ABSTRACT

Title of dissertation: MEASURE OF PARAMETERS WITH A.C.I.M. NONADJACENT TO THE CHEBYSHEV VALUE IN THE QUADRATIC FAMILY

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Dissertation directed by: Professor Michael Jakobson
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In this thesis, we consider the quadratic family $f_t(x) = tx(1-x)$, and the set Λ^+ of parameter values t for which f_t has an absolutely continuous invariant measures (a.c.i.m.). It was proven by Jakobson that Λ^+ has positive Lebesgue measure. Most of the known results about the existence and the measure of parameter values with a.c.i.m. concern a small neighborhood of the Chebyshev parameter value $t = 4$. In particular, Luzzatto and Takahasi gave an estimate on the measure of Λ^+ by showing $|\Lambda^+ \cap I^*| > 0.97|I^*|$ where $I^* = [2 - 10^{-4990}, 2]$ for the family of quadratic maps $x^2 - a$. Differently from previous works, we consider an interval of parameter not adjacent to $t = 4$, and give a lower bound for $|\Lambda^+|$ in that interval.

MEASURE OF PARAMETERS WITH A.C.I.M. NONADJACENT TO THE CHEBYSHEV VALUE IN THE QUADRATIC FAMILY

by

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Chapter 1

Basic theory, notions, and constructions. Some examples

1.1 Introduction

The quadratic family $f_t(x) = tx(1-x)$ is a family of S-unimodal maps exhibiting a wide variety of behaviors for f_t corresponding to different parameters t. This family of maps has been studied extensively and most thoroughly. For literature review we refer to [5].

One topic of interest is the abundance of parameters corresponding to maps which have absolutely continuous invariant measures. Such parameters, denoted by Λ^+, are known as the *stochastic* parameters. Topologically, Graczyk and Światek [6] and Lyubich [10] showed that the set of parameters corresponding to maps with attracting periodic orbits, which cannot have a.c.i.m. is open and dense in $(0, 4]$. Such parameters, denoted by Λ^-, are known as the *regular* parameters. This means that Λ^+, being in the complement of Λ^-, can only be a nowhere dense set. On the other hand, measure-wise, the Lebesgue measure of Λ^+ is positive ([7], [2]), and $t = 4$ is a density point of Λ^+, namely, $\lim_{\epsilon \to 0} \frac{|\Lambda^+ \cap [4-\epsilon, 4]|}{\epsilon} = 1$. In fact, Lyubich [11] showed that $\Lambda^+ \cup \Lambda^-$ takes up full measure in [0,4]. Avila and Moreira [1] showed that in the set of Λ^+, a full measure of the parameters correspond to the Collet-Eckmann maps, those are maps whose critical orbits have exponentially growing derivatives.

It is interesting to get an idea of the actual measure of Λ^+. Tucker and Wilczak [13] have computed a lower bound for the measure of Λ^-. Luzzatto and Takahasi [9] made the first attempt to find a lower bound for the measure of Λ^+ by estimating the measure of Collet-Eckmann maps in a small interval adjacent to 4. Here we work on an interval non-adjacent to 4, and provide the following result.

Theorem 1. *In the parameter interval* $\mathcal{T}_0 \approx [3.99512595000, 3.99513000706]$, *there is a set* $\mathcal{M}$ *of parameter values, such that* f_t *for* $t \in \mathcal{M}$ *has a.c.i.m. and*

$$\frac{|\mathcal{M}|}{|\mathcal{T}_0|} \geq 1.58382 * 10^{-16}. \tag{1.1}$$

The interval $\mathcal{T}_0$ is dynamically defined. The estimate given here is by no means optimal. The interval $\mathcal{T}_0$ chosen was an arbitrary choice, but similar processes can be carried out for a variety of intervals $\mathcal{T}_0$. Note that the parameter choice in our construction provides not only Collet-Eckmann maps.

We adapt methods from [7] and [8]. In [7] and [8], the inductive constructions use only $\mathcal{C}^2$ properties of unimodal maps. Here we use properties of S-unimodal maps. In particular, in our construction, the number of refinements (discussed in the text) at any step n is bounded above by $6 + 3$, whereas in [7] and [8], the number of refinements can grow with n. Our method requires some preliminary computer assisted estimates on sizes, derivatives and velocities. They constitute the base of induction.

Our approach of estimation is based on the construction of power maps. In this first chapter, we discuss the basics needed in our method of construction. At the end of this chapter, we give two examples demonstrating this method. In the second chapter, we state the algorithm for construction and then prove estimates for measures, derivatives, and distortions. This leads to the conclusion of our main theorem.

1.2 Preliminaries

For the family of quadratic maps $f_t(x) = tx(1-x)$, where $0 < t \leq 4$, explicit formula for the a.c.i.m. is only known for the case $t = 4$ (Chebyshev map). In that case, the explicit form of the invariant measure μ is given by $d\mu = \frac{1}{\pi\sqrt{x(1-x)}}dx$. It is obtained by taking a conjugacy to the full tent map and using that the full tent map has the Lebesgue measure as an invariant measure. If f_t has an attracting periodic orbit (only one can exist), such maps do not have a.c.i.m.. It is well known that for parameter values $t = 0$ to $t = 3.57025\ldots$ (Feigenbaum value), attracting periodic orbits of periods 2^k exist and they bifurcate as parameter value grows. Indifferent periodic orbit exists when the periodic orbit of period 2^k bifurcates to a periodic orbit of period 2^{k+1}. The indifferent periodic orbit plays the role of an attracting periodic orbit. We are interested in the parameter values after the Feigenbaum value.

1.2.1 S-unimodal maps

Quadratic maps are particular cases of S-unimodal maps. For the theory of S-unimodal maps, we refer to [4]. Here we give the definition and some basic properties. An S-unimodal map is a $\mathcal{C}^3$ unimodal map that has *negative Schwarzian* *derivative* on non-critical points. We say that f has negative Schwarzian derivative if

$$\mathcal{S}f = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 < 0. \tag{1.2}$$

Below are some properties of S-unimodal maps.

property 1 If $\mathcal{S}f < 0$, then $\mathcal{S}f^n < 0$ for all $n \in \mathbb{N}$.

property 2 If $\mathcal{S}f < 0$ on I, then $|f'|$ has minimum on the boundary of I.

property 3 S-unimodal maps can have at most one attracting or indifferent periodic orbit.

1.2.2 Koebe distortion principle

An important consequence of the negative Schwarzian derivative property that we will use heavily is the Koebe distortion principle. We say that $\tilde{I}$ is a τ-*scaled neighborhood* of I if each component of $\tilde{I}\backslash I$ has length of at least $\tau|I|$.

Koebe distortion principle *Let g be a diffeomorphism with negative Schwarzian derivative which maps I onto $g(I)$. Suppose $I \supset J$ and that $g(I)$ contains a τ-scaled neighborhood of $g(J)$, then*

$$(\frac{\tau}{1+\tau})^2 \leq \frac{Dg(x)}{Dg(y)} \leq (\frac{1+\tau}{\tau})^2. \quad \textit{for all} \quad x, y \quad \textit{in} \quad J \tag{1.3}$$

We say that the distortion *of g is bounded by $(\frac{1+\tau}{\tau})^2$.*

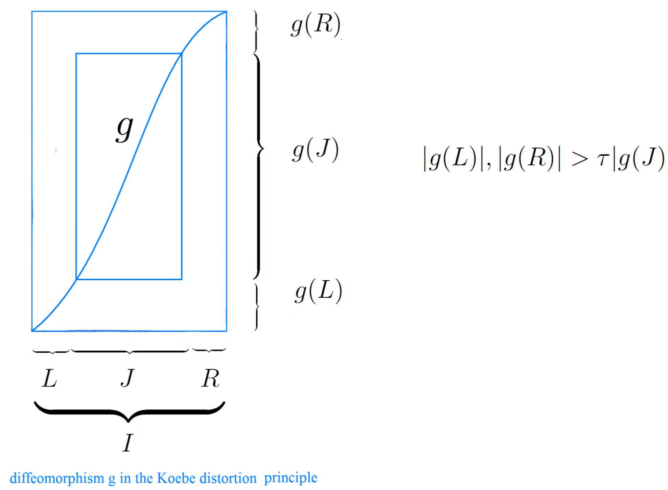


Figure 1.1: Diffeomorphism for the Koebe distortion principle

1.2.3 Power maps of f_t

We will be discussing induced (power) maps of f_t with the following properties. A power map F is defined on an interval I, and maps I into I. I is partitioned into a countable number of subintervals $I_1, I_2, \ldots$ (not necessarily in order) so that the union of the intervals has full Lebesgue measure (denoted by $I = \cup_i I_i$ (mod 0)). F restricted to each interval I_k is a power of f_t. We call the maps on each interval *branches* of F, and denote them by $f_k = F|_{I_k} = f_t^{n_k}|_{I_k} = \overbrace{f \circ f \cdots \circ f}^{n_k \text{ times}}$, where n_k is the power. In addition, f_k is either a *monotone branch* or a *critical branch.* When f_k is a monontone branch, f_k maps I_k diffeomorphically onto I. When f_k is a critical branch, f_k maps I_k into I and has one critical point. The domains in which these branches are defined are called *monotone domains* and *critical domains*, respectively.

1.2.4 Uniform extendability

For the power map F defined in the previous subsection, we define a notion of uniform extendability. If $\tilde{I}$ is a neighborhood of I, we say that F can be uniformly extended to $\tilde{I}$ if for each k there exists $\tilde{I}_k$ such that $f_k = f^{n_k}$ maps $\tilde{I}_k$ onto $\tilde{I}$ in the case where f_k is a monotone branch and $f_k = f^{n_k}$ maps $\tilde{I}_k$ onto an interval covering one end of $\tilde{I}$ in the case where f_k is a critical branch. We call $\tilde{I}_k$ the extended domain of I_k.

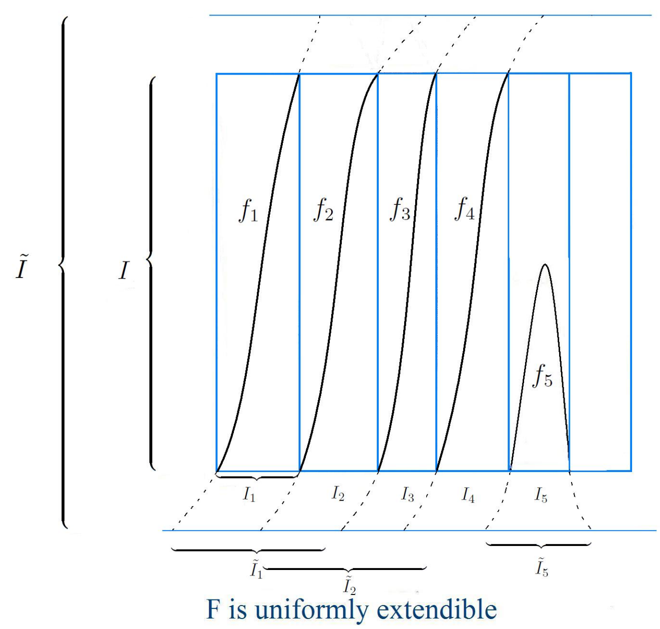


Figure 1.2: Uniform extendability for power maps

1.2.5 Folklore theorem

The existence of a.c.i.m.s for maps with countably many expanding branches relies on the Folklore theorem.

Folklore Theorem *Let F be a map defined on a countable collection of disjoint open intervals $\bigcup_{k=1}^{\infty} I_k$ in I and satisfying the following properties:*

1. $I = \bigcup_{k=1}^{\infty} I_k$ *(mod 0).*

2. $f_k = F|_{I_k}$ *extends to a* $\mathcal{C}^2$ *function on* $\boldsymbol{cl}(I_k)$ *and* $f_k(\boldsymbol{cl}(I_k)) = I$ *for each* k.

3. F *is uniformly expanding. That is, there is an* $\mathcal{R} > 1$ *independent of* k *such that* $|\frac{df_k}{dx}| \geq \mathcal{R}$ *on* $\boldsymbol{cl}(I_k)$ *for each* k.

4. F^n *has uniformly bounded distortion. That is, there exists* $K > 0$ *such that* $\frac{D(f_{k_1} \circ \cdots \circ f_{k_n})(x)}{D(f_{k_1} \circ \cdots \circ f_{k_n})(y)} < K$ *for all* x, y *in* $f_{k_n}^{-1} \circ \cdots \circ f_{k_1}^{-1}(I)$ *for any* n *and any set of indices* $k_1, \cdots, k_n$.

Then there exists an a.c.i.m. ν *with density continuous and bounded away from zero.*

See afterword in [3] for a mention of such formulation, and [5] for the proof. The first two conditions satisfy conditions of a *Markov map*. From the Koebe distortion principle, condition 4 is satisfied if the *negative Schwarzian condition* and the *uniform extendability condition* hold. The quadratic map has negative Schwarzian derivative on the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. By property of functions with negative Schwarzian derivative, the nth iterate f_t^n of the quadratic map has negative Schwarzian derivative on its non-critical points.

Our goal is to construct a power map F_t of f_t satisfying conditions of the Folklore theorem. For a given value t, there exist a fixed point $q = \frac{t-1}{t}$ of f_t, with its other preimage $q^{-1} = \frac{1}{t}$. We are interested in the interval $I = [q^{-1}, q]$ since iterates of all points except 0 and 1 will eventually fall into this interval. The power map is constructed on the interval I. If we can show that F satisfies conditions 1 through 4, then F has a.c.i.m. ν. Moreover, if

$$\sum \nu(I_k) n_k < \infty, \tag{1.4}$$

then

$$\mu(A) = \sum_k \sum_{i=0}^{n_k-1} \nu(f_t^{-i}(A) \cap I_k) \tag{1.5}$$

will give an a.c.i.m. for f_t on I.

1.3 Basic notions and constructions

1.3.1 Notations

We have already defined the interval $I = [q^{-1}, q]$ for a map f_t, where $q = \frac{t-1}{t}$ and $q^{-1} = \frac{1}{t}$. By taking further left preimages of q, it is natural to label the points $q^{-2}, q^{-3}, \ldots, q^{-k}, \ldots$. The corresponding preimages of q on the right will be $q_r^{-2}, q_r^{-3}, \ldots, q_r^{-k}, \ldots$. If f_l and f_r represent f_t restricted to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively, then $q^{-k} = f_l^{-k}(q)$ and $q_r^{-k} = f_r^{-1} \circ f_l^{-k+1}(q)$. We define intervals

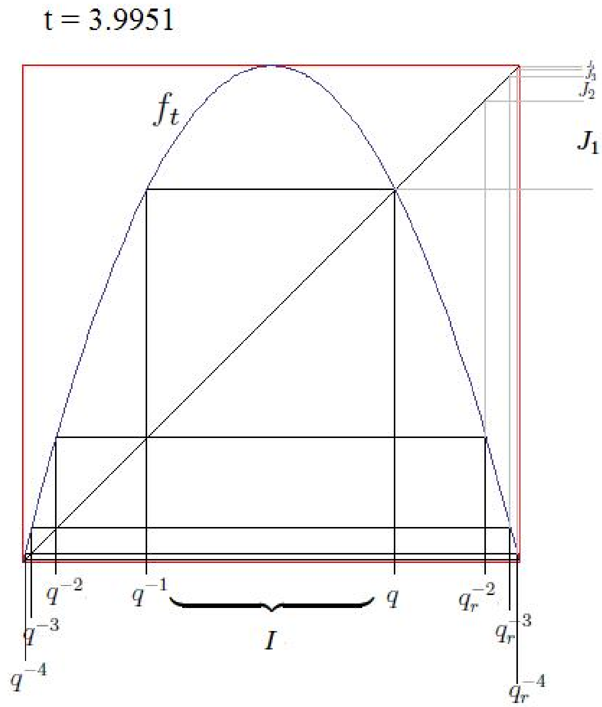


Figure 1.3: Fixed point and left and right preimages of the fixed point

$J^1 = [q, q_r^{-2}], J^2 = [q_r^{-2}, q_r^{-3}], J^3 = [q_r^{-3}, q_r^{-4}], J^4 = [q_r^{-4}, q_r^{-5}], \ldots$. The figure above shows the positions of these points and intervals in the case where t is close to 4 but not equal to the value 4. Note that in the figure, the critical value is so close to 1 that it looks as if it touches 1, but it does not actually touch 1. Each J^k is mapped by f_t^k diffeomorphically onto I. We denote such maps by g_k, so that $g_k = f_t^k|_{J^k}$ and $g_k(J^k) = I$. All intervals above vary with t, but we suppress the t for convenience. It is estimated in [8] that for large n

$$\frac{dq_r^{-n}}{dt} < cnA^{-n}, \tag{1.6}$$

where c is a constant and A is close to 4, both constants independent of n. We also know that the critical value $f_t(\frac{1}{2})$ is $\frac{t}{4}$, therefore moves at constant speed $\frac{1}{4}$ with respect to t. As t becomes larger, the range of the map covers more J^k's. Later, we will focus our investigation on the case where $f_t(\frac{1}{2}) \in J^4$.

In general, it is convenient to imagine intervals J^k's as intervals on the y-axis since later in the text we look at the interval J^k in which the critical value $f_t(\frac{1}{2})$ is positioned. We will use Δ for monotone domains and δ for critical domains. If the nature of the branch is not specified, we will just denote them by I_j's. The labeling of the indices will not have a general rule, except that δ_k^{-n} will be a preimage of δ_k and $I_{k1}, I_{k2}, I_{k3}, \ldots$ will be subintervals of I_k. For the power maps we will be considering, the leftmost and rightmost domains will always be monotone domains. We specifically refer to them as Δ_l and Δ_r, respectively.

1.3.2 First return map

For $t > 2$, we define the first return map on $I = [q^{-1}, q]$. If $f_t(\frac{1}{2}) \notin J^1$, the pullback of J^1 by f_t^{-1} consists of two intervals, namely $\Delta_1 = f_l^{-1}(J^1)$ and $\Delta_{-1} = f_r^{-1}(J^1)$. Since f_t maps Δ_1 (or Δ_{-1}) diffeomorphically onto J^1, and $g_1 = f_t|_{J_1}$ maps J^1 diffeomorphically onto I, we have that $g_1 \circ f_t|_{\Delta_1} = f_t^2|_{\Delta_1}$ (or $g_1 \circ f_t|_{\Delta_{-1}} = f_t^2|_{\Delta_{-1}}$) maps Δ_1 (or Δ_{-1}) diffeomorphically onto I. Similarly, if $f_t(\frac{1}{2}) \notin J^2$, the pullback of J^2 by f_t^{-1} consists of two intervals Δ_2 and Δ_{-2} and $g_2 \circ f_t|_{\Delta_2} = f_t^3|_{\Delta_2}$ (or $g_2 \circ f_t|_{\Delta_{-2}} = f_t^3|_{\Delta_{-2}}$) maps Δ_2 (or Δ_{-2}) diffeomorphically onto I. Δ_2 is adjacent to Δ_1. We can do the same for $J^3, J^4, \cdots$ if they do not contain $f_t(\frac{1}{2})$. There will be an interval J^N such that $f_t(\frac{1}{2}) \in J^N$. The pullback of J^N by f_t^{-1} will be one interval centered at $\frac{1}{2}$. We will call that interval δ. It will be adjacent to the intervals Δ_{N-1} and $\Delta_{-(N-1)}$. $g_N \circ f_t|_\delta = f_t^{N+1}|_\delta$ maps δ into I and has a critical value. Elements

Δ_1, Δ_2,. . .,δ,. . .,Δ_{-2},Δ_{-1} form a partition of I where we ignore common endpoints. Letting $f_k = F_0|_{\Delta_k} = f_t^{|k|+1}|_{\Delta_k}$ for $1 \leq |k| \leq N-1$ and $h_0 = F_0|_\delta = f_t^{N+1}|_\delta$, we have a power map F_0. F_0 is the *first return map* of f_t to I. The following figure is an example for the value $t = 3.989$. Again, the critical value in the figure looks as if it touches the value 1, but it actually does not.

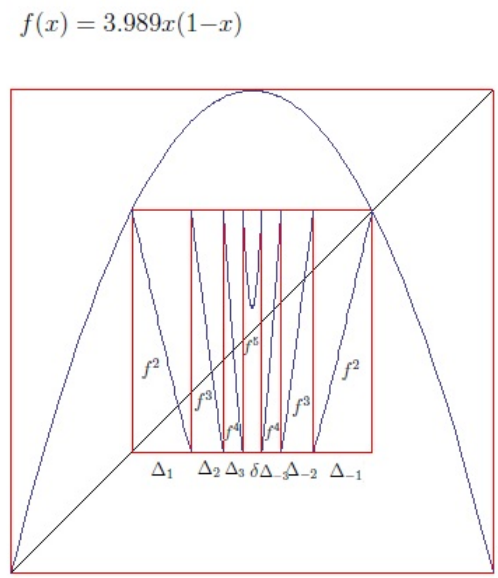


Figure 1.4: Fixed point and left and right preimages of the fixed point

If $t = 4$, the first return map will have infinitely many monotone branches with domains converging to the point $\frac{1}{2}$. If $t < 4$, there will be finitely many monotone branches on each side, and a critical branch in the center.

Due to the existence of the central critical branch, we do not automatically have a map that satisfies the conditions of the Folklore theorem. We will try to substitute

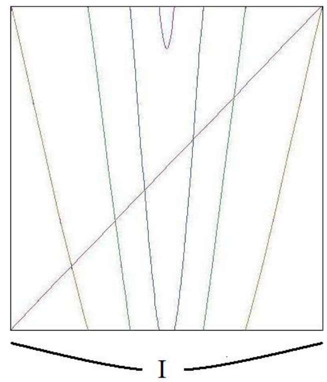


Figure 1.5: First return map

the critical branch by new branches that consist of monotone branches and critical branches with smaller domains. This is done by a series of *monotone refinements*, *parabolic pullbacks*, *critical pullbacks* and *filling-in* procedures. Our ultimate goal is to get a sequence of induced maps, where the total measure of critical domains converges to zero. In addition, we would like to ensure that the uniform extendability condition holds for a fixed extension $\tilde{I}$ of I.

1.3.3 Holes

In our inductive construction, there is always some region in the center $(\frac{1}{2})$ consisting of the central critical domain and possibly nearby domains where branches defined on these domains have not yet been fixed. We refer to these regions as central *holes*. Monotone domains in a central hole may be modified in later inductive steps. Preimages of these central regions are also considered as *holes*. Holes contain critical domains and some monotone domains whose corresponding branches may not yet satisfy the uniform extendability condition. We also use δ to denote our holes. We denote maps that map preimages of central holes to their original central hole by capital script letters $\mathcal{F}$ or $\mathcal{G}$.

We wish for the total measure of holes to converge to zero.

1.3.4 Basic procedures

Below, we will explain how the basic procedures are performed.

1.3.4.1 Monotone pullback/refinement

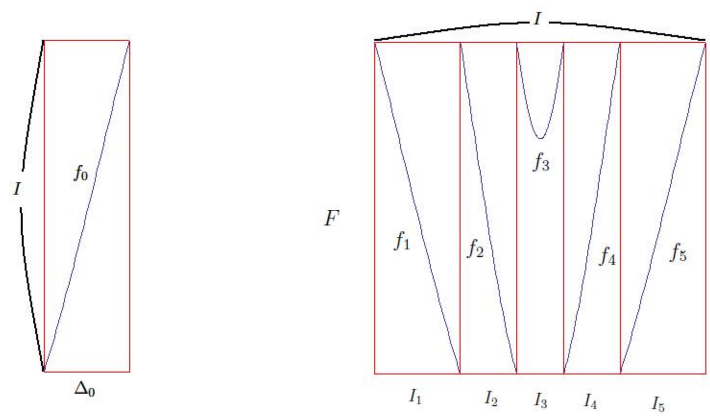
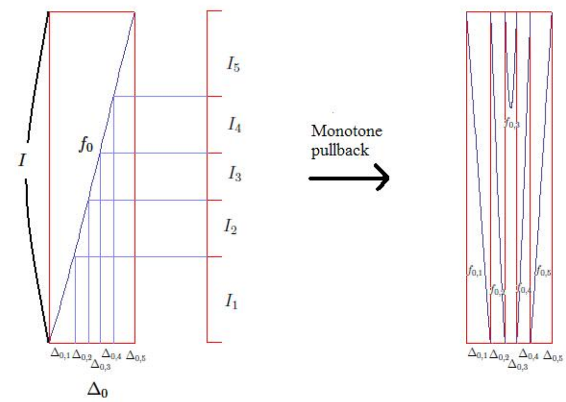


Figure 1.6: The monotone branch to be refined and the power map to pullback with

Definition 1. Let F be a power map on I and let $f_0 : \Delta_0 \to I$ be a monotone map. The *monotone pullback of* F *by* f_0^{-1} is the new power map $F \circ f_0$ on Δ_0.

More precisely, if F has branches f_k's with corresponding domains I_k's, the monotone pullback of F onto Δ_0 forms subintervals $\Delta_{01}, \Delta_{02}, \Delta_{03}, \ldots$ of Δ_0, where $\Delta_{0i} = f_0^{-1}(I_i)$, and new branches $f_{0,i} = f_i \circ f_0$. Note that $f_{0,i}$ is a monotone branch if f_i is a monotone branch and is a critical branch if f_i is a critical branch.



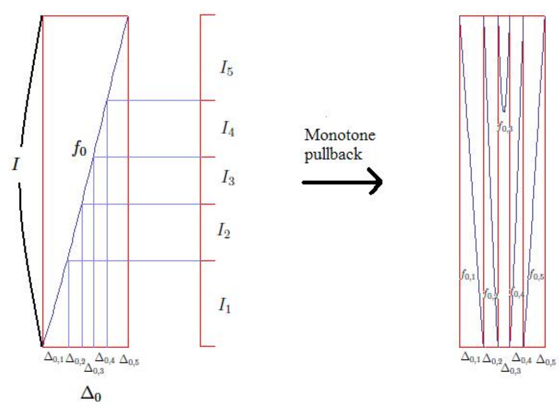


Figure 1.7: Branches after a monotone pullback

Let ξ be a partition of I into domains of F. We also consider the monotone pullback of ξ into $f_0^{-1}\xi$ as “the pullback of ξ by f_0^{-1}”.

1.3.4.2 Parabolic pullback

Definition 2. Let G be a power map on a domain J on the y-axis and let h_t be the quadratic map restricted to a neighborhood of $\frac{1}{2}$. If $h_t(\frac{1}{2})$ is in J, the *parabolic pullback of* G *by* h_t^{-1} is $G \circ h_t$.

Suppose G has branches $g_1, g_2, \cdots$ with respective domains $J^1, J^2, \cdots$. We perform parabolic pullback only in instances where $h_t(\frac{1}{2}) \in J^m$ and g_m is a monotone branch. In such cases, domains are created symmetrically on the left and right of $\frac{1}{2}$ and the central domain is $h_t^{-1}(J^m)$. Newly created branches $g_i \circ h_t$ could be either a critical branch or monotone branch again.

Let ζ be a partition of J into domains of G. Suppose $h_t^{-1}(J) = \delta$. We also consider the monotone pullback of ζ into a partition $h_t^{-1}\zeta$ of δ as "the pullback of ζ by h_t^{-1}".

1.3.4.3 Critical pullback

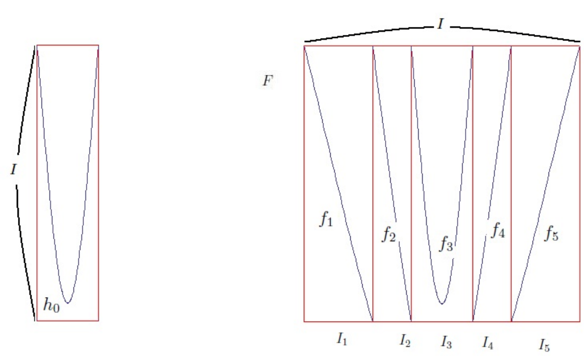


Figure 1.8: The critical branch to be refined and the power map to pullback with

Definition 3. Let F be a power map on I and let $h_0 : \delta \to I$ be the central critical branch of some power map. The *critical pullback of F by h_0^{-1}* is the new power map $F \circ h_0$ on δ.

The critical pullback is simply a combination of first a monotone pullback then a parabolic pullback. A critical pullback is always taken on the central critical branch. If F has branches f_k's with corresponding domains I_k's, we only take critical pullbacks in instances where $h_0(\frac{1}{2}) \in I_m$ and f_m is a monotone branch. In such cases, domains are created symmetrically on the left and right of $\frac{1}{2}$ and the central domain is $h_0^{-1}(I^m)$. Newly created branches $g_i \circ h_0$ could be either a critical branch or monotone branch again.

Let ξ be a partition of I into domains of F. We also consider the critical pullback of ξ into $h_0^{-1}\xi$ as "the pullback of ξ by h_0^{-1}".

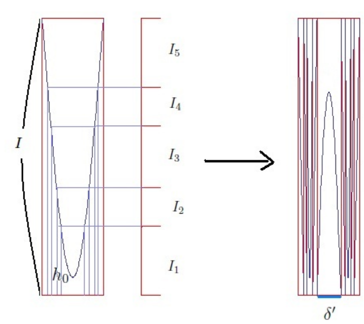


Figure 1.9: New branches after critical pullback

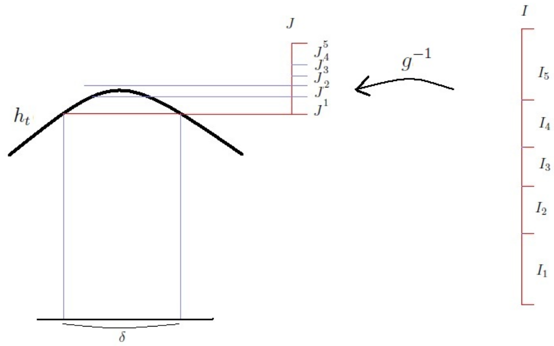


Figure 1.10: Critical pullback viewed as a monotone pullback combined with a parabolic pullback

1.3.4.4 Filling-in

Filling-in is a procedure which substitutes preimages of central holes by preimages of some partitions of central holes. A preimage of a central hole δ is represented by δ^{-n}.

Definition 4. Let $\mathcal{F} : \delta^{-n} \to \delta$ be a diffeomorphism and let H be a power map of f_t on δ. The *filling-in of* δ^{-n} *by* H is the new power map $H \circ \mathcal{F}$ on δ^{-n}.

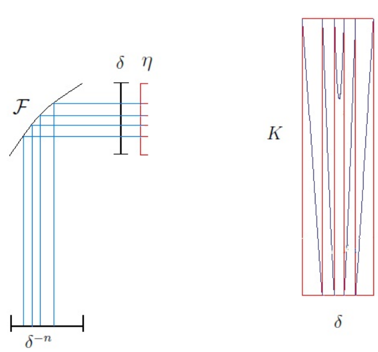


Figure 1.11: The hole and the power map to perform fill-in with

Filling-in is simply a monotone pullback performed on a smaller interval, and we distinguish it from monotone pullbacks because monotone pullbacks are performed on monotone domains and filling-ins are performed on holes.

Let η be a partition of δ into domains of H. We also consider the filling in of δ^{-n} by $\mathcal{F}^{-1}\eta$ as "the filling-in of δ^{-n} by η".

1.3.4.5 Purpose of each procedure

Each of the procedures plays an important role. Monotone pullbacks/refinements are for refinements on monotone domains that are comparatively large which in turn will have comparatively large extended domains. How refining monotone domains will give smaller extensions is explained in greater detail in the following section. Parabolic pullback is just for pulling back a partition/map from the y-axis onto the x-axis. Critical pullbacks refine the central domain. Filling-ins refine all holes other than the central hole. Both critical pullback and filling-in reduces the total measure of holes, which is one of the goals of our construction.

1.3.5 Extendability

Here we discuss the issue of extendability when performing the basic procedures. We explain ways to make our power maps extendable.

1.3.5.1 Extendability of the first return map

Let f be a diffeomorphism from Δ_1 onto J and g be a diffeomorophism from Δ_2 onto I with $J = \Delta_2$. Then $g \circ f$ is a diffeomorphism from Δ_1 onto I. A basic property of compositions is as follows.

Extendability property *Let $\tilde{\Delta}_1 \supset \Delta_1$ and $\tilde{\Delta}_2 \supset \Delta_2$. If f can be extended to a diffeomorphism from $\tilde{\Delta}_1$ onto $\tilde{J}$ and g can be extended to a diffeomorphism from $\tilde{\Delta}_2$ onto $\tilde{I}$ with $\tilde{J} \supset \tilde{\Delta}_2$, then $g \circ f$ can be extended to a diffeomorphism onto $\tilde{I}$.*

Using the above property of extendability, we will show that the interval $[f_t^N(\frac{1}{2}), f_t(\frac{1}{2})]$ is the maximal interval to which the first return map with $2N-1$ branches can be uniformly extended to. As shown in the previous subsection, each monotone branch f_k, $1 \leq |k| \leq N-1$, is given by the composition $g_{|k|} \circ f_t|_{\Delta_k}$, where each $g_{|k|} = f_t^{|k|}|_{J^{|k|}}$ is a diffeomorphism from $J^{|k|}$ onto I. The diffeomorphism $f_t^{|k|}$ on $J^{|k|}$ can be extended at most to a diffeomorphism on the interval $[f_r^{-1} \circ f_l^{-|k|+2}(\frac{1}{2}), 1]$, where $f_r^{-1} \circ f_l^{-|k|+2}(\frac{1}{2})$ is contained in $J^{|k|-1}$ and $[f_r^{-1} \circ f_l^{-|k|+2}(\frac{1}{2}), 1]$ is mapped onto $[0, f_t(\frac{1}{2})]$. Therefore $g_{|k|}$ can be extended to a diffeomorphism which maps $[f_r^{-1} \circ f_l^{-|k|+2}(\frac{1}{2}), 1]$ onto $[0, f_t(\frac{1}{2})]$. Each monotone domain Δ_k is mapped by f_t onto $J^{|k|}$, this can be extended to a diffeomorphism onto $[0, f_t(\frac{1}{2})]$. Combining the above analysis, the composition $f_k = g_{|k|} \circ f_t$ can be extended to a diffeomorphism from $[f_l^{-1} \circ f_r^{-1} \circ f_l^{-|k|+2}(\frac{1}{2}), \frac{1}{2}]$ (or $[\frac{1}{2}, f_r^{-1} \circ f_r^{-1} \circ f_l^{-|k|+2}(\frac{1}{2})]$) onto $[f_t^{|k|+1}(\frac{1}{2}), f_t(\frac{1}{2})]$. The interval $[f_t^{|k|+1}(\frac{1}{2}), f_t(\frac{1}{2})]$ is the smallest when $|k| = N-1$. Therefore the monotone branches can be uniformly extended to $[f_t^N(\frac{1}{2}), f_t(\frac{1}{2})]$. The central branch $h_0 = g_N \circ f_t|_\delta$ has image covering q. The greatest extent to which h_0 can be extended to is such that the image covers $[q, f_t(\frac{1}{2})]$. According to the definition in 1.1.4, we can conclude that the first return map can be uniformly extended to the interval $[f_t^N(\frac{1}{2}), f_t(\frac{1}{2})]$. It is the maximum possible interval of extension. If we pick $\tilde{I} = [f_t^N(\frac{1}{2}), f_t(\frac{1}{2})]$, the endpoints of the extended domains $\tilde{\Delta}_k$'s of Δ_k's and $\tilde{\delta}$ of δ excluding $\tilde{\Delta}_1$ and $\tilde{\Delta}_{-1}$ lie inside adjacent domains Δ_{k-1} and Δ_{k+1} or Δ_{N-1} and $\Delta_{-(N-1)}$, therefore inside I. The extended domains of Δ_1 and Δ_{-1} will always lie inside the extended image $\tilde{I}$ due to expanding property near the point q.

Later, $\tilde{I}$ may be chosen to be smaller than $[f_t^N(\frac{1}{2}), f_t(\frac{1}{2})]$ to accommodate more restrictions. The extended domains will then be smaller and will still satisfy the properties mentioned above.

1.3.5.2 Extendability after monotone refinement

Let F be a power map on I whose branches are uniformly extendable to $\tilde{I}$. Let $\tilde{I}_i$ be the extension of a subdomain I_i of I in the partition induced by F. Let f_0 be a monotone map on domain Δ_0 which is also extendable to $\tilde{I}$. We consider the extendability of the branches after a monotone pullback of F by f_0^{-1}. If $\tilde{I}_i \subset \tilde{I}$, then the newly created branch $f_i \circ f_0$ is also extendable to $\tilde{I}$. To guarantee uniform extendability of all new branches to $\tilde{I}$, F needs to be uniformly extendable to $\tilde{I}$ and $\tilde{I}$ needs to contain the union $\cup_k \tilde{I}_k$ of all extended domains. This will always be true in our case. Indeed, for all nonboundary branches, extensions of their domains are in I. For boundary branches, we use that their derivatives are greater than 3, and check directly that preimages of $\tilde{I}$ are contained in $\tilde{I}$.

1.3.5.3 Extendability of monotone domains after parabolic pullback or critical pullback

Since the critical pullback is a composition of a monotone pullback with a parabolic pullback, we will just give the criterion for extendability of branches after parabolic pullbacks. Let $J^{[a]}$ be a monotone domain on the y-axis mapped by $g_{[a]}$ diffeomorphically onto I. Suppose $g_{[a]}$ can be extended to a map $\tilde{g}_{[a]}$ that maps diffeomorphically onto $\tilde{I}$. The let $\tilde{J}^{[a]} = \tilde{g}_{[a]}^{-1}(\tilde{I})$. If $\tilde{J}^{[a]}$ is contained in the image of h_t, then the pullback of $g_{[a]}$ by f_l, $g_{[a]} \circ f_i|_{f_i^{-1}(J^{[a]})}$ (i=l,r) is also extendable to $\tilde{I}$. Otherwise, we perform the boundary refinement procedure defined below.

1.3.5.4 Boundary refinement

Boundary refinement is the procedure of taking a sequence of monotone refinements on boundary domains to meet the extendability criterion for a parabolic pullback.

First we define boundary partitions. Let $\hat{F}$ be a power map of f_t. We denote the map restricted to the leftmost domain Δ_l by f_l, and the map restricted to the rightmost domain Δ_r by f_r.

For t close to 4, boundary branches always satisfy the following properties. Since Δ_r is adjacent to q, within a neighborhood of q, and the derivative of f_t near q is approximately -2, f_r is always an expansion. Similarly, f_l is always an expansion. f_r is always monotonically increasing and f_l is always monotonically decreasing. Let $\hat{F}$ be is uniformly extendable to $\tilde{I}$, then f_l can be extended to a diffeomorphism $\tilde{f}_l$

on an extended domain $\tilde{\Delta}_l$ of Δ_l so that $\tilde{f}_l(\tilde{\Delta}_l) = \tilde{I}$. Similarly, $\tilde{f}_r(\tilde{\Delta}_r) = \tilde{I}$. For t close to 4 the derivative of f_t is close to -2 near q. For such t, f_r has derivative larger than 2 near q, and the right component of $\tilde{\Delta}_r \backslash \Delta_r$ has length less than $\frac{1}{2}$ the length of the right component of $\tilde{I} \backslash I$.

Consider the monotone pullback of $\hat{F}$ by f_l^{-1} onto Δ_l. We get a new map where Δ_l is refined. We denote the new map after monotone pullback by $\hat{F}_l$. The leftmost domain of this map is $f_l^{-1}(\Delta_r)$, which we denote by Δ_{lr}. We denote the branch $f_r \circ f_l$ on Δ_{lr} by f_{lr}. Since f_l has an extension $\tilde{f}_l$ that maps an extended domain $\tilde{\Delta}_l$ of Δ_l onto $\tilde{I}$ and $\tilde{I}$ includes $\tilde{\Delta}_r$, f_{lr} has an extension $\tilde{f}_{lr}$ that maps an extended domain $\tilde{\Delta}_{lr}$ of Δ_{lr} onto $\tilde{I}$. This extended domain $\tilde{\Delta}_{lr}$ is equal to $\tilde{f}_l^{-1}(\tilde{\Delta}_r)$. Since $\tilde{f}_l$ has derivative less than -2 near q^{-1}, the left component of $\tilde{\Delta}_{lr} \backslash \Delta_{lr}$ has length less than $\frac{1}{2}$ the length of the right component of $\tilde{\Delta}_r \backslash \Delta_r$.

We can consider again the monotone pullback of $\hat{F}$ by f_{lr}^{-1} onto Δ_{lr}. We denote the new map by $\hat{F}_{lr}$. The leftmost domain of this map is $f_{lr}^{-1}(\Delta_r)$ which we denote by Δ_{lrr}. The map on Δ_{lrr} is $f_r \circ f_{lr}$ which we denote by f_{lrr}. There is an extension $\tilde{f}_{lrr}$ of f_{lrr} such that $\tilde{f}_{lrr}$ maps an extended domain $\tilde{\Delta}_{lrr}$ of Δ_{lrr} onto $\tilde{I}$. $\tilde{\Delta}_{lrr}$ is equal to $f_{lr}^{-1}(\tilde{\Delta}_r)$. Since $f_{lr} = f_r \circ f_l$, f_{lr} has derivative less than -4, so the left component of $\tilde{\Delta}_{lrr} \backslash \Delta_{lrr}$ has length less than $\frac{1}{4}$ the length of the right component of $\tilde{\Delta}_r \backslash \Delta_r$.

Inductively, we can define $\Delta_{l\underbrace{r\cdots r}_{n \text{ times}}}$, $\tilde{\Delta}_{l\underbrace{r\cdots r}_{n \text{ times}}}$, $f_{l\underbrace{r\cdots r}_{n \text{ times}}}$, $\tilde{f}_{l\underbrace{r\cdots r}_{n \text{ times}}}$, and $\hat{F}_{l\underbrace{r\cdots r}_{n-1 \text{ times}}}$ by taking n consecutive monotone pullbacks of $\hat{F}$, each time on the leftmost domain. Since $f_{l\underbrace{r\cdots r}_{n-1 \text{ times}}}$ has derivative less than -2^n, the left component of $\tilde{\Delta}_{l\underbrace{r\cdots r}_{n \text{ times}}} \backslash \Delta_{l\underbrace{r\cdots r}_{n \text{ times}}}$ will have length which is less than $\frac{1}{2^n}$ times the length of the right component of $\tilde{\Delta}_r \backslash \Delta_r$. Therefore, the extended region that extends outside the left of I decreases exponentially. Extended domains of all other domains excluding Δ_r are contained in I.

A similar process can be applied to Δ_r of $\hat{F}$ to obtain $\Delta_{r\underbrace{r\cdots r}_{n \text{ times}}}$, $\tilde{\Delta}_{r\underbrace{r\cdots r}_{n \text{ times}}}$, $f_{r\underbrace{r\cdots r}_{n \text{ times}}}$, $\tilde{f}_{r\underbrace{r\cdots r}_{n \text{ times}}}$, and $\hat{F}_{r\underbrace{r\cdots r}_{n-1 \text{ times}}}$. These will give the boundary partitions which we pullback with.

Consider an interval $J^{[a]}$ on the y-axis which is mapped by $g_{[a]}$ onto I. Suppose that $g_{[a]}$ can be extended to a map $\tilde{g}_{[a]}$ that maps diffeomorphically onto $\tilde{I}$. In the case where $\tilde{J}^{[a]}$ is not contained in the image of h_t, we perform a boundary refinement which is done by a monotone pullback of $\hat{F}_{l\underbrace{r\cdots r}_{n-1 \text{ times}}}$ or $\hat{F}_{r\underbrace{r\cdots r}_{n-1 \text{ times}}}$ onto $J^{[a]}$ depending on which direction we want to shorten the extension by. A finite number of n times will be enough since as explained above, the extended length $\left|\tilde{\Delta}_{r\underbrace{r\cdots r}_{n \text{ times}}} \backslash I\right|$ decreases exponentially in size, and $\tilde{g}_{[a]}$ a has fixed distortion.

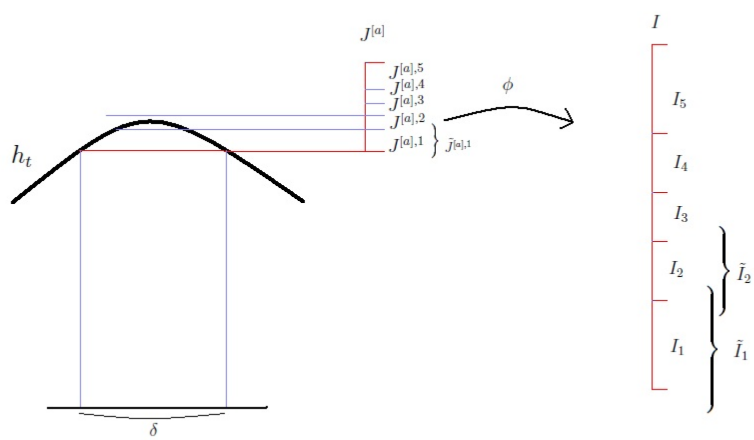


Figure 1.12: Extended domains and their pullbacks

1.3.5.5 Extendability after filling-in

Let δ be a central hole and δ^{-p} be its preimage. Let $\mathcal{F}$ be the diffeomorphism mapping δ^{-p} onto δ. Let η be a partition of δ consisting of monotone domains and smaller holes and let H be the power map on δ. Suppose that all monotone branches and all critical branches of H are uniformly extendable to $\tilde{I}$. If $\mathcal{F}$ can be extended so that its image contains the union of all extensions of monotone domains and critical domains in η, then all newly created branches in δ^{-p} will be extendable to $\tilde{I}$.

1.3.5.6 Enlargements of holes

δ, η, and $\mathcal{F}$ are defined as in the previous paragraph. Let $\tilde{\delta}$ be the union of all extensions of domains in η. To guarantee extendibility after filling-in, $\mathcal{F}$ needs to be extendable onto $\tilde{\delta}$. We define an enlargement $\hat{\delta}$ of δ as a larger interval which contains $\tilde{\delta}$. We shall define $\hat{\delta}$ below as some union of adjacent intervals large enough to contain $\tilde{\delta}$. When taking parabolic pullbacks and critical pullbacks the critical value should avoid enlargements $\hat{\delta}$ and all preimages $\hat{\delta}^{-p}$ of enlargements. That way, new monotone domains created after filling-in will again be extendable to $\tilde{I}$. All domains outside enlargements are considered to be *good domains*.

1.4 Dependence on parameter

When the critical value falls into good domains, we can take further pullbacks. These domains vary as the parameter values change. In order to estimate the measure of parameter values for which critical value falls into good domains, we need to calculate the dependence of interval partitions on the parameter.

Let $\Delta^*(t)$ be one of the good domains on the y-axis whose endpoints $y_1(t)$ and $y_2(t)$ vary continuously with respect to t. Let t_1 be the parameter where the critical value enters Δ^* and t_2 be the parameter where the critical value exits Δ^*. That is, $y_1(t_1) = \frac{t_1}{4}$ and $y_2(t_2) = \frac{t_2}{4}$. Let us define $\mathcal{T}(\Delta^*)$ as the interval $[t_1, t_2]$. Then we get the following lemma from [8].

Lemma 1. *Let $\Delta(t) = [y_1(t), y_2(t)]$ be an interval on the y-axis. Assume*

$$\left|\frac{dy_1(t)}{dt}\right|, \left|\frac{dy_2(t)}{dt}\right| < \epsilon. \tag{1.7}$$

Let $\mathcal{T}(\Delta) = [t_1, t_2]$ be the respective interval on the parameter axis, where t_1 is the time when $w(t)$ enters $\Delta(t)$ and t_2 is the time when $w(t)$ exits $\Delta(t)$. Then

$$\frac{1}{\frac{1}{4}+\epsilon} \leq \frac{|\mathcal{T}(\Delta)|}{|\Delta(t)|} \leq \frac{1}{\frac{1}{4}-\epsilon} \tag{1.8}$$

and

$$\frac{1-4\epsilon}{1+4\epsilon} \leq \frac{|\Delta(t)|}{|\Delta(t_1)|} \leq \frac{1+4\epsilon}{1-4\epsilon}. \tag{1.9}$$

for all $t \in \mathcal{T}(\Delta)$.

1.5 Transition from the phase space to the parameter space

The basic argument which allows us to estimate the portion of t such that $w(t)$ belongs to good intervals splits into 3 parts. At step n of induction we consider a parameter interval $\mathcal{T}^{(n-1)}$ such that $w(t)$ belongs to some interval $\Delta^{(n-1)}(t)$ on the y-axis. Interval $\Delta^{(n-1)}(t)$ is mapped by some branch $g_{(n-1)}$ (depending on t) onto I. By lemma 1, the length of $\mathcal{T}^{(n-1)}$ is close to $4|\Delta^{(n-1)}|$ for any $t \in \mathcal{T}^{(n-1)}$.

Part I We prove that for each $t \in \mathcal{T}^{(n-1)}$, and for k sufficiently large, $k < n$, the measure of holes in partition ξ_k is less than $C\theta^{k-14}$ for $k \geq 14$, where $\theta = 0.73$ and $C = 0.000210601$.

Part II We pullback some partition $\xi_{[sn]-3}$, $s < 1$, a few times to get a partition $\xi'_{[sn]-3}$ of I. Then we pullback $\xi'_{[sn]-3}$ onto $\Delta^{(n-1)}$ to get a partition $g^{-1}_{(n-1)}(\xi'_{[sn]-3}(t))$ of $\Delta^{(n-1)}$. Due to bounded distortion, the relative measure of holes in $g^{-1}_{(n-1)}(\xi'_{[sn]-3}(t))$ also decreases exponentially with n for each $t \in \mathcal{T}^{(n-1)}$. By lemma 1, the parameter interval corresponding to $w(t)$ belonging to a specific hole $\delta_i^{-p}(t)$ is close to $4|\delta_i^{-p}(t)|$ for any t such that $w(t)$ belongs to $\delta_i^{-p}(t)$.

Part III We show that for all $t \in \mathcal{T}^{(n-1)}$, relative measures of elements of $g^{-1}_{(n-1)}(\xi'_{[sn]-3}(t))$ in $\Delta^{(n-1)}$ remain almost the same.

Combining parts I, II, and III we get that the portion of nonadmissible parameter intervals at step n of induction decreases exponentially and get an estimate of the measure of good parameters with a.c.i.m., which proves the main theorem.

1.6 Examples

In this section, we provide two examples of specific parameter values such that respective maps have a.c.i.m..

1.6.1 The case where the critical value always falls into the sixth domain

Consider f_t where $f_t(\frac{1}{2}) \in \cdots \subset J^{4666} \subset J^{466} \subset J^{46} \subset J^4$. $J^{4\overbrace{6\cdots6}^{n+1 \text{ times}}}$ is the sixth interval of the pullback of the initial seven domain partition onto $J^{4\overbrace{6\cdots6}^{\text{n times}}}$. t will be one specific value in $[3.991749, 3.9933]$ (this is the interval for parameters t such that $f_t(\frac{1}{2}) \in J^{46}$). We denote this specific f_t as f for convenience. In this case, the critical point is preperiodic. By Misiurewicz's theorem [12] f has an a.c.i.m.. Here we give an independent proof as an example of applications of our method.

1.6.1.1 Construction of an induced map

Let F_0 be the first return map of f. Since $f(\frac{1}{2}) \in J^4$, F_0 has seven branches as discussed in chapter 1. The seven domains of the seven branches form a partition ξ_0 of I. $\xi_0 : I = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \delta_0 \cup \Delta_{-3} \cup \Delta_{-2} \cup \Delta_{-1}$, where Δ_i's are domains of monotone branches and δ_0 is the domain of the central critical branch. Branches of F_0 are denoted by $f_1 = F_0|_{\Delta_1} = f^2|_{\Delta_1}$, $f_2 = F_0|_{\Delta_2} = f^3|_{\Delta_2}$, $f_3 = F_0|_{\Delta_3} = f^4|_{\Delta_3}$, $h_0 = F_0|_{\delta_0} = f^5|_{\delta_0}$,$f_{-3} = F_0|_{\Delta_{-3}} = f^4|_{\Delta_{-3}}$, $f_{-2} = F_0|_{\Delta_{-2}} = f^3|_{\Delta_{-2}}$, and $f_{-1} = F_0|_{\Delta_{-1}} = f^2|_{\Delta_{-1}}$.

