)$ and $\tilde{e}_4 \in I(S_1)$ there exists $\tilde{l}_3$ such that $\mathcal{P}((\tilde{e}_4, \tilde{l}_3)) \in U_2(\delta)$. Moreover if dist$(\tilde{e}_4, \partial I) > 1/\chi$ then there is a neighborhood $V(\tilde{e}_4)$ of $(\tilde{e}_4, \tilde{l}_3)$ such that $\pi_{e_4,\ell_3} \circ \mathcal{P}$ maps $V$ surjectively to*

$$\{|e_4 - e_4^*| < K\delta\} \times \mathbb{T}^1.$$

*(b) Given an essential admissible surface $S_2 \subset U_2(\delta)$ and $\tilde{e}_4 \in I(S_2)$ there exists $\tilde{l}_3$ such that $\mathcal{R} \circ \mathcal{P}((\tilde{e}_4, \tilde{l}_3)) \in U_1(\delta)$. Moreover if dist$(\tilde{e}_4, \partial I) > 1/\chi$ then there is a neighborhood $V(\tilde{e}_4)$ of $(\tilde{e}_4, \tilde{l}_3)$ such that $\pi_{e_4,\ell_3} \circ \mathcal{R} \circ \mathcal{P}$ maps $V$ surjectively to*

$$\{|e_4 - e_4^{**}| < K\delta\} \times \mathbb{T}^1.$$

*(c) For points in $V(\tilde{e}_4)$ from parts (a) and (b), the particles avoid collisions before the next return and $\mu\delta \leq d \leq \frac{\mu}{\delta}$.*

Note that by Lemma 2.2.4 the diameter of $V(\tilde{e}_4)$ is $O(\delta/\chi)$.

### 2.2.7 Construction of the singular orbit.

Fix a small $\varepsilon \gg 1/\chi$. Let $S_0$ be an admissible surface such that the diameter of $S_0$ is much larger than $1/\chi$ and such that on $S_0$ we have

$$|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.$$

where $(\hat{e}_3, \hat{g}_3)$ is close to $(e_3^*, g_3^*)$. For example, we can pick a point $\boldsymbol{x} \in U_1(\delta)$ and let $\hat{w}$ be a vector in $\mathcal{K}_1(\boldsymbol{x})$ such that $\frac{\partial}{\partial \psi_3}(\hat{w}) = 0$. Then let

$$S_0 = \{(E_3, \ell_3, e_3, g_3, e_4, g_4)(\boldsymbol{x}) + a\hat{w} + (0, b, 0, 0, 0, 0)\}_{a \leq \varepsilon/\bar{K}}$$

where $\bar{K}$ is a large constant.

We wish to construct a singular orbit in $S_0$. We define $S_j$ inductively so that $S_j$ is component of $\mathcal{P}(S_{j-1}) \cap U_2(\delta)$ if $j$ is odd and $S_j$ is component of $(\mathcal{R} \circ \mathcal{P})(S_{j-1}) \cap U_1(\delta)$ if $j$ is even (we shall show below that such components exist). Let $\boldsymbol{x} = \lim_{j\to\infty} (\mathcal{R}\mathcal{P}^2)^{-j} S_{2j}$. We claim that $\boldsymbol{x}$ has singular orbit. Indeed by Lemma 2.2.1 the unscaled energy of $Q_4$ satisfies $E(j) \geq (\lambda_0 - \tilde{\delta})^{j/2}$ where $\tilde{\delta} \to 0$ as $\delta \to 0$. Accordingly the velocity of $Q_4$ during the trip $j$ is bounded from below by $c\sqrt{E(j)} \geq c(\lambda_0 - \tilde{\delta})^{j/4}$. Therefore $t_{j+1} - t_j = O((\lambda_0 - \tilde{\delta})^{-j/4})$ and so $t_* = \lim_{j\to\infty} t_j < \infty$ as needed.

It remains to show that if we can find a component of $\mathcal{P}(S_{2j})$ inside $U_2(\delta)$ and a component of $(\mathcal{R} \circ \mathcal{P}(S_{2j+1}))$ inside $U_1(\delta)$. Note that Lemma 2.2.6 allows to choose such components inside larger sets $U_2(K\delta)$ and $U_1(K\delta)$.

First note that by Lemma 2.2.3 on $\mathcal{P}(S_{2j}) \bigcap U_2(K\delta)$ and on $(\mathcal{R}\circ\mathcal{P}^2)(S_{2j}) \bigcap U_2(K\delta)$ we have $\theta_4^- = O(\mu)$. Also by Lemma 2.2.6 $e_4$ can be prescribed arbitrarily. In other words we have a good control on the orbit of $Q_4$.

In order to control the orbit of $Q_3$ note that by Lemma 2.2.4(b) the preimage of $S_{2j}$ has size $O(1/\chi)$ and so by Lemmas 2.2.2, 2.2.3 and 2.2.5 given $\varepsilon$ we have that $e_3$ and $g_3$ have oscillation less than $\varepsilon$ on $S_{2j}$ if $\mu$ is small enough. Namely part (b) of Lemma 2.2.5 shows that $e_3$ and $g_3$ have oscillation $O(1/\chi)$ on the preimage of $S_{2j}$ while Lemmas 2.2.2 and 2.2.3 show that the oscillations do not increase much after application of local and global map. Thus there exist $(\hat{e}_3, \hat{g}_3)$ such that on $S_{2j}$ we have

$$|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.$$

Also due to rescaling we have $\left|E_3 - (-\frac{1}{2})\right| = O(1/\sqrt{\chi})$. Set

$$\tilde{S}_{2j+1} = \mathcal{P}V(e'(\hat{e}_3, \hat{g}_3)), \quad \tilde{S}_{2j+2} = (\mathcal{R} \circ \mathcal{P})V(e''(\hat{e}_3, \hat{g}_3)). \tag{2.2.2}$$

Then on $\tilde{S}_{2j+1}$ we shall have

$$|e_3 - e_3^{**}| < K\varepsilon, \quad |g_3 - g_3^{**}| < K\varepsilon \text{ and } |E_3 - E_3^{**}| < K\varepsilon$$

while on $\tilde{S}_{2j+2}$ we shall have

$$|e_3 - e_3^{*}| < K^2\varepsilon, \quad |g_3 - g_3^{*}| < K^2\varepsilon \text{ and } \left|E_3 - \frac{1}{2}\right| < K/\sqrt{\chi}.$$

Denote

$$S_{2j+1} = \tilde{S}_{2j+1} \cap \{|e_4 - e''(e_3^*, g_3^*)| < \sqrt{\delta}\}, \quad S_{2j+2} = \tilde{S}_{2j+2} \cap \{|e_4 - e'(e_3^*, g_3^*)| < \sqrt{\delta}\}.$$

Taking $\varepsilon$ so small that $K^2\varepsilon < \delta$ we get that $S_{2j+1} \in U_2(\delta)$, $S_{2j+2} \in U_1(\delta)$ as needed.

Finally we use the freedom to choose the appropriate partition in the definition of $\mathcal{R}$ to ensure that $\mathcal{R}$ is continuous on the preimage of $V(e'(\hat{e}_3, \hat{g}_3))$ so that $V(e'(\hat{e}_3, \hat{g}_3))$ is a smooth surface.

**Remark 6.** In fact we do not need to use exactly $e'(\hat{e}_3, \hat{g}_3)$ and $e''(\hat{e}_3, \hat{g}_3)$ in (2.2.2). Namely any $V(e_4^\dagger)$ and $V(e_4^\ddagger)$ would do provided that

$$\left|e_4^\dagger - e_4'(\hat{e}_3, \hat{g}_3)\right| < \varepsilon, \quad \left|e_4^\ddagger - e_4''(\hat{e}_3, \hat{g}_3)\right| < \varepsilon.$$

Different choices of $e_4^\dagger$ and $e_4^\ddagger$ allow us obtain different orbits. Since such freedom exists at each step of our construction we have a Cantor set of singular orbits with a given symbolic sequence $\boldsymbol{\omega}$.

## 3.3 Hyperbolicity of the Poincaré map

### 3.3.1 Construction of invariant cones

Here we derive Lemma 2.2.4, 2.2.5 and 2.2.6 from the asymptotics of the derivative of local and global maps.

**Lemma 3.3.1.** *Suppose* $\boldsymbol{x} \in U_i(\delta)$ *and* $\mathbb{L}(\boldsymbol{x})$ *satisfies* $\theta_4^- = O(\mu), \bar{\theta}_4^+ = \pi + O(\mu)$. *Then there exist a linear functional* $\hat{\mathbf{l}}_i$ *and a vector* $\hat{u}_i$ *such that*

$$d\mathbb{L}(\boldsymbol{x}) = \frac{1}{\mu} u(\boldsymbol{x}) \otimes \mathbf{l}(\boldsymbol{x}) + B(\boldsymbol{x}) + o(1).$$

*Moreover*

$$\mathbf{l} = \hat{\mathbf{l}}_i + o(1), \quad u = \hat{u}_i + o(1), \quad B = \hat{B}_i + o(1), \ \text{as } \delta, \ \mu, \ 1/\chi \to 0,$$

This lemma is proven in Section 5.3.

We further define two new sets in the phase space:

$$\hat{U}_1(\hat{\delta}) = \{|(e_3, g_3, E_3) - \mathbf{G}_{e_4^*}(e_3^*, g_3^*, E_3^*)| < \hat{\delta}, \quad |\theta_4^+| < \hat{\delta} \text{ and } |G_3 + G_4 - (G_3^* + G_4^*)| < \hat{\delta}\},$$

$$\hat{U}_2(\hat{\delta}) = \{|(e_3, g_3, E_3) - \mathbf{G}_{e_4^{**}}(e_3^{**}, g_3^{**}, E_3^{**})| < \hat{\delta}, \quad |\theta_4^+| < \hat{\delta} \text{ and } |G_3 + G_4 - (G_3^{**} + G_4^{**})| < \hat{\delta}\}.$$

Note that if $\hat{\delta} \geq \text{Const}\delta$ then by Lemma 2.2.2 $\hat{U}_i(\hat{\delta})$ contains the part of $U_i$ consisting of the orbits which will have a close encounter with $Q_1$ during the next excursion around $Q_1$.

**Lemma 3.3.2.** (*a*) *For each* $C$ *there exists* $\tilde{C}$ *such that if* $|y(\boldsymbol{x})| \leq C$ *and* $|y(\mathbb{G}(\boldsymbol{x}))| \leq C$ *then* $Q_4$ *passes within distance* $\tilde{C}/\chi$ *from* $Q_1$.

(*b*) *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y} = \mathbb{G}(\boldsymbol{x})$ *be such that* $|y(\boldsymbol{x})| \leq C$, $|y((\boldsymbol{y}))| \leq C$ *and* $Q_4$ *passes within distance* $\tilde{C}/\chi$ *from* $Q_1$. *Then there exist linear functionals* $\bar{\mathbf{l}}(\boldsymbol{x})$ *and* $\bar{\bar{\mathbf{l}}}(\boldsymbol{x})$ *and vectorfields* $\bar{u}(\boldsymbol{y})$ *and* $\bar{\bar{u}}(\boldsymbol{y})$ *such that*

$$d\mathbb{G}(\boldsymbol{x}) = \chi^2 \bar{u}(\boldsymbol{y}) \otimes \bar{\mathbf{l}}(\boldsymbol{x}) + \chi \bar{\bar{u}}(\boldsymbol{y}) \otimes \bar{\bar{\mathbf{l}}}(\boldsymbol{x}) + O(\mu^2 \chi).$$

*Moreover there exist vector* $w_j$ *and linear functionals* $\bar{\mathbf{l}}_i, \bar{\bar{\mathbf{l}}}_i$ *such that if* $\boldsymbol{x} \in \hat{U}_i(\hat{\delta})$ *and* $\hat{\delta} \to 0$ *then*

$$\bar{\mathbf{l}}(\boldsymbol{x}) \to \bar{\mathbf{l}}_i, \quad \bar{\bar{\mathbf{l}}}(\boldsymbol{x}) \to \bar{\bar{\mathbf{l}}}_i,$$

*In addition, if* $\boldsymbol{y} \in \mathcal{R}^{-1} U_1(\delta)$ *and* $\delta \to 0$ *then*

$$span(\bar{u}(\boldsymbol{y}), \bar{\bar{u}}(\boldsymbol{y})) \to span(w_1, \tilde{w})$$

*and In addition, if* $\boldsymbol{y} \in U_2(\delta)$ *and* $\delta \to 0$ *then*

$$span(\bar{u}(\boldsymbol{y}), \bar{\bar{u}}(\boldsymbol{y})) \to span(w_2, \tilde{w})$$

*where* $\tilde{w} = \frac{\partial}{\partial \ell_3}$.

This Lemma is proven in Section 3.3.

**Lemma 3.3.3.** *The following non degeneracy conditions are satisfied.*

*(a1)* $span(\hat{u}_1, B(\hat{\mathbf{l}}_1(\tilde{w}) d\mathcal{R} w_2 - \hat{\mathbf{l}}_1(d\mathcal{R} w_2)\tilde{w}))$ *is transversal to* $Ker(\bar{\mathbf{l}}_1) \cap Ker(\bar{\bar{\mathbf{l}}}_1)$.

*(a2)* $de_4(d\mathcal{R} w_2) \neq 0$.

*(b1)* $span(\hat{u}_2, B(\hat{\mathbf{l}}_2(\tilde{w}) w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}))$ *is transversal to* $Ker(\bar{\mathbf{l}}_2) \cap Ker(\bar{\bar{\mathbf{l}}}_2)$.

*(b2)* $de_4(w_1) \neq 0$.

This Lemma is proven in Section 3.3.

**Definition 2.** We now take $\mathcal{K}_1$ to be the set of vectors which make an angle less than a small constant $\eta$ with $\text{span}(d\mathcal{R}w_2, \tilde{w}_2)$, and $\mathcal{K}_2$ to be the set of vectors which make an angle less than a small constant $\eta$ with $\text{span}(w_1, \tilde{w}_1)$.

*Proof of Lemma 2.2.4.* Consider for example the case where $\boldsymbol{x} \in U_2(\delta)$. We claim that if $\delta, \mu$ are small enough then $d\mathbb{L}(\text{span}(w_1, \tilde{w}))$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\bar{\mathbf{l}}}_2$. Indeed take $\Gamma$ such that $\mathbf{l}(\Gamma) = 0$. If $\Gamma = aw_1 + \tilde{a}\tilde{w}$ then $a\mathbf{l}(w_1) + \tilde{a}\mathbf{l}(\tilde{w}) = 0$. It follows that the direction of $\Gamma$ is close to the direction of $\hat{\Gamma} = \hat{\mathbf{l}}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}$. Next take $\tilde{\Gamma} = bw + \tilde{b}\tilde{w}$ where $b\mathbf{l}(w_1) + \tilde{b}\mathbf{l}(\tilde{w}) \neq 0$. Then the direction of $d\mathbb{L}\tilde{\Gamma}$ is close to $\hat{u}_2$ and the direction of $d\mathbb{L}(\Gamma)$ is close to $B(\hat{\Gamma})$ so our claim follows.

Thus for any plane $\Pi$ close to $\text{span}(w_1, \tilde{w})$ we have that $d\mathbb{L}(\Pi)$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\bar{\mathbf{l}}}_2$. Take any $Y \in \mathcal{K}_2$. Then either $Y$ and $w_1$ are linearly independent or $Y$ and $\tilde{w}$ are linearly independent. Hence $d\mathbb{L}(\text{span}(Y, w_1))$ or $d\mathbb{L}(\text{span}(Y, \tilde{w}))$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\bar{\mathbf{l}}}_2$. Accordingly either $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) \neq 0$ or $\bar{\bar{\mathbf{l}}}_2(d\mathbb{L}(Y)) \neq 0$. If $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) \neq 0$ then the direction of $d(\mathbb{G} \circ \mathbb{L})(Y)$ is close to $\bar{u}$. If $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) = 0$ then the direction of $d(\mathbb{G} \circ \mathbb{L})(Y)$ is close to $\bar{\bar{u}}$. In either case $d(\mathcal{R}\mathbb{G} \circ \mathbb{L})(Y) \in \mathcal{K}_1$ and $||d(\mathbb{G} \circ \mathbb{L})(Y)|| \geq c\chi||Y||$. This completes the proof in the case $\boldsymbol{x} \in U_2(\delta)$. The case where $\boldsymbol{x} \in U_1(\delta)$ is similar. □

*Proof of Lemma 2.2.5.* Part (a) follows from the definition of $\mathcal{K}_i$. Also by part (b) of Lemma 3.3.3 the map $\pi : \text{span}(w, \tilde{w}) \to \mathbb{R}^2$ given by $\pi(\Gamma) = (d\ell_3(\Gamma), de_4(\Gamma))$ is invertible. Namely if $\Gamma = aw + \tilde{a}\tilde{w}$ then

$$a = \frac{de_4(\Gamma)}{de_4(w)}, \quad \tilde{a} = dl_3(\Gamma) - adl_3(w).$$

Accordingly $\pi$ is invertible on planes close to $\text{span}(w, \tilde{w})$ proving our claim. □

To prove Lemma 2.2.6 we need two auxiliary results.

**Sublemma 3.3.4.** *Given $\tilde{e}_4$ there exists $\tilde{l}_3$ such that $\mathcal{P}(\tilde{e}_4, \tilde{l}_3) \in U_2(\delta)$.*

The proof of this Sublemma is postponed to Section 3.3.

**Sublemma 3.3.5.** *Let $\mathcal{F}$ be a map on $\mathbb{R}^2$ which fixes the origin and such that if $|\mathcal{F}(z)| < R$ then $||d\mathcal{F}(X)|| \geq \bar{\chi}||X||$. Then for each $a$ such that $|a| < R$ there exists $z$ such that $|z| < R/\bar{\chi}$ and $\mathcal{F}(z) = a$.*

*Proof of Lemma 2.2.6.* $(a)$ Similarly to the proof of Sublemma 3.3.4 it suffices to show that for each $(\bar{e}_4, \bar{l}_3)$ such that $|\bar{e}_4 - e_4^{**}| < \sqrt{\delta}$ there exist $(\hat{e}_4, \hat{l}_3)$ such that

$$\mathcal{P}(\hat{e}_4, \hat{l}_3) = (\bar{e}_4, \bar{l}_3) \tag{3.3.1}$$

since then the restrictions on $(E_3, e_3, g_3)$ and $\theta_4^-$ will be satisfied automatically. Our coordinates allow us to treat $\mathcal{P}$ as a map $\mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$. Due to Lemma 2.2.4 we can apply Sublemma 3.3.5 to the covering map $\tilde{\mathcal{P}} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\bar{\chi} = c\chi$ obtaining (3.3.1). Part $(b)$ of the lemma is similarly proven.

We give the proof of the part $(c)$ in the part $(b)$ of Lemma 5.2.2. □

### 3.3.2 Expanding directions of the global map

Estimating the derivative of the global map is the longest part of the paper. It occupies Sections 4.2–4.5.

It will be convenient to use the Delaunay coordinates $(L_3, \ell_3, G_3, g_3)$ for $Q_3$ and $(G_4, g_4)$ for $Q_4$. Delaunay coordinates are action-angle coordinates for the Kepler problem. We collect some facts about the Delaunay coordinates in Appendix A.1.

We divide the plane into several pieces by lines $x_4 = -2$ and $x_4 = -\dfrac{\chi}{2}$. Those lines cut the orbit of $Q_4$ into 4 pieces:

- $\{x_4 = -2,\ \dot{x}_4 < 0\} \to \left\{x_4 = -\dfrac{\chi}{2},\ \dot{x}_4 < 0\right\}$. We call this piece $(I)$.
- $\left\{x_4 = -\dfrac{\chi}{2},\ \dot{x}_4 < 0\right\} \to \left\{x_4 = -\dfrac{\chi}{2},\ \dot{x}_4 > 0\right\}$ turning around $Q_1$. We call it $(III)$.
- $\left\{x_4 = -\dfrac{\chi}{2},\ \dot{x}_4 > 0\right\} \to \{x_4 = -2,\ \dot{x}_4 > 0\}$. We call it $(V)$
- $\{x_4 = -2,\ \dot{x}_4 > 0\} \to \{x_4 = -2,\ \dot{x}_4 < 0\}$ turning around $Q_2$.

We composition of the first three pieces constitutes the global map. The last piece defines the local map. See Fig 3.

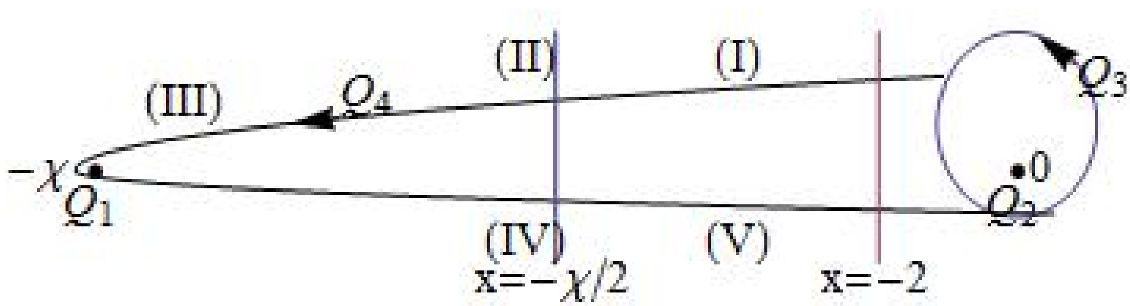


Figure 3: Poincaré sections

The line $x_4 = -\dfrac{\chi}{2}$ is convenient because if $Q_4$ is moving to the right of the line $x_4 = -\dfrac{\chi}{2}$, its motion can be treated as a hyperbolic motion focused at $Q_2$ with perturbation caused by $Q_1$ and $Q_3$. If $Q_4$ is moving to the left of this line, its motion can be treated as a hyperbolic motion focused at $Q_1$ perturbed by $Q_2$ and $Q_3$.

Since we use different guiding centers to the left and right of the line of $x_4 = -\dfrac{\chi}{2}$ we will need to change variables when $Q_4$ hits this line. This will give rise to two more matrices for the derivative of the global map: $(II)$ will correspond to the change of coordinates from right to left and $(IV)$ will correspond for the change of coordinates from left to right. Thus $d\mathbb{G} = (V)(IV)(III)(II)(I)$. In turn, each of the matrices $(II)$ and $(IV)$ will be products of three matrices corresponding to changing one variable in the times. Thus we will have $(II) = [(iii)(ii)](i)$ and $(IV) = (iii')[(ii')(i')]$.

The asymptotics of the above mentioned matrices is presented in the two propositions below.

To refer to a certain subblock of a matrix $(\sharp)$, we use the following convention:

$$(\sharp) = \left[\begin{array}{c|c} (\sharp)_{33} & (\sharp)_{34} \\ \hline (\sharp)_{43} & (\sharp)_{44} \end{array}\right].$$

Thus $(\sharp)_{33}$ is a $4 \times 4$ matrix and $(\sharp)_{44}$ is a $2 \times 2$ matrix. To refer to the $(i,j) - th$ entry of a matrix $(\sharp)$ (in the Delaunay coordinates mentioned above) we use $(\sharp)(i,j)$. For example, $(I)(1,3)$ means the derivative of $L_3$ with respect to $G_3$ when the orbit moves between sections $\{x_4 = -2\}$ and $\left\{x_4 = -\frac{\chi}{2}\right\}$.

**Proposition 3.3.6.** *Under the assumptions of Lemma 3.3.2 the matrices introduced* *above satisfy the following estimates.*

$$(I) = \left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) \\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right],$$

$$(i) = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\tilde{G}_{4R}/k_R\tilde{L}_3}{k_R^2\tilde{L}_3^2+\tilde{G}_{4R}^2}+O(\frac{1}{\chi}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & -\frac{1}{k_R^2\tilde{L}_3^2+\tilde{G}_{4R}^2}+O(\frac{1}{\chi}) & \dfrac{1}{k_R\tilde{L}_3}+O(\frac{1}{\chi}) \end{array}\right],$$

$$[(iii)(ii)] = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & \dfrac{1}{k_R} & \chi \\ O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & -\dfrac{1}{k_R\tilde{L}_3}+O(1/\chi) & -\dfrac{\chi}{\tilde{L}_3}+O(1) \end{array}\right],$$

$$
(III) = \left[\begin{array}{cccc|cc}
1+O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) \\
O(\chi) & O(1) & O(1) & O(1) & O(1) & O(1) \\
O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) \\
O(1/\chi) & O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) \\
\hline
O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(1) \\
O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(1)
\end{array}\right],
$$

$$
[(ii')(i')] =
$$

$$
\left[\begin{array}{cccc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\hline
O(1) & O(1/\chi^2) & O(1/\chi^2) & O(1/\chi^2) & \frac{k_R\chi}{\hat{L}_3^2} + O(1) & \frac{k_R\chi}{\hat{L}_3} + O(1) \\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & -\frac{1}{\hat{L}_3^2} + O(1/\chi) & -\frac{1}{\hat{L}_3} + O(1/\chi)
\end{array}\right],
$$

$$
(iii') =
$$

$$
\left[\begin{array}{cccc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\hat{G}_{4R}/(k_R)}{(k_R^2\hat{L}_3^2+G_{4R}^2)} + O(\frac{1}{\chi}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & -\frac{k_R\hat{L}_3}{k_R^2\hat{L}_3^2+\hat{G}_{4R}^2} + O(\frac{1}{\chi}) & -k_R\hat{L}_3 + O(\frac{1}{\chi})
\end{array}\right],
$$

$$
(V) = \left[\begin{array}{cccc|cc}
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
O(\chi) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\
O(\mu^2\chi) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
O(\mu^2\chi) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\
\hline
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)
\end{array}\right].
$$

*where* $k_R = 1+\mu$, $L_3 = \tilde{L}_3 + O(\mu) = \hat{L}_3 + O(\mu) = \bar{L}_3 + O(\mu)$, $G_3 = \tilde{G}_3 + O(\mu)$, $\bar{G}_3 = \hat{G}_3 + O(\mu)$. *Here* $L_3$ *and* $G_3$ *are the values of the Delaunay coordinates at the initial point and* $\bar{L}_3$ *and* $\bar{G}_3$ *are the values of the Delaunay coordinates at the final point.*

**Proposition 3.3.7.** *The* $O(1)$ *blocks in Proposition 3.3.6 can be written as a sum of continuous functions of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *and an error which vanishes in the limit* $\mu \to 0, \chi \to \infty$. *Moreover the* $O(1)$ *blocks have the following limits for orbits of interest.*

$$
(I)_{44} = \begin{bmatrix} 1 + \dfrac{\tilde{L}_4^2}{2(\tilde{L}_4^2 + \tilde{G}_4^2)} & -\frac{\tilde{L}_4}{2} \\ \dfrac{\tilde{L}_4^3}{2(\tilde{L}_4^2 + \tilde{G}_4^2)^2} & 1 - \dfrac{\tilde{L}_4^2}{2(\tilde{L}_4^2 + \tilde{G}_4^2)} \end{bmatrix}, \quad (V)_{44} = \begin{bmatrix} 1 - \dfrac{1/2\hat{L}_4^2}{\hat{L}_4^2 + \hat{G}_4^2} & -1/2\hat{L}_4 \\ \dfrac{1/2\hat{L}_4^3}{(\hat{L}_4^2 + \hat{G}_4^2)^2} & 1 + \dfrac{1/2\hat{L}_4^2}{\hat{L}_4^2 + \hat{G}_4^2} \end{bmatrix},
$$

$$
(III)_{44} = \begin{bmatrix} \dfrac{1}{2} & -\frac{L_4}{2} \\ \dfrac{3}{2L_4} & \dfrac{1}{2} \end{bmatrix}.
$$

In addition for map (I) we have

$$
((I)(5,1), (I)(6,1))^T = \left( -\frac{\tilde{G}_4 \tilde{L}_4}{2(\tilde{L}_4^2 + \tilde{G}_4^2)}, -\frac{\tilde{G}_4 \tilde{L}_4^2}{2(\tilde{L}_4^2 + \tilde{G}_4^2)^2} \right)^T.
$$

Here and below the phrase *after the first collision* means that the initial orbit has parameters $(\frac{1}{2}, e_3^{**}, g_3^{**}) + o(1)$ for $Q_3$, $G_4$ satisfies $G_4 + G_3^{**} = G_3^* + G_4^* + o(1)$ and that at the final moment the angular momentum of $Q_4$ is close to $G_4^{**}$. The phrase *after the second collision* means that the initial orbit has parameters $(\frac{1}{2}, e_3^*, g_3^*)+o(1)$ for $Q_3$, $G_4$ satisfies $G_4 + G_3^* = G_3^{**} + G_4^{**} + o(1)$ and that at the final moment the angular momentum of $Q_4$ is close to $G_4^*$.

The estimates of $(I), (III), (V)$ from Proposition 3.3.6 are proven in Sections 4.1–4.4. The estimates of $(II), (IV)$ are given in Section 4.5. Proposition 3.3.7 is proven in Section 4.3.2.

In the following, we prove Lemma 3.3.2 based on the Proposition 3.3.7.

