ABSTRACT

Title of dissertation: VACUUM ENTANGLEMENT AND BLACK HOLE ENTROPY OF GAUGE FIELDS

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Black holes in general relativity carry an entropy whose value is given by the Bekenstein-Hawking formula, but whose statistical origin remains obscure. Such horizons also possess an entanglement entropy, which has a clear statistical meaning but no a priori relation to the dynamics of gravity. For free minimally-coupled scalar and spinor fields, these two quantities are intimately related: the entanglement entropy is the one-loop correction to the black hole entropy due to renormalization of Newton's constant. For gauge fields, the entanglement entropy and the one-loop correction to the black hole entropy differ. This dissertation addresses two issues concerning the entanglement entropy of gauge fields, and its relation black hole entropy.

First, for abelian gauge fields Kabat identified a negative divergent contribution to the black hole entropy that is not part of the entanglement entropy, known as a "contact term". We show that the contact term can be attributed to an ambiguous expression for the gauge field's contribution to the Wald entropy. Moreover, in

two-dimensional de Sitter space, the contact term arises from an incorrect treatment of zero modes and is therefore unphysical. In a manifestly gauge-invariant reduced phase space quantization of two-dimensional gauge theory, the gauge field contribution to the entropy is positive, finite, and equal to the entanglement entropy. This suggests that the contact term in more than two dimensions may also be unphysical.

Second, we consider lattice gauge theory and point out that the Hilbert space corresponding to a region of space includes edge states that transform nontrivially under gauge transformations. By decomposing these edge states in irreducible representations of the gauge group, the entanglement entropy of an arbitrary state is shown to be a sum of a bulk entropy and a boundary entropy associated to the edge states. This entropy formula agrees with the two-dimensional results from the reduced phase space quantization. These results are applied to several examples, including the ground state in the strong coupling expansion of Kogut and Susskind, and the entropy of the edge states is found to be the dominant contribution to the entanglement entropy.

VACUUM ENTANGLEMENT AND BLACK HOLE ENTROPY OF GAUGE FIELDS

by

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Dedication

To my children Malcolm and Penelope, may you be as free to pursue your dreams as I have been to pursue mine.

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Chapter 1

Introduction

1.1 Black hole entropy and entanglement entropy

Since the work of Bekenstein and Hawking, it is known that black holes behave in many respects as finite-temperature systems; they radiate with a thermal spectrum [1] and obey laws analogous to the classical laws of thermodynamics [2, 3]. In particular, a black hole whose event horizon has surface area A has an entropy given by the Bekenstein-Hawking formula:

$$S_{\mathrm{BH}} = \frac{Ac^3}{4\hbar G}. \tag{1.1}$$

In what follows we use units where $c = \hbar = 1$, but it is worth taking a moment to appreciate the sight of G and $\hbar$ coexisting peacefully in the same equation. While black hole thermodynamics is well-understood classically, it is not universally agreed upon what the entropy (1.1) counts. Thus we are in the situation of understanding the macroscopic thermodynamics of black holes without knowing the microscopic statistical mechanics [4]. Since Eq. (1.1) involves the Planck length, it is natural to assume that a microscopic understanding of S_{BH} requires quantum gravity, and indeed the ability to reproduce the Bekenstein-Hawking formula as a statistical entropy is a benchmark for potential theories of quantum gravity. However we will argue that something can be understood about the statistical origin of S_{BH} without quantizing gravity.

There are a number of different proposals for a microscopic statistical origin of $S_{\rm BH}$. In loop quantum gravity, the popular view has been to ascribe the entropy to the number of possible intrinsic geometries of a black hole horizon with fixed area A [5, 6], though there are also dissenting views [7, 8]. In string theory, the black hole entropy is widely believed to count the total number of states of the black hole. This view is supported by a state counting for certain extremal black holes [9]. For a discussion of several perspectives on the statistical origin of black hole entropy, see Ref. [10].

Another entropy associated with the black hole horizon is the entanglement entropy [11, 12, 13]. Consider a spacetime with a Cauchy surface Σ and a region of space $\Omega \subset \Sigma$. For example, Ω might be the part of Σ that lies behind a black hole event horizon. For any such region Ω the Hilbert space of a quantum field theory splits as a tensor product[1]

$$\mathcal{H} = \mathcal{H}_\Omega \otimes \mathcal{H}_{\Omega^c}, \tag{1.2}$$

where $\Omega^c = \Sigma \setminus \Omega$ is the complement of Ω in Σ. For any state $|\psi\rangle$, there is a reduced density matrix given by the partial trace

$$\rho_\Omega = \mathrm{tr}_{\Omega^c} |\psi\rangle\langle\psi| \,, \tag{1.3}$$

that is necessary and sufficient to describe the outcomes of any experiments that

[1]We will see in chapter 4 that there are subtleties in defining this tensor product decomposition for gauge theories. Nevertheless it is still possible to define a reduced density matrix ρ_Ω and an associated entanglement entropy. These subtleties and more are also present in a theory in which gravity is dynamical, as described in section 5.1.

take place in the causal domain of Ω. The entanglement entropy of the state $|\psi\rangle$ is defined as the von Neumann entropy of the reduced density matrix

$$S_{\text{ent}}(\Omega) = S(\rho_\Omega) = -\,\text{tr}(\rho_\Omega \ln \rho_\Omega). \tag{1.4}$$

It is worth emphasizing the difference between the entanglement entropy and the Bekenstein-Hawking entropy. The Bekenstein-Hawking entropy (despite the appearance of $\hbar$ in its definition) is essentially a macroscopic classical quantity associated with a black hole horizon. S_{BH} is not defined as a microscopic entropy, rather it is defined by its relation to the dynamics of general relativity. By contrast, the entanglement entropy is a purely quantum quantity - in a classical system the Shannon entropy of a marginal distribution is always less than the entropy of the original distribution; in other words the classical entropy of a subsystem is always less than the entropy of the full system, so there is no analog of S_{ent} in a classical theory. Moreover, it has a clear statistical interpretation as a measure of correlations between the black hole interior and the exterior and gives a natural explanation for why a black hole should have an entropy. Since the region beyond the black hole horizon is inaccessible to outside observers, the horizon itself provides a natural coarse graining in which the degrees of freedom behind the horizon are traced out. The entanglement entropy of a black hole is therefore associated with an outside observer's inability to measure correlations of fields outside the black hole with fields behind the horizon. However S_{ent} is defined without any explicit reference to gravity, so does not have any apparent relation to the dynamics of general relativity.

This raises the question: what is the relation between the entanglement and

Bekenstein-Hawking entropies? Are they equal? Should S_{ent} be considered as only part of S_{BH}, and if so what makes up the rest of the entropy? Or are S_{ent} and S_{BH} simply unrelated? At stake is a statistical explanation of S_{BH} and an understanding of why black holes have entropy. One of the themes of this work is the relation between the Bekenstein-Hawking entropy and the entanglement entropy. We will now give some general arguments for why one might expect these two quantities to be related.

One piece of evidence for the relation between S_{BH} and S_{ent} is that the entanglement entropy of the vacuum state generically scales with the area A of the entangling surface $\partial\Omega$. In the case of a scalar field in 4 dimensions, Refs. [12, 13] found an entanglement entropy proportional to A and that diverges with the cutoff length ℓ as

$$S_{\text{ent}} \sim \frac{A}{\ell^2} + \cdots \tag{1.5}$$

where "..." represents subleading terms. The precise coefficient of the area is non-universal and depends on the cutoff procedure. If the cutoff were of order the Planck scale, $\ell \approx \ell_{\text{Planck}}$, then the entanglement entropy would be of the same order of magnitude as S_{BH}.

The area scaling of the entanglement entropy in general is made plausible by the fact that for a pure state the entanglement entropy is symmetric between a region and its complement

$$S(\Omega) = S(\Omega^c). \tag{1.6}$$

Since the surface $\partial\Omega$ is a common feature of Ω and Ω^c, it is natural to expect a scaling with the area of $\partial\Omega$. For a randomly chosen state the entanglement entropy is generically close to the maximal possible value, which is the logarithm of the dimension of the small Hilbert space [14], and hence would scale with the volume rather than area,

$$S(\Omega) = \min(\mathrm{Vol}(\Omega), \mathrm{Vol}(\Omega^c)). \tag{1.7}$$

Thus it appears that the statement (1.5) is a very special property of the ground state. Indeed, the scaling of the entanglement entropy with the area of the entangling surface appears to be a generic feature of systems with local interactions, and has become known as the area law [15]. Rigorous results in $1+1$ dimensions show that the ground state of any local, gapped Hamiltonian obeys the area law, but the problem remains open in higher dimensions [16].

Another appealing feature of the entanglement entropy is that may explain why all causal horizons, and not only black hole horizons, should have an entropy. Black hole entropy is not just about black holes; de Sitter and Rindler spacetimes also have horizons possessing a temperature and an entropy given by the Bekenstein-Hawking formula [17, 18, 19]. It has been argued that laws analogous to the laws of black hole mechanics should apply to any causal horizon, where a causal horizon is defined as the boundary of the past of any timelike curve of infinite proper length in the future direction [20]. Since any region of space has an entanglement entropy, such an entropy is naturally associated to the time slices of any causal horizon. An explanation of black hole entropy general enough to apply to arbitrary causal horizons is necessary to appeal to the argument that Einstein's equations follow from the laws of thermodynamics applied to local Rindler horizons [21].

It has also been suggested that the second law for the entanglement entropy could derive from the defining property of a causal horizon: namely that the fields behind the horizon do not influence the outside [12, 22]. Since the state of the fields outside the horizon evolve autonomously, the entropy of the fields outside the horizon can be shown to increase with time, provided the maximally mixed state of the outside evolves to itself [22]. Although the assumptions of the proof in Ref. [22] have been shown to be inconsistent [23], this observation has been used in proving the generalized second law for Rindler horizons [24].

These arguments suggest a relation between the Bekenstein-Hawking entropy and the entanglement entropy. This relation will be made more precise in section 1.2.

1.1.1 Entanglement entropy in other areas of physics

Although the entanglement entropy between regions of space was introduced to explain black hole entropy, it has become a quantity of interest in other areas of theoretical physics. By understanding the relation between $S_{\rm BH}$ and $S_{\rm ent}$ one can hope to find links between gravity and other areas of theoretical physics.

In the simulation of quantum many-body systems, the ground state is often approximated using the variational method over a certain class of states. Examples of such classes of states include Matrix Product States (MPS), Projected Entangled Pair States (PEPS) [25] , and the Multiscale Entanglement Renormalization Ansatz

(MERA) [26]. These states are distinguished by the fact that the entanglement entropy grows much more slowly than the volume scaling behavior of a randomly chosen state (MPS and PEPS satisfy an area law; MERA allows for logarithmic scaling of the entanglement entropy with length in one dimension). Thus the area law is essential for the efficient simulation of quantum many body systems (at least using existing methods).

The entanglement entropy, and particularly its subleading corrections, have been shown to be a useful probe of the structure of the ground state of condensed matter systems. These methods have been particularly useful for the study of topological phases: zero-temperature phases of matter that do not break any symmetry, and are not characterized by any order parameter [27]. It has been shown that a certain combination of entanglement entropies for different regions called the topological entanglement entropy is probe of topologically ordered phases [28, 29]. In gauge theories, entanglement entropy may also be useful in studying the deconfining phase transition [30, 31].

In the context of the anti de Sitter/conformal field theory correspondence (AdS/CFT), Ryu and Takayanagi have given a proposal for computing the entanglement entropy of conformal field theories with holographic duals described by Einstein gravity [32, 33, 34, 35]. This proposal relates the entanglement entropy of a spatial region in a conformal field theory to the area of a minimal surface in an asymptotically AdS spacetime of one higher dimension. It has therefore been argued that the entanglement entropy plays an essential role in the emergence of a classical higher dimensional spacetime from conformal field theory [36, 37]. While still conjectural, this method agrees with known results, and has also been used to derive results that have not been possible by other means, such as the entanglement entropy of strongly coupled theories.

1.2 The species puzzle and its resolution

Superficially, it appears that S_{BH} cannot be equal to the entanglement entropy, since S_{ent} depends on the cutoff, as well as the number and type of particle species, while S_{BH} is given by the Bekenstein-Hawking formula and is finite. This apparent inconsistency is known as the species puzzle.

It has been argued that gravity itself could act effectively as a cutoff that depends on the number of species in such a way to make the black hole entropy and entanglement entropies equal [38, 39, 40]. From this point of view, the divergence of the entanglement entropy is not surprising: the entanglement entropy is calculated on a fixed background without accounting for backreaction effects. Therefore one has effectively set $G = 0$ and consistency with the Bekenstein-Hawking formula demands that the entropy be infinite. But to make these arguments more precise requires a theory of quantum gravity, and it is not clear to what extent the necessary notions of causal horizon and of a tensor product decomposition of the degrees of freedom will continue to hold in such a theory.

Fortunately, one can understand the relation between the entanglement entropy and the Bekenstein-Hawking entropy within the framework of quantum field theory in curved spacetime, without taking into account the quantum dynamics of gravity. Given a quantum theory on a fixed background, the path integral over all matter fields yields a gravitational effective action that depends on the background metric and on the regulator. This action can be expanded in powers of the cutoff, and if the regulator is diffeomorphism invariant it has the form of the Einstein-Hilbert action plus higher order corrections. In other words, the same quantum fluctuations of matter that contribute to the entanglement entropy also contribute to the renormalization of $1/G$, and therefore to the Bekenstein-Hawking entropy.

To make this discussion more concrete, consider flat D-dimensional spacetime with coordinates $x^0, \ldots, x^{D-1}$. Space at a fixed time has coordinates $\vec{x} = (x^1, \ldots, x^{D-1})$, and can be divided into two regions: a "left region" $L = \{\vec{x} : x^1 < 0\}$ and "right region" $R = \{\vec{x} : x^1 > 0\}$. For a field theory satisfying the Wightman axioms, the Bisognano-Wichmann theorem [41, 42] states that the reduced density matrix for the region R is a thermal state with respect to the generator K of boosts in the (x^0, x^1) plane,

$$K = \int_R d^{D-1}x \sqrt{q}\, T_{ab}\xi^a n^b = \int_{x^1>0} d^{D-1}x \; x^1 \, T_{00} \tag{1.8}$$

where q is the determinant of the spatial metric, T_{ab} is the energy-momentum tensor, ξ^a is the boost Killing vector field ($\xi = x^1\partial_0 + x^0\partial_1$) and n^a is the normal to the surface $x^0 = 0$. Formally, the state of the region R is a Kubo-Martin-Schwinger (KMS) state of a type III von Neumann algebra, but in the presence of a suitable cutoff it can be represented as a density matrix,

$$\rho_R = \frac{e^{-2\pi K}}{Z(2\pi)}. \tag{1.9}$$

Here $Z(\beta) = \operatorname{tr} e^{-\beta K}$ is the partition function. Note that the boost generator K has dimensions of action (which in our units is dimensionless), so the associated "inverse boost temperature" β is also dimensionless.

The fact that the vacuum is thermal in the boost generator is closely related to the Unruh effect [17]. An observer with constant acceleration in Minkowski space moves along the flow of a boost generator, and this observer's proper time τ is related to the hyperbolic angle η conjugate to K by $\tau = \eta/a$. Thus a particle detector moving with constant acceleration a will respond as if exposed to a heat bath at an inverse temperature $\beta = 2\pi/a$.

The entanglement entropy of the vacuum across the surface $x^1 = 0$ is therefore equivalent to the usual thermodynamic (Gibbs/von Neumann/Shannon) entropy of the thermal state (1.9) [43]. This means that the generally difficult problem of computing the entanglement entropy reduces to the much easier problem of computing the entropy of a thermal state. The entropy of this state can be computed by varying the inverse temperature β away from 2π [44]:

$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \ln Z(\beta)\bigg|_{\beta=2\pi}. \tag{1.10}$$

Note that this formula is closely related to the "replica trick" for computing the entropy [45]. In the replica trick one computes $\mathrm{tr}(\rho^n)$ for $n \geq 1$, and analytically continues the result to non-integer n,[2] The entropy is given by

$$S = -\lim_{n\to 1}\frac{\partial}{\partial n}\,\mathrm{tr}(\rho^n). \tag{1.11}$$

The equivalence between the conical method and replica trick follows from the iden-

[2]For an argument that such an analytic continuation exists and is unique, see Ref. [46], section 2.7.

tifications $\beta = 2\pi n$ and $Z(2\pi n) = Z(2\pi)^n \operatorname{tr}(\rho^n)$.

The partition function $Z(\beta)$ can be expressed as a Euclidean path integral:

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} \tag{1.12}$$

where ϕ denotes schematically all the dynamical fields. These fields live on a manifold $C_\beta \times \mathbb{R}^{D-2}$ where C_β is the cone of angle β:

$$ds^2 = dr^2 + r^2 d\theta^2, \qquad r > 0, \qquad \theta \sim \theta + \beta. \tag{1.13}$$

For $\beta \neq 2\pi$ the cone is singular on the $D-2$-dimensional surface Σ defined as the set of points such that $r = 0$, so one has to be careful about the boundary conditions imposed there. For a scalar field, the appropriate boundary conditions for computing the entanglement entropy are Neumann boundary conditions [47]:

$$\partial_r \phi \big|_\Sigma = 0. \tag{1.14}$$

It will also be useful to take the $D-2$ transverse dimensions to be periodic with period L, so that Σ has finite area. This avoids an uninteresting infrared divergence in the entanglement entropy.