Our procedure for constructing a map that satisfies the conditions of the Folklore theorem is as follows. First, take a *critical pullback* of F_0 on the central branch $h_0 : \delta_0 \to I$ of F_0. Here h_0 can be written as the composition $g_4 \circ h|_{\delta_0}$, where h is just the parabolic map from δ_0 into J^4, and $g_4 = f^4|_{J^4}$ maps J^4 diffeomorphically onto I. If we pull back the partition ξ_0 by g_4^{-1} onto J^4, we get seven domains $J^{41} = g_4^{-1}(\Delta_{-1})$, $J^{42} = g_4^{-1}(\Delta_{-2})$, $J^{43} = g_4^{-1}(\Delta_{-3})$, $J^{44} = g_4^{-1}(\delta_0)$, $J^{45} = g_4^{-1}(\Delta_3)$, $J^{46} = g_4^{-1}(\Delta_2)$, and $J^{47} = g_4^{-1}(\Delta_1)$. Since by our assumption that $f(\frac{1}{2}) \in J^{46}$, taking a parabolic pullback of $J^{41}, \cdots, J^{47}$ by h^{-1} onto δ_0 will give 11 domains. The 11 domains include two that are preimages of δ_0 which we denote by δ_0^{-1} and one new central domain which we denote by δ_1. All others are monotone domains. We denote this partition of δ_0 into 11 domains by η_0. Next, we *fill-in* the two δ_0^{-1}'s using η_0 as a partition of δ_0, which in turn partitions δ_0^{-1} into 11 domains, including preimages of δ_0^{-1} which we denote by δ_0^{-2} and a preimage of δ_1 which we denote by δ_1^{-1}. After one critical pullback and filling-in of two holes, we denote the new map we have obtained by F_1

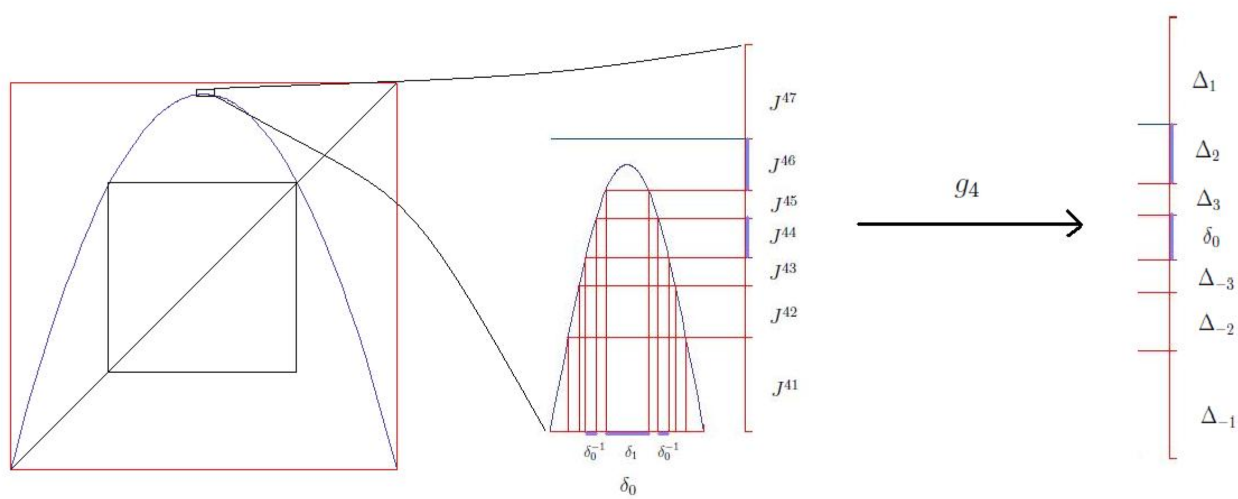


Figure 1.13: Refinement of J^4 by pullback of ξ_0

To obtain the power map F_{n+1} on I at step $n+1$, we define an inductive process. At the $n+1$th step, we have the map F_n with central branch $h_n : \delta_n \to I$ and some

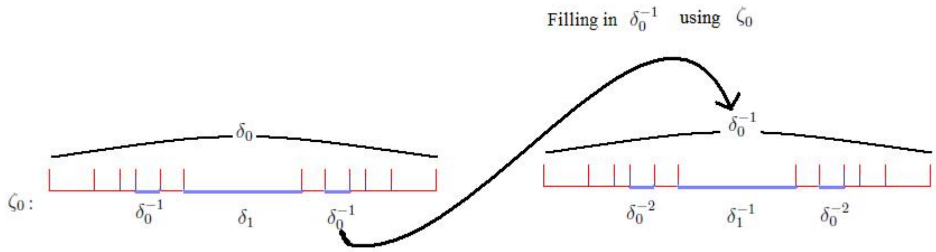


Figure 1.14: Filling-in δ_0^{-1} using ζ_0

holes δ_l^{-j}, where $0 \leq l \leq n$ and $1 \leq j \leq n+1$. First, we take a critical pullback of the first return map F_0 on the branch $h_n : \delta_n \to I$. $h_n = g_{4\underbrace{6\cdots6}_{\text{n times}}} \circ f|_{\delta_n}$, where f maps δ_n into $J^{4\overbrace{6\cdots6}^{\text{n times}}}$ and $g_{4\underbrace{6\cdots6}_{\text{n times}}} = \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4$ maps $J^{4\overbrace{6\cdots6}^{\text{n times}}}$ diffeomorphically onto I. The critical pullback of F_0 on h_n can be viewed as first taking a monotone pullback of ξ_0 onto $J^{4\overbrace{6\cdots6}^{\text{n times}}}$ to get seven subintervals $J^{4\overbrace{6\cdots6}^{\text{n times}}1}, \cdots, J^{4\overbrace{6\cdots6}^{\text{n times}}7}$, then taking a parabolic pullback of $J^{4\overbrace{6\cdots6}^{\text{n times}}1}, \cdots, J^{4\overbrace{6\cdots6}^{\text{n times}}7}$ onto δ_n. Since the critical value lies in $J^{\;4\overbrace{6\cdots6}^{\text{n+1 times}}}$, taking a parabolic pullback of $J^{4\overbrace{6\cdots6}^{\text{n times}}1}, \cdots, J^{4\overbrace{6\cdots6}^{\text{n times}}7}$ by h^{-1} onto δ_n will give 11 domains. $J^{4\overbrace{6\cdots6}^{n\text{ times}}4}$ is the preimage of δ_0, so two of the domains obtained after parabolic pullback are preimages of δ_0 which we denote by δ_0^{-1}. There will also be one new central branch formed by $h^{-1}(J^{\;4\overbrace{6\cdots6}^{n+1\text{ times}}})$ which we denote by δ_{n+1}. All other branches are monotone branches. We denote this partition of δ_n into 11 intervals by η_n. From the previous steps, holes δ_l^{-j}, where $0 \leq l \leq n$ and $1 \leq j \leq n+1$, were created as well as η_l were defined . We fill in δ_l^{-j} using η_l as a partition of δ_l. When we fill-in δ_l^{-j}, we will get 11 domains including preimages of

δ_0^{-1} which we denote by $\delta_0^{-(j+1)}$ and preimage of δ_{l+1} which we denote by δ_{l+1}^{-j}. At the $n+1$th step, we fill-in each existing hole once. After filling-in, we obtain a new map on I which we denote by F_{n+1}.

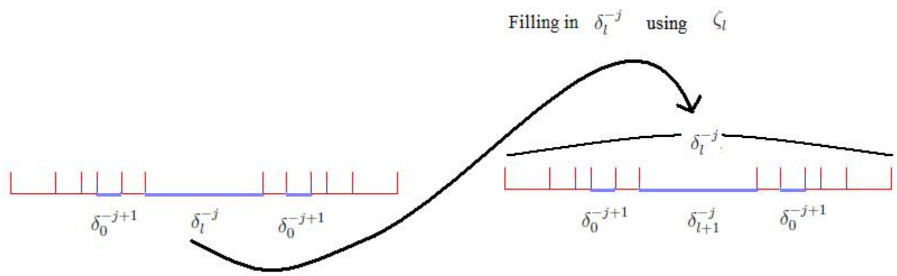


Figure 1.15: Filling-in δ_l^{-j} using ζ_l

Since the critical pullback is always performed using the initial partition, it is possible to choose an extension length e so that no boundary refinement is needed after each critical pullback. For a given e, we define $\tilde{I} = [q^{-1}-e, q+e]$. If e is chosen small enough so that the first return map is extendable to $\tilde{I}$, then we can define the extended domain of the domain of each branch in F_0. For each $i \in \{1, 2, 3, -3, -2, -1\}$ let $\tilde{\Delta}_i$ be the extended domain of Δ_i such that $f^{|i|+1}$ maps $\tilde{\Delta}_i$ diffeomorphically onto $\tilde{I}$. Let $\tilde{\delta}_0$ be the extended domain of δ_0 such that $f^5(\tilde{\delta}_0)$ covers $[q, q+e]$ on both ends of $\tilde{\delta}_0$. Endpoints of $\tilde{\delta}_0$ will lie in Δ_3 and Δ_{-3}. Endpoints of $\tilde{\Delta}_i$ will lie in the domains adjacent to Δ_i except for the left endpoint of $\tilde{\Delta}_1$ and right endpoint of $\tilde{\Delta}_{-1}$. The derivative at q is close to 2, therefore, f^2 has derivative close to 4 at q, and

$\tilde{\Delta}_1$ and $\tilde{\Delta}_{-1}$ would be contained in $\tilde{I}$. If we take a monotone pullback of the first return map F_0 to the branch $f_2 : \Delta_2 \to I$ of F_0, we will get 7 subdomains of Δ_2. Let $\hat{\xi}_0 = \xi_0 \vee f_2^{-1}(\xi_0)$ be a refined partition of ξ_0. If we pull back the partition $\hat{\xi}_0$ by g_4^{-1} onto J^4, the critical value $f(\frac{1}{2})$ will lie in $J^{466} = g_4^{-1}(f_2^{-1}(\Delta_{-2}))$. If we choose e small enough so that the left endpoint of $\tilde{\Delta}_3$ lies in $f_2^{-1}(\Delta_1)$, then $g_4^{-1}(\tilde{\Delta}_3)$ will lie in the range of f. Since the extension $\widetilde{J^{466}}$ of the pullback is equal to the pullback of the extension, we have that after the first critical pullback, the two branches adjacent to the central branch is extendable to $\tilde{I}$. Since left extensions of Δ_{-1}, Δ_{-2}, Δ_{-3}, and δ_0 are all contained in their adjacent intervals we have that $g_4^{-1}(\tilde{\Delta}_{-1})$, $g_4^{-1}(\tilde{\Delta}_{-2})$, $g_4^{-1}(\tilde{\Delta}_{-3})$, and $g_4^{-1}(\tilde{\delta}_0)$ are also contained in the image $f(\delta_0)$. After the first critical pullback, all branches are extendable to $\tilde{I}$. The arguments work exactly the same for the nth critical pullback, except instead of pulling back the partition $\hat{\xi}_0$ by g_4^{-1} onto J^4 we pull it back by $g^{-1}_{4\underbrace{6\cdots6}_{\text{n-1 times}}}$ onto $J^{4\overbrace{6\cdots6}^{\text{n-1 times}}}$. We can conclude that after each critical pullback, the new branches will be extendable to $\tilde{I}$.

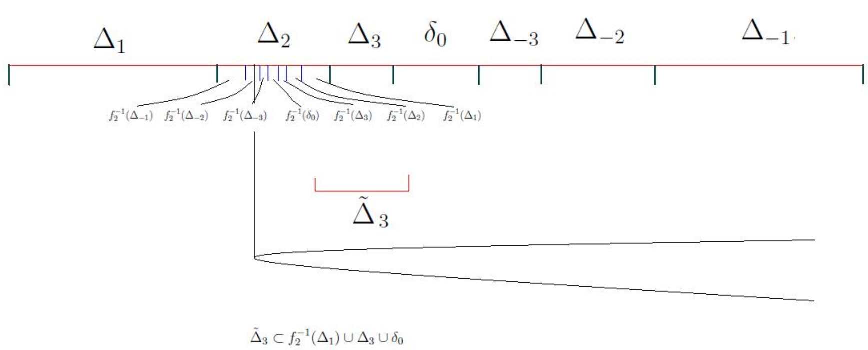


Figure 1.16: Relative position of critical value (domain sizes not to scale)

1.6.1.2 Exponential decrease of measure of holes

Next, we show that the total measure of holes in F_n decreases exponentially.

Since no boundary refinements are needed, new holes are formed from either a critical pullback or a filling in. In both of these processes, new holes lie inside original holes. To obtain the measure of holes in F_{n+1} relative to the holes in F_n, we need to obtain upper bounds for the ratios $\frac{|\delta_{n+1}|+2|\delta_0^{-1}|}{|\delta_n|}$ and $\frac{|\delta_{l+1}^{-j}|+2|\delta_0^{-j+1}|}{|\delta_l^{-j}|}$, where $\frac{|\delta_{n+1}|+2|\delta_0^{-1}|}{|\delta_n|}$ is the relative measure of new holes created in the central domain δ_n after a critical pullback, and $\frac{|\delta_{l+1}^{-j}|+2|\delta_0^{-j+1}|}{|\delta_l^{-j}|}$ is the relative measure of new holes created in δ_l^{-j} after a filling in.

For $\frac{|\delta_{l+1}|+2|\delta_0^{-1}|}{|\delta_l|}$, we need the distortion of h_n, where $h_n = g_{4\underbrace{6\cdots 6}_{\text{n times}}} \circ f = \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f$. First observe that the diffeomorphism g_4 from J^4 onto I is extendable to the interval $[1 - f(\frac{1}{2}), f(\frac{1}{2})]$. Let $\tilde{I} = [q^{-1} - e, q + e]$ as in the previous subsubsection. Since $\tilde{\Delta}_2$ and $\tilde{\Delta}_{-2}$ is contained in $\tilde{I}$ and $[1 - f(\frac{1}{2}), f(\frac{1}{2})]$, the composition $\overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 = g_{4\underbrace{6\cdots 6}_{\text{n times}}}$ is extendable to $\tilde{I}$. Let the ratio of e to $|I|$ be τ_1. By the Koebe distortion principle (1.3), we have

$$\frac{Dg_{4\underbrace{6\cdots 6}_{\text{n times}}}(x)}{Dg_{4\underbrace{6\cdots 6}_{\text{n times}}}(y)} \leq (\frac{1+\tau_1}{\tau_1})^2 = C_1 \tag{1.10}$$

for any x, y in $J^{4\overbrace{6\cdots 6}^{\text{n times}}}$, $n \in \mathbb{N}$. The following lemma is a consequence of (1.10).

Lemma 2. *For any two domains U, V in I, and any $n \in \mathbb{N}$ we have*

$$\frac{|g^{-1}_{4\underbrace{6\cdots 6}_{n\ times}}(U)|}{|g^{-1}_{4\underbrace{6\cdots 6}_{n\ times}}(V)|} \geq \frac{|U|}{|V|} \cdot \frac{1}{C_1}. \tag{1.11}$$

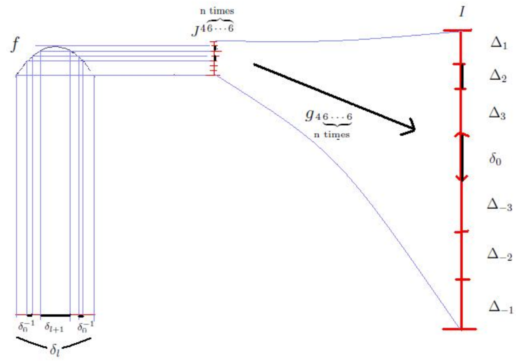


Figure 1.17: Partition of δ_l by a critical pullback of ξ_0

Since $J^{4\overbrace{6\cdots6}^{\text{n times}}1} = g^{-1}_{4\underbrace{6\cdots6}_{\text{n times}}}(\Delta_1)$, $J^{4\overbrace{6\cdots6}^{\text{n times}}2} = g^{-1}_{4\underbrace{6\cdots6}_{\text{n times}}}(\Delta_2)$, $J^{4\overbrace{6\cdots6}^{\text{n times}}3} = g^{-1}_{4\underbrace{6\cdots6}_{\text{n times}}}(\Delta_3)$, $J^{4\overbrace{6\cdots6}^{\text{n times}}4} = g^{-1}_{4\underbrace{6\cdots6}_{\text{n times}}}(\delta_0)$, $J^{4\overbrace{6\cdots6}^{\text{n times}}5} = g^{-1}_{4\underbrace{6\cdots6}_{\text{n times}}}(\Delta_{-3})$, $J^{4\overbrace{6\cdots6}^{\text{n times}}6} = g^{-1}_{4\underbrace{6\cdots6}_{\text{n times}}}(\Delta_{-2})$, by Lemma 1, there is a constant $K_1 < 1$ not depending on $n \in \mathbb{N}$ such that

$$\frac{|J^{4\overbrace{6\cdots6}^{\text{n times}}1}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}2}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}3}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}5}|}{|J^{4\overbrace{6\cdots6}^{\text{n times}}1}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}2}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}3}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}4}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}5}| + |J^{4\overbrace{6\cdots6}^{\text{n times}}6}|} \geq K_1.$$

The ratio of measures of two intervals each with an endpoint at the tip of the parabolic map will become the square root of the original ratio of measures after a parabolic pullback. To obtain an upper bound for $\frac{|\delta_{n+1}|+2|\delta_0^{-1}|}{|\delta_n|}$, we assume the worst position for $J^{4\overbrace{6\cdots6}^{\text{n times}}4}$. That is we assume that $J^{4\overbrace{6\cdots6}^{\text{n times}}4}$ is adjacent to $J^{4\overbrace{6\cdots6}^{\text{n times}}6}$. Then

$$\frac{|\delta_{n+1}|+2|\delta_0^{-1}|}{|\delta_n|}$$

$$\leq \sqrt{\frac{|J4\overbrace{6\cdots 6}^{\text{n times}}4|+|J4\overbrace{6\cdots 6}^{\text{n times}}6|}{|J4\overbrace{6\cdots 6}^{\text{n times}}1|+|J4\overbrace{6\cdots 6}^{\text{n times}}2|+|J4\overbrace{6\cdots 6}^{\text{n times}}3|+|J4\overbrace{6\cdots 6}^{\text{n times}}4|+|J4\overbrace{6\cdots 6}^{\text{n times}}5|+|J4\overbrace{6\cdots 6}^{\text{n times}}6|}}$$

$$\leq \sqrt{1-K_1}$$

If we let $K_2 = \sqrt{1-K_1} < 1$, then we have that

$$\frac{|\delta_{n+1}|+2|\delta_0^{-1}|}{|\delta_n|} \leq K_2 \tag{1.12}$$

for all $n \in \mathbb{N}$.

Next, we shall determine an upper bound for $\frac{|\delta_{l+1}^{-j}|+2|\delta_0^{-j+1}|}{|\delta_l^{-j}|}$. To do this, we analyse how δ_l^{-j} was obtained. Before that, we note that if k is fixed, critical branches which map δ_k^{-j}'s onto their image have the same height (image is the same) as the central critical branch defined on δ_k for all $j \in \mathbb{N}$.

δ_l^{-j} must be obtained from a filling in. If $l > 0$, then δ_l^{-j} was obtained from a filling in of δ_{l-1}^{-j} which was obtained from a filling in of δ_{l-2}^{-j} . If we look l steps before, we see that it came from a filling in of some interval δ_0^{-j}. Now we look at δ_0^{-j}, it was also obtained from filling-in of some $\delta_k^{-(j-1)}$.

Denote the branch on $\delta_k^{-(j-1)}$ by $h_{k,j-1}$. Since δ_0^{-j} is one of the preimages $h_{k,j-1}^{-1}(\delta_0)$, it is then easy to see that δ_l^{-j} is one of the preimages $h_{k,j-1}^{-1}(\delta_l)$ and filling

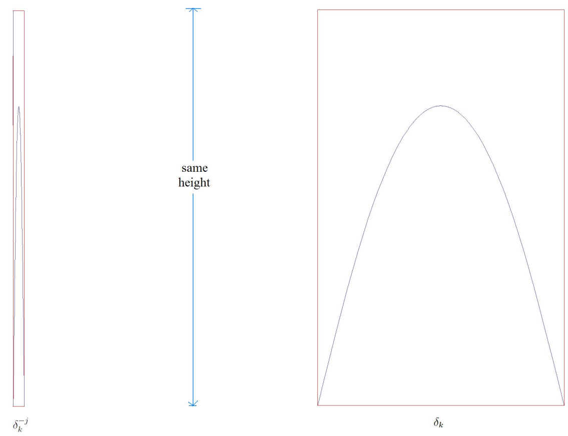


Figure 1.18: Comparing critical branch on a central critical domain with a critical branch on a preimage of the same central domain

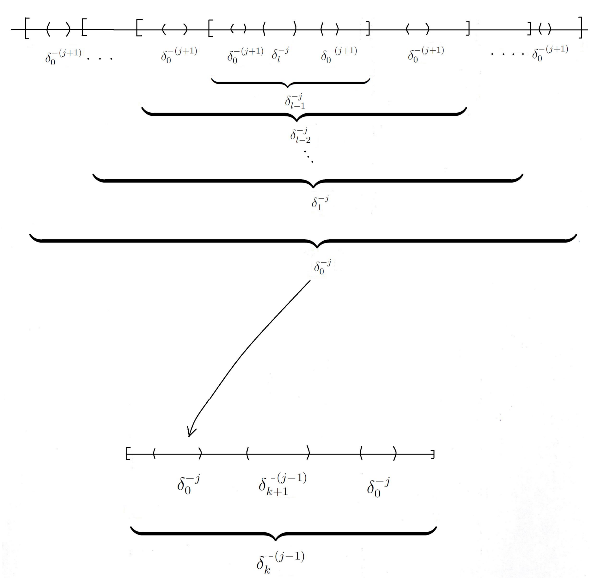


Figure 1.19: A hole δ_l^{-j} is contained in a corresponding hole δ_0^{-j}

in δ_l^{-j} means pulling back the partition η_l by $h_{k,j-1}^{-1}$ onto δ_l^{-j}. Hence, all we need is the extendability constant of $h_{k,j-1}$ on the interval δ_0^{-j}. Since we know that the height of $h_{k,j-1}$ is the same as the height of h_k, the critical value of $h_{k,j-1}$ lies in the sixth domain of ξ_0. That is, the extendability constant is greater than $\tau_2 = \frac{|\Delta_3|}{|\delta_0|}$.

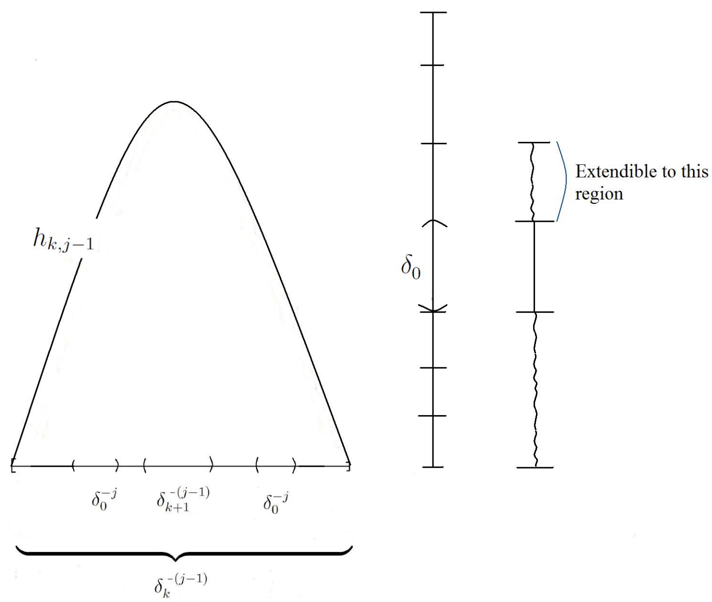


Figure 1.20: Critical value avoids a fixed neighborhood of δ_0

Again, by the Koebe Distortion Principle (1.3), we have

$$\frac{Dh_{k,j-1}(x)}{Dh_{k,j-1}(y)} \leq (\frac{1+\tau_2}{\tau_2})^2 = C_2 \tag{1.13}$$

for any x, y in δ_0^{-j}. Using (1.12), and (1.13) we get

$$\begin{aligned}
\frac{|\delta_{l+1}^{-j}| + 2|\delta_0^{-j+1}|}{|\delta_l^{-j}|} &= 1 - (1 - \frac{|\delta_{l+1}^{-j}| + 2|\delta_0^{-j+1}|}{|\delta_l^{-j}|}) \\
&= 1 - \frac{|\delta_l^{-j} \backslash \delta_{l+1}^{-j} \cup \delta_0^{-j} \cup \delta_0^{-j}|}{|\delta_l^{-j}|} \\
&\leq 1 - \frac{|h_{k,j-1}(\delta_l^{-j} \backslash \delta_{l+1}^{-j} \cup \delta_0^{-j} \cup \delta_0^{-j})|}{|h_{k,j-1}(\delta_l^{-j})|} \cdot \frac{1}{C_2} \\
&\leq 1 - (1 - K_2) \cdot \frac{1}{C_2}.
\end{aligned}$$

By letting $K_3 = 1 - (1 - K_2) \cdot \frac{1}{C_2} < 1$, we have that

$$\frac{|\delta_{l+1}^{-j}| + 2|\delta_0^{-j+1}|}{|\delta_l^{-j}|} \leq K_3 \tag{1.14}$$

for all $l \in \mathbb{N}$ and $l \in \mathbb{N}$. Let $\alpha^{(n)}$ be the total measure of holes in map F_n and let $K = \max\{K_2, K_3\}$. By (1.12) and (1.14) we have

$$\frac{\alpha^{(n+1)}}{\alpha^{(n)}} \leq K \tag{1.15}$$

for all $n \in \mathbb{N}$. We can conclude that the measure of holes decrease exponentially. The limiting map of $\{F_n\}$ which we denote by F_∞ will have infinitely many monotone branches.

1.6.1.3 Verification of summability condition

What remains is the verification of the summability condition $\sum_k |I_k| n_k < \infty$, where I_k are the branches in F_∞ and n_k is the power of each branch. We need only to look at the increase of power after each induction step. Consider again the central branch of F_n which can be written as $h_n = g_{4\underbrace{6\cdots6}_{\text{n times}}} \circ f = \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ h$. After a critical pullback, the new branches formed are $f_{-2} \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ h$, $f_{-3} \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_r$, $f_{-3} \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_l$, $h_0 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_r$, $h_0 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_l$, $f_3 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_r$, $f_3 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_l$, $f_2 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_r$, $f_2 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_l$, $f_1 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_r$, and $f_1 \circ \overbrace{f_{-2} \circ \cdots \circ f_{-2}}^{\text{n-1 times}} \circ f_2 \circ g_4 \circ f_l$, where $f_l = h|_{(0,\frac{1}{2})}$ and $f_r = h|_{(\frac{1}{2},1)}$. The power in each branch increases by at most 5.

For the filling in of δ_l^{-j}, the power also increases by at most 5, since by analysis in previous paragraphs, filling in δ_l^{-j} is replacing $h_{k,j-1}^{-1}(\delta_l)$ by partitioned domains $h_{k,j-1}^{-1}(\eta_l)$ for some k. The map $f_l \circ h_{k,j-1}$ becomes maps $f_{-2} \circ f_l \circ h_{k,j-1}$, $f_{-3} \circ f_l \circ h_{k,j-1}$, $h_0 \circ f_l \circ h_{k,j-1}$, $f_3 \circ f_l \circ h_{k,j-1}$, $f_2 \circ f_l \circ h_{k,j-1}$, or $f_1 \circ f_l \circ h_{k,j-1}$. In this case, the power increases the same way as in the critical pullback of ξ_0 on δ_l. Therefore, at the nth step, the greatest power is going to be no more than $5(n+1)$. Lengths of domains of new branches produced in the nth step will have total measure less than the total measure of holes in the $n-1$th step. Therefore we have by (1.15)

$$\sum_k |I_k| n_k \leq |I| \cdot 4 + \sum_{n=1}^{\infty} |\delta_0| \cdot K^{n-1} \cdot 5(n+1) < \infty.$$

1.6.2 Non-Misiurewicz case

We would like to construct a map that consists of an a.c.i.m. but is not in the Misiurewicz case. We start again with the first return map F_0 to $I = [q^{-1}, q]$. We would like to pick a parameter so that the forward iterates of the critical point returns arbitrarily close to the critical point. We define our inductive steps so that the total measure of holes reduces to less than some $K < 1$ times the measure of holes in the previous step. We would also like to maintain a fixed distortion for the power maps as in the previous section. We take a critical pullback of the partition ξ_0 and assume that the critical value of the central branch falls into the 6th domain of ξ_0 for most inductive steps , but occasionally at the M_kth step, the critical value will lie in the monotone domain just outside the domain δ_k, and we will pullback the partition ξ_{N_k}, where $N_k < M_k$ will be determined later.

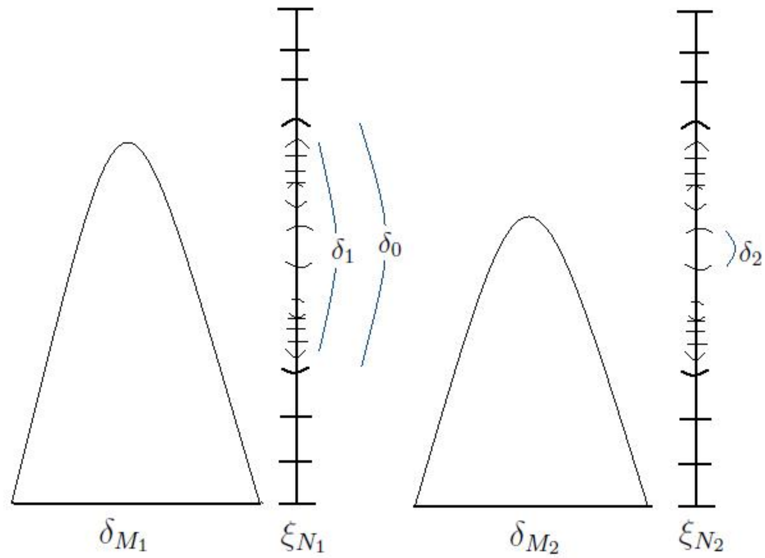


Figure 1.21: Pulling back different partitions in different specified steps

In this subsection, we will use the same set of variables as we did in the previous subsection, but their values and what they represent may differ.

F_0 has seven branches with domains Δ_1, Δ_2, Δ_3, δ_0, Δ_{-3}, Δ_{-2}, and Δ_{-1}. We denote the branches of F_0 again by $f_1 = F_0|_{\Delta_1} = f^2|_{\Delta_1}$, $f_2 = F_0|_{\Delta_2} = f^3|_{\Delta_2}$, $f_3 = F_0|_{\Delta_3} = f^4|_{\Delta_3}$, $h_0 = F_0|_{\delta_0} = f^5|_{\delta_0}$,$f_{-3} = F_0|_{\Delta_{-3}} = f^4|_{\Delta_{-3}}$, $f_{-2} = F_0|_{\Delta_{-2}} = f^3|_{\Delta_{-2}}$, and $f_{-1} = F_0|_{\Delta_{-1}} = f^2|_{\Delta_{-1}}$. Let the partition of I into the seven intervals be ξ_0. We can pull back partition ξ_0 onto each of the seven intervals. For example, $\Delta_1 = \Delta_{11} \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{14} \cup \Delta_{15} \cup \Delta_{16} \cup \Delta_{17}$ where $\Delta_{11} = f_1^{-1}(\Delta_{-1})$, $\Delta_{12} = f_1^{-1}(\Delta_{-2})$, $\Delta_{13} = f_1^{-1}(\Delta_{-3})$, $\Delta_{14} = f_1^{-1}(\delta_0)$, $\Delta_{15} = f_1^{-1}(\Delta_3)$, $\Delta_{16} = f_1^{-1}(\Delta_2)$, and $\Delta_{17} = f_1^{-1}(\Delta_1)$. For the pullback of ξ_0 onto δ_0 we have $\delta_{01} = h_{0l}^{-1}(\Delta_{-1})$, $\delta_{02} = h_{0l}^{-1}(\Delta_{-2})$, $\delta_{03} = h_{0l}^{-1}(\Delta_{-3})$, $\delta_{04} = h_{01}^{-1}(\delta_0)$, $\delta_{05} = h_{0l}^{-1}(\Delta_3)$, $\delta_{06} = h_0^{-1}(\Delta_2)$, $\delta_{07} = h_{0r}^{-1}(\Delta_3)$, $\delta_{08} = h_{0r}^{-1}(\delta_0)$, $\delta_{09} = h_{0r}^{-1}(\Delta_{-3})$, $\delta_{0(10)} = h_{0r}^{-1}(\Delta_{-2})$, and $\delta_{0(11)} = h_{0r}^{-1}(\Delta_{-1})$, where h_{0l} and h_{0r} are h_0 restricted to the left and right half of δ_0, respectively. Let $\tilde{I} = [q^{-1}-e, q+e]$, where e is some small number that is to be determined. Then we can define the extended domain $\tilde{\Delta}_i$ of Δ_i for each $i \in \{1, 2, 3, -3, -2, -1\}$ so that $f^{|i|+1}$ maps $\tilde{\Delta}_i$ diffeomorphically onto $\tilde{I}$. $\tilde{\delta}_0$ is defined so that $h_{0l}(\tilde{\delta}_{01})$ covers $[q, q+e]$ and $h_{0r}(\tilde{\delta}_{0r})$ covers $[q, q+e]$. Here $\tilde{\delta}_{0l}$ and $\tilde{\delta}_{0r}$ are the left and right half of δ_0 respectively. We can pick a number e that is small enough so that the right endpoint of $\tilde{\Delta}_1$ is contained in Δ_{21}, the right endpoint of $\tilde{\Delta}_2$ is contained in Δ_{31}, the right endpoint of $\tilde{\Delta}_3$ is contained in δ_{01}, the right endpoint of $\tilde{\delta}_0$ is contained in $\Delta_{(-3)1}$, the right endpoint of $\tilde{\Delta}_{-3}$ is contained in $\Delta_{(-2)1}$, the right endpoint of $\tilde{\Delta}_{-2}$ is contained in $\Delta_{(-1)1}$, the left endpoint of $\tilde{\Delta}_2$ is contained in Δ_{17}, the left endpoint of $\tilde{\Delta}_3$ is contained in Δ_{27}, the left endpoint of $\tilde{\delta}_0$ is contained in Δ_{37}, the left endpoint of $\tilde{\Delta}_{-3}$ is contained in $\delta_{0(11)}$, the left endpoint of $\tilde{\Delta}_{-2}$ is contained in $\Delta_{(-3)7}$, and the left endpoint of $\tilde{\Delta}_{-1}$ is contained in $\Delta_{(-2)7}$. By this choice of e, we will be able to avoid boundary refinements after each critical pullback.

Let h_n be the central critical branch of F_n, where F_n will be constructed according to rules in later description. $h_n = g_{[n]} \circ f$, where f is the parabolic map that maps δ_n into some interval $J^{[n]}$ and $g_{[n]}$ maps $J^{[n]}$ diffeomorphically onto I.

Let $\tau_1 = \frac{e}{|I|}$. Using the Koebe distortion principle (1.3), we can get

$$\frac{Dg_{[n]}(x)}{Dg_{[n]}(y)} \leq (\frac{1+\tau_1}{\tau_1})^2 = C_1 \tag{1.16}$$

for any x, y in $J^{[n]}$, and $n \in \mathbb{N}$. Similar to lemma 2 in the previous subsection, we have for any two domains U, V in I, and any $n \in \mathbb{N}$

$$\frac{|g_{[n]}^{-1}(U)|}{|g_{[n]}^{-1}(V)|} \geq \frac{|U|}{|V|} \cdot \frac{1}{C_1}. \tag{1.17}$$

Suppose that the critical value of h_n was in domain Δ_{-2} of partition ξ_0, then when we pull back partition ξ_0 by h_n^{-1} onto δ_n, there will be 11 new domains. Estimate of the ratio of total measure of new holes in δ_n to the length of δ_n is given by the same estimate as in (1.12). In the case where we pull back some partition ξ_{N_k} by $h_{M_k}^{-1}$ onto domain δ_{M_k}, since all holes in ξ_{N_k} are in δ_0 and since the critical value is positioned inside δ_0, we can obtain the following estimate.

$$\begin{aligned}
&\frac{|\text{new holes in } \delta_{M_k}|}{|\delta_{M_k}|} \\
&\leq \sqrt{1 - \text{ ratio of nonholes in the image } f(\delta_{M_k}) \cdot \frac{1}{C_1}} \\
&\leq \sqrt{1 - \frac{|\Delta_1| + |\Delta_2| + |\Delta_3|}{|\delta_0| + |\Delta_1| + |\Delta_2| + |\Delta_3|} \cdot \frac{1}{C_1}}
\end{aligned}$$

There is a constant K_2 that bounds $\sqrt{1 - \frac{|\Delta_1|+|\Delta_2|+|\Delta_3|}{|\delta_0|+|\Delta_1|+|\Delta_2|+|\Delta_3|} \cdot \frac{1}{C_1}}$ from above. Since the measure of the central critical domain decreases exponentially when we take critical pullbacks of ξ_0, it is possible to choose N_k such that $\frac{|\delta_{N_k}|}{|\delta_k|} < \frac{1}{2}$.

Then

$$\frac{Dh_{M_k}(x)}{Dh_{M_k}(y)} \leq (\frac{1+1}{1})^2 = 4 \quad \text{for any } x, y \quad \text{in} \quad \delta_{N_k}^{-1}$$

Therefore, we have as in the previous section, distortion for filling-in is the equal or less than the case in the previous section. We get a constant $K < 1$ as in (1.15).

The increase in number of iterates by a filling-in is bounded above by the increase in number of iterates of critical pullbacks in former steps. Set λ as an arbitrary number less than 1. Next, we define how M_k are chosen. Let P_k be the greatest power in the map F_{N_k}. Pick M_k so that $(K)^{M_k} \cdot P_k \leq \lambda^k$. Iterates at steps M_k to M_{k+1} cannot increase more than P_k at each step. Then we get

$$\begin{aligned}
&\sum_k |I_k| n_k \\
\leq &\sum_{k_1=0}^{M_1} (K)^{k_1} 5k_1 \\
+ &\sum_{k_2=M_1+1}^{M_2} (K)^{k_2} (5M_1 + P_1(k_2 - M_1)) \\
+ &\sum_{k_3=M_2+1}^{M_3} (K)^{k_3} (5M_1 + P_1(M_2 - M_1) + P_2(k_3 - M_2)) \\
+ &\cdots \\
\leq &\sum_{l_1=0}^{\infty} (K)^{l_1} 5l_1 + \sum_{l_2=M_1}^{\infty} (K)^{l_2} P_1(l_2 - M_1) + \sum_{l_3=M_2}^{\infty} (K)^{l_3} P_{N_2}(l_3 - M_2) + \cdots \\
\leq &(5 + (K)^{M_1} P_1 + (K)^{M_2} P_2 + \cdots) \sum_{l=0}^{\infty} (K)^l l \\
< &\infty
\end{aligned}$$

Chapter 2

Proof of the main theorem

In [7] and [8], two different algorithms were used to show positivity of measure for parameters t whose corresponding maps f_t's attain a.c.i.m.s. In this chapter, we combine the techniques of [7] and [8] with some new tools to develop a new algorithm for choosing parameters. We will show that under this algorithm, the parameters with a.c.i.m. form a set with measure greater than $1.58382 * 10^{-16} * 4.65 * 10^{-6}$.

2.1 Basic approach

We start by restricting our construction to a small parameter interval $\mathcal{T}_0$ that is close to $t = 4$ but disjoint from $t = 4$. $\mathcal{T}_0$ is chosen so that for $t \in \mathcal{T}_0$ partitions induced by power maps of f_t are dynamically equivalent up to five steps of critical pullbacks. That is, the partitioning points are preimages of q obtained by the same sequences of left and right preimages.

For each $t \in \mathcal{T}_0$, we have the partition ξ_0 of I which is the partition resulting from the first return map of f_t. We also have the partition ξ_5 which is the partition after 5 critical pullbacks by ξ_0. The critical value of the central branch of ξ_5 varies at full scale in I, whereas all branches of ξ_0 have little variation with respect to t in $\mathcal{T}_0$. This means two things. First, we need to choose subintervals from $\mathcal{T}_0$ so that critical values of the central branch of ξ_5 falls into valid domains. Second, we can refine ξ_5 with ξ_0 and obtain uniform estimates on domain sizes, derivatives and velocities of newly defined partitions for all t in $\mathcal{T}_0$. Original domain sizes, derivatives, and velocities for ξ_0 and ξ_5 are obtained numerically by Mathematica, see Appendix B.

At each inductive step n, we are confined to a finite union of admissible intervals $\cup\mathcal{T}^{(n-1)} \subset \mathcal{T}_0$. For each admissible interval $\mathcal{T}^{(n-1)}$, there is a corresponding partition $\xi_{n-1}(\mathcal{T}^{(n-1)})$ of I. For $t \in \mathcal{T}^{(n-1)}$ elements of $\xi_{n-1}(\mathcal{T}^{(n-1)})$ vary continuously. The critical value of the central branch of $\xi_{n-1}(\mathcal{T}^{(n-1)})$ varies at full scale in I for t in $\mathcal{T}^{(n-1)}$. This compels us to choose admissible subintervals $\mathcal{T}^{(n)}$'s from $\mathcal{T}^{(n-1)}$ such that the critical value of the central branch of ξ_{n-1} falls into valid domains. We always refine ξ_{n-1} with an earlier partition $\xi_{[sn]}$, $0 < s < 1$, which varies little with respect to t in $\mathcal{T}^{n-1}$. This allows us to make uniform estimates on newly defined partitions. Our algorithm is designed so that monotone branches of each partition ξ_{n-1} are uniformly extendable to some fixed interval $\tilde{I}$. We keep track of estimates on domain sizes, derivatives, and velocities.

From the algorithm, we get a sequence of collections of admissible parameter intervals $\{\mathcal{T}^{(6)}\}$, $\{\mathcal{T}^{(7)}\}$, $\cdots$, $\{\mathcal{T}^{(n)}\}$, $\cdots$, where the collection at step n is nested in the collection at step $n-1$. That is, for each $\mathcal{T}^{(n)}_{i_6 \dots i_{n-1} i_n} \in \{\mathcal{T}^{(n)}\}$, there is some $\mathcal{T}^{(n-1)}_{i_6 \dots i_{n-1}} \in \{\mathcal{T}^{(n-1)}\}$ such that $\mathcal{T}^{(n)}_{i_6 \dots i_{n-1} i_n} \subset \mathcal{T}^{(n-1)}_{i_6 \dots i_{n-1}}$.

We wish to get

$$\lim_{n\to\infty} \max_{i_6 \dots i_n}\{\text{measure of holes in } \xi_n(\mathcal{T}^{(n)}_{i_6 \dots i_n})\} = 0 \tag{2.1}$$

and

$$\lim_{n\to\infty}\sum_{i_6\ldots i_n}|\mathcal{T}^{(n)}_{i_6\ldots i_n}|=\alpha>0. \tag{2.2}$$

2.2 Preliminary construction (steps 0 through 5)

2.2.1 Initial choice of parameters

As discussed in 1.3.1, we can define right preimages $J^1 = [q, q_r^{-2}], J^2 = [q_r^{-2}, q_r^{-3}], J^3 = [q_r^{-3}, q_r^{-4}], \ldots$ of $I = [q^{-1}, q]$, depending continuously on the parameter t. According to (1.6), the rates at which the endpoints of J^n vary are relatively slow compared to the constant speed $\frac{1}{4}$ at which the critical value $\omega(t) = f_t(\frac{1}{2}) = \frac{t}{4}$ moves upward. Therefore, there are exact times t_n when the critical value enters each J^n. So when $t \in [t_n, t_{n+1}]$, $\omega(t) \in J^n$. As a primary choice of parameter values, we restrict t to $\mathcal{T}^4 := [t_4, t_5]$. Using Mathematica to solve for $f_t^4(w(t)) = q_t$ and $f_t^4(w(t)) = q_t^{-1}$, we get

$$\mathcal{T}^4 \approx [3.9826, 3.9956].$$

2.2.2 The first return map and partition ξ_0

For $t \in \mathcal{T}^4$, the first return map has 7 branches. On the left, the first return map consists of monotone domains Δ_1, Δ_2, and Δ_3 with corresponding branches denoted by $f_{0,1} = f_t^2|_{\Delta_1}$, $f_{0,2} = f_t^3|_{\Delta_2}$ and $f_{0,3} = f_t^4|_{\Delta_3}$. Symmetrically on the right are monotone domains Δ_{-1}, Δ_{-2}, and Δ_{-3} with corresponding branches denoted by $f_{0,-1} = f_t^2|_{\Delta_{-1}}$, $f_{0,-2} = f_t^3|_{\Delta_{-2}}$ and $f_{0,-3} = f_t^4|_{\Delta_{-3}}$. The central domain, denoted by δ_0, is the domain of a critical branch denoted by $h_0 = f_t^5|_{\delta_0}$. The seven domains

$\Delta_1, \Delta_2, \Delta_3, \delta_0, \Delta_{-3}, \Delta_{-2}$ and Δ_{-1} form a partition of I which we denote by ξ_0.

2.2.3 Domain Δ_y and partition $\zeta^{(0)}(\mathcal{T}^4)$

Considering J^1, J^2, J^3 and J^4 as domains on the y-axis, we define the domain Δ_y as

$$\Delta_y := J^1 \cup J^2 \cup J^3 \cup J^4. \tag{2.3}$$

The respective partition of Δ_y is denoted by $\zeta^{(0)}(\mathcal{T}^4)$. This partition of Δ_y exists for all $t \in \mathbf{int}(\mathcal{T}^4)$. Since we consider $J^1, \cdots, J^4$ as subintervals of Δ_y on the y-axis, we call $\zeta^{(0)}(\mathcal{T}^4)$ a partition of Δ_y on the y-axis. Note that the parabolic pullback of $\zeta^{(0)}(\mathcal{T}^4)$ onto I is exactly the partition ξ_0.

2.2.4 Further choice of parameter values

Using the partition ξ_0, we would like to restrict our parameter values further. J^4 is mapped by f_t^4 diffeomorphically onto I. Let $g_4 := f_t^4|_{J^4}$. If we pullback the partition ξ_0 of I by g_4^{-1} onto J^4, there will be 7 subintervals of J^4. We will label them by $J^{41}, J^{42}, \ldots, J^{47}$ from bottom to top. J^{41} is mapped by g_4 onto Δ_{-1}, J^{42} is mapped by g_4 onto Δ_{-2}, J^{43} is mapped by g_4 onto Δ_{-3}, J^{44} is mapped by g_4 onto δ_0, J^{45} is mapped by g_4 onto Δ_3, J^{46} is mapped by g_4 onto Δ_2, and J^{47} is mapped by g_4 onto Δ_1. We can obtain numerically the velocities of endpoints of $J^{41}, \cdots, J^{47}$ and get that values are always less than 0.003. Therefore, entrance and exit times of $w(t)$ to each J^{4i} exist and are unique. This is also true for more pullbacks of ξ_0, and we will not repeat this argument later. We would like to restrict parameter values so that $\omega(t) \in J^{47}$. We denote the corresponding parameter interval by $\mathcal{T}^{47}$.

$$\mathcal{T}^{47} \approx [3.9933, 3.9956].$$

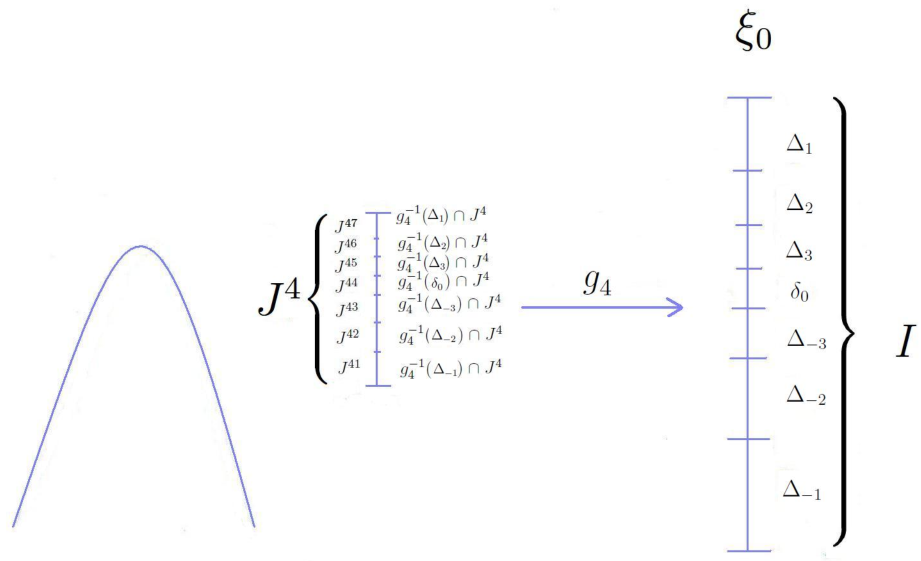


Figure 2.1: Pulling back ξ_0 by g_4^{-1} onto J^4

If we look at the first return maps of f_t's for which $t \in \mathcal{T}^{47}$, those are exactly the cases when the image of the central branch h_0 covers domains Δ_{-1} through Δ_2 and the critical value of h_0 falls into the domain Δ_1. Since

$$\Delta_y = J^1 \cup J^2 \cup J^3 \cup J^{41} \cup \cdots \cup J^{47} \text{ for all } t \in \mathcal{T}^{47}, \tag{2.4}$$

there is a corresponding partition of Δ_y which we denote by $\zeta^{(1)}(\mathcal{T}^{47})$. $\zeta^{(1)}(\mathcal{T}^{47})$ is a refinement of $\zeta^{(0)}(\mathcal{T}^4)$ for all $t \in \mathcal{T}^{47}$.

Since f_t^4 maps J^{47} diffeomorphically onto Δ_1 and f_t^2 maps Δ_1 diffeomorphically onto I, then J^{47} is mapped by f_t^6 diffeomorphically onto I. We can pull back the partition ξ_0 by $(f_t^6|_{J^{47}})^{-1}$ onto J^{47} and get 7 subintervals of J^{47}. We label them $J^{471}, J^{472}, \ldots, J^{477}$ from bottom to top. J^{471} is mapped by f_t^6 onto Δ_1, $\cdots$, J^{477} is mapped by f_t^6 onto Δ_{-1}. We make a further restriction of our parameter values so that $\omega(t) \in J^{476}$, and denote the corresponding parameter interval by $\mathcal{T}^{476}$. We obtain numerically that $\mathcal{T}^{476}$ is approximately $\mathcal{T}^{476} \approx [3.99483, 3.99513]$.

Again we have a refined partition $\zeta^{(2)}(\mathcal{T}^{476})$ of Δ_y on the y-axis.

$$\Delta_y = J^1 \cup J^2 \cup J^3 \cup J^{41} \cup \cdots \cup J^{46} \cup J^{471} \cup \cdots \cup J^{477}.$$

In general, if an interval $J^{[a]}$ on the y-axis is mapped by some diffeomorphism $g_{[a]}$ onto I, then we can pullback partition ξ_0 by $g_{[a]}^{-1}$ onto $J^{[a]}$ to form 7 subintervals which we label from bottom to top as $J^{[a]1}$, $J^{[a]2}$, $\ldots$, $J^{[a]7}$. We can also define in the parameter space the corresponding intervals $\mathcal{T}^{[a]}$ which is the interval of all t's where $w(t) \in J^{[a]}$. With this defined, we choose the interval $\mathcal{T}_0 = \mathcal{T}^{476777}$ as the set of initial parameter values to work with. We obtain numerically that $\mathcal{T}^{476777}$ is approximately $\mathcal{T}^{476777} \approx [3.99512535856, 3.99513000705]$.

$$\left|\mathcal{T}^{476777}\right| > 4.6485 * 10^{-6}. \tag{2.5}$$

Partitions $\zeta^{(3)}(\mathcal{T}^{4767})$, $\zeta^{(4)}(\mathcal{T}^{47677})$ and $\zeta^{(5)}(\mathcal{T}^{476777})$ are defined analogously to $\zeta^{(0)}(\mathcal{T}^{4})$, $\zeta^{(1)}(\mathcal{T}^{47})$ and $\zeta^{(2)}(\mathcal{T}^{476})$, where $\zeta^{(k)}(\mathcal{T}^{[a]i})$ is a refinement of $\zeta^{(k-1)}(\mathcal{T}^{[a]})$.

2.2.5 First five steps

For coherence with later construction, we define the first five steps and partitions $\xi_1, \xi_2, \cdots, \xi_5$. For all $t \in \mathcal{T}^{476777}$ we can perform all of the following steps creating dynamically equivalent partitions, *dynamically equivalent* in the sense that each branch corresponding to each domain is the same power of f_t for all $t \in \mathcal{T}^{476777}$, and branches are varying continuously.

Step 0 We create partition ξ_0 given by the first return map. Domains in ξ_0 are $\Delta_1, \Delta_2, \Delta_3, \delta_0, \Delta_{-3}, \Delta_{-2}, \Delta_{-1}$.