*Proof of Lemma 3.3.2.* $d\mathbb{G}$ is a product of several matrices. We will divide the product into three groups. The following estimates are obtained from Proposition 3.3.6 by direct computation.

$$(i)(I) = \left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) \\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right],$$

$$\boldsymbol{M} = [(ii')(i')](III)[(iii)(ii)]$$

$$= \left[\begin{array}{cccc|cc} 1+O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi) \\ O(\chi) & O(1) & O(1) & O(1) & O(1) & O(\chi) \\ O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi) \\ O(1/\chi) & O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(1/\chi^2) & O(1/\chi) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(\chi) & O(\chi^2) \\ O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(\chi) \end{array}\right],$$

$$(V)(iii') = \left[\begin{array}{cccc|cc} O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(\mu^2\chi) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu^2\chi) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right].$$

We decompose $(i)(I)$ and $(V)(iii')$ as

$$(i)(I) = \left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & 0 & 0 \\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & 0 & 0 \\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

$$\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & O(\mu) & O(\mu) \\ 0 & 1 & 0 & 0 & O(\mu) & O(\mu) \\ 0 & 0 & 1 & 0 & O(\mu) & O(\mu) \\ 0 & 0 & 0 & 1 & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right] := [b][a] \tag{3.3.2}$$

$$
(V)(iii') = \left[\begin{array}{cccc|cc}
1 & 0 & 0 & 0 & O(\mu) & O(\mu) \\
0 & 1 & 0 & 0 & O(1) & O(1) \\
0 & 0 & 1 & 0 & O(\mu) & O(\mu) \\
0 & 0 & 0 & 1 & O(\mu) & O(\mu) \\
\hline
O(1) & O(\mu^2) & O(\mu) & O(\mu) & O(1) & O(1) \\
O(1) & O(\mu^2) & O(\mu) & O(\mu) & O(1) & O(1)
\end{array}\right].
$$

$$
\left[\begin{array}{cccc|cc}
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\
O(\chi) & 1+O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\
O(\mu^2\chi) & O(\mu) & 1+O(\mu) & O(\mu) & 0 & 0 \\
O(\mu^2\chi) & O(\mu) & O(\mu) & 1+O(\mu) & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] := [d][c]
$$

Note that $[d]$ and $[a]$ are bounded so they do not change the order of magnitude of the derivative growth. On the other hand, denoting $\boldsymbol{D} = [c]\boldsymbol{M}[b]$ we obtain

$$
\boldsymbol{D} = \left[\begin{array}{cccc|cc}
O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu) & O(\mu\chi) \\
O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(1) & O(\chi) \\
O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu) & O(\mu\chi) \\
O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu) & O(\mu\chi) \\
\hline
O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\chi) & O(\chi^2) \\
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(1) & O(\chi)
\end{array}\right].
$$

Note that $\boldsymbol{D}_{44} = \boldsymbol{M}_{44}$. In particular

$$\frac{\boldsymbol{D}(5,6)}{\chi^2} = \left(\frac{k_R}{L_3^2}, \frac{k_R}{L_3}\right)(III)_{44}\begin{pmatrix} 1 \\ \\ \frac{1}{L_3} \end{pmatrix} + o(1).$$

It follows that if $\chi$ is large and $\mu$ is small then $\dfrac{\boldsymbol{D}(5,6)}{\chi^2}$ is uniformly bounded from above and below. Hence $\boldsymbol{D}$ can be represented as

$$\boldsymbol{D} = \chi^2 \bar{u}' \otimes \bar{\mathbf{l}}' + \chi \bar{\bar{u}}' \otimes \bar{\bar{\mathbf{l}}}' + O(\mu^2\chi),$$

where

$$\bar{u}' = (O(\mu/\chi), O(1/\chi), O(\mu/\chi), O(\mu/\chi), 1, O(1/\chi))^T, \quad \bar{\mathbf{l}}' = (0,0,0,0,\frac{\boldsymbol{D}(5,5)}{\boldsymbol{D}(5,6)}, 1),$$

$$\bar{\bar{u}}' = (O(\mu), 1, O(\mu), O(\mu), O(\mu), 0)^T, \quad \bar{\bar{\mathbf{l}}}' = (1, O(\mu), O(\mu), O(\mu), 0, 0)$$

and we have used the fact that $\frac{\boldsymbol{D}(5,5)}{\boldsymbol{D}(5,6)} = O\left(\frac{1}{\chi}\right)$.

Since $d\mathbb{G}$ is obtained from $\boldsymbol{D}$ by multiplying from the right and the left by bounded matrices we get

$$d\mathbb{G} = \chi^2 \bar{u} \otimes \bar{\mathbf{l}} + \chi \bar{\bar{u}} \otimes \bar{\bar{\mathbf{l}}} + O(\mu^2\chi),$$

where

$$\bar{u} = [d]\bar{u}', \;\; \bar{\bar{u}} = [d]\bar{\bar{u}}', \;\; \bar{\mathbf{l}} = \bar{\mathbf{l}}'[a], \;\; \bar{\bar{\mathbf{l}}} = \bar{\bar{\mathbf{l}}}'[a].$$

In the limit $\mu \to 0, \chi \to \infty$, we have

$$\bar{u}' = (0,0,0,0,1,0)^T, \quad \bar{\mathbf{l}}' = (0,0,0,0,0,1),$$

$$\bar{\bar{u}}' = (0,1,0,0,0,0)^T, \quad \bar{\bar{\mathbf{l}}}' = (1,0,0,0,0,0)$$

Let $[\boldsymbol{a}] = \lim_{\chi\to\infty,\mu\to 0}[a]$. Then $[\boldsymbol{a}] = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & (I)_{44} \end{bmatrix}$, where the limiting expression of $(I)_{44}$ is given in Proposition 3.3.7. This allows us to compute the limiting values of $\bar{\mathbf{l}}$ and $\bar{\bar{\mathbf{l}}}$. Similarly Proposition 3.3.7 shows that as $\chi \to \infty$, $\mu \to 0$ $\bar{\bar{u}} \to (0,1,0,0,0,0)^T$ and it allows us to compute the limiting components of $\bar{u}$ except that we do not have the exact expression for $dl_2(\bar{u})$. However we do not need to know this component because we only interested in the span of $\bar{u}$ and $\bar{\bar{u}}$ and $dl_2(\bar{u})$ can be suppressed by subtracting a suitable multiple of $\bar{\bar{u}}$. Thus the asymptotic parameters of $d\mathbb{G}$ can be summarized as follows:

$$\begin{aligned}
&\bar{\mathbf{l}} = \left(\frac{\tilde{G}_4\tilde{L}_4}{\tilde{L}_4^2+\tilde{G}_4^2}, 0, 0, 0, -\frac{1}{\tilde{L}_4^2+\tilde{G}_4^2}, 1\right), \quad \bar{\bar{\mathbf{l}}} = (1,0,0,0,0,0), \\
&w = \left(0,0,0,0,1,-\frac{\hat{L}_4}{\hat{L}_4^2+\hat{G}_4^2}\right)^T, \quad \tilde{w} = (0,1,0,0,0,0)^T.
\end{aligned} \tag{3.3.3}$$

□

### 3.3.3 Checking transversality

We study the local map numerically. The $O(1/\mu)$ part of $d\mathbb{L}$ in Lemma 3.3.1 is

**Lemma 3.3.8.** *The $O(1/\mu)$ part of the matrix $d\mathbb{L} = \dfrac{\partial(L_3,\ell_3,G_3,g_3,G_4,g_4)^+}{\partial(L_3,\ell_3,G_3,g_3,G_4,g_4)^-}$ is (using the notation of Lemma 3.3.1):*

*(a) for the first collision,*

$$\mathbf{l}_1 = [*,*,*,*,-3.34129, 2.47981].$$

$$\hat{u}_1 = [0.48639, 0.670896, -0.318336, -0.0030828, 0.202124, 0.642799].$$

($b$) *For the second collision:*

$$\mathbf{l}_2 = [*, *, *, *, -1.3908, 0.1897].$$

$$\hat{u}_2 = [-1.72492, 4.40127, 0.911991, -0.740133, 0.591504, -0.495709].$$

($c$) *If $Q_3$ and $Q_4$ switch roles after the collisions, the vectors $\hat{u}_1$ and $\hat{u}_2$ get a "$-$" sign. The computation is done using the choice of $E_3^* = -\frac{1}{2}$ and $e_3^* = \frac{1}{2}$, at Gerver's collision points.*

To check the nondegeneracy condition, it is enough to know the following.

**Lemma 3.3.9.** *If we take the directional derivative of the local map along a direction $\Gamma_i \in span\{w_{3-i}, \tilde{w}\}$, such that $\bar{\mathbf{l}}_i \cdot (d\mathbb{L}\Gamma_i) = 0$, then we have in the case $\mu = 0, \chi \to \infty$, for the both collisions i=1,2, $\dfrac{\partial E_3^+}{\partial \Gamma_i} \neq 0$, where $E_3^+$ is the energy of $Q_3$ after the close encounter with $Q_4$. These derivatives are computed in Gerver's case starting with $E_3^* = -1/2, e_3^* = 1/2$ and evaluated at Gerver's collision points. See the Appendix B.2.2 for concrete values.*

The proof of the two Lemmas are postponed to Section 5.3.

Now we can check the nondegeneracy condition.

*Proof of Lemma 3.3.3.* The items ($a2$) and ($b2$) are seen directly using (3.3.3) and Lemma 3.3.8. We focus on items ($a1$) and ($b1$). For instance, to check ($b2$), $de_4 w \neq 0$. We notice $de_4 = \dfrac{1}{e_4}\left(\dfrac{G_4}{L_4^2} dG_4 - \dfrac{G_4^2}{L_4^3} dL_4\right)$ because of $e_4 = \sqrt{1 + (G_4/L_4)^2}$. So we get $de_4 w = \dfrac{G_4}{L_4^2} \neq 0$ using (3.3.3). Item ($a2$) is proven in the same way.

We can equivalently formulate the transversality condition as follows:

$$\det \begin{pmatrix} \bar{\mathbf{l}}_i(\hat{u}_i) & \bar{\mathbf{l}}_i(\hat{B}_i(\mathbf{l}_i(\tilde{w})w_{3-i} - \mathbf{l}_i(w_{3-i})\tilde{w})) \\ \bar{\bar{\mathbf{l}}}_i(\hat{u}_i) & \bar{\bar{\mathbf{l}}}_i(\hat{B}_i(\mathbf{l}_i(\tilde{w})w_{3-i} - \mathbf{l}_i(w_{3-i})\tilde{w})) \end{pmatrix} \neq 0, \tag{3.3.4}$$

where the subscript $i = 1, 2$ indicates the first or the second collision. The case $i = 1$ is equivalent to $(a1)$ if we substitute $w_2$ by $d\mathcal{R}w_2$. The case $i = 2$ is equivalent to $(b1)$.

The computation of $\hat{u}_i$'s are done in Lemma 3.3.8. We have $\bar{\mathbf{l}}_i \cdot \hat{u}_i \neq 0$ based on these numerics.

When checking the nondegeneracy condition (3.3.4), we denote $\Gamma_i' = \mathbf{l}_i(\tilde{w})w_{3-i} - \mathbf{l}_i(w_{3-i})\tilde{w}$ and we can replace $\Gamma_i'$ by $\Gamma_i$ satisfying $\bar{\mathbf{l}}_i \cdot (d\mathbb{L}\Gamma_i) = 0$. Indeed, $d\mathbb{L}\Gamma_i$ as a vector in $span\{\hat{u}_i, \hat{B}_i\Gamma_i'\}$, can be represented as

$$d\mathbb{L}\Gamma_i = b_i\hat{u}_i + \tilde{b}_i\hat{B}_i\Gamma_i'.$$

We should have $b_i = -\bar{\mathbf{l}}_i \cdot \hat{B}_i\Gamma_i'$ and $\tilde{b}_i = \bar{\mathbf{l}}_i \cdot \hat{u}_i \neq 0$ up to a multiple of a nonzero constant in order to make sure $d\mathbb{L}\Gamma_i \in Ker\bar{\mathbf{l}}_i$. We have the following equality.

$$\det \begin{pmatrix} \bar{\mathbf{l}}_i(\hat{u}_i) & \bar{\mathbf{l}}_i(\hat{B}_i\Gamma_i') \\ \bar{\bar{\mathbf{l}}}_i(u_i) & \bar{\bar{\mathbf{l}}}_i(\hat{B}_i\Gamma_i') \end{pmatrix} = \frac{1}{\tilde{b}_i} \det \begin{pmatrix} \bar{\mathbf{l}}_i(\hat{u}_i) & \bar{\mathbf{l}}_i(d\mathbb{L}\Gamma_i) \\ \bar{\bar{\mathbf{l}}}_i(\hat{u}_i) & \bar{\bar{\mathbf{l}}}_i(d\mathbb{L}\Gamma_i) \end{pmatrix}$$

From the hypothesis of Lemma 3.3.9, we have $\bar{\mathbf{l}}_i(d\mathbb{L}\Gamma_i) = 0$. We only need to make sure $\bar{\bar{\mathbf{l}}}_i(d\mathbb{L}\Gamma_i) \neq 0$ to guarantee the nondegeneracy of the determinant. Indeed, $\bar{\bar{\mathbf{l}}}_i = (1, 0, 0, 0, 0, 0)$. So $\bar{\bar{\mathbf{l}}}_i(d\mathbb{L}\Gamma_i) \neq 0$ holds if the vector $d\mathbb{L}\Gamma_i$ has nonzero first entry. As we know $d\mathbb{L}\Gamma_i$ means to take directional derivative of the local map along the direction $\Gamma_i$. This is exactly the $\dfrac{\partial E_3^+}{\partial \Gamma_i}$ checked in Lemma 3.3.9. $\square$

### 3.3.4 The reflection and renormalization

The above calculations address two steps of collisions in Gerver's model (c.f. Appendix B.2.1 and [G1]). After two steps, we are supposed to get an ellipse of the same eccentricity but smaller semimajor (c.f. part (b) of Lemma 2.2.1) and then we zoom in the picture such that the ellipse has the original size. We call this procedure the renormalization. Notice in part (b) of Lemma 2.2.1, $\bar{\bar{g}}_3$ gets a "-" sign. In fact, after two steps of collisions in Gerver's model, the ellipse gets reflected along the $x$-axis. We should treat four steps of collisions and two renormalizations as a period. However, the calculations for the third and fourth step of collisions can be obtained from the first and the second respectively by studying the reflection carefully.

In the following, we formulate a lemma explaining the effect of reflection and then discuss the renormalization as a remark. We stress that in the calculation of the local and global map, we already take into account the renormalization. We only explain how the computations are done.

#### 3.3.4.1 The reflection

**Lemma 3.3.10.** *If we reflect the our system along the $x$-axis, then under the same assumption as Lemma 3.3.1 and 3.3.2, we have the following result for the global and local maps.*

*(a) The statement of Lemma 3.3.2 remains unchanged. The vectors and functionals $\bar{\bar{\mathrm{l}}}, w, \tilde{w}$ in (3.3.3) remain unchanged, while $\bar{\mathrm{l}}$ gets a "-" sign in its first entry.*

*(b) The statement of Lemma 3.3.1 remains the same.*

*The vectors and functionals* $\mathbf{l}_1, \mathbf{l}_2, \hat{u}_1, \hat{u}_2$ *in Lemma 3.3.8 and the vector* $\dfrac{\partial E_3^+}{\partial \Gamma_i}$ *in Lemma 3.3.9 get a "-" sign in their last four entries.*

*(c) The nondegeneracy conditions Lemma 3.3.3 hold.*

*Proof.* After the second collision, we need to apply the renormalization $\mathcal{R}$. Simultaneously, there is a reflection along the $x$-axis. As a matter of fact, we see this by comparing the smaller ellipse in Fig 2 with the rounder ellipse in Fig 1. The effect of the reflection is to give a "$-$" sign to the $G_3, g_3, G_4, g_4$ variables while keep $L_3, \ell_3$ unchanged. Therefore, if we look at the global map, the reflected matrix $d\mathbb{G}$ would be the same as the old one in the diagonal blocks $\dfrac{\partial(L_3, \ell_3)^f}{\partial(L_3, \ell_3)^i}$ and $\dfrac{\partial(G_3, g_3, G_4, g_4)^f}{\partial(G_3, g_3, G_4, g_4)^i}$ (where $i$ means "initial", $f$ means "final") while the remaining entries get a "$-$" sign. We see from (3.3.3) that after the reflection, $\bar{\mathbf{l}}$ gets a "-" sign since $G_4$ does while other vectors remains the same.

For the local map part, we notice $\mathbf{l}_i$ has the form of $\dfrac{\partial -}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-}$, $\hat{u}_i$ has the form of $\dfrac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial -}$ and $\dfrac{\partial E_3^+}{\partial \Gamma_i} = \dfrac{\partial E_3^+}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-}$ for $i = 1, 2$ and the superscripts "$\pm$" standing for entering and exiting the section $|Q_3 - Q_4| = \mu^\kappa$. As a result, we need to make the changes stated in the lemma.

To check the nondegeneracy condition, we proceed as in the proof of Lemma 3.3.3. Essentially, we need $\bar{\mathbf{l}}_i \cdot \hat{u}_i \neq 0$ and the first entry of $\dfrac{\partial E_3^+}{\partial \Gamma_i}$ is nonzero. These can be easily verified. $\square$

#### 3.3.4.2 The renormalization

The renormalization occurs after the second collision (see Fig 2). When calculating the matrices and vectors, we already take into account the renormalization. Let us explain how the information of Appendix B.2.2 is fit into the matrices. For the global map, we always choose $L_3$ to be close to 1. In $(I)$ and $(i)$, we use the information after the second collision with the renormalization $\tilde{L}_3 = 1$ and $\tilde{G}_4 = G_4/L_3,\ \tilde{G}_3 = G_3/L_3$. This is where the renormalization occurs. For the other calculations, such as $(iii')$ and $(V)$ before the first and the second collision, and $(I)$ and $(i)$ after the first collision, no renormalization is needed.

For the local map part, only the vector $\hat{u}_2$ in Lemma 3.3.8 undergoes such a rescaling. We divide the first, third, and fifth entries of $\hat{u}_2$ by $L_3$ after the second collision in Appendix B.2.2.

# Chapter 4

# The global map

## 4.1 $C^0$ estimates for global map

### 4.1.1 Equations of motion in Delaunay coordinates

We use Delaunay variables to describe both the motions of $Q_3$ and $Q_4$ (see the Appendix A.1 for an introduction). We have eight variables $(L_3, \ell_3, G_3, g_3)$ and $(L_4, \ell_4, G_4, g_4)$. We eliminate $L_4$ using the energy conservation and $\ell_4$ will play the role of independent variable.

After setting $v_{3,4} = P_{3,4}/\mu$ and dividing (1.1.1) by $\mu$ the Hamiltonian (1.1.1) takes the form

$$H = \frac{v_3^2}{2} + \frac{v_4^2}{2} - \frac{1}{|Q_3|} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{1}{|Q_4 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}. \quad (4.1.1)$$

When $Q_4$ is moving to the left of the section $\{x_4 = -\chi/2\}$, we consider the motion of $Q_3$ as elliptic motion with focus at $Q_2$, and $Q_4$ as hyperbolic motion with focus at $Q_1$, perturbed by other interactions. We can write the Hamiltonian in terms of Delaunay variables as

$$H_L = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.$$

When $Q_4$ is moving to the right of the section $\{x_4 = -\chi/2\}$, we consider the motion of $Q_3$ as an elliptic motion with focus at $Q_2$, and that of $Q_4$ as a hyperbolic motion with focus at $Q_2$ attracted by the pair $Q_2, Q_3$ which has mass $1+\mu$ plus a perturbation. For $|Q_4| \geq 2$ we have the following Taylor expansion

$$\frac{\mu}{|Q_3 - Q_4|} = \frac{\mu}{|Q_4|} + \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right).$$

Hence the Hamiltonian takes form

$$H = \frac{v_3^2}{2} + \frac{v_4^2}{2} - \frac{1}{|Q_3|} - \frac{1+\mu}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{1}{|Q_4 - (-\chi, 0)|} - \frac{\mu Q_3 \cdot Q_4}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right).$$

In terms of the corresponding Delaunay variables we have

$$H_R = -\frac{1}{2L_3^2} + \frac{(1+\mu)^2}{2L_4^2} - \frac{1}{|Q_3 + (\chi, 0)|} - \frac{1}{|Q_4 + (\chi, 0)|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right). \tag{4.1.2}$$

We shall use the following notation. The coefficients of $\frac{1}{2L_4^2}$ in the Hamiltonian will be called $k_L = 1$ and $k_R^2 = (1+\mu)^2$. The terms in the Hamiltonian containing $Q_4$ will be denoted by

$$V_R = -\frac{1}{|Q_4 + (\chi, 0)|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right), \text{ and } V_L = -\frac{1}{|Q_4|} - \frac{\mu}{|Q_3 - Q_4|}. \tag{4.1.3}$$

Here subscripts L and R mean that the corresponding expressions are used when $Q_4$ is to the left (respectively to the right) of the line $Q = -\frac{\chi}{2}$. Likewise for the terms containing $Q_3$ we define

$$\begin{aligned} U_R &= -\frac{1}{|Q_3 + (\chi, 0)|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right), \\ U_L &= -\frac{1}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}. \end{aligned} \tag{4.1.4}$$

The use of subscripts $R, L$ here is the same as above. Let us write down the full

Hamiltonian equations with the subscripts $R$ and $L$ suppressed.

$$\begin{cases} \dot{L}_3 = -\dfrac{\partial Q_3}{\partial \ell_3}\cdot\dfrac{\partial U}{\partial Q_3}, & \dot{\ell}_3 = \dfrac{1}{L_3^3}+\dfrac{\partial Q_3}{\partial L_3}\cdot\dfrac{\partial U}{\partial Q_3}, \\ \dot{G}_3 = -\dfrac{\partial Q_3}{\partial g_3}\cdot\dfrac{\partial U}{\partial Q_3}, & \dot{g}_3 = \dfrac{\partial Q_3}{\partial G_3}\cdot\dfrac{\partial U}{\partial Q_3}, \\ \dot{L}_4 = -\dfrac{\partial Q_4}{\partial \ell_4}\cdot\dfrac{\partial V}{\partial Q_4}, & \dot{l}_4 = -\dfrac{k^2}{L_4^3}+\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}, \\ \dot{G}_4 = -\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}, & \dot{g}_4 = \dfrac{\partial Q_4}{\partial G_4}\cdot\dfrac{\partial V}{\partial Q_4}. \end{cases} \tag{4.1.5}$$

Next we use the energy conservation to eliminate $L_4$. We have

$$\begin{aligned} &\frac{L_4^3}{k_R^2} = k_R L_3^3\cdot\Big(1-3L_3^2\Big(\frac{1}{|Q_3+(\chi,0)|}+\frac{1}{|Q_4+(\chi,0)|} \\ &+\frac{\mu Q_4\cdot Q_3}{|Q_4|^3}+O\left(\frac{\mu}{|Q_4|^3}\right)+O(1/\chi^2)\Big)\Big) := k_R L_3^3+W_R, \\ &\frac{L_4^3}{k_L^2} = k_L L_3^3\left(1-3L_3^2\left(\frac{1}{|Q_3+(\chi,0)|}+\frac{1}{|Q_4|}-\frac{\mu}{|Q_4-Q_3|}+O(1/\chi^2)\right)\right) \\ &:= k_L L_3^3+W_L. \end{aligned} \tag{4.1.6}$$

We use $\ell_4$ as the independent variable. Dividing (4.1.5) by $\dot{\ell}_4$ and using (4.1.6) to eliminate $L_4$ we obtain

$$\begin{cases} \dfrac{dL_3}{d\ell_4} &= (kL_3^3+W)\dfrac{\partial Q_3}{\partial \ell_3}\cdot\dfrac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}\right) \\ \dfrac{d\ell_3}{d\ell_4} &= -(kL_3^3+W)(\dfrac{1}{L_3^3}+\dfrac{\partial Q_3}{\partial L_3}\cdot\dfrac{\partial U}{\partial Q_3})\left(1+(kL_3^3+W)\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}\right) \\ \dfrac{dG_3}{d\ell_4} &= (kL_3^3+W)\dfrac{\partial Q_3}{\partial g_3}\cdot\dfrac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}\right) \\ \dfrac{dg_3}{d\ell_4} &= -(kL_3^3+W)\dfrac{\partial Q_3}{\partial G_3}\cdot\dfrac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}\right) \\ \dfrac{dG_4}{d\ell_4} &= (kL_3^3+W)\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}\left(1+(kL_3^3+W)\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}\right) \\ \dfrac{dg_4}{d\ell_4} &= -(kL_3^3+W)\dfrac{\partial Q_4}{\partial G_4}\cdot\dfrac{\partial V}{\partial Q_4}\left(1+(kL_3^3+W)\dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}\right) \end{cases} + O\left(\frac{\mu}{|Q_4|^3}+1/\chi^2\right). \tag{4.1.7}$$

We shall use the following notation $X = (L_3, \ell_3, G_3, g_3)$, $Y = (G_4, g_4)$.

### 4.1.2 A priori bounds

#### 4.1.2.1 Estimates of positions

We have the following estimates for the positions.

**Lemma 4.1.1.** *We assume that the position of $Q_3$ is bounded and the $y$-component of $Q_4$ is also bounded. Namely, suppose that*

$$|Q_3| < C \text{ and } |Q_{4y}| < C \tag{4.1.8}$$

*for some constant $C$. We also assume that the initial energy of $Q_3$ is $-1/2$.*

*(a) We have*

$$\left|\frac{\partial Q_3}{\partial X}\right| < C' \tag{4.1.9}$$

*for some constant $C'$ independent of $\mu$ and $\chi$.*

*(b) When $Q_4$ is moving to the right of the section $\{x = -\chi/2\}$ and to the left of $\{x = -2\}$, then we have the estimates*

$$|Q_4 - Q_3| = (1 + O(\mu))t, \quad |Q_4 - Q_3| \in [1, \chi/2] \text{ and } |Q_4 + (\chi, 0)| \geq \chi/2. \tag{4.1.10}$$

*(c) When $Q_4$ is moving to the left of the section $\{x = -\chi/2\}$, we have the following estimates*

$$|Q_4 - Q_1| = \chi - (1 + O(\mu))t, \quad |Q_4 - Q_1| \in (0, \chi/2] \text{ and } |Q_4|, |Q_4 - Q_3| \geq \chi/2. \tag{4.1.11}$$

The intuition behind this lemma is the following. Since the total energy of the system is zero and $Q_3$ and $Q_4$ interact only weakly with each other, then both particle have energies close to 1/2 in absolute value. Since $Q_4$ spends most of the time away from $Q_1, Q_2$ and $Q_3$ most of its energy is kinetic energy so it moves with approximately unit speed. Since it makes a little progress in $y$ direction its velocity is almost horizontal most of the time. This explains (4.1.10) and (4.1.11). To give the complete proof we have to use the Hamiltonian equations. See section 4.1.3.

**Lemma 4.1.2.** *If inequalities* (4.1.8), (4.1.10) *and* (4.1.11) *are valid and in addition*

$$1/C \leq |L_3|, |L_4| \leq C, \quad |G_3|, |G_4| < C, \tag{4.1.12}$$

*then we have*

$$\frac{\partial Q_4}{\partial \ell_4} = O(1), \quad \frac{\partial Q_4}{\partial (L_4, G_4, g_4)} = O(t), \quad \frac{\partial Q_4}{\partial g_4} \cdot Q_4 = 0 \text{ and } \frac{\partial Q_4}{\partial G_4} \cdot Q_4 = O(t)$$

*as $t \to \infty$.*

*Proof.* This follows directly from Lemma A.1.2 in Appendix A.1.4. We point out that the estimate $\frac{\partial Q_4}{\partial G_4} \cdot Q_4 = O(t)$ is nontrivial.

□

#### 4.1.2.2 Estimates of potentials

**Lemma 4.1.3.** *Under the assumptions of Lemma 4.1.2 we have the following estimates for the potentials $U, V, W$:*

$$V_R = O\left(\frac{1}{\chi} + \frac{\mu}{t^2}\right), \quad U_R = O\left(\frac{1}{\chi} + \frac{\mu}{t^2}\right), \quad t \in [2, \chi/2],$$

$$V_L = O\left(\frac{1}{\chi}\right), \quad U_L = O\left(\frac{1}{\chi}\right), \quad W_L = O\left(\frac{1}{\chi}\right), \quad t \in [\chi/2, \chi].$$

*Proof.* This follows directly from equations (4.1.3), (4.1.4) and (4.1.6). □

#### 4.1.2.3 Estimates of gradients of potentials

To take partial derivatives w.r.t. Delaunay variables, we use the formulas

$$\frac{\partial}{\partial X} = \frac{\partial Q_3}{\partial X} \cdot \frac{\partial}{\partial Q_3}, \quad \frac{\partial}{\partial Y} = \frac{\partial Q_4}{\partial Y} \cdot \frac{\partial}{\partial Q_4}.$$

**Lemma 4.1.4.** *Under the assumptions of Lemma 4.1.2 we have the following estimates for the gradients of the potentials $U, V$*

$$\frac{\partial U_R}{\partial Q_3} = O\left(\frac{1}{\chi^2} + \frac{\mu}{t^2}\right), \ \frac{\partial V_R}{\partial Q_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{t^3}\right), \ \frac{\partial Q_4}{\partial (G_4, g_4)} \frac{\partial V_R}{\partial Q_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{t^2}\right), \ t \in [2, \chi/2],$$
$$\frac{\partial U_L}{\partial Q_3} = O\left(\frac{1}{\chi^2}\right), \quad \frac{\partial V_L}{\partial Q_4} = O\left(\frac{1}{\chi^2}\right), \quad \frac{\partial Q_4}{\partial (G_4, g_4)} \frac{\partial V_L}{\partial Q_4} = O\left(\frac{1}{\chi^2}\right), \quad t \in [\chi/2, \chi]. \tag{4.1.13}$$

*Proof.* The estimates for the $\dfrac{\partial}{\partial Q_{3,4}}$ terms are straightforward. Indeed, we only need to use the fact that the derivative of functions of the form $\dfrac{1}{(1+x)^k}$ has the form of $\dfrac{-k}{(1+x)^{k+1}}$ together with the estimates in Lemma 4.1.1.

The estimates of all $\dfrac{\partial}{\partial (G_4, g_4)}$ terms are similar. We consider for instance $\frac{\partial Q_4}{\partial G_4}\frac{\partial V_R}{\partial Q_4}$. We have

$$\frac{\partial Q_4}{\partial G_4} \frac{\partial V_R}{\partial Q_4} = \frac{\partial Q_4}{\partial G_4} \frac{Q_4 + (\chi, 0)}{|Q_4 + (\chi, 0)|^3} + O\left(\mu \left|\frac{\partial Q_4}{\partial G_4}\right| |Q_4|^{-3}\right). \tag{4.1.14}$$

The second term here is $O(\mu/t^2)$ due to (4.1.10) and Lemma A.1.2(a). To handle the first term let $\dfrac{\partial Q_4}{\partial G_4} = (\boldsymbol{a}, \boldsymbol{b})$, $Q_4 = (x, y)$. Note that equations (A.1.3), (A.1.4), (4.1.8), (4.1.10) and (4.1.12) show that $x, \ell$ and $t$ are all comparable in the sense that the ratios between any two of these qualities are bounded from above and below. On the other hand Lemma A.1.2(a) tells us that $\boldsymbol{a}x+\boldsymbol{y} = O(t)$. Since $\boldsymbol{b}y = O(\boldsymbol{b}) = O(t)$ we conclude that $\boldsymbol{a}x = O(t)$ and thus $\boldsymbol{a} = O(1)$. Thus the first term in (4.1.14) is

$$\frac{\frac{\partial Q_4}{\partial G} \cdot Q_4 + \boldsymbol{a}\chi}{|Q_4 + (\chi, 0)|^3}.$$

The numerator here is $O(\chi)$ while the denominator is at least $(\chi/2)^3$. This completes the estimate of $\frac{\partial Q_4}{\partial G_4}\frac{\partial V_R}{\partial Q_4}$. Other derivatives are similar. □

Plugging the above estimates into (4.1.7) we obtain the following.