We now sketch the resolution of the species puzzle following Refs. [48, 49]. The partition function can be expressed in terms of an effective Lagrangian L_{eff},

$$\ln Z = -\int \sqrt{g}\, L_{\text{eff}}. \tag{1.15}$$

Since all matter fields have been integrated out, the effective Lagrangian depends only on the background metric. This action can be expanded in powers of the cutoff as

$$L_{\text{eff}} = \frac{1}{16\pi G_{\text{eff}}}(R - 2\Lambda_{\text{eff}} + \ldots) \tag{1.16}$$

where "..." denotes local terms of higher order in the cutoff, as well as finite and non-local terms. The entropy receives a contribution from each term in L_{eff}. The cosmological constant contribution to $\ln Z$ is proportional to the spacetime volume, and hence to β; it therefore does not contribute to the entropy. On the cone, the integral of the curvature is given by

$$\int \sqrt{g}\, R = 2(2\pi - \beta) A \tag{1.17}$$

where A is the area of Σ. Therefore the contribution of the Einstein-Hilbert term to the entropy is

$$S = \frac{A}{4G_{\text{eff}}}. \tag{1.18}$$

Higher-order local terms need to be treated carefully, since the curvature is singular at the tip of the cone, so higher order terms in the action such as R^2 are divergent. However if they are regulated by replacing the cone with an approximating smooth geometry, they are proportional to $(\beta - 2\pi)^n$ for $n > 2$, and hence do not contribute to the entropy in Rindler space [49]. When the horizon has curvature, these higher-order terms in the action also contribute to the entropy and determine the curvature-dependent corrections to the entropy density.

One can generalize this result to any static spacetime with a bifurcate Killing horizon and a regular Euclidean section [50]. This class of spacetimes includes static black hole spacetimes (such as Schwarzschild and Reissner-Nordström black holes) and de Sitter space. For such a spacetime, one can define a Hartle-Hawking state using the Euclidean path integral, and this state is thermal with respect to the generator of the time translation symmetry. Let Σ be the fixed point set of this isometry, which coincides with the bifurcation surface in the Lorentzian spacetime. We can normalize the Killing vector to have unit surface gravity, so that the geometry is regular when the periodicity β equals 2π. The Hartle-Hawking state is also regular when $\beta = 2\pi$.

The Hartle-Hawking state so defined is thermal when restricted to the region outside the Killing horizon. Its entropy can be found by varying the periodicity away from 2π, introducing a conical singularity on Σ. For infinitesimal deficit angle $2\pi - \beta$, the Riemann curvature tensor is [51]

$$R_{abcd}(x) = \overline{R}_{abcd}(x) + (2\pi - \beta)\epsilon_{ab}\epsilon_{cd}\delta_\Sigma(x). \tag{1.19}$$

Here $\overline{R}_{abcd}$ is the Riemann tensor of the smooth geometry, which can be defined on the conical geometry with $\beta \neq 2\pi$ by pulling back via the natural local embedding into the smooth geometry with $\beta = 2\pi$.

If the effective Lagrangian L_{eff} depends on the Riemann tensor algebraically[3], and all derivatives of the matter fields are symmetrized, the entropy following the conical formula is given by the Wald entropy formula [52, 53, 54, 55]:

$$S_{\text{Wald}} = -2\pi \int_\Sigma d^{D-2}x \sqrt{h}\, \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd} \tag{1.20}$$

where h is the determinant of the induced metric on Σ, and ϵ_{ab} is the binormal to Σ. The Wald entropy was originally found in generalizing the first law of black hole mechanics to higher-curvature gravity theories, and is related to the Noether charge associated with the diffeomorphism symmetry [52]. It was shown in Refs. [54, 55]

[3]This formula generalizes naturally to Lagrangians with derivatives of the Riemann tensor, but these will not be used in the present work.

that the expression (1.20) is equivalent to the entropy defined by Wald's Noether charge method.

Instead of considering the low-energy effective action that depends only on the geometry, one can consider a Wilsonian effective action at mass scale μ. The Lagrangian $L_{\text{eff}}(\mu)$ depends on the quantum fields as well as on the background metric. The entropy can be divided into two terms, a part coming from the explicit dependence of the Lagrangian on the singular curvature on Σ, and a part that comes from the change in the period of imaginary time. Thus we conjecture that the leading order part of the entropy can be expressed as

$$S = \langle S_{\text{Wald}} \rangle + S_{\text{ent}}. \tag{1.21}$$

This statement for a general quantum field theory is far from clear; in an interacting theory there is entanglement entropy across different momentum scales whose contribution to the position space entanglement entropy is unknown [56], and there are also nonlocal terms in the effective action that will also contribute to the entropy.

However we expect the leading order results to be insensitive to interactions (at least for weakly coupled theories) and nonlocal terms. We note that since the entanglement entropy is the same as the outside thermal entropy, and the Wald entropy reduces to the Bekenstein-Hawking entropy in the case of general relativity, Eq. (1.21) can be viewed as a generalization to higher curvature gravity theories of the generalized entropy that appears in the generalized second law of thermodynamics [57].

Under the renormalization group flow, the couplings in the effective Lagrangian are adjusted as μ is varied such that Z is kept fixed. Since the conical entropy formula (1.10) is defined in terms of Z, it follows that the entropy is independent of scale μ. If the relation (1.21) holds, then different scales correspond to different divisions of entropy between the entanglement entropy and the various terms in the Wald entropy. When the scale μ is set to zero, all quantum fields are integrated out, and the entropy is purely the Wald entropy of the gravitational effective action. If at some scale the Lagrangian contains no explicit dependence on the curvature, then the entropy may be purely entanglement entropy; this is the induced gravity scenario considered in Ref. [48].

To summarize, we have given a rather formal argument that the Wald entropy of the low-energy theory is given by a sum of the Wald entropy at a higher scale μ, and the entanglement entropy of all the modes below scale μ. Thus the classical entropy (of which the Bekenstein-Hawking entropy is the leading-order contribution for weakly curved horizons) is at least partly given by the entanglement entropy. Whether the entanglement entropy can account for the entire black hole entropy depends on the UV completion.

In the next section we show more explicitly how this works for free fields.

1.3 One-loop entropy of matter fields

The contribution of matter fields to the black hole entropy can be calculated at one loop order using heat kernel methods.

In the loop expansion, the Euclidean action is expanded around a saddle point

ϕ_0 as $\phi = \phi_0 + \hat{\phi}$:

$$S[\phi] = S[\phi_0] + \left\langle \hat{\phi}, (\Delta + m^2)\hat{\phi} \right\rangle \tag{1.22}$$

where $\langle \cdot, \cdot \rangle$ is an inner product on the fields, m is the mass, and Δ is an operator such that $(\Delta + m^2)\phi = 0$ gives the linearized equations of motion. The one-loop effective action is then given by the functional determinant

$$\ln Z = -\frac{1}{2} \ln \det(\Delta + m^2). \tag{1.23}$$

In the one loop approximation, one is effectively calculating the entropy in a free theory.

The determinant 1.23 can be calculated using heat kernel methods [58]. In the heat kernel regulator, the effective action is given by

$$\ln Z = \frac{1}{2} \int_{\epsilon^2}^{\infty} ds \, \frac{e^{-m^2 s}}{s} K(s) \tag{1.24}$$

where ϵ is an ultraviolet regulator with dimensions of length (analogous to $1/\mu$ of the previous section). The function $K(s)$ is the trace of the heat kernel, $K(s) \equiv \operatorname{tr} e^{-s\Delta}$, and has an asymptotic expansion as $s \to 0$,

$$K(s) \sim \frac{1}{(4\pi s)^{D/2}} \sum_{n \geq 0} a_n s^{n/2}. \tag{1.25}$$

The coefficients a_n are all integrals of local scalar curvature invariants,

$$a_n = \int d^D x \sqrt{g}\, a_n(x). \tag{1.26}$$

Since s has dimensions of (length)^2, the coefficient $a_n(x)$ must have dimensions of (length)$^{-n}$. This limits the form of the coefficients $a_n(x)$, in particular $a_0(x)$ is a constant, and $a_2(x)$ is proportional to the Ricci scalar. It is useful to define a constant c_1 such that

$$a_2(x) = c_1 R. \tag{1.27}$$

Following the arguments in the previous section, the entropy of a Rindler horizon (and the leading order curvature-independent term in the entropy density of a general horizon) is determined by the heat kernel coefficient a_2, which determines the Einstein-Hilbert term in the effective action. The coefficients c_1 for operators associated to particles of spin-s have been calculated for all spins up to 2 [46]. The results are given in table 1.1.

Spin	Field	N	c_1
0	Nonminimally coupled scalar	1	$\frac{N}{6} - \xi$
$\frac{1}{2}$	Dirac spinor	$2^{\lfloor \frac{D}{2} \rfloor - 1}$	$\frac{N}{6}$
1	Maxwell field	$D-2$	$\frac{N}{6} - 1$
$\frac{3}{2}$	Rarita-Schwinger field	$(D-3)2^{\lfloor \frac{D}{2} \rfloor - 1}$	$\frac{N}{6}$
2	Graviton	$\frac{D(D-3)}{2}$	$\frac{N}{6} - \frac{D^2-D+4}{2}$

Table 1.1: Heat kernel coefficient c_1 for fields of spin $s \leq 2$, given as a function of the number N of on-shell degrees of freedom per point. Note that in Ref. [46], an incorrect value for N was given for the Rarita-Schwinger field, since it failed to include the Nielsen-Kallosh ghost [59, 60]. However the Nielsen-Kallosh ghost is simply a (bosonic) spinor field, so its inclusion does not change the result that there is no contact term for the spin-$\frac{3}{2}$ field.

The heat kernel coefficients can each be divided into a part that is the number of on-shell degrees of freedom times 1/6; this is the entanglement entropy [46]. For scalars with $\xi \neq 0$, Yang-Mills fields, and gravitons there is an extra term in addition to the entanglement entropy. Following the language of Ref. [61], we refer to these as contact terms.

These contact terms seem to spoil the statistical interpretation of S_{BH} as entanglement entropy. Understanding these terms has been identified as a major open problem in the interpretation of black hole entropy as entanglement entropy [46]. The bulk of the present work is concerned with understanding the contact term in the case of gauge fields.

1.4 Summary of the thesis

We now summarize the contents of the thesis. Chapters 2 and 3 are based Ref. [62], written with Aron Wall, but include additional background material to make the presentation more self-contained. Chapter 4 is based on Ref. [63].

Before discussing gauge fields, in section 2.3 we consider first the case of the nonminimally coupled scalar field. The nonminimally coupled scalar Lagrangian is

$$L = \frac{1}{2}\left(\nabla_a\phi\nabla^a\phi - m^2\phi^2 - \xi R\phi^2\right). \tag{1.28}$$

The discrepancy between the entanglement entropy and the conical entropy can be attributed to the coupling of the field to the singular curvature at the tip of the cone (1.19). These contribute to the generalized entropy defined by the conical method, although they do not contribute to the entanglement entropy in flat space, which is independent of ξ [64, 65].

The contribution of the nonminimally coupled scalar to the Wald entropy is [54]:

$$S^{(\phi)}_{\text{Wald}} = -2\pi\xi \int_\Sigma d^{D-2}x\sqrt{h}\,\phi^2. \tag{1.29}$$

There are a number of independent arguments that such a contribution must appear in the entropy. First, we note that if ϕ has a vacuum expectation value ϕ_0, then the nonminimal coupling term in (1.28) takes the form of the Einstein-Hilbert term with $1/4G = -2\pi\xi\phi_0^2$. This indicates that we should view the entropy (1.29) on an equal footing with the Bekenstein-Hawking entropy.

The term (1.29) is also necessary to preserve the classical second law of black hole mechanics. This theorem states that for matter satisfying the null energy condition ($T_{ab}k^a k^b \geq 0$ for all null vectors k), the horizon area is a nondecreasing function of time [66]. Nonminimally coupled scalars do not satisfy the null energy condition, and the area is not generally increasing. However the full Wald entropy, which includes both the Bekenstein-Hawking term and the term (1.29), does increase [67].

Finally, we note that the scalar contact term is needed to obtain the correct behavior of the entropy under renormalization. As an example, Ref. [68] considered the renormalization of the entropy in a nonlinear σ-model. At high energies, the theory consists of minimally coupled scalar fields and the entropy is purely entanglement entropy. At intermediate energies, the theory is well described by a theory of a single nonminimally coupled scalar with $\xi = -1$. At this scale the entropy is a mix of entanglement entropy, the Wald entropy term (1.29), and the Bekenstein-Hawking entropy. Finally, at low energy all scalar fields are integrated out and the theory is described by Einstein gravity for which the entropy is purely the Bekenstein-Hawking entropy. The nonminimal coupling term must be included in the calculation of the entropy at intermediate scales, otherwise the entanglement entropy of the fundamental theory is undercounted.

It has been suggested that the extra term in the entropy of gauge fields should be attributed to nonminimal coupling of gauge fields to gravity, as in the case of the nonminimally coupled scalar [69]. In section 2.4, we will argue against this interpretation.

The analogy with the nonminimally coupled scalar field comes from a particular gauge-fixed Lagrangian for electromagnetism, that takes the form

$$L = -\frac{1}{2} A^a (g_{ab}\nabla^2 - R_{ab}) A^b - \bar{c}\,\nabla^2\, c \tag{1.30}$$

This Lagrangian includes a gauge-fixing term and two fermionic scalar Faddeev-Popov ghosts c and $\bar{c}$. If we apply the Wald entropy formula (1.20) to this gauge-fixed action we obtain

$$S^{(A)}_{\text{Wald}} = -\pi \int_\Sigma \sqrt{h}\, A_a A_b g_\perp^{ab} \tag{1.31}$$

where $g_\perp^{ab}$ is the inverse metric in the plane perpendicular to the horizon.

This term is not manifestly gauge-invariant, which raises questions about the gauge-invariance of the contact term. The gauge-fixed Lagrangian of course does not have a gauge symmetry, but it is invariant under Becchi-Rouet-Stora-Tyutin (BRST) symmetry, under which (1.31) is also not invariant. Furthermore, unlike the scalar field case, the contact term for gauge fields corresponds to an ambiguity in the Wald entropy formula [54]. These ambiguities vanish for classical fields at the Killing horizon, but may have nonvanishing quantum expectation values. The relevant ambiguity here is the ability to add a total derivative depending on the Riemann tensor to the Lagrangian, under which $\partial L/\partial R_{abcd}$ is not invariant. These ambiguities vanish for classical stationary field configurations, but do not vanish in quantum expectation value.

Moreover, we recall that the contact term in the entropy of the nonminimally coupled scalar field was necessary even at the classical level to preserve the second law of black hole mechanics. Unlike the nonminimally coupled scalar field, the electromagnetic field satisfies the null energy condition, so Hawking's area theorem applies and the Bekenstein-Hawking entropy increases without additional terms. The presence of such terms, even if they could be given a gauge-invariant meaning, would likely spoil the classical second law. We believe that these considerations cast serious doubt on the interpretation of the contact term for gauge fields as part of the Wald entropy.

In chapter 3, we revisit Kabat's derivation of the contact term [61]. This calculation is essentially two-dimensional: the polarizations of the gauge field transverse to the horizon contribute like scalars, and the contact term comes purely from the gauge fields polarized in the plane normal to the horizon. This two-dimensional calculation makes use of the exact scale symmetry of the cone, and we show that it depends sensitively to how the infrared is regulated. As a physically motivated choice of infrared regulator, we consider replacing Rindler space with a de Sitter space of arbitrarily large de Sitter radius. The advantage of this procedure is that the noncompact Euclidean Rindler space (the plane) is replaced with the compact Euclidean de Sitter space. The partition function can be calculated on an arbitrary compact two-dimensional manifold, and in this case the negative entropy comes from the zero modes of the Faddeev-Popov ghosts.

Thus we argue that in this case the contact term should be considered not as part of the Wald entropy, but as part of the entanglement entropy. However the relevant entanglement entropy is a negative entanglement entropy associated with states of negative norm. To see how such a negative entropy can arise in such a calculation, we point out in section 3.2.2 that the BRST symmetry does not guarantee the absence of ghosts; this must be checked on a case-by-case basis. In the case of electrodynamics, we show how inclusion of the zero modes yields a state space with a non-positive-definite inner product. Since these negative norm states are unphysical, the contact term in two dimensions is also unphysical.

In section 3.3, we calculate the physical partition function on a 2D compact manifold in the reduced phase space, without the use of gauge-fixing or ghosts, taking into account all non-perturbative effects. We find that the entropy is finite and equal to the entanglement entropy. As expected on physical grounds, there is no renormalization of $1/G$ in two dimensions.

The path integral method is rather formal, which makes obscure its relation to the states in the Hilbert space of gauge theory. In chapter 4 we instead consider the entanglement entropy in Hamiltonian lattice gauge theory. We point out that the usual definition of entanglement entropy requires some modification in the presence of a gauge symmetry. Our definition of entanglement entropy, which includes the contribution of edge states, matches the two-dimensional Euclidean results of section 3.3.

The states of gauge theory are not field configurations, but equivalence classes of field configurations under gauge transformations. In the Hamiltonian formulation, the gauge transformations are generated by constraint operators; in electrodynamics these constraints are integrals of the Gauss constraint operator,

$$C(\Lambda) = \int_{\Sigma} \sqrt{q}\,\Lambda(x)\nabla \cdot E(x), \tag{1.32}$$

where Λ is a scalar function. The space of functionals of the vector potential A, called the kinematical Hilbert space, has the tensor product structure

$$\mathcal{H}^{\text{kin}} = \mathcal{H}^{\text{kin}}_{\Omega} \otimes \mathcal{H}^{\text{kin}}_{\Omega^c}. \tag{1.33}$$

The physical Hilbert space is defined as the quotient of $\mathcal{H}^{\text{kin}}$ by all the constraint operators $C(\Lambda)$. However the constraint operators act with derivatives, so they do not act independently on the two regions. If the function Λ vanishes on $\partial\Omega$, then the constraint decomposes as a tensor product $C(\Lambda) \otimes \mathbb{1}$ and does not interfere with the tensor product structure of the kinematical Hilbert space. However the operators $C(\Lambda)$ where Λ has support on $\partial\Omega$ do not have a tensor product form and prevent the physical Hilbert space from being expressible as a tensor product. Thus for gauge theories, the decomposition of the physical Hilbert space as a tensor product does not hold.