Step 1 We take a critical pullback of ξ_0 on δ_0 and denote the new partition by $\xi_1 : I = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5 \cup \Delta_6 \cup \delta_0^{-1} \cup \Delta_7 \cup \Delta_8 \cup \delta_1 \cup \Delta_{-8} \cup \Delta_{-7} \cup \delta_0^{-1} \cup \Delta_{-6} \cup \Delta_{-5} \cup \Delta_{-4} \cup \Delta_{-3} \cup \Delta_{-2} \cup \Delta_{-1}$.

Step 2 We take a critical pullback of ξ_0 on δ_1 and denote the new partition by $\xi_2 : I = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5 \cup \Delta_6 \cup \delta_0^{-1} \cup \Delta_7 \cup \Delta_8 \cup \Delta_9 \cup \Delta_{(10)} \cup \Delta_{(11)} \cup \delta_0^{-1} \cup \Delta_{(12)} \cup \delta_2 \cup \Delta_{-(12)} \cup \delta_0^{-1} \cup \Delta_{-(11)} \cup \Delta_{-(10)} \cup \Delta_{-9} \cup \Delta_{-8} \cup \Delta_{-7} \cup \delta_0^{-1} \cup \Delta_{-6} \cup \Delta_{-5} \cup \Delta_{-4} \cup \Delta_{-3} \cup \Delta_{-2} \cup \Delta_{-1}$.

Steps 3,4,5 Similarly, we take consecutive critical pullbacks on δ_2, δ_3, δ_4 to form ξ_3, ξ_4, ξ_5.

`Remark 1.` For $t \in \mathcal{T}^{476777}$, ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 are exactly the parabolic pullbacks of $\zeta^{(1)}(\mathcal{T}^{47})$, $\zeta^{(2)}(\mathcal{T}^{476})$, $\zeta^{(3)}(\mathcal{T}^{4767})$, $\zeta^{(4)}(\mathcal{T}^{47677})$ and $\zeta^{(5)}(\mathcal{T}^{476777})$ onto I, respectively.

2.2.6 Holes and branches in ξ_5

Totally, ξ_5 consists of 65 domains. Elements of ξ_5 are monotone domains, preimages δ_0^{-1} of δ_0 and the central domain which we denote by δ_5. δ_5 is the central hole and we refer to the 10 preimages of δ_0 as the "five holes" since there are five on each side. We let $f_{5,i}$ denote monotone branches in ξ_5 and $\mathcal{F}_{5,i}$ denote the monotone maps defined on the five holes which map each hole onto δ_0. Let $\Delta^{(5)}$ be the domain J^{476777} on the y-axis and let $g_{(5)}$ be the map from $\Delta^{(5)}$ onto I. Consider the five preimages of δ_0 in $\zeta^{(5)}(\mathcal{T}^{476777})$ whose parabolic pullbacks are the five holes on the x-axis, let $\mathcal{G}_{5,i}$ denote the maps from these preimages onto δ_0.

2.2.7 Extension constant and uniform extendability of branches in ξ_5

An extended domain $\tilde{I}$ of I is chosen so that the first return map is uniformly extendable to $\tilde{I}$ for each $t \in \mathcal{T}^{476777}$. since the extension of the third branch extends a little below $q^{-1} - 0.17$, we select our extension constant to be 0.17. According to 1.3.5.1, all other branches of the first return map can then be extended below to $q^{-1} - 0.17$ and above to $q^{-1} + 0.17$.

In the following context, we speak of partitions ξ_n of I with associated branches to each domain. We would like each monotone branch outside δ_n^{re} and holes in ξ_n to be extendable to $\tilde{I}$, then we say that branches in ξ_n are uniformly extendable to $\tilde{I}$.

Lemma 3. *For $t \in \mathcal{T}^{476777}$, monotone branches in ξ_1, ξ_2, ξ_3 and ξ_4 are all uniformly extendable to $\tilde{I}$.*

Proof. First, we look at the extendability of Δ_4, Δ_5, Δ_6, δ_0^{-1}, Δ_7, and Δ_8. Since

$t \in \mathcal{T}^{476777}$, we have $t \in \mathcal{T}^{47}$ and $t \in \mathcal{T}^{476}$. The critical value $w(t)$ falls into the domain J^{47}. Whether $\Delta_4, \ldots, \Delta_8$ are extendable depends on whether $\tilde{J}^{41}$, ..., $\tilde{J}^{46}$ lie in the image of h. Here, $\tilde{J}^{41}$ is the pullback of $\tilde{\Delta}_{-1}$ by g_4^{-1}, $\tilde{J}^{42}$ is the pullback of $\tilde{\Delta}_{-2}$ by g_4^{-1}, $\tilde{J}^{43}$ is the pullback of $\tilde{\Delta}_{-3}$ by g_4^{-1}, $\tilde{J}^{44}$ is the pullback of $\tilde{\delta}_0$ by g_4^{-1}, $\tilde{J}^{45}$ is the pullback of $\tilde{\Delta}_3$ by g_4^{-1}, $\tilde{J}^{46}$ is the pullback of $\tilde{\Delta}_2$ by g_4^{-1}, and $\tilde{J}^{47}$ is the pullback of $\tilde{\Delta}_1$ by g_4^{-1}. Since we also know that $w(t) \in J^{476}$, it means $g_4(w(t))$ falls into Δ_{12} where Δ_{11}, Δ_{12}, ..., Δ_{17} are subdomains ordered from left to right of Δ_1 given by a monotone pullback of ξ_0 on Δ_1. We know that all left extensions fall into adjacent domains (see subsection 1.3.5.1), therefore $\tilde{\Delta}_{-1}$, $\tilde{\Delta}_{-2}$, $\tilde{\Delta}_{-3}$, $\tilde{\delta}_0$, and $\tilde{\Delta}_3$ are contained in the image of $g_4 \circ h_t|_{\delta_0}$. To determine whether $\tilde{\Delta}_2$ is contained in the image of $g_4 \circ h_t|_{\delta_0}$, it is enough to compare the left endpoint of $\tilde{\Delta}_2$ with right endpoint of Δ_{12}. We can obtain numerically that the left endpoint of $\tilde{\Delta}_2$ is greater than 0.34281 for all $t \in \mathcal{T}^{476777}$. The right endpoint of Δ_{12} is less than 0.294612 for all $t \in \mathcal{T}^{476777}$. This shows that $\tilde{\Delta}_2$ is always contained in the image of $g_4 \circ h_t|_{\delta_0}$.

For the extendability of Δ_9, $\Delta_{(10)}$, $\Delta_{(11)}$, δ_0^{-1}, and $\Delta_{(12)}$, arguments are the same as in the previous paragraph, except that here we need the left endpoint of $\tilde{\Delta}_3$ to be greater than the right endpoint of Δ_{21}, where Δ_{21} is the first subdomain of Δ_2 given by a monotone pullback of ξ_0 on Δ_2. For extendability of $\Delta_{(13)}$, $\Delta_{(14)}$, $\Delta_{(15)}$, δ_0^{-1}, $\Delta_{(16)}$, and $\Delta_{(17)}$, we need that the left endpoint of $\tilde{\Delta}_2$ be greater than the right endpoint of Δ_{11}, which follows from the previous paragraph. Likewise, the extendability of domains $\Delta_{(13)}$ through $\Delta_{(22)}$ follows.

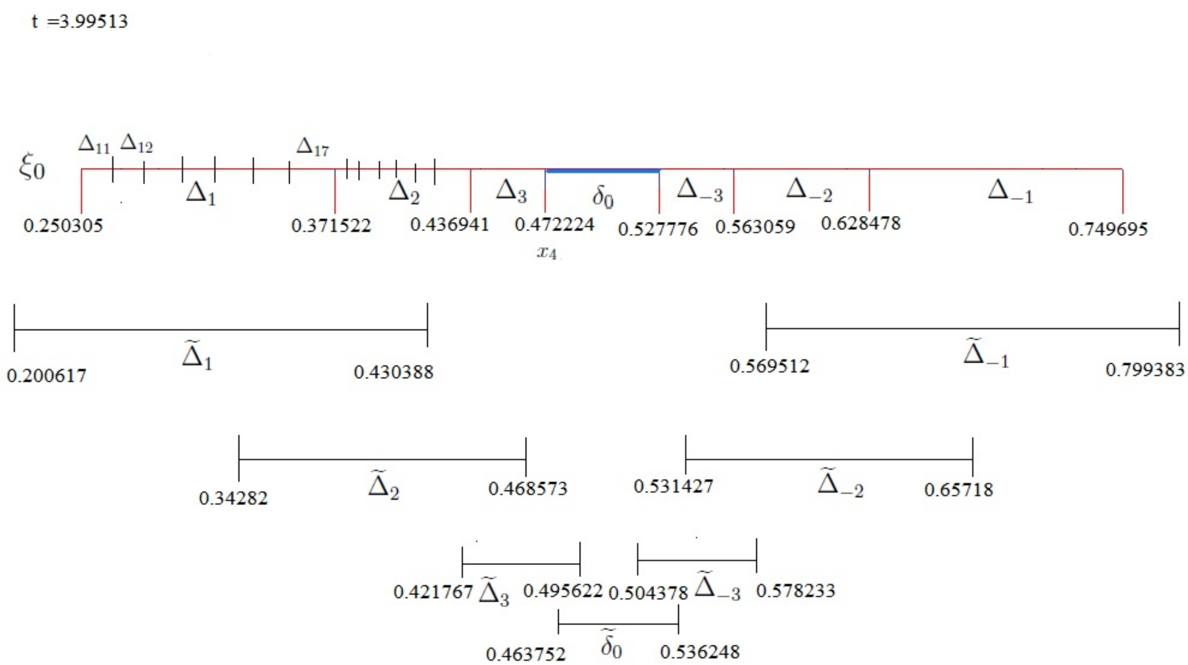


Figure 2.2: Domains of ξ_0 and respective extended domains

The above is a figure that shows the partition ξ_0 and relative positions of extensions of each domain in ξ_0 for the specific parameter value $t = 3.99513$.

□

After step 5, branches adjacent to δ_5 may not be extendable to $\tilde{I}$ when $w(t)$ is close to the lower endpoint of J^{476777}. To avoid such problems, we make an additional assumption:

$$t > 3.99512595. \tag{2.6}$$

This number was obtained by considering one of the two branches adjacent to the central branch of ξ_5 and observing at what parameters its extension falls short of 0.17.

Lemma 4. *For $t \in \mathcal{T}^{476777}$, critical branches in ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 are all uniformly* *extendable to $\tilde{I}$.*

Proof. According to 1.2.4, a critical branch is extendable to $\tilde{I}$ if it can be extended so that it covers one component of $\tilde{I}\backslash I$. From the figure above, we can see that the extended domain $\tilde{\delta}_0$ of δ_0 lies in $\Delta_3 \cup \delta_0 \cup \Delta_{-3}$. By the choice of parameter($t \in \mathcal{T}^{476777}$), the critical values of the central branches in $\xi_0, \ldots, \xi_4$ are always positioned outside the extended domain $\tilde{\delta}_0$. That makes all holes δ_0^{-p} created from the first five steps extendable to $\tilde{I}$. $\square$

We conclude that for all $t \in \mathcal{T}^{476777}$ satisfying (2.6), all branches of ξ_5 are uniformly extendable to $\tilde{I}$.

2.2.8 Enlargement of δ_0 and distortion on δ_0^{-p}

In the previous subsection, we showed that the critical value avoids extended domains $\tilde{\delta}_0^{-p}$ of δ_0^{-p} so that new critical branches formed after parabolic pullbacks are are also extendable to $\tilde{I}$. In fact, the critical value in the central branch of $\xi_0, \ldots, \xi_4$ avoids a larger neighborhood around δ_0, namely, $\hat{\delta}_0$ where

$$\hat{\delta}_0 = \Delta'_2 \cup \Delta_3 \cup \delta_0 \cup \Delta_{-3} \cup \Delta'_{-2}, \tag{2.7}$$

$$\Delta'_2 = \Delta_{22} \cup \Delta_{23} \cup \Delta_{24} \cup \Delta_{25} \cup \Delta_{26} \cup \Delta_{27}, \tag{2.8}$$

$$\Delta'_{-2} = \Delta_{(-2)1} \cup \Delta_{(-2)2} \cup \Delta_{(-2)3} \cup \Delta_{(-2)4} \cup \Delta_{(-2)5} \cup \Delta_{(-2)6}. \tag{2.9}$$

This fixed region that we avoid around δ_0^{-p} will allow us to give uniform estimates for distortion. $\hat{\delta}_0$ is called the enlargement of δ_0.

Suppose a hole δ_0^{-p} is mapped by some diffeomorphism $\mathcal{F}$ monotonically onto δ_0 and is extendable to $\hat{\delta}_0$ as defined in (2.7). Let us define $\mathcal{D}_{X \text{ over } \tilde{X}}$ as the upper bound, given by the Koebe distortion principle, of the distortion on X when extension is $\tilde{X}$. Then we have

$$\left| \frac{\frac{\partial \mathcal{F}}{\partial x}(x_0)}{\frac{\partial \mathcal{F}}{\partial x}(y_0)} \right| \leq \mathcal{D}_{\delta_0 \text{ over } \tilde{\delta}_0} := \left(1 + \frac{|\delta_0|}{\frac{1}{2}\left|\hat{\delta}_0 \setminus \delta_0\right|} \right)^2 < 2.75 \qquad \text{for } \; x_0, y_0 \in \delta_0^{-p} \tag{2.10}$$

for $t \in \mathcal{T}^{476777}$. The last number was obtained from estimates on sizes of δ_0 and $\hat{\delta}_0$.

2.2.9 Partition η_0 of δ_0

Let η_0 be the restriction of partition ξ_5 to δ_0. η_0 has 59 domains and its holes include 10 preimages of δ_0 and one central domain δ_5. The relative measure of holes $\mu_{\text{holes}}(\eta_0)$ in η_0 is between 0.166 and 0.178 for $t \in \mathcal{T}^{476777} \cap \{t > 3.99512595\}$ (see first figure in B.1.1).

Later in the algorithm, we will perform 5-step filling-ins on preimages of δ_0 defined as follows.

Definition 5. Let δ_0^{-p} be a preimage of δ_0 mapped by a diffeomorphism $\mathcal{F}$ onto δ_0. A 5-step filling-in of δ_0^{-p} is replacing δ_0^{-p} by $\mathcal{F}^{-1}(\eta_0)$.

For a 5-step filling-in of δ_0^{-p}, we can obtain an estimate for the relative measure of holes in $\mathcal{F}^{-1}(\eta_0)$ using the inequality (A.3) from the Appendix. We denote the relative measure of holes in $\mathcal{F}^{-1}(\eta_0)$ by $\mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_0))$.

$$\mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_0)) \leq \frac{\mathcal{D}_{\delta_0 \text{ over } \tilde{\delta}_0} * \mu_{\text{holes}}(\eta_0)}{1 - \mu_{\text{holes}}(\eta_0) + \mathcal{D}_{\delta_0 \text{ over } \tilde{\delta}_0} * \mu_{\text{holes}}(\eta_0)} < \frac{2.75 * 0.178}{1 - 0.178 + 2.75 * 0.178} < 0.373238 \tag{2.11}$$

The above estimate does not depend on $\mathcal{F}$ as it only depends on the fact that $\mathcal{F}$ is extendable to $\hat{\delta}_0$.

To improve the estimate for $\mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_0))$, we divide δ_0 into 5 sections and calculate a bound for each distorted ratio separately. Dividing δ_0 into sections allows us to obtain smaller distortion bounds. This method is particularly effective when the holes are in a sense "evenly scattered". We use the formula (2.12) below and the Koebe distortion principle combined to obtain the bounds.

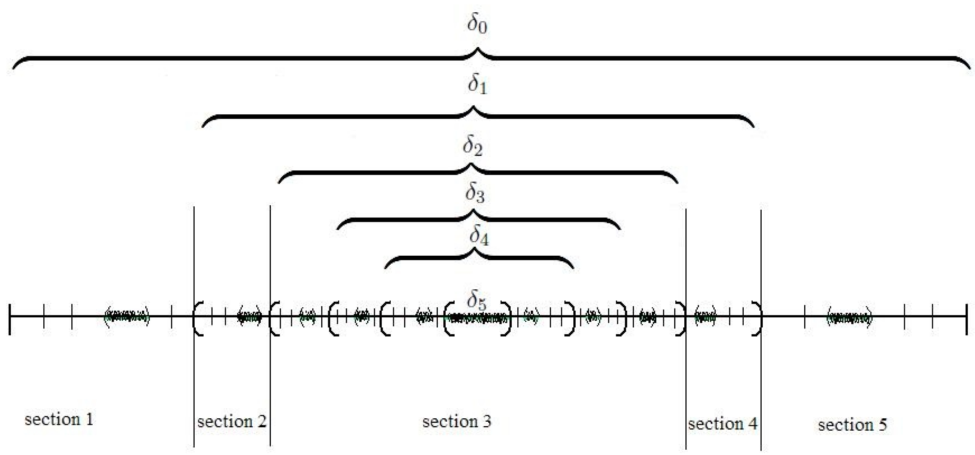


Figure 2.3: Partition of δ_0 into five sections

The sections are shown in the above figure. For each section, a distortion bound is given by formula (1.3) from the Koebe distortion principle. For example, the extended part of section one on the left is the left component $\tilde{I}\backslash I$ and the extended part of section 2 is the union of the left component of $\tilde{I}\backslash I$ with section 1. We denote the bound corresponding to section i by d_i. r_i denotes the relative measure of holes in section i and r_i' denotes the relative measure of holes in the corresponding section i of δ_0^{-k}. From (A.3), we get that

$$r_i' \leq \frac{d_i \cdot r_i}{1 - r_i + d_i \cdot r_i} \tag{2.12}$$

Table 2.1: Distortion bounds and bounds for relative measure of holes in each section

section	sections 1 and 5	sections 2 and 4	section 3
upper bound for d_i	1.44113	1.113251	1.16614
upper bound for r_i	0.145941141	0.20592	0.25624640
upper bound for $\frac{d_i \cdot r_i}{1-r_i+d_i \cdot r_i}$	0.197599	0.22702	0.286617

The bounds for d_i and r_i are valid for all $t \in \mathcal{T}^{476777}$. We can conclude that

$$\mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_0)) < 0.29. \tag{2.13}$$

This is a better estimate than (2.11).

2.2.10 Preliminary estimates

All preliminary estimates are obtained numerically from Mathematica. Sizes of domains and derivatives of branches in partitions $\xi_0, \ldots, \xi_5$ are listed in B.1.1 and B.1.2. Bounds for derivative with respect to t and variation of derivatives are listed in B.1.3 and B.1.4, respectively.

Let $\mu_{\text{holes}}(\xi)$ denote the relative measure of holes in ξ. All other notations are defined in earlier subsections of this section.

Important estimates for ξ_0 are below.

1.
$$\mu_{\text{holes}}(\xi_0) = \frac{|\delta_0|}{|I|} < 0.11123 \tag{2.14}$$

2. By the negative Schwarzian derivative property, the minimum of the absolute value of derivatives occurs on endpoints. Therefore by computing minimum at endpoints, we get the minimum derivative over each domain.
$$\left|\frac{\partial f_{0,i}}{\partial x}\right| > 3.5 \quad t \in \mathcal{T}_0, x \in \Delta_i \tag{2.15}$$

3.
$$\frac{\left|\frac{\partial f_{0,i}}{\partial t}\right|}{\left|\frac{\partial f_{0,i}}{\partial x}\right|} < 1.109 \quad t \in \mathcal{T}_0, x \in \Delta_i \tag{2.16}$$

4.
$$\frac{\left|\frac{\partial^2 f_{0,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial f_{0,i}^{-1}}{\partial z}\right|} < 50 \quad t \in \mathcal{T}_0, z \in I \tag{2.17}$$

Important estimates for ξ_5 are below.

1.
$$\mu_{\text{holes}}(\xi_5) = \frac{|\delta_5|}{|I|} < 0.0022 \quad t \in \mathcal{T}_0 \tag{2.18}$$

2.
$$\left|\frac{\partial f_{5,i}}{\partial x}\right| > 3.5 \quad t \in \mathcal{T}_0, x \in \Delta f_{5,i} \tag{2.19}$$

$\Delta f_{5,i}$ is the domain of $f_{5,i}$.

3.
$$\frac{\left|\frac{\partial f_{5,i}}{\partial t}\right|}{\left|\frac{\partial f_{5,i}}{\partial x}\right|} < 161 \quad t \in \mathcal{T}_0, x \in \Delta f_{5,i} \tag{2.20}$$

4.
$$\frac{\left|\frac{\partial^2 f_{5,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial f_{5,i}^{-1}}{\partial z}\right|} < 900000 \quad t \in \mathcal{T}_0, z \in I \tag{2.21}$$

Estimates for $g_{(5)}$ are below.

1.
$$\left|\frac{\partial g_{(5)}}{\partial y}\right| > 391005 \quad t \in \mathcal{T}_0, x \in \Delta^{(5)} \tag{2.22}$$

2. Velocities on the endpoints of $\Delta^{(5)}$ are less than 0.0019

3.
$$\frac{\left|\frac{\partial g_{(5)}}{\partial t}\right|}{\left|\frac{\partial g_{(5)}}{\partial x}\right|} < 0.00188 \quad t \in \mathcal{T}_0, x \in \Delta^{(5)} \tag{2.23}$$

4.
$$\frac{\left|\frac{\partial^2 g_{(5)}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{(5)}^{-1}}{\partial z}\right|} < 8.9 \quad t \in \mathcal{T}_0, z \in I \tag{2.24}$$

5.
$$\frac{\left|\frac{\partial^2 g_{(5)}}{\partial x^2}\right|}{\left|\frac{\partial g_{(5)}}{\partial x}\right|^2} < 1.5 \quad t \in \mathcal{T}_0, x \in \Delta^{(5)} \tag{2.25}$$

2.3 The algorithm

2.3.1 Step 6

Starting from step 6, we begin to choose subintervals $\mathcal{T}^{(6)}$'s of $\mathcal{T}^{(5)} = \mathcal{T}_0 = \mathcal{T}^{476777}$ which are admissible according to the rules of general construction. We also create new partitions $\xi_6(t)$, $\zeta^{(6)}(\Delta^{(6)})$, and $\eta_5(\Delta^{(6)})$. Domains in these partitions vary continuously when t's are in the same $\mathcal{T}^{(6)}$. We explain the algorithm below.

2.3.1.1 Starting partitions and intervals

For each t in $\mathcal{T}^{(5)}$, we have the dynamically equivalent 7 branch partition $\xi_0(t)$ whose partitioning points vary little among different t's. We also have the dynamically equivalent 65 branch partition $\xi_5(t)$ created after 5 consecutive critical pullbacks, where the central domain $\delta_5(t)$ and nearby domains vary greatly. $\Delta^{(5)}(t) = J^{476777}(t)$ is the interval on the y-axis where $w(t) \in \Delta^{(5)}(t)$ corresponds to the maps where the critical value belongs consecutively to the 7th, 6th, 7th, 7th, 7th domains after each critical pullback of $\xi_0(t)$. By construction, t is in $\mathcal{T}^{(5)}$ if and only if $w(t)$ is in $\Delta^{(5)}(t)$. We denote the lower endpoint of $\Delta^{(5)}(t)$ by $y_5(t)$, then $y_5(t)$ is exactly the image of the two endpoints of $\delta_5(t)$.

All domains and partitions depend on t, but t may be omitted in later context for convenience.

2.3.1.2 Choosing $\mathcal{T}^{(6)}$, creating $\Delta^{(6)}$ and $\zeta_1^{(6)}(\Delta^{(6)})$

At step 6 we partition $\Delta^{(5)}$ by pulling back ξ_0 onto $\Delta^{(5)}$ once. We get a partition of Δ_y which is a refinement of $\zeta^{(5)}(\mathcal{T}^{(5)})$ and we denote it by $\zeta_1^{(6)}(\Delta^{(5)})$.

When defining $\mathcal{T}^{(6)}$'s, our goal is to make each $|\mathcal{T}^{(6)}|$ small enough so that the position of points that partition $\Delta^{(5)}(t)$ varies little for t in a fixed $\mathcal{T}^{(6)}$. That is, we would like $w(t)$ to move across some small domain $\Delta^{(6)}$ when t moves across $\mathcal{T}^{(6)}$.

A domain $\Delta^{(6)}$ is considered to be small enough for parameter choice if

$$\frac{|\Delta^{(6)}(t)|}{H_5(\Delta^{(6)}(t))} < \vartheta_1 \text{ for all } t \in \mathcal{T}^{(5)}, \tag{2.26}$$

where ϑ_1 is defined in (2.104) and $H_5(\Delta) := \mathrm{dist}(\Delta, y_5)$. If $w(t) \in \Delta^{(6)}$, then the measure $|[y_5(t), w(t)]|$ would be close to $H_5(\Delta^{(6)}(t))$, and $H_5(\Delta^{(6)}(t))$ has small variation for t in $\mathcal{T}^{(5)}$.

The algorithm below defines $\mathcal{T}^{(6)}$, $\Delta^{(6)}$ and $\zeta_1^{(6)}(\Delta^{(6)})$ simultaneously.

Algorithm for defining $\mathcal{T}^{(6)}$, $\Delta^{(6)}$ and $\zeta_1^{(6)}(\Delta^{(6)})$

Consider a monotone domain Δ' in $\zeta_1^{(6)}(\Delta^{(5)})$ and above y_5 which is not any of the two monotone domains right above any preimage of δ_0 (We rule out the two domains above the preimage of δ_0 since we do not want to consider domains in the enlargement of preimages of δ_0). If $\max_{t\in\mathcal{T}^{(5)}} \frac{|\Delta'(t)|}{H_5(\Delta'(t))} < \vartheta_1$, then let $\Delta^{(6)} = \Delta'$ and $\zeta_1^{(6)}(\Delta^{(6)}) = \zeta_1^{(6)}(\Delta^{(5)})$. If Δ' does not satisfy $\max_{t\in\mathcal{T}^{(5)}} \frac{|\Delta'(t)|}{H_5(\Delta'(t))} < \vartheta_1$, then refine Δ' with ξ_0. We denote the respective partition of Δ_y by $\zeta_1^{(6)}(\Delta')$. Then pick a monotone domain Δ'' in Δ' that is not one of the two domains above the preimage of δ_0. Again we check if $\max_{t\in\mathcal{T}^{(5)}} \frac{|\Delta''(t)|}{H_5(\Delta''(t))} < \vartheta_1$. If so, let $\Delta^{(6)} = \Delta''$ and $\zeta_1^{(6)}(\Delta^{(6)}) = \zeta_1^{(6)}(\Delta')$. If not, refine Δ'' by ξ_0 and denote the new partition of Δ_y created after this refinement by $\zeta_1^{(6)}(\Delta'')$. We repeat this process until we end up with some domain Δ that is not one of the two monotone domains right above some preimage of δ_0 and satisfies $\max_{t\in\mathcal{T}^{(5)}} \frac{|\Delta(t)|}{H_5(\Delta(t))} < \vartheta_1$. As refined domains decrease exponentially in size, this process can be exhausted in finitely many steps as long as we don't always choose the domain closest to y_5. We denote a domain derived from this process by $\Delta^{(6)}$(not making a distinction between different domains). Each such domain is associated with a partition $\zeta_1^{(6)}(\Delta^{(6)})$ of Δ_y. The parameter interval corresponding to $w(t)$ being in $\Delta^{(6)}$ is denoted by $\mathcal{T}^{(6)}$.

`Remark 2.` See table in B.3.2 for sample values of $\frac{|\Delta(t)|}{H_5(\Delta(t))}$.

`Remark 3.` In step 6, we do not have to worry about monotone domains being repeatedly adjacent to y_5 after consecutive refinements since we have already put a restriction on t in (2.6). We can disregard any domain which will never contain $w(t)$ under our parameter restriction. For the remaining domains, we will argue in lemma 8 that no more than four refinements are needed to complete the algorithm in step 6. In the general step n, the number of refinements needed in $\Delta^{(n-1)}$ is always bounded above by a constant that does not depend on n. That is because the ratio of the size of $\Delta^{(n-1)}$ to the distance from $\Delta^{(n-1)}$ to y_{n-1} is bounded above, therefore we don't have to worry about a domain in $\Delta^{(n-1)}$ coming arbitrarily close to y_{n-1}.

2.3.1.3 Defining y_6 and δ_6^{re}

We would like to define $y_6(t)$ so that if $\delta_6^{\mathrm{re}}(t)$ is the interval $[h_1^{-1}(y_6(t)), h_2^{-1}(y_6(t))]$ (h_1 and h_2 are the left and right branches of the map $h(x) = tx(1-x)$ respectively), there are constants r and R such that

$$\frac{1}{3}r \leq \frac{|\delta_6^{\mathrm{re}}(t)|}{|\delta_5(t)|} \leq \frac{1}{3}R \tag{2.27}$$

for all t in $\mathcal{T}^{(6)}$ and all $\mathcal{T}^{(6)}$ in $\mathcal{T}^{(5)}$. The purpose of the inequality (2.27) will become clear in later context. The superscript **re** means that the domain is a rescaled central domain in contrast to the regular central domain obtained from a critical pullback. The ratio $\frac{|\delta_6^{\rm re}(t)|}{|\delta_5(t)|}$ could become arbitrarily close to 0, whereas by (2.27), $\frac{|\delta_6^{\rm re}(t)|}{|\delta_5(t)|}$ cannot be arbitrarily close to 0.

Now we fix any $\mathcal{T}^{(6)}$ in $\mathcal{T}^{(5)}$ which also fixes $\Delta^{(6)}$ and $\zeta_1^{(6)}(\Delta^{(6)})$. We define dynamically the point $y_6(t)$ and domain $\delta_6^{\rm re}$ through the following algorithm.

Algorithm for defining y_6 and $\delta_6^{\rm re}$

1. Let t_0 be the value in $\mathcal{T}^{(6)}$ such that the image of f_{t_0} covers completely the respective interval $\Delta^{(6)}$ on the y-axis. In other words, t_0 is the larger endpoint of $\mathcal{T}^{(6)}$.

2. Let y_6' be such that

$$|[y_6', w(t_0)]| = \frac{1}{9}|[y_5(t_0), w(t_0)]|. \tag{2.28}$$

3. y_6' belongs to a domain in partition $\zeta_1^{(6)}(\Delta^{(6)})(t_0)$ of $\Delta_y(t_0)$. If y_6' belongs to a critical domain, it has to belong to a preimage δ_0^{-p} of δ_0 since only preimages of δ_0 were created in $\zeta_1^{(6)}(\Delta^{(6)})$. In this case, we let $\delta^* = \delta_0^{-p}(t_0)$. If y_6' belongs to a monotone domain $\Delta(t_0)$, we check whether

$$\max_{t\in\mathcal{T}^{(6)}} \frac{|\Delta(t)|}{H_5(\Delta(t))} < \vartheta_2, \tag{2.29}$$

where ϑ_2 is defined in (2.103). If (2.29) is satisfied, we let $\Delta^* = \Delta$ and $\zeta_2^{(6)}(\Delta^{(6)}) = \zeta_1^{(6)}(\Delta^{(6)})$. If (2.29) is not satisfied, we take a monotone pullback of ξ_0 onto Δ. After taking a monotone pullback, we can repeat the above procedure until either y_6' lies in some monotone domain

$\Delta^*(t_0)$ such that $\max_{t\in\mathcal{T}^{(6)}} \frac{|\Delta^*(t)|}{H(\Delta^*(t))} < \vartheta_2$ or y_6' lies in some critical domain $\delta^*(t_0) = \delta_0^{-p}(t_0)$.

4. We let $y_6(t_0)$ be the upper endpoint of $\Delta^*(t_0)$ or $\delta^*(t_0)$.

5. As each $t \in \mathcal{T}^{(6)}$ has a dynamically equivalent partition $\zeta_2^{(6)}(\Delta^{(6)})$ hence dynamically equivalent domain Δ^* or δ^*, we can also define $y_6(t)$ dynamically as the upper endpoint of $\Delta^*(t)$ or $\delta^*(t)$ for all other $t \in \mathcal{T}^{(6)}$.

6. Finally, we take a parabolic pullback of $y_6(t)$ onto the x-axis, which will be two points, forming the endpoints of a rescaled central domain denoted by $\delta_6^{\mathrm{re}}(t)$.

`Remark 4.` Similar to the case with (2.26), we check (2.29) for all $t \in \mathcal{T}^{(5)}$.

`Remark 5.` The maximum number of monotone pullbacks needed depends on ϑ_2 and is calculated in lemma 7.

`Remark 6.` Since we are always taking $y_6(t_0)$ as the upper endpoint of δ^* or Δ^* containing y_6', by (2.28) we always have

$$\frac{|y_6(t_0), w(t_0)|}{|y_5(t_0), w(t_0)|} \leq \frac{1}{9}. \tag{2.30}$$

We show in 2.5.1.2 that for any $t_0 \in \mathcal{T}^{(5)}$,

$$\frac{1}{9} \cdot (1 - 0.59) \leq \frac{|y_6(t_0), w(t_0)|}{|y_5(t_0), w(t_0)|}. \tag{2.31}$$

In particular this is true for t_0 equal to the top value of any $\mathcal{T}^{(6)}$. With some more calculations we show in 2.5.1.2 that

$$\frac{1}{9} \cdot (0.3) \leq \frac{|y_6(t), w(t)|}{|y_5(t), w(t)|} \leq \frac{1}{9} \tag{2.32}$$

for all other $t \in \mathcal{T}^{(6)}$.

2.3.1.4 Boundary refinement

Consider a monotone domain Δ in $\zeta_2^{(6)}(\Delta^{(6)})(t)$ that is below y_6. It is mapped by some g onto I. Moreover, $g = f_{0,i_k} \circ \cdots \circ f_{0,i_1} \circ g_{(5)}$ where the maps in these compositions can be extended to a map onto $\tilde{I}$, therefore g can be extended to a map $\tilde{g}$ defined on the domain $\tilde{\Delta} \supset \Delta$ whose image is $\tilde{I}$. If $\tilde{\Delta}$ is not completely contained in the image of h_t, we perform a boundary refinement on this domain (boundary refinements are defined in 1.3.5.4) by pulling back the partition ξ_0.

After the boundary refinement, we denote the new partition that partitions Δ_y by $\zeta_3^{(6)}$.

`Remark 7.` When Δ is refined once, all new domains have extended domains contained in the image of h_t except for maybe the top-most domain, which is denoted by Δ_l (or Δ_r). Therefore we repeat the process only on the top-most domain until we get $\tilde{\Delta}_{l\cdots l}$ (or $\tilde{\Delta}_{rl\cdots l}$) contained in the image of h_t. We do not need to check extendability of all other subdomains of Δ since they are automatically extendable. The arguments for such are similar to 2.2.7.

`Remark 8.` Partition $\zeta_3^{(6)}(\Delta^{(6)})$ is again dynamically equivalent for all $t \in \mathcal{T}^{(6)}$.

2.3.1.5 Filling-in holes between y_5 and y_6, creating $\zeta^{(6)}(\Delta^{(6)})$

In order to bound the measure of holes in $\delta_5 \backslash \delta_6^{\mathrm{re}}$, we perform filling-ins on all holes between y_5 and y_6. Since all previous procedures consist of only refinement with ξ_0, only preimages δ_0^{-p}'s of δ_0 are created. For any preimage of δ_0, we perform a 5-step filling-in as defined in definition 5.

After performing 5-step filling-ins, preimages of δ_5 and more preimages of δ_0 are created on the y-axis. We denote this final partition of Δ_y by $\zeta^{(6)}(\Delta^{(6)})$.

2.3.1.6 Parabolic pullback onto the x-axis

After we have the partition $\zeta^{(6)}(\Delta^{(6)})$ on the y-axis, we take a parabolic pullback of $\zeta^{(6)}(\Delta^{(6)})$ onto the x-axis. If we consider domain δ_6^{re} as a hole and neglect the partition inside δ_6^{re} at this step, we have the partition $\xi_6(\Delta^{(6)})$ $\Delta^{(6)}$ will be omitted when we move on to the next inductive step. The restriction of the partition $\xi_6(\Delta^{(6)})$ to δ_5 is the partition $\eta_5(\Delta^{(6)})$. This completes the algorithm at step 6.

In later steps, we will need the 1-step filling in of δ_5 defined as follows.

Definition 6. Let δ_5^{-p} be a preimage of δ_5. Let $\mathcal{F}$ be a diffeomorphism that maps δ_5^{-p} onto δ_5, then a 1-step filling-in of δ_5^{-p} is replacing δ_5^{-p} by $\mathcal{F}^{-1}(\eta_5)$.

2.3.2 Steps 7 through 14

For steps 7 through 14, we follow the same algorithm as in step 6 to obtain $\Delta^{(7)}, \dots, \Delta^{(14)}$ and $y_7, \dots, y_{14}$. We repeat important ingredients of the algorithm below. In addition we add *lower boundary refinement* and *filling-in outside* δ_{k-1}^{re} which are procedures not present in step 6.

2.3.2.1 Inductive assumptions at step k

After step $k-1$ is completed, we have a collection of domains $\Delta^{(k-1)}$'s. If we identify one such domain as $\Delta^{(k-1),i_6\cdots i_{k-1}}$ we can backtrack a sequence of nested intervals $\Delta^{(5)} \supset \Delta^{(6),i_6} \supset \cdots \supset \Delta^{(k-1),i_6\cdots i_{k-1}}$ on the y-axis. There is also a corresponding sequence of parameter intervals $\mathcal{T}^{(5)} \supset \mathcal{T}^{(6),i_6} \supset \cdots \supset \mathcal{T}^{(k-1),i_6\cdots i_{k-1}}$ and a sequence of partitions $\zeta_1^{(5)}(\Delta^{(5)}),\ \zeta_1^{(6)}(\Delta^{(6),i_6}),\ \cdots,\ \zeta_1^{(k-1)}(\Delta^{(k-1),i_6\cdots i_{k-1}})$ of Δ_y where $\zeta_1^{(\tilde{k})}(\Delta^{(\tilde{k}),i_6\cdots i_{\tilde{k}}})$ is a refinement of $\zeta_1^{(\tilde{k}-1)}(\Delta^{(\tilde{k}-1),i_6\cdots i_{\tilde{k}-1}})$ for all $\tilde{k} < k$. There is also a sequence of points $y_5 < y_6 < \cdots < y_{k-1}$, where each y_i is continuous with respect to $t \in \mathcal{T}^{(k-1)}$.

2.3.2.2 Defining $\Delta^{(k)}$, $\mathcal{T}^{(k)}$, and $\zeta_1^{(k)}(\Delta^{(k)})$

Fix a domain $\Delta^{(k-1)}$. We check whether $\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta^{(k-1)}(t)|}{H_{k-1}(\Delta^{(k-1)}(t))} < \theta_1$, where $H_{k-1}(\Delta)$ is the distance from Δ to y_{k-1} and θ_1 is defined in (2.104). If $\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta^{(k-1)}(t)|}{H_{k-1}(\Delta^{(k-1)}(t))} < \theta_1$ then $\Delta^{(k)} = \Delta^{(k-1)}$ is the only admissible subdomain of $\Delta^{(k-1)}$. If $\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta^{(k-1)}(t)|}{H_{k-1}(\Delta^{(k-1)}(t))} > \theta_1$, then refine $\Delta^{(k-1)}$ with ξ_0. Consider the new partition of Δ_y as $\zeta_1^{(k)}(\Delta^{(k-1)})$. Consider a subdomain Δ' of $\Delta^{(k-1)}$ that is not a preimage of δ_0 or the two montone domains just above a preimage of δ_0. Then check if $\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta'(t)|}{H_{k-1}(\Delta'(t))} < \vartheta_1$. If $\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta'(t)|}{H_{k-1}(\Delta'(t))} < \vartheta_1$, then let $\Delta^{(k)} = \Delta'$ and let $\zeta_1^{(k)}(\Delta^{(k)}) = \zeta_1^{(k)}(\Delta^{(k-1)})$. If $\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta'(t)|}{H_{k-1}(\Delta'(t))} > \vartheta_1$, then refine Δ' with ξ_0 and repeat the above algorithm. We perform such an algorithm until all monotone domains $\Delta^{(k)}$ in $\Delta^{(k-1)}$ that are not the two monotone domains just above a preimage of δ_0 satisfies

$$\max_{t\in\mathcal{T}^{(k-1)}} \frac{|\Delta^{(k)}(t)|}{H_{k-1}(\Delta^{(k)}(t))} < \vartheta_1. \tag{2.33}$$

Such a domain $\Delta^{(k)}$ is considered to be an admissible domain at step k, since $w(t)$ can only belong in one of these domains. For each admissible domain $\Delta^{(k)}$, there is a corresponding admissible parameter interval $\mathcal{T}^{(k)}$ such that when $t \in \mathcal{T}^{(k)}$, we have $w(t) \in \Delta^{(k)}$.

2.3.2.3 Defining y_k and $\delta_k^{\text{re}}(t)$

y'_k is defined so that

$$|[y'_k, w(t_0)]| = \frac{1}{9}|[y_{k-1}(t_0), w(t_0)]| \qquad \text{where } t_0 \text{ is the top parameter of } \mathcal{T}^{(k)} \tag{2.34}$$

If y'_k lies in some critical domain δ (before or after refinement), then let δ^* be δ. δ^* should automatically satisfy

$$\max_{t \in \mathcal{T}^{(k-1)}} \frac{|\delta^*(t)|}{H_{k-1}(\delta^*(t))} < \vartheta_2. \tag{2.35}$$

ϑ_2 is defined as in (2.103). If y'_k lies in some monotone domain, then we refine the monotone domain with ξ_0 until y'_k lies in some critical domain δ^* or lies in a monotone domain Δ^* that satisfies

$$\max_{t \in \mathcal{T}^{(k-1)}} \frac{|\Delta^*(t)|}{H_{k-1}(\Delta^*(t))} < \vartheta_2 \tag{2.36}$$

$y_k(t_0)$ is defined as the upper endpoint of the domain $\delta^*(t_0)$ or $\Delta^*(t_0)$ containing y'_k. $y_k(t)$ is defined as dynamically the same point as $y_k(t_0)$ for all $t \in \mathcal{T}^{(k)}$. $\delta_k^{\text{re}}(t)$ is the parabolic pullback of $[y_k(t), w(t)]$ onto the x-axis.

2.3.2.4 Boundary refinement

For monotone domains in $[y_{k-1}, y_k]$ whose extended domains are not contained in the image of h_t, we perform boundary refinements.

2.3.2.5 Lower boundary refinement

For $k > 8$, we perform lower boundary refinement for monotone domains in $[y_{k-1}, y_k]$ whose lower extensions are not above y_{k-4}. That is, refining consecutively the lower boundary domain until we get that all extended domains are above y_{k-4}.

2.3.2.6 Filling-in holes between y_{k-1} and y_k

Holes that are between y_{k-1} and y_k can only be preimages of δ_0. We perform a 5-step filling-in on any such hole. The partition which we get on the y-axis is denoted by $\zeta_6^{(k)}(\Delta^{(k)})$.

2.3.2.7 Filling-in holes below y_{k-1}

Different from step 6, we perform filling-in on holes below y_{k-1}. A 1-step filling in of δ_i, $i < k$ at step k is defined inductively by previously defined partitions η_i.

Definition 7. Let δ_i^{-p} be a preimage of δ_i^{re}, $i \geq 5$. Let $\mathcal{F}$ be a diffeomorphism that maps δ_i^{-p} onto δ_i^{re}, then a 1-step filling-in of δ_i^{-p} is replacing δ_i^{-p} by $\mathcal{F}^{-1}(\eta_i)$.

The rules for filling-in below y_{k-1} are given below:

1. If there is a hole that is the preimage of δ_0, then we will perform a 5-step filling-in on that hole.

2. If there is a hole that is the preimage of $\delta_5, \cdots, \delta_{k-2}$, then we perform a 1-step filling-in.

The final partition which we get on the y-axis is denoted by $\zeta^{(k)}(\Delta^{(k)})$.

Remark 9. Notice that it is impossible to have holes that are preimages of δ_{k-1} at step k since ξ_7 has hole of highest possible order δ_5, by allowing only 1-step filling-in, creation of holes is at least two steps behind the creation of the central hole.

2.3.2.8 Parabolic pullback onto the x-axis

. We take a parabolic pullback of $\zeta^{(k)}(\Delta^{(k)})$ onto the x-axis and disregard any partition inside δ^{re}_{k-1}. We denote this partition of I by ξ_k. We consider δ^{re}_{k-1} as the rescaled central domain of ξ_k. The restriction of ξ_k to δ^{re}_{k-1} is the partition η_{k-1}, used to define 1-step filling-ins. This completes the algorithm at step k.

Remark 10. Filling-in below y_{k-1} first and then taking a parabolic pullback is equivalent to taking a parabolic pullback of $\zeta_6^{(k)}(\Delta^{(k)})$ first, then filling-in all holes outside δ^{re}_{k-1}.

2.3.3 General steps of induction after step 15

We consider all $t \in \mathcal{T}^{(n-1)}$ where $\mathcal{T}^{(n-1)}$ is an admissible interval of parameters obtained from the previous inductive step. As an inductive assumption, we assume that there is a sequence of partitions ξ_k of I, $k \leq n$, defined for all $t \in \mathcal{T}^{(n-1)}$. An interval $\Delta^{(n-1)}$ is defined on the y-axis so that $w(t)$ ranges from the bottom to the top of $\Delta^{(n-1)}$ when $t \in \mathcal{T}^{(n-1)}$.

We want to partition $\mathcal{T}^{(n-1)}$ into admissible subintervals $\mathcal{T}^{(n)}$'s.

2.3.3.1 Enlargements of holes

For later construction we need to define enlargements of domains δ_i for $i \geq 5$. We assign enlargements as follows:

$$\hat{\delta}_5 = \delta_0,\ \hat{\delta}_6 = \delta_0,\ \hat{\delta}_7 = \delta_0 \tag{2.37}$$

$$\hat{\delta}_i = \delta_{i-3} \text{ for } i \geq 8 \tag{2.38}$$

We also define $\hat{\xi}_i = \xi_0$ for $5 \leq i < 8$ and $\hat{\xi}_i = \xi_{i-3}$ for $i \geq 8$. The purpose of defining enlargements is explained in 1.3.5.6.

2.3.3.2 Defining $\Delta^{(n)}$, $\mathcal{T}^{(n)}$, and $\zeta_1^{(n)}(\Delta^{(n)})$

Fix a domain $\Delta^{(n-1)}$ created at step $n-1$. Consider the partition $\zeta^{(n-1)}(\Delta^{(n-1)})$ of Δ_y produced after the completion of step $n-1$, $\Delta^{(n-1)}$ is a domain in this partition. The algorithm for choosing $\mathcal{T}^{(n)}$ and $\Delta^{(n)}$ is exactly the same as in steps 7 through 14. Consider $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta^{(n-1)}(t)|}{H_{n-1}(\Delta^{(n-1)}(t))}$, where $H_{n-1}(\Delta(t))$ is the distance from $\Delta(t)$ to $y_{n-1}(t)$. If $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta^{(n-1)}(t)|}{H_{n-1}(\Delta^{(n-1)}(t))} < \vartheta_1$, then let $\Delta^{(n)} = \Delta^{(n-1)}$ and $\Delta^{(n)}$ would be the only admissible subdomain of $\Delta^{(n-1)}$. In this case, let $\zeta_1^{(n)}(\Delta^{(n)}) = \zeta^{(n-1)}(\Delta^{(n-1)})$. If $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta^{(n-1)}(t)|}{H_{n-1}(\Delta^{(n-1)}(t))} > \vartheta_1$, we pullback partition $\xi_{[\frac{n}{3}]}$ onto the interval $\Delta^{(n-1)}$ and get a new partition of Δ_y which we denote by $\zeta_1^{(n)}(\Delta^{(n-1)})$. Consider a monotone domain Δ' in $\zeta_1^{(n)}(\Delta^{(n-1)})$ that is outside the union of enlargements of the central hole and preimages of enlargements of holes in $\xi_{[\frac{n}{3}]}$. Then we check $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta'(t)|}{H_{n-1}(\Delta'(t))}$. If $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta'(t)|}{H_{n-1}(\Delta'(t))} < \vartheta_1$, we let $\Delta^{(n)} = \Delta'$. We consider the corresponding parameter interval as an admissible parameter interval $\mathcal{T}^{(n)}$ and let $\zeta_1^{(n)}(\Delta^{(n)}) = \zeta_1^{(n)}(\Delta') := \zeta_1^{(n)}(\Delta^{(n-1)})$. If

$\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta'(t)|}{H_{n-1}(\Delta'(t))} > \vartheta_1$, we take a pullback of $\xi_{[\frac{n}{3}]}$ onto Δ' which forms a new partition of Δ_y which we denote by $\zeta_1^{(n)}(\Delta')$. We consider a monotone subdomain Δ'' of Δ' that is outside the union of the enlargement of the central hole and preimages of enlargements of holes in $\xi_{[\frac{n}{3}]}$. If $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta''(t)|}{H_{n-1}(\Delta''(t))} < \vartheta_1$, we let $\Delta^{(n)} = \Delta''$, $\zeta_1^{(n)}(\Delta^{(n)}) = \zeta_1^{(n)}(\Delta'') := \zeta_1^{(n)}(\Delta')$ and consider the corresponding parameter interval as an admissible parameter interval $\mathcal{T}^{(n)}$, otherwise, we repeat the argument again. After we have obtained some final $\mathcal{T}^{(n)}$ and $\Delta^{(n)}$ such that

$$\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta^{(n)}(t)|}{H_{n-1}(\Delta^{(n)}(t))} < \vartheta_1. \tag{2.39}$$

As in the case of step 6, the variation of $w(t)$ is small with respect to the size of $|[y_{n-1}, w(t)]|$ for $t \in \mathcal{T}^{(n)}$ as in the case of step 6. Completion of this part of the algorithm will give a partition $\zeta_1^{(n)}(\Delta^{(n)})$ of Δ_y.

2.3.3.3 Defining y_n and δ_n^{re}

The algorithm for defining y_n is the same as the algorithm for defining y_6. We fix the parameter value $t_0^{(n)} \in \mathcal{T}^{(n)}$ as the parameter for which the image of quadratic map covers the whole domain $\Delta^{(n)}$. We set y_n' so that $\frac{|[y_n', w(t_0^{(n)})]|}{|[y_{n-1}(t_0^{(n)}), w(t_0^{(n)})]|} = \theta_0^2 = \frac{1}{9}$. If y_{n+1}' lies in a critical domain δ, then let $\delta^* = \delta$. If y_n' lies in a monotone domain Δ, we check to see if $\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta(t)|}{H_{n-1}(\Delta(t))} < \vartheta_2$. If so, we let $\Delta^* = \Delta$. If not, then we refine Δ by pulling back the partition $\xi_{[\frac{n}{3}]}$ onto Δ. We repeat the process until y_n' lies in some critical domain δ^* or some monotone domain Δ^* which satisfies

$$\max_{t\in\mathcal{T}^{(n-1)}} \frac{|\Delta^*(t)|}{H_{n-1}(\Delta^*(t))} < \vartheta_2, \tag{2.40}$$

where ϑ_2 is defined in (2.103). Choose $y_n(t_0^{(n)})$ as the upper endpoint of the Δ^* or δ^* for which y_n' lies in. For all other $t \in \mathcal{T}^{(n)}$, we define $y_n(t)$ as dynamically the same point as $y_n(t_0^{(n)})$. After this step, we get a partition $\zeta_2^{(n)}(\Delta^{(n)})$ of Δ_y.

The parabolic preimages of y_n form endpoints of the rescaled central domain δ_n^{re} on the x-axis.

2.3.3.4 Boundary refinement

For monotone domains between y_{n-1} and y_n whose extended domains are not contained in the image of h_t, we perform boundary refinements with $\xi_{[\frac{n}{3}]}$. After this step, the partition we have of Δ_y is denoted by $\zeta_3^{(n)}(\Delta^{(n)})$.

2.3.3.5 Lower boundary refinement

For monotone domains in $[y_{n-1}, y_n]$ whose extended domains extend below y_{n-4}, we perform boundary refinements with $\xi_{[\frac{n}{3}]}$. After this step, the partition we have of Δ_y is denoted by $\zeta_4^{(n)}(\Delta^{(n)})$.

2.3.3.6 Filling-in of holes in $[y_{n-1}, y_n]$

For holes between y_{n-1} and y_n we perform filling-in according to the following rules.

- For holes that are preimages of δ_0, we perform a 5-step filling-in, and that's it.
- For all other holes, we perform a 1-step filling-in. If this is a first filling-in at step n, we repeat the process one more time for holes created here.