**Lemma 4.1.5.** *Under the assumptions of Lemma 4.1.2 we have the following estimates on the RHS of* (4.1.7).

*(a) When $Q_4$ is moving to the right of the section $\{x = -\chi/2\}$ and to the left of the section $\{x = -2\}$, we have $t \in [2, \chi/2]$ and*

$$\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{t^2}\right), \quad \frac{d\ell_3}{d\ell_4} = O(1).$$

*(b) When $Q_4$ is moving to the left of the section $\{x = -\chi/2\}$, we have $t \in [\chi/2, \chi]$ and*

$$\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2}\right), \quad \frac{d\ell_3}{d\ell_4} = O(1).$$

In Section 4.3 we will need the following bounds on the second derivatives.

**Lemma 4.1.6.** *Under the assumptions of Lemma 4.1.2 we have the following esti-*

*mates for the second derivatives.*

$$\frac{\partial^2 U_R}{\partial Q_3^2} = O\left(\frac{1}{\chi^3}+\frac{\mu}{t^2}\right),\ \frac{\partial^2 V_R}{\partial Q_4^2} = O\left(\frac{1}{\chi^3}+\frac{\mu}{t^4}\right),\ \frac{\partial^2 (U_R,V_R)}{\partial Q_3\partial Q_4} = O\left(\frac{\mu}{t^3}\right),\ t\in[2,\chi/2]$$
$$\frac{\partial^2 U_L}{\partial Q_3^2} = O\left(\frac{1}{\chi^3}\right),\quad \frac{\partial^2 V_L}{\partial Q_4^2} = O\left(\frac{1}{\chi^3}\right),\quad \frac{\partial^2 (U_L,V_L)}{\partial Q_3\partial Q_4} = O\left(\frac{1}{\chi^3}\right)\quad t\in[\chi/2,\chi]. \tag{4.1.15}$$

We omit the proof since it is again direct computation.

### 4.1.3 Proof of Lemma 4.1.1

*proof of Lemma 4.1.1.* We first impose an assumption

$$\frac{1}{2}|L_3(0)| \le |L_3(t)| \le 2|L_3(0)|.$$

This is always correct if the time $t$ is small due to the continuity of the Hamiltonian flow. Then using formula (A.1.3), we find $|Q_4| \ge \frac{1}{4}L_3(0)^2|\ell_4|$. Using the estimate of potentials in Lemma A.1.2 and Lemma 4.1.5, we get $\frac{dL_3}{d\ell_4} = O\left(\frac{1}{\chi^2}+\frac{\mu}{t^2}\right)$, where $t$ is actually $|Q_4|$ without using Lemma 4.1.1. Then we integrate the equation w.r.t. $\ell_4$, we find the oscillation of $L_3$ is $O(\mu)$ for $\ell_4 \in [2, 2+\delta)$ and $\delta$ small. As a result our assumptions on $L_3$ always hold and we can integrate over time of order $\chi$ and get that the oscillation of $L_3$ is $O(\mu)$. Similar argument holds for other variables $G_3, g_3, G_4, g_4$. So we get that the oscillations of the variables $G_3, g_3, G_4, g_4$ are $O(\mu)$.

To show part (a), we notice $\frac{\partial Q_3}{\partial X}$ depends on $\ell_3, g_3$ periodically according to equation (A.1.1). So it is enough to bound $L_3$ and $G_3$. This follows from the above argument.

To show part (b) and (c), we use the Hamiltonian equation $\dot{\ell}_4 = \dfrac{-1}{L_4^3} + O(\mu)$. The $O(\mu)$ oscillation estimates above together with the equation A.1.3 show that $|Q_4|$ grows linearly. This completes the proof of the Lemma 4.1.1. $\square$

### 4.1.4 Avoiding collisions

Here we exclude the possibility of collisions. The only possible collisions may occur for the pair $Q_3, Q_4$ and the pair $Q_1, Q_4$. The fact that $Q_3$ and $Q_4$ do not collide to each other will be shown in Lemma 5.2.2 in Section 5.3. Now we prove there is no collision between $Q_4$ and $Q_1$.

**Lemma 4.1.7.** *If the angular momentum $G_4$ has $O(1)$ change when evaluated on the section $\{x_4 = -2\}$ after applying the global map compared with the value before applying the global map, then there is no collision between $Q_4$ and $Q_1$.*

*Proof.* Suppose there is such a collision, we derive some estimates. Consider the Hamiltonian equations for the $Q_4$ part. Let us write the Hamiltonian equations as

$$Y' = \mathcal{V},$$

where $Y = (G_4, g_4)$ and $\mathcal{V}$ is the RHS of the corresponding Hamiltonian equations.

We run the orbit coming to a collision backward so that we can compare two orbits exiting collisions. If we write the equation as $Y'_{in} = \mathcal{V}_{in}$ orbit coming to collision with time arrow reversed and $Y'_{out} = \mathcal{V}_{out}$ for orbit exiting collision, we have $\mathcal{V}_{in} - \mathcal{V}_{out} = O\left(\dfrac{\mu}{|Q_4 - Q_3|^2}\right)$. This difference is created by the motion of $Q_3$.

We have

$$(Y_{in} - Y_{out})' = \frac{\partial \mathcal{V}}{\partial Y}(Y_{in} - Y_{out}) + O\left(\frac{\mu}{|Q_4 - Q_3|^2}\right)$$

We integrate the equation along an orbit starting from $Q_1$ and ending at $\{x_4 = -\chi/2\}$. The initial condition is $Y_{in} - Y_{out} = 0$ since $G_4, g_4$ are conserved quantities for Kepler motion and they assume the same values before and after the $Q_4, Q_1$ collision. Using the fact that $\int_{\ell_4^i}^{\ell_4^f} \frac{\partial \mathcal{V}}{\partial Y} d\ell_4 = O(1)$, and $\int_{\ell_4^i}^{\ell_4^f} O\left(\frac{\mu}{|Q_4 - Q_3|^2}\right) d\ell_4 = O(\mu/\chi)$, we have

$$Y_{in} - Y_{out} = O(\mu/\chi) \tag{4.1.16}$$

in the section $\{x_4 = -\chi/2\}$.

Let us see what happens to the angular momentum of $Q_4$ when measured w.r.t. $Q_2$. From equation (4.1.17) and the Appendix A.1, we have the formula

$$G_{4R}/k_R = G_{4L} + v_{4y}\chi.$$

Here $v_{4y}$ is the $y$ component of the velocity of $Q_4$ measured in the section $\{x_4 = -\chi/2\}$. Using the equation (A.1.5) in the Appendix A.1.2, equation (A.1.3) and equation (4.1.16) that we obtained just now, we have $v_{4y,in} - v_{4y,out} = O(\mu/\chi)$. This means, if we measure the angular momentum of $Q_4$ w.r.t. $Q_2$ in the section $\{x = -\chi/2\}$, we have

$$G_{4R,in} - G_{4R,out} = O(\mu)$$

However, as we know, if we measure the angular momentum of $Q_4$ w.r.t. $Q_2$ along orbits after close encounter with $Q_3$ and before the next close encounter, in Gerver's construction, the angular momentum differ by $O(1)$. The difference is still

$O(1)$ when measured in the section $\{x = -\chi/2\}$ according to Lemma 2.2.3. This is a contradiction. So collision between $Q_4$ and $Q_1$ is excluded.

$\square$

### 4.1.5 $C^0$ estimates, Proof of Lemma 2.2.3

*Proof of Lemma 2.2.3.* We use Lemma 4.1.5.

$$\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{t^2}\right),$$

For part $(a)$ of the lemma, we integrate the equations of $\dfrac{dL_3}{d\ell_4}, \dfrac{dG_3}{d\ell_4}, \dfrac{dg_3}{d\ell_4}$, over $t \in [2, \chi]$ twice since $Q_4$ moves far from $Q_2$ then comes back. Therefore we get
$2\displaystyle\int_2^{\chi} \frac{\mu}{t^2} + \frac{1}{\chi^2} dt = O(\mu)$ after integration.
For part $(b)$ of the lemma, we integrate the equations of $\dfrac{dG_4}{d\ell_4}, \dfrac{dg_4}{d\ell_4}$, over $t \in [2, \chi/2]$. We get $O(\mu)$ again after integration. $\square$

The next lemma gives more information about the $Q_4$ part of the orbit than Lemma 2.2.3.

**Lemma 4.1.8.** *Under the same hypothesis as Lemma 4.1.1, we have:*

*(a) when $Q_4$ is moving to the right of the section $\{x = -\chi/2\}$, we have*

$$\tan g_4 = \pm \frac{G_4}{L_4} + O(\mu/t + 1/\chi), \quad u \gtrless 0, \quad |t| \to \infty.$$

*(b) when $Q_4$ is moving to the left of the section $\{x = -\chi/2\}$, then $G, g = O(1/\chi)$ as $\chi \to \infty$.*

*Proof.* We prove part $(b)$ first. We have shown that there is no collision between $Q_4$ and $Q_1$ in Section 4.1.4. When $Q_4$ and $Q_1$ are close, their motion is a Kepler motion perturbed by $Q_2$ and $Q_3$. If we neglect the perturbation. The angle formed by the two asymptotes are $O(1/\chi)$ since the vertical drift of $Q_4$ is $O(1)$ when its horizontal distance to $Q_1$ becomes $O(\chi)$. In terms of Delaunay variables, this angle is $2\arctan\dfrac{G}{L}$. This shows $G = O(1/\chi)$ since $L$ is close to 1. The argument of periapsis $g$ is also $O(1/\chi)$, since the $x$-axis lies inside the $O(1/\chi)$ angle formed by the two asymptotes. If we introduce the perturbations from $Q_2$ and $Q_3$, then using Lemma 4.1.5, after integrating over time $\chi$, the perturbations from $Q_2$ and $Q_3$ give rise to an $O(1/\chi)$ oscillation of $G_4, g_4$.

Then consider part $(a)$. This condition $g = \pm\arctan G/L$ means horizontal asymptote (see equation (A.1.5)). We want to allow the asymptotes to be slightly tilted. Part $(b)$ shows that when measured at the section $\{x = -\chi/2\}$, we have $|G_{4L}|, |g_{4L}| = O(1/\chi)$, the slope of the asymptotes of $Q_4$ expressed using the coordinates system of the left of the section $\{x = -\chi/2\}$ is $g_{4L} \pm \arctan\dfrac{G_{4L}}{L_{4L}} = O(1/\chi)$. When passing through the section $\{x = -\chi/2\}$, we need to express the asymptote using the coordinates system of the right. This is $g_{4R} - \arctan\dfrac{G_{4R}}{L_{4R}} = O(1/\chi)$. Then we want to see the propagation of the error as $Q_4$ moves towards $Q_2$. In the right case.

$$\frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{t^2}\right), \quad \text{as } \chi \to \infty.$$

Notice here $t$ goes from $\chi/2$ to 2 if $Q_4$ goes from the section $\{x = -\chi/2\}$ to $\{x = -2\}$. We define a new variable $s = \chi/2 - t$. Suppose when $s = 0$, we have $t = \chi/2$, we want to know the behavior of $G, g$ at time $s = T$, then we have

$$g(T) - g(0),\ G(T) - G(0) = \int_0^T \frac{\mu}{t^2} + \frac{1}{\chi^2} ds = O\left(\frac{\mu}{\chi/2 - T}\right) + O(1/\chi) \quad \text{as } \chi \to \infty.$$

We identify $\chi/2 - T = |t|$ to get part $(a)$ of the lemma. $\square$

### 4.1.6 Choosing angular momentum

*Proof of the Sublemma 3.3.4.* The Poincaré map $\mathcal{P}$ restricted on the admissible surface $S_0$ is a function of two variables. If we fix $e_4 = \tilde{e}_4$, then $\mathcal{P}$ becomes a function of one variable $\ell_3$. By working out the vectors and functionals of Lemma 3.3.1 and Lemma 3.3.2 in (3.3.3), we see that $\dfrac{\partial}{\partial \ell_3}$ does not lie in the $Ker\mathbf{l}_i$. Therefore $d\mathbb{L}(\boldsymbol{x})\dfrac{\partial}{\partial \ell_3} = \dfrac{c(\boldsymbol{x})}{\mu}u_i(\boldsymbol{x}) + O(1)$, for some number $c$ depending on $\boldsymbol{x}$ smoothly. We also have $\bar{\mathbf{l}}_i(\mathbb{L}(\boldsymbol{x}))\cdot u_i(\boldsymbol{x}) \neq 0$ (This is done when checking the nondegeneracy condition). In Lemma (3.3.3), we see that $\bar{\mathbf{l}}_i$ contains nonzero $\partial/\partial e_4$ component. This implies, the projection of $\mathcal{P} = \mathbb{G}\circ\mathbb{L}$ to the $e_4$ component, i.e. $\pi_{e_4}\mathcal{P}(\ell_3, \tilde{e}_4) : \mathbb{T}^1 \to \mathbb{R}^1$ as a function of $\ell_3$ with $e_4 = \tilde{e}_4$ fixed is strongly expanding with derivative $O(\chi^2/\mu)$.

Since we have the relation $e = \sqrt{1 + 2(G/L)^2}$, we study $G_4$ instead of $e_4$. We denote by $M_R$ the angular momentum of $Q_4$ measured w.r.t. $Q_2$ and by $M_L$ that measured w.r.t. $Q_1$. We have

$$M_L = M_R - v_4 \times (-\chi, 0) = M_R - v_{4y}\chi, \tag{4.1.17}$$

where $v_4$ and $v_{4y}$ are velocity and the $y$ component of velocity of $Q_4$ respectively. When $Q_4$ is moving to the left of $x_4 = -\chi/2$, the angular momentum $M_L$ is almost conserved. We can adjust $v_{4y}$ to make $M_L$ negative or positive. The physical meaning is, by varying $\psi_3$, we can make the orbit of $Q_4$ turn around $Q_1$ clockwisely, or anti-clockwisely. If we increase the closest distance between $Q_4$ and $Q_1$ from 0, in the first case, $Q_4$ tends to stay in the upper half plane, and in the second case, it tends to stay in the lower half plane whose image in the section $\{x_4 = -2, \dot{x}_4 > 0\}$ can be below any prescribed $y$ coordinate.

It follows from the proof of Lemma 4.1.7 that the range of $\pi_{e_4}\mathcal{P}(\ell_3, \tilde{e}_4)$ contains a point in a $O(\mu)$ neighborhood of $\tilde{e}_4$. Indeed, if $Q_4$ collides with $Q_1$, the $G_4, L_4$ variables of the returning orbit deviate only by $O(\mu)$ from its initial values. Then it follows from the strong expansion of the map $\pi_{e_4}\mathcal{P}(\ell_3, \tilde{e}_4)$ that a $C$ neighborhoods of $\bar{e}_4$ is covered if $\ell_3$ varies in a $C\mu/\chi^2$ neighborhood. We choose $C$ large enough to cover a $\sqrt{\delta}$ neighborhood of $e_4^{**}$. The function $\pi_{e_4}\mathcal{P}(\ell_3, \tilde{e}_4)$ is continuous since the Poincaré map $\mathcal{P}$ is. Then we use the Intermediate Value Theorem to find $\ell_3$ such that $|\bar{e}_4 - e_4^{**}| < \sqrt{\delta}$.

Since $e_4$ changes substantially $Q_4$ must pass close to $Q_1$ and hence $\mathbb{L}(\tilde{e}_4, \tilde{l}_3)$ must have $\theta_4^+$ small. Therefore by Lemma 2.2.2 $\mathbb{L}(\tilde{e}_4, \tilde{l}_3)$ has $(E_3, e_3, g_3)$ close to $\mathbf{G}_{\tilde{e}_4,2,4}(E_3(\tilde{e}_4, \tilde{l}_3), e_3(\tilde{e}_4, \tilde{l}_3), g_3(\tilde{e}_4, \tilde{l}_3))$. It follows that

$$|E_3 - E_3^{**}| < K\delta, \quad |e_3 - e_3^{**}| < K\delta, \quad |g_3 - g_3^{**}| < K\delta.$$

Next Lemma 2.2.3 shows that after the application of $\mathbb{G}$, $(E_3, e_3, g_3)$ change little and $\theta_4^-$ becomes $O(\mu)$. $\square$

*Proof of Sublemma 3.3.5.* Without the loss of generality we may assume that $a = (r, 0)$. Let $V(z)$ be the direction field defined by the condition that the direction of $d\mathcal{F}(V(z))$ is parallel to $(1, 0)$. Let $\gamma(t)$ be the integral curve of $V$ passing through the origin and parameterized by the arclength. Then $\mathcal{F}(\gamma(t))$ has form $(\sigma(t), 0)$ where $\sigma(0) = 0$ and $|\dot{\sigma}(t)| \geq \bar{\chi}$ as long as $|\sigma| < R$. Now the statement follows easily. $\square$

## 4.2 Derivatives of the Poincaré map

In computing $C^1$ asymptotics of both local and global maps we will need formulas for the derivatives of Poincaré maps between two sections. Here we give the formulas for such derivatives for the later reference.

Recall our use of notations. $X$ denotes $Q_3$ part of our system and $Y$ denotes $Q_4$ part. Thus

$$X = (L_3, \ell_3, G_3, g_3), \quad Y = (G_4, g_4).$$

$(X, Y)^i$ will denote the orbit parameters at the initial section and $(X, Y)^f$ will denote the orbit parameters at the final section. Likewise we denote by $\ell_4^i$ the initial "time" when $Q_4$ crosses some section, and by $\ell_4^f$ final "time" when $Q_4$ arrives at the next. We abbreviate the RHS of (4.1.7)) as

$$X' = \mathcal{U}, \quad Y' = \mathcal{V}.$$

Here $'$ is the derivative w.r.t. $\ell_4$. We also denote $Z = (X, Y)$ and $\mathcal{W} = (\mathcal{U}, \mathcal{V})$ to simplify the notations further.

Suppose that we want to compute the derivative of the Poincaré map between the sections $S^i$ and $S^f$. Assume that on $S^i$ we have $\ell_4 = \ell_4^i(Z^i)$ and on $S^f$ we have $\ell_4 = \ell_4^f(Z^f)$. We want to compute the derivative $\mathcal{D}$ of the Poincaré map along the orbit starting from $(Z_*^i, \ell_*^i)$ and ending at $(Z_*^f, \ell_*^f)$. We have $\mathcal{D} = dF_3 dF_2 dF_1$ where $F_1$ is the Poincaré map between $S^i$ and $\{\ell_4 = \ell_*^i\}$, $F_2$ is the flow map between the times $\ell_*^i$ and $\ell_*^f$, and $F_3$ is the Poincaré map between $\{\ell_4 = \ell_*^f\}$ and $S^f$. We have $F_1 = \Phi(Z^i, \ell_4(Z^i), \ell_*^i)$ where $\Phi(Z, a, b)$ denotes the flow map starting from $Z$ at time

$a$ and ending at time $b$. Since

$$\frac{\partial \Phi}{\partial Z}(Z_*^i, \ell_*^i, \ell_*^i) = Id, \quad \frac{\partial \Phi}{\partial a} = -\mathcal{W}$$

we have $dF_1 = Id - \mathcal{W}(\ell_4^i) \otimes \frac{D\ell_4^i}{DZ^i}$. Inverting the time we get

$$dF_3 = \left( Id - \mathcal{W}(\ell_4^f) \otimes \frac{D\ell_4^f}{DZ^f} \right)^{-1}.$$

Finally $dF_2 = \frac{DZ(\ell_*^f)}{DZ(\ell_*^i)}$ is just the fundamental solution of the variational equation between the times $\ell_*^i$ and $\ell_*^f$. Thus we get

$$\mathcal{D} = \left( Id - \mathcal{W}(\ell_4^f) \otimes \frac{D\ell_4^f}{DZ^f} \right)^{-1} \frac{DZ(\ell_4^f)}{DZ(\ell_4^i)} \left( Id - \mathcal{W}(\ell_4^i) \otimes \frac{D\ell_4^i}{DZ^i} \right). \tag{4.2.1}$$

Next, we study the fundamental solution $\frac{DZ(\ell_*^f)}{DZ(\ell_*^i)}$ of the variational equation. We consider $Q_3$ and $Q_4$ individually. The variational equation takes form

$$\left(\frac{\partial X}{\partial X(\ell_*^i)}\right)' = \frac{\partial \mathcal{U}}{\partial X}\frac{\partial X}{\partial X(\ell_*^i)} + \frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial X(\ell_*^i)}, \quad \left(\frac{\partial X}{\partial Y(\ell_*^i)}\right)' = \frac{\partial \mathcal{U}}{\partial X}\frac{\partial X}{\partial Y(\ell_*^i)} + \frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial Y(\ell_*^i)},$$
$$\left(\frac{\partial Y}{\partial X(\ell_*^i)}\right)' = \frac{\partial \mathcal{V}}{\partial Y}\frac{\partial Y}{\partial X(\ell_*^i)} + \frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial X(\ell_*^i)}, \quad \left(\frac{\partial Y}{\partial Y(\ell_*^i)}\right)' = \frac{\partial \mathcal{V}}{\partial Y}\frac{\partial Y}{\partial Y(\ell_*^i)} + \frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial Y(\ell_*^i)},$$

Using the Duhamel principle we see that the solution of the variational equation should satisfy

$$\frac{\partial X(\ell_*^f)}{\partial X(\ell_*^i)} = \mathbb{U}(\ell_*^i, \ell_*^f) + \int_{\ell_*^i}^{\ell_*^f} \mathbb{U}(\ell_4, \ell_*^f)\frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial X(\ell_*^i)} d\ell_4, \quad \frac{\partial X}{\partial Y(\ell_4^i)} = \int_{\ell_*^i}^{\ell_*^f} \mathbb{U}(\ell_4, \ell_*^f)\frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial Y(\ell_*^i)} d\ell_4,$$
$$\frac{\partial Y}{\partial Y(\ell_4^i)} = \mathbb{V}(\ell_*^i, \ell_*^f) + \int_{\ell_*^i}^{\ell_*^f} \mathbb{V}(\ell_4, \ell_*^f)\frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial Y(\ell_*^i)} d\ell_4, \quad \frac{\partial Y}{\partial X(\ell_4^i)} = \mathbb{U}(\ell_4, \ell_*^f)\frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial X(\ell_*^i)} d\ell_4 \tag{4.2.2}$$

where $\mathbb{U}$ and $\mathbb{V}$ denote the fundamental solutions of

$$\mathbb{U}' = \frac{\partial \mathcal{U}}{\partial X}\mathbb{U} \text{ and } \mathbb{V}' = \frac{\partial \mathcal{V}}{\partial Y}\mathbb{V}$$

respectively.

## 4.3 Variational equation

The next step in the proof is the $C^1$ analysis of the global map. It occupies sections 4.3-4.5. We shall work under the assumptions of Lemma 3.3.2. In particular we will use the estimates of Section 4.1 and Appendix A.1.

The plan of the proof of Proposition 3.3.6 is the following. Matrices (I), (III) and (V) are treated in Sections 4.3 and 4.4. Namely, in Sections 4.3 we study the variational equation while in Section 4.4 we describe the contribution of the boundary terms. Finally in Section 4.5 we compute matrices (II) and (IV) which describe the change of variables between the Delaunay coordinates with different centers which are used to the left and to the right of the line $x = -\dfrac{\chi}{2}$.

### 4.3.1 Estimates of the coefficients

**Lemma 4.3.1.** *We have the following estimates for the RHS of the variational equation.*

*(a) When $Q_4$ is moving to the right of the section $\{x = -\chi/2\}$, we have $t \in [2, \chi/2]$ and*

$$\left[\begin{array}{c|c} \dfrac{\partial \mathcal{U}_R}{\partial X} & \dfrac{\partial \mathcal{U}_R}{\partial Y} \\ \hline \dfrac{\partial \mathcal{V}_R}{\partial X} & \dfrac{\partial \mathcal{V}_R}{\partial Y} \end{array}\right] =$$

$$
O\left(\left[\begin{array}{cccc|cc}
\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi}+\frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2} \\
\hline
\frac{1}{\chi}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi}+\frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi}+\frac{\mu}{|Q_4|^2} & \frac{1}{\chi}+\frac{\mu}{|Q_4|^2}
\end{array}\right]\right)
$$

*In addition we have*

$$
\frac{\partial \mathcal{V}}{\partial Y} = -\frac{1}{\chi}\left[\begin{array}{cc}
\frac{\xi L^4 \mathrm{sign}(\dot{x_4})}{(G^2+L^2)(1-\xi)^3} & \frac{\xi L^3}{(1-\xi)^3} \\
\frac{-\xi L^5}{(G^2+L^2)^2(1-\xi)^3} & \frac{-\xi L^4 \mathrm{sign}(\dot{x_4})}{(G^2+L^2)(1-\xi)^3}
\end{array}\right] + O\left(\frac{\mu}{\chi}+\frac{\mu}{|Q_4|^2}\right),
$$

$$
\frac{\partial \mathcal{V}}{\partial L_3} = -\frac{1}{\chi}\left(\frac{-\xi G_4 L_4^3 \mathrm{sign}(\dot{x}_4)}{(L_4^2+G_4^2)(1-\xi)^3}, \frac{\xi G_4 L_4^4}{(L^2+G_4^2)^2(1-\xi)^3}\right)^T + O\left(\frac{\mu}{\chi}+\frac{\mu}{|Q_4|^2}\right),
$$

*where* $\xi = \dfrac{|Q_4|}{\chi} = \dfrac{|Q_4 - Q_2|}{\chi}$.

*(b) When* $Q_4$ *is moving to the left of the section* $x = -\chi/2$, *we have* $t \in [\chi/2, \chi]$

*and*

$$
\left[\begin{array}{c|c}
\frac{\partial \mathcal{U}_L}{\partial X} & \frac{\partial \mathcal{U}_L}{\partial Y} \\
\hline
\frac{\partial \mathcal{V}_L}{\partial X} & \frac{\partial \mathcal{V}_L}{\partial Y}
\end{array}\right] = O\left(\left[\begin{array}{cccc|cc}
\frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} \\
\frac{1}{\chi} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} \\
\frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} \\
\frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} \\
\hline
\frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{1}{\chi} & \frac{1}{\chi} \\
\frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{1}{\chi} & \frac{1}{\chi}
\end{array}\right]\right)
$$

*In addition we have*

$$\frac{\partial \mathcal{V}}{\partial Y} = -\frac{1}{\chi}\begin{bmatrix} \frac{\xi L^2 \text{sign}(\dot{x_4})}{(1-\xi)^3} & \frac{\xi L^3}{(1-\xi)^3} \\ \frac{-\xi L}{(1-\xi)^3} & \frac{-\xi L^2 \text{sign}(\dot{x_4})}{(1-\xi)^3} \end{bmatrix} + O\left(\frac{\mu}{\chi}\right),$$

*where* $\xi = \dfrac{|Q_4 - Q_1|}{\chi}$.

*Proof.* (a) We estimate the four blocks of the derivative matrix separately.

• We begin with $\dfrac{\partial \mathcal{U}_R}{\partial X}$ part. We consider first the partial derivatives of $\ell_3'$ since it is the largest component of $\mathcal{U}$. Opening the brackets in the second line of (4.1.7) we get

$$\frac{d\ell_3}{d\ell_4} = -k + \frac{1}{L_3^3}W + kL_3^3\frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3} + k^2L_3^3\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + 2kW\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^3}\right). \tag{4.3.1}$$

Note that by (4.1.6)

$$W_R = k_R 3L_3^5\left(\frac{1}{|Q_3 + (\chi, 0)|} + \frac{1}{|Q_4 + (\chi, 0)|} + \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3}\right) + O\left(\frac{\mu}{|Q_4|^3}\right) = O\left(\frac{1}{\chi} + \frac{\mu}{|Q_4|^2}\right) \tag{4.3.2}$$

Observe that the RHS of (4.3.1) depends on $L_3$ in three ways. First, in contains several terms of the form $L_3^m$. Second, $Q_3$ depends on $L_3$ via (A.1.2). Third, $Q_4$ depends on $L_3$ via (A.1.5) and $L_4$ depends on $L_3$ via (4.1.6). In particular we need to consider the contribution to $\dfrac{\partial}{\partial L_3}\dfrac{d\ell_3}{d\ell_4}$ coming from

$$\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4} = \frac{\partial L_4}{\partial L_3}\frac{\partial Q_4}{\partial L_4}\frac{\partial}{\partial Q_4}.$$

By Lemma A.1.2 and equation 4.1.10 we have $\dfrac{\partial Q_4}{\partial L_4} = O(|Q_4|)$. Therefore the main contribution to (2,1) entry is $O\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right)$ and it comes from $\frac{\partial W_R}{\partial Q_4}\frac{\partial Q_4}{\partial L_4}\frac{\partial L_4}{\partial L_3}$, $W_R\frac{\partial}{\partial L_3}\frac{1}{L_3^3}$ and $\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\left(k^2L_3^3\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)$.

For the $(2,2),(2,3),(2,4)$ entries, the computations are similar. We need to act $\frac{\partial}{\partial \ell_3},\frac{\partial}{\partial G_3},\frac{\partial}{\partial g_3}$ on (4.3.1). (4.1.6) and (4.3.2) show that the contribution coming from $\frac{\partial L_4}{\partial(\ell_3,G_3,g_3)}$ is $O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)$. It remains to consider the contribution coming from $\frac{\partial Q_3}{\partial(\ell_3,G_3,g_3)}\frac{\partial}{\partial Q_3}$. Now the bound for $(2,2),(2,3)$ and $(2,4)$ entries follows directly from Lemmas 4.1.1, 4.1.3, 4.1.4, and 4.1.6.

Next, consider $(1,1)$ entry. We need to estimate

$$\frac{\partial}{\partial L_3}\left((kL_3^3+W)\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\right).$$

Using the Leibniz rule we see that the leading term comes from $\frac{\partial}{\partial L_3}\left(kL_3^3\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\right)$ which is of order $O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)$. The estimates for other entries of the $\frac{\partial \mathcal{U}_R}{\partial X}$ part are similar to the $(1,1)$ entry. This completes the analysis of $\frac{\partial \mathcal{U}_R}{\partial X}$.