The solution is that when a gauge theory is quantized on a manifold with boundary, the constraints have to be relaxed at the boundary. By relaxing the constraint at the boundary, states that would otherwise be pure gauge are treated as physical states, known as edge states. The edge states are well-known from the quantum Hall effect, where the boundary is a physical barrier at which the fields must satisfy some imposed boundary condition. In the entanglement entropy the entangling surface $\partial\Omega$ is not a physical barrier, and no boundary conditions are imposed there. Nevertheless, the entanglement entropy is sensitive to the existence of edge states, and in fact behaves much like a thermal entropy of a physical edge [70, 71]

Edge states also appear in 2+1-dimensional quantum gravity [72]. In 2+1 dimensions, general relativity has no local degrees of freedom, and can be expressed as a Chern-Simons theory [73, 74]. It has been argued that the edge states are of the right number to account for the black hole entropy [75]. In 2+1 gravity, the edge states are necessary to account for the Bekenstein-Hawking entropy, since the theory has no bulk degrees of freedom.

In chapter 4, we show how the edge states are related to the entanglement entropy in the setting of Hamiltonian lattice gauge theory. The lattice is convenient because it allows the entanglement entropy to be regulated while keeping exact gauge symmetry. Not only does the definition of the Hilbert space require the introduction of edge states, but the gauge transformations on the boundary become physical symmetries of the density matrix. These symmetries allow the entanglement entropy to be decomposed into a sum of three terms: one is associated with non-local correlations of the gauge field, and two are associated with the edge states. The edge state part of the entropy is localized to the boundary, and in the cases considered in section 4.2, the edge states give the dominant contribution to the entropy.

In chapter 5, results are summarized and several areas of future work are described.

Chapter 2

Conical entropy and Wald entropy

In this chapter we investigate the relation between the contact terms in the conical entropy and the Wald entropy. We begin in section 2.1 by introducing the Wald entropy, focussing on the ambiguities that arise in its definition. In section 2.2, we briefly introduce aspects of the heat kernel regularization that will be used in the subsequent sections. In section 2.3, we consider the case of a nonminimally coupled scalar field, and show that the one loop correction to the conical entropy is equal to the sum of the entanglement entropy and the expectation value of the contribution of the scalar field to the Wald entropy. In section 2.4, we show that a similar property holds for the contact term of abelian gauge fields: the one loop correction to the conical entropy is equal to the sum of entanglement entropy and the gauge field contribution to the Wald entropy. However this property applies only if we choose a particular non-gauge-invariant form of the Wald entropy.

2.1 Wald entropy

The entropy of a bifurcate Killing horizon can be calculated in a D-dimensional diffeomorphism-invariant classical theory by the Wald Noether charge method [52, 55]. We now sketch how the entropy is defined in Wald's method.

Let $L[\phi]$ be a Lagrangian locally and covariantly constructed from a set of dynamical fields schematically denoted by ϕ. Following Ref. [52] it will be convenient to adopt the language of differential forms and view $L[\phi]$ as a differential D-form. If the fields are varied by the action of a diffeomorphism, the variation of the fields and of the Lagrangian are given by the Lie derivative:

$$\delta\phi = \mathcal{L}_\xi \phi, \qquad \delta L[\phi] = \mathcal{L}_\xi L[\phi] = d(\xi \cdot L[\phi]), \tag{2.1}$$

where "$\cdot$" denotes contraction of a vector field with a differential form (the interior product). Here we have used Cartan's formula, $\mathcal{L}_\xi \alpha = \xi \cdot d\alpha + d(\xi \cdot \alpha)$ for α a differential form, and the fact that the exterior derivative of a D-form vanishes.

After an integration by parts the variation of the Lagrangian takes the form

$$\delta L = E[\phi] \cdot \delta\phi + d\Theta[\phi, \delta\phi], \tag{2.2}$$

where $E[\phi]$ are the equations of motion. $\Theta[\phi, \delta\phi]$ is a $(D-1)$-form linear in the variation $\delta\phi$ called the symplectic potential. It carries the information about the symplectic structure of the theory (in mechanics, it would be of the form $p_i \delta q^i$).

Associated to any diffeomorphism generated by a vector field ξ, there is a conserved Noether current:

$$J_\xi[\phi] = \Theta[\phi, \mathcal{L}_\xi \phi] - \xi \cdot L[\phi]. \tag{2.3}$$

Since $J_\xi[\phi]$ is locally and covariantly constructed from the fields, and closed for any vector field ξ, one can always find a Noether charge $D-2$-form $Q_\xi[\phi]$ such that $J_\xi[\phi] = dQ_\xi[\phi]$ [76]. For a stationary spacetime with a bifurcate Killing horizon, the Wald entropy is the integral over the bifurcation surface Σ of the Noether charge associated to the Killing vector χ:

$$S_{\text{Wald}} = 2\pi \int_{\Sigma} Q_{\chi}[\phi], \tag{2.4}$$

where where χ is normalized to have unit surface gravity. The charge Q constructed in this way depends on the vector field χ, but it can be expressed in a form that depends only on the metric and the dynamical fields [52, 54]. It was shown in Ref. [54] that this form of the entropy yields the same result evaluated on any cross-section of the horizon, not just the bifurcation surface Σ.

This definition of the entropy is motivated by fact that S_{Wald} satisfies the first law of black hole mechanics. Consider a static black hole solution with constant surface gravity κ. Let E be the total energy, defined as the value of the Hamiltonian generating motion along the timelike Killing vector field χ. Under a (not necessarily static) perturbation of the static solution satisfying the linearized equations of motion, the change in energy is given by

$$\delta E = \frac{\kappa}{2\pi} \delta S_{\text{Wald}}. \tag{2.5}$$

We recognize this as the first law of thermodynamics, with $\kappa/2\pi$ the physical temperature of a black hole horizon. This first law extends also to the case of stationary but nonstatic (rotating) and charged black holes, with the addition of work terms in the first law, corresponding to the change in the black hole's angular momentum and charges.

The Noether charge is subject to several ambiguities in its definition. These are all related to the ambiguity in solving an equation of the form $d\alpha = \beta$ for α: this can be done only up to the addition of a closed form $\alpha \to \alpha + \gamma$ where $d\gamma = 0$.

1. The Lagrangian can be modified by a total derivative $L \rightarrow L + d\mu$ where μ is a $D-1$ form. This modification does not change the classical equations of motion, and therefore defines an equivalent classical theory. This change of L induces changes to Θ, J and Q:

$$\Theta \rightarrow \Theta + \delta\mu, \tag{2.6}$$

$$J \rightarrow J + \mathcal{L}_\xi \mu - \xi \cdot d\mu = J + d(\xi \cdot \mu), \tag{2.7}$$

$$Q \rightarrow Q + \xi \cdot \mu. \tag{2.8}$$

   When Q is evaluated on the bifurcation surface, $\xi = 0$ and the additional term vanishes.

2. The symplectic potential is defined only up to the addition of a closed $D-1$ form linear in $\delta\phi$. By the result of Ref. [76], this closed form is also exact, so the ambiguity is of the form $\Theta \rightarrow \Theta + dY$ where Y is a $D-2$ form linear in $\delta\phi$. This changes Q by $Q \rightarrow Q + dY$. The entropy is defined as the integral of Q over the (closed) bifurcation surface Σ, and hence the ambiguity vanishes upon integration.

3. The Noether charge Q is defined only up to the addition of a closed form. Again, by the result of Ref. [76], the form is also exact so the ambiguity takes the form $Q \rightarrow Q + dZ$. This also vanishes when integrating over Σ.

If the Lagrangian L does not depend on derivatives of the Riemann tensor, and all derivatives of the matter fields are symmetrized, the classical black hole entropy is given by differentiating L with respect to the Riemann tensor [54]:

$$S_{\text{Wald}} = -2\pi \int_{\Sigma} d^{D-2}x \sqrt{h}\, \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}, \tag{2.9}$$

where h is the pullback of the metric to the $D-2$ dimensional bifurcation surface Σ and ϵ_{ab} is the binormal to the slice. This formula was proven to be equivalent to the Noether charge entropy (2.4) on stationary Killing horizons, but has the advantage that (2.9) gives the same value when evaluated on any cross-section of the Killing horizon. We note that the formula (2.9) appeared earlier in Ref. [53], where it was derived via Euclidean techniques.

This formula for the entropy is also subject to ambiguities, related to the ambiguities in the Noether charge (2.4). In particular, it changes if we add to L a total derivative that depends on the Riemann tensor. These ambiguities can also be shown to vanish for classical fields at a Killing horizon [54], and are therefore not important for the classical first law.

The Wald entropy is classical, and we are interested in the full entropy as defined by the conical entropy formula (1.10). For a classical theory, the conical entropy is equivalent to the Wald entropy [77], while for minimally coupled scalar and spinor fields it equals the entanglement entropy [43]. It is thus natural to conjecture, in accordance with the arguments of Refs. [77, 78], that the leading order contribution to the entropy for a general quantum field theory the conical entropy is given by the sum

$$S_{\text{cone}} = \langle S_{\text{Wald}} \rangle + S_{\text{ent}}, \tag{2.10}$$

where S_{Wald} is the form of the Wald entropy given by (2.9). The Wald entropy contains both a zero-loop classical part, and a contribution from the coupling of fields to the curvature at the tip of the cone. For general relativity with minimally coupled matter, the right-hand side of Eq. (2.10) is the generalized entropy, which is conjectured to obey the generalized second law [23].[1]

Since S_{cone} is defined in terms of the renormalized effective action $-\ln Z$, it must be independent of the renormalization scale. Therefore an important consistency check of Eq. 2.10 is whether the generalized entropy is also invariant under the renormalization group flow. This is nontrivial, since the terms S_{Wald} and S_{ent} depend explicitly on the renormalization scale: the latter because of the ultraviolet divergence of $-\text{tr}(\rho \ln \rho)$, and the former because of the RG flow of the coupling constants such as $1/G$, and (in some cases) divergent products of fields such as ϕ^2. In order for Eq. (2.10) to hold, the renormalization of the entanglement entropy must match the renormalization of the Wald entropy, when both are regulated in the same way. We will now check this using the heat kernel regulator for the nonminimally coupled scalar and for Maxwell theory.

[1]The generalized second law has been proven in various regimes [23, 24, 79] for fields minimally coupled to general relativity, where $S_{\text{Wald}} = A/4G$. For higher curvature gravity and nonminimal couplings, it is not even known whether the theory obeys a classical second law, except for special cases such as $f(R)$ gravity [21], nonminimally coupled scalars [67], and first order perturbations to Lovelock horizons [80]. However, it is also known that the Wald entropy can decrease in classical mergers of Lovelock black holes [54, 81, 82].

2.2 The heat kernel method

To calculate the one-loop effective action, we will follow the heat kernel method [58, 83]. At the one loop level, the effective action reduces to a functional determinant of an operator $\Delta + m^2$:

$$\ln Z = -\frac{1}{2} \ln \det(\Delta + m^2). \tag{2.11}$$

We will assume that Δ is an operator of Laplace type acting on a vector bundle over M of the form:

$$\Delta = -(g^{ab}\nabla_a\nabla_b + E) \tag{2.12}$$

where $E : V \to V$ is a matrix-valued function on M.

To calculate the functional determinant of the operator Δ we introduce the heat kernel, which is the kernel of the operator $e^{-s\Delta}$. The heat kernel is the function $K(s, x, y)$ defined for $s > 0$ and $x, y \in M$ as the solution of the differential equation

$$\partial_s K(s, x, y) = -\Delta_x K(s, x, y), \tag{2.13}$$

$$K(0, x, y) = \delta(x - y)/\sqrt{g}. \tag{2.14}$$

In the case where Δ is the Laplacian, K is simply the Green's function for the heat equation.

The trace of the heat kernel takes the form

$$K(s) = \operatorname{tr} e^{-s\Delta} = \int_M d^D x \sqrt{g} K(s, x, x), \tag{2.15}$$

and encodes information about the spectrum of Δ. Let us suppose that M is compact. Then the spectrum of Δ is bounded below, discrete, and without accumulation points, i.e. it is given by a non-decreasing sequence

$$\sigma(\Delta) = \{\lambda_1, \lambda_2, \cdots, \}. \tag{2.16}$$

In terms of the eigenvalues, the trace of the heat kernel is given by

$$K(s) = \mathrm{tr}\, e^{-s\Delta} = \sum_n e^{-s\lambda_n}. \tag{2.17}$$

Note that Eq. (2.17) can be viewed as the Laplace transform of the spectral measure

$$\mu(X) = \#\{\lambda_n : \lambda_n \in X\} \tag{2.18}$$

and therefore completely characterizes the spectrum whenever the inverse Laplace transform of $K(s)$ exists.

The heat kernel is a particularly useful representation of the spectrum of a differential operator because it can be used to relate properties of the spectrum to geometric invariants of M. In particular, these invariants are encoded in the asymptotic expansion of $K(s)$ as $s \to 0$:

$$K(s) \sim \frac{1}{(4\pi s)^{D/2}} \sum_{n \geq 0} a_n s^{n/2}. \tag{2.19}$$

The coefficients a_n are all integrals of local geometric invariants $a_n = \int d^D x \sqrt{g}\, a_n(x)$ depending on the metric and the matrix E that enters the definition of Δ (2.12). The first two nonzero coefficients are [58]

$$a_0(x) = \mathrm{tr}_V(\mathbb{1}), \tag{2.20}$$

$$a_2(x) = \mathrm{tr}_V(E + \frac{1}{6}R). \tag{2.21}$$

For manifolds without boundary, the odd coefficients vanish, which explains the absence of a_1. Note that s has dimensions of length^2, so the coefficient a_n has dimensions of length^{-n}. Thus the higher even heat kernel coefficients a_{2n} correspond to R^n type terms in the heat kernel, and hence in the effective action.

The partition function of a theory with kinetic operator $\Delta + m^2$ is given by

$$\ln Z = \frac{1}{2} \int_{\epsilon^2}^{\infty} ds \, \frac{e^{-m^2 s}}{s} K(s). \tag{2.22}$$

This equation holds in the limit $\epsilon \to 0$ up to terms that diverge in this limit. This can be seen by expanding $\ln Z$ and $K(s)$ as a sum over eigenvalues and using the identity

$$-\ln \lambda = \int_0^{\infty} \frac{e^{-\lambda s} - e^{-s}}{s} ds. \tag{2.23}$$

To obtain an expansion of $\ln Z$ in terms of local geometric quantities, we can substitute the asymptotic expansion of the heat kernel in (2.22). The contribution to the coefficient of R (the Einstein-Hilbert term) in $\ln Z$ is the integral

$$\frac{1}{2} \int_{\epsilon^2}^{\infty} ds \, \frac{e^{-m^2 s}}{(4\pi s)^{D/2}} \int d^D x \sqrt{g} a_2(x). \tag{2.24}$$

If the metric is the only background field, then $a_2(x) = c_1 R$ for some constant c_1. This contribution to the effective action can be written in the form of a local term in L_{eff} where $\ln Z = -\int \sqrt{g} d^D x L_{\text{eff}}$, given by

$$-\frac{c_1}{2} \int_{\epsilon^2}^{\infty} ds \, \frac{e^{-m^2 s}}{(4\pi s)^{D/2}} R. \tag{2.25}$$

The coefficient of R in the effective action is the difference between the effective $(16\pi G)^{-1}$ and the bare $(16\pi G)^{-1}$,

$$\delta \left(\frac{1}{16\pi G} \right) = \frac{c_1}{2} \int_{\epsilon^2}^{\infty} ds \, \frac{e^{-m^2 s}}{(4\pi s)^{D/2}}. \tag{2.26}$$

This corresponds to a contribution to the Bekenstein-Hawking entropy of

$$\delta S_{BH} = \frac{A}{4}\delta\left(\frac{1}{G}\right) = 2\pi c_1 \int_{\epsilon^2}^{\infty} ds\, \frac{e^{-m^2 s}}{(4\pi s)^{D/2}}. \tag{2.27}$$

When there are multiple non-interacting fields, their contribution to $1/G_{\text{eff}}$ and to S_{BH} are added.

Note that the expression for the effective action is only valid under analytic continuation, and the expression (2.22) may diverge, even if the partition function is not divergent. We illustrate this point with a simple example.

2.2.1 Example: harmonic oscillator

To illustrate the heat kernel method, we consider the simplest possible example of a $0+1$ scalar field theory (i.e. a harmonic oscillator). Here the partition function can be computed canonically as

$$Z = \operatorname{tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta m(n+1/2)} = \frac{e^{-\beta m/2}}{1-e^{-\beta m}}. \tag{2.28}$$

We can then write $\ln Z$ in a Taylor expansion valid for $\beta > 0$,

$$\ln Z = -\frac{\beta m}{2} + \sum_{n=1}^{\infty} \frac{e^{-\beta m n}}{n}. \tag{2.29}$$

Now consider the heat kernel evaluation of $\ln Z$. The operator in the heat kernel is $-\partial_t^2$ on a circle of circumference β. This has the eigenvalues

$$\lambda_n = \left(\frac{2\pi n}{\beta}\right)^2, \qquad n \in \mathbb{Z}. \tag{2.30}$$

The trace of the heat kernel can be computed from the formula (2.17)

$$K(s) = \sum_{n\in\mathbb{Z}} e^{-s\lambda_n} = \frac{\beta}{\sqrt{4\pi s}} \sum_{n\in\mathbb{Z}} e^{-\beta^2 n^2/4s} \tag{2.31}$$

where we have made use of the Poisson summation formula [2]. Eq. (2.31) can integrated term by term to obtain the partition function

$$\ln(Z) = \frac{1}{2}\int_{\epsilon^2}^{\infty} ds\, \frac{e^{-m^2 s}}{s} K(s) = \frac{\beta}{2\sqrt{4\pi}} \sum_n \int_{\epsilon^2}^{\infty} ds\ s^{-3/2} e^{-\beta^2 n^2/4s - m^2 s} \tag{2.33}$$

For $n \neq 0$ this integral is convergent as $\epsilon \to 0$, and is given by

$$\int_0^{\infty} ds\ s^{-3/2} e^{-\beta^2 n^2/4s - m^2 s} = \frac{\sqrt{4\pi} e^{-\beta m n}}{\beta n} \tag{2.34}$$

For $n = 0$ the integrand is of order $s^{-3/2}$ for small s, and hence diverges. This divergence is spurious, since we know the correct result (2.29) is finite. However we can can recognize that the integral looks like a representation of the Gamma function

$$\int_0^{\infty} dx\ x^{z-1} e^{-x} = \Gamma(z), \qquad \mathrm{re}(z) > 0 \tag{2.35}$$

and analytically continue to $z = -1/2$, giving[3]

$$\int_0^{\infty} dx\ x^{-3/2} e^{-x} = \Gamma(-1/2) = -\sqrt{4\pi}. \tag{2.36}$$

We therefore have (substituting (2.34) and (2.36) into (2.33)) we have

$$\ln Z = -\frac{\beta m}{2} + \sum_{n=1}^{\infty} \frac{e^{-\beta m n}}{n} \tag{2.37}$$

[2] The Poisson summation formula states that $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \tilde{f}(n)$, where $\tilde{f}$ is the Fourier transform of f in the "signal processing" convention

$$\tilde{f}(\nu) = \int dt e^{-2\pi i \nu t} f(t). \tag{2.32}$$

[3] The same result can be obtained by analytically continuing Eq. (2.34) to $n = 0$. We thank Ted Jacobson for pointing this out.

which agrees with the canonical result (2.29).