The final partition of Δ_y on the y-axis is denoted by $\zeta_5^{(n)}(\Delta^{(n)})$.

2.3.3.7 Filling-in outside δ_{n-1}

For each hole below y_{n-1}, we perform a 1-step or a 5-step filling-in (depending on whether or not the hole is a preimage of δ_0). The final partition of Δ_y on the y-axis is denoted by $\zeta^{(n)}(\Delta^{(n)})$.

2.3.3.8 Parabolic pullback onto $\delta_{n-1}^{\text{re}} \backslash \delta_n^{\text{re}}$

We take a parabolic pullback of the partition $\zeta^{(n)}(\Delta^{(n)})$ onto the x-axis. We neglect any partition inside δ_n^{re} and this forms the final partition $\xi_n(\Delta^{(n)})$ of I on the x-axis. The restriction of the partition $\xi_n(\Delta^{(n)})$ to δ_{n-1}^{re} is denoted by the partition $\eta_{n-1}(\Delta^{(n)})$.

2.4 Structure of the phase domains, parameter intervals and maps at step n

We have described our algorithm for constructing the partition for each inductive step. Now we look at some structures that we get as a consequence of the algorithm.

2.4.1 Nested sequence of collection of parameter intervals

Up to step n, we have a finite collection of admissible parameter intervals $\{\mathcal{T}^{(n)}\}$ whose elements are mutually disjoint except for maybe endpoints of $\mathcal{T}^{(n)}$.

Each parameter interval $\mathcal{T}^{(n)}$ is contained in an admissible parameter interval $\mathcal{T}^{(n-1)}$ from step $n-1$. We can index admissible parameter intervals by $i_6 \cdots i_n$ to show its inclusion relation, $\mathcal{T}^{(n)}_{i_6 \cdots i_{n-1} i_n} \subset \mathcal{T}^{(n-1)}_{i_6 \cdots i_{n-1}} \subset \cdots \subset \mathcal{T}^{(6)}_{i_6} \subset \mathcal{T}^{(5)} = \mathcal{T}^{476777}$. If we are looking at one fixed interval $\mathcal{T}^{(n-1)}_{i_6 \cdots i_{n-1}}$ and its subintervals $\mathcal{T}^{(n)}_{i_6 \cdots i_{n-1} j}$, we omit the index of $\mathcal{T}^{(n-1)}$ for simplicity. Therefore, we use expressions such as $\bigcup_i \mathcal{T}^{(n)}_i \subset \mathcal{T}^{(n-1)}$ when we actually mean $\bigcup_j \mathcal{T}^{(n)}_{i_6 \cdots i_{n-1} j} \subset \mathcal{T}^{(n-1)}_{i_6 \cdots i_{n-1}}$.

2.4.2 Parameter-induced partition of $\Delta^{(n-1)}$

The intervals $\mathcal{T}^{(n)}_i$ and their complement in $\mathcal{T}^{(n-1)}$ form a partition of $\mathcal{T}^{(n-1)}$. We consider respective partition of $\Delta^{(n-1)}$ in the phase space. This partition is obtained by the pullback of $\hat{\xi}_{[\frac{n}{3}]}$ which depends continuously on t in $\mathcal{T}^{[\frac{n}{3}]}$, therefore also depends continuously on t in a smaller parameter interval $\mathcal{T}^{(n-1)}$. The non-admissible domains in $\Delta^{(n-1)}$ are hence decided by holes and preimages of holes in $\hat{\xi}_{[\frac{n}{3}]}$. Since this partition of $\Delta^{(n-1)}$ into subintervals $\Delta^{(n)}_i$ and its complement decides admissible parameter intervals, we call this the parameter-induced partition of $\Delta^{(n-1)}$. This is to distinguish it from the partitions that define the power maps. Note that the parameter induced partition is a partition in the phase space.

2.4.3 Phase partition

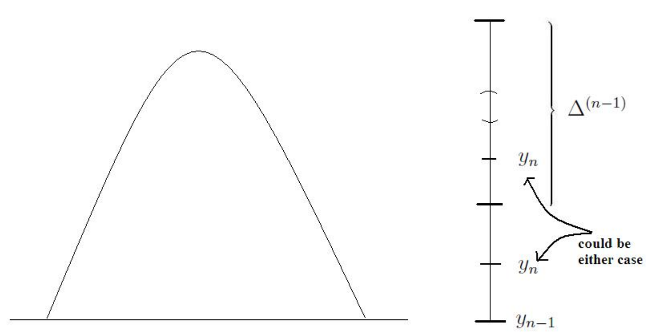
In the phase space, there are two other partitions, the partition ξ_n of $I = [q_t^{-1}, q_t]$ on the x-axis and the partition $\zeta^{(n)}$ of Δ_y on the y-axis. The branches corresponding to ξ_n defines the power map at the nth step of induction. Both ξ_n

and $\zeta^{(n)}$ vary continuously with $t \in \mathcal{T}^{(n)}$, but does not vary continuously with t in the larger parameter interval $\mathcal{T}^{(n-1)}$ containing $\mathcal{T}^{(n)}$. Therefore, we write ξ_n as $\xi_n(\Delta^{(n)})$ or $\xi_n(\mathcal{T}^{(n)})$ and $\zeta^{(n)}$ as $\zeta^{(n)}(\Delta^{(n)})$ or $\zeta^{(n)}(\mathcal{T}^{(n)})$ to specify this dependence.

Partition $\zeta^{(n)}(\mathcal{T}^{(n)})$ is a refinement of $\zeta^{(n-1)}(\mathcal{T}^{(n-1)})$ for $t \in \mathcal{T}^{(n)}$. The parabolic pullback of $\zeta^{(n)}(\Delta^{(n)})$ gives exactly the part of the partition $\xi_n(\mathcal{T}^{(n)})$ when neglecting the partition in δ_n^{re}. All monotone domains outside holes of ξ_n remain intact after step n.

2.4.4 Monotone maps and maps on holes

We write out possible forms of compositions for maps defined on domains in ξ_n and $\zeta^{(n)}$. For the partition $\zeta^{(n)}$ of Δ_y, $\Delta^{(n)}$ denotes the domain that contains the critical value. The monotone branch on $\Delta^{(n)}$ is the topmost branch which we will consider on the y-axis. $\Delta^{(n)}$ is always contained in $\Delta^{(n-1)}$. For the other branches in $\zeta^{(n)}$, we distinguish the ones above y_n from the ones below y_n. Notice that y_n could be inside or below $\Delta^{(n-1)}$.



Hence, on the y-axis, we discuss maps that are defined on domains of the following possible cases.

1. The case where y_n is in $\Delta^{(n-1)}$.
   (a) Monotone domain $\Delta^{(n)}$ containing the critical value
   (b) Monotone domains $\bar{\Delta}_i$ in $\Delta^{(n-1)}$, above y_n
   (c) Monotone domains Δ_i in $\Delta^{(n-1)}$, below y_n
   (d) Holes in $\Delta^{(n-1)}$, above y_n
   (e) Holes in $\Delta^{(n-1)}$ below y_n
   (f) Monotone domains Δ_i below $\Delta^{(n-1)}$, above y_{n-1}
   (g) Holes below $\Delta^{(n-1)}$, above y_{n-1}
   (h) Monotone domains Δ_i below y_{n-1}
   (i) Holes below y_{n-1}
2. The case where y_n is below $\Delta^{(n-1)}$.
   (a) Monotone domain $\Delta^{(n)}$ containing the critical value
   (b) Monotone domains $\bar{\Delta}_i$ in $\Delta^{(n-1)}$
   (c) Holes in $\Delta^{(n-1)}$
   (d) Monotone domains $\bar{\Delta}_i$ below $\Delta^{(n-1)}$, above y_n
   (e) Holes below $\Delta^{(n-1)}$ above y_n
   (f) Monotone domains Δ_i below $\Delta^{(n-1)}$, above y_{n-1} and below y_n
   (g) Holes below $\Delta^{(n-1)}$, above y_{n-1} and below y_n

(h) Monotone domains Δ_i below y_{n-1}

(i) Holes below y_{n-1}

Notations used below are described as follows. When choosing parameters at each step n, we pullback the partition $\hat{\xi}_{[\frac{n}{3}]} = \xi_{[\frac{n}{3}]-3}$ onto $\Delta^{(n-1)}$ until all monotone domains are sufficiently small. (i.e. satisfying (2.39)). These monotone domains are the admissible domains for which the critical value may possibly fall into. We denote monotone maps on $\Delta^{(n)}$ by $g_{(n)}$. The remaining domains are holes corresponding to parameter values which we throw away in the parameter space. Maps on these holes are denoted by $\mathcal{G}_{(n),i} : \delta^{(n)} \to \delta_m^{\text{re}}$. Hence $g_{(n),i}$ and $\mathcal{G}_{(n),i}$ are maps defined for parameter choice or in other terms, are maps defined on the parameter-induced partition of $\Delta^{(n-1)}$ as described in 2.4.2.

For the actual partition on the phase space, we first pullback $\xi_{[\frac{n}{3}]}$ so that the domain containing the critical value is sufficiently small. Then we pullback $\xi_{[\frac{n}{3}]}$ until the monotone domain containing y_n' is sufficiently small. We define y_n to be the upper endpoint of the final domain containing y_n'. Monotone maps above y_n are denoted by $\bar{g}_{n,i}$. Maps on holes above y_n are denoted by $\bar{\mathcal{G}}_{n,i}$. We do not perform boundary refinement on monotone domains above y_n at step n. We do not fill-in any holes above y_n at step n. For monotone domains below y_n, we perform boundary refinements if needed. For each hole in $[y_{n-1}, y_n]$, we take two 1-step filling-ins, one 1-step filling-in followed by a 5-step filling-in or one 5-step filling-in depending on what rescaled central domain the hole is the preimage of. After refinement and filling-in, the monotone maps on domains in $[y_{n-1}, y_n]$ are denoted by $g_{n,i}$'s and maps on holes in $[y_{n-1}, y_n]$ are denoted by $\mathcal{G}_{n,i}$'s. Monotone domains below y_{n-1} remain unchanged. Holes below y_{n-1} are filled in once. We use $g_{n,i}$'s and $\mathcal{G}_{n,i}$'s to denote maps on domains below y_{n-1} as well. Then take a parabolic pullback of $g_{n,i}$'s and $\mathcal{G}_{n,i}$'s onto the x-axis to form $f_{n,i}$'s and $\mathcal{F}_{n,i}$'s which are monotone maps and maps on holes, respectively, in ξ_n.

In general, compositions that result from monotone refinements are expressed in the following form.

$$\overbrace{f_{[\frac{n}{3}],k_{s''}} \circ \cdots \circ f_{[\frac{n}{3}],k_1}}^{\text{boundary refinements below } y_n} \circ \overbrace{f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1}}^{\text{refinements on domain containing } y'_n} \circ \overbrace{\hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1}}^{\text{refinements on domains containing critical value}} \tag{2.41}$$

$\hat{f}_{[\frac{n}{3}]}$ are monotone branches of $\hat{\xi}_{[\frac{n}{3}]}$. The following are expressions of maps of step n written as compositions of maps from steps before n.

2.4.4.1 Branches on the y-axis

Monotone domain $\Delta^{(n)}$ containing the critical value

$$g_{(n)} : \Delta^{(n)} \to I$$

For each value t, there is only one $\Delta^{(n)}$ containing the critical value. It was obtained by refining $\Delta^{(n-1)}$ with $\xi_{[\frac{n}{3}]}$ and avoiding enlargements of holes in $\xi_{[\frac{n}{3}]}$ or equivalently, refining with $\hat{\xi}_{[\frac{n}{3}]}$. $\hat{\xi}_{[\frac{n}{3}]}$ is $\xi_{[\frac{n}{3}]-3}$ in most cases, other cases are better, so we write

$$g_{(n)} = \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.42}$$

Monotone domains $\bar{\Delta}_i$ in $\Delta^{(n-1)}$, above y_n

$$\bar{g}_{n,i} : \bar{\Delta}_i \to I$$

$\bar{\Delta}_i$ may be some monotone domain created from the refinements for obtaining $\Delta^{(n)}$, then refined further when obtaining y_n, so we have

$$\bar{g}_{n,i} = f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.43}$$

No boundary refinements are performed since domains $\bar{\Delta}_i$ are above y_n. Hence, extended domains of these branches may not be in the image of h_t.

Monotone domains Δ_i in $\Delta^{(n-1)}$, below y_n

$$g_{n,i} : \Delta_i \to I$$

$$g_{n,i} = f_{[\frac{n}{3}],k_{s''}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.44}$$

The last compositions come from possible boundary refinements for domains Δ_i below y_n.

Holes in $\Delta^{(n-1)}$, above y_n

$$\bar{\mathcal{G}}_{n,i} : \delta_m^{-p} \to \delta_m^{\mathrm{re}}$$

These are monotone maps that map preimages of central holes to their respective rescaled central domains.

Case 1: This is the case when the holes are created after refinements when obtaining $\Delta^{(n)}$. In the two forms below, the first form gives the compositions for the map on the central hole after the last refinement, and the second form gives the composition for the maps on holes other than the central hole after the last refinement. We will see maps on holes in these two forms many times.

$$\bar{\mathcal{G}}_{n,i} = \hat{f}_{[\frac{n}{3}],i_{s-1}} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.45}$$

$$\bar{\mathcal{G}}_{n,i} = \mathcal{F}_{[\frac{n}{3}]-3,i_s} \circ \hat{f}_{[\frac{n}{3}],i_{s-1}} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.46}$$

Case 2: This is the case when the holes are created after refinements to obtain $\Delta^{(n)}$ and also after refinements to obtain y_n.

$$\bar{\mathcal{G}}_{n,i} = f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.47}$$

$$\bar{\mathcal{G}}_{n,i} = \mathcal{F}_{[\frac{n}{3}],j_{s'}} \circ f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.48}$$

Filling-ins are not performed above y_n at step n.

Holes in $\Delta^{(n-1)}$ below y_n

We use $\mathcal{G}^{\text{temp}}_{n,i}$ to denote maps on holes after all possible refinements because holes below y_n will be filled in.

$$\mathcal{G}^{\text{temp}}_{n,i} : \delta_m^{-p} \to \delta_m^{\text{re}}$$

Case 1

$$\mathcal{G}^{\text{temp}}_{n,i} = \hat{f}_{[\frac{n}{3}],i_{s-1}} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.49}$$

$$\mathcal{G}^{\text{temp}}_{n,i} = \mathcal{F}_{[\frac{n}{3}]-3,i_s} \circ \hat{f}_{[\frac{n}{3}],i_{s-1}} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.50}$$

Case 2

$$\mathcal{G}^{\text{temp}}_{n,i} = f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \tag{2.51}$$

$$\mathcal{G}_{n,i}^{\text{temp}} = \mathcal{F}_{[\frac{n}{3}],j_{s'}} \circ f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \quad (2.52)$$

Case 3

Due to boundary refinements, there can also be additional compositions.

$$\mathcal{G}_{n,i}^{\text{temp}} = f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \quad (2.53)$$

$$\mathcal{G}_{n,i}^{\text{temp}} = \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \quad (2.54)$$

After a first filling-in, we get some monotone branches

$$g_{n,i} = f_{m+1,l} \circ \mathcal{G}_{n,j}^{\text{temp}} \qquad m \leq [\frac{n}{3}]. \quad (2.55)$$

m is less than or equal to $[\frac{n}{3}]$ because $\mathcal{G}_{n,i}^{\text{temp}}$ are maps on holes created from refinements by $\xi_{[\frac{n}{3}]}$ or earlier partitions. Plugging in (2.49) through (2.54), form (2.55) can be written into the following detailed forms.

$$g_{n,i} = f_{m+1,l} \circ \hat{f}_{[\frac{n}{3}],i_{s-1}} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}$$

$$g_{n,i} = f_{m+1,l} \circ \mathcal{F}_{[\frac{n}{3}]-3,i_s} \circ \hat{f}_{[\frac{n}{3}],i_{s-1}} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}$$

$$g_{n,i} = f_{m+1,l} \circ f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}$$

$$g_{n,i} = f_{m+1,l} \circ \mathcal{F}_{[\frac{n}{3}],j_{s'}} \circ f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}$$

$$g_{n,i} = f_{m+1,l} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}$$

$$\begin{aligned} g_{n,i} = {} & f_{m+1,l} \circ \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \\ & \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \end{aligned}$$

After a first filling-in, we also get new maps on holes which we denote by $\mathcal{G}_{n,i}^{\text{temp2}}$ because such holes are filled in a second time.

$$\mathcal{G}_{n,i}^{\text{temp2}} = \mathcal{G}_{n,j}^{\text{temp}} \tag{2.56}$$

or

$$\mathcal{G}_{n,i}^{\text{temp2}} = \mathcal{F}_{m+1,l} \circ \mathcal{G}_{n,j}^{\text{temp}} \tag{2.57}$$

where, $m \leq [\frac{n}{3}]$ or $[\frac{n}{3}]-3$. Possible compositions are exactly the same as those of monotone branches except $f_{m+1,l}$ is replaced by $\mathcal{F}_{m+1,l}$.

After a second filling-in, we get more monotone branches

$$g_{n,i} = f_{\bar{m}+1,l} \circ \mathcal{G}_{n,j}^{\text{temp2}} \qquad \tilde{m} \leq m+1 \tag{2.58}$$

and more maps on holes

$$\mathcal{G}_{n,i} = \mathcal{G}_{n,j}^{\text{temp2}} \tag{2.59}$$

or

$$\mathcal{G}_{n,i} = \mathcal{F}_{\bar{m}+1,l} \circ \mathcal{G}_{n,j}^{\text{temp2}} \tag{2.60}$$

where $\bar{m} \leq m+1$. Final expressions would have the most general form

$$\begin{aligned} g_{n,i} =& f_{\bar{m}+1,l_2} \circ \mathcal{F}_{m+1,l_1} \circ \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \\ & \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_{n,i} =& \mathcal{F}_{\bar{m}+1,l_2} \circ \mathcal{F}_{m+1,l_1} \circ \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \\ & \cdots \circ f_{[\frac{n}{3}],j_1} \circ \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}. \end{aligned}$$

Monotone domains $\bar{\Delta}_i$ below $\Delta^{(n-1)}$, above y_n

Monotone domains $\bar{\Delta}_i$ below $\Delta^{(n-1)}$ either come from monotone domains from previous inductive steps or monotone domains created after refinements when obtaining y_n. No boundary refinements are performed on monotone domains above y_n at step n. The composition is just

$$\bar{g}_{n,i} = f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j} \tag{2.61}$$

Note how we use $\bar{g}_{n-1,j}$ here instead of $g_{(n-1)}$ as in (2.43) since $\bar{\Delta}_i$ is not in $\Delta^{(n-1)}$ anymore.

Holes below $\Delta^{(n-1)}$ above y_n

For maps on holes below $\Delta^{(n-1)}$ and above y_n, the composition for $\bar{\mathcal{G}}_{n,i}$ has a form similar to (2.61).

$$\bar{\mathcal{G}}_{n,i} = f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j}$$

or

$$= \mathcal{F}_{[\frac{n}{3}],j_{s'}} \circ f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j}$$

Holes above y_n do not get filled in at step n.

Monotone domains Δ_i below $\Delta^{(n-1)}$, below y_n and above y_{n-1}

For domains Δ_i below $\Delta^{(n-1)}$ and below y_n, we add possible boundary refinements to compositions.

$$g_{n,i} = f_{[\frac{n}{3}],k_{s''}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j} \tag{2.62}$$

Holes below $\Delta^{(n-1)}$, below y_n and above y_{n-1}

For maps on holes below $\Delta^{(n-1)}$ and below y_n, we use the temporary notation $\mathcal{G}_{n,i}^{\text{temp}}$ because we will fill in these holes.

Case 1

$$\mathcal{G}_{n,i}^{\text{temp}} = f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j} \tag{2.63}$$

or

$$\mathcal{G}_{n,i}^{\text{temp}} = \mathcal{F}_{[\frac{n}{3}],j_{s'}} \circ f_{[\frac{n}{3}],j_{s'-1}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j} \tag{2.64}$$

Case 2

$$\mathcal{G}_{n,i}^{\text{temp}} = f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j} \tag{2.65}$$

or

$$\mathcal{G}_{n,i}^{\text{temp}} = \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j} \tag{2.66}$$

After one filling-in, we have some new monotone branches

$$g_{n,i} = f_{m+1,l} \circ \mathcal{G}_{n,j}^{\text{temp}} \tag{2.67}$$

$m \leq [\frac{n}{3}]$. We also have maps on holes that are temporarily expressed as $\mathcal{G}_{n,i}^{temp2}$ before a second filling-in.

$$\mathcal{G}_{n,i}^{\text{temp2}} = \mathcal{G}_{n,j}^{\text{temp}} \tag{2.68}$$

or

$$\mathcal{G}_{n,i}^{\text{temp2}} = \mathcal{F}_{m+1,l} \circ \mathcal{G}_{n,j}^{\text{temp}} \tag{2.69}$$

After a second filling-in, we have some more new monotone branches

$$g_{n,i} = f_{\bar{m}+1,l} \circ \mathcal{G}_{n,j}^{\text{temp2}} \tag{2.70}$$

$\bar{m} \leq m + 1$.

We have final maps on holes

$$\mathcal{G}_{n,i} = \mathcal{G}_{n,j}^{\text{temp2}} \tag{2.71}$$

or

$$\mathcal{G}_{n,i} = \mathcal{F}_{\bar{m}+1,l} \circ \mathcal{G}_{n,j}^{\text{temp2}} \tag{2.72}$$

Writing out the composition, we would have the general forms

$$g_{n,i} = f_{\bar{m}+1,l_2} \circ \mathcal{F}_{m+1,l_1} \circ \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j}$$

and

$$\mathcal{G}_{n,i} = \mathcal{F}_{\bar{m}+1,l_2} \circ \mathcal{F}_{m+1,l_1} \circ \mathcal{F}_{[\frac{n}{3}],k_{s''}} \circ f_{[\frac{n}{3}],k_{s''-1}} \circ \cdots \circ f_{[\frac{n}{3}],k_1} \circ f_{[\frac{n}{3}],j_{s'}} \circ \cdots \circ f_{[\frac{n}{3}],j_1} \circ \bar{g}_{n-1,j}.$$

Monotone domains Δ_i below y_{n-1}

These branches come from earlier inductive steps and they remain the same as in step $n-1$.

$$g_{n,i} = g_{n-1,j} \tag{2.73}$$

Holes below y_{n-1}

For each hole below y_{n-1} we perform a 1-step filling-in. Suppose that the hole we fill-in is a preimage of $\delta_{\tilde{n}}^{\text{re}}$ for some $\tilde{n} \leq n-1$, then new monotone branches are formed by compositions with $f_{\tilde{n},l}$'s.

$$g_{n,i} = f_{\tilde{n},l} \circ \mathcal{G}_{n-1,j} \qquad \tilde{n} \leq n-1 \tag{2.74}$$

We also have new holes and maps on these holes are denoted by

$$\mathcal{G}_{n,i} = \mathcal{F}_{\tilde{n},l} \circ \mathcal{G}_{n-1,j} \tag{2.75}$$

$\tilde{n} \leq n-1$.

2.4.4.2 Branches on the x-axis

Domains on the x-axis are split into domains inside $\delta_{n-1}^{\mathrm{re}}$ and domains outside $\delta_{n-1}^{\mathrm{re}}$.

Let $f_{n,i}$ represent a monotone branch in partition ξ_n. A monotone branch $f_{n,i}$ is simply the composition $g_{n,i} \circ h_t$ where $g_{n,i}$ is a monotone branch that maps some domain Δ^i in $\zeta^{(n)}(\Delta^{(n)})$ onto I and $h_t(x) = tx(1-x)$. Similarly, maps on holes in ξ_n are represented by $\mathcal{F}_{n,i} = \mathcal{G}_{n,i} \circ h_t$

Maps defined on domains inside $\delta_{n-1}^{\mathbf{re}} \backslash \delta_n^{\mathbf{re}}$

Monotone domains in $\delta_{n-1}^{\mathrm{re}} \backslash \delta_n^{\mathrm{re}}$ can be expressed as

$$f_{n,i} = g_{n,i} \circ h_t, \tag{2.76}$$

where $g_{n,i}$ is a monotone map defined on a monotone domain in $[y_{n-1}, y_n]$. Maps on holes in $\delta_{n-1}^{\mathrm{re}} \backslash \delta_n^{\mathrm{re}}$ can be expressed as

$$\mathcal{F}_{n,i} = \mathcal{G}_{n,i} \circ h_t, \tag{2.77}$$

where $\mathcal{G}_{n,i}$ is a monotone map defined on a hole in $[y_{n-1}, y_n]$.

Maps defined on domains outside $\delta_{n-1}^{\mathbf{re}}$

Monotone branches outside $\delta_{n-1}^{\mathrm{re}}$ were formed in previous steps, they remain the same as before.

$$f_{n,i} = f_{n-1,j} \tag{2.78}$$

Holes get a 1-step filling in, forming new monotone branches

$$f_{n,i} = f_{\tilde{n},j'} \circ f_{n-1,j} \quad \text{where } \tilde{n} \leq n-1, \tag{2.79}$$

and new maps on holes

$$\mathcal{F}_{n,i} = \mathcal{F}_{\tilde{n},j'} \circ f_{n-1,j} \quad \text{where } \tilde{n} \leq n-1. \tag{2.80}$$

2.5 Estimates on the measure of holes, domain sizes, derivatives and velocities

We fix the following parameter values.

1. $\epsilon_0 := 0.003$
2. $\vartheta_1 := 0.0098$
3. $\vartheta_2 := 0.6 * \frac{1}{8}$

2.5.1 Step 6

We derive properties for step 6 as a result of the algorithm at step 6.

(I) Velocities of partition points in the parameter-induced partition of $\Delta^{(5)}$ are less than ϵ_0. Velocities of partition points in the phase partition ζ of $\Delta^{(5)}$ are less than ϵ_0. Velocities of partition points in η_5 are less than $\frac{1}{4\left|\delta_6^{\mathrm{re}}\right|}$.

(II) $\left|\mathcal{T}^{(6)}\right| \leq \frac{1}{\frac{1}{4}-\epsilon_0}\left|\Delta^{(6)}\right|$, $\left|\Delta^{(6)}\right| \leq H_5(\Delta^{(6)})\vartheta_1 \leq |[y_5(t), w(t)]|\,\vartheta_1 < \left|\Delta^{(5)}\right|\vartheta_1$

(III)

$$\frac{1}{3}\sqrt{0.3}\,|\delta_5| \leq |\delta_6^{\mathrm{re}}| \leq \frac{1}{3}\,|\delta_5|\,. \tag{2.81}$$

(IV) No more than 5 pullbacks are needed to achieve $\frac{\left|\Delta^{(6)}\right|}{H_5(\Delta^{(6)})} < \vartheta_1$.

(V) No more than 5 pullbacks are needed to achieve $\frac{\left|\Delta^{i}\right|}{H_5(\Delta^{i})} < \vartheta_2$, where Δ^i is a monotone domain containing y_6'.

(VI) No more than 2 boundary refinements are needed.

(VII) $\mu_{\text{holes}}(\eta_5) < 0.526667$, $\mu_{\text{holes}}(\xi_6) < 0.0189$, where η_5 is the partition ξ_6 restricted to δ_5.

(VIII) For $g_{(6)}$, $\mathcal{G}_{(6)}$, $\bar{g}_{6,i}$, $\bar{\mathcal{G}}_{6,i}$, $g_{6,i}$ and $\mathcal{G}_{6,i}$ defined in 2.4.4, we have

$g_{(6)} = \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$

$\mathcal{G}_{(6),i} = \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$

$\bar{g}_{6,i} = f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

$\bar{\mathcal{G}}_{6,i} = f_{0,i_{s-1}} \circ \cdots \circ f_{0,i_1} \circ g_{(5)}$ for $s \leq 5$ or $\bar{\mathcal{G}}_{6,i} = f_{0,j_{s'-1}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

$g_{6,i} = f_{5,k} \circ f_{0,1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

$\mathcal{G}_{6,i} = \hat{f}_{0,i_{s-1}} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$ or $\mathcal{G}_{6,i} = f_{0,j_{s'-1}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$ or $\mathcal{G}_{6,i} = f_{0,k_{s''-1}} \circ \cdots \circ f_{0,k_1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$ for $s \leq 5$, $1 \leq s + s' \leq 5$, $s'' \leq 2$ or

$\mathcal{G}_{6,i} = \mathcal{F}_{5,k} \circ \hat{f}_{0,i_{s-1}} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$ or $\mathcal{G}_{6,i} = \mathcal{F}_{5,k} \circ f_{0,j_{s'-1}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$ or $\mathcal{G}_{6,i} = \mathcal{F}_{5,k} \circ f_{0,k_{s''-1}} \circ$

$\cdots \circ f_{0,k_1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$ for $s \leq 5$, $1 \leq s+s' \leq 5$, $s'' \leq 2$

(IX) Monotone branches $f_{6,i}$ in ξ_6 are extendible to $\tilde{I}$. Maps $\mathcal{F}_{6,i}$ on holes are extendible to the enlargements of the holes.

(X) $$\left|\frac{\partial g_{(6)}}{\partial x}\right| \geq \max\left\{391005 * 3.5, \frac{|I|}{\left|\Delta^{(6)}\right|} * \frac{1}{15.6}\right\}$$

$$\left|\frac{\partial \mathcal{G}_{(6),i}}{\partial x}\right| \geq \max\left\{391005, \frac{|\delta_0|}{\left|\Delta^{(5)}\right| * (\text{worst distorted ratio of } \delta_0^{-1} \text{ in } \Delta^{(5)})} * \frac{1}{1.3035}\right\} = 391005$$

$$\left|\frac{\partial \bar{g}_{6,i}}{\partial x}\right| \geq \max\left\{391005 * 3.5, \frac{|I|}{|[y_6(t), w(t)]|} * \frac{1}{15.6}\right\}$$

$$\left|\frac{\partial \bar{\mathcal{G}}_{6,i}}{\partial x}\right| \geq \max\left\{391005, \frac{|\delta_0|}{|[y_6(t), w(t)]|} * \frac{1}{2.75}\right\}$$

$$\left|\frac{\partial g_{6,i}}{\partial x}\right| \geq \max\left\{391005 * 3.5, \frac{|I|}{|[y_5(t), y_6(t)]|} * \frac{1}{15.6}\right\}$$

$$\left|\frac{\partial \mathcal{G}_{6,i}}{\partial x}\right| \geq \max\left\{391005, \min\left\{\frac{|\delta_0|}{|[y_5(t), y_6(t)]|} * \frac{1}{2.75}, \frac{|\delta_5|}{|[y_5(t), y_6(t)]|} * \frac{1}{1.1}\right\}\right\}$$

$$\left|\frac{\partial f_{6,i}}{\partial x}\right| \geq 109 * 3.5$$

$$\left|\frac{\partial \mathcal{F}_{6,i}}{\partial x}\right| \geq 109$$

(XI) $$\frac{\left|\frac{\partial^2 \mathcal{G}_{(6),i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{G}_{(6),i}^{-1}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 g_{(6)}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{(6)}^{-1}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 \bar{\mathcal{G}}_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{6,i}^{-1}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 \bar{g}_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \bar{g}_{6,i}^{-1}}{\partial z}\right|} \leq 211.23$$

$$\frac{\left|\frac{\partial^2 \mathcal{G}_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{G}_{6,i}^{-1}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 g_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{6,i}^{-1}}{\partial z}\right|} < 902421$$

$$\frac{\left|\frac{\partial^2 \mathcal{F}_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{F}_{6,i}^{-1}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 f_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial f_{6,i}^{-1}}{\partial z}\right|} < \frac{1.38}{\left|\delta_6^{\mathrm{re}}\right|^2} < 2.9 * 10^8$$

2.5.1.1 Bounds for velocities of partitioning points of the parameter-induced partition and phase partitions of $\Delta^{(5)}$

Here we show that the velocities of partitioning points of the parameter-induced partition and/or the phase partition is less than $\epsilon_0 = 0.003$. All partitioning points of the parameter-induced partition (discussed in 2.4.2) and phase partitions (discussed in 2.4.3) of $\Delta^{(5)}$ are formed by a finite number of monotone pullbacks of ξ_0 onto or into $\Delta^{(5)}$.

Lemma 5. *Let Δ be any monotone domain either in the parameter-induced partition of $\Delta^{(5)}$ or phase partitions of $\Delta^{(5)}$, then*

$$\left|\frac{dx_1^{(6)}(t)}{dt}\right|, \left|\frac{dx_2^{(6)}(t)}{dt}\right| < 0.003 =: \epsilon_0 \tag{2.82}$$

where x_1 and x_2 are endpoints of Δ.

Proof. First note that Δ must be mapped by some monotone map g onto I. Here, g could be $g_{(6)}$, $\bar{g}_{6,i}$ or $g_{6,i}$. Since

$$g(t, x_1(t)) = q_t^{-1}(\text{ or } q_t, \text{ doesn't matter}) \tag{2.83}$$

we have

$$\frac{\partial g}{\partial t}(t, x_1(t)) + \frac{\partial g}{\partial x}(t, x_1(t))\frac{dx_1(t)}{dt} = \frac{-1}{t^2}. \tag{2.84}$$

Then the velocity of the endpoint x_1 of Δ satisfies the inequality

$$\left|\frac{dx_1(t)}{dt}\right| \leq \frac{\frac{1}{t^2}}{\left|\frac{\partial g}{\partial x}(t, x_1(t))\right|} + \frac{\left|\frac{\partial g}{\partial t}(t, x_1(t))\right|}{\left|\frac{\partial g}{\partial x}(t, x_1(t))\right|}. \tag{2.85}$$

According to 2.4.4.1, $g_{(6)}$, $\bar{g}_{6,i}$ or $g_{6,i}$ can be written as compositions of $g_{(5)}$ and branches of ξ_0 or ξ_5. The case that gives the worst value for $\frac{\left|\frac{\partial g}{\partial t}(t,x_1(t))\right|}{\left|\frac{\partial g}{\partial x}(t,x_1(t))\right|}$ above is when g has the form

$$g = f_{5,j} \circ f_{0,i_r} \circ \cdots \circ f_{0,i_1} \circ g_{(5)}. \tag{2.86}$$

Using (2.164) and preliminary estimates from 2.2.10, we get

$$\begin{aligned}
\frac{\left|\frac{\partial\left(f_{0,i_r}\circ\cdots\circ f_{0,i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{0,i_r}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|} &\leq \frac{\left|\frac{\partial\left(f_{0,i_{r-1}}\circ\cdots\circ f_{0,i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{0,i_{r-1}}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|} + \frac{1}{\left|\frac{\partial\left(f_{0,i_{r-1}}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|}\frac{\left|\frac{\partial f_{0,i_r}}{\partial t}\right|}{\left|\frac{\partial f_{0,i_r}}{\partial x}\right|} \\
&\leq \frac{\left|\frac{\partial\left(f_{0,i_{r-2}}\circ\cdots\circ f_{0,i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{0,i_{r-2}}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|} + \frac{1}{\left|\frac{\partial\left(f_{0,i_{r-2}}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|}\frac{\left|\frac{\partial f_{0,i_{r-1}}}{\partial t}\right|}{\left|\frac{\partial f_{0,i_{r-1}}}{\partial x}\right|} + \frac{1}{\left|\frac{\partial\left(f_{0,i_{r-1}}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|}\frac{\left|\frac{\partial f_{0,i_r}}{\partial t}\right|}{\left|\frac{\partial f_{0,i_r}}{\partial x}\right|} \\
&\vdots \\
&< \left(1 + \frac{1}{3.5} + \frac{1}{3.5^2} + \cdots + \frac{1}{3.5^{r-1}}\right) * \frac{\left|\frac{\partial f_{0,i}}{\partial t}\right|}{\left|\frac{\partial f_{0,i}}{\partial x}\right|} \\
&< 1.4 * 1.109 < 1.5527.
\end{aligned} \tag{2.87}$$

Combining (2.87) and (2.20), we get

$$\begin{aligned}
\frac{\left|\frac{\partial(f_{5,k}\circ f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial t}\right|}{\left|\frac{\partial(f_{5,k}\circ f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial x}\right|} &\leq \frac{\left|\frac{\partial(f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial t}\right|}{\left|\frac{\partial(f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial x}\right|} + \frac{1}{\left|\frac{\partial(f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial x}\right|} \cdot \frac{\left|\frac{\partial f_{5,k}}{\partial t}\right|}{\left|\frac{\partial f_{5,k}}{\partial x}\right|} \\
&\leq 1.5527 + \frac{1}{3.5} * 161 \\
&< 48.
\end{aligned} \tag{2.88}$$

Combining (2.88) and (2.23), we get

$$\begin{aligned}
\frac{\left|\frac{\partial(f_{5,k}\circ f_{0,i_r}\circ\cdots\circ f_{0,i_1}\circ g_{(5)})}{\partial t}\right|}{\left|\frac{\partial(f_{5,k}\circ f_{0,i_r}\circ\cdots\circ f_{0,i_1}\circ g_{(5)})}{\partial x}\right|} &\leq \frac{\left|\frac{\partial g_{(5)}}{\partial t}\right|}{\left|\frac{\partial g_{(5)}}{\partial x}\right|} + \frac{1}{\left|\frac{\partial g_{(5)}}{\partial x}\right|} \cdot \frac{\left|\frac{\partial(f_{5,k}\circ f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial t}\right|}{\left|\frac{\partial(f_{5,k}\circ f_{0,i_r}\circ\cdots\circ f_{0,i_1})}{\partial x}\right|} \\
&\leq 0.0019 + \frac{1}{391005} * 48 \\
&< 0.00202277.
\end{aligned} \tag{2.89}$$

Since g is the composition of $g_{(5)}$ and at least one monotone branch from ξ_5, g has derivative greater than $391005 * 3.5$, so the first term of (2.85) is relatively small. We have

$$\left|\frac{dx_1(t)}{dt}\right| \leq 0.000000047 + 0.00202277 < \epsilon_0 \tag{2.90}$$

as desired. x_1 can be replaced by x_2. □

As a corollary of lemma 5, we estimate the relative shifts of $y_5(t)$ and $y_6(t)$. $y_5(t)$ and $y_6(t)$ are defined in 2.3.1.1 and 2.3.1.2, respectively.

`Corollary 1.` *Let $w(t)$ be in $\Delta^{(6)}$ satisfying (2.26), and $\mathcal{T}^{(6)} = \mathcal{T}(\Delta^{(6)})$ be the parameter interval such that when $t \in \mathcal{T}^{(6)}$, we have $w(t) \in \Delta^{(6)}$. If t_0 is the top endpoint of $\mathcal{T}^{(6)}$ and t is any other value in $\mathcal{T}^{(6)}$, then*

$$\frac{|y_6(t) - y_6(t_0)|}{H_5(t_0)} < \epsilon_0 \frac{|4(w(t) - w(t_0))|}{H_5(t_0)} < \frac{4\epsilon_0}{1 - 4\epsilon_0}\vartheta_1. \tag{2.91}$$

and

$$\frac{|y_5(t) - y_5(t_0)|}{H_5(t_0)} < \epsilon_0 \frac{|4(w(t) - w(t_0))|}{H_5(t_0)} < \frac{4\epsilon_0}{1 - 4\epsilon_0}\vartheta_1. \tag{2.92}$$

where y_5 and y_6 are as defined in the algorithm.

Proof. By (1.8) and lemma 5 we have

$$\frac{1}{\frac{1}{4} + \epsilon_0} < \frac{|\mathcal{T}(\Delta^{(6)})|}{|\Delta^{(6)}(t_0)|} < \frac{1}{\frac{1}{4} - \epsilon_0} \tag{2.93}$$

We know

$$w(t) - w(t_0) = \frac{1}{4}(t - t_0). \tag{2.94}$$

Combining (2.94) and (2.93), we have

$$w(t) - w(t_0) < \frac{\frac{1}{4}}{\frac{1}{4} - \epsilon_0}|\Delta^{(6)}(t_0)|. \tag{2.95}$$

Then by (2.26),

$$\frac{w(t)-w(t_0)}{H_5(t_0)} \leq \frac{1}{1-4\epsilon_0}\frac{|\Delta^{(6)}(t_0)|}{H_5(t_0)} < \frac{1}{1-4\epsilon_0}\frac{|\Delta^{(6)}(t_0)|}{H_5(\Delta^{(6)}(t_0))} < \frac{1}{1-4\epsilon_0}\vartheta_1, \tag{2.96}$$

where $H_5(t) = |[y_5(t), w(t)]|$. By lemma 5, we have $|y_6(t) - y_6(t_0)| < \epsilon_0|t - t_0|$. Then by (2.94) and (2.96), we get (2.91). Similarly, we get (2.92). $\square$

The corollary above shows that the shift of $y_5(t)$ and $y_6(t)$ is relatively small when t is restricted to a small interval whose size is controlled by the parameter ϑ_1.

2.5.1.2 Estimating the shift from y_6' to y_6 and calculations for $\frac{|\delta_6^{\text{re}}(t)|}{|\delta_5(t)|}$ (Defining ϑ_1 and ϑ_2)

Let δ_6^{re} be the parabolic pullback of $[y_6(t), w(t)]$ onto the x-axis.

Lemma 6. *Based on the algorithm given in 2.3.1, if we assign* $\vartheta_1 := 0.0098$ *and* $\vartheta_2 := 0.6 * \frac{1}{8}$ *then*

$$\frac{\sqrt{0.3}}{3}\,|\delta_5(t)| \leq |\delta_6^{re}(t)| \leq \frac{1}{3}\,|\delta_5(t)| \tag{2.97}$$

or equivalently

$$\frac{0.3}{9}\,|[y_5(t), w(t)]| \leq |[y_6(t), w(t)]| \leq \frac{1}{9}\,|[y_5(t), w(t)]| \tag{2.98}$$

for all $t \in \mathcal{T}^{(6)}$

Proof. To prove the lemma, we first prove some inequality for some specific parameter value. Then, using the small variation of each dynamically defined point, we prove the inequality for all $t \in \mathcal{T}^{(6)}$.

For the top value t_0 of each $\mathcal{T}^{(6)}$, we first find $r(t_0)$ and $R(t_0)$ so that

$$\frac{1}{3}r(t_0) \leq \frac{|\delta_6^{\mathrm{re}}(t_0)|}{|\delta_5(t_0)|} \leq R(t_0)\frac{1}{3}. \tag{2.99}$$

From (2.30) we have $R(t_0) = 1$.

The lower bound of $|\delta_6^{\mathrm{re}}(t_0)|$ depends on the distance from $y_6(t_0)$ to $w(t_0)$ which in turn depends on the shift from y_6' to $y_6(t_0)$. The shift from y_6' to $y_6(t_0)$ is bounded above by the size of $\delta^*(t_0)$ or $\Delta^*(t_0)$ which contains y_6'. Since we can always refine monotone domains when y_6' falls in a monotone domain, $r(t_0)$ is determined by the worst possible value of the ratio of $\delta^*(t_0) = \delta_0^{-p}(t_0)$ over $[y_5(t_0), w(t_0)]$.

y_6' is in a hole $\delta^*(t_0)$

When y_6' lies in δ_0^{-p}, $y_6(t_0)$ is defined as the upper endpoint of δ_0^{-p}. The domain δ_0^{-p} is mapped by some diffeomorphism G monotonically onto δ_0. This map can be extended to $\tilde{G}$ where the extended image is $\tilde{I} = [q^{-1} - 0.17, q + 0.17]$. The image of $\tilde{G} \circ h$ will cover at least domains Δ_{-3} and Δ_{-2}' as defined in (2.9). Consider Y as the pullback of $\delta_0 \cup \Delta_{-3} \cup \Delta_{-2}'$ by $\tilde{G}^{-1}$ into $\Delta^{(5)}$.

Then

$$\frac{|[y_6(t_0), w(t_0)]|}{\frac{1}{9}|[y_5(t_0), w(t_0)]|} = \frac{|[y_6(t_0), w(t_0)]|}{|[y_6', w(t_0)]|} = 1 - \frac{|[y_6', y_6(t_0)]|}{|[y_6', w(t_0)]|} \geq 1 - \frac{|\delta_0^{-p}|}{|Y|} \tag{2.100}$$

We let $\delta_X = \delta_0$, $X = \delta_0 \cup \Delta_{-3} \cup \Delta_{-2}'$, and $\tilde{X} = \tilde{I}$ and apply (A.3). We get that letting $\hat{\hat{X}} = X$ gives the better upper bound for distorted ratio.

$$\frac{|\delta_0^{-p}|}{|Y|} < 0.59. \tag{2.101}$$

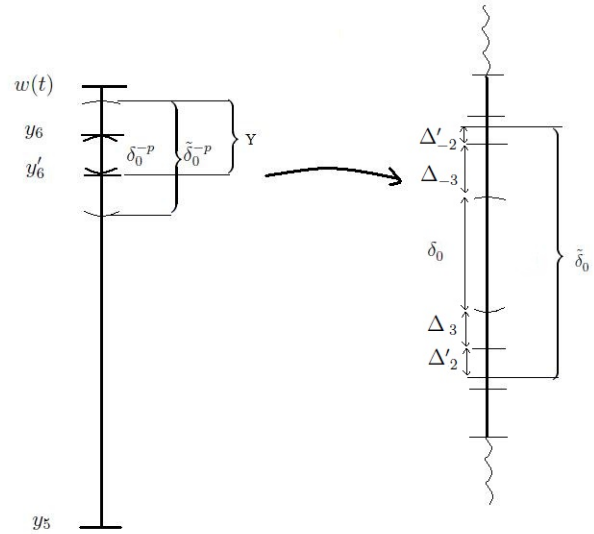


Figure 2.4: Y as the pullback of $\delta_0 \cup \Delta_{-3} \cup \Delta'_{-2}$ by $\tilde{G}^{-1}$ into $\Delta^{(5)}$

So $\frac{|[y_6(t_0),w(t_0)]|}{|[y_5(t_0),w(t_0)]|} > (1-k)\cdot\frac{1}{9}$ where $k = 0.59$. Then $\frac{|\delta_6^{\rm re}(t_0)|}{|\delta_5(t_0)|} > \sqrt{1-0.6}\cdot\frac{1}{3} > 0.63\cdot\frac{1}{3}$. So we can let

$$r(t_0) = 0.63. \tag{2.102}$$

y_6' is in a monotone domains Δ^*

We would also like the left hand side of (2.99) to hold for the case when y_6' falls into a monotone domain Δ^*. This can be done since ϑ_2 is chosen to be sufficiently small. If we have

$$\frac{|\Delta^*|}{H_5(\Delta^*)} < 0.6 * \frac{1}{8} =: \vartheta_2 \tag{2.103}$$

that will imply

$$
\begin{aligned}
\frac{|[y_6', y_6(t_0)]|}{|[y_6', w(t_0)]|} &< \frac{|\Delta^*(t_0)|}{\text{measure of } \Delta^*(t_0) \text{ and the region up to } w(t_0)} \\
&< \left(\frac{H_5(\Delta^*(t_0))}{\text{measure of } \Delta^*(t_0) \text{ and the region up to } w(t_0)}\right)\left(\frac{|\Delta^*(t_0)|}{H_5(\Delta^*(t_0))}\right) \\
&< \left(\frac{|[y_5(t_0), y_6']|}{|[y_6', w(t_0)]|}\right)\left(\frac{|\Delta^*(t_0)|}{H_5(\Delta^*(t_0))}\right) \\
&= 8\left(\frac{|\Delta^*(t_0)|}{H_5(\Delta^*(t_0))}\right) < 0.6.
\end{aligned}
$$

The equality follows from (2.28). We can plug this into (2.100) and derive the left hand side of (2.99) as we did for the case where y_6' is in a hole $\delta^*(t_0)$.

Left inequality of (2.97)

For general $t \in \mathcal{T}^{(6)}$, we apply (2.96) and (2.91) to get

$$
\begin{aligned}
|[y_6(t), w(t)]| \geq& |[y_6(t_0), w(t_0)]| - |[y_6(t), y_6(t_0)]| - |[w(t), w(t_0)]| \\
\geq& \frac{1}{9} r(t_0)^2 H_5(t_0) - \left(\frac{4\epsilon_0}{1-4\epsilon_0}\right)\vartheta_1 H_5(t_0) - \left(\frac{1}{1-4\epsilon_0}\right)\vartheta_1 H_5(t_0) \\
\geq& \frac{1}{9}\left(1 - 0.6 - 9 \cdot \left(\frac{1+4\epsilon_0}{1-4\epsilon_0}\right)\vartheta_1\right) H_5(t_0).
\end{aligned}
$$

For

$$
\vartheta_1 := 0.0098 \tag{2.104}
$$

and $\epsilon_0 = 0.003$, we have $9\left(\frac{1+4\epsilon_0}{1-4\epsilon_0}\right)\vartheta_1 < 0.1$. Then, since t_0 is the top value of $\mathcal{T}^{(6)}$ and $w(t)$ moves faster than $y_5(t)$, we get

$$
\begin{aligned}
|[y_6(t), w(t)]| \geq& \frac{1}{9}\left(1 - 0.6 - 0.1\right) H_5(t_0) \\
\geq& \frac{1}{9}(0.3) H_5(t).
\end{aligned}
$$

So for all t, we get

$$
\frac{|\delta_6^{\text{re}}(t)|}{|\delta_5(t)|} \geq \frac{1}{3} \cdot \sqrt{0.3}. \tag{2.105}
$$

We can define

$$r = \sqrt{0.3} \tag{2.106}$$

Right inequality of (2.97)

As t_0 is the top parameter of $\mathcal{T}^{(6)}$, we have $w(t_0) > w(t)$. Using (2.91) and (2.92), we get

$$\begin{aligned}
\frac{|[y_6(t), w(t)]|}{|[y_5(t), w(t)]|} &= \frac{w(t) - y_6(t)}{w(t) - y_5(t)} \\
&= \frac{(w(t) - w(t_0)) + (w(t_0) - y_6(t_0)) + (y_6(t_0) - y_6(t))}{(w(t) - w(t_0)) + (w(t_0) - y_5(t_0)) + (y_5(t_0) - y_5(t))} \\
&= \frac{\frac{w(t)-w(t_0)}{H_5(t_0)}\left(1 + \frac{y_6(t_0)-y_6(t)}{w(t)-w(t_0)}\right) + \frac{w(t_0)-y_6(t_0)}{H_5(t_0)}}{\frac{w(t)-w(t_0)}{H_5(t_0)}\left(1 + \frac{y_5(t_0)-y_5(t)}{w(t)-w(t_0)}\right) + \frac{w(t_0)-y_5(t_0)}{H_5(t_0)}} \\
&\leq \frac{\frac{w(t)-w(t_0)}{H_5(t_0)}(1 - 4\epsilon_0) + \frac{1}{9}}{\frac{w(t)-w(t_0)}{H_5(t_0)}(1 + 4\epsilon_0) + 1} \\
&\leq \frac{1}{9}
\end{aligned}$$

The last inequality is true because $(1 - 4\epsilon_0) \geq \frac{1}{9}(1 + 4\epsilon_0)$ for $\epsilon_0 = 0.003$ and $w(t) - w(t_0)$ is negative. So for all $t \in \mathcal{T}^{(6)}$,

$$\frac{|\delta_6^{\mathrm{re}}|}{|\delta_5|} \leq \frac{1}{3}. \tag{2.107}$$

This shows the right hand side of (2.97).

$\square$

2.5.1.3 Maximum number of monotone pullbacks for step 6 is less than 5

In our algorithm, we perform monotone refinements by ξ_0 when defining $\mathcal{T}^{(6)}$'s (or $\Delta^{(6)}$'s) and $y_6(t)$ so that (2.26) is satisfied for $\Delta^{(6)}(t)$ containing $w(t)$ and (2.29) is satisfied for Δ such that $\Delta(t_0)$ contains y_6'. Now we discuss the number of monotone pullbacks needed in these two procedures.

Lemma 7. *If we create $\Delta^{(6)}$ and $y_6(t)$ according to our algorithm in 2.3.1, the number of monotone refinements needed in defining $\Delta^{(6)}$ and $y_6(t)$ summed together will not exceed five.*

Proof. This lemma is justified by numeric computations. In (2.6) we made an extra assumption on the parameters at the initial steps in order for all branches of ξ_5 to be extendable. Now we find some $t_* > 3.99512595$ which is Markov, meaning $w(t_*)$ is a preimage of q_{t_*}. Since $w(3.99512595)$ lies in $\Delta^{(5)14}(t) = g_5^{-1}(\Delta_1(t) \cap f_1^{-1}(\delta_0(t)))$ for all $t \in \mathcal{T}^{(5)}$, it makes sense to choose t_* such that $w(t_*)$ is the upper endpoint of $\Delta^{(5)14}(t_*)$.

$$t_* \approx 3.99512600657. \tag{2.108}$$

We check (2.26) and (2.29) for domains $\Delta^{(5)14}$ and above.