- Next, we consider $\frac{\partial \mathcal{V}_R}{\partial Y}$.

Using the Leibniz rule again we see that the main contribution to the derivatives of $\mathcal{V}$ comes from differentiating $\begin{bmatrix} L_3^3\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4} \\ -L_3^3\frac{\partial Q_4}{\partial G_4}\cdot\frac{\partial V}{\partial Q_4}\end{bmatrix}$

Consider the $(5,5)$ entry. The main contribution to this entry comes from

$$\frac{\partial}{\partial G_4}\left(L_3^3\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)=L_3^3\left(\frac{\partial^2 Q_4}{\partial G_4\partial g_4}\cdot\frac{\partial V}{\partial Q_4}+\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial^2 V}{\partial Q_4^2}\cdot\frac{\partial Q_4}{\partial G_4}\right).$$

By Lemmas 4.1.4 and 4.1.6 the first term is $|Q_4|\cdot O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^3}\right)=O\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right)$ and the second term is $|Q_4|^2\cdot O\left(\frac{1}{\chi^3}+\frac{\mu}{|Q_4|^4}\right)=O\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right)$. This gives the desired upper bound of the $(5,5)$ entry. Notice that $O(1/\chi)$ term comes from

$L_3^3 \frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial \tilde{V}}{\partial Q_4}\right)$ where $\tilde{V} = -\frac{1}{|Q_4+(\chi,0)|}$. Thus we need to find the asymptotics of

$$L_3^3 \frac{\partial}{\partial G_4}\left(\frac{\frac{\partial Q_4}{\partial g_4}\cdot(Q_4+(\chi,0))}{|Q_4+(\chi,0)|^3}\right). \tag{4.3.3}$$

Let $\frac{\partial Q_4}{\partial g_4} = (\boldsymbol{a},\boldsymbol{b})$. Arguing in the same way as in the estimation of (4.1.14) we see that $\boldsymbol{a} = O(1)$. Accordingly the numerator in (4.3.3) is $O(\chi)$ so if we differentiate the denominator of (4.3.3) the resulting fraction will be of order $O(\chi)O(\chi^{-3}) = O(\chi^{-2})$. Hence $O(1/\chi)$ term comes from

$$L_3^3 \frac{\frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot(Q_4+(\chi,0))\right)}{|Q_4+(\chi,0)|^3}.$$

The numerator here equals to

$$\frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot Q_4\right)+\frac{\partial^2 Q_4}{\partial G_4 \partial g_4}\cdot(\chi,0).$$

The first term is $O(\chi)$ due to Lemma A.1.2(a) so the main contribution comes from the second term. Using Lemma A.1.3 we see that $(5,5)$ entry equals to

$$-\frac{L_3^3 L_4^2}{\sqrt{L_4^2+G_4^2}}\frac{\chi \sinh u}{|Q_4+(\chi,0)|^3}+O\left(\frac{\mu}{\chi}+\frac{\mu}{|Q_4|^2}\right).$$

Recall that $L_3 = L_4(1+o(1))$ (due to (4.1.6)) and $\sinh u = \operatorname{sign}(u)\frac{|\ell_4| L_4}{\sqrt{L_4^2+G_4^2}}$ (due to (A.1.4)). Since Lemma 4.1.1 implies that $|Q_4| = |\ell_4|/L_4^2(1+o(1))$ we obtain that $O(1/\chi)$-term in $(5,5)$ is asymptotic to

$$-\frac{L^4 \mathrm{sign}(u)}{L^2+G^2}\frac{\chi|Q_4|}{(\chi-|Q_4|)^3}.$$

Since $u$ and $\dot{x}_4$ have opposite signs we obtain the asymptotics of $O(1/\chi)$-term claimed in part (a) of the Lemma 4.3.1. The analysis of other entries of $\frac{\partial \mathcal{V}_R}{\partial Y}$ is similar.

• Next, consider the $\dfrac{\partial \mathcal{U}_R}{\partial Y}$ term.

The analysis of $(2,5)$ entry is similar to the analysis of $(2,2)$ entry except that $\dfrac{\partial}{\partial G_4}\left(k^2 L_3^3 \dfrac{\partial Q_4}{\partial L_4}\dfrac{\partial V}{\partial Q_4}\right)$ contains the term $k^2 L_3^3 \dfrac{\partial^2 Q_4}{\partial L_4 \partial G_4}\dfrac{\partial V}{\partial Q_4}$ which is of order $O(1/\chi)$ due to Lemmas 4.1.6 and A.1.3 which provides the leading contribution for large $t$. The analysis of $(2,6)$ is similar to $(2,5)$.

The estimate of the remaining entries of $\dfrac{\partial \mathcal{U}_R}{\partial Y}$ is similar to the analysis of $(1,1)$ entry.

• Thus to complete the proof of (a) it remains to consider $\dfrac{\partial \mathcal{V}}{\partial X}$. We begin with $(5,1)$ entry. We need to act by $\dfrac{\partial}{\partial L_3} + \dfrac{\partial L_4}{\partial L_3}\dfrac{\partial}{\partial L_4}$ on

$$(kL_3^3 + W)\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\left(1 + (kL_3^3 + W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right).$$

The leading term for the estimate of $(5,1)$ comes from

$$\left(\frac{\partial}{\partial L_3} + \frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right)\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) = \frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) + O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}\right) = O\left(\frac{1}{\chi} + \frac{\mu}{|Q_4|^2}\right)$$

Observe that $O(1/\chi)$ term here comes from $\dfrac{\partial}{\partial L_4}\left(\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}\right)$ which can be analyzed in the same way as $(5,5)$ term. The analysis of $(6,1)$ is the same as of $(5,1)$.

The $(5,2)$ entry is equal to $\left(\dfrac{\partial}{\partial \ell_3} + \dfrac{\partial L_4}{\partial \ell_3}\dfrac{\partial}{\partial L_4}\right)\left[\left(\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}\right)\Gamma\right]$ where

$$\Gamma = kL_3^3 + W + k^2 L_3^6 \frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + 2kL_3^3 W\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + W^2\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}.$$

Now the estimate of the $(5,2)$ entry follows from the following estimates

$$\Gamma = O(1), \quad \left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) = O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}\right),$$

$$\left(\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right)\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)=\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial}{\partial \ell_3}\frac{\partial V}{\partial Q_4}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{L_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)$$

$$=O\left(\frac{\mu}{|Q_4|^2}+\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right)\right)=O\left(\frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2}\right),$$

and

$$\left(\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right)\Gamma=O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right).$$

The remaining entries of $\dfrac{\partial \mathcal{V}}{\partial X}$ are similar to the $(5,2)$ entry. This completes the proof of part (a).

(b)• The estimate of $\dfrac{\partial \mathcal{V}_L}{\partial Y}$ and $\dfrac{\partial \mathcal{U}_L}{\partial X}$ are the same as in part (a) however, now $|Q_4|$ is of order $\chi$ so $O(\mu/|Q_4|^2)$ is dominated by other terms. In addition to compute the leading part we need to use part (c) Lemma A.1.3 rather than part (b). Moreover, in order to be able to use the formulas of that Lemma we need to shift the origin to $Q_1$. Therefore the coordinates of $Q_2$ become $(\chi, 0)$. Then we have

$$\frac{\partial \mathcal{V}_L}{\partial Y}=L_3^3\begin{bmatrix}\dfrac{\partial^2 Q_4}{\partial G\partial g}\cdot\dfrac{(-\chi,0)}{|Q_4-(\chi,0)|^3} & \dfrac{\partial^2 Q_4}{\partial g^2}\cdot\dfrac{(-\chi,0)}{|Q_4-(\chi,0)|^3}\\ -\dfrac{\partial^2 Q_4}{\partial G^2}\cdot\dfrac{(-\chi,0)}{|Q_4-(\chi,0)|^3} & -\dfrac{\partial^2 Q_4}{\partial G\partial g}\cdot\dfrac{(-\chi,0)}{|Q_4-(\chi,0)|^3}\end{bmatrix}+O\left(\frac{\mu}{\chi}\right).$$

Now the asymptotic expression of $\dfrac{\partial \mathcal{V}_L}{\partial Y}$ follows directly from Lemma A.1.3(c). We point out the subtle point that the "$-$" sign in front of the matrices of $\dfrac{\partial V}{\partial Y}$ and $\dfrac{\partial V}{\partial L_3}$ comes from the fact that the new time $\ell_4$ that we are using satisfies $\dfrac{d\ell_4}{dt}=-\dfrac{1}{L_4^3}+o(1)$ as $\mu\to 0, \chi\to\infty$.

• Next, we consider the $\dfrac{\partial \mathcal{U}_L}{\partial Y}$ term.

First consider $(1,5)$. We need to find $G_4$ derivative of

$$\left[\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\right](kL_3^3+W)\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right).$$

Differentiating the first factor we get using Lemma 4.1.6

$$\frac{\partial}{\partial G_4}\left(\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\right)=\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial^2 U}{\partial Q_3\partial Q_4}\frac{\partial Q_4}{\partial G_4}=O\left(\frac{\mu}{\chi^2}\right). \tag{4.3.4}$$

When we differentiate the product of the remaining factors then the main contribution comes from

$$\frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)=\frac{\partial^2 Q_4}{\partial L_4\partial G_4}\cdot\frac{\partial V}{\partial Q_4}+\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial}{\partial G_4}\left(\frac{\partial V}{\partial Q_4}\right). \tag{4.3.5}$$

To bound the last expression we use Lemma A.1.3. Namely, the second derivative $\dfrac{\partial^2 Q_4}{\partial G_4\partial L_4}=O(1)+\ell_4(0,1)$, is almost vertical for $\ell_4\in[\chi/2,\chi]$, and $\dfrac{\partial V_L}{\partial Q_4}=\dfrac{Q_4}{|Q_4|^3}+\dfrac{\mu(Q_4-Q_3)}{|Q_4-Q_3|^3}$ is almost horizontal. This shows that $\dfrac{\partial^2 Q_4}{\partial G_4\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4}=\dfrac{1}{\chi^2}$. The main contribution to the second summand in (4.3.5) comes from $\dfrac{\partial}{\partial G_4}\left(\nabla\left(\frac{1}{Q_4}\right)\right)$. Using Lemma A.1.2, we get

$$\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial}{\partial G_4}\left(\nabla\left(\frac{1}{Q_4}\right)\right)=(\ell_4(1,0)+O(1))\left(\frac{-Id}{|Q_4|^3}+3\frac{Q_4\otimes Q_4}{|Q_4|^5}\right)(\ell(0,1)+O(1))=\frac{1}{\chi^2}$$

for $\ell_4\in[\chi/2,\chi]$. Since $\dfrac{\partial Q_3}{\partial \ell_3}\cdot\dfrac{\partial U}{\partial Q_3}=O(1/\chi^2)$ we get the required estimate for $(1,5)$ entry.

The estimates of other $\dfrac{\partial \mathcal{U}_L}{\partial Y}$ terms are similar to the estimate of $(1,5)$ entry, except for $(2,5)$ and $(2,6)$ entries which are different because $\dfrac{d\ell_3}{d\ell_4}$ is larger than the other coordinates of $\mathcal{U}$.

Now consider $(2,5)$. We need to compute

$$\begin{aligned}
&-\frac{\partial}{\partial G_4}\left((kL_3^3+W)(\frac{1}{L_3^3}+\frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3})\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\right)\\
&=-\frac{\partial}{\partial G_4}\left(k+\frac{1}{L_3^3}W+kL_3^3\frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3}+k^2L_3^3\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}+2kW\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}+\frac{1}{\chi^3}\right)\\
&=0+\frac{1}{\chi^2}+\frac{\mu}{\chi^2}+\frac{1}{\chi^2}+\frac{1}{\chi^3}+\frac{1}{\chi^3}=O\left(\frac{1}{\chi^2}\right)
\end{aligned} \tag{4.3.6}$$

where the analysis of the leading terms is similar to (4.3.4), (4.3.5).

- Finally, we consider $\frac{\partial \mathcal{V}_L}{\partial X}$. We begin with $(5,1)$. We need to compute

$$\left[\frac{\partial}{\partial L_3}+\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right]\left(\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)\Gamma\right)$$

where

$$\Gamma = kL_3^3 + W + k^2L_3^6\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + 2kL_3^3W\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + W^2\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}.$$

The main contribution to $\left[\frac{\partial}{\partial L_3}+\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right]\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)$ comes from

$$\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) = \frac{\partial L_4}{\partial L_3}\frac{\partial^2 Q_4}{\partial L_4\partial g_4}\cdot\frac{\partial V}{\partial Q_4} + \frac{\partial L_4}{\partial L_3}\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial^2 V}{\partial Q_4^2}\frac{\partial Q_4}{\partial L_4}.$$

The two summands above can be estimated by $O(1/\chi^2)$ by the argument used to bound (4.3.5). Next a direct calculation shows that $\Gamma = O(1)$, $\left[\frac{\partial}{\partial L_3}+\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right]\Gamma = O(1)$ while $\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) = O(1/\chi^2)$ by Lemma 4.1.4 This gives the required bound for the $(5,1)$ entry. The bound for the $(6,1)$ entry is similar.

Next, consider $(5,2)$. It equals to

$$\left[\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right]\left(\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)\Gamma\right)$$

The main contribution to $\left[\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right]\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)$ comes from $\frac{\partial}{\partial \ell_3}\left(\frac{\partial Q_4}{\partial g_4}\cdot\nabla\left(\frac{\mu}{|Q_4-Q_3|}\right)\right)$ and it is of order $O\left(\frac{\mu}{\chi^2}\right)$. On the other hand the main contribution to $\left[\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right]\Gamma$ comes from $\frac{\partial W}{\partial \ell_3}$ and it is of order $O\left(\frac{1}{\chi^2}\right)$. Combining this with $C^0$ bounds mentioned used in the analysis of $(5,1)$ we obtain the required estimate on the $(5,2)$ entry. The remaining entries of $\frac{\partial \mathcal{V}_L}{\partial X}$ are similar to $(5,2)$. □

### 4.3.2 Estimates of the solutions

We integrate the variational equations to get the $\frac{\partial(X,Y)(\ell_4^f)}{\partial(X,Y)(\ell_4^i)}$ in equation (4.2.1).

Recall that map (I) describes the transition between sections $\{x=-2\}$ and $\left\{x=-\frac{\chi}{2}\right\}$, map (III) describes the transition between sections $\left\{x=-\frac{\chi}{2},\ \dot{x}<0\right\}$ and $\left\{x=-\frac{\chi}{2},\ \dot{x}>0\right\}$ and map (V) describes the transition between sections $\left\{x=-\frac{\chi}{2}\right\}$, and $\{x=-2\}$.

**Lemma 4.3.2.** *The following estimates are valid*

*(a) For maps (I) and (V),*

$$\frac{\partial(X,Y)(\ell_4^f)}{\partial(X,Y)(\ell_4^i)}=\left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu)\\ O(1) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)\\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu)\\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu)\\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)\\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right]. \tag{4.3.7}$$

*(b) For map (III),*

$$\frac{\partial(X,Y)(\ell_4^f)}{\partial(X,Y)(\ell_4^i)}=\left[\begin{array}{cccc|cc} 1+O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi})\\ O(1) & 1+O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi})\\ O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & 1+O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi})\\ O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & 1+O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi})\\ \hline O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(1) & O(1)\\ O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(1) & O(1) \end{array}\right]. \tag{4.3.8}$$

*(c)* $\dfrac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^i)}$ *has the following limits as* $\mu \to 0, \chi \to \infty$

$$\textit{Map (I):} \begin{bmatrix} 1+\dfrac{\tilde{L}_4^2}{2(\tilde{L}_4^2+\tilde{G}_4^2)} & -\frac{\tilde{L}_4}{2} \\ \dfrac{\tilde{L}_4^3}{2(\tilde{L}_4^2+\tilde{G}_4^2)^2} & 1-\dfrac{\tilde{L}_4^2}{2(\tilde{L}_4^2+\tilde{G}_4^2)} \end{bmatrix}, \quad \textit{Map (V):} \begin{bmatrix} 1-\dfrac{\hat{L}_4^2}{2(\hat{L}_4^2+\hat{G}_4^2)} & -\frac{\hat{L}_4}{2} \\ \dfrac{\hat{L}_4^3}{2(\hat{L}_4^2+\hat{G}_4^2)^2} & 1+\dfrac{\hat{L}_4^2}{2(\hat{L}_4^2+\hat{G}_4^2)} \end{bmatrix}$$

$$\textit{Map (III):} \begin{bmatrix} \dfrac{1}{2} & -\frac{L_4}{2} \\ \dfrac{3}{2L_4} & \dfrac{1}{2} \end{bmatrix}.$$

*In addition for map (I) we have* $\dfrac{\partial Y}{\partial L_3} \to \left( -\dfrac{\tilde{G}_4\tilde{L}_4}{2(\tilde{L}_4^2+\tilde{G}_4^2)}, -\dfrac{\tilde{G}_4\tilde{L}_4^2}{2(\tilde{L}_4^2+\tilde{G}_4^2)^2} \right)^T.$

*Proof.* (a) We divide the proof into several steps.

*Step 1.* Keeping in mind the integrals

$$\int_0^{\chi/2} \frac{1}{\chi} dt = O(1), \text{ and } \int_0^{\chi/2} \frac{\mu}{|Q_4|^2(t)} dt = O(\mu)$$

we conclude using the Gromwall Inequality that if $(\delta X, \delta Y)(\ell_4^i) = O(1)$ then $(\delta X, \delta Y)(t) = O(1)$ for all $t \in [\ell_4^i, \ell_4^f]$.

*Step 2.* Plugging the estimate of step 1 back into the variational equation we see that $(\delta L_3, \delta G_3, \delta g_3)(t) - (\delta L_3, \delta G_3, \delta g_3)(0) = O(\mu)$. This proves the required bound for $(\delta L_3, \delta G_3, \delta g_3)$.

*Step 3.* Steps 1 and 2 imply that

$$\delta \dot{Y}(t) = \frac{\partial \mathcal{V}}{\partial Y}(t)\delta Y(t) + \frac{\partial \mathcal{V}}{\partial L_3}(t)\delta L_3(0) + O\left(\frac{\mu}{1+t^2}\right).$$

We treat this as a nonhomogeneous linear equation for $\delta Y$. Thus

$$\delta Y(t) = \mathbb{V}(0,t)\delta Y(0) + \left(\int_0^t \mathbb{V}(s,t)\frac{\partial \mathcal{V}}{\partial L_3}(s)ds\right)\delta L_3(0) + \int_0^t O\left(||\mathbb{V}(s,t)||\frac{\mu}{1+s^2}\right)ds \tag{4.3.9}$$

where $\mathbb{V}(s,t)$ denotes the fundamental solution of the corresponding homogeneous equation. (4.3.9) immediately implies the required bound for $\delta Y$.

*Step 4.* Plugging the estimates of steps 2 and 3 into the equation for $\delta\ell_3$ we see that if $(\delta L_3, \delta G_3, \delta g_3)(0) = 0$ and hence $(\delta L_3, \delta G_3, \delta g_3)(t) = O(\mu)$ for all $t$ then $\delta\dot{\ell}_3 = O\left(\frac{\mu}{1+t^2}\right)$ proving the required bound for $\delta\ell_3$.

(b) We use the same steps as in part (a). On step 1 we show that $(\delta X, \delta Y)(t) = O(1)$ for all $t$. On step 2 we conclude that $(\delta L_3, \delta G_3, \delta g_3)(t) - (\delta L_3, \delta G_3, \delta g_3)(0) = O(1/\chi)$. On step 3 we prove the result of part (b) for $\delta Y$. On step 4 we use the results of step 3 to show that if $\delta X(0) = 0$ then $(\delta L_3, \delta G_3, \delta g_3)(t) = O(\mu/\chi)$ and $\delta\ell_3(t) = O(1/\chi)$.

To prove (c) we need to find the asymptotics of $\mathbb{V}$. Consider map (I) first. $\mathbb{V}$ satisfies

$$\dot{\mathbb{V}} = \frac{\partial \mathcal{V}}{\partial Y}\mathbb{V}.$$

By already established part (a) $\mathbb{V} = O(1)$ so the above equation can be rewritten as

$$\dot{\mathbb{V}} = \frac{\xi L^2}{\chi(1-\xi)^3} A\mathbb{V} + O\left(\frac{\mu}{t^2+1} + \frac{\mu}{\chi}\right).$$

where $A = \begin{bmatrix} -\dfrac{L^2}{(G^2+L^2)} & L \\ -\dfrac{L^3}{(G^2+L^2)^2} & \dfrac{L^2}{(G^2+L^2)} \end{bmatrix}$. Now Gronwall Lemma gives $\mathbb{V} \approx \tilde{\mathbb{V}}$ where $\tilde{\mathbb{V}}$ is the fundamental solution of $\dot{\tilde{\mathbb{V}}} = \frac{\xi L^2}{\chi(1-\xi)^3} A\tilde{\mathbb{V}}$. Using $\xi$ as the independent variable we get $\dfrac{d\tilde{\mathbb{V}}}{d\xi} = -\dfrac{\xi}{(1-\xi)^3} A\tilde{\mathbb{V}}$. Note that $\xi(\ell_4^i) = o(1)$, $\xi(\ell_4^f) = \dfrac{1}{2} + o(1)$. Making a further time change $d\tau = \dfrac{\xi d\xi}{(1-\xi)^3}$ we obtain the constant coefficient linear equation $\dfrac{d\tilde{\mathbb{V}}}{d\tau} = -A\tilde{\mathbb{V}}$. Observe that $\mathrm{Tr}(A) = \det(A) = 0$ and so $A^2 = 0$.

Therefore

$$\tilde{\mathbb{V}}(\sigma,\tau) = Id - (\tau-\sigma)A. \tag{4.3.10}$$

Since $\tau = \dfrac{\xi^2}{2(1-\xi)^2}$ we have $\tau(0)=0$, $\tau\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$. Plugging this into (4.3.10) we get the claimed asymptotics for map (I). The analysis of map (V) is similar. To analyze map (III) we split

$$\frac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^i)} = \frac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^m)}\frac{\partial Y(\ell_4^m)}{\partial Y(\ell_4^i)}$$

where $\ell_4^m = \dfrac{\ell_4^i + \ell_4^f}{2}$. Using the argument presented above we obtain

$$\frac{\partial Y(\ell_4^m)}{\partial Y(\ell_4^i)} = \begin{bmatrix} \frac{3}{2} & -\frac{L}{2} \\ \\ \frac{1}{2L} & \frac{1}{2} \end{bmatrix}, \quad \frac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^m)} = \begin{bmatrix} \frac{1}{2} & -\frac{L}{2} \\ \\ \frac{1}{2L} & \frac{3}{2} \end{bmatrix}.$$

Multiplying the above matrices we obtain the required asymptotics for map (III).

Next using the same argument as in analysis of $\dfrac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^i)}$ we obtain $\dfrac{\partial Y}{\partial L_3} \approx \mathbb{W}$ where

$$\dot{\mathbb{W}} = \frac{\xi L^2}{\chi(1-\xi)^3}\left[A\mathbb{W} + \left(\frac{GL}{(L^2+G^2)}, \frac{GL^2}{(L^2+G^2)^2}\right)^T\right].$$

In terms of the new time this equation reads

$$\frac{d\mathbb{W}}{d\tau} = -\left[A\mathbb{W} + \left(\frac{GL}{(L^2+G^2)}, \frac{GL^2}{(L^2+G^2)^2}\right)^T\right].$$

Solving this equation using (4.3.10) and initial condition $(0,0)^T$, we obtain the asymptotics of $\dfrac{\partial Y}{\partial L_3}$. $\square$

## 4.4 Boundary contributions and the proof of Proposition 3.3.6

According to (4.2.1) we need to work out the boundary contributions in order to complete the proof of Proposition 3.3.6.

### 4.4.1 Dependence of $\ell_4$ on variables $(X,Y)$

To use the formula (4.2.1) we need to work out $(\mathcal{U},\mathcal{V})(\ell_4^i)\otimes\dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ and $(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial \ell_4^f}{\partial (X,Y)^f}$. Consider $x_4$ component of $Q_4$ (see equation (A.1.5)).

$$x_4=\cos g_4(L_4^2\sinh u_4-e_4)-\sin g_4(L_4G_4\cosh u_4).$$

For fixed $x_4=-\chi/2$ or $-2$, we can solve $\ell_4$ as a function of $L_4,G_4,g_4$. From the calculations in the Appendix A.1.2, Lemma A.1.2, and the implicit function theorem, we get

$$\text{for the section } x_4=-\chi/2,\ \left(\frac{\partial \ell_4}{\partial L_4},\frac{\partial \ell_4}{\partial G_4},\frac{\partial \ell_4}{\partial g_4}\right)\Big|_{x_4=-\chi/2}=(O(\chi),O(1),O(1)),$$
$$\text{for the section } x_4=-2,\ \left(\frac{\partial \ell_4}{\partial L_4},\frac{\partial \ell_4}{\partial G_4},\frac{\partial \ell_4}{\partial g_4}\right)\Big|_{x_4=-2}=(O(1),O(1),O(1)).$$

Using equation (4.1.6) which relates $L_4$ to $L_3$, we obtain for the section $\{x_4=-\chi/2\}$,

$$\begin{aligned}\frac{\partial \ell_4}{\partial (X,Y)}\Big|_{x_4=-\chi/2}&=(O(\chi),O(1/\chi),O(1/\chi),O(1/\chi),O(1),O(1)),\\ (\mathcal{U},\mathcal{V})\Big|_{x_4=-\chi/2}&=(O(1/\chi^2),O(1),O(1/\chi^2),O(1/\chi^2),O(1/\chi^2),O(1/\chi^2))^T,\end{aligned}\tag{4.4.1}$$

For the section $\{x_4=-2\}$,

$$\begin{aligned}\frac{\partial \ell_4}{\partial L_3}\Big|_{x_4=-2}&=(O(1),O(\mu),O(\mu),O(\mu),O(1),O(1)),\\ (\mathcal{U},\mathcal{V})\Big|_{x_4=-2}&=(O(\mu),O(1),O(\mu),O(\mu),O(\mu),O(\mu))^T.\end{aligned}\tag{4.4.2}$$

The matrix $(\mathcal{U},\mathcal{V}) \otimes \dfrac{\partial \ell_4}{\partial (X,Y)}\Big|_{x_4=-\chi/2}$ has rank 1 and the only nonzero eigenvalue is $O(1/\chi)$, and $(\mathcal{U},\mathcal{V}) \otimes \dfrac{\partial \ell_4}{\partial (X,Y)}\Big|_{x_4=-2}$ has rank 1 and the only nonzero eigenvalue is $O(\mu)$. So the inversion appearing in (4.2.1) is valid.

### 4.4.2 Asymptotics of matrices $(I), (III), (V)$ from the Proposition 3.3.6

Here we complete the computations of matrices (I), (III) and (V).

**The boundary contribution to** $(I)$**.** In this case, $\ell_4^i$ stands for the section $\{x_4 = -2\}$ and $\ell_4^f$ stands for the section $\{x_4 = -\chi/2\}$. So we use equation (4.4.2) to form $(\mathcal{U},\mathcal{V})(\ell_4^i) \otimes \dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ in equation (4.2.1) and equation (4.4.1) to form $(\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \dfrac{\partial \ell_4^f}{\partial (X,Y)^f}$. We have

$$\left(\mathrm{Id} - (\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial (X,Y)^f}\right)^{-1} =$$
$$\left[\begin{array}{cccc|cc}
1+O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^2) & O(1/\chi^2) \\
O(\chi) & 1+O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1) & O(1) \\
O(1/\chi) & O(1/\chi^3) & 1+O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^2) & O(1/\chi^2) \\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & 1+O(1/\chi^3) & O(1/\chi^2) & O(1/\chi^2) \\
\hline
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & 1+O(1/\chi^2) & O(1/\chi^2) \\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^2) & 1+O(1/\chi^2)
\end{array}\right] \tag{4.4.3}$$

Now we use equation (4.2.1) and Lemma 4.3.2 to obtain the asymptotics of the matrix $(I)$ stated in Proposition 3.3.6.

**The boundary contribution to** $(III)$

This time we use equation (4.4.1) to form both $(\mathcal{U},\mathcal{V})(\ell_4^i) \otimes \dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ and

$(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial\ell_4^f}{\partial(X,Y)^f}$ in equation (4.2.1).

The matrix $\left(\mathrm{Id}-(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial\ell_4^f}{\partial(X,Y)^f}\right)^{-1}$ has the same form as (4.4.3). Now we use equation (4.2.1) and Lemma 4.3.2 to obtain the asymptotics of the matrix $(III)$ stated in Proposition 3.3.6.

**The boundary contribution to** $(V)$

This time we use equation (4.4.1) to form $(\mathcal{U},\mathcal{V})(\ell_4^i)\otimes\dfrac{\partial\ell_4^i}{\partial(X,Y)^i}$ and equation (4.4.2) to form $(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial\ell_4^f}{\partial(X,Y)^f}$ in equation (4.2.1).

The matrix $\left(\mathrm{Id}-(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial\ell_4^f}{\partial(X,Y)^f}\right)^{-1}$ has the form

$$\left[\begin{array}{cccc|cc}
1+O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu)\\
O(1) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)\\
O(\mu) & O(\mu^2) & 1+O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu)\\
O(\mu) & O(\mu^2) & O(\mu^2) & 1+O(\mu^2) & O(\mu) & O(\mu)\\
\hline
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & 1+O(\mu) & O(\mu)\\
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & 1+O(\mu)
\end{array}\right]$$

Now we use equation (4.2.1) and Lemma 4.3.2 to obtain the asymptotics of the matrix $(V)$ stated in Proposition 3.3.6.

## 4.5 Switching foci

Recall that we treat the motion of $Q_4$ as a Kepler motion focused at $Q_2$ when it is moving to the right of the section $\{x=-\chi/2\}$ and treat it as a Kepler motion focused at $Q_1$ when it is moving to the left of the section $\{x=-\chi/2\}$. Therefore, we need to make a change of coordinates when $Q_4$ crosses the section $\{x_4 = -\chi/2\}$. These are described by the matrices $(II)$ and $(IV)$. Under this coordinate change the $Q_3$ part of the Delaunay variables does not change. The change of $G_4$ is given by the difference of angular momentums w.r.t. different reference points ($Q_1$ or $Q_2$). To handle it we introduce an auxiliary variable $v_{4y}$-the $y$ component of the velocity of $Q_4$. Relating $g_4$ with respect to the different reference points to $v_{4y}$ we complete the computation.