The moral of this story is that the heat kernel method may introduce divergences, even when computing perfectly finite quantities such as the partition function of a harmonic oscillator. It is only after subtracting these divergent parts (for example, via analytic continuation) that the correct results can be obtained.

2.3 Nonminimally coupled scalar field

An illustrative example of a field theory with a consistent contact term is the nonminimally coupled scalar field [64]. Its Lagrangian is

$$L = \frac{1}{2}\left(\nabla_a\phi\nabla^a\phi - m^2\phi^2 - \xi R\phi^2\right). \tag{2.38}$$

The partition function of this theory is related to the functional determinant of the operator $\Delta_0^\xi = (-\nabla^2 + \xi R)$. Following the formula (2.21), the second heat kernel coefficient is

$$a_2(x) = (1/6 - \xi)R(x) \tag{2.39}$$

i.e. $c_1 = 1/6 - \xi$. Note that for positive ξ the nonminimal coupling term antiscreens $1/G$, and for $\xi > 1/6$ there is net antiscreening[4].

The coefficient in the conical entropy divergence (2.27) is given by $c_1 = 1/6 - \xi$. The value of $1/6$ comes from the usual entanglement entropy divergence in S_{ent}, which to leading order in the curvature expansion is independent of ξ. As pointed out in Refs. [84, 85], there is also a contribution to the conical entropy from the

[4] In $D = 4$, a conformally coupled scalar field has $\xi = 1/6$, and therefore does not renormalize $1/G$. This apparent coincidence does not hold in $D \neq 4$.

Wald entropy (2.10),

$$\left\langle S_{\text{Wald}}^{(\phi)} \right\rangle = -2\pi\xi \int d^{D-2}x \sqrt{h} \left\langle \phi^2 \right\rangle . \tag{2.40}$$

The ξ-dependent divergence in c_1 is a contact term coming from the divergence in $\langle \phi^2 \rangle$. Divergences of this type are not associated with the entanglement entropy. Instead they correspond to particle loops that interact with the curvature at the conic singularity.[5] Although ϕ^2 is an intrinsically positive quantity, the coupling ξ can take either sign. For positive ξ, for which $1/G$ is antiscreened, the contribution of the nonminimal coupling to the entropy is negative.

We will now show explicitly how the Wald entropy term (2.40) appears as the ξ-dependent term in the conical entropy formula, using a representation of the heat kernel as a sum over single-particle paths. In this representation, the Wald term arises from paths that begin and end at the conical singularity.

Let K_S^ξ denote the heat kernel associated to the operator Δ_0^ξ. The heat kernel can be expressed as a Schwinger path integral over paths $x^a(s)$ through the Euclidean spacetime with s as the "time" parameter:

$$K_{\mathrm{S}}^{\xi}(s,x,y) = \int_{x(0)=x}^{x(s)=y} \mathcal{D}x e^{-\int_0^s ds' \frac{1}{4}\dot{x}^a(s')\dot{x}_a(s') + \xi R}. \tag{2.41}$$

To evaluate the entropy by the conical entropy formula, we are interested in the heat kernel for first-order variations of β away from 2π. The conical deficit introduces a

[5] This is not necessarily inconsistent with the hypothesis [48] that the horizon entropy ultimately comes entirely from entanglement entropy. It could be that the nonminimal coupling term is induced by entanglement at an even higher energy scale, as in the $O(N)$ model considered in Ref. [68].

singular curvature at the tip, given by [51]

$$R_{\text{tip}}(x) = 2(2\pi - \beta)\delta_\Sigma(x). \tag{2.42}$$

The Schwinger path integral depends on β through both the change of angular periodicity, and the introduction of curvature at the tip. To first order in $\beta - 2\pi$, these two contributions are independent and we can write the trace of the heat kernel as a sum of paths that do not interact with the singularity, and those that do [64]:

$$K_S^\xi(s) = K_S^\xi(s)|_{\partial_n \phi = 0} + K_{\text{tip}}(s). \tag{2.43}$$

The first term is the heat kernel with Neumann boundary conditions $\partial_n \phi = 0$ at the tip of the cone, whose contribution to S_{cone} is the entanglement entropy S_{ent} [47]. The second term is

$$K_{\text{tip}}(s) = \int d^D x \sqrt{g} \int_{x(0)=x}^{x(s)=x} \mathcal{D}x \int_0^s ds' \left(-\xi R_{\text{tip}}(x(s'))\right) e^{-\int_0^s ds'' \frac{1}{4}\dot{x}^a(s'')\dot{x}_a(s'') + \xi R} \tag{2.44}$$

$$= -2\xi(2\pi - \beta) \int d^D x \sqrt{g} \int_\Sigma d^{D-2} y \sqrt{h} \int_0^s ds' K_S^\xi(s', x, y) K_S^\xi(s - s', y, x) \tag{2.45}$$

$$= -2\xi(2\pi - \beta) \int_\Sigma d^{D-2} y \sqrt{h}\, s\, K_S^\xi(s, y, y). \tag{2.46}$$

where we have used the heat kernel identity

$$K_S^\xi(s, x, x) = \int_M d^D y \sqrt{g} K_S^\xi(s', x, y) K_S^\xi(s - s', y, x). \tag{2.47}$$

The contribution to the effective action is

$$\ln Z_{\text{tip}} = \frac{1}{2}\int_{\epsilon^2}^{\infty} ds\,\frac{e^{-m^2 s}}{s} K_{\text{tip}}(s) \tag{2.48}$$

$$= -\xi(2\pi-\beta)\int_\Sigma d^{D-2}y\sqrt{h}\int_{\epsilon^2}^{\infty} ds\, e^{-m^2 s} K^{\xi}_{\text{S}}(s,y,y). \tag{2.49}$$

We can identify in this last expression the expectation value of ϕ^2 in heat kernel regularization:

$$\left\langle \phi^2(y) \right\rangle = \int_{\epsilon^2}^{\infty} ds\, e^{-m^2 s} K_S(s,y,y). \tag{2.50}$$

We therefore have

$$\ln Z_{\text{tip}} = -\xi(2\pi-\beta)\int_\Sigma d^{D-2}y\sqrt{h}\left\langle \phi(y)^2 \right\rangle, \tag{2.51}$$

and the contribution to the conical entropy is

$$S_{\text{tip}} = (1-\beta\partial_\beta)\ln Z_{\text{tip}}\big|_{\beta=2\pi} = -2\pi\xi\int_\Sigma d^{D-2}y\sqrt{h}\left\langle \phi(y)^2 \right\rangle. \tag{2.52}$$

This is precisely the same as the expectation value of the scalar contribution to the Wald entropy (2.9). So we see that the conjecture (2.10) holds in the case of the nonminimally coupled scalar field.

2.4 Maxwell field

It is tempting to interpret the Maxwell contact term in the same way, as a contribution coming from the Wald entropy, just as in the case of the nonminimally coupled scalar field. We will see that this interpretation gives the correct value for the contact term, but that there are apparent problems with gauge invariance.

In Ref. [61], the thermal entropy of Maxwell fields in Rindler space was obtained from the partition function Z on the cone. The Euclidean action for the

Maxwell field includes (fermionic-scalar) ghosts and a gauge fixing term. In Feynman gauge:

$$I = \int d^D x \sqrt{g} \left[\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} (\nabla_a A^a)^2 - \bar{c}\, \nabla^2\, c \right]. \tag{2.53}$$

To express the one-loop effective action as a determinant, we integrate by parts:

$$I = \int d^D x \sqrt{g} \left[-\frac{1}{2} A^a (g_{ab} \nabla^2 - R_{ab}) A^b - \bar{c}\, \nabla^2\, c \right]. \tag{2.54}$$

The gauge-fixed vector field A_a has D degrees of freedom, while the two Faddeev-Popov ghosts c and $\bar{c}$ each represent -1 degrees of freedom.

Note that the ghosts c and $\bar{c}$ do not couple to the gauge field A_a. Because the ghosts decouple, they do not influence any scattering processes in electrodynamics and so are usually neglected. However, we are interested in the dependence of the partition function on the background geometry, which depends on the coupling of the fields to gravity. Since the ghosts couple to gravity, they cannot be neglected in calculating the conical entropy. The ghosts are also important when considering field theory at finite temperature, for a similar reason: in that case the ghost determinant depends on the inverse temperature β, which is the periodicity of the imaginary time coordinate.

The partition function can then be expressed as a functional determinant

$$\ln Z = -\frac{1}{2} \ln \det \Delta_1 + \ln \det \Delta_0. \tag{2.55}$$

where

$$\Delta_0 = -\nabla^2, \qquad \Delta_1 = -g_{ab} \nabla^2 + R_{ab}. \tag{2.56}$$

The ghosts are minimally coupled and so do not contribute a contact term. The operator Δ_1 is the Laplace-de Rham operator, and it differs from the Laplace-Beltrami operator ∇^2 by a Weitzenböck term proportional to the Ricci tensor [86]. We will now show how this curvature dependence is what leads to the contact term.

The manipulations leading to Eq. (2.52) can be repeated for any theory in which the action depends on the Riemann tensor. On the conical manifold, the singular part of the Riemann tensor is given by [51]

$$R^{\text{tip}}_{abcd}(x) = (2\pi - \beta)\epsilon_{ab}\epsilon_{cd}\delta_\Sigma(x). \tag{2.57}$$

The calculation proceeds much as in the case of the nonminimally coupled scalar field. The result is that

$$S_{\text{tip}} = -2\pi \int_\Sigma d^{D-2}x\sqrt{h}\left\langle \frac{\partial L}{\partial R_{abcd}} \right\rangle \epsilon_{ab}\epsilon_{cd}. \tag{2.58}$$

Here the ambiguity of the Wald entropy formula under integration by parts has been fixed by the need to express the Lagrangian in the form $A\Delta_1 A$ to employ the heat kernel method.

Calculating the Wald entropy (2.9) by differentiating the action (2.54) with respect to the curvature, we obtain [87]

$$S^{(A)}_{\text{Wald}} = -\pi \int_\Sigma d^{(D-2)}x\,\sqrt{h}\,A_a A_b g^{ab}_\perp, \tag{2.59}$$

where $g^{ab}_\perp$ is the inverse metric projected onto the directions perpendicular to the horizon. The expectation value of Eq. (2.59) is the same as the contribution of two nonminimally coupled scalars with $\xi = 1/2$ [69]. This gives a contribution $c_1 = -1$, in exact agreement with the contact term. Thus at first sight it appears that the antiscreening of Newton's constant can be explained physically through a divergence in the Wald entropy.

But this interpretation is problematic, because the A^2 term is not gauge-invariant. This lack of gauge invariance is just a consequence of the gauge-fixing of the action (2.54). However an avatar of the original gauge invariance remains in the form of the fermionic BRST symmetry $\mathbf{s}$ relating the ghosts to the unphysical vector modes:

$$\mathbf{s}A_a = \nabla_a c, \qquad \mathbf{s}\bar{c} = \nabla_a A^a, \qquad \mathbf{s}c = 0. \tag{2.60}$$

BRST symmetry guarantees that the expectation values of BRST-invariant operators are independent of the choice of gauge. The operator appearing in Eq. (2.59) is not BRST-invariant, but instead transforms as

$$\mathbf{s}(A_a A_b g_\perp^{ab}) = 2A_a\, \nabla_b c\, g_\perp^{ab}. \tag{2.61}$$

This may help explain the results of Ref. [88], where it was found that the contact term depends on the parameter ξ of the R_ξ gauge using ζ-function regularization in $D = 4$ (though not in $D = 2$).

The A^2 term (2.59) is an ambiguity of the type discussed in Ref. [54] in the definition of the Wald entropy, since it can be removed by adding a total derivative to the Lagrangian; the Wald entropy (2.9) of the original Maxwell Lagrangian $\frac{1}{4}F_{ab}F^{ab}$ vanishes. Classically, ambiguity terms such as $A_a A_b g_\perp^{ab}$ vanish on the Killing horizon for stationary field configurations. The requirement that A be stationary at the bifurcation surface is

$$0 = (\mathcal{L}_\xi A)^b = \xi^a \nabla_a A^b - A^a \nabla_a \xi^b = -A^a \epsilon_a{}^b \tag{2.62}$$

where we have used the fact that $\xi = 0$ and $\nabla_a \xi_b = \epsilon_{ab}$ at the bifurcation surface. The binormal ϵ_{ab} is related to $g_\perp$ by

$$g^\perp_{ab} = -\epsilon_{ac}\epsilon^c{}_b, \tag{2.63}$$

so the Wald entropy term is proportional to

$$g^\perp_{ab} A^a A^b = -\epsilon_{ac}\epsilon^c{}_b A^a A^b = 0. \tag{2.64}$$

Hence the Wald entropy vanishes because the Killing flow acts as a boost near the bifurcation surface, and the requirement that the vector A be boost-invariant constrains it to lie in the plane of the horizon. However the requirement that A be boost-invariant is not gauge-invariant. In the quantum theory this ambiguity term can (and does) have a nonzero expectation value.

Finally, we point out that since Maxwell fields (coupled to general relativity) satisfy the null energy condition, there is a classical second law in which the horizon entropy is given by the Bekenstein-Hawking area term alone. The addition of Eq. (2.59) to the entropy seems likely to spoil this result. This is in contrast with the nonminimally coupled scalar field, for which the inclusion of the Wald entropy term $-2\pi\xi\phi^2$ is necessary for the classical second law [67].

Chapter 3

Entropy of gauge fields in two dimensions

The arguments of chapter 2 suggest an interpretation of the contact term of electrodynamics as a non-gauge-invariant form of contribution of the gauge field to the Wald entropy. Obviously physics must be gauge-invariant, so this raises the question of what went wrong in the calculation of the entropy. We therefore revisit the original calculation of the contact term [61]. This calculation took place on a two-dimensional cone; so we concentrate on the case of two dimensions. The contact term on a two-dimensional cone is found to depend sensitively on the way in which the ultraviolet and infrared are regulated. We consider a regularization in which the Euclidean spacetime is made compact and the conical singularity is smoothed out. Much as the entanglement entropy can be viewed either as part of the Wald entropy or as entanglement entropy depending on scale at which the theory is described, we find that the contact term, which was described in chapter 2 as a term in the Wald entropy also has an interpretation as negative entanglement entropy coming from the zero modes of ghosts. Thus like the interpretation in terms of the Wald entropy, the entanglement entropy description of the contact term does not have a gauge-invariant description. In section 3.2.2, we review BRST symmetry, whose role is to ensure cancellation between unphysical states, and show how BRST symmetry can fail to cancel all negative norm states. We argue that this is further evidence the contact term in two dimensions is unphysical.

Fortunately, two-dimensional electrodynamics is sufficiently simple that it can be quantized exactly without the need for ghosts. This is carried out in section 3.3, and we find no sign of the contact term; the entropy defined by the conical method is positive and equal to the entanglement entropy. This quantization generalizes easily to Yang-Mills theory with any compact gauge group. The only difference in the non-abelian case is a new contribution reminiscent of the contact term; however, unlike the contact term, this new term is positive and has a clear statistical interpretation as part of the entanglement entropy.

3.1 Derivation of the Kabat contact term

We now summarize the calculation of Ref. [61] that led to the puzzling contact term of electromagnetism.

Recall that the partition function can be expressed as a functional determinant

$$\ln Z = -\frac{1}{2}\ln\det\Delta_1 + \ln\det\Delta_0. \tag{3.1}$$

where

$$\Delta_0 = -\nabla^2, \qquad \Delta_1 = -g_{ab}\nabla^2 + R_{ab}. \tag{3.2}$$

To compute this functional determinant (3.1) we first compute the scalar heat kernel K_S. Let ϕ_n be a complete set of modes for Δ_0:

$$-\nabla^2\phi_n = \lambda_n\phi_n. \tag{3.3}$$

The heat kernel is given by

$$K_S(s,x,y) = \sum_n e^{-s\lambda_n}\phi_n(x)\phi_n(y). \tag{3.4}$$

Although we have written the heat kernel in the case of a discrete spectrum, the results generalize naturally to the case of continuous spectrum.