Number of monotone refinements in defining $\mathcal{T}^{(6)}$

When we choose $\Delta^{(6)}$, we only need to consider admissible domains above $\Delta^{(5)14}$. For each monotone domain Δ obtained from consecutive pullbacks of ξ_0 onto the y-axis, ratio's $\frac{|\Delta|}{H_5(\Delta)}$ can be obtained numerically. The charts in B.3.2 give values of $\frac{|\Delta|}{H_5(\Delta)}$ for monotone domains and their refinements. From (2.104), we have $\vartheta_1 = 0.0098$. We can conclude from the chart that for domains above $\Delta^{(5)14}$, at most 5 monotone refinements are needed to achieve (2.26). In particular, 4 monotone refinements are needed for the domain at the very top of $\Delta^{(5)}$.

Number of monotone refinements when defining y_6

The domain $\Delta^{(6)}$ containing $w(t)$ satisfies (2.26). Since $\vartheta_1 < \frac{1}{9}$, by (2.28) we know that it cannot contain y_6'. Some domain other than $\Delta^{(6)}$ contains y_6'. Since $w(t_0)$ is always above $w(t_*)$ and by lemma 5, the variation of $y_5(t)$ is small compared to variation of $w(t)$, that means $y_6' > y_5(t_0) + \frac{8|[y_5(t_0), w(t_0)]|}{9} \geq y_5(t_0) + \frac{8|[y_5(t_*), w(t_*)]|}{9}$. Domain $g_5^{-1}(\Delta_1(t) \cap f_1^{-1}(\Delta_{-3}(t)))$ contains $y_5(t) + \frac{8|[y_5(t_*), w(t_*)]|}{9}$ for all $t \in \mathcal{T}^{(5)}$. It suffices to look at all monotone domains above $\Delta^{(5)13} = g_5^{-1}(\Delta_1 \cap f_1^{-1}(\Delta_{-3}))$ to check for inequality (2.29), where $\vartheta_2 = 0.075$.

$\square$

2.5.1.4 Number of boundary refinements for step 6 is less than 2

Lemma 8. *No more than two boundary refinements are needed on monotone domains Δ in $[y_5, y_6]$ at step 6.*

Proof. The argument uses (2.28) and the right hand side of the inequality (2.32). We consider two cases:

Case 1: Δ is adjacent to y_6

Since Δ and y_6 are defined dynamically, this condition holds for all $t \in \mathcal{T}^{(6)}$

once it holds for one specific t in $\mathcal{T}^{(6)}$. $\Delta(t_0)$ adjacent to $y_6(t_0)$ means that $y_6' \in \Delta(t_0)$. From (2.29), we have $\frac{|\Delta|}{H_5(\Delta)} < \vartheta_2 = \frac{1}{8} * 0.6$ for all $t \in \mathcal{T}^{(6)}$. Let us make the following assumption:

$$\frac{|\text{top component of } \tilde{\Delta}_{l\ldots l} \backslash \Delta_{l\ldots l}|}{|\Delta|} < 0.47 \tag{2.109}$$

for all $t \in \mathcal{T}^{(6)}$

Combining (2.109) and (2.29) we get

$$\begin{aligned}
&\frac{|\text{top component of } \tilde{\Delta}_{l\ldots l}(t) \backslash \Delta_{l\ldots l}(t)|}{|[y_5(t), y_6(t)]|} \\
=&\frac{|\text{top component of } \tilde{\Delta}_{l\ldots l}(t) \backslash \Delta_{l\ldots l}(t)|}{|\Delta(t)|} \cdot \frac{|\Delta(t)|}{H_5(\Delta(t)) + |\Delta(t)|} \\
<&0.47 * \frac{\vartheta_2}{1+\vartheta_2} < 0.47 * \frac{\frac{0.6}{8}}{1+\frac{0.6}{8}} < \frac{1}{9} * 0.3.
\end{aligned} \tag{2.110}$$

From (2.110) and (2.32) we get

$$\frac{|\text{top component of } \tilde{\Delta}_{l\ldots l}(t) \backslash \Delta_{l\ldots l}(t)|}{|[y_5(t), y_6(t)]|} < \frac{1}{9} * 0.3 \leq \frac{|[y_6(t), w(t)]|}{|[y_5(t), w(t)]|} \leq \frac{|[y_6(t), w(t)]|}{|[y_5(t), y_6(t)]|} \tag{2.111}$$

which implies that the extended domain $\tilde{\Delta}_{l\ldots l}$ lies below $w(t)$ for all $t \in \mathcal{T}^{(6)}$. From numerical results in B.14, we get that it only takes one refinement to get condition (2.109) to hold.

Case 2: Δ is not adjacent to y_6

Let $z(t)$ be the upper endpoint of $\Delta(t)$, then Δ not adjacent to y_6 implies $z(t_0) \leq y_6'$. However, this does not imply $z(t) \leq y_6'$ for all $t \in \mathcal{T}^{(6)}$. Still we can make estimates since $z(t)$ and $z(t_0)$ are close. Similar to (2.91), we have

$$\frac{|z(t) - z(t_0)|}{H_5(t_0)} < \frac{4\epsilon_0}{1 - 4\epsilon_0}\vartheta_1. \tag{2.112}$$

We use (2.112), (2.92), and (2.96) to get

$$\begin{aligned}
\frac{|[z(t),w(t)]|}{|[y_5(t),z(t)]|} &\geq \frac{|[z(t_0),w(t_0)]|-|[w(t),w(t_0)]|-|[z(t),z(t_0)]|}{|[y_5(t_0),z(t_0)]|+|[y_5(t_0),y_5(t)]|+|[z(t),z(t_0)]|}\\
&\geq \frac{|[y_6',w(t_0)]|-|[w(t),w(t_0)]|-|[z(t),z(t_0)]|}{|[y_5(t_0),y_6']|+|[y_5(t_0),y_5(t)]|+|[z(t),z(t_0)]|}\\
&= \frac{\frac{|[y_6',w(t_0)]|}{H_5(t_0)}-\frac{|[w(t),w(t_0)]|}{H_5(t_0)}-\frac{|[z(t),z(t_0)]|}{H_5(t_0)}}{\frac{|[y_5(t_0),y_6']|}{H_5(t_0)}+\frac{|[y_5(t_0),y_5(t)]|}{H_5(t_0)}+\frac{|[z(t),z(t_0)]|}{H_5(t_0)}}\\
&\geq \frac{\frac{1}{9}-\frac{1+4\epsilon_0}{1-4\epsilon_0}\vartheta_1}{\frac{8}{9}+2*\frac{4\epsilon_0}{1-4\epsilon_0}\vartheta_1} > 0.11.
\end{aligned}$$

If we have

$$\frac{|\text{top component of } \tilde{\Delta}_{l\ldots l}\backslash\Delta_{l\ldots l}|}{|\Delta|} < 0.11 \tag{2.113}$$

then

$$\frac{|\text{top component of } \tilde{\Delta}_{l\ldots l}\backslash\Delta_{l\ldots l}|}{|[y_5(t),z(t)]|} < \frac{|\text{top component of } \tilde{\Delta}_{l\ldots l}\backslash\Delta_{l\ldots l}|}{|\Delta|} < 0.11 < \frac{|[z(t),w(t)]|}{|[y_5(t),z(t)]|} \tag{2.114}$$

which implies that the extended domain $\tilde{\Delta}_{l\ldots l}$ lies below $w(t)$ for all $t \in \mathcal{T}^{(6)}$.

From the table in B.14, we see that (2.113) can still be achieved within two refinements.

□

2.5.1.5 Estimates on the relative measure of holes in the phase space

Let $\mu_{\text{holes}}(\xi_6)$ denote the relative measure of holes in ξ_6 and let $\mu_{\text{holes}}(\eta_5)$ denote the relative measure of holes in η_5.

From (2.13), we have

$$\frac{\text{measure of holes in } \delta_0^{-p} \text{ after 5-step filling-in of } \delta_0^{-p}}{|\delta_0^{-p}|} = \mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_0)) < 0.29. \tag{2.115}$$

This will give an estimate for the measure of holes in $\delta_5 \backslash \delta_6^{\mathrm{re}}$ after 5-step filling-ins on all preimages of δ_0 in $\delta_5 \backslash \delta_6^{\mathrm{re}}$.

$$\frac{\text{measure of holes in } \delta_5 \backslash \delta_6^{\mathrm{re}} \text{ of } \xi_6}{|\delta_5 \backslash \delta_6^{\mathrm{re}}|} < 0.29 \tag{2.116}$$

Combining (2.107), (2.116), and (A.9) we get

$$\begin{aligned} \mu_{\text{holes}}(\eta_5) <& \frac{|\delta_6^{\mathrm{re}}|}{|\delta_5|} + \left(1 - \frac{|\delta_6^{\mathrm{re}}|}{|\delta_5|}\right) \frac{\text{measure of holes in } \delta_5 \backslash \delta_6^{\mathrm{re}}}{|\delta_5 \backslash \delta_6^{\mathrm{re}}|} \\ <& \frac{1}{3} + \frac{2}{3} * (0.29) \\ <& 0.5267. \end{aligned} \tag{2.117}$$

For the measure of holes outside δ_5, we have the numeric bound

$$\frac{\text{measure of holes in } I \backslash \delta_5 \text{ of } \xi_5}{|I \backslash \delta_5|} < 0.01776 \tag{2.118}$$

for all $t \in \mathcal{T}^{(5)} \cap \{t > 3.99512595\}$. For the measure of δ_5 with respect to the measure of I, we also have an upper bound

$$\frac{|\delta_5|}{|I|} < 0.0022. \tag{2.119}$$

(2.119) can be observed from the table in B.1.1 on relative sizes of domains. Combining (2.117), (2.118), (2.119) and (A.10), we get

$$\begin{aligned} \mu_{\text{holes}}(\xi_6) =& \frac{|I \backslash \delta_5|}{|I|} \frac{\text{measure of holes in } I \backslash \delta_5 \text{ of } \xi_5}{|I \backslash \delta_5|} + \frac{|\delta_5|}{|I|} \frac{\text{measure of holes in } \delta_5 \text{ after step 6}}{|\delta_5|} \\ <& (1 - 0.0022) * 0.01776 + 0.0022 * 0.5267 \\ <& 0.0189 \end{aligned} \tag{2.120}$$

2.5.1.6 Possible compositions

Here we repeat the possible compositions for the maps as discussed in 2.4.4.1 and 2.4.4.2, but write out possible compositions particularly for step 6. We give possible compositions with additional information on the maximum possible number of refinements for $g_{(6)}$, $\mathcal{G}_{(6)}$, $\bar{g}_{6,i}$, $\bar{\mathcal{G}}_{6,i}$, $g_{6,i}$, $\mathcal{G}_{6,i}$, $f_{6,i}$ and $\mathcal{F}_{6,i}$.

Let $\hat{f}_{0,i}$ denote the branches of admissible domains in $\hat{\xi}_0$. $g_{(6)}$ and $\mathcal{G}_{(6)}$ are maps on domains of the parameter-induced partition of $\Delta^{(5)}$. The number of monotone refinements needed to form $\Delta^{(6)}$ is less than or equal to 5, therefore we have

$$g_{(6)} = \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)} \qquad 1 \leq s \leq 5 \tag{2.121}$$

$$\mathcal{G}_{(6),i} = \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)} \qquad 1 \leq s \leq 5 \tag{2.122}$$

$\bar{g}_{6,i}$ and $\bar{\mathcal{G}}_{6,i}$ are maps on domains above y_6 of the partitions $\zeta^{(6)}(\Delta^{(6)})$. The number of monotone refinements needed to achieve (2.29) is less than or equal to 5, therefore we have

$\bar{g}_{6,i} = f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

$\bar{\mathcal{G}}_{6,i} = f_{0,j_{s'-1}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

$g_{6,i}$ and $\mathcal{G}_{6,i}$ are maps on domains below y_6 of the partitions $\zeta^{(6)}(\Delta^{(6)})$. In addition to compositions that form $\bar{g}_{6,i}$ and $\bar{\mathcal{G}}_{6,i}$, $f_{0,1}$ is due to possible boundary refinements and $f_{5,k}$ or $\mathcal{F}_{5,k}$ are due to a filling-in.

$g_{6,i} = f_{5,k} \circ f_{0,1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

$\mathcal{G}_{6,i} = f_{0,1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$ or

$\mathcal{G}_{6,i} = \mathcal{F}_{5,k} \circ f_{0,1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \leq s \leq 5$, $s + s' \leq 5$

From (2.76) and (2.77), we have for maps $f_{6,i}$ and $\mathcal{F}_{6,i}$ on domains in $\delta_5 \backslash \delta_6^{\mathrm{re}}$,

$f_{6,i} = g_{6,i} \circ h$ where $g_{6,i}$ is a monotone branch defined on $[y_5(t), w(t)]$

$\mathcal{F}_{6,i} = \mathcal{G}_{6,i} \circ h$, where $\mathcal{G}_{6,i}$ is a monotone branch defined on $[y_5(t), w(t)]$.

2.5.1.7 Extendability and extensions

Lemma 9. *All monotone branches $f_{6,i}$ in ξ_6 are extendable to $\tilde{I}$.*

Proof. All monotone branches from partition ξ_5 are uniformly extendable to $\tilde{I}$, therefore we only have to show extendability for newly created monotone branches. New monotone branches are created in two ways, from monotone refinements and from filling-ins.

Monotone branches created from monotone refinements are extendable to $\tilde{I}$ because we perform boundary refinements if they are not.

Monotone branches created from filling-ins are extendable to $\tilde{I}$ by the following arguments. Since filling-in first, then taking parabolic pullback, and taking parabolic pullback, then filling-in are equivalent, for convenience here, we will consider all filling-ins from the perspective that all filling-ins are done after a parabolic pullback, which means all filling-ins are performed on the x-axis. The only holes that are filled-in at step 6 are preimages δ_0^{-1} of δ_0 inside δ_5. They are mapped by some diffeomorphism $\mathcal{F}$ onto δ_0 and can be extended onto the enlargement $\hat{\delta}_0$ due to our choice of parameters(critical value avoids two monotone domains on top of each δ_0^{-1}

on the y-axis).

If we fill-in δ_0^{-1} by η_0, all new monotone branches in δ_0^{-1} will be extendable to $\tilde{I}$ if $\hat{\delta}_0$ contains all extensions of monotone domains in η_0, which is true from observation on extended domains of monotone domains in ξ_5. □

Since $(1 + \frac{|I|}{\frac{1}{2}|\tilde{I}\backslash I|})^2 < (1 + \frac{1}{2*0.17})^2 < 15.6$, we have

`Corollary 2.` *Distortion on monotone branches in ξ_6 is less than* 15.6.

Lemma 10. *All maps on preimages of δ_0, $\mathcal{F}_{6,i} : \delta_0^{-p} \to \delta_0$, in ξ_6 are extendable to $\hat{\delta}_0$. All maps on preimages of δ_5, $\mathcal{F}_{6,i} : \delta_5^{-1} \to \delta_5$, in ξ_6 are extendable to $\hat{\delta}_0$.*

Proof. We know precisely that the newly created holes in step 6 are either preimages of δ_0 or preimages of δ_5, both obtained by filling-in of δ_0^{-1} with η_0. As in the proof of the previous lemma, each such δ_0^{-1} is mapped by some diffeomorphism $\mathcal{F}$ onto δ_0 and can be extended to a map $\tilde{\mathcal{F}}$ that maps onto the enlargement $\hat{\delta}_0$. The central domain of η_0 is δ_5, so this shows that $\mathcal{F}_{6,i} : \delta_5^{-1} \to \delta_5$ are extendable to $\hat{\delta}_0$.

Consider $\hat{\delta}_0^{-1}$ as $\tilde{\mathcal{F}}_{5,j}^{-1}(\hat{\delta}_0)$. $\hat{\delta}_0^{-1}$'s are all contained in δ_0 and hence in $\hat{\delta}_0$. Since $\mathcal{F}_{6,i} = \mathcal{F}_{5,j} \circ \mathcal{F}$, this shows $\mathcal{F}_{6,i} : \delta_0^{-p} \to \delta_0$ are extendable to $\hat{\delta}_0$. □

Lemma 11. *The union of extensions of monotone domains in η_5, denoted by $\tilde{\delta}_5$, is contained in δ_0.*

Proof. The union of the extensions of monotone domains in η_5 is contained in the union of δ_5 and the two monotone domains adjacent to δ_5 which is well within δ_0. □

Due to lemma 11, we define the enlargement $\hat{\delta}_5$ of δ_5 to be δ_0.

2.5.1.8 Derivatives

Our requirement for derivatives is very low. All we need is to show that all derivatives on the monotone branches on the x-axis are greater than 3.5. Compositions of monotone branches make derivatives greater, which is better. Parabolic pullbacks make derivatives smaller, but as long as the increase compensates for the decrease, we can still prove that derivatives are still greater than 3.5. From 2.2.10, we have $\left|\frac{\partial f_{0,i}}{\partial x}\right|, \left|\frac{\partial f_{5,i}}{\partial x}\right| \geq 3.5$, $\left|\frac{\partial \mathcal{F}_{5,i}}{\partial x}\right| \geq 20$ and $\left|\frac{\partial g_{(5)}}{\partial x}\right| \geq 391005$. The worst case for monotone maps on the y-axis at step 6 is when $g_{(5)}$ composes with a monotone branch in ξ_0 just once. In this case $g_{(6)}$, $\bar{g}_{6,i}$, or $g_{6,i}$ is $f_{0,j} \circ g_{(5)}$.

$$\begin{aligned}\left|\frac{\partial g_{6,i}}{\partial x}\right| &\geq \left|\frac{\partial f_{0,j}}{\partial x}\right| \cdot \left|\frac{\partial g_{(5)}}{\partial x}\right| \\ &\geq 3.5 * 391005 \end{aligned} \tag{2.123}$$

The worst case for maps on holes on the y-axis at step 6 is when the hole is just a preimage of δ_0 or δ_5 and $\mathcal{G}_{(6),i}$, $\bar{\mathcal{G}}_{6,i}$, or $\mathcal{G}_{6,i}$ is just $g_{(5)}$.

$$\begin{aligned}\left|\frac{\partial \mathcal{G}_{6,i}}{\partial x}\right| &\geq \left|\frac{\partial g_{(5)}}{\partial x}\right| \\ &\geq 391005 \end{aligned} \tag{2.124}$$

Another way to estimate derivatives is to take the length of the image divided by the length of the domain divided by the worst possible distortion. We use distortion from lemma 2 and distortion on holes to get

$$\left|\frac{\partial g_{(6)}}{\partial x}\right| \geq \max\left\{391005 * 3.5, \frac{|I|}{\left|\Delta^{(6)}\right|} * \frac{1}{15.6}\right\} \geq \frac{|I|}{\left|\Delta^{(6)}\right|} * \frac{1}{15.6} > 2.6 * 10^6$$

$$\left|\frac{\partial \mathcal{G}_{(6),i}}{\partial x}\right| \geq \max\left\{391005, \frac{|\delta_0|}{\left|\Delta^{(5)}\right| * (\text{worst distorted ratio of } \delta_0^{-1} \text{ in } \Delta^{(5)})} * \frac{1}{1.3035}\right\} = 391005$$

$$\left|\frac{\partial \bar{g}_{6,i}}{\partial x}\right| \geq \max\left\{391005 * 3.5, \frac{|I|}{|[y_6(t), w(t)]|} * \frac{1}{15.6}\right\} = 391005 * 3.5$$

$$\left|\frac{\partial\bar{\mathcal{G}}_{6,i}}{\partial x}\right| \geq \max\left\{391005, \frac{|\delta_0|}{|[y_6(t),w(t)]|} * \frac{1}{1.3035}\right\} = 391005$$
$$\left|\frac{\partial g_{6,i}}{\partial x}\right| \geq \max\left\{391005 * 3.5, \frac{|I|}{|[y_5(t),y_6(t)]|} * \frac{1}{15.6}\right\} = 391005 * 3.5$$
$$\left|\frac{\partial\mathcal{G}_{6,i}}{\partial x}\right| \geq \max\left\{391005, \min\left\{\frac{|\delta_0|}{|[y_5(t),y_6(t)]|} * \frac{1}{2.75}, \frac{|\delta_5|}{|[y_5(t),y_6(t)]|} * \frac{1}{1.1}\right\}\right\} = 391005$$

Now we consider derivatives for $f_{6,i}$ and $\mathcal{F}_{6,i}$. When considering maps on the x-axis, we only consider maps outside δ_6^{re}. For x outside δ_6^{re}, we have $\left|\frac{\partial h}{\partial x}(x)\right| \geq t\,|\delta_6^{\mathrm{re}}|$. So

$$\begin{aligned}\left|\frac{\partial f_{6,i}}{\partial x}\right| &\geq \left|\frac{\partial g_{6,j}}{\partial x}\right| \cdot \left|\frac{\partial h}{\partial x}\right| \\ &\geq 391005 * 3.5 * t * \frac{1}{3}\sqrt{0.3} * |\delta_5| \\ &\geq 391005 * 3.5 * t * \frac{1}{3}\sqrt{0.3} * 0.00038 \\ &> 3.5.\end{aligned}$$

Estimates are similar for $\mathcal{F}_{6,i}$.

2.5.1.9 Variation of derivatives

We estimate variation of derivatives for maps $g_{(6)}$, $\mathcal{G}_{(6)}$, $\bar{g}_{6,i}$, $\bar{\mathcal{G}}_{6,i}$, $g_{6,i}$, $\mathcal{G}_{6,i}$, $f_{6,i}$ and $\mathcal{F}_{6,i}$ with forms given in 2.5.1.6. We use (2.199) and preliminary estimates in 2.2.10.

$g_{(6)}$ **and** $\mathcal{G}_{(6)}$

$g_{(6)}$ has the form (2.121). From (2.199) and table in B.1.5, we have

$$\begin{aligned}\frac{\left|\frac{\partial^2 g_{(6)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(6)}^{-1}}{\partial z}\right|} &\le \frac{\left|\frac{\partial^2 g_{(5)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(5)}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 g_{(5)}}{\partial x^2}\right|}{\left|\frac{\partial g_{(5)}}{\partial x}\right|^2} \cdot \frac{\left|\frac{\partial(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial t}\right|}{\left|\frac{\partial(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial x}\right|} + \frac{\left|\frac{\partial^2(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial z}\right|} \\ &\le 8.9 + 1.5 * 1.5527 + 200 \\ &< 211.23 \end{aligned} \tag{2.125}$$

Since $\mathcal{G}_{(6),i} = \hat{f}_{0,i_{s-1}} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $\mathcal{G}_{(6),i}$ has similar or better estimates.

$\bar{g}_{6,i}$ **and** $\bar{\mathcal{G}}_{6,i}$

$\bar{g}_{6,i} = f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$

$\bar{\mathcal{G}}_{6,i} = f_{0,j_{s'-1}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$

Since $1 \le s + s' \le 5$, estimates are the same as $g_{(6)}$ and $\mathcal{G}_{(6),i}$.

$g_{6,i}$ **and** $\mathcal{G}_{6,i}$

The worst possible cases for $g_{6,i}$ and $\mathcal{G}_{6,i}$ are

$g_{6,i} = f_{5,k} \circ f_{0,1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \le s \le 5$, $s + s' \le 5$

$\mathcal{G}_{6,i} = \mathcal{F}_{5,k} \circ f_{0,1} \circ f_{0,j_{s'}} \circ \cdots \circ f_{0,j_1} \circ \hat{f}_{0,i_s} \circ \cdots \circ \hat{f}_{0,i_1} \circ g_{(5)}$, $1 \le s \le 5$, $s + s' \le 5$

$$\frac{\left|\frac{\partial^2(f_{5,k}\circ f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial(f_{5,k}\circ f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial z}\right|}$$

$$\leq\frac{\left|\frac{\partial^2(f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial(f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial z}\right|}$$

$$+\frac{\left|\frac{\partial^2(f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial x^2}\right|}{\left|\frac{\partial(f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial f_{5,k}}{\partial t}\right|}{\left|\frac{\partial f_{5,k}}{\partial x}\right|}+\frac{\left|\frac{\partial^2 f_{5,k}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{5,k}^{-1}}{\partial z}\right|}$$

$$\leq(200+12*1.5527+200)+12*161+900,000<902,340 \tag{2.126}$$

where the bound for $\frac{\left|\frac{\partial^2(f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial x^2}\right|}{\left|\frac{\partial(f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial x}\right|^2}$ uses Corollary 8. Using (2.126), (2.24),(2.25) and (2.88), we get

$$\frac{\left|\frac{\partial^2 g_{6,i}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{6,i}^{-1}}{\partial z}\right|}\leq\frac{\left|\frac{\partial^2 g_{(5)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(5)}^{-1}}{\partial z}\right|}+\frac{\left|\frac{\partial^2 g_{(5)}}{\partial x^2}\right|}{\left|\frac{\partial g_{(5)}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial(f_{5,k}\circ f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial t}\right|}{\left|\frac{\partial(f_{5,k}\circ f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})}{\partial x}\right|}$$

$$+\frac{\left|\frac{\partial(f_{5,k}\circ f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial(f_{5,k}\circ f_{0,1}\circ f_{0,j_{s'}}\circ\cdots\circ f_{0,j_1}\circ\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{0,i_1})^{-1}}{\partial z}\right|}$$

$$\leq 8.9+1.5*48+902,340 \tag{2.127}$$

$$<902,421. \tag{2.128}$$

$\mathcal{G}_{6,i}$ has similar or better estimate.

$f_{6,i}$ **and** $\mathcal{F}_{6,i}$ **in** $\delta_5 \backslash \delta_6^{\mathrm{re}}$

$f_{6,i} = g_{6,i} \circ h$ where $g_{6,i}$ is in $[y_5(t), w(t)]$

$\mathcal{F}_{6,i} = \mathcal{G}_{6,i} \circ h$, where $\mathcal{G}_{6,i}$ is in $[y_5(t), w(t)]$.

We use

$$\frac{\left|\frac{\partial^2 h^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial h^{-1}}{\partial z}\right|} \leq \frac{1}{t} + \frac{2(\frac{1}{4} - \frac{1}{4}\left|\delta_6^{\mathrm{re}}\right|^2)}{t\left|\delta_6^{\mathrm{re}}\right|^2}. \tag{2.129}$$

$$\begin{aligned}
\frac{\left|\frac{\partial^2 f_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial f_{6,i}^{-1}}{\partial z}\right|} &\leq \frac{\left|\frac{\partial^2 h^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial h^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 h}{\partial x^2}\right|}{\left|\frac{\partial h}{\partial x}\right|^2} \cdot \frac{\left|\frac{\partial g_{6,i}}{\partial t}\right|}{\left|\frac{\partial g_{6,i}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 g_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{6,i}^{-1}}{\partial z}\right|} \\
&\leq \frac{1}{t} + \frac{2(\frac{1}{4} - \frac{1}{4}\left|\delta_6^{\mathrm{re}}\right|^2)}{t\left|\delta_6^{\mathrm{re}}\right|^2} + \frac{2}{t\left|\delta_6^{\mathrm{re}}\right|^2} \cdot \frac{\left|\frac{\partial g_{6,i}}{\partial t}\right|}{\left|\frac{\partial g_{6,i}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 g_{6,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{6,i}^{-1}}{\partial z}\right|} \\
&\leq \frac{1}{3.99} + \frac{1}{3.99} * \frac{1}{\left|\delta_6^{\mathrm{re}}\right|^2} * \frac{1}{2} + \frac{2}{3.99} * \frac{1}{\left|\delta_6^{\mathrm{re}}\right|^2} * 0.0021 + 902421 \\
&\leq \frac{1}{3.99} * \frac{1}{\left|\delta_6^{\mathrm{re}}\right|^2} * \frac{1}{2} + \frac{1}{3.99} * \frac{1}{\left|\delta_6^{\mathrm{re}}\right|^2} * \frac{1}{2} + \frac{2}{3.99} * \frac{1}{\left|\delta_6^{\mathrm{re}}\right|^2} * 0.0021 + \frac{9}{8}\frac{1}{\left|\delta_6^{\mathrm{re}}\right|^2} \\
&< \frac{1.38}{\left|\delta_6^{\mathrm{re}}\right|^2} \\
&< 2.9 * 10^8
\end{aligned} \tag{2.130}$$

Estimates for $\mathcal{F}_{6,i}$ in $\delta_5 \backslash \delta_6^{\mathrm{re}}$ are similar.

$f_{6,i}$ **and** $\mathcal{F}_{6,i}$ **outside** δ_5

$f_{6,i} = f_{5,j}$

$\mathcal{F}_{6,i} = \mathcal{F}_{5,j}$ where map $\mathcal{F}_{5,j}$'s are the maps on the five holes in ξ_5.

Estimates remain the same as in step 5.

2.5.2 Steps 7 through 14

For steps 7 through 14 we pullback the same partition, ξ_0 or $\hat{\xi}_0$, as we did in step 6. Therefore, some estimates are the same as in step 6. The difference between steps 7 through 14 and step 6 is that Δ^{k-1} is no longer adjacent to y_{k-1} as Δ_5 was adjacent to y_5.

(I) Velocities on partitioning points of $\Delta^{(k-1)}$ and Δ_y are less than $\epsilon_0 = 0.003$.

(II) For each step k and each rescaled central domain δ_k^{re}, we have

$$\frac{1}{3}\sqrt{0.3} \leq \frac{|\delta_k^{\text{re}}(t)|}{|\delta_{k-1}^{\text{re}}(t)|} \leq \frac{1}{3}, \tag{2.131}$$

or equivalently

$$\frac{1}{9}0.3 \leq \frac{|[y_k(t), w(t)]|}{|[y_{k-1}(t), w(t)]|} \leq \frac{1}{9}, \tag{2.132}$$

(III)

$$\begin{aligned}
\left|\mathcal{T}^{(k)}\right| &\leq \frac{1}{\frac{1}{4}-\epsilon_0}\left|\Delta^{(k)}\right| \\
&\leq \frac{1}{\frac{1}{4}-\epsilon_0}\vartheta_1 H_{k-1}(\Delta^{(k)}) \\
&\leq \frac{1}{\frac{1}{4}-\epsilon_0}\vartheta_1 \left|[y_{k-1}(t), w(t)]\right| \\
&\leq \frac{1}{\frac{1}{4}-\epsilon_0}\vartheta_1 \left(\frac{1}{9}\right)^{k-6} \left|[y_5(t), w(t)]\right| \\
&\leq \frac{1}{\frac{1}{4}-\epsilon_0}\vartheta_1 \left(\frac{1}{9}\right)^{k-6} \frac{t}{4}\left|\delta_5\right|^2
\end{aligned} \tag{2.133}$$

(IV) The number of monotone refinements needed is no more than 5.

(V) $\mu_{\text{holes}}(\eta_{k-1}) < 0.5267$. $\mu_{\text{holes}}(\xi_k) < 0.0189 * (0.57)^{k-6}$.

A list of more complete properties for the general step n is in the next section.

2.5.2.1 Number of monotone refinements in creating $\Delta^{(k)}$ is less than or equal to 5

Lemma 12. *Let K be greater than 6. If equations (2.132) and (2.33) hold for all steps $k \leq K-1$, and*

$$\frac{\left|\Delta^{(K)}\right|}{\left|\Delta^{(K-1)}\right|} < 0.023, \tag{2.134}$$

then inequality (2.33) holds for $k = K$.

Proof. We have

$$\frac{|\Delta^{(k-1)}(t)|}{\operatorname{dist}(\Delta^{(k-1)}(t), y_{k-2}(t))} < \vartheta_1 \qquad t \in \mathcal{T}^{(k-1)} \tag{2.135}$$

and

$$\frac{|[y_{k-1}(t), w(t)]|}{|[y_{k-2}(t), w(t)]|} \geq \frac{1}{9} \cdot 0.3 \qquad t \in \mathcal{T}^{(k-1)} \tag{2.136}$$

for all $k \leq K-1$. This gives

$$\begin{aligned}
&\frac{|\Delta^{(K-1)}(t)|}{\operatorname{dist}(\Delta^{(K-1)}(t), y_{K-1}(t))} \\
=&\frac{\left|\Delta^{(K-1)}(t)\right|}{|\Delta^{(K-1)}(t)| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)} \cdot \frac{\left|\Delta^{(K-1)}(t)\right| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)}{\operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-1}(t)\right)} \\
<&\frac{\left|\Delta^{(K-1)}(t)\right|}{|\Delta^{(K-1)}(t)| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)} \cdot \frac{\left|\Delta^{(K-1)}(t)\right| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)}{|\Delta^{(K-1)}(t)| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-1}(t)\right) - |\Delta^{(K-1)}|} \\
<&\frac{\frac{\left|\Delta^{(K-1)}(t)\right|}{\operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)}}{\frac{\left|\Delta^{(K-1)}(t)\right|}{\operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)} + 1} \cdot \frac{1}{\frac{\left|\Delta^{(K-1)}(t)\right| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-1}(t)\right)}{\left|\Delta^{(K-1)}(t)\right| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)} - \frac{\left|\Delta^{(K-1)}(t)\right|}{\left|\Delta^{(K-1)}(t)\right| + \operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-2}(t)\right)}} \\
<&\frac{\frac{\vartheta_1}{1+\vartheta_1}}{\frac{1}{9} \cdot 0.3 - \frac{\vartheta_1}{1+\vartheta_1}} \\
<&0.42
\end{aligned} \tag{2.137}$$

for $t \in \mathcal{T}^{(K-1)}$.

Combining (2.134) and (2.137), we get

$$\frac{\left|\Delta^{(K)}(t)\right|}{\operatorname{dist}\left(\Delta^{(K)}(t), y_{K-1}(t)\right)} < \frac{\left|\Delta^{(K)}(t)\right|}{\operatorname{dist}\left(\Delta^{(K-1)}(t), y_{K-1}(t)\right)} < 0.023 * 0.42 < 0.0098 = \vartheta_1 \tag{2.138}$$

□

At steps 6 to 14 we still pullback initial partition ξ_0, so the estimates of step 6 prove that the number of refinements needed to achieve (2.134) is less than 5.

`Corollary 3.` *The number of monotone pullbacks needed to create $\Delta^{(k)}$ is no more than 5.*

2.5.2.2 Relative measure of holes in η_{k-1} and ξ_k

Since the algorithm inside $\delta_{k-1}^{\mathrm{re}}$ for step k, $7 \leq k \leq 14$, is exactly the same as in step 6, we can obtain the same estimate as in (2.117). By (2.131) and (2.13), we get

$$\begin{aligned}
\mu_{\text{holes}}(\eta_{k-1}) &= \frac{\text{measure of holes in } \delta_{k-1}^{\mathrm{re}} \text{ after step k}}{|\delta_{k-1}^{\mathrm{re}}|} \\
&= \frac{|\delta_k^{\mathrm{re}}| + |\text{ holes between } \delta_{k-1}^{\mathrm{re}} \text{ and } \delta_k^{\mathrm{re}}|}{|\delta_{k-1}^{\mathrm{re}}|} \\
&< 0.5267
\end{aligned} \tag{2.139}$$

for all $t \in \mathcal{T}^{(k)}$. Using (A.3), we get that

$$\begin{aligned}
\mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_{k-1})) &= \frac{\text{measure of holes in } \delta_{k-1}^{-p} \text{ after 1 step filling-in}}{|\delta_{k-1}^{-p}|} \\
&< \frac{0.526667 * \mathcal{D}}{1 - 0.526667 + 0.526667 * \mathcal{D}} \\
&\approx 0.57 =: \chi_0
\end{aligned} \tag{2.140}$$

where $\mathcal{D}$ is defined in (2.182). From (2.139) and (2.116) and the algorithm at step k, we get that the total measure of holes will become less than $\max\{0.53, 0.57\}$ the measure of holes in step $k-1$. If ξ_k is the partition of I we get after step k, we have

$$\mu_{\text{holes}}(\xi_k) \leq (0.57)^{k-6}\mu_{\text{holes}}(\xi_6) \leq 0.0189 * (0.57)^{k-6} \tag{2.141}$$

where the last value is obtained from (2.120). For $k = 14$, we have

$$\mu_{\text{holes}}(\xi_{14}) \leq 0.0189 * (0.57)^8 < 0.000210601. \tag{2.142}$$

2.5.3 Steps n larger than 15

2.5.3.1 Estimates at step n

Let $n \geq 15$. We consider a list of estimates and properties that we assume to be true for $k \leq n-1$, and prove that all properties will again hold true at step n. The properties are listed in the order that they can be concluded after the previous ones are shown.

(I) Velocities of the endpoints of the domains Δ of $\zeta^{(n)}$ above y_{n-1}. If $\Delta = [x_1(t), x_2(t)]$ is an element of $\zeta^{(n)}$ above y_{n-1}, then

$$\left|\frac{dx_i}{dt}\right| < \epsilon_0 = 0.003. \tag{2.143}$$

(II) Sizes of central domains. Sizes of rescaled central domains satisfy

$$\frac{1}{3}\sqrt{0.3}|\delta^{\text{re}}_{n-1}| \leq |\delta^{\text{re}}_n| \leq \frac{1}{3}|\delta^{\text{re}}_{n-1}|, \tag{2.144}$$

or equivalently,

$$\frac{0.3}{9}\left|[y_{n-1}(t), w(t)]\right| \leq \left|[y_n(t), w(t)]\right| \leq \frac{1}{9}\left|[y_{n-1}(t), w(t)]\right|. \tag{2.145}$$

(III) Distortions on holes. Since the enlargement $\hat{\delta}_n^{\mathrm{re}}$ of δ_n^{re} is defined as $\delta_{n-3}^{\mathrm{re}}$ for $n \geq 8$, distortion on preimages δ_n^{-p} of δ_n^{re}, $n \geq 8$, is less than

$$\left(1 + \frac{|\delta_n^{\mathrm{re}}|}{1 + \frac{1}{2}\left|\hat{\delta}_n^{\mathrm{re}} \backslash \delta_n^{\mathrm{re}}\right|}\right)^2 < 1.16 =: \mathcal{D}. \tag{2.146}$$

(IV) Size of $\Delta^{(n)}$. The size of $\Delta^{(n)}$ satisfies

$$|\Delta^{(n)}| < H_{n-1}(\Delta^{(n)})\vartheta_1 < |[y_{n-1}(t), w(t)]|\,\vartheta_1 \leq \left(\frac{1}{9}\right)^{n-6} |[y_5(t), w(t)]|\vartheta_1. \tag{2.147}$$

$\Delta^{(n)}$ is not necessarily strictly contained in $\Delta^{(n-1)}$, since $\Delta^{(n)}$ could be exactly the domain $\Delta^{(n-1)}$.

(V) Extendability and expansion of maps. Elements of partitions ξ_n on the x-axis are domains of good maps $f_{n,i} : \Delta \to I$ and domains of $\mathcal{F}_{n,i} : \delta_m^{-p} \to \delta_m^{\mathrm{re}}$, $m < n$.

Maps $f_{n,i}$ are extendable to $\tilde{f}_{n,i} : \tilde{\Delta} \to \tilde{I}$ and $\mathcal{F}_{n,i}$ are extendable to $\tilde{\mathcal{F}}_{n,i} : \tilde{\delta}_m^{-p} \to \hat{\delta}_m^{\mathrm{re}}$ where $\hat{\delta}_m^{\mathrm{re}} = \delta_{m-3}^{\mathrm{re}}$ for $m \geq 8$ and $\hat{\delta}_m^{\mathrm{re}}$ as defined in 2.2.8 and 2.3.3.1 for $m \leq 7$. Derivatives of all maps satisfy

$$\left|\frac{df_{n,i}}{dx}\right|, \left|\frac{d\mathcal{F}_{n,i}}{dx}\right| > 3.5. \tag{2.148}$$

(VI) Number of monotone pullbacks No more than 6+3 monotone refinements are needed in each step n

(VII) Measure of holes.

a)Measure of holes in $\delta^{\mathrm{re}}_{n-1}$ after step n satisfies

$$\mu_{\mathrm{holes}}(\eta_{n-1}) < 0.613. \tag{2.149}$$

b)Measure of holes in partition ξ_n satisfies

$$\mu_{\mathrm{holes}}(\xi_n) < \mu_{\mathrm{holes}}(\xi_{14}) \cdot (\chi')^{n-14} < 0.000210601 * (0.73)^{n-14}. \tag{2.150}$$

(VIII) Ratio of derivatives

$$\frac{\left|\frac{\partial g_{(n)}}{\partial t}\right|}{\left|\frac{\partial g_{(n)}}{\partial x}\right|}, \frac{\left|\frac{\partial \mathcal{G}_{(n),i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{G}_{(n),i}}{\partial x}\right|} \leq 0.003. \tag{2.151}$$

$$\frac{\left|\frac{\partial \bar{g}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \bar{g}_{n,i}}{\partial x}\right|}, \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n,i}}{\partial x}\right|} \leq 0.001909 + 1.16 * \frac{t}{16}\left(\frac{1}{3}\right)^{-4} \sum_{k=16}^{n} \left(\frac{1}{3}\right)^{\frac{4k}{3}} < 0.003 \tag{2.152}$$

For branches $g_{n,i}$ or $\mathcal{G}_{n,i}$ above y_{n-1},

$$\frac{\left|\frac{\partial g_{n,i}}{\partial t}\right|}{\left|\frac{\partial g_{n,i}}{\partial x}\right|}, \frac{\left|\frac{\partial \mathcal{G}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{G}_{n,i}}{\partial x}\right|} \leq 0.003 \tag{2.153}$$

$$\frac{\left|\frac{\partial f_{n,i}}{\partial t}\right|}{\left|\frac{\partial f_{n,i}}{\partial x}\right|}, \frac{\left|\frac{\partial \mathcal{F}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{F}_{n,i}}{\partial x}\right|} \leq \frac{1}{4\left|\delta^{\mathrm{re}}_{n}\right|} \tag{2.154}$$

(VIII) Variation of derivatives. As in [7] Lemma 5 we have

$$\frac{\left|\frac{\partial^2 g^{-1}_{(n)}}{\partial t \partial z}\right|}{\left|\frac{\partial g^{-1}_{(n)}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 \mathcal{G}^{-1}_{(n),i}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{G}^{-1}_{(n),i}}{\partial z}\right|} \leq \frac{3 * (n \bmod 3) + 3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} \tag{2.155}$$

$$\frac{\left|\frac{\partial^2 \bar{g}^{-1}_{n,i}}{\partial t \partial z}\right|}{\left|\frac{\partial \bar{g}^{-1}_{n,i}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 \bar{\mathcal{G}}^{-1}_{n,i}}{\partial t \partial z}\right|}{\left|\frac{\partial \bar{\mathcal{G}}^{-1}_{n,i}}{\partial z}\right|} \leq \frac{3 * (n \bmod 3) + 3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|^2} \tag{2.156}$$

$$\frac{\left|\frac{\partial^2 g^{-1}_{n,i}}{\partial t \partial z}\right|}{\left|\frac{\partial g^{-1}_{n,i}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 \mathcal{G}^{-1}_{n,i}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{G}^{-1}_{n,i}}{\partial z}\right|} \leq \frac{1.3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2} \tag{2.157}$$

$$\frac{\left|\frac{\partial^2 f_{n,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial f_{n,i}^{-1}}{\partial z}\right|}, \frac{\left|\frac{\partial^2 \mathcal{F}_{n,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{F}_{n,i}^{-1}}{\partial z}\right|} \le \frac{1}{|\delta_n^{\mathrm{re}}|^2} \tag{2.158}$$

2.5.3.2 Velocity estimates for partitioning points in the parameter-induced partition of $\Delta^{(n-1)}$ and partitions $\zeta^{(n)}(\Delta^{(n)})$

This is done in a similar way as in step 6.

When we consider velocities of the partitioning points of ξ_n and $\zeta^{(n)}(\Delta^{(n)})$, it suffices to consider velocities on endpoints of monotone domains. That is, we do not need to consider velocities of endpoints of rescaled critical domains or their preimages because of the following.

Lemma 13. *For any hole at any step of construction, there is an adjacent monotone branch on Δ mapped onto I.*

Proof. For the initial 7-branch partition, the central hole δ_0 is adjacent to two monotone domains Δ_3 and Δ_{-3}. Suppose up to step n each central hole is adjacent to two good branches. Consider the new central hole at step $n+1$. When we choose parameter we are choosing the position of the critical value $w(t)$. Then for each hole δ^{-k} on the y-axis we consider its enlargement $\hat{\delta}^{-k}$. By construction the boundary domains of $\hat{\delta}^{-k}$ are monotone domains. Construction implies that only monotone domains can be adjacent to the new central hole. Then monotone domains will be adjacent to any preimage of the new central branch.

□

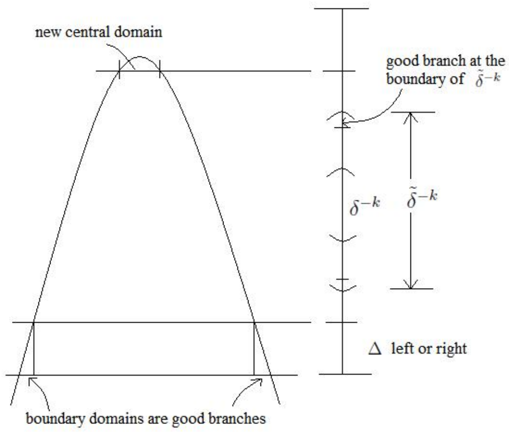


Figure 2.5: Domains adjacent to rescaled central domains are monotone domains

Basic approach for calculating velocities

a) Any monotone domain $\Delta(t) = [z_1(t), z_2(t)]$ is mapped by some map g onto $I = [q_t^{-1}, q_t] = [\frac{1}{t}, \frac{t-1}{t}]$. Therefore $g(t, z_i(t)) = q_t$ or q_t^{-1}. By chain rule, we have

$$\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \cdot \frac{dz_i(t)}{dt} = \frac{dq_t}{dt} \tag{2.159}$$

$$\frac{dz_i(t)}{dt} = \frac{\frac{dq_t}{dt}}{\frac{\partial g}{\partial x}} - \frac{\frac{\partial g}{\partial t}}{\frac{\partial g}{\partial x}} \tag{2.160}$$

We use formula (2.160) for velocity estimates on endpoints of monotone domains. $\left|\frac{dq_t}{dt}\right| = \frac{1}{t^2} \approx \frac{1}{16}$. g is a composition of maps with derivatives greater than 3.5. As powers grow, $\frac{\partial g}{\partial x}$ approaches ∞ and the term $\frac{\frac{dq_t}{dt}}{\frac{\partial g}{\partial x}}$ becomes irrelevant, so we can estimate $\frac{\frac{\partial g}{\partial t}}{\frac{\partial g}{\partial x}}$ instead.

b) We use the inductive assumptions that for $k < n$,

1. For any monotone branch $f_{k,i}$ in ξ_k,

$$\left|\frac{\partial f_{k,i}}{\partial x}\right| \geq 3.5 \tag{2.161}$$

2. For any monotone branch $f_{k,i}$ on ξ_k,

$$\left|\frac{\frac{\partial f_{k,i}}{\partial t}}{\frac{\partial f_{k,i}}{\partial x}}\right| \leq \frac{1}{4|\delta_k^{\mathrm{re}}|} \tag{2.162}$$

3. For branches $\bar{g}_{k,i}$'s and $\bar{\mathcal{G}}_{k,i}$'s on the y-axis defined above y_{k-1},

$$\left|\frac{\frac{\partial \bar{g}_{k,i}}{\partial t}}{\frac{\partial \bar{g}_{k,i}}{\partial x}}\right|, \left|\frac{\frac{\partial \bar{\mathcal{G}}_{k,i}}{\partial t}}{\frac{\partial \bar{\mathcal{G}}_{k,i}}{\partial x}}\right| \leq 0.001909 + 1.16 * \frac{t}{16}\left(\frac{1}{3}\right)^{-4}\sum_{l=16}^{k}\left(\frac{1}{3}\right)^{\frac{4l}{3}} < 0.003 \tag{2.163}$$

c) We use the following inequality given by the chain rule for inductive estimates on derivatives.

$$\left|\frac{\frac{\partial(\varphi_1\circ\varphi_2)}{\partial t}}{\frac{\partial(\varphi_1\circ\varphi_2)}{\partial x}}\right| \leq \left|\frac{\frac{\partial\varphi_2}{\partial t}}{\frac{\partial\varphi_2}{\partial x}}\right| + \left|\frac{1}{\frac{\partial\varphi_2}{\partial x}}\right| \cdot \left|\frac{\frac{\partial\varphi_1}{\partial t}}{\frac{\partial\varphi_1}{\partial x}}\right| \tag{2.164}$$

Calculations of velocity bounds

The monotone maps g in $\zeta^{(n)}(\Delta^{(n)})$ could be $g_{(n)}$, $\bar{g}_{6,i}$ or $g_{6,i}$. The monotone maps g in the parameter-induced partition of $\Delta^{(n-1)}$ are just $g_{(n)}$. Possible expressions for monotone maps g are as discussed in 2.4.4. Since the maps we are considering here are all above y_{n-1}, we have the following two worst cases.

Case 1: $g = f_{\tilde{m}+1,k_2} \circ \mathcal{F}_{m+1,k_1} \circ f_{[\frac{n}{3}],i_{r-1}} \circ \cdots \circ f_{[\frac{n}{3}],i_1} \circ \bar{g}_{n-1,i}$

Case 2: $g = f_{\tilde{m}+1,k_2} \circ \mathcal{F}_{m+1,k_1} \circ \bar{\mathcal{G}}_{n-1,i}$

where $\tilde{m} \leq m+1$ and $m \leq [\frac{n}{3}]$.

In other cases the member of compositions is less and respective estimates are better. Note here that at this point, there is no restriction on r. However, we will show later that the maximum number of refinements is bounded. That is shown after we prove some other properties. The properties are proven under the assumption that velocity is small, which is why we need to prove small velocity before knowing a bound for r.