### 4.5.1 From the right to the left

We have $(II) = \dfrac{\partial(L_3, \ell_3, G_3, g_3, G_{4L}, g_{4L})}{\partial(L_3, \ell_3, G_3, g_3, G_{4R}, g_{4R})}\Big|_{x_4=\chi/2} = (iii)(ii)(i)$ where matrices $(i), (ii)$ and $(iii)$ correspond to the following coordinate changes.

$$(G, g)_{4R}\left(\frac{\chi}{2}\right) \xrightarrow{(i)} (G, v_y)_{4R}\left(\frac{\chi}{2}\right) \xrightarrow{(ii)} (G, v)_{yL}\left(\frac{\chi}{2}\right) \xrightarrow{(iii)} (G, g)_{4L}\left(\frac{\chi}{2}\right).$$

*proof of matrices (i) and (iii)(ii) in Proposition 3.3.6.* $(i)$ is given by the relation

$$G_{4R} = G_{4R}, \quad v_{4y} = \frac{-\dfrac{1}{L_{4R}}\sinh u_{4R}\sin g_{4R} + \dfrac{G_{4R}}{L_{4R}^2}\cos g_{4R}\cosh u_{4R}}{1 - e_{4R}\cosh u_{4R}} < 0, \quad L_{4R} = k_R L_3 - \frac{W_R}{3L_3^2}.$$

where last relation follows from (4.1.6). Recall that by Lemma 4.1.8 $g_{4R} = \arctan\dfrac{G_{4R}}{L_{4R}} + O(1/\chi)$. In addition (4.5.1) below and the fact that $G_{4R}$ and $G_{4L}$ are $O(1)$ implies $v_{4y} = O(1/\chi)$. Now the asymptotics of (i) is obtain by direct computation. We compute $\frac{dv_{4y}}{dL_3}$ the other derivatives are similar but easier. We have $\frac{dv_{4y}}{dL_3} = \frac{dv_{4y}}{dL_{4R}}\frac{\partial L_{4R}}{\partial L_3}$. The second term is $k_R + O(1/\chi)$. On the other hand

$$\frac{dv_{4y}}{dL_4} = \frac{\frac{\partial}{\partial L_{4R}}\left(-\dfrac{1}{L_{4R}}\sinh u_{4R}\sin g_{4R} + \dfrac{G_{4R}}{L_{4R}^2}\cos g_{4R}\cosh u_{4R}\right)}{1 - e_{4R}\cosh u_{4R}} + v_{4R}\frac{\frac{\partial e_{4R}}{\partial L_{4R}}\cosh u_{4R}}{1 - e_{4R}\cosh u_{4R}} + \frac{\partial v_{4R}}{\partial \ell_{4R}}\frac{\partial \ell_{4R}}{\partial L_{4R}}.$$

The main contribution comes from the first term which equals

$$-\frac{G_{4R}}{L_{4R}(L_{4R}^2+G_{4R}^2)}+O(1/\chi).$$

The second term is $O(1/\chi)$ since $v_{4R}=O(1/\chi)$. Next rewriting

$$v_{4y}=\frac{-\frac{1}{L_{4R}}\tanh u_{4R}\sin g_{4R}+\frac{G_{4R}}{L_{4R}^2}\cos g_{4R}}{(1/\cosh u_{4R})-e_{4R}}$$

we see that

$$\frac{\partial v_{4y}}{\partial \ell_{4R}}\frac{\partial \ell_{4R}}{\partial L_{4R}}=O(1/\chi^2)\times O(\chi)=O(1/\chi)$$

since $\frac{\partial \ell_{4R}}{\partial L_{4R}}=O(\chi)$ by (4.4.1).

$(ii)$ is given by

$$G_L=G_R/k_R+\chi v_{4y}. \tag{4.5.1}$$

Here $G_{4R}$ and $v_{4y}$ are independent variables so the computation of the derivative of (ii) is straightforward.

To compute the derivative of $(iii)$ we use the relation

$$G_L=G_L,\qquad v_{4y}=\frac{-\frac{1}{L_{4L}}\sinh u_{4L}\sin g_{4L}+\frac{G_{4L}}{L_{4L}^2}\cos g_{4L}\cosh u_{4L}}{1-e_{4L}\cosh u_{4L}}.$$

where $u_L<0$. Arguing the same way as for (i) and using the fact that by Lemma 4.1.8, $G_L, g_L=O(1/\chi)$, $-\sinh u_L, \cosh u_L\simeq\dfrac{\ell_{4L}}{e_L}$ we obtain

$$\delta v_{4y}=-\frac{\delta G_{4L}}{k_R^2L_3^2}-\frac{\delta g_{4L}}{k_RL_3}+HOT$$

Hence

$$\delta g_{4R}=-\frac{\delta G_{4L}}{k_RL_3}-k_RL_3\delta v_{4y}+HOT=-\frac{(\delta G_{4R}/k_R)+\chi\delta v_{4y}}{k_RL_3}+HOT$$

completing the proof of the lemma.

□

### 4.5.2 From the left to the right

At this step we need to compute

$$(IV)=\frac{\partial(L_3,\ell_3,G_3,g_3,G_{4R},g_{4R})}{\partial(L_3,\ell_3,G_3,g_3,G_{4L},g_{4L})}\Big|_{x_4=\chi/2}=(iii')(ii')(i').$$

where the matrices $(iii')$, $(ii')$ and $(i')$ correspond to the following changes of variables.

$$(G,g)_L\left(\frac{\chi}{2}\right)\xrightarrow{(i')}(G,v_{4y})_L\left(\frac{\chi}{2}\right)\xrightarrow{(ii')}(G,v_{4y})_R\left(\frac{\chi}{2}\right)\xrightarrow{(iii')}(G,g)_R\left(\frac{\chi}{2}\right).$$

*proof of matrices (iii') and (ii')(i') in Proposition 3.3.6.* $(i')$ is given by

$$v_{4y}=\frac{-\dfrac{1}{L_{4L}}\sinh u_{4L}\sin g_{4L}+\dfrac{G_{4L}}{L_{4L}^2}\cos g_{4L}\cosh u_{4L}}{1-e_{4L}\cosh u_{4L}}<0.$$

Here $u_L>0$ and $G_{4L},g_{4L}=O(1/\chi)$.

$(ii')$ is given by

$$G_R/k_R=G_L-\chi v_{4yL}.$$

Now the analysis is similar to Subsection 4.5.1. In particular the main contribution to $[(ii')(i')]_{44}$ comes from

$$\frac{\partial(G_{4R},v_{4y})}{\partial(G_{4L}g_{4L})}=\frac{\partial(G_{4R},v_{4y})}{\partial(G_{4L}v_{4y})}\frac{\partial(G_{4L},v_{4y})}{\partial(G_{4L}g_{4L})}=\begin{bmatrix}k_R & -k_R\chi\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\dfrac{1}{L_3^2}+O\left(\frac{1}{\chi}\right) & \dfrac{1}{L_3}+O\left(\frac{1}{\chi}\right)\end{bmatrix}.$$

The analysis of 43 part is similar.

$(iii')$ is given by

$$G_R=G_R,\qquad v_{4y}=\frac{-\dfrac{1}{L_{4R}}\sinh u_{4R}\sin g_{4R}+\dfrac{G_{4R}}{L_{4R}^2}\cos g_{4R}\cosh u_{4R}}{1-e_{4R}\cosh u_{4R}}<0.$$

Here $u_{4R} < 0$, and by Lemma 4.1.8 $\tan g_{4R} = -\dfrac{G_{4R}}{L_{4R}} + O(1/\chi)$. To get the asymptotics of the derivative we first show that similarly to Subsection 4.5.1, we have

$$dv_{4y} = \left(\frac{G_{4R}}{L_3(k_R^2 L_3^2 + G_{4R}^2)} + O(1/\chi), O(1/\chi^2), O(1/\chi^2), O(1/\chi^2), -\frac{1}{k_R^2 L_3^2 + G_{4R}^2}, -\frac{1}{k_R L_3}\right)$$

and then take the inverse. $\square$

# Chapter 5

# The local map

## 5.1 Approaching close encounter

In this paper we choose to separate local and global map by section $\{x_4 = -2\}$. We could have use instead $\{x_4 = -10\}$, or $\{x_4 = -100\}$. Our first goal is to show that the arbitrariness of this choice does not change the asymptotics of derivative of the local map (we have already seen in Sections 4.3.2 and 4.4 that it does not in change the asymptotics of the derivative of the global map).

We choose the section $\{|Q_3 - Q_4| = \mu^\kappa\}$, $1/3 < \kappa < 1/2$. Outside the section the orbits are treated as perturbed Kepler motions and inside the section the orbits are treated as two body scattering. We shall estimate the errors of this approximation. We break the orbit into three pieces: from $\{x_4 = -2, \dot{x}_4 > 0\}$ to $\{|Q_3^- - Q_4^-| = \mu^\kappa\}$, from $\{|Q_3^- - Q_4^-| = \mu^\kappa\}$ to $\{|Q_3^+ - Q_4^+| = \mu^\kappa\}$ and from $\{|Q_3^+ - Q_4^+| = \mu^\kappa\}$ to $\{x_4 = -2, \dot{x}_4 > 0\}$.

In this section we consider the two pieces of orbit outside the section $\{|Q_3 - Q_4| = \mu^\kappa\}$. The Hamiltonian that we use is (4.1.1). Then we convert the Cartesian coordinates to Delaunay coordinates. The resulting Hamiltonian is

$$H_L = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4 + (\chi, 0)|} - \frac{1}{|Q_3 + (\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.$$

The difference with the Hamiltonian (4.1.2) is that we do not do the Taylor expansion to the potential $-\dfrac{1}{|Q_3-Q_4|}$.

The next lemma and the remark after it tell us that we can neglect those two pieces.

**Lemma 5.1.1.** *Consider the orbits satisfying the conditions of Lemma 3.3.1. For the pieces of orbit from* $x_4=-2, \dot{x}_4>0$ *to* $|Q_3^- - Q_4^-| = \mu^\kappa$ *and from* $|Q_3^+ - Q_4^+| = \mu^\kappa$ *to* $x_4=-2, \dot{x}_4>0$, $1/3<\kappa<1/2$ *the derivative matrices have the following form in Delaunay coordinates*

$$\frac{\partial(X,Y)^-}{\partial(X,Y)(-2)}, \frac{\partial(X,Y)(-2)}{\partial(X,Y)^+} = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ O(1) & O(1) & O(1) & O(1) & O(1) & O(1) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] + O(\mu^{1-2\kappa} + 1/\chi^3).$$

*Proof.* The proof follows the plan in Section 4.2. We first consider the integration of the variational equation. We treat the orbit as Kepler motions perturbed by $Q_1$ and interaction between $Q_3$ and $Q_4$. Consider first the perturbation coming from the interaction of $Q_3$ and $Q_4$. The contribution of this interaction to the variational equation is of order $\dfrac{\mu}{|Q_3-Q_4|^3}$. If we integrate the variational equation along an orbit such that $|Q_3-Q_4|$ goes from $-2$ to $\mu^\kappa$, then the contribution has the order

$$O\left(\int_{-2}^{\mu^\kappa} \frac{\mu}{|t|^3} dt\right) = O(\mu^{1-2\kappa}). \tag{5.1.1}$$

Similar consideration shows that the perturbation from $Q_1$ is $O(1/\chi^3)$.

On the other hand absence of perturbation, all Delaunay variables except $\ell_3$

are constants of motion. In particular, the solutions to the variational equations have the form

$$\mathrm{Id} - \frac{3\Delta\ell}{L^4} e_{2,1} + O(\mu^{1-2\kappa} + 1/\chi^3)$$

where $\Delta\ell$ is the time it takes to go from one section to the next and $e_{2,1}$ means a $6 \times 6$ matrix whose $(2,1)$ entry is 1 and others are 0.

Next we compute the boundary contributions. The analysis is the same as Section 4.4. The derivative is given by formula (4.2.1). We need to work out $(\mathcal{U},\mathcal{V})(\ell_4^i) \otimes \dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ and $(\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \dfrac{\partial \ell_4^f}{\partial (X,Y)^f}$. In both cases we have

$$(\mathcal{U},\mathcal{V}) = (0,1,0,0,0,0) + O(\mu^{1-2\kappa}).$$

For the section $\{x_4 = -2\}$, we use (4.4.2). For the section $\{|Q_3 - Q_4| = \mu^\kappa\}$, we have

$$\frac{\partial \ell_4}{\partial (X,Y)} = -\left(\frac{\partial |Q_3 - Q_4|}{\partial \ell_4}\right)^{-1} \frac{\partial |Q_3 - Q_4|}{\partial (X,Y)} = -\frac{(Q_3 - Q_4)\cdot \frac{\partial (Q_3 - Q_4)}{\partial (X,Y)}}{(Q_3 - Q_4)\cdot \frac{\partial (Q_3 - Q_4)}{\partial \ell_4}} \tag{5.1.2}$$

We will prove in Lemma 5.2.2(c) below that the angle formed by $Q_3 - Q_4$ and $v_3 - v_4$ is $O\left(\mu^{1-\kappa}\right)$ (the proof of Lemma 5.2.2 does not rely on section 5.1). Thus in (5.1.2) we can replace $Q_3 - Q_4$ by $v_3 - v_4$ making $O\left(\mu^{1-\kappa}\right)$ mistake. Hence

$$\frac{\partial \ell_4}{\partial (X,Y)} = \frac{(v_3 - v_4)\cdot \frac{\partial (Q_3 - Q_4)}{\partial (X,Y)}}{(v_3 - v_4)\cdot \frac{\partial Q_4}{\partial \ell_4}} + O(\mu^{1-\kappa}),$$

Note that $\frac{\partial Q_4}{\partial \ell_4}$ is parallel to $v_4$. Using the information about $v_3$ and $v_4$ from Appendix B.2.1 we see that $\langle v_3, v_4\rangle \neq \langle v_4, v_4\rangle$. Therefore the denominator in (5.1.2) is bounded away from zero and so

$$\frac{\partial \ell_4}{\partial (X,Y)} = (O(1), O(1), O(1), O(1), O(1), O(1)).$$

We also need to make sure the second component $\frac{\partial \ell_4}{\partial \ell_3}$ is not close to 1, so that $\mathrm{Id} - (\mathcal{U}, \mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial (X,Y)^f}$ is invertible when $|Q_3 - Q_4| = \mu^\kappa$ serves as the final section. In fact, due to (4.1.7), $\frac{\partial \ell_4}{\partial \ell_3} \simeq -1$. Using formula (4.2.1), we get the asymptotics stated in the lemma. □

**Remark 7.** Using the explicit value of the vectors $\bar{\mathbf{l}}_2,\ \bar{\mathbf{l}}_3,\ w,\ \tilde{w}$ in equations (3.3.3), we find that in the limit $\mu \to 0, \chi \to \infty$

$$\left(\frac{\partial (X,Y)^-}{\partial (X,Y)(-2)}\right) \mathrm{span}\{w, \tilde{w}\} = \mathrm{span}\{w, \tilde{w}\}$$

and

$$\bar{\mathbf{l}}_2 \left(\frac{\partial (X,Y)(-2)}{\partial (X,Y)^+}\right) = \bar{\mathbf{l}}_2, \quad \bar{\mathbf{l}}_3 \left(\frac{\partial (X,Y)(-2)}{\partial (X,Y)^+}\right) = \bar{\mathbf{l}}_3$$

This tells us that we can neglect the two matrices corresponding to the pieces of orbit from $x_4 = -2, \dot{x}_4 > 0$ to $|Q_3^- - Q_4^-| = \mu^\kappa$ and from $|Q_3^+ - Q_4^+| = \mu^\kappa$ to $x_4 = -2, \dot{x}_4 > 0$. We thus have the identification

$$d\mathbb{L} = \frac{\partial (L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial (L_3, \ell_3, G_3, g_3, G_4, g_4)^-} + O(\mu^{1-2\kappa})$$

where $(L_3, \ell_3, G_3, g_3, G_4, g_4)^\pm$ denote the Delaunay variables measured on the section $\{|Q_3^\pm - Q_4^\pm| = \mu^\kappa\}$.

## 5.2 $C^0$ estimate for the local map

In Sections 5.2 and 5.3 we consider the piece of orbit from $|Q_3^- - Q_4^-| = \mu^\kappa$ to $|Q_3^+ - Q_4^+| = \mu^\kappa$. Because of Remark 7, we simply write $d\mathbb{L}$ to stand for the derivative for this piece.

It is convenient to use the coordinates of relative motion and the motion of mass center. We define

$$v_\pm = v_3 \pm v_4, \quad Q_\pm = \frac{Q_3 \pm Q_4}{2}. \tag{5.2.1}$$

Here "-" refers to the relative motion and "+" refers to the center of mass motion. To study the relative motion, we make the following rescaling:

$$q_- := Q_-/\mu, \quad \tau := t/\mu \text{ and } v_- \text{ remains unchanged.} \tag{5.2.2}$$

In this way, we zoom in the picture of $Q_3$ and $Q_4$ by a factor $1/\mu$.

Then we have the following lemma.

**Lemma 5.2.1.** *(a) Inside the sphere $|Q_-| = \mu^\kappa, 1/3 < \kappa < 1/2$, the motion of the center of mass is a Kepler motion focused at $Q_2$ perturbed by $O(\mu^{2\kappa})$.*

$$\dot{Q}_+ = \frac{v_+}{2}, \quad \dot{v}_+ = -\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa}). \tag{5.2.3}$$

*(b) In the rescaled variables, the relative motion is a Kepler motion focused at the origin perturbed by $O(\mu^{1+2\kappa})$.*

$$q'_- = \frac{v_-}{2}, \quad v'_- = \frac{q_-}{2|q_-|^3} + O(\mu^{1+2\kappa}), \tag{5.2.4}$$

*where we use "′" to stand for the derivative w.r.t. the new time $\tau$.*

*Proof.* Note that (5.2.1) preserves the symplectic form.

$$dv_3 \wedge dQ_3 + dv_4 \wedge dQ_4 = dv_- \wedge dQ_- + dv_+ \wedge dQ_+,$$

The Hamiltonian becomes

$$H = \frac{|v_-|^2}{4} - \frac{\mu}{2|Q_-|} + \frac{|v_+|^2}{4} - \frac{1}{|Q_+ + Q_-|} - \frac{1}{|Q_+ - Q_-|} - \frac{1}{|Q_+ + Q_- + (\chi, 0)|} - \frac{1}{|Q_+ - Q_- + (\chi, 0)|}$$
$$= \frac{|v_-|^2}{4} - \frac{\mu}{2|Q_-|} + \frac{|v_+|^2}{4} - \frac{2}{|Q_+|} + \frac{|Q_-|^2}{2|Q_+|^3} - \frac{3|Q_+ \cdot Q_-|^2}{2|Q_+|^5} + O(\mu^{3\kappa}) + O(1/\chi), \tag{5.2.5}$$

where the $O(\mu^{3\kappa})$ includes the $|Q_-|^3$ and higher order terms. In the following, we drop $O(1/\chi)$ terms since $1/\chi \ll \mu$. So the Hamiltonian equations for the motion of the mass center part are

$$\dot{Q}_+ = \frac{v_+}{2}, \quad \dot{v}_+ = -\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa})$$

proving part (a) of the lemma.

Next, we study the relative motion. From equation (5.2.5), we get the equations of motion for the mass center part is

$$\dot{Q}_- = \frac{v_-}{2}, \quad \dot{v}_- = -\frac{\mu Q_-}{2|Q_-|^3} - \frac{Q_-}{|Q_+|^3} + \frac{3|Q_+ \cdot Q_-|Q_+}{|Q_+|^5} + O(\mu^{2\kappa}),$$

as $\mu \to 0$, where $O(\mu^{2\kappa})$ includes quadratic and higher order terms of $|Q_-|$. After making the rescaling according to (5.2.2) the equations for the relative motion part become

$$q_-' = \frac{v_-}{2}, \quad v_-' = \frac{q_-}{2|q_-|^3} + \frac{\mu^2 q_-}{|Q_+|^3} - \frac{3\mu^2|Q_+ \cdot q_-|Q_+}{|Q_+|^5} + O(\mu^{1+2\kappa}). \tag{5.2.6}$$

$\square$

Lemma 5.2.1 implies the following $C^0$ estimate.

Let $v_{3,4}^-, Q_{3,4}^-$ be the velocities and positions measured at the time when the orbit of the system enters $|Q_3 - Q_4| = \mu^\kappa$ and $v_{3,4}^+, Q_{3,4}^+$ be the velocities and positions measured at the time when the orbit of the system exits $|Q_3 - Q_4| = \mu^\kappa$, $1/3 < \kappa < 1/2$.

**Lemma 5.2.2.** *(a) We have the following equations*

$$\begin{cases} v_3^+ = \dfrac{1}{2}R(\alpha)(v_3^- - v_4^-) + \dfrac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{\kappa-1}), \\ v_4^+ = -\dfrac{1}{2}R(\alpha)(v_3^- - v_4^-) + \dfrac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{\kappa-1}), \\ Q_3^+ + Q_4^+ = Q_3^- + Q_4^- + O(\mu^k), \\ |Q_3^- - Q_4^-| = \mu^\kappa, \quad |Q_3^+ - Q_4^+| = \mu^\kappa, \end{cases} \tag{5.2.7}$$

*where* $R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

$$\alpha = \pi + 2\arctan\left(\frac{G_{in}}{\mu L'_{in}}\right), \quad \text{and } \frac{1}{4L'^2_{in}} = \frac{v_-^2}{4} - \frac{\mu}{2|Q_-|}, \quad G_{in} = 2v_- \times Q_-.$$

*(b) We have* $L'_{in} = O(1)$*. If* $\alpha$ *is bounded away from* $0$ *and* $2\pi$ *by an angle independent of* $\mu$ *then* $G_{in} = O(\mu)$ *and the closest distance between* $Q_3$ *and* $Q_4$ *is bounded away from zero by* $\delta\mu$ *and from above by* $\mu/\delta$ *for some* $\delta > 0$ *independent of* $\mu$*.*

*(c) If* $\alpha$ *is bounded away from* $\pi$ *by an angle independent of* $\mu$*, the angle formed by* $Q_-$ *and* $v_-$ *is* $O(\mu^{1-\kappa})$*.*

*(d) The time interval during which the orbit stays in the sphere* $|Q_-| = \mu^\kappa$ *is*

$$\Delta t = \mu \Delta\tau = O(\mu^\kappa).$$

*Proof.* In the proof, we omit the subscript *in* standing for the variables *inside* the sphere $|Q_-| = \mu^\kappa$ without leading to confusion.

The idea of the proof is to treat the relative motion as a perturbation of Kepler motion and then approximate the relative velocities by their asymptotic values for the Kepler motion.

Part $(d)$ is the easiest. The radius of the sphere $|Q_-| = \mu^\kappa$ is $\mu^\kappa$. The relative velocity is $O(1)$ and it gets larger when $Q_-$ gets closer to the origin. So the total time for the relative motion to stay inside the sphere is $O(\mu^\kappa)$.

Fix a small number $\delta_1$. Below we derive several estimates valid for the first $\delta_1$ units of time the orbit spends in the set $|Q_-| \le \mu^k$. We then show that $\Delta t \ll \delta_1$. It will be convenient to measure time from the orbit enters the set $|Q_-| < \mu^k$.

Using the formula in the Appendix A.1.1, we decompose the Hamiltonian (5.2.5) as $H = H_{rel} + \mathfrak{h}(Q_+, v_+)$ where

$$H_{rel} = \frac{\mu^2}{4L^2} + \frac{|Q_-|^2}{2|Q_+|^2} - \frac{|Q_+ \cdot Q_-|^2}{2|Q_+|^5} + O(\mu^{3\kappa}),\ \text{as } \mu \to 0,$$

and $\mathfrak{h}$ depends only on $Q_+$ and $v_+$.

Note that $H$ is preserved and $\dot{\mathfrak{h}} = O(1)$ which implies that $\dfrac{L}{\mu}$ is $O(1)$ and moreover that ratio does not change much for $t \in [0, \delta_1]$. Using the identity $\dfrac{\mu^2}{4L^2} = \dfrac{v_-^2}{4} - \dfrac{\mu}{2|Q_-|}$ we see that initially $\dfrac{L}{\mu}$ is uniformly bounded from below for the orbits from Lemma 2.2.2. Thus there is a constant $\delta_2$ such that for $t \in [0, \delta_1]$ we have $\delta_2 \mu \le L(t) \le \dfrac{\mu}{\delta_2}$.

Expressing the Cartesian variables via Delaunay variables (c.f. equation (A.1.3) in Section A.1.2) we have up to rotation by angle $g$

$$q_1 = \frac{1}{\mu} L^2 (\cosh u - e) = O(\mu^\kappa), \quad q_2 = \frac{1}{\mu} LG \sinh u = O(\mu^\kappa) \tag{5.2.8}$$

where $u - e \sinh u = \ell$. We have

$$\dot{G} = \frac{\partial H}{\partial Q_-}\frac{\partial Q_-}{\partial g} = O(|Q_-|^2) = O(\mu^{2\kappa}). \tag{5.2.9}$$

Since $G = 2v_- \times Q_-$ we conclude that $G = O(\mu^\kappa)$ and hence $G(t) = O(\mu^\kappa)$ for all $t \in [0, \delta_1]$. This shows that $e = O(\mu^{\kappa-1})$. Now equation (5.2.8) shows that $\cosh u = O(\mu^{\kappa-1})$ and so $\ell = O(\mu^{\kappa-1})$. Next

$$\dot{\ell} = -\frac{\partial H}{\partial L} = -\frac{\mu^2}{2L^3} - \frac{\partial H_{rel}}{\partial Q_-}\frac{\partial Q_-}{\partial L} = -\frac{\mu^2}{2L^3} + O(\mu^\kappa)O(\mu^{\kappa-1}) = -\frac{\mu^2}{2L^3} + O(\mu^{2\kappa-1}).$$

Since the leading term here is at least $\frac{\delta_2^3}{2\mu}$ while $\ell = O(\mu^{\kappa-1})$ we obtain part (d) of the lemma. In particular the estimates derived above are valid for the time the orbits spends in $|Q_-| \leq \mu^\kappa$. Next

$$\dot{g} = -\frac{\partial H}{\partial G} = -\frac{\partial H}{\partial Q_-}\frac{\partial Q_-}{\partial G} = O(\mu^\kappa)O(\mu^{\kappa-1}) = O(\mu^{2\kappa-1}). \tag{5.2.10}$$

Integrating over time $\Delta t = O(\mu^\kappa)$ we get $|g_-^+ - g_-^-| = O(\mu^{3\kappa-1})$. Therefore $g$ and $\arctan\frac{G}{L}$ change by $O(\mu^{3\kappa-1})$.

We are now ready to derive the first two equations of (5.2.7). Let us denote till the end of the proof $\phi = \arctan\frac{G}{L}$, $\gamma = \frac{(1/2)-\kappa}{3}$. Recall (see (A.1.3)) that

$$p_1 = \tilde{p}_1 \cos g + \tilde{p}_2 \sin g, \quad p_2 = -\tilde{p}_1 \sin g + \tilde{p}_2 \cos g \text{ where} \tag{5.2.11}$$

$$\tilde{p}_1 = \frac{\mu}{L}\frac{\sinh u}{1 - e\cosh u}, \quad \tilde{p}_2 = \frac{\mu G}{L^2}\frac{\cosh u}{1 - e\cosh u}.$$

Consider two cases.

(I) $G \leq \mu^{\kappa+\gamma}$. In this case on the boundary of the sphere $|Q_-| = \mu^\kappa$ we have $\ell > \delta_3\mu^{-\gamma}$ for some constant $\delta_3$. Thus

$$\frac{p_2}{p_1} = \frac{\frac{\mu G}{L^2}\cosh u \cos g + \frac{\mu}{L}\sinh u \sin g}{-\frac{\mu G}{L^2}\cosh u \sin g + \frac{\mu}{L}\sinh u \cos g} = \frac{\frac{G}{L} \pm \tan g}{\pm 1 - \frac{G}{L}\tan g} + O(e^{-2|u|}) = \tan(g \pm \phi) + O(\mu^{-2\gamma}).$$

where the plus sign is taken if $u > 0$ and the minus sign is taken if $u < 0$. Is arctan is globally Lipshitz, this completes the proof in case (I).

(II) $G > \mu^{\kappa+\gamma}$. In this case $\dfrac{G}{L} \gg 1$ and so it suffices to show that $\dfrac{p_2}{p_1}$ (or $\dfrac{p_1}{p_2}$) changes little during the time the orbit is inside the sphere. Consider first the case where $|g^-| > \dfrac{\pi}{4}$ so $\sin g$ is bounded from below. Then

$$\frac{p_2}{p_1} = \cot g + O(\mu^{1-(\kappa+\gamma)})$$

proving the claim of part (a) in that case. The case $|g^-| \leq \dfrac{\pi}{4}$ is similar but we need to consider $\dfrac{p_1}{p_2}$. This completes the proof in case (II).

Combining equation (5.2.3) and Lemma 5.2.1(c) we obtain

$$Q_+^+ = Q_+^- + O(\mu^\kappa). \tag{5.2.12}$$

We also have $Q_-^+ = Q_-^- + O(\mu^\kappa)$ according to the definition of the sections $\{|Q_-^\pm| = \mu^\kappa\}$. This proves the last two equation in (5.2.7). Plugging (5.2.12) into (5.2.3) we see that

$$v_+^+ = v_+^- + O(\mu^\kappa).$$

This completes the proof of part (a).

The first claim of part (b) has already been established. The estimate of $G$ follows from the formula for $\alpha$. The estimate of the closest distance follows from the fact that if $\alpha$ is bounded away from 0 and $2\pi$ then the $Q_-$ orbit of $Q_-(t)$ is a small perturbation of Kepler motion and for Kepler motion the closest distance is of order $G$. We integrate the $\dot{G}$ equation (5.2.9) over time $O(\mu^\kappa)$ to get the total variation $\Delta G$ is at most $\mu^{3\kappa}$, which is much smaller than $\mu$. So $G$ is bounded away from 0 by a quantity of order $O(\mu)$.