To compute the heat kernel of the vector Laplacian, we construct a complete set of eigenfunctions of the operator Δ_1:

$$(-g_{ab}\nabla^2 + R_{ab})A^b = \lambda_n A^a, \tag{3.5}$$

and define the vector heat kernel

$$K_V(s,x,y)_{ab} = \sum_n e^{-s\lambda_n}A_{na}(x)A_{nb}(y). \tag{3.6}$$

In two dimensions, the vector modes can be written in terms of the scalar eigenfunctions as

$$\frac{1}{\sqrt{\lambda_n}}\nabla_a\phi_n, \qquad \frac{1}{\sqrt{\lambda_n}}\epsilon_{ab}\nabla^b\phi_n. \tag{3.7}$$

The vector heat kernel at coincident points and with the vector indices contracted can be expressed in terms of the scalar heat kernel as

$$\begin{aligned}
K_V(s,x,x)^a_a &= \sum_n \frac{e^{-s\lambda_n}}{\lambda_n}[2\nabla_a\phi_n\nabla^a\phi_n] && (3.8)\\
&= \sum_n \frac{e^{-s\lambda_n}}{\lambda_n}[-2\phi_n\nabla^2\phi_n + 2\nabla_a(\phi_n\nabla^a\phi_n)] && (3.9)\\
&= \sum_n e^{-s\lambda_n}[2\phi_n^2 + \frac{1}{\lambda_n}\nabla^2(\phi_n^2)] && (3.10)\\
&= 2K_S(s,x,x) + \int_s^\infty ds'\nabla^2 K_S(s',x,x). && (3.11)
\end{aligned}$$

In the heat kernel regularization, the partition function (3.1) is given by

$$\ln Z = \frac{1}{2}\int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{s} K_M(s) \tag{3.12}$$

where the mass m is an infrared regulator. Here we have defined $K_M(s)$ as the trace of the "Maxwell heat kernel":

$$K_M(s) = \int d^2x\sqrt{g}\,(K_V(s,x,x) - 2K_S(s,x,x)) \tag{3.13}$$

$$= \int d^2x\sqrt{g}\int_s^{\infty} ds' \nabla^2 K_S(s',x,x) \tag{3.14}$$

using (3.11) in the last line. It seems tempting to move the integration with respect to s past the Laplacian, turning this expression into the integral of a total derivative. However this would not be valid as K_S behaves as $1/s$ for large s, so the s integral would be ill-defined.

To evaluate $K_M(s)$ on a cone of angle β, Kabat exploits rotation and scale symmetry to write the scalar heat kernel in the form

$$K_S(s',x,x) = f(r^2/s')/s' \tag{3.15}$$

so that Eq. (3.14) becomes

$$K_M(s) = \beta\int rdr\int_s^{\infty} ds' \frac{1}{r}\partial_r r\partial_r s' f(r^2/s'). \tag{3.16}$$

From the formula 3.15, the r-derivatives can be traded for s-derivatives using the formula $r\partial_r K = -2\partial_s(sK)$. Both integrals can then be carried out, yielding

$$K_M(s) = -2\beta f(r^2/s')\Big|_{s'=s}^{s'=\infty}\Big|_{r=0}^{r=\infty}. \tag{3.17}$$

When $r^2 \gg s$, the heat kernel on a cone takes the same form as on the plane, $f(r^2/s) \approx \frac{1}{4\pi}$. When $r^2 \ll s$, the heat kernel is very sensitive to the conical singularity, and $f(r^2/s) \approx \frac{1}{2\beta}$. In Eq. (3.17), there are two contributions from $r = 0$ that each give $\frac{1}{2\beta}$, and a contribution from $r = \infty$, $s' = s$ that yields $\frac{1}{4\pi}$. But there is also a contribution from $s = \infty$, $r = \infty$ that depends on the order in which the limits are taken. If we take the limit $s \to \infty$ first, we find the same result as Kabat:

$$K_M(s) = -\frac{1}{2\pi}(2\pi - \beta) = -\frac{1}{4\pi}\int \sqrt{g}R. \tag{3.18}$$

If instead we take the $r \to \infty$ limit first, the Maxwell heat kernel vanishes identically, and we obtain no contact term.

The partition function associated to (3.18) is given by

$$\ln Z = -\frac{1}{4\pi}(2\pi - \beta)\int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{s}, \tag{3.19}$$

from which we easily find the entropy using (1.10):

$$S_{\text{cone}} = -2\pi \int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{4\pi s}. \tag{3.20}$$

This corresponds to a term $c_1 = -1$ in the conical entropy (2.27).

We note that the Maxwell heat kernel on the cone (3.18) is independent of s. Thus it enters the heat kernel in the same way as $(\beta/2\pi - 1)$ zero modes would, although this coefficient is not in general an integer. This suggests that the calculation may depend on the way that zero modes are handled. Indeed, the dependence on the order of limits $s \to \infty$ and $r \to \infty$ shows that the calculation is sensitive to the far infrared. Taking the limit $s \to \infty$ first corresponds to allowing paths a sufficient amount of Schwinger proper time to probe the boundary at large r.

In section 3.2 we will repeat the contact term calculation on a smooth compact space without boundary or singularities. We will see that the contact term does indeed arise from zero modes.

3.1.1 Extension to higher dimensions

We now see how the contact term in higher dimensions comes from the two-dimensional contact term. We consider a spacetime that is a product of a two-dimensional cone M and a manifold N that we will take to be a flat $D-2$ torus. The Maxwell heat kernel of the product spacetime is

$$K_M(M \times N) = K_V(M \times N) - 2K_S(M \times N) \tag{3.21}$$

where we have changed the notation slightly to make the dependence on the manifold explicit, but suppressed the dependence on s. The heat kernel of a product manifold is given by

$$K_S(M \times N) = K_S(M)K_S(N), \tag{3.22}$$

$$K_V(M \times N) = K_V(M)K_S(N) + K_S(M)K_V(N). \tag{3.23}$$

The equation for the vector field reflects the fact that a vector on $M \times N$ can be decomposed into a component pointing along M and a component along N. Because N is a torus, the vector simply acts like $D-2$ scalars: $K_V(N) = (D-2)K_S(N)$. We therefore have

$$K_M(M \times N) = K_M(M)K_S(N) + (D-2)K_S(M \times N). \tag{3.24}$$

Thus the Maxwell heat kernel in higher dimensions is just the sum of the two dimensional Maxwell heat kernel (which consists entirely of the contact term) and $(D-2)$ scalars that describe the propagating degrees of freedom of the Maxwell field. Thus the contact term in $D > 2$ arises directly from the two-dimensional contact term.

3.2 2D Maxwell theory on a compact spacetime

Because Kabat derived the contact term on a manifold with a conical singularity and a boundary at infinity, one might wonder whether the result comes from the improper treatment of these boundaries. To show that this is not the case, in this section we will re-derive the Kabat contact term for smooth compact orientable two-dimensional Euclidean manifolds. However the interpretation is different: the contact term in the entropy arises from zero modes of ghosts, explaining its negative sign.

When calculating the entropy on smooth manifolds, we will replace the conical singularity with a smooth cap, smearing out the curvature over some finite radius r_0 [77, 51, 89]. Because of approximate translation symmetry near the horizon, to first order in the angle deficit $2\pi - \beta$ and in the limit that $r_0 \to 0$, the heat kernel does not depend on the details of the smoothing. Formally therefore, the replacement of the conical singularity with the smoothed tip should have no consequences, and indeed this is what we will find.

To compute the effective action, we use the trace of the Maxwell heat kernel (3.13)

$$K_M(s) = \mathrm{tr}(e^{-s\Delta_1}) - 2\,\mathrm{tr}(e^{-s\Delta_0}). \tag{3.25}$$

Both operators Δ_0 and Δ_1 are cases of the Hodge Laplacian acting on p-forms:

$$\Delta_p = d\delta + \delta d \tag{3.26}$$

where d is the exterior derivative and δ is the codifferential.

By the Hodge decomposition, any $1-$form A can be expressed as

$$A = d\phi + \delta\psi + B \tag{3.27}$$

where ϕ is a 0-form (scalar), ψ is a 2-form and B is a harmonic 1-form i.e. $\Delta_1 B = 0$. By Eq. (3.27), the spectrum of Δ_1 is the union of the spectrum of Δ_0 and Δ_2 up to zero modes. Moreover, by Hodge duality, the spectra of Δ_0 and Δ_2 are equivalent on orientable manifolds. In terms of the heat kernels, this implies that

$$K_V(s) = 2K_S(s) + b_1 - b_0 - b_2 \tag{3.28}$$

where $b_p = \dim \Delta_p$ is p^{th} Betti number, which counts the number of $p-$form zero modes. On a connected orientable manifold, $b_0 = b_2 = 1$ and b_1 is twice the genus, so we have

$$K_V(s) = 2K_S(s) - \chi \tag{3.29}$$

where $\chi = b_0 - b_1 + b_2$ is the Euler characteristic.

Subtracing the two scalar ghosts from the vector heat kernel (3.29), we find the Maxwell heat kernel

$$K_M(s) = -\chi = -\frac{1}{4\pi}\int d^2x\sqrt{g}R. \tag{3.30}$$

where we have used the Gauss-Bonnet theorem. Now note the similarity between this result and Kabat's result for the cone: the right-hand side of Eq. (3.30) and Eq. (3.18) are the same.

To find $\ln Z$ in terms of the heat kernel, we again introduce an ultraviolet cutoff length ϵ and an infrared regulating mass m,

$$\ln Z = \frac{1}{2}\int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{s} K_M(s) \tag{3.31}$$

$$= -\int \sqrt{g}\left(\frac{1}{2}\int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{4\pi s} R\right). \tag{3.32}$$

By comparison with equation (2.25), this corresponds to $c_1 = -1$ in the effective action.

We have therefore confirmed the presence of the contact term for compact manifolds. But more importantly, we have elucidated the origin of the contact term: it arises from the difference in the number of degrees of freedom in the vector zero modes as compared to the ghost zero modes.[1]

As an example, let us consider the effective action on de Sitter space dS_2, for which the Euclidean geometry is the sphere S_2. All modes cancel except for the two zero modes of the ghosts c and $\bar{c}$, since there are no vector zero modes on the sphere. These modes are ghosts and contribute negatively to the entropy $S = (1 - \beta\partial_\beta)\ln Z$. The β variation of the sphere (which corresponds to deforming

[1]One may wonder how these zero modes could possibly give rise to a logarithmic divergence in $1/G$, considering that a finite number of modes cannot give rise to an ultraviolet divergence. The explanation is that when taking the determinant of a dimensionful operator Δ, one must insert a dimensionful parameter μ so that the partition function $Z = \det(\mu^{-2}\Delta)$ is dimensionless. Although conceptually μ has no necessary relationship to an ultraviolet cutoff Λ on short-distance modes, since both μ and Λ are dimensionful parameters needed to make the path integral well defined, one may as well identify $\Lambda = \mu$. In any case both parameters must be varied in order to perform an RG flow. The distinction between these two conceptually distinct reasons for renormalization is obscured by the heat kernel regulator ϵ, which does not distinguish between them.

it into a "football" with two smoothed out conical caps) vanishes, because Z depends only on the topology, not β. This leaves only the $\ln Z$ term, which is negative.

Thus we rederive the contact term, but now the interpretation is that it comes from negative entanglement entropy due to ghosts. But this should immediately arouse suspicion! The sole purpose of ghosts is to cancel out unphysical vector modes, and the ghost zero modes are extra fields which are not associated with any such vector modes.

A similar calculation was carried out in Ref. [65] in which all vector and ghost zero modes were omitted, leading to a trivial partition function $Z = 1$. This prescription removes the contact term; however it neglects nonperturbative contributions to Z that will be considered in section 3.3.

3.2.1 Problems with the naïve Maxwell heat kernel

Let us go back to the original justification for the ghosts. In the Faddeev-Popov trick, one takes a path integral of the form

$$\int \mathcal{D}A\, e^{-S(A)}, \tag{3.33}$$

and inserts the "identity"

$$\int \mathcal{D}\alpha\, \delta(G(A^\alpha)) \det \left[\frac{\delta G(A^\alpha)}{\delta \alpha}\right] = 1. \tag{3.34}$$

Here $G(A) = \nabla_a A^a$ is the Lorenz gauge-fixing condition, and $A^\alpha = A + \nabla\alpha$. This assumes that for every A there is exactly one α such that $G(A^\alpha) = \nabla_a A^a + \nabla^2\alpha = 0$. However if α satisfies this condition, then so does $\alpha + c$ where c is a spacetime constant. This means that we should integrate over equivalence classes of functions α under the relation $\alpha \sim \alpha + c$, in other words the determinant in Eq. 3.34 should not include zero modes; hence the ghost zero modes are spurious.

On manifolds with handles, the vector zero modes must also be treated with great care. If the gauge group of the Maxwell theory is $\mathbb{R}$, there will be infrared divergences coming from these winding modes. For a U(1) gauge field, this infrared divergence is replaced with an integral over the moduli space of flat connections, which has finite volume. These zero modes must be exluded from the one-loop determinant and handled separately. It is also necessary to sum over nontrivial U(1) bundles.

3.2.2 BRST symmetry

There is an additional problem that the BRST state space is not the same as the physical Hilbert space of the canonical Maxwell theory, but contains extra degrees of freedom with negative norm states. To see how this arises, we recall some aspects of BRST symmetry [90]. Recall that the Lagrangian is invariant under the BRST symmetry

$$\mathbf{s}A_a = \nabla_a c, \qquad \mathbf{s}\bar{c} = \nabla_a A^a, \qquad \mathbf{s}c = 0. \tag{3.35}$$

Note that $\mathbf{s}$ is a nilpotent fermionic symmetry:

$$\mathbf{s}^2 = 0, \qquad \mathbf{s}(AB) = (\mathbf{s}A)B + (-1)^{f_A} A(\mathbf{s}B), \tag{3.36}$$

where f_A is the fermion number of A. It is also invariant under the bosonic ghost number symmetry $\mathbf{g}$, under which c carries ghost number 1 and $\bar{c}$ carries ghost number -1,

$$\mathbf{g}c = c, \qquad \mathbf{g}\bar{c} = -\bar{c}, \qquad \mathbf{g}A_a = 0. \tag{3.37}$$

Associated to the symmetries $\mathbf{s}$ and $\mathbf{g}$ are conserved Noether charges denoted Q and G and satisfying

$$[Q, \Phi] = i\mathbf{s}\Phi, \qquad [G, \Phi] = i\mathbf{g}\Phi \tag{3.38}$$

where here $[A, B]$ denotes the graded commutator: it is the anticommutator if A and B are both fermionic, otherwise it is the commutator. The nilpotence of the symmetry $\mathbf{s}$ implies that $Q^2 = 0$. Since the symmetry $\mathbf{s}$ always increases the ghost number by one, we have $[G, Q] = Q$.

To define (pseudo-)unitary representations of this algebra, we also need to impose self-adjointness relations on the fields. Following Refs. [91, 92], we make the choice in which c is Hermitian and $\bar{c}$ is anti-Hermitian:

$$c^\dagger = c, \qquad \bar{c}^\dagger = -\bar{c}. \tag{3.39}$$

This leads to a Hermitian Q, and an anti-Hermitian G. We therefore have a representation of the BRST charge-ghost number algebra:

$$Q^2 = 0, \qquad [G, Q] = Q, \qquad Q = Q^\dagger, \qquad G = -G^\dagger. \tag{3.40}$$

The space of states is not positive definite, and contains a large number of negative norm states. The physical Hilbert space must be a subspace on which the inner product is positive-definite. Since Q is Hermitian and nilpotent, Q restricted to this subspace must be zero. We therefore consider a subspace of $\ker Q$. Furthermore, the fact that Q is Hermitian implies that $\operatorname{im} Q$ is orthogonal to $\ker Q$, and hence consists of null vectors. This leads us to consider the BRST cohomology:

$$\mathcal{H}_{\mathrm{BRST}} = \ker Q / \operatorname{im} Q. \tag{3.41}$$

It is not guaranteed that this space has a positive-definite inner product, but when it does, it defines a suitable Hilbert space of physical states.

The irreducible representations of this algebra are completely classified [93]. These will be expressed by giving matrix representations of Q and G, as well as the matrix J that defines the non-positive-definite inner product $\langle \cdot | \cdot \rangle$ via

$$\langle a | b \rangle = (a | J | b) \tag{3.42}$$

where $(\cdot|\cdot)$ is the standard Euclidean inner product. The irreducible pseudounitary representations are given by:

- **Singlet**

$$Q = 0, \quad G = 0, \quad J = \pm 1. \tag{3.43}$$

- **Non-null doublet**

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.44}$$

  where $k \in \mathbb{Z}/2$, $k \neq 0$.

- **Null doublet at ghost number $\pm\frac{1}{2}$**

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad J = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.45}$$

- **Quartet**

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} k-1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 1-k \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.46}$$

where $k \in \mathbb{Z}/2$.

When passing to the cohomology, we see that the quartet and null doublet representations drop out, but the singlets and non-null doublets survive. Thus the cohomology has a positive inner product if and only if the representation is free of negative-norm singlets and non-null doublets.

As a consequence of BRST symmetry, the partition function graded by ghost number,

$$Z = \mathrm{tr}(-1)^G e^{-\beta H}, \tag{3.47}$$

is the same whether calculated in the theory with ghosts, or on the BRST cohomology. If the inner product on the cohomology space is positive-definite, then the partition function can be interpreted as a canonical partition function of a unitary theory.

To illustrate why the ghosts do not cancel in the case of Maxwell theory we now consider the simplest possible example.

3.2.3 One-dimensional Maxwell theory

To understand why the naive prescription

$$\ln Z_{\text{Maxwell}} = \ln Z_{\text{Vector}} - 2 \ln Z_{\text{Scalar}} \tag{3.48}$$

fails, we consider Maxwell theory in zero spatial dimensions and one Euclidean time dimension with period β. This describes Maxwell theory on a point at finite temperature - a completely trivial theory. In one dimension the partition function for the scalar and the vector field are the same, so the naive prescription leads to

$$\ln Z_{\text{Maxwell}} = -\ln Z_{\text{Scalar}}. \tag{3.49}$$

The degrees of freedom are the vector potential A_0, and the ghosts $c, \bar{c}$.

In order to avoid infrared divergences, we need to introduce a regulator such as a mass, as was done in the heat kernel calculation. Of course the presence of a mass breaks BRST symmetry, but one may hope that when we take $m \to 0$ the symmetry is restored in such a way as to make all unphysical contributions to the partition function cancel in the limit. However, we will see that this is not the case.

Regulating with a mass, states of this theory can be labelled by an integer occupation number for A_0 and occupation numbers in $\{0, 1\}$ for the ghosts. The action of the BRST charge operator Q is to annihilate a quantum of A_0 and to increase the ghost number by one. The kernel of Q consists of those states with zero occupation number in A_0, or ghost number one, and modding out by the image of Q leaves two states: a state of energy $m/2$ and a state of energy $-m/2$ differing by one unit of ghost number. The partition function is therefore

$$Z_{\text{Maxwell}} = e^{\beta m/2} - e^{-\beta m/2} = 2\sinh(\beta m/2) = Z_{\text{Scalar}}^{-1} \tag{3.50}$$

which is in complete agreement with the naive prescription. However as $m \to 0$, this gives $Z \to 0$ and hence $\ln Z \to -\infty$; our apparently trivial theory has a negative divergence in the effective action.