First we consider case 1. We compute separately estimates for $f_{[\frac{n}{3}],i_r} \circ \cdots \circ f_{[\frac{n}{3}],i_1}$ and $f_{\tilde{m}+1,k_2} \circ \mathcal{F}_{m+1,k_1}$. Using (2.164) repeatedly, we get

$$
\begin{aligned}
&\frac{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_r}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_r}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}\\
\leq&\frac{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}+\frac{1}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}\cdot\frac{\left|\frac{\partial f_{[\frac{n}{3}],i_r}}{\partial t}\right|}{\left|\frac{\partial f_{[\frac{n}{3}],i_r}}{\partial x}\right|}\\
\leq&\frac{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-2}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-2}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}+\frac{1}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-2}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}\cdot\frac{\left|\frac{\partial f_{[\frac{n}{3}],i_{r-1}}}{\partial t}\right|}{\left|\frac{\partial f_{[\frac{n}{3}],i_{r-1}}}{\partial x}\right|}+\frac{1}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}\cdot\frac{\left|\frac{\partial f_{[\frac{n}{3}],i_r}}{\partial t}\right|}{\left|\frac{\partial f_{[\frac{n}{3}],i_r}}{\partial x}\right|}\\
<&\left(1+\frac{1}{3.5}+\frac{1}{3.5^2}+\cdots\right)\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|}\\
\leq&1.4\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|}
\end{aligned}
\tag{2.165}
$$

$$
\begin{aligned}
\frac{\left|\frac{\partial\left(f_{\tilde{m}+1,k_2}\circ\mathcal{F}_{m+1,k_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde{m}+1,k_2}\circ\mathcal{F}_{m+1,k_1}\right)}{\partial x}\right|}\leq&\frac{\left|\frac{\partial\mathcal{F}_{m+1,k_1}}{\partial t}\right|}{\left|\frac{\partial\mathcal{F}_{m+1,k_1}}{\partial x}\right|}+\frac{1}{\left|\frac{\partial\mathcal{F}_{m+1,k_1}}{\partial x}\right|}\cdot\frac{\left|\frac{\partial f_{\tilde{m}+1,k_2}}{\partial t}\right|}{\left|\frac{\partial f_{\tilde{m}+1,k_2}}{\partial x}\right|}\\
\leq&\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|}+\frac{1}{3.5}*\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}\\
\leq&\frac{1}{4\left|\delta_{[\frac{n}{3}]+2}\right|}
\end{aligned}
\tag{2.166}
$$

Combining (2.165) and (2.166), we get

$$\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|} \le \frac{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|} + \frac{1}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}\cdot\frac{\left|\frac{\partial f_{[\frac{n}{3}]+2,i_r}}{\partial t}\right|}{\left|\frac{\partial f_{[\frac{n}{3}]+2,i_r}}{\partial x}\right|}$$

$$\begin{aligned}
&< \frac{1.4}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|} + \frac{1}{3.5}\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}\\
&\le \frac{1.4}{9*4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|} + \frac{1}{3.5}\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}\\
&< \frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|} \qquad (2.167)
\end{aligned}$$

Using that the branch $\bar g_{n-1,j}$ is always above y_{n-1}, we can estimate the derivative of $\bar g_{n-1,j}$ using that the worst possible distortion is 15.6.

$$\left|\frac{\partial \bar g_{n-1,j}}{\partial x}\right| \ge \frac{|I|}{|[y_{n-1}(t),w(t)]|}*\frac{1}{15.6} = \frac{|I|}{\frac{t}{4}\left|\delta^{\mathrm{re}}_{n-1}\right|^2}*\frac{1}{15.6}. \qquad (2.168)$$

$$\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\circ\bar g_{n-1,i}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\circ\bar g_{n-1,i}\right)}{\partial x}\right|}$$

$$\begin{aligned}
&\le \frac{\left|\frac{\partial \bar g_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar g_{n-1,i}}{\partial x}\right|} + \frac{1}{\left|\frac{\partial \bar g_{n-1,i}}{\partial x}\right|}\cdot\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ f_{[\frac{n}{3}],i_{r-1}}\circ\cdots\circ f_{[\frac{n}{3}],i_1}\right)}{\partial x}\right|}\\
&\le \frac{\left|\frac{\partial \bar g_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar g_{n-1,i}}{\partial x}\right|} + \frac{15.6\frac{t}{4}\left|\delta^{\mathrm{re}}_{n-1}\right|^2}{|I|}\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}\frac{|\delta_5|}{|\delta_5|}\\
&< 0.002 + \frac{t\,|\delta_5|}{|I|}\left(\frac{1}{3}\right)^{n-1-[\frac{n}{3}]-2}\left(\frac{1}{3}\right)^{n-1-5}\\
&< 0.002 + \frac{t\,|\delta_5|}{|I|}3^9\left(\frac{1}{3}\right)^{\frac{5n}{3}} \qquad (2.169)
\end{aligned}$$

Case 2 is a little bit worse since the estimate for the derivative of $\bar{\mathcal{G}}_{n-1,i}$ is worse than that of $\bar{g}_{n-1,i}$.

$$\left|\frac{\partial \bar{\mathcal{G}}_{n-1,j}}{\partial x}\right| \geq \frac{\left|\delta^{\mathrm{re}}_{[\frac{n-1}{3}]}\right|}{|[y_{n-1}(t), w(t)]|} \frac{1}{\text{distortion on } \delta^{\mathrm{re}}_{[\frac{n-1}{3}]}} \geq \frac{\left|\delta^{\mathrm{re}}_{[\frac{n-1}{3}]}\right|}{\frac{t}{4}\left|\delta^{\mathrm{re}}_{n-1}\right|^2} * \frac{1}{1.16}. \quad (2.170)$$

$$\begin{aligned}
\frac{\left|\frac{\partial(f_{\tilde{m}+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ\bar{\mathcal{G}}_{n-1,i})}{\partial t}\right|}{\left|\frac{\partial(f_{\tilde{m}+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ\bar{\mathcal{G}}_{n-1,i})}{\partial x}\right|} &\leq \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial x}\right|} + \frac{1}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial x}\right|} \cdot \frac{\left|\frac{\partial f_{\tilde{m}+1,k_2}\circ\mathcal{F}_{m+1,k_1}}{\partial t}\right|}{\left|\frac{\partial f_{\tilde{m}+1,k_2}\circ\mathcal{F}_{m+1,k_1}}{\partial x}\right|} \\
&\leq \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial x}\right|} + 1.16 * \frac{\frac{t}{4}\left|\delta^{\mathrm{re}}_{n-1}\right|^2}{\left|\delta^{\mathrm{re}}_{[\frac{n-1}{3}]}\right|} \cdot \frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|} \\
&\leq \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial x}\right|} + 1.16 * \frac{t}{16}\left(\frac{1}{3}\right)^{n-1-[\frac{n-1}{3}]}\left(\frac{1}{3}\right)^{n-1-[\frac{n}{3}]-2} \\
&\leq \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial x}\right|} + 1.16 * \frac{t}{16}\left(\frac{1}{3}\right)^{n-1-\frac{n-1}{3}}\left(\frac{1}{3}\right)^{n-1-\frac{n}{3}-2} \\
&\leq \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,i}}{\partial x}\right|} + 1.16 * \frac{t}{16}\left(\frac{1}{3}\right)^{-4+\frac{1}{3}}\left(\left(\frac{1}{3}\right)^{\frac{4}{3}}\right)^{n}
\end{aligned} \quad (2.171)$$

Using the assumption (2.163), we get

$$\begin{aligned}
\frac{\left|\frac{\partial g}{\partial t}\right|}{\left|\frac{\partial g}{\partial x}\right|} &\leq \max\left\{\frac{\left|\frac{\partial \bar{g}_{n-1,j}}{\partial t}\right|}{\left|\frac{\partial \bar{g}_{n-1,j}}{\partial x}\right|}, \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,j}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,j}}{\partial x}\right|}\right\} + 1.16 * \frac{t}{16}\left(\frac{1}{3}\right)^{-4}\left(\left(\frac{1}{3}\right)^{\frac{4}{3}}\right)^{n} \\
&\leq 0.0021
\end{aligned} \quad (2.172)$$

Since g is the composition of maps f with derivatives greater than 3.5 and map $\bar{g}_{n-1,j}$ or $\bar{G}_{n-1,j}$, derivative of g is greater than the derivative of $\bar{g}_{n-1,j}$ or

$\bar{G}_{n-1,j}$. We have

$$\frac{1}{\left|\frac{\partial g}{\partial x}\right|} \leq \frac{1}{\left|\frac{\partial \bar{g}_{n-1,j}}{\partial x}\right|}, \frac{1}{\left|\frac{\partial \bar{\mathcal{G}}_{n-1,j}}{\partial x}\right|} \leq 10^{-10}$$

for $n \geq 16$. Finally, let $\Delta = [z_1(t), z_2(t)]$, then

$$\left|\frac{dz_1(t)}{dt}\right| \leq \frac{\frac{1}{t^2}}{\left|\frac{\partial g}{\partial x}\right|} + \frac{\left|\frac{\partial g}{\partial t}\right|}{\left|\frac{\partial g}{\partial x}\right|} < 10^{-10} + 0.0021 < \epsilon_0 \tag{2.173}$$

2.5.3.3 Estimating shift from y'_n to y_n

We will now show that the shift from y'_n to y_n satisfies

$$\frac{|[y'_n, y_n(t_0)]|}{|[y_{n-1}(t_0), w(t_0)]|} < \frac{1}{9} \cdot 0.6 \tag{2.174}$$

in either the case when y'_n falls into a critical domain δ^* or the case when y'_n falls into a montone domain Δ^* which satisfies (2.40). From this, we can show

$$\frac{1}{3}(\sqrt{0.3}) < \frac{|\delta_n^{\text{re}}(t)|}{|\delta_{n-1}^{\text{re}}(t)|} < \frac{1}{3} \tag{2.175}$$

for all $t \in \mathcal{T}^{(n)}$.

We imitate calculations from 2.5.1.2, except here, δ^* could also be δ_i^{-p} for $5 \leq i \leq [\frac{n}{3}]$. First consider the case when y'_n is in δ^*. If δ^* is δ_0 or δ_0^{-p}, we have already stated in 2.5.1.2 that numerical calculations give

$$\frac{|\delta^*|}{|\delta^* \cup \text{upper half of } \hat{\delta}^*|} < 0.59. \tag{2.176}$$

By the choice of parameters, the critical value is outside the following enlargements of preimages $\delta_5^{-p}, \delta_6^{-p}, \ldots$, namely δ_0^{-p} for $\delta_5^{-p}, \delta_6^{-p}, \delta_7^{-p}$, δ_5^{-p} for δ_8^{-p} and in general δ_{n-3}^{-p} for δ_n^{-p}. As ratios $\frac{|\delta_n^{-p}|}{|\delta_{n-3}^{-p}|}$ and $\frac{|\delta_i^{-p}|}{|\delta_0^{-p}|}$ for $i = 5, 6, 7$ are much larger than $\frac{|\delta_0^{-p}|}{|\hat{\delta}_0^{-p}|}$, we get that in all other cases, estimate (2.176) is less than 0.59. That implies (2.176) in all cases when y'_n belongs to a hole. (2.176) will give

$$\frac{|[y'_n, y_n(t_0)]|}{|[y'_n, w(t_0)]|} < 0.6 \tag{2.177}$$

which is equivalent to (2.174). Next we consider the case when y'_n is in Δ^*. Since Δ^* satisfies (2.40), we have

$$\frac{|[y'_n, y_n(t_0)]|}{|[y_{n-1}(t_0), y'_n]|} < \vartheta_2 = 0.6 \cdot \frac{1}{8} \tag{2.178}$$

which is also equivalent to (2.174).

Arguments to show (2.175) are exactly the same as in 2.5.1.2. This is where we need velocities from general step n.

2.5.3.4 Size of $\mathcal{T}^{(n)}$

Using (1.8) and (2.39), we get

$$\begin{aligned}
\left|\mathcal{T}^{(n)}\right| &\le \frac{1}{\frac{1}{4} - \epsilon_0} \left|\Delta^{(n)}\right| \\
&\le \frac{1}{\frac{1}{4} - \epsilon_0} H_{n-1}(\Delta^{(n)}) \vartheta_1 \\
&\le \frac{1}{\frac{1}{4} - \epsilon_0} \left|[y_{n-1}(t), w(t)]\right| \vartheta_1 \\
&\le \frac{1}{\frac{1}{4} - \epsilon_0} \frac{t}{4} \left|\delta^{\mathrm{re}}_{n-1}\right|^2 \vartheta_1
\end{aligned} \tag{2.179}$$

2.5.3.5 Extendability of maps

As corollaries of the algorithm defined, we have

`Corollary 4.` *All monotone branches $f_{n,i}$ in ξ_n and all monotone branches g in $\zeta^{(n)}(\Delta^{(n)})$ can be extended to maps onto $\tilde{I}$*

Proof. **Monotone branches in $\delta^{\mathbf{re}}_{n-1}\backslash\delta^{\mathbf{re}}_n$**

Monotone domains in $\delta^{\text{re}}_{n-1}\backslash\delta^{\text{re}}_n$ are extendable since we perform boundary refinement on any non-extendable monotone branches.

Monotone branches from filling-in outside $\delta^{\mathbf{re}}_{n-1}$

Newly created monotone domains outside δ^{re}_{n-1} are those from filling-in. Monotone domains created from filling-in are always extendable since we always avoid an enlargement of holes when doing parabolic pullback. By the lower boundary refinements we performed in each step, we guarantee that extended domains of monotone domains in $\delta^{\text{re}}_i\backslash\delta^{\text{re}}_{i+1}$ are always inside $\hat{\delta}^{\text{re}}_i$.

Monotone branches on the y axis

If we have by induction that any previous maps created on the y-axis are uniformly extendable to $\tilde{I}$ and any previous monotone maps on the x-axis are uniformly extendable to $\tilde{I}$. Then compositions of monotone maps extendable to $\tilde{I}$ are still extendable to $\tilde{I}$ (see 1.3.5.2).

$\square$

`Corollary 5.` *All maps $\mathcal{F}_{n,i}$ on holes in ξ_n that are preimages of δ^{re}_m can be extended to maps onto $\hat{\delta}^{re}_m$. All maps $\mathcal{G}$ on holes in $\zeta^{(n)}(\Delta^{(n)})$ that are preimages of δ^{re}_m can be extended to maps onto $\hat{\delta}^{re}_m$.*

2.5.3.6 Distortion on holes

We derive the distortion bound $\mathcal{D}_i = (1 + \frac{|\delta_i|}{\frac{1}{2}|\hat{\delta}_i\backslash\delta_i|})^2$ according to (1.3). To compute $\mathcal{D}_i$, we need to use the right hand side of the inequality (2.131). Taking the ratio of the largest possible value for $|\delta_i|$ to the smallest possible value for $|\hat{\delta}_i|$ for all values $t \in \mathcal{T}^{(5)}$, we get

$$\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_7 < 1.10 \tag{2.180}$$

$$\mathcal{D}_i < 1.16, \text{ for } i \geq 8 \tag{2.181}$$

We take the maximum of all distortion bounds and denote it by $\mathcal{D}$. Let

$$\mathcal{D} = 1.16. \tag{2.182}$$

2.5.3.7 Expansiveness of $f_{n,i}$ and $\mathcal{F}_{n,i}$

We show (2.148). We need to show for new domains created in $\delta^{\text{re}}_{n-1} \backslash \delta^{\text{re}}_n$, that their maps have derivatives greater than 3.5. For domains outside δ^{re}_{n-1}, their maps are just compositions of maps with derivatives greater than 3.5.

We use $\left|\frac{\partial \mathcal{F}_{n,i}}{\partial x}\right| = \left|\frac{\partial \mathcal{G}_{n,i}}{\partial x}\right| \cdot \left|\frac{\partial h}{\partial x}\right|$ and $\left|\frac{\partial h}{\partial x}\right| \geq t\,|\delta^{\text{re}}_n|$.

For $n \geq 15$,

$$\begin{aligned}\left|\frac{\partial \mathcal{G}_{n,i}}{\partial x}\right| &\geq \max\left\{391005, \frac{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|}{|[y_{n-1}, y_n]|} \cdot \frac{1}{1.16}\right\} \\ &\geq \frac{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|}{|[y_{n-1}, y_n]|} \cdot \frac{1}{1.16}\end{aligned} \tag{2.183}$$

$$\begin{aligned}\left|\frac{\partial \mathcal{F}_{n,i}}{\partial x}\right| &= \left|\frac{\partial \mathcal{G}_{n,i}}{\partial x}\right| \cdot \left|\frac{\partial h}{\partial x}\right| \\ &\geq \frac{\left|\delta_{[\frac{n}{3}]+2}\right|}{\frac{8}{9}\frac{t}{4}\left|\delta^{\text{re}}_{n-1}\right|^2} * \frac{1}{1.16} * t\,|\delta^{\text{re}}_n| \\ &\geq \frac{3^{n-1-[\frac{n}{3}]-2}}{\frac{2}{9}} * \frac{1}{1.16} * \frac{\sqrt{0.3}}{3}\end{aligned} \tag{2.184}$$

2.5.3.8 Number of monotone refinements in defining $\Delta^{(n)}$ is less than or equal to 5

Admissible domains

The partition $\hat{\xi}_{[\frac{n}{3}]}$ associated to $\xi_{[\frac{n}{3}]}$ is defined as a partition whose union of holes contains all enlargements of holes in $\xi_{[\frac{n}{3}]}$. $\hat{\xi}_{[\frac{n}{3}]}$ is usually $\xi_{[\frac{n}{3}]-3}$ except for the first steps. For steps n greater than 24 we start pulling back ξ_8, ξ_9, $\cdots$ whose associated partitions are $\hat{\xi}_8 = \xi_5$, $\hat{\xi}_9 = \xi_6$, $\cdots$. Admissible domains of ξ_i are non-hole domains of $\hat{\xi}_i$. Therefore, we are actually checking the domain sizes in $\hat{\xi}_{[\frac{n}{3}]}$ at step n.

Number of pullbacks

When $\hat{\xi}_i$ is ξ_0, a maximum number of five pullbacks are needed. When n is greater than 25, the partition $\hat{\xi}_{[\frac{n}{3}]}$ that we pullback for parameter choice is not ξ_0 anymore, but additional domains all lie inside δ_0. We see from the table in B.3.1 that $\frac{|\delta_0|}{\text{dist}(\delta_0, q^{-1})}$ is less than $\frac{|\Delta_{-1}|}{\text{dist}(\Delta_{-1}, q^{-1})}$. So for all admissible domains Δ in δ_0, we have $\frac{|\Delta|}{\text{dist}(\Delta, q^{-1})}$ is less than $\frac{|\Delta_{-1}|}{\text{dist}(\Delta_{-1}, q^{-1})}$. Also, distortion on $\Delta \cup$ (domains below Δ) is also less than distortion on $\Delta_{-1} \cup$ (domains belowΔ_{-1}). So the maximum number of pullbacks needed for the additional domains in δ_0 will be less than 5.

Domains that do not need refinement

We can comment on one other thing for domains inside δ_6^{re}. Since $\frac{|\delta_6^{\text{re}}|}{\text{dist}(\delta_6^{\text{re}}, q^{-1})} < \frac{\frac{1}{3}|\delta_5|}{\frac{1}{2}(|I| - \frac{1}{3}|\delta_5|)} < \frac{\frac{1}{3} * 0.0011}{\frac{1}{2}(1 - \frac{1}{3} * 0.0011)} \approx 0.000733602$ and distortion on $\delta_6^{\text{re}} \cup$(lower half of I) $<$

6.12194, their product 0.00449107 is less than ϑ_1. Therefore, no refinements are needed on the domains that lie inside δ_6^{re}.

2.5.3.9 Number of monotone refinements in defining y_n is less than or equal to 6

When we define y_n, first we define non-dynamically the point y'_n. If y'_n is contained in a hole δ^*, then we use arguments as in 2.5.3.3. If y'_n is contained in a monotone domain, then, we refine the monotone domain until y'_n is in a hole or (2.40) is satisfied.

Lemma 14. *The number of refinements needed in a general step n to define y_n is no more than* 6*.*

Proof. We prove this by splitting into cases of where y'_n could be.

The case where y'_n is in $\Delta^{(n-1)}$

If y'_n is in $\Delta^{(n-1)}$, the arguments are the same as the previous subsection except we replace ϑ_2 by ϑ_1, which is better.

The case where y'_n is below $\Delta^{(n-1)}$

Let Δ be the starting monotone domain containing y'_n. $\Delta = [z_1, z_2]$.

Case 1: $y := \frac{|[y_{n-1},z_1]|}{|[y_{n-1},w]|} < 0.334 =: \vartheta_3$

Show that $\Delta \cap g^{-1}(\Delta_1)$ is always below y'_n if $y := \frac{|[y_{n-1},z_1]|}{|[y_{n-1},w]|} < 0.334 =: \vartheta_3$

Taking into account distortion, we get $\frac{|\Delta \cap g^{-1}(\Delta_1)|}{|\Delta|} < 0.83$. Using (A.9),

$$\begin{aligned}\frac{|[y_{n-1}, z_1] \cup (\Delta \cap g^{-1}(\Delta_1))|}{|[y_{n-1}, w]|} &\leq y + (1-y) * 0.833 \\ &< 0.334 + (1 - 0.334) * 0.833 \\ &< \frac{8}{9} = \frac{|[y_{n-1}, y'_n]|}{|[y_{n-1}, w]|}.\end{aligned}$$

This shows that $\Delta \cap g^{-1}(\Delta_1)$ is always below y'_n if $y := \frac{|[y_{n-1}, z_1]|}{|[y_{n-1}, w]|} < 0.334 =: \vartheta_3$.

Let Δ' be the domain in Δ containing y'_n, we will show $\frac{|\Delta'|}{H_{n-1}(\Delta')} < 8.5$

$$\begin{aligned}&\frac{|\Delta'|}{H_{n-1}(\Delta')} < \\ &\max\left\{\frac{|g^{-1}(\Delta_2)|}{|g^{-1}(\Delta_1)|}, \frac{|g^{-1}(\Delta_3)|}{|g^{-1}(\Delta_1 \cup \Delta_2)|}, \frac{|g^{-1}(\delta_0)|}{|g^{-1}(\Delta_1 \cup \Delta_2 \cup \Delta_3)|}, \frac{|g^{-1}(\Delta_{-3})|}{|g^{-1}(\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \delta_0)|},\right. \\ &\left.\frac{|g^{-1}(\Delta_{-2})|}{|g^{-1}(\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \delta_0 \cup \Delta_{-3})|}, \frac{|g^{-1}(\Delta_{-1})|}{|g^{-1}(\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \delta_0 \cup \Delta_{-3} \cup \Delta_{-2})|}\right\} \\ &< \frac{|g^{-1}(\Delta_2)|}{|g^{-1}(\Delta_1)|} \\ &< 15.6 * \frac{|\Delta_2|}{|\Delta_1|} \\ &< 15.6 * 0.54 \\ &< 8.5\end{aligned}$$

If after several refinements we get a domain $\Delta^* \subset \Delta'$ containing y'_n such that $\frac{|\Delta^*|}{|\Delta'|} < \frac{1}{8} * 0.6 * \frac{1}{8.5} \approx 0.0088$, then

$$\frac{|\Delta^*|}{H_{n-1}(\Delta^*)} < \frac{|\Delta^*|}{|\Delta'|}\frac{|\Delta'|}{H_{n-1}(\Delta^*)} < \frac{|\Delta^*|}{|\Delta'|}\frac{|\Delta'|}{H_{n-1}(\Delta')} < \frac{1}{8} * 0.6 = \vartheta_2 \quad (2.185)$$

We check by computation that in order to get $\frac{|\Delta^*|}{|\Delta'|} < \frac{1}{8} * 0.6 * \frac{1}{8.5} \approx 0.0088$, we need no more than 5 refinements. That means a total of no more than 5+1 refinements are needed.

Case 2: $y \geq 0.334$

It is immediate that if $y \geq 0.334$, $\frac{|\Delta|}{H_{n-1}(\Delta)} < 2$.

If there is a domain Δ^* in Δ such that $\frac{|\Delta^*|}{|\Delta'|} < \frac{1}{8} * 0.6 * \frac{1}{2} = 0.375$, then

$$\frac{|\Delta^*|}{H_{n-1}(\Delta^*)} < \frac{|\Delta^*|}{|\Delta|} \frac{|\Delta|}{H_{n-1}(\Delta^*)} < \frac{|\Delta^*|}{|\Delta|} \frac{|\Delta|}{H_{n-1}(\Delta)} < \frac{1}{8} * 0.6 = \vartheta_2 \quad (2.186)$$

We check by computation that in order to get $\frac{|\Delta^*|}{|\Delta|} < \frac{1}{8} * 0.6 * \frac{1}{2} = 0.375$ we need no more than 4 refinements. That means a total of no more than 4+1 refinements are needed.

$\square$

2.5.3.10 Number of boundary refinements for monotone domains in $\delta_{n-1}^{\text{re}} \backslash \delta_n^{\text{re}}$ is less than or equal to 3

We consider a monotone domain Δ between y_{n-1} and y_n. If extension of Δ is not in the image of h, we ask how many boundary refinements are needed in order for all refined domains to have extensions in the image of h. In lemma 8 we showed that the number of boundary refinements needed in step 6 is no more than 2. We argued by considering two separate cases. We obtained that if (2.109) and (2.113) in the two separate cases are satisfied, respectively, then we have that extension of $\Delta_{l\ldots l}$ is in the image of h. Different from step 6, we estimate $\frac{\text{top component of } \tilde{\Delta}_{l\ldots l} \backslash \Delta_{l\ldots l}}{\Delta}$

by evaluating $\frac{|\tilde{\Delta}_{1\cdots1}\backslash I|}{|I|}$ ($\Delta_{1\cdots1}$'s are the first subdomains of Δ_1 after consecutive refinements on the first domain) and multiplying that by distortion on $\tilde{\Delta}_{1\cdots1} \cup I$. This is because monotone domains Δ are formed by pullbacks of $\xi_{[\frac{n}{3}]}$ where monotone domains are not just the domains Δ_1, Δ_2 and Δ_3 anymore. From numerical calculations, we have

$$\frac{|\tilde{\Delta}_{111}\backslash I|}{|I|} < 0.0066 \tag{2.187}$$

and distortion is less than 15.6. Therefore the distorted ratio always satifies (2.109) and (2.113), which means a maximum of three boundary refinements are needed.

2.5.3.11 Simplifying compositions

Since we only need to perform refinements on larger domains, we are mostly composing branches corresponding to larger domains such as Δ_1, Δ_2 and Δ_3. The compositions will boil down to the following cases.

`Corollary 6.` *Compositions $f_{n,i_s} \circ \cdots \circ f_{n,i_1}$ defined specifically from the refinement processes in our algorithm can be simplified to one of the following forms.*

$$f_{n,i_s} \circ f_{0,i_{s-1}} \circ \cdots \circ f_{0,i_1} \qquad s \leq 5 \tag{2.188}$$

$$f_{n,i_2} \circ f_{5,i_1} \tag{2.189}$$

$$f_{n,i_1} \tag{2.190}$$

This corollary is a consequence of remark 17 below.

2.5.3.12 Estimating relative sizes of holes at step n of induction

Here we prove the estimate for $\mu_{\text{holes}}(\eta_{n-1})$, where η_{n-1} is the restriction of the partition ξ_n to the rescaled central domain δ^{re}_{n-1} and $\mu_{\text{holes}}(\eta_{n-1})$ denotes the relative measure of holes in η_{n-1}.

Lemma 15. *Let $N \geq 15$. Suppose (2.149) holds for $15 \leq n \leq N-1$, (2.144) holds for $n = N$ and (2.131) holds for $6 \leq k \leq 14$, then*

$$\mu_{holes}(\eta_{N-1}) \leq 0.613 =: \chi. \tag{2.191}$$

Proof. By the algorithm in 2.3.3, partition η_{N-1} is formed by first constructing a new rescaled central domain δ^{re}_N inside δ^{re}_{N-1}, then filling-in holes in $\delta^{\text{re}}_{N-1} \backslash \delta^{\text{re}}_N$. According to the assumption, the rescaled central domain δ^{re}_N satisfies (2.175). The filling-ins could be composed of two 1-step filling-ins, one 1-step filling-in followed by a 5-step filling-in, or just one 5-step filling-in. 5-step filling-ins are performed on preimages of δ_0 and the relative measure of holes in a given hole after a 5-step filling is given by (2.13). One step filling-ins are performed on preimages of δ^{re}_k where $5 \leq k \leq [\frac{N}{3}] + 1$ and the relative measure of holes after one such filling-in is given by

$$\mu_{\text{holes}}(\mathcal{F}^{-1}(\eta_k)) < \frac{\chi * \mathcal{D}}{1 - \chi + \chi * \mathcal{D}} \approx 0.73 =: \chi' \tag{2.192}$$

where $\mathcal{D}$ is the uniform upper bound for distortions on δ_i's. Since χ' is greater than 0.29, the worst case possible for filling-ins in $\delta^{\text{re}}_{N-1} \backslash \delta^{\text{re}}_N$ is the case where all holes undergo two 1-step filling-in. So we get

$$\frac{\text{measure of holes in } \delta^{\text{re}}_{N-1} \backslash \delta^{\text{re}}_N \text{ after filling-in}}{\left|\delta^{\text{re}}_{N-1} \backslash \delta^{\text{re}}_N\right|} \leq (\chi')^2 \tag{2.193}$$

Combining inequalities (2.193), (2.175) for $n = N$ and using (A.9), we get

$$\mu_{\text{holes}}(\eta_{N-1}) = \frac{|\delta_N^{\text{re}}| + |\text{ holes between } \delta_{N-1}^{\text{re}} \text{ and } \delta_N^{\text{re}}|}{|\delta_{N-1}^{\text{re}}|} < \frac{1}{3} + \frac{2}{3}(\chi')^2 < \chi \tag{2.194}$$

□

`Remark 11.` χ was chosen by solving for $\frac{1}{3} + \frac{2}{3}\left(\frac{\chi * \mathcal{D}}{1-\chi+\chi*\mathcal{D}}\right)^2 = \chi$, which is approximately 0.613. Any number greater than that works.

We can conclude that χ depends on the number of 1-step filling-ins we assign in the algorithm.

Since after step n, the measure of holes inside δ_{n-1}^{re} reduces to less than $\chi * \left|\delta_{n-1}^{\text{re}}\right|$ and outside δ_{n-1}^{re} we perform a 1-step filling-in which reduces the measure of holes to less than χ' times the original measure of holes, we can conclude the following.

`Corollary 7.`

$$\mu_{holes}(\xi_n) < \chi' * \mu_{holes}(\xi_{n-1}). \tag{2.195}$$

So for $n \geq 15$ we have (2.150). That proves that the measure of holes will decrease to zero.

2.5.3.13 Estimating derivatives $\frac{\left|\frac{\partial g_{(n)}}{\partial t}\right|}{\left|\frac{\partial g_{(n)}}{\partial x}\right|}, \frac{\left|\frac{\partial \mathcal{G}_{(n),i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{G}_{(n),i}}{\partial x}\right|}, \frac{\left|\frac{\partial \bar{g}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \bar{g}_{n,i}}{\partial x}\right|}, \frac{\left|\frac{\partial \bar{\mathcal{G}}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \bar{\mathcal{G}}_{n,i}}{\partial x}\right|}, \frac{\left|\frac{\partial g_{n,i}}{\partial t}\right|}{\left|\frac{\partial g_{n,i}}{\partial x}\right|}, \frac{\left|\frac{\partial \mathcal{G}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{G}_{n,i}}{\partial x}\right|}$ on the y-axis

The estimate for these derivatives follow the same spirit as estimates in 2.5.3.2. All can be shown to be less than ϵ_0 when the maps are above y_{n-1}.

2.5.3.14 Estimating derivatives $\frac{\left|\frac{\partial f_{n,i}}{\partial t}\right|}{\left|\frac{\partial f_{n,i}}{\partial x}\right|}$, $\frac{\left|\frac{\partial \mathcal{F}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{F}_{n,i}}{\partial x}\right|}$ on the x-axis

We would like to show (2.154). We assume (2.154) holds in earlier steps. We use

$$\frac{\left|\frac{\partial h}{\partial t}\right|}{\left|\frac{\partial h}{\partial x}\right|} < \frac{\frac{1}{4} - \frac{1}{4}\left|\delta_n^{\mathrm{re}}\right|^2}{t\left|\delta_n^{\mathrm{re}}\right|} \tag{2.196}$$

for x outside δ_n^{re}.

For $f_{n,i}$ and $\mathcal{F}_{n,i}$ in $\delta_{n-1}^{\mathbf{re}} \backslash \delta_n^{\mathbf{re}}$.

$f_{n,i} = g_{n,i} \circ h$ where $g_{n,i}$ is in $[y_{n-1}(t), y_n(t)]$

$\mathcal{F}_{n,i} = \mathcal{G}_{n,i} \circ h$, where $\mathcal{G}_{n,i}$ is in $[y_{n-1}(t), y_n(t)]$

$$\begin{aligned}
\frac{\left|\frac{\partial f_{n,i}}{\partial t}\right|}{\left|\frac{\partial f_{n,i}}{\partial x}\right|} &\le \frac{\left|\frac{\partial h}{\partial t}\right|}{\left|\frac{\partial h}{\partial x}\right|} + \frac{1}{\left|\frac{\partial h}{\partial x}\right|} \cdot \frac{\left|\frac{\partial g_{n,i}}{\partial t}\right|}{\left|\frac{\partial g_{n,i}}{\partial x}\right|} \\
&\le \frac{\left(\frac{1}{4} - \frac{1}{4}\left|\delta_n^{\mathrm{re}}\right|^2\right)}{t\left|\delta_n^{\mathrm{re}}\right|} + \frac{1}{t\left|\delta_n^{\mathrm{re}}\right|} \cdot \frac{\left|\frac{\partial g_{n,i}}{\partial t}\right|}{\left|\frac{\partial g_{n,i}}{\partial x}\right|} \\
&\le \frac{\left(\frac{1}{4} - \frac{1}{4}\left|\delta_n^{\mathrm{re}}\right|^2\right)}{t\left|\delta_n^{\mathrm{re}}\right|} + \frac{1}{t\left|\delta_n^{\mathrm{re}}\right|} * 0.003 \\
&\le \frac{1}{4\left|\delta_n^{\mathrm{re}}\right|}
\end{aligned} \tag{2.197}$$

Similarly, $\frac{\left|\frac{\partial \mathcal{F}_{n,i}}{\partial t}\right|}{\left|\frac{\partial \mathcal{F}_{n,i}}{\partial x}\right|} \le \frac{1}{4\left|\delta_n^{\mathrm{re}}\right|}$.

For $f_{n,i}$ and $\mathcal{F}_{n,i}$ outside $\delta_{n-1}^{\mathbf{re}}$.

We assume the worst possible case, which is the case when the new branches come from filling-in of $\delta_{n-1}^{\mathrm{re}}$.

$f_{n,i} = f_{n-1,l} \circ \mathcal{F}_{n-1,j}$ where map $\mathcal{F}_{n-1,j}$ are maps on holes outside $\delta_{n-1}^{\mathrm{re}}$ in ξ_{n-1}.

$\mathcal{F}_{n,i} = \mathcal{F}_{n-1,l} \circ \mathcal{F}_{n-1,j}$ where map $\mathcal{F}_{n-1,j}$ are maps on holes outside $\delta_{n-1}^{\mathrm{re}}$ in ξ_{n-1}..

$$
\begin{aligned}
\frac{\left|\frac{\partial f_{n,i}}{\partial t}\right|}{\left|\frac{\partial f_{n,i}}{\partial x}\right|} &\leq \frac{\left|\frac{\partial \mathcal{F}_{n-1,j}}{\partial t}\right|}{\left|\frac{\partial \mathcal{F}_{n-1,j}}{\partial x}\right|} + \frac{1}{\left|\frac{\partial \mathcal{F}_{n-1,j}}{\partial x}\right|} \cdot \frac{\left|\frac{\partial f_{n-1,l}}{\partial t}\right|}{\left|\frac{\partial f_{n-1,l}}{\partial x}\right|} \\
&\leq \frac{1}{4\left|\delta_{n-1}^{\mathrm{re}}\right|} + \frac{1}{3.5} * \frac{1}{4\left|\delta_{n-1}^{\mathrm{re}}\right|} \\
&< \frac{1}{4\left|\delta_{n}^{\mathrm{re}}\right|}
\end{aligned} \tag{2.198}
$$

2.5.3.15 Variation of derivatives

We show (2.155), (2.156), (2.157) and (2.158). We refer to the value $\frac{\frac{\partial}{\partial t}\frac{\partial \varphi^{-1}}{\partial z}}{\frac{\partial \varphi^{-1}}{\partial z}}$ as the variation of derivative of φ. We constantly use the composition formula below. Let $\varphi(t,x) = \varphi_2(t, \varphi_1(t,x))$, then

$$
\begin{aligned}
\frac{\frac{\partial}{\partial t}\frac{\partial \varphi^{-1}}{\partial z}}{\frac{\partial \varphi^{-1}}{\partial z}} &= \frac{\frac{\partial^2 \varphi_1^{-1}}{\partial t \partial z}(t, \varphi_2^{-1}(t,z))}{\frac{\partial \varphi_1^{-1}}{\partial z}(t, \varphi_2^{-1}(t,z))} + \frac{\frac{\partial^2 \varphi_1^{-1}}{\partial z^2}(t, \varphi_2^{-1}(t,z))}{\frac{\partial \varphi_1^{-1}}{\partial z}(t, \varphi_2^{-1}(t,z))} \cdot \frac{\partial \varphi_2^{-1}}{\partial t}(t,z) + \frac{\frac{\partial^2 \varphi_2^{-1}}{\partial t \partial z}(t,z)}{\frac{\partial \varphi_2^{-1}}{\partial z}(t,z)} \\
&= \frac{\frac{\partial^2 \varphi_1^{-1}}{\partial t \partial z}(t, \varphi_2^{-1}(t,z))}{\frac{\partial \varphi_1^{-1}}{\partial z}(t, \varphi_2^{-1}(t,z))} + \frac{\frac{\partial^2 \varphi_1}{\partial x^2}(t, \varphi_1^{-1}(t, \varphi_2^{-1}(t,z)))}{\frac{\partial \varphi_1}{\partial x}(t, \varphi_1^{-1}(t, \varphi_2^{-1}(t,z)))} \cdot \frac{\frac{\partial \varphi_2}{\partial t}}{\frac{\partial \varphi_2}{\partial x}}(t, \varphi_2^{-1}(t,z)) + \frac{\frac{\partial^2 \varphi_2^{-1}}{\partial t \partial z}(t,z)}{\frac{\partial \varphi_2^{-1}}{\partial z}(t,z)}
\end{aligned} \tag{2.199}
$$

Due to the second term in (2.199), we need the following lemma to estimate $\left|\frac{\frac{\partial^2 \varphi}{\partial x^2}}{\left(\frac{\partial \varphi}{\partial x}\right)^2}\right|$.

Lemma 16. *Let $\varphi : \Delta \to J$ be a map satisfying the negative Schwarzian derivative condition. Suppose φ can be extended to $\tilde{\varphi}$ which maps onto an extension $\tilde{J}$ of J, where the extension on each end has length e. Then*

$$
\left|\frac{\varphi''(z)}{(\varphi'(z))^2}\right| \leq \frac{2}{e} \tag{2.200}
$$

for any $z \in \Delta$.

Proof. We can assume that the derivatives of φ on Δ are all positive, since if derivatives are all negative, then we prove for $\psi(x) = -\varphi(x)$. Let z be any point in Δ. Assume first that z is not a boundary point of Δ. Choose two points x and y such that $z \in [x, y] \subset \Delta$. By (1.3), we have

$$\left(1 + \frac{|\varphi(x) - \varphi(y)|}{e}\right)^2 \geq \frac{|\varphi'(x)|}{|\varphi'(y)|}. \tag{2.201}$$

By the mean value theorem and some basic calculations we have

$$\begin{aligned}
\frac{|\varphi'(x)|}{|\varphi'(y)|} &= e^{\log \frac{\varphi'(x)}{\varphi'(y)}} \\
&= e^{\log \varphi'(x) - \log \varphi'(y)} \\
&= e^{\frac{\varphi''(\theta)}{\varphi'(\theta)}(x-y)} \\
&= e^{\frac{|\varphi''(\theta)|}{|\varphi'(\theta)|} \frac{|\varphi(x)-\varphi(y)|}{|\varphi'(\tilde{\theta})|}}
\end{aligned}$$

where θ and $\tilde{\theta}$ are in (x, y).

$$\begin{aligned}
1 + 2\frac{|\varphi(x) - \varphi(y)|}{e} + \left(\frac{|\varphi(x) - \varphi(y)|}{e}\right)^2 &= \left(1 + \frac{|\varphi(x) - \varphi(y)|}{e}\right)^2 \\
&\geq e^{\frac{|\varphi''(\theta)|}{|\varphi'(\theta)|} \frac{|\varphi(x)-\varphi(y)|}{|\varphi'(\tilde{\theta})|}} \\
&= 1 + \frac{|\varphi''(\theta)|}{|\varphi'(\theta)|} \frac{|\varphi(x) - \varphi(y)|}{|\varphi'(\tilde{\theta})|} + \mathrm{O}(|\varphi(x) - \varphi(y)|^2).
\end{aligned}$$

Then we have,

$$\frac{2}{e} + \frac{|\varphi(x) - \varphi(y)|}{e} \geq \frac{|\varphi''(\theta)|}{|\varphi'(\theta)|} \frac{1}{|\varphi'(\tilde{\theta})|} + \mathrm{O}(|\varphi(x) - \varphi(y)|) \tag{2.202}$$

Let $x \to z^-$ and $y \to z^+$. Then

$$\frac{2}{e} \geq \frac{|\varphi''(z)|}{|\varphi'(z)|^2}. \tag{2.203}$$

If z is a boundary point of Δ, choose $[x, y] \subset \tilde{\Delta}$ where $\tilde{\Delta} = \tilde{\varphi}^{-1}(\tilde{J})$. e should be replaced by a smaller extension value varying with x or y. As x and y tend to z, the extension value will again converge to e, and the same result holds. □

`Remark 12.` From the proof, we can see that we should be able to obtain better estimates if z does not lie on the boundary of Δ.

Corollary 8. *Let $f_{j_1}, \cdots, f_{j_r}$ be monotone branches in $\xi_{j_1}, \cdots, \xi_{j_r}$ respectively, then*

$$\frac{\left|\frac{\partial^2\left(f_{j_r}\circ\cdots\circ f_{j_1}\right)}{\partial x^2}\right|}{\left|\frac{\partial\left(f_{j_r}\circ\cdots\circ f_{j_1}\right)}{\partial x}\right|^2} < \frac{2}{0.17} < 12 \tag{2.204}$$

Let g be any monotone map from a domain on the y-axis onto I at any step n,

$$\frac{\left|\frac{\partial^2 g}{\partial x^2}\right|}{\left|\frac{\partial g}{\partial x}\right|^2} < \frac{2}{0.17} < 12 \tag{2.205}$$

Estimates for $g_{(n)}$ and $\mathcal{G}_{(n),i}$.

For $n = 6$, we have (2.125). For $6 < n < 24$,

$$\begin{aligned}
\frac{\left|\frac{\partial^2 g_{(n)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(n)}^{-1}}{\partial z}\right|} &\le \frac{\left|\frac{\partial^2 g_{(n-1)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(n-1)}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 g_{(n-1)}}{\partial x^2}\right|}{\left|\frac{\partial g_{(n-1)}}{\partial x}\right|^2} \cdot \frac{\left|\frac{\partial\left(f_{0,i_s}\circ\cdots\circ f_{0,i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{0,i_s}\circ\cdots\circ f_{0,i_1}\right)}{\partial x}\right|} + \frac{\left|\frac{\partial^2\left(f_{0,i_s}\circ\cdots\circ f_{0,i_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{0,i_s}\circ\cdots\circ f_{0,i_1}\right)^{-1}}{\partial z}\right|} \\
&\le \frac{\left|\frac{\partial^2 g_{(n-1)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(n-1)}^{-1}}{\partial z}\right|} + 12 * 1.5527 + 200 \\
&\le \frac{\left|\frac{\partial^2 g_{(6)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(6)}^{-1}}{\partial z}\right|} + (n-6) * (12 * 1.5527 + 200) \\
&\le 212 + (n-6) * (12 * 1.5527 + 200)
\end{aligned} \tag{2.206}$$

In particular,

$$\begin{aligned}
\frac{\left|\frac{\partial^2 g_{(23)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(23)}^{-1}}{\partial z}\right|} &\le 212 + (23-6) * (12 * 1.5527 + 200) \\
&< 3929 \\
&< \frac{10^{-6}}{|\delta_5|^2}
\end{aligned} \tag{2.207}$$

For $n \ge 24$, we have $g_{(n)} = \hat{f}_{[\frac{n}{3}],i_s} \circ \cdots \circ \hat{f}_{[\frac{n}{3}],i_1} \circ g_{(n-1)}$, which by corollary 6 can be simplified to one of the following cases,

$$g_{(n)} = \hat{f}_{[\frac{n}{3}],i_s} \circ f_{0,i_{s-1}} \circ \cdots \circ f_{0,i_1} \circ g_{(n-1)} \tag{2.208}$$

$$g_{(n)} = \hat{f}_{[\frac{n}{3}],i_2} \circ f_{5,i_1} \circ g_{(n-1)} \tag{2.209}$$

$$g_{(n)} = \hat{f}_{[\frac{n}{3}],i_1} \tag{2.210}$$

The worst case in terms of estimates for the variation of derivatives is of the form (2.209). First we estimate $\frac{\left|\frac{\partial^2 (f_{[\frac{n}{3}]-3,i_2}\circ f_{5,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial (f_{[\frac{n}{3}]-3,i_2}\circ f_{5,i_1})^{-1}}{\partial z}\right|}$.

When $[\frac{n}{3}]-3=5$, we have

$$\begin{aligned}
\frac{\left|\frac{\partial^2 (f_{5,i_2}\circ f_{5,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial (f_{5,i_2}\circ f_{5,i_1})^{-1}}{\partial z}\right|} \le & \frac{\left|\frac{\partial^2 f_{5,i_1}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{5,i_1}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 f_{5,i_1}}{\partial x^2}\right|}{\left|\frac{\partial f_{5,i_1}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial f_{5,i_2}}{\partial t}\right|}{\left|\frac{\partial f_{5,i_2}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 f_{5,i_2}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{5,i_2}^{-1}}{\partial z}\right|} \\
\le & 900,000 + 12*161 + 900,000 \\
< & 1801920 \\
< & 2.19\frac{1}{|\delta_5|^2}
\end{aligned} \tag{2.211}$$

When $[\frac{n}{3}]-3=6$, we have

$$\begin{aligned}
\frac{\left|\frac{\partial^2 (f_{6,i_2}\circ f_{5,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial (f_{6,i_2}\circ f_{5,i_1})^{-1}}{\partial z}\right|} \le & \frac{\left|\frac{\partial^2 f_{5,i_1}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{5,i_1}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 f_{5,i_1}}{\partial x^2}\right|}{\left|\frac{\partial f_{5,i_1}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial f_{6,i_2}}{\partial t}\right|}{\left|\frac{\partial f_{6,i_2}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 f_{6,i_2}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{6,i_2}^{-1}}{\partial z}\right|} \\
\le & 900,000 + 12*161 + 1.38\frac{1}{|\delta_6|^2} \\
\le & 0.122\frac{1}{|\delta_6^{\mathrm{re}}|^2} + 1.38\frac{1}{|\delta_6^{\mathrm{re}}|^2} \\
< & 1.51\frac{1}{|\delta_6^{\mathrm{re}}|^2}
\end{aligned} \tag{2.212}$$

Starting from $[\frac{n}{3}]-3=7$, we have a general formula. For $[\frac{n}{3}]-3\ge 7$

$$\begin{aligned}
\frac{\left|\frac{\partial^2 (f_{[\frac{n}{3}]-3,i_2}\circ f_{5,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial (f_{[\frac{n}{3}]-3,i_2}\circ f_{5,i_1})^{-1}}{\partial z}\right|} \le & \frac{\left|\frac{\partial^2 f_{5,i_1}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{5,i_1}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 f_{5,i_1}}{\partial x^2}\right|}{\left|\frac{\partial f_{5,i_1}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial f_{[\frac{n}{3}]-3,i_2}}{\partial t}\right|}{\left|\frac{\partial f_{[\frac{n}{3}]-3,i_2}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 f_{[\frac{n}{3}]-3,i_2}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial f_{[\frac{n}{3}]-3,i_2}^{-1}}{\partial z}\right|} \\
\le & 900,000 + 12*161 + \frac{1}{\left|\delta_{[\frac{n}{3}]-3}^{\mathrm{re}}\right|^2} \\
\le & 0.014\frac{1}{|\delta_7^{\mathrm{re}}|^2} + \frac{1}{\left|\delta_{[\frac{n}{3}]-3}^{\mathrm{re}}\right|^2} \\
< & 1.014\frac{1}{\left|\delta_{[\frac{n}{3}]-3}^{\mathrm{re}}\right|^2}
\end{aligned} \tag{2.213}$$

Then we can estimate $\frac{\left|\frac{\partial^2 g_{(n)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(n)}^{-1}}{\partial z}\right|}$ for $n\ge 24$. Using (2.207) and (2.211), we get

$$\begin{aligned}
\frac{\left|\frac{\partial^2 g_{(24)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(24)}^{-1}}{\partial z}\right|} \le & \frac{\left|\frac{\partial^2 g_{(23)}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial g_{(23)}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 g_{(23)}}{\partial x^2}\right|}{\left|\frac{\partial g_{(23)}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial (f_{5,i_2}\circ f_{5,i_1})}{\partial t}\right|}{\left|\frac{\partial (f_{5,i_2}\circ f_{5,i_1})}{\partial x}\right|} + \frac{\left|\frac{\partial^2 (f_{5,i_2}\circ f_{5,i_1})^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial (f_{5,i_2}\circ f_{5,i_1})^{-1}}{\partial z}\right|} \\
\le & \frac{10^{-6}}{|\delta_5|^2} + 12*\frac{1}{4|\delta_5|} + 2.19\frac{1}{|\delta_5|^2} \\
\le & 3\frac{1}{|\delta_5|^2}
\end{aligned} \tag{2.214}$$

Assume as an inductive assumption that $\frac{\left|\frac{\partial^2 g_{(k)}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{(k)}^{-1}}{\partial z}\right|} \leq \frac{3*(k \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{k}{3}]-3}\right|^2}$ for $k \leq n-1$, then from (2.211),(2.212), and (2.213) we get

$$
\begin{aligned}
\frac{\left|\frac{\partial^2 g_{(n)}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{(n)}^{-1}}{\partial z}\right|} &\leq \frac{\left|\frac{\partial^2 g_{(n-1)}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{(n-1)}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 g_{(n-1)}}{\partial x^2}\right|}{\left|\frac{\partial g_{(n-1)}}{\partial x}\right|^2} \cdot \frac{\left|\frac{\partial (f_{[\frac{n}{3}]-3,i_2} \circ f_{5,i_1})}{\partial t}\right|}{\left|\frac{\partial (f_{[\frac{n}{3}]-3,i_2} \circ f_{5,i_1})}{\partial x}\right|} + \frac{\left|\frac{\partial^2 (f_{[\frac{n}{3}]-3,i_2} \circ f_{5,i_1})^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial (f_{[\frac{n}{3}]-3,i_2} \circ f_{5,i_1})^{-1}}{\partial z}\right|} \\
&\leq \frac{3*(n-1 \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{n-1}{3}]-3}\right|^2} + 12 * \frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|} + 2.19 \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} \\
&\leq \frac{3*(n-1 \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{n-1}{3}]-3}\right|^2} + 0.0004 * \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} + 2.19 \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} \\
&\leq \begin{cases} \frac{3*(n-1 \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} + 3\frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2}, & \text{if } [\frac{n}{3}] = [\frac{n-1}{3}] \\ \frac{3*2+3}{9\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} + 0.0004 * \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2} + 2.19\frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2}, & \text{if } [\frac{n}{3}] = [\frac{n-1}{3}]+1 \end{cases} \\
&\leq \begin{cases} \frac{3*(n \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2}, & \text{if } [\frac{n}{3}] = [\frac{n-1}{3}] \\ \frac{3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2}, & \text{if } [\frac{n}{3}] = [\frac{n-1}{3}]+1 \end{cases} \\
&= \frac{3*(n \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]-3}\right|^2}.
\end{aligned} \tag{2.215}
$$

Estimates for $\bar{g}_{n,i}$ and $\bar{\mathcal{G}}_{n,i}$ above y_n.

$$\bar{g}_{n,i} = f_{[\frac{n}{3}],i_s} \circ \cdots \circ f_{[\frac{n}{3}],i_1} \circ \bar{g}_{n-1,j}$$

or

$$\bar{g}_{n,i} = f_{[\frac{n}{3}],i_s} \circ \cdots \circ f_{[\frac{n}{3}],i_1} \circ g_{(n-1)}.$$

So the estimates should be the same as for $g_{(n)}$.

Estimates for $g_{n,i}$ and $\mathcal{G}_{n,i}$ above y_{n-1}.