Finally part (c) follows since we know $G = \mu^\kappa |v_-| \sin \measuredangle(v_-, Q_-) = O(\mu)$. $\square$

*Proof of Lemma 2.2.2.* Letting $\mu = 0$ in the first two equations of (5.2.7) we obtain the equations of elastic collisions. Namely, both the kinetic energy conservation

$$|v_3^+|^2 + |v_4^+|^2 = |v_3^-|^2 + |v_4^-|^2$$

and momentum conservation

$$v_3^+ + v_4^+ = v_3^- + v_4^-$$

laws hold. On the other hand, the Gerver's map $\mathbf{G}$ in Lemma 2.2.2 is also defined through elastic collisions. As a result, Lemma 5.2.2 says actually the same thing as Lemma 2.2.2 up to a change of variables going from Cartesian to the set of variables $E_3, \ell_3, G_3, g_3, G_4, g_4$. $\square$

## 5.3 Derivative of the local map

### 5.3.1 Justifying the asymptotics

Here we give the proof of Lemma 3.3.1. Our goal is to show that the main contribution to the derivative comes from differentiating the main term in Lemma 5.2.2.

*Proof of Lemma 3.3.1.* Since the transformation from Delaunay to Cartesian variables is symplectic and the norms of the transformation matrices are independent of $\mu$, it is sufficient to prove the lemma in terms of Cartesian coordinates. To go to the coordinates system used in Lemma 3.3.1, we only need to multiply the Cartesian derivative matrix by $O(1)$ matrices, namely, by $\dfrac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(Q_3, v_3, Q_4, v_4)^+}$ on the left and by $\dfrac{\partial(Q_3, v_3, Q_4, v_4)^-}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-}$ on the right. This does not change the form of the $d\mathbb{L}$ stated in Lemma 3.3.1.

As before we use the formula (4.2.1). We need to consider the integration of the variational equations and also the boundary contribution. The proof is organized as follows. The main part of the proof is to study the relative motion part, while controlling the motion of the mass center is easier.

Using $\ell$ as the time variable the equations for relative motion take the following form (recall that the scale for $\ell$ is $O(\mu^{\kappa-1})$):

$$\begin{cases} \dfrac{\partial L}{\partial \ell} = -2\mu^{-2}L^3\dfrac{\partial H}{\partial \ell}\left(1 - 2\mu^{-2}L^3\dfrac{\partial H}{\partial L} + \dots\right) = O(\mu^{2+\kappa}), \\ \dfrac{\partial G}{\partial \ell} = -2\mu^{-2}L^3\dfrac{\partial H}{\partial g}\left(1 - 2\mu^{-2}L^3\dfrac{\partial H}{\partial L} + \dots\right) = O(\mu^{1+2\kappa}), \\ \dfrac{\partial g}{\partial \ell} = 2\mu^{-2}L^3\dfrac{\partial H}{\partial G}\left(1 - 2\mu^{-2}L^3\dfrac{\partial H}{\partial L} + \dots\right) = O(\mu^{2\kappa}), \end{cases} \tag{5.3.1}$$

where $\dots$ denote the lower order terms. The estimates of the last two equations follow from (5.2.9) and (5.2.10) while the estimate of the first equation is similar to the last two.

Then we analyze the variational equations.

$$\begin{bmatrix} \dfrac{d\delta L}{d\ell} \\ \dfrac{d\delta G}{d\ell} \\ \dfrac{d\delta g}{d\ell} \end{bmatrix} = O\left(\begin{bmatrix} \mu^{1+\kappa} & \mu^{1+\kappa} & \mu^{1+2\kappa} \\ \mu^{1+\kappa} & \mu^{2\kappa} & \mu^{1+2\kappa} \\ \mu^{2\kappa-1} & \mu^{2\kappa-1} & \mu^{2\kappa} \end{bmatrix}\right)\begin{bmatrix} \delta L \\ \delta G \\ \delta g \end{bmatrix} + O\left(\begin{bmatrix} \mu^{2+\kappa} & 0 \\ \mu^{1+2\kappa} & 0 \\ \mu^{2\kappa} & 0 \end{bmatrix}\right)\begin{bmatrix} \delta Q_+ \\ \delta v_+ \end{bmatrix}. \tag{5.3.2}$$

In the following, we first set $\delta Q_+ = 0$ and work with the fundamental solution of the homogeneous equation. Then we will prove that $\delta Q_+$ is negligible.

Introducing $\delta g' = \dfrac{\delta g}{\mu^{3\kappa-2}}$ we need to find the asymptotics of the fundamental solution of

$$\begin{bmatrix} \dfrac{d\delta L}{d\ell} \\ \dfrac{d\delta G}{d\ell} \\ \dfrac{d\delta g'}{d\ell} \end{bmatrix} = O\left(\begin{bmatrix} \mu^{1+\kappa} & \mu^{1+\kappa} & \mu^{5\kappa-1} \\ \mu^{1+\kappa} & \mu^{2\kappa} & \mu^{5\kappa-1} \\ \mu^{1-\kappa} & \mu^{1-\kappa} & \mu^{2\kappa} \end{bmatrix}\right) \begin{bmatrix} \delta L \\ \delta G \\ \delta g' \end{bmatrix} \tag{5.3.3}$$

Integrating this equation over time $\mu^{\kappa-1}$ we see that the fundamental solution is $O(1)$. Now arguing the same way as in section 4.3.2 we see that the fundamental solution takes form

$$Id + O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa} & \mu^{6\kappa-2} \\ \mu^{2\kappa} & \mu^{3\kappa-1} & \mu^{6\kappa-2} \\ 1 & 1 & \mu^{3\kappa-1} \end{bmatrix}\right). \tag{5.3.4}$$

In the following it is convenient to use variables $L' = \mu L$, $G$ and $g$. In these variables fundamental solution part of the variational equation is

$$Id + O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{bmatrix}\right). \tag{5.3.5}$$

Next, we compute the boundary contribution. In terms of the Delaunay variables inside the sphere $|Q_-| = \mu^\kappa$, we have

$$\frac{\partial \ell}{\partial(L', G, g)} = -\left(\frac{\partial |Q_-|}{\partial \ell}\right)^{-1} \frac{\partial |Q_-|}{\partial(L', G, g)} = (O(\mu^{\kappa-1}), O(\mu^{\kappa-2}), 0). \tag{5.3.6}$$

Indeed, due to (5.2.8) we have $\dfrac{\partial |Q_-|}{\partial g} = 0$, $\dfrac{\partial |Q_-|}{\partial \ell} = O(\mu)$, $\dfrac{\partial |Q_-|}{\partial L'} = O(\mu^\kappa)$ and

$\dfrac{\partial|Q_-|}{\partial G} = O(\mu^{\kappa-1})$. Combining this with (5.3.1) we get

$$\left(\frac{\partial L'}{\partial \ell}, \frac{\partial G}{\partial \ell}, \frac{\partial g}{\partial \ell}\right) \otimes \frac{\partial \ell}{\partial (L', G, g)} = O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix}\right). \tag{5.3.7}$$

Using (4.2.1) we obtain the derivative matrix

$$\begin{aligned}\frac{\partial(L', G, g)^+}{\partial(L', G, g)^-} &= \left(Id + O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix}\right)\right)^{-1} \\ &\cdot \left(Id + O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{bmatrix}\right)\right)\left(Id - O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix}\right)\right) \\ &= Id + O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{bmatrix}\right) := Id + P.\end{aligned} \tag{5.3.8}$$

We are now ready to compute the relative motion part of the derivative of the Poincaré map. For the space variables, we are only interested in the angle $\theta := \arctan\left(\dfrac{q_2}{q_1}\right)$ since the length $|(q_1, q_2)|$ is fixed when restricted on the sphere.

We split the derivative matrix as follows:

$$\frac{\partial(\theta_-, v_-)^+}{\partial(\theta_-, v_-)^-} = \frac{\partial(\theta_-, v_-)^+}{\partial(L', G, g)^+}\frac{\partial(L', G, g)^+}{\partial(L', G, g)^-}\frac{\partial(L', G, g)^-}{\partial(\theta_-, v_-)^-} = \tag{5.3.9}$$

$$\frac{\partial(\theta_-, v_-)^+}{\partial(L', G, g)^+}\frac{\partial(L', G, g)^-}{\partial(\theta_-, v_-)^-} + \frac{\partial(\theta_-, v_-)^+}{\partial(L', G, g)^+}P\frac{\partial(L', G, g)^-}{\partial(\theta_-, v_-)^-} = I + II.$$

Using equations (5.2.8) and (5.2.11) we obtain

$$\frac{\partial(\theta_-, v_-)^+}{\partial(L', G, g)^+} = O\left(\begin{bmatrix} 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \end{bmatrix}\right). \tag{5.3.10}$$

Next, we consider the first term in (5.3.9).

$$I = \frac{\partial(\theta_-, v_-)^+}{\partial L'^+} \otimes \frac{\partial L'^-}{\partial(\theta_-, v_-)^-} + \frac{\partial(\theta_-, v_-)^+}{\partial G^+} \otimes \frac{\partial G^-}{\partial(\theta_-, v_-)^-} + \frac{\partial(\theta_-, v_-)^+}{\partial g^+} \otimes \frac{\partial g^-}{\partial(\theta_-, v_-)^-}. \tag{5.3.11}$$

Using the expressions

$$\frac{1}{4L'^2} = \frac{v_-^2}{4} - \frac{\mu}{2|Q_-|}, \quad G = v_- \times Q_- = |v_-| \cdot |Q_-| \sin \measuredangle(v_-, Q_-)$$

we see that

$$\frac{\partial L'^-}{\partial(\theta_-, v_-)^-} = O(1), \quad \frac{\partial G^-}{\partial(\theta_-, v_-)^-} = (O(\mu^\kappa), O(\mu^\kappa)). \tag{5.3.12}$$

Next, we have $\dfrac{\partial(\theta_-, v_-)^+}{\partial g^+} = (O(1), O(1))$ from equations (5.2.8) and (5.2.11). To obtain the derivatives of $g$ we use the fact that

$$\frac{p_2}{p_1} = \frac{\sin g \sinh u \pm \frac{G}{\mu L'} \cos g \cosh u}{\cos g \sinh u \mp \frac{G}{\mu L'} \sin g \cosh u} = \frac{\tan g \pm \frac{G}{\mu L'}}{1 \mp \frac{G}{\mu L'} \tan g} + e^{-2|u|} E(G/\mu L', g, u),$$

where $E$ is a smooth function satisfying $\dfrac{\partial E}{\partial g} = O(1)$ as $\ell \to \infty$. Therefore we get

$$g = \arctan\left(\frac{p_2}{p_1} - e^{-2|u|} E(G/\mu L', g)\right) \mp \arctan \frac{G}{\mu L'} \text{ as } \ell \to \infty.$$

We choose the + when considering the incoming orbit parameters. Thus

$$\frac{\partial g}{\partial(\theta_-, v_-)}\left(1 + O(e^{-2|u|})\right) = \frac{\partial \arctan \frac{p_2}{p_1}}{\partial(\theta_-, v_-)} + \frac{\partial \arctan \frac{G}{\mu L'}}{\partial L'} \frac{\partial L'}{\partial(\theta_-, v_-)} + \frac{\partial \arctan \frac{G}{\mu L'}}{\partial G} \frac{\partial G}{\partial(\theta_-, v_-)} + O(e^{-2|u|}).$$

Hence

$$\frac{\partial g}{\partial(\theta_-, v_-)} = O\left(\frac{1}{\mu}\right)\frac{\partial G}{\partial(\theta_-, v_-)} + O(1), \tag{5.3.13}$$

where the $1/\mu$ comes from $\dfrac{\partial \arctan \dfrac{G}{\mu L'}}{\partial G}$ and all other terms are $O(1)$ or even smaller.

Therefore

$$I = \frac{1}{\mu}\left(\mu\frac{\partial(\theta_-, v_-)^+}{\partial G^+} + \mu\frac{\partial \arctan \dfrac{G^-}{\mu L'^-}}{\partial G^-}\frac{\partial(\theta_-, v_-)^+}{\partial g^+}\right) \otimes \frac{\partial G^-}{\partial(\theta_-, v_-)^-}$$
$$+\left(\frac{\partial(\theta_-, v_-)^+}{\partial L'^+} \otimes \frac{\partial L'^-}{\partial(\theta_-, v_-)^-} + \frac{\partial(\theta_-, v_-)^+}{\partial g^+} \otimes \left(\frac{\partial \arctan \frac{p_2^-}{p_1^-}}{\partial(\theta_-, v_-)} + \frac{\partial \arctan \frac{G^-}{\mu L'^-}}{\partial L'^-}\frac{\partial L'^-}{\partial(\theta_-, v_-)}\right)\right). \tag{5.3.14}$$

Since the expression in parenthesis of the first term is $O(1)$, I has the rate of growth required in Lemma 3.3.1.

Now we study the second term in (5.3.9)

$$
\begin{aligned}
II &= O\left(\begin{bmatrix} 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \end{bmatrix}\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{bmatrix}\right)\frac{\partial(L', G, g)^-}{\partial(\theta_-, v_-)^-} \\
&= O\left(\begin{bmatrix} \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{bmatrix}\right)\frac{\partial(L', G, g)^-}{\partial(\theta_-, v_-)^-} \\
&= O\left(\begin{bmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{bmatrix}\right)\otimes\frac{\partial L'^-}{\partial(\theta_-, v_-)^-} \\
&+ O\left(\begin{bmatrix} \mu^{3\kappa-2} \\ \mu^{3\kappa-2} \\ \mu^{3\kappa-2} \end{bmatrix}\right)\otimes\frac{\partial G^-}{\partial(\theta_-, v_-)^-} + O\left(\begin{bmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{bmatrix}\right)\otimes\frac{\partial g^-}{\partial(\theta_-, v_-)^-}
\end{aligned}
\tag{5.3.15}
$$

where we use the assumption that $\kappa < 1/2$, which implies $\mu^{2\kappa} < \mu^{3\kappa-1}$ and $\mu^{2\kappa-1} < \mu^{3\kappa-2}$. The first summand in (5.3.15) is $O(\mu^{3\kappa-1})$.Therefore (5.3.13) implies that

$$
II = \frac{1}{\mu}\begin{bmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{bmatrix}\otimes\frac{\partial G^-}{\partial(\theta_-, v_-)^-} + O(\mu^{3\kappa-1}). \tag{5.3.16}
$$

Now we combine (5.3.14) and (5.3.16) to get

$$\frac{\partial(\theta_-,v_-)^+}{\partial(\theta_-,v_-)^-}=\frac{1}{\mu}\left(\mu\frac{\partial(\theta_-,v_-)^+}{\partial G^+}+\mu\frac{\partial \arctan\dfrac{G^-}{\mu L'^-}}{\partial G^-}\frac{\partial(\theta_-,v_-)^+}{\partial g^+}+O(\mu^{3\kappa-1})\right)\otimes\frac{\partial G^-}{\partial(\theta_-,v_-)^-}$$
$$+\left(\frac{\partial(\theta_-,v_-)^+}{\partial L'^+}\otimes\frac{\partial L'^-}{\partial(\theta_-,v_-)^-}+\frac{\partial(\theta_-,v_-)^+}{\partial g^+}\otimes\left(\frac{\partial \arctan\frac{p_2^-}{p_1^-}}{\partial(\theta_-,v_-)^-}+\frac{\partial\arctan\frac{G^-}{\mu L'^-}}{\partial L'^-}\frac{\partial L'^-}{\partial(\theta_-,v_-)^-}\right)+O(\mu^{3\kappa-1})\right) \tag{5.3.17}$$

(5.3.17) has the structure stated in the lemma. In (5.3.17), we use the variable $\theta_-$ for the relative position $Q_-$ and we have $\dfrac{\partial G^-}{\partial(\theta_-,v_-)^-}=O(\mu^\kappa)$. To get back to $Q_-$, i.e. to obtain $\dfrac{\partial(Q_-,v_-)^+}{\partial(Q_-,v_-)^-}$, we use $Q_-=\mu^\kappa(\cos\theta_-,\sin\theta_-)$. So we have the estimate $\dfrac{\partial Q_-^+}{\partial(L_-,G_-,g_-)^+}=O(\mu^\kappa)\dfrac{\partial\theta_-^+}{\partial(L_-,G_-,g_-)^+}=O(\mu^\kappa)$. To get $\dfrac{\partial-}{\partial Q_-^-}$, we use the transformation from polar coordinates to Cartesian, $\dfrac{\partial-}{\partial Q_-^-}=\dfrac{\partial-}{\partial(r_-,\theta_-)^-}\dfrac{\partial(r_-,\theta_-)^-}{\partial Q_-^-}$, where $r_-=|Q_-^-|=\mu^\kappa$, therefore we have $\dfrac{\partial r_-^-}{\partial Q_-^-}=0$ and $\dfrac{\partial-}{\partial Q_-^-}=\dfrac{1}{\mu^\kappa}\dfrac{\partial-}{\partial\theta_-^-}(-\sin\theta_-^-,\cos\theta_-^-)$. So we have the estimate $\dfrac{\partial G^-}{\partial Q_-^-}=O(1)$, and $\dfrac{\partial L'^-}{\partial Q_-^-}=\dfrac{\partial L'^-}{\partial\theta_-^-}=0$ since in the expression $\dfrac{1}{4L'^2}=\dfrac{v_-^2}{4}-\dfrac{\mu}{2|Q_-|}$, the angle $\theta_-$ plays no role. Finally, we have $\dfrac{\partial\arctan\frac{p_2^-}{p_1^-}}{\partial Q_-^-}=0$. So we get

$$\frac{\partial(Q_-,v_-)^+}{\partial(Q_-,v_-)^-}=\frac{1}{\mu}(O(\mu^\kappa)_{1\times2},O(1)_{1\times2})\otimes(O(1)_{1\times2},O(\mu^\kappa)_{1\times2})+O(1)_{4\times4}+O(\mu^{3\kappa-1}).$$

It remains to show that other entries of the derivative matrix are $O(1)$. Consider the following decomposition

$$\begin{aligned}
&\frac{\partial(Q_-, v_-, Q_+, v_+)^+}{\partial(Q_-, v_-, Q_+, v_+)^-} = \frac{\partial(Q_-, v_-, Q_+, v_+)^+}{\partial(L', G, g, Q_+, v_+)^+}\frac{\partial(L', G, g, Q_+, v_+)^+}{\partial(L', G, g, Q_+, v_+)(\ell^f)} \\
&\frac{\partial(L', G, g, Q_+, v_+)(\ell^f)}{\partial(L', G, g, Q_+, v_+)(\ell^i)}\frac{\partial(L', G, g, Q_+, v_+)(\ell^i)}{\partial(L', G, g, Q_+, v_+)^-}\frac{\partial(L', G, g, Q_+, v_+)^-}{\partial(Q_-, v_-, Q_+, v_+)^-} \\
&:= \begin{bmatrix} M & 0 \\ 0 & Id \end{bmatrix}\begin{bmatrix} A & 0 \\ B & Id \end{bmatrix}\begin{bmatrix} C & D \\ E & F \end{bmatrix}\begin{bmatrix} A' & 0 \\ B' & Id \end{bmatrix}\begin{bmatrix} N & 0 \\ 0 & Id \end{bmatrix} \\
&= \begin{bmatrix} MACA'N + MADB'N & MAD \\ (BC+E)A'N + (BD+F)B'N & BD+F \end{bmatrix}
\end{aligned} \tag{5.3.18}$$

We have already computed $M$, $A$, $C$, $A'$ and $N$ (see (5.3.10), (5.3.7), (5.3.12), (5.3.13)), where $C$ is (5.3.5) and $ACA' = Id + P$ is (5.3.8). We still need to compute $B, B', D, E, F$.

From the Hamiltonian (5.2.5), we have $\dot{\ell} = -\dfrac{1}{2\mu L'^3} + O(\mu^{2\kappa})$. We need to supplement (5.3.1) and (5.3.2) by the following equations.

$$\begin{aligned}
\frac{dQ_+}{d\ell} &= -\frac{v_+}{2}(2\mu L'^3)(1 + O(\mu^{2\kappa+1})) = O(\mu) \\
\frac{dv_+}{d\ell} &= \left(\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa})\right)(2\mu L'^3)(1 + O(\mu^{2\kappa+1})) = O(\mu).
\end{aligned} \tag{5.3.19}$$

$$\begin{bmatrix} \dfrac{d\delta Q_+}{d\ell} \\ \dfrac{d\delta v_+}{d\ell} \end{bmatrix} = \begin{bmatrix} \mu^{2\kappa+2} & \mu \\ \mu & 0 \end{bmatrix}\begin{bmatrix} \delta Q_+ \\ \delta v_+ \end{bmatrix} + \begin{bmatrix} \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2} \\ \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2} \end{bmatrix}\begin{bmatrix} \delta L' \\ \delta G \\ \delta g \end{bmatrix}. \tag{5.3.20}$$

It follows from (5.3.6) and (5.3.19) that

$$B, B' = O\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}\right) \otimes O\left([\mu^{\kappa-1}, \mu^{2\kappa-2}, 0]\right).$$

Next, we obtain

$$D = O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-1} \end{bmatrix}\right) E = O\left(\begin{bmatrix} \mu^{\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \end{bmatrix}\right), \ F = Id + O\left(\begin{bmatrix} \mu^{3\kappa} & \mu^{\kappa} \\ \mu^{\kappa} & \mu^{3\kappa} \end{bmatrix}\right).$$

It is a straightforward computation that $CA' \gg DB'$, so $ADB'$ is only a small perturbation to the $P$ part in $ACA' = Id + P$ in (5.3.8), and therefore $MACA'N + MADB'N$ in (5.3.18) has the same structure as $MACA'N$ obtained in (5.3.14) and (5.3.15). We also have the entry $BD + F = \begin{bmatrix} 1+\mu^{5\kappa-1} & \mu^{\kappa} \\ \mu^{\kappa} & 1+\mu^{5\kappa-1} \end{bmatrix}$. The entry

$$(BC+E)A'N + (BD+F)B'N = \frac{1}{\mu}[O(\mu^{2\mu})]_{1\times 3} \otimes \frac{\partial G^-}{\partial(\theta, v)^-_-} + O(\mu^{\kappa}). \qquad (5.3.21)$$

Finally, we have that the entry $MAD = [O(\mu^{3\kappa-1})]_{3\times 2}$.

This estimate of the matrix in (5.3.18) is enough to conclude the Lemma. $\square$

The above proof actually gives us more information. Now we use the Delaunay variables $(L_3, \ell_3, G_3, g_3, G_4, g_4)^{\pm}$ as the orbit parameters *outside* the sphere $|Q_-| = \mu^{\kappa}$ and add a subscript *in* to the Delaunay variables *inside* the sphere. We relate Lemma 5.2.2 to the above proof.

**Corollary 1.** *The derivative of the local map has the following form*

$$d\mathbb{L} = \frac{1}{\mu}(\hat{u} + O(\mu^{3\kappa-1})) \otimes \mathbf{1} + \hat{B} + O(\mu^{3\kappa-1}),$$

*where* $\hat{u}, \mathbf{1}$ *and* $\hat{B}$ *are computed by discarding the* $O(\mu^{3\kappa-1})$ *and* $O(\mu^{\kappa})$ *errors in* (5.2.7). *In particular,*

$$\hat{u} = \frac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(Q_3, v_3, Q_4, v_4)^+} \frac{\partial(Q_3, v_3, Q_4, v_4)^+}{\partial \alpha} \left(\mu \frac{\partial \alpha}{\partial G_{in}}\right),$$

$$\mathbf{l} = \frac{\partial G_{in}}{\partial(Q_3, v_3, Q_4, v_4)^-}\frac{\partial(Q_3, v_3, Q_4, v_4)^-}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-}.$$

*Proof.* We first discard the $O(\mu^{3\kappa-1})$ and $O(\mu^\kappa)$ errors in (5.2.7), and rewrite (5.2.7) in terms of the coordinates of the relative motion and motion of mass center as

$$Q_-^+ = 0,\ v_-^+ = R(\alpha)v_-^- = R(\alpha/2+g)(|v_-^-|, 0), \quad Q_+^+ = Q_+^-,\ v_+^+ = v_+^-.$$

The derivative matrix is block diagonalized. We get identity for the derivative of the motion of the mass center part, which agrees with the entry $BD + F$ in (5.3.18) in the limit $\mu \to 0$. Then we only need to focus on relative motion part. On the other hand, our computation of (5.3.14) is based on formula (5.2.8), where the velocity can be written as $v_-^+ = R(\alpha/2+g)(1/L', 0) + O(e^{-2|u|}),\ |u| \to \infty$. We also have $1/L' = |v_-^-|$ as $\mu \to 0$. Moreover, in (5.3.14), we have $\dfrac{\partial \theta_-^+}{\partial(\theta_-, v_-)^-}$ is of the same order as $\dfrac{\partial v_-^+}{\partial(\theta_-, v_-)^-}$, which implies $\dfrac{\partial Q_-^+}{\partial(\theta_-, v_-)^-} = O(\mu^\kappa)\dfrac{\partial v_-^+}{\partial(\theta_-, v_-)^-}$ as $\mu \to 0$ since $Q_-^+ = \mu^\kappa(\cos\theta_-^+, \cos\theta_-^+)$. So we only consider $\dfrac{\partial v_-^+}{\partial(\theta_-, v_-)^-}$ in the following.

Then compute the derivative

$$\frac{\partial v_-^+}{\partial(\theta_-, v_-)^-} = \frac{\partial v_-^+}{\partial \alpha}\frac{\partial \alpha}{\partial G_{in}} \otimes \frac{\partial G_{in}}{\partial(\theta_-, v_-)^-} + \frac{\partial v_-^+}{\partial \alpha}\frac{\partial \alpha}{\partial L'_{in}} \otimes \frac{\partial L'_{in}}{\partial(\theta_-, v_-)^-} + \frac{\bar\partial v_-^+}{\partial(\theta_-, v_-)^-}, \tag{5.3.22}$$

where in the last summand we use $\bar\partial/\partial$ to stand for the partial derivative w.r.t. the *explicit* dependence on $v_-^-$. Notice $\dfrac{\partial \alpha}{\partial G_{in}} = O(1/\mu)$ and $\dfrac{\partial \alpha}{\partial L'_{in}} = O(1)$. This matrix has the form of $\dfrac{1}{\mu}\hat u \otimes \mathbf{l} + \hat B$ up to a change of variables to Delaunay variables outside the sphere.

We notice $g = \alpha/2 + \arctan \dfrac{p_2^-}{p_1^-}$ and $v_-^- = (p_1^-, p_2^-)$. We express

$$\frac{\bar\partial v_-^+}{\partial(\theta_-, v_-)^-} = \frac{\bar\partial v_-^+}{\partial \arctan \frac{p_2^-}{p_1^-}} \frac{\partial \arctan \frac{p_2^-}{p_1^-}}{\partial(\theta_-, v_-)^-} + \frac{\bar\partial v_-^+}{\partial |v_-^-|} \frac{\partial |v_-^-|}{\partial(\theta_-, v_-)^-}.$$

Then we identify $\dfrac{1}{\mu}\left(\mu \dfrac{\partial(\theta_-, v_-)^+}{\partial G^+} + \mu \dfrac{\partial \arctan \frac{G^-}{\mu L'^-}}{\partial G^-} \dfrac{\partial(\theta_-, v_-)^+}{\partial g^+}\right)$ in (5.3.14) with $\dfrac{\partial v_-^+}{\partial \alpha}\dfrac{\partial \alpha}{\partial G}$. Then we identify $\dfrac{\partial(\theta_-, v_-)^+}{\partial L'^+}$ in (5.3.14) with $\dfrac{\partial v_-^+}{\partial \alpha}\dfrac{\partial \alpha}{\partial L'} + \dfrac{\bar\partial v_-^+}{\partial |v_-^-|}\dfrac{d|v_-^-|}{dL'}$ and finally, we identify $\dfrac{\partial(\theta_-, v_-)^+}{\partial g^+} \otimes \dfrac{\partial \arctan \frac{p_2^-}{p_1^-}}{\partial(\theta_-, v_-)}$ in (5.3.14) with $\dfrac{\bar\partial v_-^+}{\partial \arctan \frac{p_2^-}{p_1^-}} \dfrac{\partial \arctan \frac{p_2^-}{p_1^-}}{\partial(\theta_-, v_-)^-}$ and $\dfrac{\partial(\theta_-, v_-)^+}{\partial g^+} \otimes \dfrac{\partial \arctan \frac{G^-}{\mu L'^-}}{\partial L'^-} \dfrac{\partial L'^-}{\partial(\theta_-, v_-)}$ with $\dfrac{\bar\partial v_-^+}{\partial |v_-^-|} \dfrac{\partial |v_-^-|}{\partial(\theta_-, v_-)^-}$ using $1/L' = |v_-^-|$. Now we have shown that the formal derivative in (5.2.7) agrees with the derivative we obtained in the proof of Lemma 3.3.1. □

**Corollary 2.** *If we take derivative along a direction $\Gamma = \gamma'(0) = O(1)$, where $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^6$ is a curve parameterized by $\psi$, and $\dfrac{\partial G_{in}^-}{\partial \psi} = O(\mu)$ in the following set of equations*

$$\begin{cases} |v_3^+|^2 + |v_3^+|^2 = |v_3^-|^2 + |v_3^-|^2 + o(1), \\ v_3^+ + v_3^+ = v_3^- + v_3^- + o(1), \\ Q_3^+ + Q_4^+ = Q_3^- + Q_4^- + o(1), \end{cases}$$

*obtained from equation* (5.2.7)*, then the $o(1)$ terms are also $o(1)$ in the $C^1$ sense. Namely, we can drop the $o(1)$ terms when we take the derivative $\dfrac{d}{d\psi}$.*

*Proof.* For the motion of the mass center, it follows from Corollary 1 that

$$\frac{\partial(Q_+,v_+)^+}{\partial(Q_-,v_-,Q_+,v_+)^-} = \frac{1}{\mu}\frac{\partial(Q_+,v_+)^+}{\partial\alpha}\otimes \mathbf{l} + (0_{4\times 4}, Id_{4\times 4}) + o(1).$$ We already obtained that $\frac{\partial(Q_+,v_+)^+}{\partial\alpha} = O(\mu^{2\kappa})$ in equation (5.3.21). Our assumption $\frac{\partial G_{in}^-}{\partial\psi} = \mathbf{l}\cdot\Gamma = O(\mu)$ suppresses the $1/\mu$ term. We are only left with $Id$ with a small perturbation. This proves the motion of mass center part of the Corollary.