But it is clear what has gone wrong - the Hilbert space $\ker Q/\operatorname{im} Q$ contains two states of differing ghost number. One of these states must have non-zero ghost number, and so is clearly unphysical. By projecting out the ghost state, we arrive at a theory with the one-dimensional Hilbert space and a partition function

$$Z_{\text{Maxwell}} = e^{\beta m/2} \tag{3.51}$$

which is the correct partition function for a trivial theory and gives $Z = 1$ as $m \to 0$.

This problem also arises on any spatially compact manifold, but for specificity consider a static Lorentzian manifold (of any dimension) taking the form $\Sigma \times \mathbb{R}_{\text{time}}$. Let q be the determinant of the spatial metric, and t be the time coordinate on $\mathbb{R}$. On any $t =$ const. time slice, the following pair of canonically-conjugate spatially constant ghost modes

$$c_0 = \int_\Sigma d^{D-1}x\sqrt{q}\, c, \quad \dot{\bar{c}}_0 = \int_\Sigma d^{D-1}x\sqrt{q}\, \frac{d\bar{c}}{dt} \tag{3.52}$$

are BRST-trivial, i.e. they are not paired by BRST symmetry with any other modes. This means that the BRST-trivial ghost modes remain in the "physical" state space despite the fact that they include negative norm states, and do not correspond to any modes in the canonical Maxwell theory. These spurious degrees of freedom are similar to the extra ghosts that arise when BRST-quantizing the zero mode of a string, and which are normally cured by imposing Siegel gauge [94] (see also the discussion in section 4.2 of Ref. [95]).

These problems invalidate the "vector minus two scalars" derivation of the contact term. Rather than try to resolve these issues here, we will instead quantize the two-dimensional theory using the reduced phase space of gauge-invariant canonical degrees of freedom. We will see that the contact term is absent.

3.3 Reduced phase space quantization

Two dimensional Maxwell theory has no local degrees of freedom, but there are still global degrees of freedom. In fact, the system is exactly solvable without the introduction of gauge fixing or ghosts (for a review see Ref. [96]).

On a 2-dimensional orientable Euclidean manifold the Maxwell action is

$$I = \int d^2x \sqrt{g} \frac{1}{2} F^2, \tag{3.53}$$

where we define $F = \frac{1}{2\sqrt{g}} F_{ab} \epsilon^{ab}$. In order to perform a canonical analysis, we will start by assuming that the manifold can be foliated by circles (i.e. is a sphere or torus); this assumption will be lifted at the end of this section.

At a fixed time, the configuration degrees of freedom are the gauge equivalence classes of a U(1) connection A_a on the circle. These equivalence classes are parameterized by a single degree of freedom, the Wilson loop around the circle,

$$A = \oint A_a dx^a. \tag{3.54}$$

Note that the action (3.53) depends on the metric only via the volume form. By choosing a coordinate $x \in [0, 1]$ parameterizing the circle, and a coordinate t that measures the elapsed spacetime volume, the volume element becomes $\sqrt{g} d^2x = dtdx$. Then we can reduce the phase space by imposing Coulomb gauge, in which A_x is constant, and $A_t = 0$. Maxwell theory in two dimensions then reduces to the free particle with Hamiltonian

$$H = \frac{1}{2}E^2 \tag{3.55}$$

where the electric field E is canonically conjugate to A:[2]

$$\{A, E\} = 1. \tag{3.56}$$

All the relevant information about the manifold is encoded in the total volume V, and the boundary conditions imposed on A at $t = 0$ and $t = V$.

To quantize the theory, we simply replace the Poisson brackets by commutators, giving a free particle. For a theory with gauge group $\mathbb{R}$, A can take any real value, but for a U(1) gauge theory A is periodic:

$$A \sim A + \frac{2\pi}{q}, \tag{3.57}$$

where q the minimal charge, so the free particle lives on a circle. This implies that the electric field is quantized as

$$E \in q\mathbb{Z}. \tag{3.58}$$

To compute the partition function, we first need to specify the topology of the Euclidean manifold, which determines the boundary conditions for A. We first consider the torus, for which the appropriate boundary conditions are the periodic ones:

$$A(0) = A(V). \tag{3.59}$$

[2] Although on-shell $E = F$, off-shell it is important to distinguish between the momentum E and the velocity F. The former is conserved and the latter fluctuates.

The partition function is

$$Z = \operatorname{tr} e^{-VH} = \sum_{E \in q\mathbb{Z}} e^{-\frac{1}{2}VE^2}. \tag{3.60}$$

We can also compute Z by the Euclidean path integral. Because the action is quadratic, we can factor the partition function into a sum over classical paths times a contribution from fluctuations about the classical paths,

$$Z = \sum_{n \in \mathbb{Z}} e^{-S[A_n]} \times Z_{\text{fluctuations}} \tag{3.61}$$

where A_n is the classical path that wraps around the circle with winding number n:

$$A(t) = \frac{2\pi n}{qV} t, \qquad F = \frac{2\pi n}{qV} \tag{3.62}$$

which is the familiar quantization of magnetic flux. The fluctuations can be calculated from the Euclidean free particle propagator on the plane,

$$U(\Delta x, \Delta\tau) = \sqrt{\frac{1}{2\pi\Delta\tau}} e^{-(\Delta x)^2/2\Delta\tau}, \tag{3.63}$$

yielding

$$Z_{\text{fluctuations}} = \int_0^{2\pi/q} dA\, U(0, V) = \sqrt{\frac{2\pi}{q^2 V}}. \tag{3.64}$$

Combining this result with the classical action, the partition function is

$$Z = \sqrt{\frac{2\pi}{q^2 V}} \sum_{F \in (2\pi/qV)\mathbb{Z}} e^{-\frac{1}{2}VF^2}. \tag{3.65}$$

While the formulae for the partition function Eq. (3.60) and Eq. (3.65) have a similar form, the quantization of E (electric quantization) and of F (magnetic quantization) are completely different in nature. The electric quantization condition is quantum kinematical effect arising from the finite radius of the circle, whereas the magnetic quantization condition is a classical topological result that makes use of the equation of motion. Nevertheless, Eq. (3.60) and Eq. (3.65) can be shown to be equal by the Poisson summation formula.

When the spacetime manifold is a sphere, the circle shrinks to a point at $t = 0$ and $t = V$, leading to the boundary conditions

$$A(0) = A(V) = 0. \tag{3.66}$$

The partition function on the sphere is then given by

$$Z = \langle\psi|\, e^{-VH}\, |\psi\rangle \tag{3.67}$$

where $|\psi\rangle$ is the (unnormalizable) wavefunction given in the E basis by $\psi(E) = 1$ and in the A basis by $\psi(A) = \sqrt{2\pi/q}\,\delta(A)$.[3] The result is identical to Eq. (3.60), showing that the partition function of 2D Maxwell theory does not distinguish between a sphere and a torus of the same volume.

In fact, we can generalize this result to Euclidean manifolds of arbitrary genus by sewing together manifolds with boundary. Using the same boundary condition (3.67) as the sphere, one can show that manifold of volume V with the topology of a disk produces the state $\psi(E) = e^{-\frac{1}{2}VE^2}$. These disks can be sewn together using manifolds with three spatial boundaries ("pairs of pants"). If we consider a pair of pants in the limit in which the volume vanishes, it can be viewed as a wavefunction of the electric fields on each of its three boundaries, given by

[3] Although this normalization is the most natural, if one were to choose a different normalization of $|\psi\rangle$, this would be equivalent to a finite shift of $1/G$. However, this cannot produce a logarithmic divergence in $1/G$, so the choice of normalization does not affect the c_1 coefficient.

$\psi(E_1, E_2, E_3) = \delta(E_1, E_2)\delta(E_2, E_3)$, where the normalization factor is fixed by the requirement that one recover the partition function of the sphere when sewing the pants to three disks. By sewing together an arbitrary number of pants and disks, we find that the partition function for an arbitrary two-dimensional closed Euclidean manifold without boundary depends only on the volume,

$$Z = \sum_{E \in q\mathbb{Z}} e^{-\frac{1}{2}VE^2}. \tag{3.68}$$

The conical entropy is easily calculated from Eq. (3.60). Since the volume of the Euclidean manifold is linear in the deficit angle β, the formula (1.10) for the entropy yields

$$S = (1 - V\partial_V) \ln Z = -\sum_{E \in q\mathbb{Z}} p(E) \ln p(E) \tag{3.69}$$

where $p(E)$ is the probability of measuring a given value of E locally, $p(E) = Z^{-1}e^{-\frac{1}{2}VE^2}$. This entropy is manifestly positive, and has an obvious statistical interpretation: the only local observable is E, and this is constant over space. Therefore observers on different sides of the horizon measuring E will find perfect correlation of their measurement results; the degree to which their states are entangled is given by the entropy in Eq. (3.69). We conclude that in two dimensions, the conical entropy of a gauge field coincides with its entanglement entropy. Note that this entropy vanishes in the large volume limit $q^2V \to \infty$.

The results of this section can be generalized immediately to $(D-1)$-form electromagnetism in D dimensions. Since the action depends only on the total spacetime volume, the dimension is irrelevant.

3.3.1 Topological Susceptibility

In Ref. [97], it was proposed that the contact term is related to the topological susceptibility χ_{t}, which measures the response of F to the introduction of a source term $i(\frac{q}{2\pi})\theta \int \sqrt{g}F$,

$$\chi_{\mathrm{t}} = -\frac{1}{V}\frac{\partial^2}{\partial\theta^2}\ln Z\bigg|_{\theta=0} = Z^{-1}\sqrt{\frac{2\pi}{q^2 V}}\sum_{F\in\frac{2\pi}{qV}\mathbb{Z}}\left(\frac{q}{2\pi}\right)^2 VF^2 e^{-\frac{1}{2}VF^2} = \left(\frac{q}{2\pi}\right)^2 V\left\langle F^2\right\rangle . \tag{3.70}$$

The topological susceptibility has properties reminiscent of Kabat's contact term: in particular the contribution from the electromagnetic field has a sign opposite to all possible matter terms (which contribute negatively). In Ref. [97] it was conjectured that the "wrong sign" term in the topological susceptibility is responsible for the negative contact term in the entropy. We will now show that, although the entropy remains manifestly positive, the topological susceptibility does contribute to the entropy with a negative sign.

To see how the susceptibility appears in the entropy, we can compute the entropy from the partition function (3.65):

$$S = (1 - V\partial_V)\ln Z = \ln Z + \frac{1}{2} - \frac{1}{2}\left(\frac{2\pi}{q}\right)^2\chi_{\mathrm{t}}. \tag{3.71}$$

The first term is proportional to the free energy. The 1/2 comes from the fluctuation term (3.64). The last term comes from differentiating the sum over nontrivial bundles, and is proportional to the topological susceptibility. It appears that the χ_{t} term could make the entropy negative, but this is not the case. At small V, the $\ln Z$ term is positive and dominates the entropy. As V increases, χ_{t} increases to $(q/2\pi)^2$ in the large volume limit, and its negative contribution to the entropy exactly cancels with the 1/2 contribution from the fluctuations.

3.3.2 Yang-Mills theory

The partition function of non-abelian gauge theory is also known exactly in 2D and is given by the generalization of Eq. (3.68),

$$Z = \sum_R (\dim R)^\chi e^{-\frac{1}{2}q^2 V C_2(R)} \tag{3.72}$$

where the sum extends over all irreducible unitary representations R of the gauge group, and C_2 is the quadratic Casimir. In the U(1) theory, the representations are labelled by integers, with $\dim(R_n) = 1$ and $C_2(R_n) = n^2$. The dependence on the Euler characteristic χ is reminiscent of the contact term, but the contribution of this term to the entropy is finite and positive:

$$S = -\sum_R p(R) \ln p(R) + \sum_R p(R) \ln \dim R \tag{3.73}$$

where $p(R)$ is the probability distribution over representations, $p(R) \propto e^{-\frac{1}{2}q^2 V C_2(R)}$. We can see that the entropy is positive, and is equal to the entanglement entropy derived in Ref. [63]. Note that the extra term involving $\ln \dim R$ comes from the χ-dependence of the effective action. It therefore appears in the entanglement entropy of de Sitter space, for which the Euclidean spacetime is the sphere ($\chi = 2$), but not in the thermal entropy of circle, for which the Euclidean spacetime is a torus ($\chi = 0$). In chapter 4 we will see that terms of this form play an important role in the entanglement entropy in lattice gauge theory.

Chapter 4

Entanglement entropy in lattice gauge theory

We have so far been discussing the entanglement entropy associated to regions of space in quantum field theory. Implicit in this discussion is the ability to associate to each region of space a tensor factor of the Hilbert space. In gauge theories, such an association is complicated by the fact that gauge-invariant states are not precisely localizable in space. The result is that the Hilbert space corresponding to a region of space includes edge states that contribute to the entanglement entropy [98, 99]. These edge states are similar to the "would-be pure gauge" degrees of freedom in (2+1)-dimensional quantum gravity [75]. Recall that in 2+1 gravity there are no local degrees of freedom in the bulk, yet the usual thermodynamic arguments suggest that black holes have an entropy proportional to the length of the horizon. In the approach of Ref. [75], the horizon has local degrees of freedom and it is these degrees of freedom whose entropy is given by the Bekenstein-Hawking area law.

It is well-known that the entanglement entropy is ultraviolet divergent, so a regulator is needed in its definition. We have seen in chapters 2 and 3 that standard quantum field theory methods involving ghosts run into problems with gauge invariance. In this chapter we consider the entanglement entropy for lattice gauge theories. This allows us to regulate the entanglement entropy while keeping manifest gauge invariance.

Entanglement entropy has been considered before for certain classes of states in lattice gauge theory [100, 99, 101, 102, 103]. Here we adopt the Hamiltonian formulation of lattice gauge theory, rather than the replica method [45] that has typically been used in numerical calculations of the entanglement entropy. The replica method relates the entanglement entropy of the ground state of a given theory to the partition function computed on an n-sheeted cover of the Euclidean spacetime. Our results do not use the replica method, so they do not require the state to be expressed as a Euclidean path integral, though we agree with results obtained using the replica method where the latter is applicable.

Closely related to lattice gauge theory is loop quantum gravity, which is formulated as an SU(2) lattice gauge theory on a superposition of lattices. Although we will not discuss loop quantum gravity, entanglement entropy in loop quantum gravity was discussed in Refs. [104, 7], and we expect the techniques of this chapter to generalize easily to a superposition of lattices. We note also that the Hilbert space of edge states in SU(2) lattice gauge theory is closely related to the Hilbert space of the SU(2) Chern-Simons theory whose states are counted in the loop quantum gravity derivation of black hole entropy [5, 6].

We now briefly summarize our result. Consider a lattice whose set of nodes N is divided into two disjoint sets A and B whose union is all of N. In a lattice theory where the degrees of freedom live on the nodes, the Hilbert space associated with a set of nodes is simply the tensor product of the Hilbert spaces of each individual node. This leads to a tensor product decomposition of the whole Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. In a lattice gauge theory, the degrees of freedom live on the links, so there is not such a simple tensor product decomposition. However, following Ref. [99] we can define a Hilbert space $\mathcal{H}_A$ by splitting the links that cross the boundary. Along each link l with one end point in A and one end point in B, we insert a new vertex on the boundary and divide the link into two smaller links, one associated with region A and one associated with region B. The Hilbert space $\mathcal{H}_A$ then consists of functionals of the connection on the links in A that are invariant under gauge transformations that act on the nodes in the interior of A, but not on the boundary nodes. In restricting the gauge symmetry, degrees of freedom that were previously pure gauge are promoted to physical degrees of freedom. The new degrees of freedom are edge states that are associated with the boundary vertices and transform nontrivially under gauge transformations acting on the boundary. They are the lattice analogue of the continuum edge states studied in Ref. [98].

The Hilbert space $\mathcal{H}$ is not equal to $\mathcal{H}_A \otimes \mathcal{H}_B$, since the former is invariant under all gauge transformations, and the latter is invariant under only those gauge transformations that act trivially on the boundary. Thus instead of an isomorphism of Hilbert spaces, we have the embedding

$$\mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B. \tag{4.1}$$

The entanglement entropy of any state in $\mathcal{H}$ can be defined by embedding the state into $\mathcal{H}_A \otimes \mathcal{H}_B$. Letting ρ denote a state, represented as a density matrix in $\mathcal{H}_A \otimes \mathcal{H}_B$, the reduced density matrix of system A is the partial trace $\rho_A = \mathrm{tr}_B(\rho)$, and the entanglement entropy is

$$S_{\mathrm{ent}} = S(\rho_A) \tag{4.2}$$

where the function $S(\rho)$ is the von Neumann entropy,

$$S(\rho) = -\mathrm{tr}(\rho \ln \rho). \tag{4.3}$$

The states in $\mathcal{H}$ are invariant under all gauge transformations, including those acting on the boundary. The reduced density matrix ρ_A associated with a gauge-invariant state is then also invariant under the group of boundary gauge transformations, which acts nontrivially on $\mathcal{H}_A$. When decomposed into irreducible representations of the group of boundary gauge transformations, the matrix ρ_A takes the form of a direct sum of tensor products. Using properties of the von Neumann entropy under direct sum and tensor product, we decompose the entanglement entropy of a generic state in lattice gauge theory as a sum of three positive terms (4.32):

- the Shannon entropy of the distribution of boundary link representations found in Ref. [99],
- the weighted average of the logarithm of the dimension of the boundary representations found in Ref. [7], and
- a third term that captures nonlocal correlations of the bulk field.

The first two terms are purely local to the boundary, and together they capture the entropy of the edge states. Our result directly generalizes the results of Refs. [99, 7]. We will derive the result in Sec. 4.1 and give several applications to specific states of lattice gauge theory in Sec. 4.2.