According to 2.4.4.1 and corollary 6, the compositions of $g_{n,i}$ has the following form: $g_{n,i} = f_{\tilde{m}+1,k_3} \circ \mathcal{F}_{m+1,k_2} \circ \mathcal{F}_{[\frac{n}{3}],k_1} \circ f_{0,1} \circ f_{0,1} \circ f_{[\frac{n}{3}],i_2} \circ f_{5,i_1} \circ \bar{g}_{n-1,j}$, where $m \leq [\frac{n}{3}]$ and $\tilde{m} \leq m+1$, or $g_{n,i} = f_{\tilde{m}+1,k_2} \circ \mathcal{F}_{m+1,k_1} \circ \bar{\mathcal{G}}_{n-1,j}$. To estimate the variation of derivative of $f_{\tilde{m}+1,k_3} \circ \mathcal{F}_{m+1,k_2} \circ \mathcal{F}_{[\frac{n}{3}],k_1} \circ f_{0,1} \circ f_{0,1} \circ f_{[\frac{n}{3}],i_2} \circ f_{5,i_1} \circ \bar{g}_{n-1,j}$, we first estimate the variation of derivative for $\mathcal{F}_{m+1,k_2} \circ \mathcal{F}_{[\frac{n}{3}],k_1}$. We have

$$\begin{aligned}
\frac{\left|\frac{\partial^2\left(\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial z}\right|} &\leq \frac{\left|\frac{\partial^2\mathcal{F}^{-1}_{[\frac{n}{3}],k_1}}{\partial t\partial z}\right|}{\left|\frac{\partial\mathcal{F}^{-1}_{[\frac{n}{3}],k_1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2\mathcal{F}_{[\frac{n}{3}],k_1}}{\partial x^2}\right|}{\left|\frac{\partial\mathcal{F}_{[\frac{n}{3}],k_1}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial\mathcal{F}_{m+1,k_2}}{\partial t}\right|}{\left|\frac{\partial\mathcal{F}_{m+1,k_2}}{\partial x}\right|} + \frac{\left|\frac{\partial^2\mathcal{F}^{-1}_{m+1,k_2}}{\partial t\partial z}\right|}{\left|\frac{\partial\mathcal{F}^{-1}_{m+1,k_2}}{\partial z}\right|} \\
&\leq \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|^2} + \frac{2}{e}\cdot\frac{1}{4\left|\delta^{\mathrm{re}}_{m+1}\right|} + \frac{1}{\left|\delta^{\mathrm{re}}_{m+1}\right|^2} \\
&\leq \frac{1}{9}\cdot\frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|^2} + \frac{2}{13\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|}\cdot\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|} + \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|^2} \\
&\leq \frac{1}{9}\cdot\frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|^2} + \frac{2}{13*3\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|}\cdot\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|} + \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|^2} \\
&\leq \frac{1.13}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|^2}
\end{aligned} \tag{2.216}$$

Then we estimate the variation of derivative for $f_{\tilde{m}+1,k_3}$ composed with $\mathcal{F}_{m+1,k_2} \circ \mathcal{F}_{[\frac{n}{3}],k_1}$. We have

$$\begin{aligned}
&\frac{\left|\frac{\partial^2\left(f_{\tilde{m}+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde{m}+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial z}\right|} \\
\leq&\frac{\left|\frac{\partial^2\left(\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2\left(\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)}{\partial x^2}\right|}{\left|\frac{\partial\left(\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial f_{\tilde{m}+1,k_3}}{\partial t}\right|}{\left|\frac{\partial f_{\tilde{m}+1,k_3}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 f^{-1}_{\tilde{m}+1,k_3}}{\partial t\partial z}\right|}{\left|\frac{\partial f^{-1}_{\tilde{m}+1,k_3}}{\partial z}\right|} \\
\leq&\frac{1.13}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|^2} + \frac{2}{e}\cdot\frac{1}{4\left|\delta^{\mathrm{re}}_{\tilde{m}+1}\right|} + \frac{1}{\left|\delta^{\mathrm{re}}_{\tilde{m}+1}\right|^2} \\
\leq&\frac{1.13}{9}\cdot\frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2} + \frac{2}{13\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+1}\right|}\cdot\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|} + \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2} \\
\leq&\frac{1.13}{9}\cdot\frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2} + \frac{2}{13*3\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}\cdot\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|} + \frac{1}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2} \\
\leq&\frac{1.14}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2}.
\end{aligned} \tag{2.217}$$

Then we estimate the variation of derivative for $f_{\tilde{m}+1,k_3} \circ \mathcal{F}_{m+1,k_2} \circ \mathcal{F}_{[\frac{n}{3}],k_1}$ composed

with $f_{0,1} \circ f_{0,1}$. We have

$$
\begin{aligned}
&\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\right)^{-1}}{\partial z}\right|}\\
\leq&\frac{\left|\frac{\partial^2\left(f_{0,1}\circ f_{0,1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{0,1}\circ f_{0,1}\right)^{-1}}{\partial z}\right|}+\frac{\left|\frac{\partial^2\left(f_{0,1}\circ f_{0,1}\right)}{\partial x^2}\right|}{\left|\frac{\partial\left(f_{0,1}\circ f_{0,1}\right)}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)}{\partial x}\right|}+\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\right)^{-1}}{\partial z}\right|}\\
\leq&200+12*\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}+\frac{1.14}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2}\\
\leq&\frac{1.15}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2}.
\end{aligned}
\tag{2.218}
$$

Then we estimate the variation of derivative for $f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}$ composed with $f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}$. The estimate for the variation of derivative of $f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}$ comes from (2.213).

$$
\begin{aligned}
&\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)^{-1}}{\partial z}\right|}\\
\leq&\frac{\left|\frac{\partial^2\left(f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)^{-1}}{\partial z}\right|}+\frac{\left|\frac{\partial^2\left(f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)}{\partial x^2}\right|}{\left|\frac{\partial\left(f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\right)}{\partial x}\right|}\\
+&\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\right)^{-1}}{\partial z}\right|}\\
\leq&\frac{1.014}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]}\right|^2}+12*\frac{1}{4\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|}+\frac{1.15}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2}\\
\leq&\frac{1.16}{\left|\delta^{\mathrm{re}}_{[\frac{n}{3}]+2}\right|^2}.
\end{aligned}
\tag{2.219}
$$

Finally, we estimate the variation of derivative for $f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}$ composed with $\bar g_{n-1,j}$. Bounds for the variation of derivative of $\bar g_{k,j}$ comes from the inductive assumption that $\frac{\left|\frac{\partial^2\bar g_{k,j}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\bar g_{k,j}^{-1}}{\partial z}\right|}\leq\frac{3*(k \bmod 3)+3}{\left|\delta^{\mathrm{re}}_{[\frac{k}{3}]}\right|^2}$ for $k\leq n-1$.

We have

$$\begin{aligned}
&\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\circ\bar g_{n-1,j}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\circ\bar g_{n-1,j}\right)^{-1}}{\partial z}\right|}\\
&\leq\frac{\left|\frac{\partial^2\bar g^{-1}_{n-1,j}}{\partial t\partial z}\right|}{\left|\frac{\partial\bar g^{-1}_{n-1,j}}{\partial z}\right|}+\frac{\left|\frac{\partial^2\bar g_{n-1,j}}{\partial x^2}\right|}{\left|\frac{\partial\bar g_{n-1,j}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)}{\partial x}\right|}\\
&+\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_3}\circ\mathcal{F}_{m+1,k_2}\circ\mathcal{F}_{[\frac{n}{3}],k_1}\circ f_{0,1}\circ f_{0,1}\circ f_{[\frac{n}{3}],i_2}\circ f_{5,i_1}\right)^{-1}}{\partial z}\right|}\\
&\leq\frac{3*(n-1\bmod 3)+3}{\left|\delta^{\text{re}}_{[\frac{n-1}{3}]}\right|^2}+12*\frac{1}{4\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|}+\frac{1.16}{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|^2}\\
&\leq\frac{1.3}{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|^2}.
\end{aligned}\tag{2.220}$$

Now lets consider the other case. Estimates for $\bar{\mathcal{G}}_{n-1,j}$ come from the inductive assumption that $\frac{\left|\frac{\partial^2\bar{\mathcal{G}}^{-1}_{k,j}}{\partial t\partial z}\right|}{\left|\frac{\partial\bar{\mathcal{G}}^{-1}_{k,j}}{\partial z}\right|}\leq\frac{3*(k\bmod 3)+3}{\left|\delta^{\text{re}}_{[\frac{k}{3}]}\right|^2}$ for $k\leq n-1$. We have

$$\begin{aligned}
&\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ\bar{\mathcal{G}}_{n-1,j}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\circ\bar{\mathcal{G}}_{n-1,j}\right)^{-1}}{\partial z}\right|}\\
&\leq\frac{\left|\frac{\partial^2\bar{\mathcal{G}}^{-1}_{n-1,j}}{\partial t\partial z}\right|}{\left|\frac{\partial\bar{\mathcal{G}}^{-1}_{n-1,j}}{\partial z}\right|}+\frac{\left|\frac{\partial^2\bar{\mathcal{G}}_{n-1,j}}{\partial x^2}\right|}{\left|\frac{\partial\bar{\mathcal{G}}_{n-1,j}}{\partial x}\right|^2}\cdot\frac{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\right)}{\partial t}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\right)}{\partial x}\right|}+\frac{\left|\frac{\partial^2\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\right)^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\left(f_{\tilde m+1,k_2}\circ\mathcal{F}_{m+1,k_1}\right)^{-1}}{\partial z}\right|}\\
&\leq\frac{3*(n-1\bmod 3)+3}{\left|\delta^{\text{re}}_{[\frac{n-1}{3}]}\right|^2}+\frac{2}{13\left|\delta^{\text{re}}_{[\frac{n-1}{3}]}\right|}*\frac{1}{4\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|}+\frac{1.16}{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|^2}\\
&\leq\frac{1.3}{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|^2}
\end{aligned}\tag{2.221}$$

From (2.220) and (2.221), we can conclude that

$$\frac{\left|\frac{\partial^2 g^{-1}_{n,i}}{\partial t\partial z}\right|}{\left|\frac{\partial g^{-1}_{n,i}}{\partial z}\right|}\leq\frac{1.3}{\left|\delta^{\text{re}}_{[\frac{n}{3}]+2}\right|^2}\tag{2.222}$$

Similar estimates can be derived for $\mathcal{G}_{n,i}$.

Estimates for $f_{n,i}$ and $\mathcal{F}_{n,i}$ whose domains are in $\delta^{\text{re}}_{n-1}\backslash\delta^{\text{re}}_{n}$.

$f_{n,i}=g_{n,i}\circ h$ where the domains of $g_{n,i}$'s are in $[y_{n-1}(t),y_n(t)]$.

$\mathcal{F}_{n,i} = \mathcal{G}_{n,i} \circ h$, where the domains of $\mathcal{G}_{n,i}$'s are in $[y_{n-1}(t), y_n(t)]$.

We use

$$\frac{\left|\frac{\partial^2 h^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial h^{-1}}{\partial z}\right|} = \frac{\left|\frac{\partial^2 h}{\partial t \partial x}(t, h^{-1}(t,z)) + \frac{\partial^2 h}{\partial x^2}(t, h^{-1}(t,z)) \frac{\frac{\partial h}{\partial t}(t,h^{-1}(t,z))}{\frac{\partial h}{\partial x}(t,h^{-1}(t,z))}\right|}{\left|\frac{\partial h}{\partial x}(t, h^{-1}(t,z))\right|} \leq \frac{1}{t} + \frac{2(\frac{1}{4} - \frac{1}{4}\left|\delta_n^{\mathrm{re}}\right|^2)}{t\left|\delta_n^{\mathrm{re}}\right|^2} \tag{2.223}$$

for x outside δ_n^{re}.

$$\begin{aligned}
\frac{\left|\frac{\partial^2 f_{n,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial f_{n,i}^{-1}}{\partial z}\right|} &\leq \frac{\left|\frac{\partial^2 h^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial h^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 h}{\partial x^2}\right|}{\left|\frac{\partial h}{\partial x}\right|^2} \cdot \frac{\left|\frac{\partial g_{n,i}}{\partial t}\right|}{\left|\frac{\partial g_{n,i}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 g_{n,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{n,i}^{-1}}{\partial z}\right|} \\
&\leq \frac{1}{t} + \frac{2(\frac{1}{4} - \frac{1}{4}\left|\delta_n^{\mathrm{re}}\right|^2)}{t\left|\delta_n^{\mathrm{re}}\right|^2} + \frac{2}{t\left|\delta_n^{\mathrm{re}}\right|^2} \cdot \frac{\left|\frac{\partial g_{n,i}}{\partial t}\right|}{\left|\frac{\partial g_{n,i}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 g_{n,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial g_{n,i}^{-1}}{\partial z}\right|} \\
&\leq \frac{1}{t} + \frac{2(\frac{1}{4} - \frac{1}{4}\left|\delta_n^{\mathrm{re}}\right|^2)}{t\left|\delta_n^{\mathrm{re}}\right|^2} + \frac{2}{t\left|\delta_n^{\mathrm{re}}\right|^2} * 0.003 + \frac{1.3}{\left|\delta_{[\frac{n}{3}]+2}^{\mathrm{re}}\right|^2} \\
&< \frac{1}{\left|\delta_n^{\mathrm{re}}\right|^2}
\end{aligned} \tag{2.224}$$

Estimates for $f_{n,i}$ and $\mathcal{F}_{n,i}$ whose domains are outside $\delta_{n-1}^{\mathrm{re}}$.

$f_{n,i} = f_{n-1,l} \circ \mathcal{F}_{n-1,j}$ where the domains of the maps $\mathcal{F}_{n-1,j}$ is outside $\delta_{n-1}^{\mathrm{re}}$.

$\mathcal{F}_{n,i} = \mathcal{F}_{n-1,l} \circ \mathcal{F}_{n-1,j}$ where the domains of the maps $\mathcal{F}_{n-1,j}$ are outside $\delta_{n-1}^{\mathrm{re}}$.

$$\begin{aligned}
\frac{\left|\frac{\partial^2 \mathcal{F}_{n,i}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{F}_{n,i}^{-1}}{\partial z}\right|} &\leq \frac{\left|\frac{\partial^2 \mathcal{F}_{n-1,j}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{F}_{n-1,j}^{-1}}{\partial z}\right|} + \frac{\left|\frac{\partial^2 \mathcal{F}_{n-1,j}}{\partial x^2}\right|}{\left|\frac{\partial \mathcal{F}_{n-1,j}}{\partial x}\right|^2} \cdot \frac{\left|\frac{\partial \mathcal{F}_{n-1,l}}{\partial t}\right|}{\left|\frac{\partial \mathcal{F}_{n-1,l}}{\partial x}\right|} + \frac{\left|\frac{\partial^2 \mathcal{F}_{n-1,l}^{-1}}{\partial t \partial z}\right|}{\left|\frac{\partial \mathcal{F}_{n-1,l}^{-1}}{\partial z}\right|} \\
&\leq \frac{1.3}{\left|\delta_{n-1}^{\mathrm{re}}\right|^2} + \frac{2}{13\left|\delta_{n-1}^{\mathrm{re}}\right|} \cdot \frac{1}{4\left|\delta_{n-1}^{\mathrm{re}}\right|} + \frac{1.3}{\left|\delta_{n-1}^{\mathrm{re}}\right|^2} \\
&\leq \frac{1}{\left|\delta_n^{\mathrm{re}}\right|^2}
\end{aligned} \tag{2.225}$$

2.6 Admissible domains and admissible parameter values

2.6.1 Step 6

2.6.1.1 Total measure of $\cup\mathcal{T}^{(6)}$

We can consider admissible intervals in the phase space either from the perspective of a partition on I or from the perspective of a partition on $\Delta^{(5)}$. Both notions are interchangable by a diffeomorphism $g_{(5)}$ that maps $\Delta^{(5)}$ onto I. On the parameter interval, we say that an interval is admissible if t traversing through the interval corresponds to $w(t)$ traversing through an admissible interval in I.

When defining $\mathcal{T}^{(6)}$, we always performed refinements by pulling back the seven branch partition ξ_0. Therefore it is natural to label monotone domains and refined monotone domains by $\Delta_{a_1 \cdots a_j}$ where $1 \leq j \leq 5$ and $a_1, \ldots, a_j \in \{1, 2, 3, 5, 6, 7\}$. The index j does not exceed 5 since we do not need to perform more than 4 monotone refinements. Within these monotone intervals, we define admissible intervals as the following:

Definition 8. A monotone domain $\Delta_{a_1 \cdots a_j}$ in I is an *admissible domain at step 6* if

1. subindices $a_1, \ldots, a_j$ do not equal to 5 or 6.
2. $\frac{|g_5^{-1}(\Delta_{a_1 \cdots a_j})|}{H_5(g_5^{-1}(\Delta_{a_1 \cdots a_j}))} < \vartheta_1$ and $\frac{|g_5^{-1}(\Delta_{a_1 \cdots a_{j-1}})|}{H_5(g_5^{-1}(\Delta_{a_1 \cdots a_{j-1}}))} \geq \vartheta_1$ (when $j \geq 2$)
3. $a_1 \ldots a_j \succ 114$ in lexicographical ordering.

We pullback admissible intervals in I by g_5^{-1} into $\Delta^{(5)}$. Then they become admissible intervals in $\Delta^{(5)}$. Such definition comes directly from the algorithm for defining $\Delta^{(6)}$. $\Delta^{(6)}$'s are exactly the admissible domains in $\Delta^{(5)}$ at step 6.

We do not need to avoid domains with subindices $a_1 \ldots a_{j-1}2$ or $a_1 \ldots a_{j-1}3$ since when $g_5(w(t))$ falls into such domains, the image of $g_5 \circ f$ does not contain the δ_0^{-p} represented by subindex $a_1 \dots a_{j-1}4$.

Definition 9. A parameter interval $\mathcal{T}'$ in $\mathcal{T}^{(5)}$ is an *admissible parameter interval at step 6* if $t \in \mathcal{T}'$ corresponds to $g_5(w(t)) \in \Delta'$ for some admissible domain Δ' in I.

By our definition of admissible intervals, all admissible intervals are disjoint except at endpoints. We collect the maximal possible collection of admissible intervals $\cup \mathcal{T}^{(6)}$.

Now we state the numerical results on the measure of admissible intervals and admissible parameter intervals.

1. Under the algorithm at step 6, there are 135 admissible domains.

2. The total measure of admissible domains in I relative to the measure of I is bounded below by 0.196180 and bounded above by 0.196195.

3. The total measure of admissible parameters in $\mathcal{T}^{(5)}$ at step 6 is $9.1443 * 10^{-7}$. If we divide that by the measure of $\mathcal{T}^{(5)}$ which is $4.64851 * 10^{-6}$, we get

$$\frac{|\cup \mathcal{T}^{(6)}|}{|\mathcal{T}^{(5)}|} \geq 0.196714646. \tag{2.226}$$

2.6.2 Measure of admissible domains for general step $n > 6$

Admissible intervals in $\Delta^{(n-1)}$ are monotone domains $\Delta^{(n)}$ in the parameter-induced partition (defined in 2.4.2) of $\Delta^{(n-1)}$. Non-admissible intervals $\delta^{(n)}$ are the holes in the parameter-induced partition. We denote the relative measure of non-admissible intervals in each $\Delta^{(n-1)}$ by $\mathcal{H}_n$.

$$\mathcal{H}_n(t) = \frac{\left|\bigcup_i \delta_i^{(n)}(t)\right|}{\left|\Delta^{(n-1)}(t)\right|} \tag{2.227}$$

As described in the previous subsection, we can also consider admissible intervals from the perspective of a partition on I on the x-axis. By estimating relative measures on the

x-axis and considering distortions, we get the following bounds for $\mathcal{H}_n(t)$. We use some techniques to lower the bounds of $\mathcal{H}_n(t)$ in order to get a better final estimate in (2.262).

$$\mathcal{H}_n(t) < \frac{0.773247352 * (15/13) * (1.29)^{n-6}}{1 - 0.773247352 + 0.773247352 * (15/13) * (1.29)^{n-6}} \qquad \text{for } 6 < n < 15 \tag{2.228}$$

$$\mathcal{H}_n(t) < 0.7265 \qquad \text{for } 1 < [\frac{n}{3}] - 3 < 5 \tag{2.229}$$

$$\mathcal{H}_n(t) < 0.1716 \qquad \text{for } [\frac{n}{3}] - 3 = 5 \tag{2.230}$$

$$\mathcal{H}_n(t) < 0.171126 \qquad \text{for } [\frac{n}{3}] - 3 = 6 \tag{2.231}$$

$$\mathcal{H}_n(t) < 0.171126 * (0.57)^{[\frac{n}{3}]-3-6} \qquad \text{for } 6 < [\frac{n}{3}] - 3 < 15 \tag{2.232}$$

$$\mathcal{H}_n(t) < 0.171126 * (0.57)^8 * (0.73)^{[\frac{n}{3}]-3-14} \qquad \text{for } [\frac{n}{3}] - 3 \geq 15 \tag{2.233}$$

2.6.2.1 Calculations for inequalities (2.229) through (2.233)

The algorithm for constructing the parameter-induced partition of $\Delta^{(n-1)}$ requires pullbacks of $\hat{\xi}_{[\frac{n}{3}]} = \xi_{[\frac{n}{3}]-3}$ onto or into $\Delta^{(n-1)}$ until all monotone domains $\Delta^{(n)}$ satisfy (2.39). Lemma 12, which makes use of the fact that $\Delta^{(n-1)}$ is always a fixed-proportional-to-size-$|\Delta^{(n-1)}|$ distance away from y_{n-1}, proves that (2.134) will imply (2.39). In practice, we are not able to check actual measures after pullback onto each $\Delta^{(n-1)}$, since $\Delta^{(n-1)}$'s were not obtained explicitly in the previous step but only estimated for their total measures. Therefore we use estimates which take distortion into account. Let $\Delta = [x_1, x_2]$ be a given monotone domain in I and g be the monotone map that maps $\Delta^{(n-1)}$ onto I, depicted in the figure below.

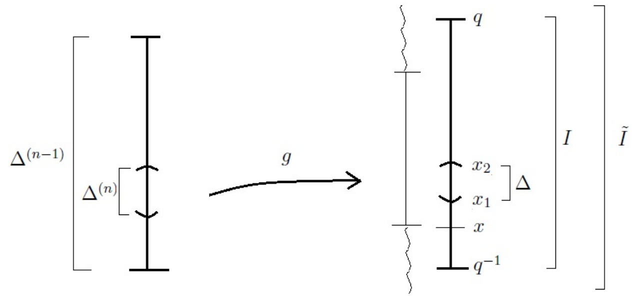


Figure 2.6: Δ as the image of $\Delta^{(n)}$ under mapping g

With reference to the figure, we define the following.

$$\text{Ratio}(\Delta, x) := \frac{x_2 - x_1}{2 * \left|\frac{1}{2} - x\right|} \tag{2.234}$$

$$\text{Dist}(x) := \left(1 + \frac{2 * \left|\frac{1}{2} - x\right|}{\min\{|[q^{-1}, x]| + 0.17, |[x, q]| + 0.17\}}\right)^2 \tag{2.235}$$

$$\text{MinDistRatio}(\Delta) := \min_{x \in \text{ smaller component of } I \setminus \Delta} \frac{\text{Ratio}(\Delta, x) * \text{Dist}(x)}{1 - \text{Ratio}(\Delta, x) + \text{Ratio}(\Delta, x) * \text{Dist}(x)} \tag{2.236}$$

MinDistRatio is a function which gives an upper bound to $\frac{\left|g^{-1}(\Delta)\right|}{\left|\Delta^{(n-1)}\right|}$ when pulling back by ξ and using A.2 and (A.3). We have

$$\frac{\left|g^{-1}(\Delta)\right|}{\left|\Delta^{(n-1)}\right|} < \text{MinDistRatio}(\Delta). \tag{2.237}$$

The following algorithm determines a worst possible partition ξ' on I, worst in the sense that it has the maximum possible measure of non-admissible domains, using MinDistRatio(Δ) as an upper bound for $\frac{\left|g^{-1}(\Delta)\right|}{\left|\Delta^{(n-1)}\right|}$.

1. A partition of I (starting with ξ) is considered on the x-axis. Consider each monotone domain Δ in the partition. Determine MinDistRatio(Δ) for all monotone domains in the partition.

2. We check if MinDistRatio(Δ) < 0.023. If so, then we do not partition the domain further, if not, we partition the domain by ξ. We do this for all monotone domains and go back to step one to repeat the procedure until all monotone domains satisfy MinDistRatio(Δ) < 0.023.

The resulting partition ξ' on I is pulled back onto $\Delta^{(n-1)}$ to get a worst possible parameter-induced partition of $\Delta^{(n-1)}$. The previous algorithm implies the following lemma 17 which specifies the number of refinements needed to get the parameter induced partition.

Lemma 17. *If we pullback the partition ξ_k, $k \geq 6$, in the above algorithm, then the partition ξ'_k which we obtain will coincide with ξ'_0 outside δ_0 and outside preimages of δ_0 in ξ'_0. Inside δ_0 and outside holes of ξ''_5, the partition ξ'_k will coincide with ξ''_5, where ξ''_5 is the refinement of ξ_5 inside δ_0 by ξ_0 using the above algorithm. Inside holes of ξ''_5 and preimages of δ_0 in ξ'_0, domains do not need extra refinements.*

Proof. We check numerically that the sizes of holes in ξ'_0 or $\xi''_5|_{\delta_0}$ after the above pullbacks are small enough to satisfy

$$\text{MinDistRatio}(\delta) < 0.023. \tag{2.238}$$

Monotone domains contained in holes of ξ'_0 and $\xi''_5|_{\delta_0}$ will be smaller than the holes they are contained in. □

We get the following corollary.

Corollary 9. *The maximum number of refinements needed to obtain ξ'_k is determined by the maximum amount of refinements needed to obtain ξ'_0 and ξ'_5.*

Obtaining (2.229)

Using ξ_0 as the partition that we pull back in the above algorithm, we obtain a partition ξ_0' of I. ξ_0' has 859 domains with central domain δ_0. The relative measure of holes in ξ_0' is less than 0.36 for all parameters in $\mathcal{T}^{476777}$. When we consider the distortion on I, which is the big number 15.6, and apply (A.3) directly, we get that the relative measure of non-admissible domains in $\Delta^{(n-1)}$ after step n for $7 \leq n \leq 23$ is bounded above by 0.898. To improve this estimate, we use the method of dividing into sections as used in 2.2.9 for domains outside δ_0. The sections and their respective ratios and distortions are listed in B.4. The first table in B.4 shows that the distorted relative measure of holes in $I \backslash \delta_0$ is less than 0.5 for partition ξ_0'.

$$\mu_{\text{holes}}(g^{-1}(\xi_0'|_{I\backslash\delta_0})) < 0.5 =: b \tag{2.239}$$

Using A.2 and A.3, we get

$$\frac{\left|g^{-1}(\delta_0)\right|}{\left|\Delta^{(n-1)}\right|} \leq 0.4524 =: a \tag{2.240}$$

Using (A.9), we get

$$\mathcal{H}_n(t) \leq \frac{\left|g^{-1}(\delta_0)\right|}{\left|\Delta^{(n-1)}\right|} + \frac{\left|g^{-1}(I\backslash\delta_0)\right|}{\left|\Delta^{(n-1)}\right|} * \mu_{\text{holes}}(g^{-1}(\xi_0'|_{I\backslash\delta_0})) \leq a + (1-a) * b \tag{2.241}$$

for $7 \leq n \leq 23$. Combining (2.240) and (2.241), we get (2.229).

Obtaining (2.230)

For steps n where $[\frac{n}{3}] - 3 = 5$, we pullback with ξ_5 in the algorithm to obtain ξ_5'. ξ_5' has 13761 domains. The union of domains 4038 through 9214 is δ_0. Sections and their respective ratios and distortions are listed in the second table in B.4. The second table in B.4 shows that the distorted relative measure of holes outside δ_0 is less than 0.1,

$$\mu_{\text{holes}}(g^{-1}(\xi_5'|_{I\backslash\delta_0})) < 0.1, \tag{2.242}$$

and the distorted relative measure of holes inside δ_0 is less than 0.25,

$$\mu_{\text{holes}}(g^{-1}(\xi_5'|_{\delta_0})) < 0.25. \tag{2.243}$$

By (A.10), we get

$$\mathcal{H}_n(t) < a * 0.25 + (1-a) * 0.1 < 0.16786 \tag{2.244}$$

for $24 \le n \le 26$. Therefore we have (2.230).

Obtaining (2.231)

For steps n where $[\frac{n}{3}] - 3 = 6$, we pullback ξ_6. Since ξ_6 changes with parameters, we do not obtain each partition ξ_6' as we did for the earlier steps, it would involve consideration of several hundred cases. Instead, we take $\xi_0'|_{I\backslash\delta_0}$ and estimate the relative measure of holes in $I\backslash\delta_0$ after filling-in each preimage of δ_0 by $\xi_6|_{\delta_0}$.

$$\mu_{\text{holes}}(\xi_6|_{\delta_0}) \le \frac{|\text{five holes}|}{|\delta_0|} + \frac{|\delta_5|}{|\delta_0|} * \left(\frac{1}{3} + \frac{2}{3} * 0.29\right) < 0.168 =: f \tag{2.245}$$

where bounds for $\frac{|\text{five holes}|}{|\delta_0|}$ and $\frac{|\delta_5|}{|\delta_0|}$ are obtained numerically.

$$\mu_{\text{holes}}(\mathcal{F}^{-1}(\xi_6|_{\delta_0})) < \frac{f * D_{\delta_0}}{1 - f + f * D_{\delta_0}} < 0.209 =: f' \tag{2.246}$$

where $\mathcal{F}$ denotes the maps from δ_0^{-p}'s in $I\backslash\delta_0$ onto δ_0. D_{δ_0} is the distortion on δ_0 when image extension is $\tilde{I}$.

$$\begin{aligned} \mu_{\text{holes}}(g^{-1}(\xi_6'|_{I\backslash\delta_0})) &\le \mu_{\text{holes}}(\xi_0'|_{I\backslash\delta_0}) * \max_{\mathcal{F}} \mu_{\text{holes}}\left(\mathcal{F}^{-1}\left(\xi_6|_{\delta_0}\right)\right) \\ &\le b * f' \end{aligned} \tag{2.247}$$

$$\begin{aligned} \mu_{\text{holes}}(\xi_6'|_{\delta_0}) &\le \frac{|\text{five holes}| + |\delta_5|}{|\delta_0|} + \mu_{\text{holes}}(\xi_5'') * \mu_{\text{holes}}(\mathcal{F}^{-1}(\xi_6|_{\delta_0})) \\ &\le 0.178 + 0.129 * f' \\ &< 0.205 =: e \end{aligned} \tag{2.248}$$

$$\mu_{\text{holes}}(g^{-1}(\xi_6'|_{\delta_0})) \leq \frac{e * D_{\delta_0}}{1 - e + e * D_{\delta_0}} < 0.252 =: e' \tag{2.249}$$

Combining (2.247), (2.249) and (A.10), we get

$$H_n(t) < a * e' + (1 - a) * b * f' < 0.171126 \tag{2.250}$$

which gives (2.231).

Obtaining (2.232)

For steps n where $6 < [\frac{n}{3}] - 3 \leq 14$, we have $\mu_{\text{holes}}(\xi_{[\frac{n}{3}]-3}) \leq 0.57 * \mu_{\text{holes}}(\xi_{[\frac{n}{3}]-3-1})$ from (2.140). By lemma 17, ξ_{k+1}' is what we get after filling-in of holes in ξ_k'. Since the way of filling-in ξ_k' is decided by the way of filling-in in ξ_k, the relative measure of holes after filling-in is the same as in (2.140).

$$\mu_{\text{holes}}(\xi_{k+1}') \leq \mu_{\text{holes}}(\xi_k') * (0.57) \tag{2.251}$$

for $7 \leq k \leq 14$. By (2.251), we get (2.232).

Obtaining (2.233)

For steps n where $15 \leq [\frac{n}{3}]$, we have $\mu_{\text{holes}}(\xi_{[\frac{n}{3}]-3}) \leq 0.73 * \mu_{\text{holes}}(\xi_{[\frac{n}{3}]-3-1})$. With the same arguments as for (2.251), we get

$$\mu_{\text{holes}}(\xi_{k+1}') \leq \mu_{\text{holes}}(\xi_k') * (0.73), \tag{2.252}$$

and hence (2.233).

2.6.3 Measure of admissible parameters

For each admissible domain $\Delta^{(n)}$ in $\Delta^{(n-1)}$, there is a corresponding admissible parameter interval $\mathcal{T}^{(n)}$. Similarly, we have non-admissible parameter intervals $\mathcal{T}(\delta^{(n)})$'s that correspond to non-admissible domains $\delta^{(n)}$ in $\Delta^{(n-1)}$. We denote the relative measure of admissible parameter intervals by

$$\mathcal{M}_n := \frac{|\bigcup_{\mathcal{T}^{(n)} \subset \mathcal{T}^{(n-1)}} \mathcal{T}^{(n)}|}{|\mathcal{T}^{(n-1)}|}. \tag{2.253}$$

and relative measure of non-admissible parameter intervals by

$$\mathcal{M}_n^c := \frac{|\mathcal{T}^{(n-1)} \setminus \bigcup_{\mathcal{T}^{(n)} \subset \mathcal{T}^{(n-1)}} \mathcal{T}^{(n)}|}{|\mathcal{T}^{(n-1)}|}. \tag{2.254}$$

The following lemma follows from Gronwall's inequality and the fact that the central domain is larger for greater parameter values.

Lemma 18. *Let δ_m^{-p} be mapped by $\mathcal{G}$ onto the rescaled central domain δ_m^{re}. Suppose*

$$\frac{\left|\frac{\partial^2 \mathcal{G}^{-1}}{\partial t \partial z}(t,z)\right|}{\left|\frac{\partial \mathcal{G}^{-1}}{\partial z}(t,z)\right|} < C \textit{ for all } z \in \delta_m^{re}(t) \textit{ for all } t \in \mathcal{T}. \tag{2.255}$$

Then

$$\left|\delta_m^{-p}(t)\right| \leq e^{C|\mathcal{T}|} \left|\delta_m^{-p}(t_{top})\right| \textit{ for all } t \in \mathcal{T} \tag{2.256}$$

where t_{top} is the top value of $\mathcal{T}$.

Similarly

Lemma 19. *Let Δ be mapped by g onto I. Suppose*

$$\frac{\left|\frac{\partial^2 g^{-1}}{\partial t \partial z}(t,z)\right|}{\left|\frac{\partial g^{-1}}{\partial z}(t,z)\right|} < C \textit{ for all } z \in I(t) \textit{ for all } t \in \mathcal{T}. \tag{2.257}$$

Then

$$|\Delta(t)| \geq e^{-C|\mathcal{T}|} \left|\Delta(t_{bottom})\right| \textit{ for all } t \in \mathcal{T} \tag{2.258}$$

where t_{bottom} is the bottom value of $\mathcal{T}$.

Theorem 2. *Let $\mathcal{H}_n(t)$ be defined as in (2.227). Let $\mathcal{M}_n^c$ be the relative measure of non-admissible parameters in $\mathcal{T}^{(n-1)}$. Then*

$$\begin{aligned}
\mathcal{M}_n^c &\leq \mathcal{H}_n(t_{top}^{(n-1)}) * \frac{1+4\epsilon_0}{1-4\epsilon_0} * exp\left(\max_{t \in \mathcal{T}^{(n-1)}} \frac{\left| \frac{\partial^2 \mathcal{G}_{(n),i}^{-1}}{\partial t \partial z} \right|}{\left| \frac{\partial \mathcal{G}_{(n),i}^{-1}}{\partial z} \right|} \cdot \left| \mathcal{T}^{(n-1)} \right| \right) \\
&\leq \mathcal{H}_n(t_{top}^{(n-1)}) * \frac{1+4\epsilon_0}{1-4\epsilon_0} * exp\left(\frac{8}{\left| \delta_{[\frac{n}{3}]-3}^{re} \right|^2} \cdot \frac{1}{\frac{1}{4}-\epsilon_0} \frac{t}{4} \left| \delta_{n-2}^{re} \right|^2 \vartheta_1 \right) \\
&\leq \mathcal{H}_n(t_{top}^{(n-1)}) * \frac{1+4\epsilon_0}{1-4\epsilon_0} * exp\left(\frac{8}{9^{n-[\frac{n}{3}]+1}} \cdot \frac{\vartheta_1}{\frac{1}{4}-\epsilon_0} \right) \qquad (2.259)
\end{aligned}$$

for $n \geq 24$.

Before proving this, we incorporate computer estimates. Our initial parameter interval $\mathcal{T}_0$ was described in the first five steps. Then at steps 6 through 23 the relative measure of admissible parameters follows from the last two columns of the table in B.1.5 (we use the better estimate). By multiplying these numbers we get at step 23 the measure of admissible parameters is greater than

$$\prod_{n=6}^{23} \mathcal{M}_n > 1.00614 * 10^{-15} =: \mathcal{X} \qquad (2.260)$$

Starting at step 24, we delete no more than

$$\mathcal{H}_n(t_{\text{top}}^{(n-1)}) * \frac{1+4\epsilon_0}{1-4\epsilon_0} * exp\left(\frac{8}{9^{n-[\frac{n}{3}]+1}} \cdot \frac{\vartheta_1}{\frac{1}{4}-\epsilon_0} \right) \qquad (2.261)$$

at each step n. Then we get that the relative measure of admissible parameters is greater than

$$\mathcal{X} \prod_{n=24}^{\infty} \mathcal{M}_n > \mathcal{X} \prod_{n=24}^{\infty} \left(1 - \mathcal{H}_n(t_{\text{top}}^{(n-1)}) * \frac{1+4\epsilon_0}{1-4\epsilon_0} * exp\left(\frac{8}{9^{n-[\frac{n}{3}]+1}} \cdot \frac{\vartheta_1}{\frac{1}{4}-\epsilon_0} \right) \right).$$

We combine that with bounds for $\mathcal{H}_n$ in (2.229) through (2.233) and get the following corollary.

`Corollary 10.` *Let $\mathcal{M}_n$ be the relative measure of admissible parameters at step n. Then*

$$\prod_{n=6}^{\infty} \mathcal{M}_n > 1.58382 * 10^{-16}. \tag{2.262}$$

Proof of theorem 2. From 2.5.3.2, we have that velocities of endpoints of $\delta^{(n)}$'s are less than $\epsilon_0 := 0.003$. By (1.8), we get

$$\frac{4}{1+4\epsilon_0}|\delta^{(n)}(t)| < |\mathcal{T}(\delta^{(n)})| < \frac{4}{1-4\epsilon_0}|\delta^{(n)}(t)| \text{ for all } t \in \mathcal{T}(\delta^{(n)}) \tag{2.263}$$

and

$$\frac{4}{1+4\epsilon_0}|\Delta^{(n-1)}(t)| < |\mathcal{T}^{(n-1)}| < \frac{4}{1-4\epsilon_0}|\Delta^{(n-1)}(t)| \text{ for all } t \in \mathcal{T}^{(n-1)} \tag{2.264}$$

Let $t_{\text{top}}^{(n-1)}$ be the top value of $\mathcal{T}^{(n-1)}$. From lemma 19, we get that for any non-admissible domain $\delta^{(n)} \subset \Delta^{(n-1)}$ and $t \in \mathcal{T}(\delta^{(n)})$, we have

$$\left|\delta^{(n)}(t)\right| \leq \left|\delta^{(n)}(t_{\text{top}}^{(n-1)})\right| * exp\left(\left(\max_{t\in\mathcal{T}^{(n-1)}} \max_{z\in\mathcal{G}_{(n),i}(\delta^{(n)}(t))} \frac{\left|\frac{\partial^2\mathcal{G}_{(n),i}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\mathcal{G}_{(n),i}^{-1}}{\partial z}\right|}\right)\cdot\left|\mathcal{T}^{(n-1)}\right|\right). \tag{2.265}$$

From (2.263), (2.264), and (2.265), we get

$$\begin{aligned}
\frac{\left|\bigcup_i \mathcal{T}(\delta_i^{(n)})\right|}{\left|\mathcal{T}^{(n-1)}\right|} &= \sum_i \frac{\left|\mathcal{T}(\delta_i^{(n)})\right|}{\left|\mathcal{T}^{(n-1)}\right|} \\
&< \frac{\frac{1}{4}+\epsilon_0}{\frac{1}{4}-\epsilon_0} * \sum_i \frac{\left|\delta_i^{(n)}(t_i)\right|}{\left|\Delta^{(n-1)}(t_{\text{top}}^{(n-1)})\right|} \qquad t_i \in \mathcal{T}(\delta_i^{(n)}) \\
&< \frac{\frac{1}{4}+\epsilon_0}{\frac{1}{4}-\epsilon_0} * exp\left(\left(\max_{t\in\mathcal{T}^{(n-1)}} \max_{z\in\mathcal{G}_{(n),i}(\delta^{(n)}(t))} \frac{\left|\frac{\partial^2\mathcal{G}_{(n),i}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\mathcal{G}_{(n),i}^{-1}}{\partial z}\right|}\right)\cdot\left|\mathcal{T}^{(n-1)}\right|\right) * \sum_i \frac{\left|\delta_i^{(n)}(t_{\text{top}}^{(n-1)})\right|}{\left|\Delta^{(n-1)}(t_{\text{top}}^{(n-1)})\right|} \\
&= \frac{\frac{1}{4}+\epsilon_0}{\frac{1}{4}-\epsilon_0} * exp\left(\left(\max_{t\in\mathcal{T}^{(n-1)}} \max_{z\in\mathcal{G}_{(n),i}(\delta^{(n)}(t))} \frac{\left|\frac{\partial^2\mathcal{G}_{(n),i}^{-1}}{\partial t\partial z}\right|}{\left|\frac{\partial\mathcal{G}_{(n),i}^{-1}}{\partial z}\right|}\right)\cdot\left|\mathcal{T}^{(n-1)}\right|\right) * \frac{\left|\bigcup_i \delta_i^{(n)}(t_{\text{top}}^{(n-1)})\right|}{\left|\Delta^{(n-1)}(t_{\text{top}}^{(n-1)})\right|}.
\end{aligned}$$

Using estimates from (2.155) and (2.179), we get (2.259). □

That finishes the proof of the main theorem except for the summability condition.

2.7 Summability condition

According to section 1.2.5, we need to show the summablility condition (1.4) for the power maps of f_t constructed through the given algorithm. Then we can conclude that f_t has an a.c.i.m. given by (1.5) for $t \in \bigcap_n \left(\cup \mathcal{T}^{(n)}\right)$. Let us define the following notations:

- $N_x(k)$: the maximum number of iterates of branches in ξ_k.
- $\Delta\bar{N}_y(k)$: the maximum increase in the number of iterates of branches defined on the y-axis above y_{k-1}, at step k.
- $\Delta\bar{N}_x(k)$: the maximum increase in the number of iterates of branches defined on the x-axis inside δ^{re}_{k-1}, at step k.
- $\Delta\bar{\bar{N}}_x(k)$: the maximum increase in the number of iterates of branches defined on the x-axis outside δ^{re}_{k-1}, at step k.

The maximum number of iterates for initial partitions are calculated directly to be

$$\begin{aligned} N_y(0) &\leq 4 \\ N_x(0) &\leq 5 \\ N_y(5) &\leq 18 \\ N_x(5) &\leq 19 \end{aligned} \tag{2.266}$$

For general n, we have the following lemma.

Lemma 20. *Given $0 < \epsilon_1 < 1$ there is a constant N_{ϵ_1} such that*

$$N_x(n) \leq N_{\epsilon_1} * (1 + \epsilon_1)^n \tag{2.267}$$

for all $n \geq 6$.

Proof. Fix ϵ_1. Assume the inductive assumption that for $k \leq K-1$, we have

$$N_x(k) \leq N_{\epsilon_1} * (1+\epsilon_1)^k, \tag{2.268}$$

where N_{ϵ_1} is to be chosen later. We will show (2.268) for $k = K$.

By construction, we pullback elements of partition $\xi_{[\frac{k}{3}]}$ into $\zeta^{(k-1)}(\Delta^{(k-1)})$ at step k. According to 2.4.4, the worst possible cases of maps g's on domains above y_{k-1} are

$g = f_{[\frac{k}{3}]+2,j} \circ f_{0,1} \circ f_{0,1} \circ f_{[\frac{k}{3}],i_s} \circ f_{0,i_{s-1}} \circ \cdots \circ f_{0,i_1} \circ g_{k-1,i}$ and

$g = f_{[\frac{k}{3}]+2,j} \circ f_{0,1} \circ f_{0,1} \circ f_{[\frac{k}{3}],i_2} \circ f_{5,i_1} \circ g_{k-1,i}$.

Therefore, the change in number of iterates above y_{k-1} is given by the maximum possible sum of the number of iterates of the maps which we compose with.

$$\begin{aligned}\Delta\bar{N}_y(k) &< N_x\left(\left[\frac{k}{3}\right]+2\right) + 2*2 + N_x\left(\left[\frac{k}{3}\right]\right) + \max\left\{4*N_x\left(0\right), N_x\left(5\right)\right\} \\ &< N_x\left(\left[\frac{k}{3}\right]+2\right) + 4 + N_x\left(\left[\frac{k}{3}\right]\right) + 20. \end{aligned}\tag{2.269}$$

for any step k. By (2.268) and (2.269) we get,

$$\begin{aligned}\Delta\bar{N}_y(K) &< N_{\epsilon_1} * (1+\epsilon_1)^{[\frac{K}{3}]+2} + 4 + N_{\epsilon_1} * (1+\epsilon_1)^{[\frac{K}{3}]} + 20 \\ &\leq N_{\epsilon_1} * \left(\frac{1}{(1+\epsilon_1)^{\frac{2(K-1)}{3}-[\frac{K}{3}]-2}} + \frac{1}{(1+\epsilon_1)^{\frac{2(K-1)}{3}-[\frac{K}{3}]}} + \frac{24}{(1+\epsilon_1)^{\frac{2(K-1)}{3}}}\right) * (1+\epsilon_1)^{\frac{2(K-1)}{3}}\end{aligned}\tag{2.270}$$

We choose K_0 sufficiently large so that

$$\left(\frac{1}{(1+\epsilon_1)^{\frac{2(K_0-1)}{3}-[\frac{K_0}{3}]-2}} + \frac{1}{(1+\epsilon_1)^{\frac{2(K_0-1)}{3}-[\frac{K_0}{3}]}} + \frac{24}{(1+\epsilon_1)^{\frac{2(K_0-1)}{3}}}\right) < \epsilon_1 \tag{2.271}$$

Then for $K \geq K_0$

$$\Delta\bar{N}_y(K) < N_{\epsilon_1} * (1+\epsilon_1)^{\frac{2(K-1)}{3}}. \tag{2.272}$$

Since the parabolic pullback of $\zeta^{(k)}(\Delta^{(k)})$ onto I includes all partitions of and in fact more partitions than ξ_k, we have

$$\Delta\bar{N}_x(K) \leq \Delta\bar{N}_y(K) < N_{\epsilon_1} * (1+\epsilon_1)^{\frac{2(K-1)}{3}}. \tag{2.273}$$

Outside δ^{re}_{K-1}, the increase of iterates comes from the 1-step or 5-step filling-in on each hole. When we fill-in a hole δ_i^{-p} that is the preimage of δ_i^{re}, $i = 0$ or $5 \le i \le K-1$, the increase of the number of iterates will be no more than $\Delta\bar{N}_x(i)$. Therefore, the worst cases for the increase in the number of iterates would be when we fill-in holes that are preimages of δ^{re}_{K-1}. This gives

$$\Delta\bar{\bar{N}}_x(K) \le \Delta\bar{N}_x(K-1) < N_{\epsilon_1} * (1+\epsilon_1)^{\frac{2(K-2)}{3}}. \tag{2.274}$$

Since $\max\{\Delta\bar{\bar{N}}_x(K), \Delta\bar{N}_x(K)\}$ will provided an upper bound for the maximum increase of iterates for any branch created on the x-axis in step K, we have from (2.272), (2.273) and (2.271) that

$$\begin{aligned} N_x(K) &\le N_x(K-1) + \max\{\Delta\bar{\bar{N}}_x(K), \Delta\bar{N}_x(K)\} \\ &\le N_{\epsilon_1} * (1+\epsilon_1)^{K-1} + N_{\epsilon_1} * (1+\epsilon_1)^{\frac{2(K-1)}{3}} \\ &\le N_{\epsilon_1} * (1+\epsilon_1)^{K-1} * \left(1 + \frac{1}{(1+\epsilon_1)^{K-1-\frac{2(K-1)}{3}}}\right) \\ &\le N_{\epsilon_1} * (1+\epsilon_1)^{K} \end{aligned} \tag{2.275}$$

for $K \ge K_0$. If we set $N_{\epsilon_1} := N_x(K_0)$, then (2.268) will hold for all K. $\square$

Since monotone branches in $\delta^{\text{re}}_{k-1} \backslash \delta^{\text{re}}_k$ of ξ_k will not change after step k, monotone branches of the limiting power map with power greater than $N_x(k)$ has domain inside δ^{re}_k. Combining this with (2.150), we get

$$\sum_i n_i\,|I_i| < N_y(0) * |I| + \sum_{k=5}^{\infty} N_y(k) * \mu_{\text{holes}}(\xi_k) \tag{2.276}$$

$$\le N_y(0) * |I| + \sum_{k=5}^{\infty} (1+\epsilon_1)^k * 0.000210601 * (0.73)^{k-14} \tag{2.277}$$

As ϵ_1 can be chosen to be arbitrarily small, we choose

$$(1+\epsilon_1) * 0.73 < 1.$$

Then $\sum_i n_i |I_i|$ converges.

2.7.0.1 Decay of correlations

As a consequence of lemma 20, we have decay of correlations at polynomial rate.

Lemma 21. *For any $p > 0$, there is some K_p, such that for any $K \geq K_p$, the measure of monotone domains in the power maps constructed by our algorithm with the number of iterates of the original map greater than K is less than $C\frac{1}{K^p}$ for some fixed constant $C = C(p)$.*

Proof. From lemma 20, we have for arbitrarily small ϵ an N_ϵ such that the maximum number of iterates of f_t of branches in ξ_n is less than $N_\epsilon * (1+\epsilon)^n$ for all n. Choose ϵ_p so that $0.73 * (\epsilon_p + 1)^p < 1$. Then choose n_p so that $N_{\frac{\epsilon_p}{2}} * (1+\frac{\epsilon_p}{2})^{n_p} < (1+\epsilon_p)^{n_p}$. Let $K_p = N_{\frac{\epsilon_p}{2}} * (1+\frac{\epsilon_p}{2})^{n_p}$. For any $K \geq K_p$, we have one of $K = [N_{\frac{\epsilon}{2}}(1+\frac{\epsilon_p}{2})^n]+1$, $K = [N_{\frac{\epsilon}{2}}(1+\frac{\epsilon_p}{2})^n]+2$, $\cdots$, or $K = [N_{\frac{\epsilon_p}{2}}(1+\frac{\epsilon_p}{2})^{n+1}]$, for some $n > n_p$, which means $N_{\frac{\epsilon}{2}}(1+\frac{\epsilon_p}{2})^n \leq K \leq N_{\frac{\epsilon}{2}}(1+\frac{\epsilon_p}{2})^{n+1}$ for some $n > n_p$. The measure of domains with the maximum number of iterates greater than K will be less than

$$
\begin{aligned}
& C_1 * (0.73)^n \\
<& C_1 * \frac{1}{(1+\epsilon_p)^{np}} \\
<& C_1 * \left(\frac{1}{N_{\frac{\epsilon_p}{2}} * (1+\frac{\epsilon_p}{2})^n}\right)^p \\
=& C_1 * \left(N_{\frac{\epsilon_p}{2}} * (1+\frac{\epsilon_p}{2})\right)^p * \left(\frac{1}{N_{\frac{\epsilon_p}{2}} * (1+\frac{\epsilon_p}{2})^{n+1}}\right)^p \\
\leq& \frac{C_1 * \left(N_{\frac{\epsilon_p}{2}}(1+\frac{\epsilon_p}{2})\right)^p}{K^p},
\end{aligned}
\tag{2.278}
$$

where $C_1 = 0.000210601$. Letting $C = C_1 * \left(N_{\frac{\epsilon_p}{2}}(1+\frac{\epsilon_p}{2})\right)^p$ proves the claim. $\square$

By the theorem of L-S Young [14], lemma 21 implies polynomial decay of correlations. As mentioned in [8], there exists parameter values in construction such as the one explained here that the decay of correlations is slower than exponential decay.