For the energy conservation part, we use the Hamiltonian (5.2.5). It is enough to show $|v_-^+|^2 = |v_-^-|^2 + o(1)$ with $C^1$ small perturbation if we take $\psi$ derivative, since we already proved the lemma for the velocity of the mass center. In (5.2.5), the terms involving only $Q_+, v_+$ are handled using the result of the previous paragraph. The term $-\frac{\mu}{|Q_-|}$ vanishes after taking $\psi$ derivative due to $|Q_-| = \mu^\kappa$. All the remaining terms has $Q_-$ to the power 2 or higher. We have $\frac{\partial Q_-^-}{\partial\psi} = O(1)$ since $\Gamma = O(1)$. We also have $\frac{\partial Q_-^+}{\partial\psi} = O(1)$ since we have $d\mathbb{L}\Gamma = O(1)$. As a result, after taking the $\psi$ derivative, any term involving $Q_-$ is of order $O(\mu^\kappa)$. This completes the proof of the energy conservation part. □

### 5.3.2 Proof of the Lemma 3.3.8

In the following, we first try to work out the $O(1/\mu)$ term in the local map. We need that $span\{w_i, \tilde{w}\}$ does not lie in $Ker\mathbf{l}_{3-i}$ in order to check the nondegeneracy condition. Any vector in $span\{w_i, \tilde{w}\}$ has the form of $(0, *, 0, 0, *, *)$. We will pick a vector in the span of the form $(0, 0, 0, 0, *, *)$ to show it does not lie in the $Ker\mathbf{l}_{3-i}$. For this purpose, we only need to work out $\frac{\partial G_{in}}{\partial G_4}, \frac{\partial G_{in}}{\partial g_4}$ for the functional $\mathbf{l}$.

*Proof of Lemma 3.3.8.* The proof is done numerically.

**Before collision, $\mathbf{l}=\dfrac{\partial G_{in}}{\partial -}$.**

To simplify the computation, we treat $L_3^-,\ell_3^-,G_3^-,g_3^-$ as fixed. This will be enough for us to check that $span\{w,\tilde{w}\}$ does not lie in the $Ker\mathbf{l}$ since the first four entries of $w$ are zero according to equation (3.3.3). When we consider $\dfrac{\partial G_{in}}{\partial G_4^-},\dfrac{\partial G_{in}}{\partial g_4^-}$, we see from the expression of $G_{in}$ in Lemma 5.2.2, up to a term of order $\mu^\kappa$, we have

$$\left(\frac{\partial G_{in}}{\partial G_4^-},\frac{\partial G_{in}}{\partial g_4^-}\right)=(v_3^- - v_4^-)\times\left(\frac{\partial}{\partial G_4^-},\frac{\partial}{\partial g_4^-}\right)Q_4+(v_3^- - v_4^-)\times\left(\frac{\partial Q_4}{\partial \ell_4^-}\right)\cdot\left(\frac{\partial \ell_4^-}{\partial G_4^-},\frac{\partial \ell_4^-}{\partial g_4^-}\right)+O(\mu^\kappa).$$

We need to eliminate $\ell_4$ using the relation $|Q_3-Q_4|=\mu^\kappa$.

$$\left(\frac{\partial \ell_4^-}{\partial G_4^-},\frac{\partial \ell_4^-}{\partial g_4^-}\right)=-\left(\frac{\partial|Q_3-Q_4|}{\partial \ell_4^-}\right)^{-1}\left(\frac{\partial|Q_3-Q_4|}{\partial G_4^-},\frac{\partial|Q_3-Q_4|}{\partial g_4^-}\right)$$
$$=-\frac{(Q_3-Q_4)\cdot\left(\dfrac{\partial Q_4}{\partial G_4^-},\dfrac{\partial Q_4}{\partial g_4^-}\right)}{(Q_3-Q_4)\cdot\dfrac{\partial Q_4}{\partial \ell_4^-}}=-\frac{(v_3^- - v_4^-)\cdot\left(\dfrac{\partial Q_4}{\partial G_4^-},\dfrac{\partial Q_4}{\partial g_4^-}\right)}{(v_3^- - v_4^-)\cdot\dfrac{\partial Q_4}{\partial \ell_4^-}}+O(\mu^{1-\kappa}).$$

Here we replace $Q_3^- - Q_4^-$ by $v_3^- - v_4^-$ using the fact that the two vectors form an angle of order $O(\mu^{1-\kappa})$.

So
$$\left(\frac{\partial G_{in}}{\partial G_4^-},\frac{\partial G_{in}}{\partial g_4^-}\right)=(v_3^- - v_4^-)\times\left(\frac{\partial}{\partial G_4^-},\frac{\partial}{\partial g_4^-}\right)Q_4$$
$$-(v_3^- - v_4^-)\times\frac{\partial Q_4}{\partial \ell_4^-}\left(\frac{(v_3^- - v_4^-)\cdot(\dfrac{\partial Q_4}{\partial G_4^-},\dfrac{\partial Q_4}{\partial g_4^-})}{(v_3^- - v_4^-)\cdot\dfrac{\partial Q_4}{\partial \ell_4^-}}\right)+O(\mu^\kappa).$$

We use mathematica and the formulas and data in the Appendix B.2.2 to work out $\dfrac{\partial G_{in}}{\partial -}$ in the lemma.

**After collision,$\hat{u}=\dfrac{\partial -}{\partial \alpha}$.**

In equation (5.2.7), we let $\mu\to 0$. We need to eliminate $\ell_4^+$ using the condition

$|Q_3^+ - Q_4^+| = \mu^\kappa$. This is nothing but equation (5.1.2). We apply the implicit function theorem to (5.2.7) when $\mu = 0$ to obtain

$$\left(\frac{\partial(Q_3^+, v_3^+, Q_4^+, v_4^+)}{\partial(X^+, Y^+)} + \frac{\partial(Q_3^+, v_3^+, Q_4^+, v_4^+)}{\partial \ell_4^+} \otimes \frac{\partial \ell_4^+}{\partial(X^+, Y^+)}\right) \cdot \frac{\partial(X^+, Y^+)}{\partial \alpha}$$
$$= \frac{1}{2}\left(0, 0, R\left(\frac{\pi}{2} + \alpha\right)(v_3^- - v_4^-), 0, 0, -R\left(\frac{\pi}{2} + \alpha\right)(v_3^- - v_4^-)\right)^T$$
$$= \frac{1}{2}\left(0, 0, R\left(\frac{\pi}{2}\right)(v_3^+ - v_4^+), 0, 0, -R\left(\frac{\pi}{2}\right)(v_3^+ - v_4^+)\right)^T.$$

where $R(\pi/2+\alpha) = \dfrac{dR(\alpha)}{d\alpha}$ and the formula for $\dfrac{\partial \ell_4^+}{\partial(X^+, Y^+)}$ is given in (5.1.2). Again use mathematica and these formulas to work out the $\dfrac{\partial -}{\partial \alpha}$ in the lemma. To obtain a symbolic sequence with any order of symbols $3, 4$ as claimed in the main Theorem, we notice that the only difference is that the outgoing relative velocity changes sign $(v_3^+ - v_4^+) \to -(v_3^+ - v_4^+)$. So we only need to send $\hat{u} \to -\hat{u}$. We point out that we renormalize the vector $\hat{u}_2$ according to the discussion in Section 3.3.4. $\square$

### 5.3.3 Proof of the Lemma 3.3.9

Recall in Lemma 3.3.1 and 3.3.2 we have the form of local map and global map

$$d\mathbb{L} = \frac{1}{\mu} u \otimes \mathbf{l} + B + O(\mu^\kappa), \quad d\mathbb{G} = \chi^2 \bar{u} \otimes \bar{\mathbf{l}} + \chi \bar{\bar{u}} \otimes \bar{\bar{\mathbf{l}}} + O(\mu^2 \chi),$$

where we suppress the subscript $i$ standing for the first or second collision. Moreover, in the limit $\chi \to \infty, \mu \to 0$,

$$span\{\bar{u}, \bar{\bar{u}}\} \to span\{w_i, \tilde{w}\}, \quad \mathbf{l} \to \mathbf{l}_i,\ \bar{\mathbf{l}} \to \bar{\mathbf{l}}_i,\ \bar{\bar{\mathbf{l}}} \to \bar{\bar{\mathbf{l}}}_i, \quad i = 1, 2.$$

We first have the following abstract lemma that reduces the study of the local map of the $\mu > 0$ case to $\mu = 0$ case.

**Lemma 5.3.1.** *Suppose the vector $\tilde{\Gamma} \in span\{\bar{u}, \bar{\bar{u}}\}$ satisfies $\bar{\mathbf{l}} \cdot d\mathbb{L}\tilde{\Gamma} = 0$ when $\mu > 0$. We normalize $\tilde{\Gamma}$ such that its $\ell^\infty$ norm is 1. Then in the limit $\mu \to 0$, the directional derivative of the local map along the direction $\tilde{\Gamma}$ has a limit*

$$d\mathbb{L}\tilde{\Gamma} \to \Delta_i, \text{ for } \Delta \in \mathbb{R}^n,$$

*and for the vector $\Gamma_i = \lim_\mu \tilde{\Gamma} \in span\{w_{3-i}, \tilde{w}\}$ satisfying $\bar{\mathbf{l}}_i \cdot \Delta_i = 0$ and $\mathbf{l}_i \cdot \Gamma_i = 0$.*

*Proof.* Now let us consider $\mu > 0, \chi = \infty$. After the action of the local map, we obtain a plane spanned by $u$ and $B\Gamma'$ up to an error of order $O(\mu^{3\kappa-1})$ where $\Gamma' = \mathbf{l}(\bar{u})\bar{\bar{u}} - \mathbf{l}(\bar{\bar{u}})\bar{u} \in Ker\mathbf{l}$.

We want to find a vector $\tilde{\Gamma} \in span\{\bar{u}, \bar{\bar{u}}\}$ such that the directional derivative

$$d\mathbb{L}\tilde{\Gamma} \in Ker\bar{\mathbf{l}}$$

Suppose $d\mathbb{L}\tilde{\Gamma} = bu + \tilde{b}B\Gamma' \in span\{u, B\Gamma'\}$, then $\bar{\mathbf{l}} \cdot d\mathbb{L}\tilde{\Gamma} = b\bar{\mathbf{l}} \cdot u + \tilde{b}\bar{\mathbf{l}} \cdot B\Gamma' = 0$ gives $b = -\bar{\mathbf{l}} \cdot B\Gamma'$ and $\tilde{b} = \bar{\mathbf{l}} \cdot u$ up to a multiple of a nonzero constant. Due to the normalization of $\tilde{\Gamma}$, we know $|b|$ and $|\tilde{b}|$ are bounded.

$$d\mathbb{L}\tilde{\Gamma} = \frac{1}{\mu} u \otimes \mathbf{l} \cdot \tilde{\Gamma} + B\tilde{\Gamma} + O(\mu^\kappa) = bu + \tilde{b}B\Gamma' + O(\mu^{3\kappa-1}).$$

The limit $\lim bu + \tilde{b}B\Gamma'$ exists since each term is written down explicitly and has a limit. Therefore we have the convergence $d\mathbb{L}\tilde{\Gamma} \to \Delta_i$. Moreover, $\bar{\mathbf{l}}_i \cdot \Delta_i = 0$ since $\bar{\mathbf{l}} \cdot d\mathbb{L}\tilde{\Gamma} = 0$ and $\bar{\mathbf{l}} \to \bar{\mathbf{l}}_i$. We get the following by making comparison,

$$\frac{1}{\mu} u \otimes \mathbf{l} \cdot \tilde{\Gamma} = bu + O(\mu^\kappa), \quad B\tilde{\Gamma} = \tilde{b}B\Gamma' + O(\mu^\kappa).$$

This implies

$$\frac{1}{\mu}\mathbf{l} \cdot \tilde{\Gamma} = b + O(\mu^\kappa).$$

Therefore $\mathbf{l} \cdot \tilde{\Gamma} = b\mu \to 0$ as $\mu \to 0$. Finally, we see that $\lim_{\mu} \tilde{\Gamma} = \Gamma_i$. The reason is that $\Gamma_i$ is determined by $Ker\mathbf{l}_i \cap span\{w_{3-i}, \tilde{w}\}$ and $\tilde{\Gamma}$ lies in the cone $span\{\bar{u}, \bar{\bar{u}}\} \cap \{x : \mathbf{l} \cdot x = b\mu\}$. Obviously, the former is the limit of the latter. $\square$

In the following, it is more convenient for us to change setting to polar coordinates.

#### 5.3.3.1 Equations to solve the elastic collision in polar coordinates

We need the following quantities.

$\psi$: polar angle, related to $u$ by $\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$ for ellipse.

$E$: energy. $e$: eccentricity, $e = \sqrt{1+2EM^2}$.

$G$: angular momentum, $g$: argument of periapsis. We have the formula $r = \frac{G^2}{1 - e\cos\psi}$ for conic sections in which the perigee lies on the axis $\psi = \pi$. The superscript $\pm$ means before or after collision, evaluated on the sphere $|Q_3 - Q_4| = \mu^\kappa$. The subscript $3, 4$ stand for $Q_3$ or $Q_4$.

**Lemma 5.3.2.** *We choose the positive $y$ axis as the axis $\psi = 0$. Then the equa-*

*tions* (5.2.7) *can be rewritten as follows using polar coordinates,*

$$\begin{cases} E_3^+ + E_4^+ = E_3^- + E_4^- + o(1), \\ G_3^+ + G_4^+ = G_3^- + G_4^- + o(1), \\ \dfrac{e_3^+}{G_3^+}\cos(\psi_3^+ + g_3^+) + \dfrac{e_4^+}{G_4^+}\cos(\psi_4^- - g_4^-) = \dfrac{e_3^-}{G_3^-}\cos(\psi_3^- + g_3^-) + \dfrac{e_4^-}{G_4^-}\cos(\psi_4^- - g_4^-) + o(1), \\ \vec{r}_3^+ + \vec{r}_4^+ = \vec{r}_3^- + \vec{r}_4^- + o(1), \\ |\vec{r}_3^- - \vec{r}_4^-| = \mu^\kappa, \quad |\vec{r}_3^+ - \vec{r}_4^+| = \mu^\kappa, \end{cases} \tag{5.3.23}$$

*where we have* $\begin{cases} r_3^\pm = \dfrac{(G_3^\pm)^2}{1 - e_3^\pm \sin(\psi_3^\pm + g_3^\pm)} + o(1), \\ r_4^\pm = \dfrac{(G_4^\pm)^2}{1 - e_4^\pm \sin(\psi_4^\pm - g_4^\pm)} + o(1). \end{cases}$ *as* $\mu \to 0$.

*Proof.* The equations for $r_{3,4}^\pm$ in the equations (5.3.23) are obtained from the polar coordinates representation of conic section after proper rotation, where $g_{3,4}^\pm$ in the Gerver's case can be found in the Appendix B.2.1 and B.2.2.

The first three equations here are energy conservation and momentum conservation. The energy conservation is straightforward. Now we focus on the momentum conservation. The position vector is $\vec{r} = r\hat{e}_r$. Then the velocity is $\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi$. Momentum conservation means $\dot{\vec{r}}_3^- + \dot{\vec{r}}_4^- = \dot{\vec{r}}_3^+ + \dot{\vec{r}}_4^+$.

Componentwisely, we have

$$\dot{r}_3^- + \dot{r}_4^- = \dot{r}_3^+ + \dot{r}_4^+, \quad r_3^-\dot{\psi}_3^- + r_4^-\dot{\psi}_4^- = r_3^+\dot{\psi}_3^+ + r_4^+\dot{\psi}_4^+.$$

Using the fact that $r_3^- = r_4^- = r_3^+ = r_4^+$ and $r^2\dot{\psi} = G$, we obtain from the second equation that

$$G_3^- + G_4^- = G_3^+ + G_4^+.$$

This is the angular momentum conservation. For the other conserved quantity, consider $r = \dfrac{G^2}{1 - e\cos\psi}$. So we get

$$\dot r = -\frac{G^2}{(1-e\cos\psi)^2} e \sin\psi \dot\psi = -\frac{r^2}{G^2} e \sin\psi \frac{G}{r^2} = -\frac{e}{G}\sin\psi.$$

To obtain the last equation in the equations (5.3.23), we simply replace $\sin\psi$ here by $\cos(\psi + g_{3,4}^{\pm})$. □

**Lemma 5.3.3.** *Under the same assumption as Corollary 2 and if in addition we use the equations $|\vec r_3^{\pm} - \vec r_4^{\pm}| = \mu^{\kappa}$, then we have in the limit $\mu \to 0$ that*

$$\frac{dr_3^+}{d\psi} = \frac{dr_4^+}{d\psi}, \quad \frac{dr_3^-}{d\psi} = \frac{dr_4^-}{d\psi}, \quad \frac{d\psi_3^+}{d\psi} = \frac{d\psi_4^+}{d\psi}, \quad \frac{d\psi_3^-}{d\psi} = \frac{d\psi_4^-}{d\psi}.$$

*Moreover, in the equations of $r_{3,4}^{\pm}$ of* (5.3.23), *the $o(1)$ terms are also $C^1$ small when taking the $\psi$ derivative.*

*Proof.* To prove the statement about $r_{3,4}^{\pm}$ equations in (5.3.23), we use the Hamiltonian (4.1.1). The $r_{3,4}^{\pm}$ of (5.3.23) solve the Hamiltonian system (4.1.1) in terms of polar coordinates. The estimate (5.1.1) shows the $\dfrac{-\mu}{|Q_3 - Q_4|}$ gives small perturbation to the variational equations. The two $O(1/\chi)$ terms in (4.1.1) are also small. This shows that the perturbations to Kepler motion is $C^1$ small.

Then we consider the derivatives $\dfrac{\partial r_{3,4}^{\pm}}{\partial\psi}$. We consider first the case of "$-$". We use the condition for the Poincaré section $|\vec r_3 - \vec r_4| = \mu^{\kappa}$, to get

$$(\vec r_3 - \vec r_4)\cdot \frac{d}{d\psi}(\vec r_3 - \vec r_4) = 0.$$

This implies $(\vec{r}_3-\vec{r}_4)\perp \dfrac{d}{d\psi}(\vec{r}_3-\vec{r}_4)$.

We also know the angular momentum for the relative motion is $G_{in}=(\dot{\vec{r}}_3-\dot{\vec{r}}_4)\times(\vec{r}_3-\vec{r}_4)=O(\mu)$, which implies $\dot{\vec{r}}_3-\dot{\vec{r}}_4$ is almost parallel with $\vec{r}_3-\vec{r}_4$. When taking derivative along $\Gamma$, parameterized by $\psi$, we require $\dfrac{\partial G_{in}^-}{\partial\psi}=O(\mu)$. This implies

$$\left(\frac{d}{d\psi}(\dot{\vec{r}}_3-\dot{\vec{r}}_4)\right)\times(\vec{r}_3-\vec{r}_4)+(\dot{\vec{r}}_3-\dot{\vec{r}}_4)\times\left(\frac{d}{d\psi}(\vec{r}_3-\vec{r}_4)\right)=O(\mu).$$

Then we take limit $\mu\to 0$, using $|\vec{r}_3-\vec{r}_4|=\mu^\kappa$ we get $(\dot{\vec{r}}_3-\dot{\vec{r}}_4)\times\left(\dfrac{d}{d\psi}(\vec{r}_3-\vec{r}_4)\right)=0$. Since $\dfrac{d}{d\psi}(\vec{r}_3-\vec{r}_4)$ is almost perpendicular to $(\dot{\vec{r}}_3-\dot{\vec{r}}_4)$ using the above analysis, and the perpendicular relation becomes exact in the limit $\mu\to 0$, we get $\dfrac{d}{d\psi}(\vec{r}_3-\vec{r}_4)=0$. We write $\vec{r}_i=r_i\hat{e}_{r_i}$, $i=3,4$, then $\dfrac{d}{d\psi}\vec{r}_i=\dfrac{dr_i}{d\psi}\hat{e}_{r_i}+r_i\dfrac{d\psi_i}{d\psi}\hat{e}_{\psi_i}$. In the limit $\mu\to 0$, we have $r_3\hat{e}_{r_3}=r_4\hat{e}_{r_4},\psi_3\hat{e}_{\psi_3}=\psi_4\hat{e}_{\psi_4}$, So the two components of $\dfrac{d}{d\psi}(\vec{r}_3-\vec{r}_4)=0$ implies $\dfrac{dr_3}{d\psi}=\dfrac{dr_4}{d\psi},\quad \dfrac{d\psi_3}{d\psi}=\dfrac{d\psi_4}{d\psi}$.

The lemma is now proved for variables with "−". To repeat the above argument for "+" variables, we first need to establish $\dfrac{\partial G_{in}^-}{\partial\psi}=O(\mu)$. Indeed, we use equation (5.3.8) and (5.3.18) to get $\dfrac{\partial G_{in}^+}{\partial\psi}=\dfrac{\partial G_{in}^+}{\partial(L',G_{in},g,Q_+,v_+)^-}\dfrac{\partial(L',G_{in},g,Q_+,v_+)^-}{\partial\psi}=O(\mu^{3\kappa},1,\mu^{3\kappa},\mu^{3\kappa}_{1\times2},\mu^{3\kappa}_{1\times2})\cdot O(1,\mu,1,1_{1\times2},1_{1\times2})=O(\mu)$.

It remains to show $\left(\dfrac{d}{d\psi}(\dot{\vec{r}}_3-\dot{\vec{r}}_4)\right)=O(1)$ in the " + " case. We know it is true in the "-" case. Then the "+" case follows, since the directional derivative of the local map $d\mathbb{L}\Gamma$ is bounded due to our choice of $\Gamma$. □

Using this lemma, we get the following set of equations from equation (5.3.23) by taking limit $\mu\to 0$, which are valid not only in the $C^0$ sense but also in the $C^1$ sense when taking $\psi$ derivative. (The $C^1$-ness of the first three equations are established in Corollary 2.)

$$\begin{cases} E_3^+ + E_4^+ = E_3^- + E_4^-, \\ \\ G_3^+ + G_4^+ = G_3^- + G_4^-, \\ \dfrac{e_3^+}{G_3^+}\cos(\psi_3^+ + g_3^+) + \dfrac{e_4^+}{G_4^+}\cos(\psi_4^- - g_4^-) = \dfrac{e_3^-}{G_3^-}\cos(\psi_3^- + g_3^-) + \dfrac{e_4^-}{G_4^-}\cos(\psi_4^- - g_4^-), \\ \dfrac{(G_3^+)^2}{1 - e_3^+ \sin(\psi_3^+ + g_3^+)} = \dfrac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)}, \\ \psi_3^+ = \psi_3^- (:= \psi), \\ \dfrac{(G_3^+)^2}{1 - e_3^+ \sin(\psi_3^+ + g_3^+)} = \dfrac{(G_4^+)^2}{1 - e_4^+ \sin(\psi_4^+ - g_4^+)}, \\ \dfrac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)} = \dfrac{(G_4^-)^2}{1 - e_4^- \sin(\psi_4^- - g_4^-)}, \\ \psi_3^- = \psi_4^- (= \psi), \\ \\ \psi_3^+ = \psi_4^+ (= \psi), \end{cases} \tag{5.3.24}$$

where the fourth and fifth equations are $Q_3^+ + Q_4^+ = Q_3^- + Q_4^-$, which implies $r_3^+ = r_3^-$ and $\psi_3^+ = \psi_3^-$ using Lemma 5.3.3. The sixth and seventh equations are in fact $r_3^- = r_4^-$ and $r_3^+ = r_4^+$.

We set the total energy to be zero. So we get $E_4^\pm = -E_3^\pm$. This eliminates $E_4^\pm$. Then we also eliminate $\psi_{3,4}^\pm$ by setting them to be $\psi$.

*Proof of the Lemma 3.3.9.* We take directional derivative along a direction $\Gamma \in Ker\mathbf{l} \cap span\{w, \tilde{w}\}$. Since we have in Delaunay coordinates $w = (0, 1, 0, 0, 0, 0)$ and $\tilde{w} = (0, 0, 0, 0, 1, a)$, where $a = \dfrac{-L_4^-}{(L_4^-)^2 + (G_4^-)^2}$ from equations (3.3.3). They have the same form in our polar coordinates using the formula $\tan\dfrac{\psi}{2} = \sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{u}{2}$ relating $\psi$ and $\ell$ through $u$. Our $\Gamma$ has the form of $(0, 1, 0, 0, c, ca)$. Moreover, the constant $c$ will be fixed by the $r_3^- = r_4^-$ condition, i.e. the seventh equation in (5.3.24) since that is an equation involving only incoming orbit parameters and an equation talking about exact collision.

Due to the special form of $\Gamma$, we consider $\gamma$ lying in the intersection of the hyperplances $E_4^- = -E_3^- = const$, $G_3 = const$, $g_3 = const$, where the constants are fixed by Gerver's values in the Appendix.

We write the remaining equations (the second, third, fifth and sixth) in (5.3.3) formally as $\mathbf{F}(Z^+, Z^-) = 0$, where in $Z^+ = (E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)$ and $Z^- = (E_3^-, \psi, G_3^-, g_3^-, G_4^-, g_4^-)$.

We have

$$\frac{\partial \mathbf{F}}{\partial Z^+}\frac{\partial Z^+}{\partial \psi} + \frac{\partial \mathbf{F}}{\partial Z^-}\Gamma = 0.$$

However, we have only four equations of $\mathbf{F}$ while 5 variables $(E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)$ in $Z^+$. To decide $\dfrac{\partial Z^+}{\partial \psi}$, we need one more condition $\bar{\mathbf{l}} \cdot \dfrac{\partial Z^+}{\partial \psi} = 0$, where $\bar{\mathbf{l}} = \left(\frac{G_4^+ L_4^+}{(L_4^+)^2 + (G_4^+)^2}, 0, 0, 0, \frac{-1}{(L_4^+)^2 + (G_4^+)^2}, 1\right)$ from equations (3.3.3). So we form a matrix of $5 \times 5$ by $\begin{bmatrix} \bar{\mathbf{l}} \\ \dfrac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix}$. Then we get

$$\begin{bmatrix} \bar{\mathbf{l}} \\ \dfrac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix} \frac{\partial Z^+}{\partial \psi} = -\begin{bmatrix} 0 \\ \dfrac{\partial \mathbf{F}}{\partial Z^-}\Gamma \end{bmatrix}, \quad \frac{\partial Z^+}{\partial \psi} = -\begin{bmatrix} \bar{\mathbf{l}} \\ \dfrac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \dfrac{\partial \mathbf{F}}{\partial Z^-}\Gamma \end{bmatrix}.$$

We use numerics to complete the computation. We only need the entry $\dfrac{\partial E_3^+}{\partial \psi}$ to prove the Lemma 3.3.9. It turns out this number is 1.07507 for the first collision and $-1.66364$ for the second after numerical computation. Both are nonzero. Therefore we complete the proof of Lemma 3.3.9. $\square$

## A.1 Delaunay coordinates

### A.1.1 Elliptic motion

The material of this section could be found in [Al]. Consider the two-body problem with Hamiltonian

$$H(P,Q) = \frac{|P|^2}{2m} - \frac{k}{|Q|}, \quad (P,Q) \in \mathbb{R}^4.$$

This system is integrable in the Liouville-Arnold sense when $H < 0$. So we can introduce the action-angle variables $(L, \ell, G, g)$ in which the Hamiltonian can be written as

$$H(L, \ell, G, g) = -\frac{mk^2}{2L^2}, \quad (L, \ell, G, g) \in T^*\mathbb{T}^2.$$

The Hamiltonian equations are

$$\dot{L} = \dot{G} = \dot{g} = 0, \quad \dot{\ell} = \frac{mk^2}{L^3}.$$

We introduce the following notation $E$-energy, $M$-angular momentum, $e$-eccentricity, $a$-semimajor axis, $b$-semiminor axis. Then we have the following relations which explain the physical and geometrical meaning of the Delaunay coordinates.

$$a = \frac{L^2}{mk}, \; b = \frac{LG}{mk}, \; E = -\frac{k}{2a}, \; M = G, \; e = \sqrt{1 - \left(\frac{G}{L}\right)^2}.$$

Moreover, $g$ is the argument of periapsis and $\ell$ is called the mean anomaly, and $\ell$ can be related to the polar angle $\psi$ through the equations

$$\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{u}{2}, \quad u - e\sin u = \ell.$$

We also have the Kepler's law $\dfrac{a^3}{T^2} = \dfrac{1}{(2\pi)^2}$ which relates the semimajor $a$ and the period $T$ of the ellipse.

Denoting particle's position by $(q_1, q_2)$ and its momentum $(p_1, p_2)$ we have the following formulas in case $g = 0$.

$$\begin{cases} q_1 = a(\cos u - e), \\ q_2 = a\sqrt{1-e^2}\sin u, \end{cases} \qquad \begin{cases} p_1 = -\sqrt{mk}a^{-1/2}\dfrac{\sin u}{1 - e\cos u}, \\ p_2 = \sqrt{mk}a^{-1/2}\dfrac{\sqrt{1-e^2}\cos u}{1 - e\cos u}, \end{cases}$$

where $u$ and $l$ are related by $u - e\sin u = \ell$.

Expressing $e$ and $a$ in terms of Delaunay coordinates we obtain the following

$$\begin{aligned} q_1 &= \frac{L^2}{mk}\left(\cos u - \sqrt{1 - \frac{G^2}{L^2}}\right), \quad q_2 = \frac{LG}{mk}\sin u. \\ p_1 &= -\frac{mk}{L}\frac{\sin u}{1 - \sqrt{1 - \frac{G^2}{L^2}}\cos u}, \quad p_2 = \frac{mk}{L^2}\frac{G\cos u}{1 - \sqrt{1 - \frac{G^2}{L^2}}\cos u}. \end{aligned} \tag{A.1.1}$$

Here $g$ does not enter because the argument of perihelion is chosen to be zero. In general case, we need to rotate the $(q_1, q_2)$ and $(p_1, p_2)$ using the matrix $\begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix}$.