4.1 Decomposition of the entanglement entropy

We first review the kinematics of Hamiltonian lattice gauge theory [105] and spin network states [106]. Consider a lattice consisting of a set N of nodes and a set L of oriented links, and let G be a gauge group that is either a compact Lie group, or a discrete group. A field configuration is an assignment of group elements u_l to links, and a gauge transformation is an assignment of group elements g_n to nodes, which acts on u_l as

$$u_l \to g_{t(l)} \circ u_l \circ g_{s(l)}^{-1} \tag{4.4}$$

where $s(l)$ and $t(l)$ are respectively the nodes at the source and target of the link l. The Hilbert space $\mathcal{H}$ consists of square-integrable functionals of the holonomies u_l that are invariant under gauge transformations. An orthonormal basis for $\mathcal{H}$ is given by a generalization of the spin network states [107, 108, 106]. A spin network consists of an assignment of irreducible representations $R = \{r_l : l \in L\}$ to each link, and intertwiners $I = \{i_n : n \in N\}$ to each node. Each intertwiner i_n is a G-invariant linear map between representation spaces

$$i_n : \left(\bigotimes_{l:t(l)=n} r_l \right) \to \left(\bigotimes_{l:s(l)=n} r_l \right) \tag{4.5}$$

The spin network state $|S\rangle$ associated with a spin network S is the functional obtained by taking the representation r_l of the group element on each link l, multiplying by $\sqrt{\dim(r)}$, and contracting the free indices with the intertwiners i_n [106],

$$\langle S|U\rangle = \left(\bigotimes_{l\in L} \sqrt{\dim(r_l)}\, r_l(u_l) \right) \circ \left(\bigotimes_{n\in N} i_n \right). \tag{4.6}$$

The intertwiners are chosen to be orthonormal in the inner product,

$$\langle i_1, i_2 \rangle = \mathrm{tr}(i_1 i_2^\dagger) \tag{4.7}$$

so that the resulting spin network states form an orthonormal basis of $\mathcal{H}$ [109].

We now describe the tensor product decomposition of the Hilbert space, which was described in Ref. [99] for Abelian lattice gauge theories. Let A be a region of space, which on the lattice will mean a subset of the nodes. The configuration space of $\mathcal{H}$ consists of holonomies on all links of the lattice. The links can be divided into three sets: L_A is the set of links with both end points in A, L_B is the set of links with both end points in B and L_∂ is the set of links that cross the boundary. In order to partition the degrees of freedom of the boundary links between $\mathcal{H}_A$ and $\mathcal{H}_B$, we split each boundary link into two at the boundary, such for each link in L_∂ there is a new link in $L_{\partial A}$ and one in $L_{\partial B}$. The Hilbert space $\mathcal{H}_A$ is then defined as the square-integrable functions of the holonomies $\{u_l : l \in L_A \cup L_{\partial A}\}$ invariant under gauge transformations acting at nodes in the interior of A (but *not* under gauge transformations acting on the boundary). Then for each link $l \in L_\partial$, the holonomy u_l can be obtained as a product of a holonomy in $L_{\partial A}$ and one in $L_{\partial B}$, and we define the product map

$$\pi : U_{\partial A} \times U_{\partial B} \to U_\partial. \tag{4.8}$$

The pullback map π^* then gives an embedding

$$\pi^* : \mathcal{H} \to \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}. \tag{4.9}$$

For example, if there is just one boundary link l split into links l_1 and l_2 such that

$t(l_1) = s(l_2)$, then the map π is given by

$$\pi(u_1, u_2) = u_2 u_1, \tag{4.10}$$

and the pullback $\pi^*\psi$ of a function $\psi : G \to \mathbb{C}$ is given by

$$(\pi^*\psi)(u_1, u_2) = \psi(u_1 u_2). \tag{4.11}$$

This embedding preserves the norm of the state, a fact which follows from the G-invariance and normalization of the Haar measure.

To specify a spin network S, we specify all its representations and intertwiners

$$S = (R_A, R_B, R_\partial, I_A, I_B). \tag{4.12}$$

Just as the space $\mathcal{H}$ is spanned by spin network states, the space $\mathcal{H}_A$ is spanned by open spin network states [110]. An open spin network S_A is specified by

$$S_A = (R_A, R_\partial, I_A, M) \tag{4.13}$$

where $M = \{m_l : l \in L_\partial\}$ is a set of vectors in the boundary representation spaces, $m_l \in r_l$ if the link l points inward at the boundary, or in the dual representation $m_l \in \bar{r}_l$ if the link l points outward (for unitary representations, the dual representation $\bar{r}$ and complex conjugate representation r^* coincide). The open spin network state $|S_A\rangle$ is defined just as in Eq. (4.6), except that the extra free indices associated with the boundary vertices are contracted with the vectors m_l. The open spin network states form an orthonormal basis of $\mathcal{H}_A$ provided the vectors m_l and intertwiners i_n are chosen to be orthonormal.

As shown in Ref. [7], under the embedding π^*, the spin network state $|S\rangle$ maps to

$$\pi^* |S\rangle = \prod_{l \in L_{\partial A}} \frac{1}{\sqrt{\dim(r_l)}} \sum_{m_l} |S_A\rangle \otimes |S_B\rangle \tag{4.14}$$

where S_A is given by (4.13), and $S_B = (R_B, R_\partial, I_B, M^*)$, where the vectors M^* are dual (complex conjugate) to the vectors M. In Eq. (4.14), m_l ranges over an orthonormal basis of r_l. This decomposition follows from inserting a resolution of unity at each point where a link crosses the boundary, and the factors of $1/\sqrt{\dim(r_l)}$ arise to cancel the extra factors of $\sqrt{\dim(r_l)}$ in Eq. (4.6) that come from splitting the boundary links.

We now consider an arbitrary gauge-invariant state $|\psi\rangle$ expressed in the spin network basis,

$$|\psi\rangle = \sum_S \psi(S) |S\rangle \,. \tag{4.15}$$

Using the decomposition (4.14), the reduced density matrix for region A is

$$\rho_A = \sum_{\substack{R_A, R'_A, I_A, I'_A, \\ R_\partial, R_B, I_B, M}} \frac{\psi(S)\psi(S')^*}{\prod_{l \in L_{\partial A}} \dim(r_l)} |S_A\rangle\langle S'_A| \tag{4.16}$$

where S and S_A are given by (4.12) and (4.13), and S' and S'_A are given by

$$S' = (R'_A, R_B, R_\partial, I'_A, I_B), \qquad S'_A = (R'_A, R_\partial, I'_A, M). \tag{4.17}$$

The sums over intertwiners in Eq. (4.16) are taken over an orthonormal basis of the space of intertwiners compatible with the representations incident on each node. In the case where there is no such intertwiner, the sum is zero.

There are two features worthy of note about Eq. (4.16). First, the set of representations R_∂ is always the same for S_A and S'_A. This means that the matrix

ρ_A has no off-diagonal terms that mix different boundary representations. Second, the coefficients in Eq. (4.16) are independent of M, so within each representation the M degrees of freedom are in a maximally mixed state.

This structure of the reduced density matrix can be seen from group theory. The boundary gauge transformations form a group G^n where n is the number of boundary links, and is represented unitarily on $\mathcal{H}_A$. Such a representation $R(g)$ can always be written as a direct sum of irreducible representations:

$$R(g) = \bigoplus_r r(g) \otimes \mathbb{1}_{n(r)} \tag{4.18}$$

where $g \in G^n$, r runs over all irreducible representations of G^n, and $n(r)$ is the multiplicity with which the irreducible representation r appears in the representation R. The reduced density matrix ρ_A comes from a gauge-invariant state, so it must commute with the representation R. To commute with $R(g)$ for all g, ρ_A must take the form

$$\rho_A = \bigoplus_r \frac{\mathbb{1}_{\dim(r)}}{\dim(r)} \otimes \rho_A(r) \tag{4.19}$$

where $\rho_A(r)$ is a density matrix of dimension $n(r)$.

To see more explicitly how the density matrix decomposes into representations, it is useful to divide the Hilbert space $\mathcal{H}_A$ into an edge Hilbert space and a bulk Hilbert space such that the states of the boundary Hilbert space are labeled by (R_∂, M), and the bulk Hilbert space is labeled by (R_A, I_A). This decomposition of the Hilbert space is

$$\mathcal{H}_A = \bigoplus_{R_\partial} \left[\left(\bigotimes_{l \in L_\partial} r_l \right) \otimes \mathcal{H}_A(R_\partial) \right] \tag{4.20}$$

where $\mathcal{H}_A(R_\partial)$ is spanned by states $|R_A, I_A\rangle$ that are compatible with the assignment of representations R_∂ to the boundary.

In the decomposition (4.20) the open spin network states $|S_A\rangle$ and $|S'_A\rangle$ can be written as

$$|S_A\rangle \quad = |R_\partial\rangle \otimes |M\rangle \otimes |R_A, I_A\rangle \,, \tag{4.21}$$

$$|S'_A\rangle \quad = |R_\partial\rangle \otimes |M\rangle \otimes |R'_A, I'_A\rangle \,. \tag{4.22}$$

so that their outer product takes the form

$$|S_A\rangle\langle S'_A| = |R_\partial\rangle\langle R_\partial| \otimes |M\rangle\langle M| \otimes |R_A, I_A\rangle\langle R'_A, I'_A| \,. \tag{4.23}$$

Substituting Eq. (4.23) into the reduced density matrix (4.16) and rearranging terms yields

$$\rho_A = \sum_{R_\partial} p(R_\partial)\, |R_\partial\rangle\langle R_\partial| \otimes \left(\sum_M \frac{|M\rangle\langle M|}{\prod_{l \in L_{\partial A}} \dim(r_l)} \right) \otimes \rho_A(R_\partial) \tag{4.24}$$

where $p(R_\partial)$ is the probability of distribution of representations on the boundary,

$$p(R_\partial) = \sum_{R_A, R_B, I_A, I_B} |\psi(S)|^2 \,, \tag{4.25}$$

and $\rho_A(R_\partial)$ is the reduced density matrix,

$$\rho_A(R_\partial) = \sum_{\substack{R_A, R'_A, R_B, \\ I_A, I'_A, I_B}} \frac{\psi(S)\psi(S')^*}{p(R_\partial)} \, |R_A, I_A\rangle\langle R'_A, I'_A| \,. \tag{4.26}$$

The factor $p(R_\partial)$ is included in the definition of $\rho_A(R_\partial)$ to maintain the unit trace condition.

Since the same R_∂ appears in both the ket and the bra in the first tensor factor in Eq. (4.24), the state does indeed lie in the direct sum Hilbert space (4.20).

Moreover, the second tensor factor in Eq. (4.24) is proportional to the identity matrix, so the density matrix ρ_A can equivalently be written

$$\rho_A = \bigoplus_{R_\partial} p(R_\partial) \left[\left(\bigotimes_{l \in L_\partial} \frac{\mathbb{1}_{r_l}}{\dim(r_l)} \right) \otimes \rho_A(R_\partial) \right]. \tag{4.27}$$

The structure of the reduced density matrix (4.27) allows us to simplify the entanglement entropy by using properties of the von Neumann entropy under direct sum and direct product. Let p_n be positive real numbers summing to one, and ρ_n density matrices on Hilbert space $\mathcal{H}_n$. The von Neumann entropy of a weighted direct sum is

$$S\left(\bigoplus_n p_n \rho_n \right) = H(p_n) + \langle S(\rho_n) \rangle \tag{4.28}$$

where $\langle \cdot \rangle$ denotes expectation value with respect to the probability distribution p_n, and $H(p_n)$ is the Shannon entropy of this distribution (the classical analogue of the von Neumann entropy),

$$H(p_n) = -\sum_n p_n \ln p_n. \tag{4.29}$$

Under a tensor product, the von Neumann entropy is additive,

$$S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2). \tag{4.30}$$

Finally, the maximally mixed state of dimension n has entropy $\ln n$,

$$S(\mathbb{1}_n/n) = \ln n. \tag{4.31}$$

Applying the properties of the von Neumann entropy (4.28), (4.30) and (4.31) to the reduced density matrix ρ_A (4.27) gives the entropy as the sum of three positive terms,

$$S(\rho_A) = H(p(R_\partial)) + \sum_{l \in L_{\partial A}} \langle \ln \dim(r_l) \rangle + \langle S(\rho_A(R_\partial)) \rangle \tag{4.32}$$

where $\langle\cdot\rangle$ denotes expectation value with respect to the probability distribution $p(R_\partial)$. Eq. (4.32) is the main result of this chapter. Individual terms in this expression have appeared before: the first term appeared in Ref. [99] where it was derived for a specific class of states (see section 4.2.1), and the second term appeared in Ref. [7] as the entropy of a single spin network state.

The first two terms of Eq. (4.32) depend only on the distribution of the boundary representations, and in this sense are purely local. The second term is a sum over boundary links, and so is extensive on the boundary. The first term is not extensive, but will be approximately extensive as long as the correlations between different representations are local. The effect of correlations is always to decrease the entropy, so we can obtain an extensive upper bound by neglecting these correlations. If we assume that the statistics of each edge are the same (which would be the case for states with a discrete translation and rotation symmetry, such as the ground state of a translation- and rotation-invariant Hamiltonian) then the upper bound depends only on the probability distribution of representation on each edge $p(r)$ and is given by

$$S_{\text{boundary}} \leq n\Big(H(p) + \langle \ln \dim(r) \rangle\Big), \tag{4.33}$$

with n the number of boundary links.

In principle either of the local terms can be larger. For example, in a state sharply peaked on spin networks with high-dimension representations, the second term will dominate. In a state that is a superposition of many spin networks with low-dimension representations, the first term will dominate. In particular, an Abelian theory has only one-dimensional representations so the second term in Eq. (4.32) vanishes.

The third term in Eq. (4.32) is the most difficult to characterize. It includes the effects of correlations between distantly separated degrees of freedom, and in general it is not bounded by the area of the boundary. However we will see that, for the classes of states considered in Sec. 4.2, this term is either vanishing or much smaller than the local terms.

4.2 Examples

We now consider several examples of states whose entanglement entropy can be calculated using our method.

4.2.1 Electric string states

Reference [99] considers a class of states in $\mathbb{Z}_2$ lattice gauge theory. This theory has just two irreducible representations: a trivial representation and an alternating representation. The associated spin network states are "electric string states" where the two representations are interpreted as the presence or absence of electric strings along the edges. The states considered are of the form

$$|\alpha\rangle = \frac{1}{\mathcal{N}} \sum_S e^{-\frac{\alpha}{2} L(S)} |S\rangle \tag{4.34}$$

where α is a real parameter, $L(S)$ is the total length of electric string (i.e. the number of alternating representations), and $\mathcal{N}$ is a normalization factor.

For such a state, we now show that the entanglement entropy is given entirely by the Shannon entropy of the representations on the boundary [the first term in Eq. (4.32)]. Since the gauge group is Abelian, the second term in Eq. (4.32) vanishes. Now consider fixing the set of representations on the boundary, R_∂. The total length of electric strings is the sum of the strings in A, the strings in B, and those crossing the boundary: $L(S) = L(S_A) + L(S_B) + L(\partial A)$. For a fixed set of boundary representations, the reduced density matrix is

$$\begin{aligned} \rho_A(R_\partial) &\propto \sum_{S_A, S'_A} e^{-\frac{\alpha}{2}L(S_A) - \frac{\alpha}{2}L(S'_A)} \left|S_A\right\rangle\!\left\langle S'_A\right| \qquad (4.35) \\ &= |\psi\rangle\!\langle\psi| \qquad (4.36) \end{aligned}$$

where

$$|\psi\rangle = \sum_{S_A} e^{-\frac{\alpha}{2}L(S_A)} \left|S_A\right\rangle . \qquad (4.37)$$

This is a pure state, so $S(\rho_A(R_\partial)) = 0$.

Here we rederive the result that for the states in Eq. (4.34) the entropy is just the Shannon entropy of the string end points. The entropy for this special class of states was originally derived in Ref. [99], and on that basis it was conjectured that the Shannon entropy of the string end points is a good approximation to the full entanglement entropy of the ground state in lattice gauge theory. The advantage of the Shannon entropy over the full entropy is that it depends only on the probability distribution of representations on the boundary, and is more easily computed in computer simulations.

Here we have proven that the Shannon entropy is a lower bound to the entropy that appears generically for all states, and not just states of the special form (4.34). Moreover we have characterized precisely the difference between the Shannon entropy and the full entanglement entropy. The fact that Ref. [99] finds good agreement between the Shannon entropy and the full entropy for the true ground state indicates that the third term in Eq. (4.32) is subleading for this state.

For a non-Abelian gauge theory we can improve on the Shannon entropy as an approximation of the full entropy by including the log-dimension term [the second term of Eq. (4.32)]. This term also depends only on the distribution of boundary representations and should therefore also be easy to compute in computer simulations. Reference [99] also noted the similarity of the Shannon entropy to the log-dimension term that appeared in Ref. [7]. Here we have shown that these two terms are distinct contributions to the entanglement entropy.

4.2.2 Topological phase ground state

In Ref. [29], the limit $\alpha \to 0$ was considered, in which the state approaches a superposition of spin network states in which every spin network has an equal amplitude. Every configuration of string end points on the boundary of a region is equally probable, but gauge invariance requires the total number of string end points crossing each connected component to be even. For a region with n boundary edges and whose boundary has k connected components, the entropy is

$$S = (n - k) \ln 2. \tag{4.38}$$

For a macroscopic region, n becomes large while k stays constant so the entropy is approximately extensive on the boundary, with small nonextensive corrections.

The deviation from extensivity of the entanglement entropy is captured by the topological entanglement entropy. Given a pair of regions A and B,[1] the topological entanglement entropy is (following Ref. [29], but closely related to the definition in Ref. [28])

$$S_{\text{top}} = S(A) + S(B) - S(A \cup B) - S(A \cap B). \tag{4.39}$$

Note that terms proportional to the volume or to the surface area [such as the term proportional to n in Eq. (4.38)] do not contribute to the topological entanglement entropy. This is because the volume and surface area obey the inclusion-exclusion principle,

$$f(A) + f(B) = f(A \cup B) + f(A \cap B). \tag{4.40}$$

where f is a function measuring either the volume or the surface area[2]. However the k-dependent term does contribute to the topological entanglement entropy. If one considers a set of regions A, B as in Ref. [29] such that A and B are each topologically disks, $A \cap B$ has two connected components, and $A \cup B$ is topologically an annulus, we find a topological entanglement entropy of

$$S_{\text{top}} = 2 \ln 2 \tag{4.41}$$

in agreement with the result of Ref. [29].

[1] We are now allowing B to be an arbitrary region, not necessarily the complement of A.

[2] Note that the Euler characteristic also obeys the inclusion-exclusion principle. This means that (counterintuitively) terms in the entanglement entropy proportional to the Euler characteristic do not contribute to the topological entanglement entropy.

4.2.3 Strong coupling limit

We now consider the entanglement entropy of the ground state of the SU(2) Kogut-Susskind Hamiltonian [105] in the limit of strong coupling, $g \gg 1$. Consider a hypercubic lattice in dimension $d \geq 2$. The Kogut-Susskind Hamiltonian is a sum of electric and magnetic parts,

$$H = H_E + H_B. \tag{4.42}$$

We will work with a rescaled version of this Hamiltonian, but the ground state and therefore its entanglement entropy are not sensitive to this rescaling. The electric part is diagonal in the spin network basis, and is given by

$$H_E \left| S \right\rangle = \sum_{l \in L} j_l (j_l + 1) \left| S \right\rangle \tag{4.43}$$

where j_l is the spin of the representation r_l. The state of lowest energy for H_E is the spin network state in which all edges are in the $j = 0$ representation. We will denote this state by $|0\rangle$.

The magnetic part of the Hamiltonian is not diagonal in the spin network basis, but can be expressed as a functional of the holonomies,

$$H_B = 3\lambda \sum_{\square} [\mathrm{tr}(u_\square) + \mathrm{h.c.}] \tag{4.44}$$

where $\square$ is the set of all plaquettes (closed loops containing exactly four links), and $\mathrm{tr}(u_\square)$ is the associated Wilson loop operator in the fundamental representation $j = \frac{1}{2}$. The parameter λ is related to the gauge coupling g by $\lambda \sim g^{-4}$. The operator $\mathrm{tr}(u_\square)$ acts on the trivial spin network as

$$\mathrm{tr}(u_\square) \left| 0 \right\rangle = \left| \square \right\rangle \tag{4.45}$$

where $|\Box\rangle$ is the spin network state in which each edge around the plaquette $\Box$ is assigned the $j = \frac{1}{2}$ representation, and all other edges are assigned the trivial representation.

For strong coupling $g \gg 1$, so $\lambda \ll 1$ and we can use perturbation theory to calculate the ground state, treating H_B as a perturbation of H_E. We will be interested in computing the entropy to order λ^2, so we compute the ground state to order λ^2:

$$
\begin{aligned}
|\Omega\rangle \quad &= \quad \left(1 - \tfrac{1}{2} N_{\Box} \lambda^2\right) |0\rangle + \lambda \sum_{\Box} |\Box\rangle \\
&\quad + \lambda^2 \left(\sum_{\Box, \Box'} |\Box\Box'\rangle + \sum_{\Box\!\Box} c_{\Box\!\Box} |\Box\!\Box\rangle + \sum_{\boxed{}} c_{\boxed{}} |\boxed{}\rangle \right) \\
&\quad + O(\lambda^3).
\end{aligned} \tag{4.46}
$$

Here $N_{\Box}$ is the total number of plaquettes in the lattice, ensuring that the state is normalized to order λ^2. The state $|\Box\Box'\rangle$ denotes the spin network state of two nonintersecting single-plaquette Wilson loops around the plaquettes $\Box$ and $\Box'$. The state $|\Box\!\Box\rangle$ denotes a spin network state with support on two intersecting plaquettes with outer links in the $j = \frac{1}{2}$ and an intermediate link with $j = 1$, and the state $|\boxed{}\rangle$ is a spin network of a single loop encircling two plaquettes in the $j = \frac{1}{2}$ representation. The numbers $c_{\Box\!\Box}$ and $c_{\boxed{}}$ are constants of order unity that are irrelevant for the entanglement entropy.

To describe the way different spin networks intersect the region A, we will write $\Box \in A$, $\Box\!\Box \in A$ to indicate spin networks that lie entirely in region A. We can divide the single-plaquette spin networks into those within A, those within B, and those intersecting the boundary. The numbers of plaquettes of each type are given by $N_\Box(A)$, $N_\Box(B)$, and $N_\Box(\partial)$, respectively, with

$$N_\Box = N_\Box(A) + N_\Box(B) + N_\Box(\partial). \tag{4.47}$$

To calculate $N_\Box(\partial)$, we note that a single-plaquette loop, if it intersects the boundary at all must intersect an even number of times. We will assume that the region A is chosen so that single-plaquette loops can intersect either twice or not at all. Let n be the number of boundary links of region A. To count the number of ways a single plaquette can intersect the boundary, we fix one of the links intersecting the boundary, and after doing so there are $2(d-1)$ different orientations the plaquette can take. This overcounts by a factor of 2, since the loop intersects the boundary twice, and so there are

$$N_\Box(\partial) = n(d-1) \tag{4.48}$$

ways a single plaquette can intersect the boundary.

We now compute the entanglement entropy of the state (4.46) by calculating each term of Eq. (4.32) in turn. To find the probability distribution of representations on the boundary we note that the probability of a two-plaquette state is $O(\lambda^4)$ and therefore negligible. Thus the only states contributing to this distribution are the trivial spin network and the single-plaquette spin networks. The number of different possible sets of representations R_∂ is $N_\Box(\partial)$ and each has probability λ^2, with the probability of having no intersections given by $1 - N_\Box(\partial)\lambda^2$. The entropy of this probability distribution is

$$H(p(R_\partial)) \;=\; n(d-1)\lambda^2(-\ln\lambda^2 + 1) + O(\lambda^3) \tag{4.49}$$

and since each single-plaquette spin network intersects in two $j = \frac{1}{2}$ links, the second term of Eq. 4.32 is

$$\sum_{l \in L_\partial} \langle \ln(2j_l + 1) \rangle = n(d-1)\lambda^2 2 \ln 2. \tag{4.50}$$

We now consider the entropy of the density matrices $\rho_A(R_\partial)$. Since we are taking an expectation value, we only need to consider sets of boundary representations with probability of order λ^2 or larger. This means either there is no intersection with the boundary, or a single plaquette $\Box$ intersecting the boundary. In the latter case, the only matrix element of ρ_A compatible with the assignment of representations to the boundary and probability at least order λ^2 is $\lambda^2 |\Box\rangle\langle\Box|$. Thus $\rho_A(R_\partial)$ is a pure state to order λ^2 and so contributes no entropy.

In the case where there is no plaquette intersecting the boundary, we need to know the state $\rho_A(R_\partial)$ to order λ^2. A short calculation shows that

$$\rho_A(R_\partial) = |\psi\rangle\langle\psi| + O(\lambda^3) \tag{4.51}$$

where

$$\begin{aligned} |\psi\rangle &= (1 - \tfrac{1}{2} N_\Box(A)\lambda^2)\,|0\rangle + \lambda \sum_{\Box \in A} |\Box\rangle \qquad (4.52) \\ &\quad + \lambda^2 \left(\sum_{\Box, \Box' \in A} |\Box\Box'\rangle + \sum_{\Box\!\Box \in A} c_{\Box\!\Box}\, |\Box\!\Box\rangle + \sum_{\sqsubset\!\sqsupset \in A} c_{\sqsubset\!\sqsupset}\, |\sqsubset\!\sqsupset\rangle \right). \end{aligned}$$

Since this is a pure state, its entropy is zero to order λ^2.

Combining the terms in the previous paragraphs the entanglement entropy at first nonvanishing order in the strong coupling expansion is

$$S = n(d-1)\lambda^2(-\ln \lambda^2 + 1 + 2\ln 2) + O(\lambda^3). \tag{4.53}$$

It is extensive in the boundary area (proportional to n). The entropy is also proportional to $(d-1)$, which is the number of polarizations of the gauge field. This factor is to be expected for weak coupling, where free field theory is a good approximation. It is not clear why this factor should appear also at strong coupling.

4.3 Summary

We have given a formula for the entanglement entropy of an arbitrary state in lattice gauge theory as a sum of three terms [Eq. (4.32)]. Two of these terms are local to the boundary and have appeared before in the literature [99, 7]; the other captures nonlocal correlations between bulk degrees of freedom. Our result extends the result of Ref. [99], which proposed that the Shannon entropy of the boundary representations [the first term of Eq. (4.32)] is an approximation to the entanglement entropy that depends only on the statistics of boundary observables. Our results prove that the Shannon entropy is a lower bound, and we give an improvement of this lower bound for non-Abelian gauge theories [the second term of Eq. (4.32)] that also depends only on the statistics of boundary observables. Moreover, a precise expression is given for the difference between the local part of the entropy and the full entropy [the third term of Eq. (4.32)].

We have verified several results for entanglement entropy of specific states that appeared already in the literature, and considered also the entanglement entropy of the ground state of the Kogut-Susskind Hamiltonian for SU(2) lattice gauge theory to first nonvanishing order in the strong coupling expansion. While at this leading order only the local terms contribute to the entropy, at higher order all terms will contribute. This agrees with field theory calculations of the entropy, where the entropy density is found to diverge as the horizon is approached. While we expect the dominant contribution to entanglement entropy to come from states localized near the boundary, there should be a finite contribution from correlations at a distance of more than one lattice spacing.

We note that there is no sign of a negative contact term in this result. Instead, this result for lattice gauge theory agrees with the results of the reduced phase space calculation (3.73). Both the Shannon entropy term and the non-abelian correction appear in two-dimensional Yang Mills, but the bulk term vanishes due to the absence of bulk degrees of freedom.

Chapter 5

Conclusion

Gauge theory presents subtleties for defining the entanglement entropy of a region of space. In the lattice regularization considered in chapter 4, The Hilbert space associated to a region with a boundary contains edge states, and they contribute to the entropy of entanglement. In fact the gauge symmetry allows the total entropy to be written as a sum of a bulk contribution and a boundary contribution: the latter automatically satisfies the area law, and is the dominant contribution for several physically interesting classes of states.

The lattice result for the entanglement entropy agrees completely with the exact result for two dimensional gauge theory calculated using the conical method in section 3.3. In two dimensions, the entropy consists only of the boundary terms because there are no bulk degrees of freedom. When the physical Wilson loop is regulated by introducing a finite minimum charge and is normalized correctly, the partition function of two-dimensional electrodynamics depends only on the Euclidean volume, not the curvature. The contribution of the Maxwell field to the effective action is finite, and vanishes as the Euclidean volume goes to infinity; in two dimensions, Maxwell fields do not renormalize $1/G$. The Kabat contact term does not appear when the theory is quantized using the true physical degrees of freedom. Hence the gauge-dependent term in the Wald entropy discussed in chapter

2 is absent, and the conical entropy formula agrees exactly with the entanglement entropy.

This result is in disagreement with the partition function computed from the Maxwell heat kernel $K_M = K_V - 2K_S$, which we have calculated for compact manifolds in section 3.2. Although we confirm the existence of the contact term in this model, the model is unphysical because it includes contributions from spurious ghosts identified in section 3.2.1. The path integral contains a contribution from ghost zero modes, and the canonical phase space contains a pair of spatially constant BRST-invariant ghost modes. Additionally, the infrared divergence of the vector zero modes was treated in an unphysical way by introducing a small mass. This is physically incorrect since gauge fields cannot be given masses without introducing an extra degree of freedom. The physically correct infrared regulator is invariance under large U(1) gauge transformations, and this gives a different result for the partition function.

Since a noncompact manifold can be viewed as the limit of an infinitely large compact manifold, the absence of the contact term ought to manifest somehow in this limit as well. Since a noncompact manifold has a continuous spectrum, it is harder to see the effects of the zero mode prescription. However, in section 3.1 it was observed that the derivation of the contact term for the cone is sensitive in the infrared to an order of limits: if one takes $r \to \infty$ before taking $s \to \infty$, the contact term does not appear. Thus the calculation on the cone is also sensitive to the prescription for dealing with the infrared aspects of the theory.

We have argued that the contribution of a quantum field to the entropy of a black hole is generally the sum of two terms: the expectation value of the field's Wald entropy, and the entanglement entropy. The renormalization group flow mixes these terms in such a way that their sum remains constant; whether the entropy of a given mode is counted as Wald entropy or entanglement entropy (or both) depends on the renormalization group scale. In the case of the gauge field, it was shown in chapter 2 that the Kabat contact term can also be described either as Wald entropy or as entanglement entropy. However the description in terms of Wald entropy is not gauge invariant; in two dimensions this seems to be a reflection of its origin as a statistical entropy of unphysical negative norm states.

5.1 Future work

Although the conclusion that the contact term is absent is confined to the case of $D = 2$, the absence of the contact term in $D = 2$ suggests that the $D > 2$ calculations should also be revisited. Since in Kabat's derivation, the contact term in the higher-dimensional heat kernel just comes from the product of the contact term in the two-dimensional Maxwell heat kernel times the $D-2$-dimensional scalar heat kernel, one might think that the contact term will also be absent in higher dimensions. However, in higher dimensions the contact term no longer arises solely because of zero modes, so the analysis will be different. Since $1/G$ is power-law divergent in $D > 2$, the results may also depend on ones choice of renormalization scheme, as well as the choice of gauge [88].

The arguments of chapter 2 have exposed a new source of ambiguity in the

Wald entropy when applied to a quantum theory. Of course, the Wald entropy formula was derived for a classical theory of gravity. It would therefore be of great interest to have a quantum version of the arguments of [52].

There is also the hope that a the entanglement entropy may help to resolve the ambiguities in the classical Wald entropy. While suitable expressions for the Wald entropy are known for Killing horizons, the correct generalization to arbitrary dynamical horizons is known only for a small class of theories [54]. In principle, the entanglement entropy can be defined even for dynamical horizons. Therefore, one could come up with a candidate expression for the Wald entropy of an effective theory by expressing the entanglement entropy in terms of the coefficients in the gravitational effective action. In practice, this requires going beyond the conical entropy formula, which can only compute the entanglement entropy across the bifurcation surface for static spacetimes. However, first steps toward methods for computing the entanglement entropy for more general spacetime regions have been taken recently [111].

Another puzzling aspect of the entropy of gauge fields is the claim [61] that the gauge-fixed theory does not admit a Hamiltonian formulation in Rindler space. Possibly related is the fact that nonminimally coupled scalar fields with $\xi > 0$ (the sign for which the scalar contact term is negative) also apparently do not admit a Hamiltonian formulation [47]. The reason for the absence of a Hamiltonian formulation in both cases is related to the non-existence of square-integrable mode solutions satisfying the prescribed boundary conditions at the horizon. One approach to this issue might be to introduce ghosts at the level of the Hamiltonian formalism [93] rather than at the level of the path integral.

In this work we have considered entanglement entropy for free theories, ignoring interactions. In a free theory, modes decouple and the entanglement entropy can be expressed as a sum over modes. To study the entanglement entropy at a given scale, one can simply suppress the contribution from modes above that scale. In an interacting theory, the modes are coupled so that the theory exhibits also a nontrivial entanglement in momentum space [56]. The renormalization group can be viewed as a partial trace in momentum space, where the high-momentum modes are integrated out to yield a theory at lower momentum scales. Therefore to study the position space entanglement entropy in an interacting theory, we need to consider not only entanglement with modes beyond the horizon, but also with modes above the cutoff scale. These two tensor product decompositions are not compatible with each other, as a consequence of the uncertainty principle. Therefore it is not clear how one should even define entanglement entropy in an interacting theory with a momentum-space cutoff. An approach that has been successful in the discrete setting is to use a position-space renormalization group, such as MERA [26]. It would be particularly interesting to apply these methods to lattice gauge theory (see [112] for some related work). Not only would this give a more realistic class of states than those considered in chapter 4, it would allow us to see how the boundary and bulk terms in the entanglement entropy behave under renormalization group flow.

While we have argued against the analogy between the nonminimal coupling term in the Wald entropy and the contact term for gauge fields, there is one argument for the presence of the nonminimal coupling term in the entropy of the black hole that has not been addressed in the gauge theory case. Namely, the argument that in models where the nonminimal coupling is induced, its contribution to the entropy corresponds to entanglement entropy at a higher energy scale [68]. To see whether an analogous argument would apply to gauge theory, one could consider a model in which the gauge symmetry is emergent, and compare the entropy in the effective gauge theory with the entanglement entropy of the fundamental theory. Some models with emergent gauge symmetry have been proposed in Refs. [113, 114, 115]. Unfortunately these models break Lorentz invariance, so techniques based on the conical method would not apply.

We would like to extend our derivation of entanglement entropy of edge states to the case of gravity. In gravity, there are two constraints that replace Gauss' law: a vector constraints that generates diffeomorphisms within a timeslice, and a scalar constraint that generates normal deformations of the timeslice. Because the action of the vector constraint moves the entangling surface, it seems that one must partially fix the diffeomorphism symmetry by selecting an entangling surface. Additionally, the scalar constraint of gravity is an elliptic equation, unlike the first-order Gauss constraint, which may lead to further technical complications. These problems are related to the background independent nature of gravity; in order to calculate entanglement entropy one must specify the entangling surface in an invariant way. Ideally one would fix this surface to be the causal horizon; but this condition poses a huge technical challenge, as it requires solving for the full future evolution of the spacetime.

Finally, it is tempting to speculate on the relation between the contact term studied in chapters 2 and 3 and the edge states studied in chapter 4. We note that the contact term for the gauge field is equal to the partition function of a single scalar field restricted to the bifurcation surface, but with the opposite sign (this fact for the nonminimal scalar contact term was pointed out in Ref. [84]). The electric string endpoints considered in chapter 4 consist of a single degree of freedom per point of the horizon. This suggests that a possible interpretation of the negative contact term is that it is due to "missing" edge states. Also suggestive is the fact that the contact term of the graviton is also a negative integer $-(D^2-D+4)/2$ times the partition function of a scalar field on the bifurcation surface. This coefficient depends quadratically on the dimension, which is consistent with the fact that the number of degrees of freedom as well as the number of conserved charges both grow quadratically with D. One could therefore hope that the edge states will also shed light on the contact terms in the case of gravity.

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