Notice that the equation (A.1.1) describes an ellipse with one focus at the origin and the other focus on the negative $x$-axis. We want to be consistent with [G1], i.e. we want $g = \pi/2$ to correspond to the "vertical" ellipse with one focus at the origin and the other focus on the positive $y$-axis (see Appendix B.2.2). Therefore we rotate the picture clockwisely. So we use the Delaunay coordinates which are related to the Cartesian ones through the equation

$$\begin{aligned} q_1 =& \frac{1}{mk}\left(L^2\left(\cos u - \sqrt{1 - \frac{G^2}{L^2}}\right)\cos g + LG\sin u\sin g\right), \\ q_2 =& \frac{1}{mk}\left(-L^2\left(\cos u - \sqrt{1 - \frac{G^2}{L^2}}\right)\sin g + LG\sin u\cos g\right). \end{aligned} \tag{A.1.2}$$

### A.1.2 Hyperbolic motion

The above formulas can also be used to describe hyperbolic motion, where we need to replace "$\sin \to \sinh, \cos \to \cosh$" (c.f.[Al, F]). Namely, we have

$$\begin{aligned}
q_1 &= \frac{L^2}{mk}\left(\cosh u - \sqrt{1+\frac{G^2}{L^2}}\right), \quad q_2 = \frac{LG}{mk}\sinh u, \\
p_1 &= -\frac{mk}{L}\frac{\sinh u}{1-\sqrt{1+\frac{G^2}{L^2}}\cosh u}, \quad p_2 = -\frac{mk}{L^2}\frac{G\cosh u}{1-\sqrt{1+\frac{G^2}{L^2}}\cosh u}.
\end{aligned} \tag{A.1.3}$$

where $u$ and $l$ are related by

$$u - e\sinh u = \ell, \text{ where } e = \sqrt{1+\left(\frac{G}{L}\right)^2}. \tag{A.1.4}$$

This hyperbola is symmetric w.r.t. the $x$-axis, opens to the right and the particle moves clockwisely on it when $u$ increases ($\ell$ decreases). When the particle moves to the right of $x = -\frac{\chi}{2}$ line we have a hyperbola opening to the left and the particle moves anti-clockwisely. To achieve this we first reflect $(q_1, q_2)$ around the $y$-axis, then rotate it by an angle $g$. If we restrict $|g| < \pi/2$, then the particle moves anti-clockwisely on the hyperbola as $u$ increases ($\ell$ decreases) due to the reflection. Thus we have

$$\begin{aligned}
q_1 =& -\frac{1}{mk}\left(\cos g L^2(\cosh u - e) + \sin g LG\sinh u\right), \\
q_2 =& \frac{1}{mk}\left(-\sin g L^2(\cosh u - e) + \cos g LG\sinh u\right).
\end{aligned} \tag{A.1.5}$$

If the incoming asymptote is horizontal, then the particle comes from the left, and as $u$ tends to $-\infty$, the $y$-coordinate is bounded and $x$-coordinate is negative. In this case we have $\tan g = -\frac{G}{L}$, $g \in (-\pi/2, 0)$.

If the outgoing asymptote is horizontal, then the particle escapes to the left, and as $u$ tends to $+\infty$, the $y$-coordinate is bounded and $x$-coordinate is negative. In this case we have $\tan g = +\frac{G}{L}, g \in (0, \pi/2)$.

When the particle $Q_4$ is moving to the left of the section $\{x = -\chi/2\}$, we treat the motion as hyperbolic motion focused at $Q_1$. We move the origin to $Q_1$. The hyperbola opens to the right. The orbit has the following parametrization

$$\begin{aligned} q_1 =& \frac{1}{mk}\left(\cos gL^2(\cosh u - e) - \sin gLG \sinh u\right), \\ q_2 =& \frac{1}{mk}(\sin gL^2(\cosh u - e) + \cos gLG \sinh u). \end{aligned} \tag{A.1.6}$$

### A.1.3 Large $\ell$ asymptotics: auxiliary results

In the remaining part of Appendix A.1 we study the first and second order derivatives of $Q_4$ w.r.t. the Delaunay variables $(L, \ell, G, g)_4$. These computations are used in our proof. The next lemma allows us to simplify the computations. Since the hyperbolic motion approaches a linear motion, this lemma shows that, we can replace $u$ by $\ln(\mp\ell/e)$ when taking first and second order derivatives.

**Lemma A.1.1.** *Let $u$ be the function of $\ell, G$ and $L$ given by* (A.1.4)*. Then we can approximate $u$ by* $\ln(\mp\ell/e)$ *in the following sense.*

$$u \mp \ln\frac{\mp\ell}{e} = O(\ln|\ell|/\ell), \quad \frac{\partial u}{\partial \ell} = \pm 1/\ell + O(1/\ell^2),$$
$$\left(\frac{\partial}{\partial L}, \frac{\partial}{\partial G}\right)(u \pm \ln e) = O(1/|\ell|), \quad \left(\frac{\partial}{\partial L}, \frac{\partial}{\partial G}\right)^2 (u \pm \ln e) = O(1/|\ell|),$$

*Here the first sign is taken if $u > 0$ and the second sign is taken then $u < 0$. The estimates above are uniform as long as $|G| \le K$, $1/K \le L \le K$, $\ell > \ell_0$ and the implied constants in $O(\cdot)$ depend on $K$ and $\ell_0$.*

*Proof.* We see from formula (A.1.4) that $\sinh u \simeq \cosh u = -\frac{\ell}{e} + O(\ln|\ell|)$ when $u > 0$ and $\sinh u \simeq -\cosh u \simeq -\frac{\ell}{e} + O(\ln|\ell|)$ when $u < 0$ and $|u|$ large enough.

This proves $C^0$ estimate.

Now we consider the first order derivatives. We assume that $u > 0$ to fix the notation. Differentiating (A.1.4) with respect to $\ell$ we get

$$\frac{\partial u}{\partial \ell} - e\cosh u \frac{\partial u}{\partial \ell} = 1, \quad \frac{\partial u}{\partial \ell} = 1/\ell + O(1/\ell^2).$$

Next, we differentiate (A.1.4) with respect to $L$ to obtain

$$\frac{\partial u}{\partial L} - \frac{\partial e}{\partial L}\sinh u - e\cosh u \frac{\partial u}{\partial L} = 0.$$

Therefore,

$$\frac{\partial u}{\partial L} = \frac{\sinh u}{1 - e\cosh u}\frac{\partial e}{\partial L} = -\frac{1}{e}\frac{\partial e}{\partial L} + O(e^{-|u|}) = -\frac{\partial}{\partial L}\ln(e) + O(1/|\ell|).$$

The same argument holds for $\dfrac{\partial}{\partial G}$. This proves $C^1$ part of the Lemma.

Now we consider second order derivatives. We take $\dfrac{\partial^2}{\partial L^2}$ as example. Combining

$$\frac{\partial^2 u}{\partial L^2} - \frac{\partial^2 e}{\partial L^2}\sinh u - 2\cosh u \frac{\partial e}{\partial L}\frac{\partial u}{\partial L} - e\cosh u \frac{\partial^2 u}{\partial L^2} - e\sinh u \left(\frac{\partial u}{\partial L}\right)^2 = 0.$$

with $C^1$ estimate proven above we get

$$\frac{\partial^2 u}{\partial L^2} = -\frac{1}{e}\frac{\partial^2 e}{\partial L^2} - \frac{2\partial e}{e\partial L}\frac{\partial u}{\partial L} + \left(\frac{\partial u}{\partial L}\right)^2 + O\left(\frac{1}{\ell}\right)$$
$$= -\frac{1}{e}\frac{\partial^2 e}{\partial L^2} + \left(\frac{1}{e}\frac{\partial e}{\partial L}\right)^2 + O\left(\frac{1}{\ell}\right) = \frac{\partial^2}{\partial L^2}\ln e + O\left(\frac{1}{\ell}\right).$$

This concludes the $C^2$ part of the lemma. □

In the estimate of the derivatives presented in the next two subsections we shall often use the following facts. Let $f = \ln e$. Then

$$f_G = \frac{G}{L^2 + G^2}, \quad f_L = -\frac{G^2}{L(L^2 + G^2)}, \tag{A.1.7}$$

$$(f)_{GG} = \frac{L^2 - G^2}{(L^2 + G^2)^2}, \quad f_{LG} = -\frac{2GL}{(L^2 + G^2)^2}. \tag{A.1.8}$$

### A.1.4 First order derivatives

In the following computations, we assume for simplicity that $m = k = 1$. To get the general case we only need to divide positions by $mk$.

**Lemma A.1.2.** *Under the same conditions as in Lemma A.1.1 we have the following result for the first order derivatives*

$$(a)\qquad \left|\frac{\partial Q_4}{\partial \ell_4}\right| = O(1), \quad \left|\frac{\partial Q_4}{\partial (L_4, G_4, g_4)}\right| = O(\ell), \quad \frac{\partial Q_4}{\partial g_4}\cdot Q_4 = 0, .$$

*In addition*

$$\frac{\partial Q_4}{\partial G_4}\cdot Q_4 = O_{C^2(L,G,g)}(\ell).$$

*(b) If in addition we have* $\left|g \mp \arctan\dfrac{G}{L}\right| \leq C/\ell$ *where* $-$ *sign is taken for* $u > 0$ *and* $+$ *sign is taken for* $u < 0$ *then we have the following bounds for* (A.1.5)

$$\frac{\partial Q_4}{\partial G} = \sinh u\left(0, \frac{L^2}{\sqrt{L^2+G^2}}\right) + O(1), \quad \frac{\partial Q_4}{\partial L} = -\sinh u\left(2\sqrt{L^2+G^2}, \frac{GL}{\sqrt{L^2+G^2}}\right) + O(1).$$

*(c) If in addition to the conditions of Lemma A.1.1 we have* $G, g = O(1/\chi)$ *and* $\ell = O(\chi)$*, then we have the following bounds for* (A.1.6)

$$\frac{\partial Q_4}{\partial G} = \sinh u(0,1) + O(1), \quad \frac{\partial Q_4}{\partial L} = \sinh u(2,0) + O(1).$$

**Remark 8.** The assumptions of the lemma and the next lemma hold due to Lemma 4.1.8.

*Proof.* We consider only the case $u > 0$. We have

$$Q_4 = O(1) - \sinh u(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG), \text{ as } \ell \to \infty. \qquad \text{(A.1.9)}$$

Now the first three estimates of part (a) follow easily. Next

$$\frac{\partial Q_4}{\partial G} = -(\cosh u)u'_G(\cos gL^2+\sin gLG, \sin gL^2-\cos gLG)-\sinh u(\sin gL, -\cos gL)+O(1).$$

Using Lemma A.1.1 we obtain

$$Q_4 \cdot \frac{\partial Q_4}{\partial G} = \frac{1}{2}(\sinh 2u)u'_G|(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG)|^2$$

$$+ (\sinh u)^2(\sin gL, -\cos gL) \cdot (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) + O(\ell)$$

$$= \frac{1}{2}(\sinh 2u)(-\ln e)'_G(L^4 + L^2G^2) + L^2G(\sinh u)^2 + O(\ell) = O(\ell)$$

where the last equality relies on (A.1.7).

We prove (b) in the + case, the - case being similar. Assume first that $g$ is exactly equal to $\arctan \dfrac{G}{L}$. Using (A.1.9) and (A.1.7) we obtain

$$\frac{\partial Q_4}{\partial G} = (\cosh u)f_G(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG)$$

$$- \sinh u(\sin gL, -\cos gL) + O(1)$$

$$= \sinh u\left(\frac{G}{L^2+G^2}\left(\frac{L^3+LG^2}{\sqrt{L^2+G^2}}, 0\right) - \left(\frac{GL}{\sqrt{L^2+G^2}}, -\frac{L^2}{\sqrt{L^2+G^2}}\right)\right) + O(1)$$

$$= \sinh u\left(0, \frac{L^2}{\sqrt{L^2+G^2}}\right) + O(1).$$

$$\begin{aligned}\frac{\partial Q_4}{\partial L} &= (\cosh u)f_L(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) \\ &- \sinh u(2\cos gL + \sin gG, 2\sin gL - \cos gG) + O(1)\end{aligned} \tag{A.1.10}$$

$$= -\sinh u\left(\frac{G^2/L}{L^2+G^2}\left(\frac{L^3+LG^2}{\sqrt{L^2+G^2}}, 0\right) + \left(\frac{2L^2+G^2}{\sqrt{L^2+G^2}}, \frac{GL}{\sqrt{L^2+G^2}}\right)\right) + O(1)$$

$$= -\sinh u\left(2\sqrt{L^2+G^2}, \frac{GL}{\sqrt{L^2+G^2}}\right) + O(1).$$

This proves (b) under the assumption $g = \arctan \dfrac{G}{L}$. If $\left|g - \arctan \dfrac{G}{L}\right| < \frac{C}{|\ell|}$ then we get an additional $O(1)$ error in the above computation which does not change the final result.

Part (c) follows from part (b) since both $g$ and $\arctan\dfrac{G}{L}$ are $O(1/\ell)$. $\square$

### A.1.5 Second order derivatives

The following estimates of the second order derivatives are used in integrating the variational equation.

**Lemma A.1.3.** *We have the following information for the second order derivatives of $Q_4$ w.r.t. the Delaunay variables.*

*(a) Under the conditions of Lemma A.1.2(a) we have*

$$\frac{\partial^2 Q_4}{\partial g_4^2} = -Q_4, \quad \frac{\partial^2 Q_4}{\partial g_4 \partial G_4} \perp \frac{\partial Q_4}{\partial G_4}, \quad \left(\frac{\partial}{\partial G_4}, \frac{\partial}{\partial g_4}\right)\left(\frac{\partial |Q_4|^2}{\partial g_4}\right) = (0,0), \quad \frac{\partial^2 Q_4}{\partial G_4^2} = O(\ell).$$

*In addition* $\dfrac{\partial^2 Q_4}{\partial L_4^2} = O(\ell)$.

*(b) Under the conditions of Lemma A.1.2(b) we have we have*

$$\frac{\partial^2 Q_4}{\partial G_4^2} = \frac{L^2}{(L^2+G^2)^{3/2}}(L\cosh u, -2G\sinh u) + O(1),$$
$$\frac{\partial^2 Q_4}{\partial g_4 \partial G_4} = \left(-\frac{L^2\sinh u}{\sqrt{L^2+G^2}}, 0\right) + O(1),$$
$$\frac{\partial^2 Q_4}{\partial g_4 \partial L_4} = \left(\frac{GL\sinh u}{\sqrt{L^2+G^2}}, -2\sqrt{L^2+G^2}\cosh u\right) + O(1),$$
$$\frac{\partial^2 Q_4}{\partial G \partial L} = \frac{L}{(L^2+G^2)^{3/2}}\big(-LG\cosh u, (L^2+3G^2)\sinh u\big) + O(1).$$

*(c) Under the conditions of Lemma A.1.2(c) we have*

$$\frac{\partial^2 Q_4}{\partial G_4^2} = -\cosh u(1,0) + O(1), \quad \frac{\partial^2 Q_4}{\partial g \partial G} = -L\sinh u(1,0) + O(1),$$
$$\frac{\partial^2 Q_4}{\partial g \partial L} = L\sinh u(0,2) + O(1), \quad \frac{\partial^2 Q_4}{\partial G \partial L} = \cosh u(0,1) + O(1).$$

*Proof.* The estimate $\dfrac{\partial^2 Q_4}{\partial G_4^2} = O(\ell)$ follows immediately from Lemma A.1.2. The estimate $\dfrac{\partial^2 Q_4}{\partial L_4^2} = O(\ell)$ follows immediately from (A.1.5) (or (A.1.6)).

The estimates of the derivatives involving $g_4$ are relatively easy since the dependence of $Q_4$ on $g_4$ is through a rotation. We consider $\frac{\partial^2 Q_4}{\partial L_4 \partial g_4}$, for example, the other derivatives are similar. Differentiating (A.1.10) with respect to $g$ and using (A.1.7) we get

$$\frac{\partial^2 Q_4}{\partial L_4 \partial g_4} = \cosh u f_L(-L^2 \sin g + LG \cos g, L^2 \cos g + LG \sin g)$$

$$- \sinh u(-2L \sin g + G \cos g, 2L \cos g + G \sin G) + O(1)$$

$$= - \sinh u \frac{G^2}{L(L^2+G^2)} \left( \frac{-L^2 G + L^2 G}{\sqrt{L^2+G^2}}, \frac{L^3 + LG^2}{\sqrt{L^2+G^2}} \right) - \sinh u \left( \frac{-2LG + LG}{\sqrt{L^2+G^2}}, \frac{2L^2 + G^2}{\sqrt{L^2+G^2}} \right) + O(1)$$

$$= - \sinh u \left( 0, \frac{G^2}{\sqrt{L^2+G^2}} \right) - \sinh u \left( - \frac{LG}{\sqrt{L^2+G^2}}, \frac{2L^2+G^2}{\sqrt{L^2+G^2}} \right) + O(1)$$

$$= \sinh u \left( \frac{LG}{\sqrt{L^2+G^2}}, -2\sqrt{L^2+G^2} \right) + O(1).$$

Next, we compute $\frac{\partial^2 Q_4}{\partial G_4 \partial L_4}$ and $\frac{\partial^2 Q_4}{\partial G_4^2}$. We consider only the case $u > 0$ and take the $+$ sign. The other cases are similar.

As in the proof of Lemma A.1.2 it suffices to consider the case $g = \arctan \frac{G}{L}$. Differentiating the expression for $\frac{\partial Q_4}{\partial G_4}$ and using Lemma A.1.1, (A.1.7) and (A.1.8) we obtain

$$\frac{\partial^2 Q_4}{\partial G_4^2} = -L(\sinh u((\ln e)_G)^2 - \cosh u(\ln e)_{GG})(\cos gL + \sin gG, \sin gL - \cos gG)$$

$$+ 2L \cosh u(\ln e)_G(\sin g, -\cos g) + O(1)$$

$$= L \sinh u \left( \frac{L^2 - 2G^2}{(L^2+G^2)^2} \right) \left( \frac{L^2}{(L^2+G^2)^{1/2}} + \frac{G^2}{(L^2+G^2)^{1/2}}, 0 \right),$$

$$+ 2L \sinh u \frac{G}{L^2+G^2} \left( \frac{G}{(L^2+G^2)^{1/2}}, - \frac{L}{(L^2+G^2)^{1/2}} \right) + O(1)$$

$$= \frac{L^2}{(L^2+G^2)^{3/2}} \sinh u(L, -2G) + O(1)$$

proving the estimate for $\frac{\partial^2 Q_4}{\partial G_4^2}$. Next,

$$\frac{\partial^2 Q_4}{\partial G_4 \partial L_4} = -(\sinh u)_{LG}(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) - (\sinh u)_L(\sin gL, -\cos gL)$$
$$- (\sinh u)_G(2\cos gL + \sin gG, 2\sin gL - \cos gG) - \sinh u(\sin g, -\cos g) + O(1)$$
$$= -(\sinh u(\ln e)_L(\ln e)_G - \cosh u(\ln e)_{GL})(L(L^2+G^2)^{1/2}, 0)$$
$$+ \cosh u(\ln e)_L\left(\frac{GL}{(L^2+G^2)^{1/2}}, -\frac{L^2}{(L^2+G^2)^{1/2}}\right) + \cosh u(\ln e)_G\left(\frac{2L^2+G^2}{(L^2+G^2)^{1/2}}, \frac{GL}{(L^2+G^2)^{1/2}}\right)$$
$$- \sinh u\left(\frac{G}{(L^2+G^2)^{1/2}}, -\frac{L}{(L^2+G^2)^{1/2}}\right) + O(1)$$
$$= \frac{L}{(L^2+G^2)^{3/2}}\sinh u\left(-LG, L^2+3G^2\right) + O(1).$$

Part (c) follows from part (b) as in Lemma A.1.2. □

## B.2 Gerver's mechanism

### B.2.1 Gerver's result in [G1]

We summarize the result of [G1] in the following table. Recall that the Gerver scenario deals with the limiting case $\chi \to \infty, \mu \to 0$. Accordingly $Q_1$ disappears at infinity and there is no interaction between $Q_3$ and $Q_4$. Hence both particles perform Kepler motions. The shape of each Kepler orbit is characterized by energy, angular momentum and the argument of periapsis. In Gerver's scenario, the incoming and outgoing asymptotes of the hyperbola are always horizontal and the semimajor of the ellipse is always vertical. So we only need to describe on the energy and angular momentum.

| | 1st collision | @$(-\varepsilon_0\varepsilon_1, \varepsilon_0+\varepsilon_1)$ | 2nd collision | @$(\varepsilon_0^2, 0)$ |
|---|---|---|---|---|
| | $Q_3$ | $Q_4$ | $Q_3$ | $Q_4$ |
| energy | $-1/2$ | $1/2$ | $-1/2 \to -\frac{\varepsilon_1^2}{2\varepsilon_0^2}$ | $1/2 \to \frac{\varepsilon_1^2}{2\varepsilon_0^2}$ |
| angular momentum | $\varepsilon_1 \to -\varepsilon_0$ | $p_1 \to -p_2$ | $-\varepsilon_0$ | $\sqrt{2}\varepsilon_0$ |
| eccentricity | $\varepsilon_0 \to \varepsilon_1$ | | $\varepsilon_1 \to \varepsilon_0$ | |
| semimajor | $1$ | $-1$ | $1 \to (\frac{\varepsilon_0}{\varepsilon_1})^2$ | $1 \to -\frac{\varepsilon_1^2}{\varepsilon_0^2}$ |
| semiminor | $\varepsilon_1 \to \varepsilon_0$ | $p_1 \to p_2$ | $\varepsilon_0 \to \frac{\varepsilon_0^2}{\varepsilon_1}$ | $\sqrt{2}\varepsilon_0 \to \sqrt{2}\varepsilon_1$ |

Here

$$p_{1,2} = \frac{-Y \pm \sqrt{Y^2 + 4(X+R)}}{2}, \quad R = \sqrt{X^2+Y^2}.$$

and $(X, Y)$ stands for the point where collision occurs (the parenthesis after @ in the table). We will call the two points the Gerver's collision points.

In the above table $\varepsilon_0$ is a free parameter and $\varepsilon_1 = \sqrt{1-\varepsilon_0^2}$.

At the collision points, the velocities of the particles are the following.

For the first collision,

$$v_3^- = \left(\frac{-\varepsilon_1^2}{\varepsilon_0\varepsilon_1+1}, \frac{-\varepsilon_0}{\varepsilon_0\varepsilon_1+1}\right), \quad v_4^- = \left(1-\frac{Y}{Rp_1}, \frac{1}{Rp_1}\right).$$

$$v_3^+ = \left(\frac{\varepsilon_0^2}{\varepsilon_0\varepsilon_1+1}, \frac{\varepsilon_1}{\varepsilon_0\varepsilon_1+1}\right), \quad v_4^+ = \left(-1+\frac{Y}{Rp_2}, -\frac{1}{Rp_2}\right).$$

For the second collision,

$$v_3^- = \left(\frac{-\varepsilon_1}{\varepsilon_0}, \frac{-1}{\varepsilon_0}\right), \; v_4^- = \left(1, \frac{\sqrt{2}}{\varepsilon_0}\right), \quad v_3^+ = \left(1, \frac{-1}{\varepsilon_0}\right), \; v_4^+ = \left(\frac{-\varepsilon_1}{\varepsilon_0}, \frac{\sqrt{2}}{\varepsilon_0}\right).$$

### B.2.2 Numerical information for a particularly chosen $\varepsilon_0 = 1/2$

For the first collision $e_3 : \frac{1}{2} \to \frac{\sqrt{3}}{2}$.

We want to figure out the Delaunay coordinates $(L, u, G, g)$ for both $Q_3$ and $Q_4$. (Here we replace $\ell$ by $u$ for convenience.) The first collision point is

$$(X, Y) = (-\varepsilon_0\varepsilon_1, \varepsilon_0+\varepsilon_1) = \left(-\frac{\sqrt{3}}{4}, \frac{1+\sqrt{3}}{2}\right).$$

Before collision

$$(L, u, G, g)_3^- = \left(1, -\frac{5\pi}{6}, \frac{\sqrt{3}}{2}, \pi/2\right), \quad (L, u, G, g)_4^- = (1, 1.40034, p_1, -\arctan p_1),$$

where

$$p_1^- = \frac{-Y+\sqrt{Y^2+4(X+R)}}{2} = \frac{-(\varepsilon_0+\varepsilon_1)+\sqrt{5+2\varepsilon_0\varepsilon_1}}{2} = 0.52798125.$$

After collision

$$(L,u,G,g)_3^+ = \left(1, \frac{2\pi}{3}, -\frac{1}{2}, \pi/2\right), \quad (L,u,G,g)_4^+ = (1, 0.515747, -p_2, -\arctan p_2),$$

where

$$p_2^+ = \frac{-Y - \sqrt{Y^2 + 4(X+R)}}{2} = \frac{-(\varepsilon_0 + \varepsilon_1) - \sqrt{5 + 2\varepsilon_0\varepsilon_1}}{2} = -1.894006654.$$

For the second collision $e_3 : \dfrac{\sqrt{3}}{2} \to \dfrac{1}{2}$.

The collision point is $(X,Y) = (\varepsilon_0^2, 0) = \left(\dfrac{1}{4}, 0\right)$.

Before collision

$$(L,u,G,g)_3^- = \left(1, -\frac{\pi}{6}, -\frac{1}{2}, \pi/2\right), \quad (L,u,G,g)_4^- = \left(1, 0.20273, \sqrt{2}/2, -\arctan\frac{\sqrt{2}}{2}\right).$$

After collision

$$(L,u,G,g)_3^+ = \left(\frac{1}{\sqrt{3}}, \frac{\pi}{3}, -\frac{1}{2}, -\pi/2\right), \quad (L,u,G,g)_4^+ = \left(\frac{1}{\sqrt{3}}, -0.45815, \sqrt{2}/2, \arctan\frac{\sqrt{6}}{2}\right).$$

### B.2.3 Control the shape of the ellipse

*proof of Lemma 2.2.1.* Using Lemma 2.2.2, we only need to control the shape of the ellipse in the case studied by Gerver, i.e. $\mu = 1/\chi = 0$. We use the Lemma 5.3.3.1 again. The idea of the computation is similar to the proof of Lemma 3.3.9. The only difference is, we replace the two conditions used in the proof of Lemma 3.3.9 by the following: the incoming and outgoing asymptotes of the hyperbola are assumed to be horizontal, i.e. we substitute $g_4^- = -\arctan\dfrac{G_4^-}{L_4^-}$ and $g_4^+ = \arctan\dfrac{G_4^+}{L_4^+}$. We can still compute the derivatives of $E_3^+, G_3^+, g_3^+$ w.r.t. $\psi$ for the second collision. Then we use the formula $e_3 = \sqrt{1 - 2G_3^2 E_3}$ to obtain $de_3 = -\dfrac{2G_3E_3dG_3 + G_3^2dE_3}{\sqrt{1-2G_3^2E_3}}$.

So we first obtain the two entries $\dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_2} = -0.158494$ and $\dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_2} = 0.369599$. The meanings of the two entries are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the *second* collision.

We need more work to figure out the two entries $\dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_1}$ and $\dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_1}$, which are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the *first* collision. We use the relation

$$\frac{\partial \bar{\bar{e}}_3}{\partial \psi_1} = \frac{\partial \bar{\bar{e}}_3}{\partial \bar{E}_3^+}\frac{\partial \bar{E}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{e}}_3}{\partial \bar{G}_3^+}\frac{\partial \bar{G}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{e}}_3}{\partial \bar{g}_3^+}\frac{\partial \bar{g}_3^+}{\partial \psi_1}.$$

The reason is, if we vary $\psi_1$ in the first collision, then this will vary the shape of the ellipse after the collision, i.e. $\bar{E}^+, \bar{G}_3^+, \bar{g}_3^+$. We notice that the quantities $E_3, G_3, g_3$ after the first collision is the same as those before the second collision. we replace some of the $\bar{E}_3^+, \bar{G}_3^+, \bar{g}_3^+$ by $\bar{\bar{E}}_3^-, \bar{\bar{G}}_3^-, \bar{\bar{g}}_3^-$ to obtain the following form

$$\frac{\partial \bar{\bar{e}}_3}{\partial \psi_1} = \frac{\partial \bar{\bar{e}}_3}{\partial \bar{\bar{E}}_3^-}\frac{\partial \bar{E}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{e}}_3}{\partial \bar{\bar{G}}_3^-}\frac{\partial \bar{G}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{e}}_3}{\partial \bar{\bar{g}}_3^-}\frac{\partial \bar{g}_3^+}{\partial \psi_1}.$$

Similarly we have

$$\frac{\partial \bar{\bar{g}}_3}{\partial \psi_1} = \frac{\partial \bar{\bar{g}}_3}{\partial \bar{\bar{E}}_3^-}\frac{\partial \bar{E}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{g}}_3}{\partial \bar{\bar{G}}_3^-}\frac{\partial \bar{G}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{g}}_3}{\partial \bar{\bar{g}}_3^-}\frac{\partial \bar{g}_3^+}{\partial \psi_1}.$$

As the proof of Lemma 3.3.9, we immediately obtain $(\dfrac{\partial \bar{E}_3^+}{\partial \psi_1}, \dfrac{\partial \bar{G}_3^+}{\partial \psi_1}, \dfrac{\partial \bar{g}_3^+}{\partial \psi_1})$. To obtain the remaining $\left(\dfrac{\partial}{\partial \bar{\bar{E}}_3^-}, \dfrac{\partial}{\partial \bar{\bar{G}}_3^-}, \dfrac{\partial}{\partial \bar{\bar{g}}_3^-}\right)(\bar{\bar{e}}_3, \bar{\bar{g}}_3)$, we need more work when computing the second collision. Now we consider the *second collision* only.

This time, we consider in equation (5.3.23) the variables $E_3^-, G_3^-, g_3^-$ as variables as contrast to treating them as constants as we did in the proof of Lemma 3.3.9. We still denote the implicit function as $\mathbf{F} = 0$. We obtain the $7 \times 7$ matrix

$\dfrac{\partial \mathbf{F}}{\partial(E_3^+, G_3^+, g_3^+, E_4^+, G_4^+, G_4^-, r)}$. Therefore we can compute the derivatives $\dfrac{\partial(E_3^+, G_3^+, g_3^+, E_4^+, G_4^+, G_4^-, r)}{\partial(\psi, E_3^-, G_3^-, g_3^-)}$ using the implicit function theorem in the following way

$$\frac{\partial \mathbf{F}}{\partial(E_3^+, G_3^+, g_3^+, E_4^+, G_4^+, G_4^-, r)} \cdot \frac{\partial(E_3^+, G_3^+, g_3^+, E_4^+, G_4^+, G_4^-, r)}{\partial(\psi, E_3^-, G_3^-, g_3^-)} = -\frac{\partial \mathbf{F}}{\partial(\psi, E_3^-, G_3^-, g_3^-)}.$$

This is enough for us to work out $\dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_1}$ and $\dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_1}$. It turns out that the resulting matrix is

$$\begin{bmatrix} \dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_1} & \dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_1} \\ \dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_2} & \dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_2} \end{bmatrix} = \begin{bmatrix} 0.620725 & 2.9253 \\ -0.158494 & 0 \end{bmatrix},$$

which is obviously nondegenerate. $\square$

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