Appendix A

A.1 Distortion estimates

Let χ be a diffeomorphism that maps the interval Y onto the interval X. Let $Y = Y_1 \cup Y_2$ be a partition of Y, $X_1 = \chi(Y_1)$, and $X_2 = \chi(Y_2)$. Suppose $\frac{|X_1|}{|X_2|} = \alpha$ and $\frac{|Y_1|}{|Y_2|} = k\alpha$. If there is some constant $\mathcal{D}$ such that $\frac{D\chi(y_1)}{D\chi(y_2)} \leq \mathcal{D}$ for all $y_1,\ y_2 \in Y$, then $\frac{1}{\mathcal{D}} \leq k \leq \mathcal{D}$, which gives

$$\frac{|Y_1|}{|Y|} = \frac{k\alpha}{1+k\alpha} \leq \frac{\mathcal{D}\alpha}{1+\mathcal{D}\alpha}. \tag{A.1}$$

and

$$\frac{|Y_1|}{|Y|} = \frac{k\alpha}{1+k\alpha} \geq \frac{\frac{1}{\mathcal{D}}\alpha}{1+\frac{1}{\mathcal{D}}\alpha}. \tag{A.2}$$

On the other hand, if $\frac{|X_1|}{|X|} = \gamma$, then $\frac{|X_1|}{|X_2|} = \frac{\gamma}{1-\gamma}$. From (A.1) we obtain

$$\frac{|Y_1|}{|Y|} \leq \frac{\mathcal{D}(\frac{\gamma}{1-\gamma})}{1+\mathcal{D}(\frac{\gamma}{1-\gamma})} = \frac{\mathcal{D}\gamma}{(1-\gamma)+\mathcal{D}\gamma} \tag{A.3}$$

and

$$\frac{|Y_1|}{|Y|} \geq \frac{\frac{1}{\mathcal{D}}(\frac{\gamma}{1-\gamma})}{1+\frac{1}{\mathcal{D}}(\frac{\gamma}{1-\gamma})} = \frac{\gamma}{\mathcal{D}(1-\gamma)+\gamma}. \tag{A.4}$$

A.2 Minimizing distorted ratios I

We frequently use the following technique for obtaining the best (smallest) ratio when taking into account distortion bounds. Suppose χ is a diffeomorphism that maps the interval Y onto the interval X. Moreover χ can be extended to a diffeomorphism $\tilde{\chi}$ from $\tilde{Y} \supset Y$ onto $\tilde{X} \supset X$. If $\tilde{X}$ is a τ-neighborhood of X, then from (1.3) , we get $\frac{D\chi(y_1)}{D\chi(y_2)} \leq (1+\frac{1}{\tau})^2 =: \mathcal{D}$. Suppose there is a domain $\delta_X \subset X$ such that $\frac{|\delta_X|}{|X|} = \gamma$, then to

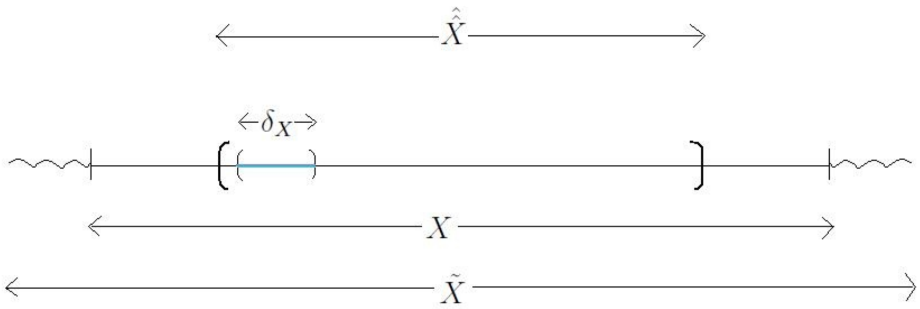


Figure A.1: Minimizing distorted ratio by adjusting the intermediate domain

estimate $\frac{|\chi^{-1}(\delta_x)|}{|Y|}$, we can use (A.3) and get the upper bound $\frac{\gamma\cdot\mathcal{D}}{1-\gamma+\gamma\cdot\mathcal{D}}$. Or, we can pick an intermediate domain $\hat{\hat{X}} = [z_1, z_2]$ such that $\delta_X \subset \hat{\hat{X}} \subset X$. This will give a new extension constant

$$\tau' = \frac{\min\{|\text{left component of } \tilde{X}\backslash\hat{\hat{X}}|, |\text{right component of } \tilde{X}\backslash\hat{\hat{X}}|\}}{|\hat{\hat{X}}|}. \tag{A.5}$$

The new distortion bound given by (1.3) is

$$\mathcal{D}' = (1+\frac{1}{\tau'})^2 = \left(1 + \frac{|\hat{\hat{X}}|}{\min\{|\text{left component of } \tilde{X}\backslash\hat{\hat{X}}|, |\text{right component of } \tilde{X}\backslash\hat{\hat{X}}|\}}\right)^2 \tag{A.6}$$

By (A.3), we get

$$\frac{|\chi^{-1}(\delta_x)|}{|Y|} < \frac{\frac{|\delta_X|}{|\hat{\hat{X}}|}\cdot\mathcal{D}'}{1-\frac{|\delta_X|}{|\hat{\hat{X}}|}+\frac{|\delta_X|}{|\hat{\hat{X}}|}\cdot\mathcal{D}'}. \tag{A.7}$$

We can adjust $\hat{\hat{X}}$ so that $\frac{\frac{|\delta_X|}{|\hat{\hat{X}}|}\cdot\mathcal{D}'}{1-\frac{|\delta_X|}{|\hat{\hat{X}}|}+\frac{|\delta_X|}{|\hat{\hat{X}}|}\cdot\mathcal{D}'}$ is minimized.

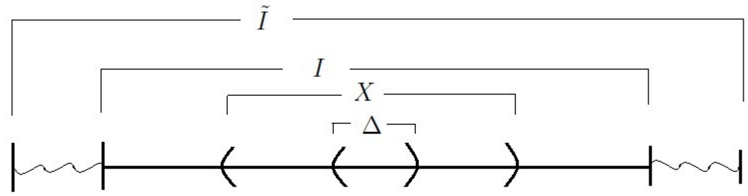


Figure A.2: Minimizing distorted ratio by repeatedly choosing intermediate domains

A.3 Minimizing distorted ratios II

On the basis of A.2, we can improve the estimate for distorted ratios even more. Define $\mathcal{D}_{X \text{ over } \tilde{X}}$ as the upper bound of the distortion on X when extension is $\tilde{X}$ given by the Koebe distortion principle. Then

$$\text{distorted ratio of } \frac{|\Delta|}{|I|} = \frac{\left|\mathcal{F}^{-1}(\Delta)\right|}{|\mathcal{F}^{-1}(I)|} \leq \frac{|X|}{|I|} * \mathcal{D}_{I \text{ over } \tilde{I}} * \frac{|\Delta|}{|X|} * \mathcal{D}_{X \text{ over } \tilde{I}} \leq \frac{|\Delta|}{|I|} * \mathcal{D}_{I \text{ over } \tilde{I}} \tag{A.8}$$

Therefore defining intermediate intervals gives better bounds.

A.4 Simple arithmetic

This is very simple arithmetic, but we use it many times so we write it down here to simplify the calculations in the text. Let $0 \leq A \leq A' < 1$ and $0 \leq \chi \leq \chi' < 1$, then

$$\begin{aligned} A + (1 - A)\,\chi =& A\,(1 - \chi) + \chi \\ \leq& A'\,(1 - \chi) + \chi \\ =& A' + (1 - A')\chi \\ <& A' + (1 - A')\chi' \end{aligned} \tag{A.9}$$

Let $0 \leq A \leq A' < 1$, $0 \leq \chi \leq \chi' < 1$, $0 \leq \psi \leq \psi' < 1$ and $\chi' < \psi'$, then

$$\begin{aligned} A\psi + (1-A)\,\chi =& A\psi' + (1-A)\,\chi' \\ =& A\left(\psi' - \chi'\right) + \chi' \\ \leq& A'\left(\psi' - \chi'\right) + \chi' \\ <& A'\psi' + (1-A')\chi' \end{aligned} \tag{A.10}$$

Appendix B

All estimates here are obtained using Mathematica. Most estimates are obtained for parameter values approximately at the two endpoints of $\mathcal{T}_0 = [t_{\text{bottom}}, t_{\text{top}}]$. This is sufficient because from graphing these values as functions of t, we observe that the graphs are monotone.

B.1 Estimates for ξ_0 and ξ_5

Since ξ_0 and ξ_5 are symmetric partitions, we only provide estimates for the first half of the domains.

B.1.1 Relative sizes of domains

Table B.1: Relative sizes of domains in ξ_5

t	t_{bottom}	t_{top}
$\frac{\lvert\Delta_1\rvert}{\lvert I\rvert}$	0.2427319087	0.2427306095
$\frac{\lvert\Delta_2\rvert}{\lvert I\rvert}$	0.1309998911	0.1309975736
$\frac{\lvert\Delta_3\rvert}{\lvert I\rvert}$	0.07065822374	0.07065293974
$\frac{\lvert\Delta_4\rvert}{\lvert I\rvert}$	0.01004307132	0.01004097488
$\frac{\lvert\Delta_5\rvert}{\lvert I\rvert}$	0.005404021765	0.005402410542
$\frac{\lvert\Delta_6\rvert}{\lvert I\rvert}$	0.002998582113	0.002997323711
$\frac{\lvert\delta_0^{-1}\rvert}{\lvert I\rvert}$	0.004953891000	0.004952576925
$\frac{\lvert\Delta_7\rvert}{\lvert I\rvert}$	0.003382907318	0.003380821552
$\frac{\lvert\Delta_8\rvert}{\lvert I\rvert}$	0.007167250156	0.007161326857
$\frac{\lvert\Delta_9\rvert}{\lvert I\rvert}$	0.004271401491	0.004265621070
$\frac{\lvert\Delta_{(10)}\rvert}{\lvert I\rvert}$	0.002515416726	0.002510447359
$\frac{\lvert\Delta_{(11)}\rvert}{\lvert I\rvert}$	0.001493126335	0.001489215309
$\frac{\lvert\delta_0^{-1}\rvert}{\lvert I\rvert}$	0.002695390798	0.002686328996
$\frac{\lvert\Delta_{(12)}\rvert}{\lvert I\rvert}$	0.002105055444	0.002093411296
$\frac{\lvert\Delta_{(13)}\rvert}{\lvert I\rvert}$	0.001201827266	0.001192314156
$\frac{\lvert\Delta_{(14)}\rvert}{\lvert I\rvert}$	0.0006818043795	0.0006749125945

$\frac{\lvert\Delta_{(15)}\rvert}{\lvert I\rvert}$	0.0003898465847	0.0003852279662
$\frac{\lvert\delta_0^{-1}\rvert}{\lvert I\rvert}$	0.0006620850197	0.0006529792462
$\frac{\lvert\Delta_{(16)}\rvert}{\lvert I\rvert}$	0.0004642227615	0.0004563669112
$\frac{\lvert\Delta_{(17)}\rvert}{\lvert I\rvert}$	0.001006865975	0.0009841148679
$\frac{\lvert\Delta_{(18)}\rvert}{\lvert I\rvert}$	0.0006018800761	0.0005824543224
$\frac{\lvert\Delta_{(19)}\rvert}{\lvert I\rvert}$	0.0003466195253	0.0003323382542
$\frac{\lvert\Delta_{(20)}\rvert}{\lvert I\rvert}$	0.0001999401332	0.0001903082403
$\frac{\lvert\delta_0^{-1}\rvert}{\lvert I\rvert}$	0.0003429122408	0.0003234190773
$\frac{\lvert\Delta_{(21)}\rvert}{\lvert I\rvert}$	0.0002434628705	0.0002265727424
$\frac{\lvert\Delta_{(22)}\rvert}{\lvert I\rvert}$	0.0005405182143	0.0004898067834
$\frac{\lvert\Delta_{(23)}\rvert}{\lvert I\rvert}$	0.0003363503340	0.0002905431268
$\frac{\lvert\Delta_{(24)}\rvert}{\lvert I\rvert}$	0.0002015655023	0.0001659570049
$\frac{\lvert\Delta_{(25)}\rvert}{\lvert I\rvert}$	0.0001202404555	0.00009508442368
$\frac{\lvert\delta_0^{-1}\rvert}{\lvert I\rvert}$	0.0002164648905	0.0001616696455
$\frac{\lvert\Delta_{(26)}\rvert}{\lvert I\rvert}$	0.0001660506337	0.0001133122945
$\frac{\lvert\Delta_{(27)}\rvert}{\lvert I\rvert}$	0.0004734182300	0.0002450917575
$\frac{\lvert\delta_5\rvert}{\lvert I\rvert}$	0.0007675737511	0.002151890379

Figure B.1.1 graphs the relative measure of holes in ξ_5 restricted to $I\backslash\delta_5$ as a function of t.

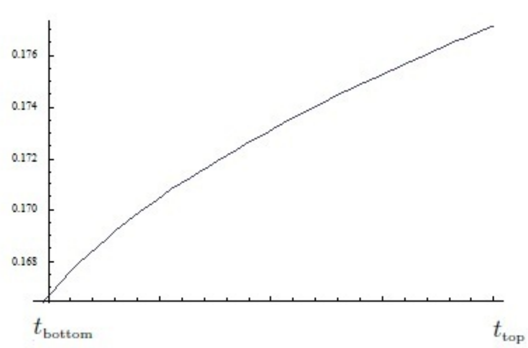


Figure B.1: Relative measure of holes in η_0 as a function of parameter t

B.1.2 Derivatives

By property of functions with negative Schwarzian derivative, the minimum of the absolute value of the derivative occurs on the endpoints.

Table B.2: Minimum derivatives of monotone branches in ξ_5

t	t_{bottom}	t_{top}
$\min_{x\in\Delta_1} \left\| \frac{\partial f_{5,1}}{\partial x} \right\|$	3.550344958	3.550374917
$\min_{x\in\Delta_2} \left\| \frac{\partial f_{5,2}}{\partial x} \right\|$	6.723459232	6.723682199
$\min_{x\in\Delta_3} \left\| \frac{\partial f_{5,3}}{\partial x} \right\|$	11.72819466	11.73013718
$\min_{x\in\Delta_4} \left\| \frac{\partial f_{5,4}}{\partial x} \right\|$	86.87310503	86.89533073
$\min_{x\in\Delta_5} \left\| \frac{\partial f_{5,5}}{\partial x} \right\|$	160.5497500	160.6061824
$\min_{x\in\Delta_6} \left\| \frac{\partial f_{5,6}}{\partial x} \right\|$	272.1965563	272.3434811
first hole		
$\min_{x\in\Delta_7} \left\| \frac{\partial f_{5,7}}{\partial x} \right\|$	253.2091857	253.3735781
$\min_{x\in\Delta_8} \left\| \frac{\partial f_{5,8}}{\partial x} \right\|$	115.0400218	115.1652419
$\min_{x\in\Delta_9} \left\| \frac{\partial f_{5,9}}{\partial x} \right\|$	193.6785762	194.0053214
$\min_{x\in\Delta_{(10)}} \left\| \frac{\partial f_{5,(10)}}{\partial x} \right\|$	331.2450544	332.0156282
$\min_{x\in\Delta_{(11)}} \left\| \frac{\partial f_{5,(11)}}{\partial x} \right\|$	531.1810758	532.7770510
second hole		
$\min_{x\in\Delta_{(12)}} \left\| \frac{\partial f_{5,(12)}}{\partial x} \right\|$	402.7311018	405.4987128
$\min_{x\in\Delta_{(13)}} \left\| \frac{\partial f_{5,(13)}}{\partial x} \right\|$	701.9768611	708.4823797
$\min_{x\in\Delta_{(14)}} \left\| \frac{\partial f_{5,(14)}}{\partial x} \right\|$	1248.211624	1262.263100
$\min_{x\in\Delta_{(15)}} \left\| \frac{\partial f_{5,(15)}}{\partial x} \right\|$	2071.551280	2098.096375
third hole		
$\min_{x\in\Delta_{(16)}} \left\| \frac{\partial f_{5,(16)}}{\partial x} \right\|$	1863.962889	1893.613157
$\min_{x\in\Delta_{(17)}} \left\| \frac{\partial f_{5,(17)}}{\partial x} \right\|$	812.6257973	835.8831673
$\min_{x\in\Delta_{(18)}} \left\| \frac{\partial f_{5,(18)}}{\partial x} \right\|$	1388.217406	1442.191933
$\min_{x\in\Delta_{(19)}} \left\| \frac{\partial f_{5,(19)}}{\partial x} \right\|$	2441.862586	2557.873810
$\min_{x\in\Delta_{(20)}} \left\| \frac{\partial f_{5,(20)}}{\partial x} \right\|$	4025.403248	4242.651494
fourth hole		
$\min_{x\in\Delta_{(21)}} \left\| \frac{\partial f_{5,(21)}}{\partial x} \right\|$	3574.163547	3817.617767
$\min_{x\in\Delta_{(22)}} \left\| \frac{\partial f_{5,(22)}}{\partial x} \right\|$	1481.907922	1676.960973
$\min_{x\in\Delta_{(23)}} \left\| \frac{\partial f_{5,(23)}}{\partial x} \right\|$	2429.255414	2889.129726
$\min_{x\in\Delta_{(24)}} \left\| \frac{\partial f_{5,(24)}}{\partial x} \right\|$	4117.551441	5120.475467
$\min_{x\in\Delta_{(25)}} \left\| \frac{\partial f_{5,(25)}}{\partial x} \right\|$	6593.336572	8489.935758
fifth hole		
$\min_{x\in\Delta_{(26)}} \left\| \frac{\partial f_{5,(26)}}{\partial x} \right\|$	5185.817708	7634.923726
$\min_{x\in\Delta_{(27)}} \left\| \frac{\partial f_{5,(27)}}{\partial x} \right\|$	1194.970643	3350.229588
δ_5		

Table B.3: Minimum derivatives of maps on holes in ξ_5

t	t_{bottom}	t_{top}
$\min_{x\in\text{first hole}} \left\lvert\frac{\partial\mathcal{F}_{5,1}}{\partial x}\right\rvert$	21.5	21.5
$\min_{x\in\text{second hole}} \left\lvert\frac{\partial\mathcal{F}_{5,2}}{\partial x}\right\rvert$	37	37
$\min_{x\in\text{third hole}} \left\lvert\frac{\partial\mathcal{F}_{5,3}}{\partial x}\right\rvert$	159	160
$\min_{x\in\text{fourth hole}} \left\lvert\frac{\partial\mathcal{F}_{5,4}}{\partial x}\right\rvert$	300	325
$\min_{x\in\text{fifth hole}} \left\lvert\frac{\partial\mathcal{F}_{5,5}}{\partial x}\right\rvert$	460	650

B.1.3 Velocities

This is for $t \approx t_{\text{bottom}}$

Table B.4: Velocities compared with ratio of derivatives of endpoints of monotone domains for the bottom parameter

	$\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial x}}$		$\frac{dx(t)}{dt}$	
Δ	left endpoint of Δ	right endpoint of Δ	left endpoint of Δ	right endpoint of Δ
Δ_1	0.04691305788	0.2098046492	-0.06265274890	-0.1921578221
Δ_2	0.1833134070	0.4703299944	-0.1921582611	-0.4610116203
Δ_3	0.4563422466	1.107998824	-0.4610125742	-1.102657049
Δ_4	1.101991496	1.360261390	-1.102663703	-1.359539846
Δ_5	1.359181607	1.557697813	-1.359542576	-1.557307073
Δ_6	1.557115130	1.712355790	-1.557310194	-1.712124748
first hole				
Δ_7	1.873180686	2.129928542	-1.873426996	-2.129696013
Δ_8	2.129229951	2.859278703	-2.129691727	-2.858733312
Δ_9	2.858467109	3.571474464	-2.858739036	-3.571149860
$\Delta_{(10)}$	3.570996072	4.184574665	-3.571156987	-4.184384000
$\Delta_{(11)}$	4.184298808	4.668154514	-4.184392301	-4.668033953
second hole				
$\Delta_{(12)}$	5.789089567	7.233381363	-5.789230595	-7.233223494
$\Delta_{(13)}$	7.233162025	8.417863700	-7.233237643	-8.417771406
$\Delta_{(14)}$	8.417745193	9.278989877	-8.417787505	-9.278935629
$\Delta_{(15)}$	9.278929767	9.859145529	-9.278952827	-9.859107579
third hole				
$\Delta_{(16)}$	11.00251042	11.99280134	-11.00253953	-11.99276388
$\Delta_{(17)}$	11.99267607	14.88960255	-11.99273627	-14.88951968
$\Delta_{(18)}$	14.88951209	17.39938947	-14.88954706	-17.39933601

$\Delta_{(19)}$	17.39934704	19.27031749	-17.39936675	-19.27027655
$\Delta_{(20)}$	19.27030030	20.37224043	-19.27031292	-20.37293944
fourth hole				
$\Delta_{(21)}$	23.16206023	25.47200468	-23.16207435	-25.47191555
$\Delta_{(22)}$	25.47165138	32.70658570	-25.47168019	-32.70652074
$\Delta_{(23)}$	32.70656075	39.72259010	-32.70658044	-39.72242540
$\Delta_{(24)}$	39.72258387	45.56999831	-39.72262588	-45.57001621
$\Delta_{(25)}$	45.57001340	49.95686962	-45.57000665	-49.95689000
fifth hole				
$\Delta_{(26)}$	60.41266814	71.93685307	-60.41265954	-71.93687061
$\Delta_{(27)}$	71.93685973	155.5888650	-71.93692639	-155.5889281
δ_5				

Table B.5: Velocities compared with ratio of derivatives of endpoints of holes for the bottom parameter

	$\frac{\frac{\partial \mathcal{F}}{\partial t}}{\frac{\partial \mathcal{F}}{\partial x}}$	
δ_0^{-1}	left endpoint of δ_0^{-1}	right endpoint of δ_0^{-1}
first hole	1.65	2
second hole	4.5	5.8
third hole	9.8	11
fourth hole	20.1	23.2
fifth hole	50	61

This is for $t \approx t_{\text{top}}$

Table B.6: Velocities compared with ratio of derivatives of endpoints of monotone domains for the top parameter

	$\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial x}}$		$\frac{dx(t)}{dt}$	
Δ	left endpoint of Δ	right endpoint of Δ	left endpoint of Δ	right endpoint of Δ
Δ_1	0.04691279046	0.2098029741	-0.06265246566	-0.1921562559
Δ_2	0.1833113595	0.4703147173	-0.1921562559	-0.4609965396
Δ_3	0.4563260782	1.107817863	-0.4609965396	-1.102476709
Δ_4	1.101804164	1.359917399	-1.102476709	-1.359196388
Δ_5	1.358835003	1.557156619	-1.359196388	-1.556766520
Δ_6	1.556570993	1.711441025	-1.556766520	-1.711210975
first hole				
Δ_7	1.871981055	2.128576243	-1.872228328	-2.128344475
Δ_8	2.127882066	2.856207363	-2.128344475	-2.855663340

Δ_9	2.855390665	3.565532308	-2.855663340	-3.565209366
$\Delta_{(10)}$	3.565047501	4.174979348	-3.565209366	-4.174790645
$\Delta_{(11)}$	4.174696063	4.654317395	-4.174790645	-4.654199799
second hole				
$\Delta_{(12)}$	5.763104191	7.184623098	-5.763247721	-7.184468590
$\Delta_{(13)}$	7.184391148	8.341520636	-7.184468590	-8.341432204
$\Delta_{(14)}$	8.341387880	9.176939224	-8.341432204	-9.176889589
$\Delta_{(15)}$	9.176864711	9.735422487	-9.176889589	-9.735392626
third hole				
$\Delta_{(16)}$	10.83272057	11.77543960	-10.83275366	-11.77540806
$\Delta_{(17)}$	11.77534514	14.48049458	-11.77540806	-14.48041962
$\Delta_{(18)}$	14.48038205	16.75635452	-14.48041962	-16.75631108
$\Delta_{(19)}$	16.75628930	18.40687523	-16.75631108	-18.40685074
$\Delta_{(20)}$	18.40683846	19.50827507	-18.40685074	-19.50826030
fourth hole				
$\Delta_{(21)}$	21.70555335	23.57312764	-21.70556976	-23.57311197
$\Delta_{(22)}$	23.57308070	28.95228316	-23.57311197	-28.95224580
$\Delta_{(23)}$	28.95222707	33.48332006	-28.95224580	-33.48329838
$\Delta_{(24)}$	33.48328751	36.77019997	-33.48329838	-36.77018774
$\Delta_{(25)}$	36.77018160	38.96205579	-36.77018774	-38.96204841
fifth hole				
$\Delta_{(26)}$	43.35079340	47.07072420	-43.35080160	-47.07071636
$\Delta_{(27)}$	47.07070072	57.79458380	-47.07071636	-57.79456510
δ_5				

Table B.7: Velocities compared with ratio of derivatives of endpoints of holes for the bottom parameter

	$\frac{\frac{\partial \mathcal{F}}{\partial t}}{\frac{\partial \mathcal{F}}{\partial x}}$	
δ_0^{-1}	left endpoint of δ_0^{-1}	right endpoint of δ_0^{-1}
first hole	1.65	2
second hole	4.5	5.8
third hole	9.7	10.9
fourth hole	19.5	21.7
fifth hole	39	43.5

For $t \approx t_{\mathrm{bottom}}$

Table B.8: Velocities compared with ratio of derivatives of endpoints of monotone domains for the bottom parameter on the y-axis

	$\frac{\frac{\partial g_5}{\partial t}}{\frac{\partial g_5}{\partial y}}$		$\frac{dy(t)}{dt}$	
Δ	lower endpoint of $\Delta^{(5)}$	upper endpoint of $\Delta^{(5)}$	lower endpoint of $\Delta^{(5)}$	upper endpoint of $\Delta^{(5)}$
	-0.00187040473	-0.00186108386	0.00187024454	0.00186124318

B.1.4 Variation of derivatives

Let $f_{5,i}$ be monotone branches in ξ_5, we obtain upper bounds for $\left|\frac{\frac{\partial}{\partial t}\frac{\partial f_{5,i}^{-1}}{\partial x}}{\frac{\partial f_{5,i}^{-1}}{\partial x}}\right|$ for x over the interval Δ and t over the parameter interval $\mathcal{T}_0$ as follows.

Table B.9: Upper bounds for mixed derivatives for monotone branches

Upper bounds for $\frac{\frac{\partial}{\partial t}\frac{\partial f_{5,i}^{-1}}{\partial x}}{\frac{\partial f_{5,i}^{-1}}{\partial x}}$												
domain Δ	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	δ_0^{-1}	Δ_7	Δ_8	Δ_9	$\Delta_{(10)}$	$\Delta_{(11)}$
Upper bound of $\frac{\frac{\partial}{\partial t}\frac{\partial f_{5,i}^{-1}}{\partial x}}{\frac{\partial f_{5,i}^{-1}}{\partial x}}$	2.2	8.5	41	65	87	135		160	250	420	600	750
domain Δ	δ_0^{-1}	$\Delta_{(12)}$	$\Delta_{(13)}$	$\Delta_{(14)}$	$\Delta_{(15)}$	δ_0^{-1}	$\Delta_{(16)}$	$\Delta_{(17)}$	$\Delta_{(18)}$	$\Delta_{(19)}$	$\Delta_{(20)}$	δ_0^{-1}
Upper bound of $\frac{\frac{\partial}{\partial t}\frac{\partial f_{5,i}^{-1}}{\partial x}}{\frac{\partial f_{5,i}^{-1}}{\partial x}}$		1700	2300	2800	3200		4700	7500	10000	12000	14000	
domain Δ	$\Delta_{(21)}$	$\Delta_{(22)}$	$\Delta_{(23)}$	$\Delta_{(24)}$	$\Delta_{(25)}$	δ_0^{-1}	$\Delta_{(26)}$	$\Delta_{(27)}$	δ_5			
Upper bound of $\frac{\frac{\partial}{\partial t}\frac{\partial f_{5,i}^{-1}}{\partial x}}{\frac{\partial f_{5,i}^{-1}}{\partial x}}$	21000	34000	55000	70000	82000		170000	900000				

Let $\mathcal{F}_{5,i}$'s map δ_0^{-1}'s to δ_0. δ_0^{-1}'s are the "five holes" in ξ_5.

Table B.10: Upper bounds for mixed derivatives for maps on holes

Upper bounds for $\frac{\frac{\partial}{\partial t}\frac{\partial \mathcal{F}_{5,i}^{-1}}{\partial x}}{\frac{\partial \mathcal{F}_{5,i}^{-1}}{\partial x}}$ **over the interval** δ_0^{-1}					
δ_0^{-1}	first hole	second hole	third hole	forth hole	fifth hole

Upper bounds for	$\frac{\frac{\partial}{\partial t}\frac{\partial \mathcal{F}_{5,i}^{-1}}{\partial x}}{\frac{\partial \mathcal{F}_{5,i}^{-1}}{\partial x}}$		125	1100	4000	17500	120000

For $t \approx t_{\text{top}}$,

Table B.11: Upper bounds for mixed derivatives for the initial monotone branch on the y-axis

$\frac{\frac{\partial}{\partial t}\frac{\partial g_5^{-1}}{\partial y}}{\frac{\partial g_5^{-1}}{\partial y}}$	
lower endpoint of $\Delta^{(5)}$	upper endpoint of $\Delta^{(5)}$
-8.9	-7.9

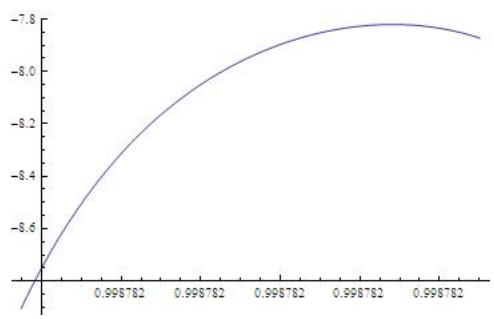


Figure B.2: Mixed derivative for z ranging over $\Delta^{(5)}$

Let $\mathcal{G}_{5,i}$'s map δ_0^{-1}'s to δ_0. δ_0^{-1}'s are the "five holes" in $\zeta^{(5)}$.

Table B.12: Upper bounds for mixed derivatives for the maps on holes on the y-axis

Upper bounds for $\frac{\frac{\partial}{\partial t}\frac{\partial \mathcal{G}_{5,i}^{-1}}{\partial x}}{\frac{\partial \mathcal{G}_{5,i}^{-1}}{\partial x}}$ **over the interval** δ_0^{-1}							
δ_0^{-1}			first hole	second hole	third hole	forth hole	fifth hole
Upper bound for	$\frac{\frac{\partial}{\partial t}\frac{\partial \mathcal{G}_{5,i}^{-1}}{\partial x}}{\frac{\partial \mathcal{G}_{5,i}^{-1}}{\partial x}}$		1.22	2.47	6.5	6.24	7.04

B.1.5 Bounds for initial partitions

This summarizes estimates for ξ_0 and ξ_5.

Table B.13: Overall bounds for derivatives for the initial maps

ξ_0	
lower bound for $\left\lvert \frac{\partial f_{0,i}}{\partial x} \right\rvert$	3.5
upper bound for $\frac{\left\lvert \frac{\partial f_{0,i}}{\partial t} \right\rvert}{\left\lvert \frac{\partial f_{0,i}}{\partial x} \right\rvert}$	1.109
upper bound for $\frac{\left\lvert \frac{\partial^2 f_{0,i}^{-1}}{\partial t \partial z} \right\rvert}{\left\lvert \frac{\partial f_{0,i}^{-1}}{\partial z} \right\rvert}$	50
upper bound for $\frac{\left\lvert \frac{\partial (f_{0,i_s} \circ \cdots \circ f_{0,i})}{\partial t} \right\rvert}{\left\lvert \frac{\partial f_{0,i_s} \circ \cdots \circ f_{0,i})}{\partial x} \right\rvert}$	$1.5527\ (1.109 * (1 + \frac{1}{3.5} + \frac{1}{3.5^2} + \cdots))$
upper bound for $\frac{\left\lvert \frac{\partial^2 (f_{0,i_s} \circ \cdots \circ f_{0,i})^{-1}}{\partial t \partial z} \right\rvert}{\left\lvert \frac{\partial (f_{0,i_s} \circ \cdots \circ f_{0,i})^{-1}}{\partial z} \right\rvert}$, $s \le 6$	200
ξ_5, $x - axis$	
lower bound for $\left\lvert \frac{\partial f_{5,i}}{\partial x} \right\rvert$	85 (this is for the 4th to the 32th domain)
upper bound for $\frac{\left\lvert \frac{\partial f_{5,i}}{\partial t} \right\rvert}{\left\lvert \frac{\partial f_{5,i}}{\partial x} \right\rvert}$	160
upper bound for $\frac{\left\lvert \frac{\partial^2 f_{5,i}^{-1}}{\partial t \partial z} \right\rvert}{\left\lvert \frac{\partial f_{5,i}^{-1}}{\partial z} \right\rvert}$	$900,000$
lower bound for $\left\lvert \frac{\partial \mathcal{F}_{5,i}}{\partial x} \right\rvert$	20
upper bound for $\frac{\left\lvert \frac{\partial^2 \mathcal{F}_{5,i}}{\partial x^2} \right\rvert}{\left\lvert \frac{\partial \mathcal{F}_{5,i}}{\partial x} \right\rvert^2}$	4
upper bound for $\frac{\left\lvert \frac{\partial \mathcal{F}_{5,i}}{\partial t} \right\rvert}{\left\lvert \frac{\partial \mathcal{F}_{5,i}}{\partial x} \right\rvert}$	50
upper bound for $\frac{\left\lvert \frac{\partial^2 \mathcal{F}_{5,i}^{-1}}{\partial t \partial z} \right\rvert}{\left\lvert \frac{\partial \mathcal{F}_{5,i}^{-1}}{\partial z} \right\rvert}$	$61,000$
ξ_5, $y - axis$	
lower bound for $\left\lvert \frac{\partial g_{(5)}}{\partial x} \right\rvert$	391005
upper bound for $\frac{\left\lvert \frac{\partial g_5}{\partial x}(x_0) \right\rvert}{\left\lvert \frac{\partial g_{(5)}}{\partial x}(y_0) \right\rvert}$	$\frac{15}{13} \approx 1.15385$
upper bound for $\frac{\left\lvert \frac{\partial^2 g_{(5)}}{\partial x^2} \right\rvert}{\left\lvert \frac{\partial g_{(5)}}{\partial x} \right\rvert^2}$	1.5
upper bound for $\frac{\left\lvert \frac{\partial g_{(5)}}{\partial t} \right\rvert}{\left\lvert \frac{\partial g_{(5)}}{\partial x} \right\rvert}$	0.0019
upper bound for $\frac{\left\lvert \frac{\partial^2 g_{(5)}^{-1}}{\partial t \partial z} \right\rvert}{\left\lvert \frac{\partial g_{(5)}^{-1}}{\partial z} \right\rvert}$	8.9
lower bound for $\left\lvert \frac{\partial \mathcal{G}_{5,i}}{\partial x} \right\rvert$	37

upper bound for $\frac{\left\lvert\frac{\partial \mathcal{G}_{5,i}}{\partial t}\right\rvert}{\left\lvert\frac{\partial \mathcal{G}_{5,i}}{\partial x}\right\rvert}$	0.0025
upper bound for $\frac{\left\lvert\frac{\partial^2 \mathcal{G}_{5,i}^{-1}}{\partial t \partial z}\right\rvert}{\left\lvert\frac{\partial \mathcal{G}_{5,i}^{-1}}{\partial z}\right\rvert}$	8

B.2 Extensions and refined extensions

The extensions of domains in ξ_0 will give the maximum number of boundary refinements needed. Values in this chart are upper bounds over all $t \in \mathcal{T}^{476777}$

Table B.14: Upper bounds for distorted ratios of sizes of extended domains to sizes of corresponding domains

r	d :=distortion on :	$d * r$
$\frac{\lvert\text{left component of } \tilde{\Delta}_1 \backslash \Delta_1\rvert}{\lvert\Delta_1\rvert} < 0.409908$	(left component of $\tilde{\Delta}_1 \backslash \Delta_1) \cup \Delta_1 < 5.85896$	2.40163
$\frac{\lvert\text{right component of } \tilde{\Delta}_1 \backslash \Delta_1\rvert}{\lvert\Delta_1\rvert} < 0.486451$	(right component of $\tilde{\Delta}_1 \backslash \Delta_1) \cup \Delta_1 < 4.24323$	2.07412
$\frac{\lvert\text{left component of } \tilde{\Delta}_{11} \backslash \Delta_{11}\rvert}{\lvert\Delta_1\rvert} < 0.105975$	(left component of $\tilde{\Delta}_{11} \backslash \Delta_{11}) \cup \Delta_1 < 3.43389$	0.363906
$\frac{\lvert\text{right component of } \tilde{\Delta}_{17} \backslash \Delta_{17}\rvert}{\lvert\Delta_1\rvert} < 0.120445$	(right component of $\tilde{\Delta}_{17} \backslash \Delta_{17}) \cup \Delta_1 < 3.23616$	0.389779
$\frac{\lvert\text{left component of } \tilde{\Delta}_{111} \backslash \Delta_{111}\rvert}{\lvert\Delta_1\rvert} < 0.0268014$	(left component of $\tilde{\Delta}_{111} \backslash \Delta_{111}) \cup \Delta_1 < 3.05$	0.0817443
$\frac{\lvert\text{right component of } \tilde{\Delta}_{177} \backslash \Delta_{177}\rvert}{\lvert\Delta_1\rvert} < 0.03016$	(right component of $\tilde{\Delta}_{177} \backslash \Delta_{177}) \cup \Delta_1 < 3.00866$	0.0907141
$\frac{\lvert\text{left component of } \tilde{\Delta}_2 \backslash \Delta_2\rvert}{\lvert\Delta_2\rvert} < 0.438737$	(left component of $\tilde{\Delta}_2 \backslash \Delta_2) \cup \Delta_2 < 1.84564$	0.809751
$\frac{\lvert\text{right component of } \tilde{\Delta}_2 \backslash \Delta_2\rvert}{\lvert\Delta_2\rvert} < 0.483555$	(right component of $\tilde{\Delta}_2 \backslash \Delta_2) \cup \Delta_2 < 1.77762$	0.859577
$\frac{\lvert\text{left component of } \tilde{\Delta}_{21} \backslash \Delta_{21}\rvert}{\lvert\Delta_2\rvert} < 0.111057$	(left component of $\tilde{\Delta}_{21} \backslash \Delta_{21}) \cup \Delta_2 < 1.57749$	0.175191
$\frac{\lvert\text{right component of } \tilde{\Delta}_{27} \backslash \Delta_{27}\rvert}{\lvert\Delta_2\rvert} < 0.117951$	(right component of $\tilde{\Delta}_{27} \backslash \Delta_{27}) \cup \Delta_2 < 1.56536$	0.184636
$\frac{\lvert\text{left component of } \tilde{\Delta}_{211} \backslash \Delta_{211}\rvert}{\lvert\Delta_2\rvert} < 0.027951$	(left component of $\tilde{\Delta}_{211} \backslash \Delta_{211}) \cup \Delta_2 < 1.51877$	0.0424511
$\frac{\lvert\text{right component of } \tilde{\Delta}_{277} \backslash \Delta_{277}\rvert}{\lvert\Delta_2\rvert} < 0.03016$	(right component of $\tilde{\Delta}_{277} \backslash \Delta_{277}) \cup \Delta_2 < 1.51604$	0.0447295
$\frac{\lvert\text{left component of } \tilde{\Delta}_3 \backslash \Delta_3\rvert}{\lvert\Delta_3\rvert} < 0.430055$	(left component of $\tilde{\Delta}_3 \backslash \Delta_3) \cup \Delta_3 < 1.31740$	0.566554
$\frac{\lvert\text{right component of } \tilde{\Delta}_3 \backslash \Delta_3\rvert}{\lvert\Delta_3\rvert} < 0.6639$	(right component of $\tilde{\Delta}_3 \backslash \Delta_3) \cup \Delta_3 < 1.35640$	0.900514
$\frac{\lvert\text{left component of } \tilde{\Delta}_{31} \backslash \Delta_{31}\rvert}{\lvert\Delta_3\rvert} < 0.108727$	(left component of $\tilde{\Delta}_{31} \backslash \Delta_{31}) \cup \Delta_3 < 1.23409$	0.134179
$\frac{\lvert\text{right component of } \tilde{\Delta}_{37} \backslash \Delta_{37}\rvert}{\lvert\Delta_3\rvert} < 0.127995$	(right component of $\tilde{\Delta}_{37} \backslash \Delta_{37}) \cup \Delta_3 < 1.23568$	0.158161
$\frac{\lvert\text{left component of } \tilde{\Delta}_{311} \backslash \Delta_{311}\rvert}{\lvert\Delta_3\rvert} < 0.0273644$	(left component of $\tilde{\Delta}_{311} \backslash \Delta_{311}) \cup \Delta_3 < 1.21425$	0.0332272
$\frac{\lvert\text{right component of } \tilde{\Delta}_{377} \backslash \Delta_{377}\rvert}{\lvert\Delta_3\rvert} < 0.031502$	(right component of $\tilde{\Delta}_{377} \backslash \Delta_{377}) \cup \Delta_3 < 1.15815$	0.036484

B.3

B.3.1 Primary ratios

Table B.15: Overall bounds for derivatives for the initial maps

Ratios on I		
t	t_{bottom}	t_{top}
$\frac{\lvert\Delta_{-1}\rvert}{\text{dist}(\Delta_{-1},q^{-1})}$	0.3205362640	0.3205340300
$\frac{\lvert\Delta_{-2}\rvert}{\text{dist}(\Delta_{-2},q^{-1})}$	0.2091753153	0.2091704753
$\frac{\lvert\Delta_{-3}\rvert}{\text{dist}(\Delta_{-3},q^{-1})}$	0.1271721842	0.1271607979
$\frac{\lvert\delta_0\rvert}{\text{dist}(\delta_0,q^{-1})}$	0.2502761679	0.2503206110
$\frac{\lvert\Delta_3\rvert}{\text{dist}(\Delta_3,q^{-1})}$	0.1890611386	0.1890490012
$\frac{\lvert\Delta_2\rvert}{\text{dist}(\Delta_2,q^{-1})}$	0.5396895251	0.5396829591

B.3.2 Selected ratios $\frac{|\Delta|}{H_5(\Delta)}$

Here, we let $\Delta_7 = \Delta_{-1}$, $\Delta_6 = \Delta_{-2}$, $\Delta_5 = \Delta_{-3}$ to show the order they appear on the y-axis. Other subscripts will also be given according to the order they appear on the y-axis. Let g_5 be the diffeomorphism that maps $\Delta^{(5)}$ onto I.

Table B.16: Ratio of domain sizes to partial of remaining domains

Ratios on I		
t	t_{bottom}	t_{top}
$\frac{\lvert\Delta_7\rvert}{\text{dist}(\Delta_7,q^{-1})}$	0.32053663	0.3205340300298
$\frac{\lvert\Delta_6\rvert}{\text{dist}(\Delta_6,q^{-1})}$	0.20917610	0.2091704753206
$\frac{\lvert\Delta_5\rvert}{\text{dist}(\Delta_5,q^{-1})}$	0.12717403	0.1271607979076
$\frac{\lvert\delta_0\rvert}{\text{dist}(\delta_0,q^{-1})}$	0.25026897	0.2503206110132
$\frac{\lvert\Delta_3\rvert}{\text{dist}(\Delta_3,q^{-1})}$	0.18906310	0.1890490011995
$\frac{\lvert\Delta_2\rvert}{\text{dist}(\Delta_2,q^{-1})}$	0.53969059	0.5396829590522
$\frac{\lvert\Delta_{77}\rvert}{\text{dist}(\Delta_{77},q^{-1})}$	0.061977625	0.06197692151387
$\frac{\lvert\Delta_{71}\rvert}{\text{dist}(\Delta_{71},q^{-1})}$	0.083906928	0.08390559199038
$\frac{\lvert\Delta_{37}\rvert}{\text{dist}(\Delta_{37},q^{-1})}$	0.043995279	0.04398951322458
$\frac{\lvert\Delta_{31}\rvert}{\text{dist}(\Delta_{31},q^{-1})}$	0.045789189	0.04578733117880

$\frac{\|\Delta_{27}\|}{\mathrm{dist}(\Delta_{27},q^{-1})}$	0.098793277	0.09879052896388
$\frac{\|\Delta_{21}\|}{\mathrm{dist}(\Delta_{21},q^{-1})}$	0.13337261	0.1333709148367
$\frac{\|\Delta_{17}\|}{\mathrm{dist}(\Delta_{17},q^{-1})}$	0.35459170	0.3545879467640
$\frac{\|\Delta_{12}\|}{\mathrm{dist}(\Delta_{12},q^{-1})}$	0.52020847	0.5202009580464

Table B.17: Ratio of domain sizes to partial of remaining domains on the y-axis

Ratios on $\Delta^{(5)}$		
t	t_{bottom}	t_{top}
$\frac{\|g_5^{-1}(\Delta_7)\|}{H_5(g_5^{-1}(\Delta_7))}$	0.33541197	0.3354098997771
$\frac{\|g_5^{-1}(\Delta_6)\|}{H_5(g_5^{-1}(\Delta_6))}$	0.20504858	0.2050435703931
$\frac{\|g_5^{-1}(\Delta_5)\|}{H_5(g_5^{-1}(\Delta_5))}$	0.12228690	0.1222745479508
$\frac{\|g_5^{-1}(\delta_0)\|}{H_5(g_5^{-1}(\delta_0))}$	0.23732558	0.2373747405086
$\frac{\|g_5^{-1}(\Delta_3)\|}{H_5(g_5^{-1}(\Delta_3))}$	0.17844517	0.1784321639675
$\frac{\|g_5^{-1}(\Delta_2)\|}{H_5(g_5^{-1}(\Delta_2))}$	0.50626223	0.5062560261036
$\frac{\|g_5^{-1}(\Delta_{77})\|}{H_5(g_5^{-1}(\Delta_{77}))}$	0.067344702	0.06734403693652
$\frac{\|g_5^{-1}(\Delta_{71})\|}{H_5(g_5^{-1}(\Delta_{71}))}$	0.084545160	0.08454397745428
$\frac{\|g_5^{-1}(\Delta_{37})\|}{H_5(g_5^{-1}(\Delta_{37}))}$	0.041707790	0.04170237694486
$\frac{\|g_5^{-1}(\Delta_{31})\|}{H_5(g_5^{-1}(\Delta_{31}))}$	0.043337254	0.04333556027423
$\frac{\|g_5^{-1}(\Delta_{27})\|}{H_5(g_5^{-1}(\Delta_{27}))}$	0.093372605	0.09337015582215
$\frac{\|g_5^{-1}(\Delta_{21})\|}{H_5(g_5^{-1}(\Delta_{21}))}$	0.12646672	0.1264652977255
$\frac{\|g_5^{-1}(\Delta_{17})\|}{H_5(g_5^{-1}(\Delta_{17}))}$	0.33650744	0.3365043713358
$\frac{\|g_5^{-1}(\Delta_{12})\|}{H_5(g_5^{-1}(\Delta_{12}))}$	0.50692626	0.5069194318305

Table B.18: More ratio of domain sizes to partial of remaining domains

Ratios on I		
t	t_{bottom}	t_{top}
$\frac{\|\Delta_{777}\|}{\mathrm{dist}(\Delta_{777},q^{-1})}$	0.014677608	0.01467738117096
$\frac{\|\Delta_{772}\|}{\mathrm{dist}(\Delta_{772},q^{-1})}$	0.0078377722	0.007837528334170
$\frac{\|\Delta_{771}\|}{\mathrm{dist}(\Delta_{771},q^{-1})}$	0.015826017	0.01582571222937
$\frac{\|\Delta_{737}\|}{\mathrm{dist}(\Delta_{737},q^{-1})}$	0.0050770231	0.005076253134255
$\frac{\|\Delta_{727}\|}{\mathrm{dist}(\Delta_{727},q^{-1})}$	0.0094806979	0.009480300321950
$\frac{\|\Delta_{717}\|}{\mathrm{dist}(\Delta_{717},q^{-1})}$	0.019724964	0.01972453216366
$\frac{\|\Delta_{377}\|}{\mathrm{dist}(\Delta_{377},q^{-1})}$	0.010988716	0.01098691333112

$\frac{\lvert\Delta_{371}\rvert}{\mathrm{dist}(\Delta_{371},q^{-1})}$	0.010831911	0.01083071327374
$\frac{\lvert\Delta_{337}\rvert}{\mathrm{dist}(\Delta_{337},q^{-1})}$	0.0030897953	0.003089179866715
$\frac{\lvert\Delta_{327}\rvert}{\mathrm{dist}(\Delta_{327},q^{-1})}$	0.0056348362	0.005634374374564
$\frac{\lvert\Delta_{317}\rvert}{\mathrm{dist}(\Delta_{317},q^{-1})}$	0.011216713	0.01121613285142
$\frac{\lvert\Delta_{311}\rvert}{\mathrm{dist}(\Delta_{311},q^{-1})}$	0.011447672	0.01144720261362
$\frac{\lvert\Delta_{277}\rvert}{\mathrm{dist}(\Delta_{277},q^{-1})}$	0.023273862	0.02327302474135
$\frac{\lvert\Delta_{272}\rvert}{\mathrm{dist}(\Delta_{272},q^{-1})}$	0.012260892	0.01226034520501
$\frac{\lvert\Delta_{271}\rvert}{\mathrm{dist}(\Delta_{271},q^{-1})}$	0.024740046	0.02473928014316
$\frac{\lvert\Delta_{237}\rvert}{\mathrm{dist}(\Delta_{237},q^{-1})}$	0.0078405532	0.007839341900395
$\frac{\lvert\Delta_{231}\rvert}{\mathrm{dist}(\Delta_{231},q^{-1})}$	0.0073433283	0.007342901066202
$\frac{\lvert\Delta_{227}\rvert}{\mathrm{dist}(\Delta_{227},q^{-1})}$	0.014664821	0.01466418789082
$\frac{\lvert\Delta_{217}\rvert}{\mathrm{dist}(\Delta_{217},q^{-1})}$	0.030690287	0.03068965729156
$\frac{\lvert\Delta_{177}\rvert}{\mathrm{dist}(\Delta_{177},q^{-1})}$	0.071028946	0.07102791382318
$\frac{\lvert\Delta_{171}\rvert}{\mathrm{dist}(\Delta_{171},q^{-1})}$	0.088604528	0.08860298707742
$\frac{\lvert\Delta_{137}\rvert}{\mathrm{dist}(\Delta_{137},q^{-1})}$	0.042985456	0.04297976529638
$\frac{\lvert\Delta_{131}\rvert}{\mathrm{dist}(\Delta_{131},q^{-1})}$	0.044538903	0.04453706606865
$\frac{\lvert\Delta_{127}\rvert}{\mathrm{dist}(\Delta_{127},q^{-1})}$	0.095882311	0.09587959619373
$\frac{\lvert\Delta_{117}\rvert}{\mathrm{dist}(\Delta_{117},q^{-1})}$	0.34291367	0.3429100228299
$\frac{\lvert\Delta_{7777}\rvert}{\mathrm{dist}(\Delta_{7777},q^{-1})}$	0.0036343166	0.003634316656083
$\frac{\lvert\Delta_{1777}\rvert}{\mathrm{dist}(\Delta_{1777},q^{-1})}$	0.016977990	0.01697799017366

Table B.19: More ratio of domain sizes to partial of remaining domains on the y-axis

Ratios on $\Delta^{(5)}$		
t	t_{bottom}	t_{top}
$\frac{\lvert g_5^{-1}(\Delta_{777})\rvert}{H_5(g_5^{-1}\Delta_{777}))}$	0.016106136	0.01610590840944
$\frac{\lvert g_5^{-1}(\Delta_{772})\rvert}{H_5(g_5^{-1}\Delta_{772}))}$	0.0084509762	0.008450725325239
$\frac{\lvert g_5^{-1}(\Delta_{771})\rvert}{H_5(g_5^{-1}\Delta_{771}))}$	0.016974444	0.01697414336095
$\frac{\lvert g_5^{-1}(\Delta_{737})\rvert}{H_5(g_5^{-1}\Delta_{737}))}$	0.0052526462	0.005251854412446
$\frac{\lvert g_5^{-1}(\Delta_{727})\rvert}{H_5(g_5^{-1}\Delta_{727}))}$	0.0097417664	0.009741372135499
$\frac{\lvert g_5^{-1}(\Delta_{717})\rvert}{H_5(g_5^{-1}\Delta_{717}))}$	0.020021280	0.02002087732676
$\frac{\lvert g_5^{-1}(\Delta_{377})\rvert}{H_5(g_5^{-1}\Delta_{377}))}$	0.010429497	0.01042779682981
$\frac{\lvert g_5^{-1}(\Delta_{371})\rvert}{H_5(g_5^{-1}\Delta_{371}))}$	0.010273920	0.01027279602420
$\frac{\lvert g_5^{-1}(\Delta_{337})\rvert}{H_5(g_5^{-1}\Delta_{337}))}$	0.0029287563	0.002928176877694
$\frac{\lvert g_5^{-1}(\Delta_{327})\rvert}{H_5(g_5^{-1}\Delta_{327}))}$	0.0053401109	0.005339680622473
$\frac{\lvert g_5^{-1}(\Delta_{317})\rvert}{H_5(g_5^{-1}\Delta_{317}))}$	0.010626426	0.01062589142954
$\frac{\lvert g_5^{-1}(\Delta_{311})\rvert}{H_5(g_5^{-1}\Delta_{311}))}$	0.010843463	0.01084303409416
$\frac{\lvert g_5^{-1}(\Delta_{277})\rvert}{H_5(g_5^{-1}\Delta_{277}))}$	0.022037915	0.02203715421925

$\frac{\lvert g_5^{-1}(\Delta_{272})\rvert}{H_5(g_5^{-1}\Delta_{272}))}$	0.011613686	0.01161318538644
$\frac{\lvert g_5^{-1}(\Delta_{271})\rvert}{H_5(g_5^{-1}\Delta_{271}))}$	0.023428273	0.02342758275824
$\frac{\lvert g_5^{-1}(\Delta_{237})\rvert}{H_5(g_5^{-1}\Delta_{237}))}$	0.0074378572	0.007436719453193
$\frac{\lvert g_5^{-1}(\Delta_{231})\rvert}{H_5(g_5^{-1}\Delta_{231}))}$	0.0069684446	0.006968049379401
$\frac{\lvert g_5^{-1}(\Delta_{227})\rvert}{H_5(g_5^{-1}\Delta_{227}))}$	0.013915875	0.01391529483451
$\frac{\lvert g_5^{-1}(\Delta_{217})\rvert}{H_5(g_5^{-1}\Delta_{217}))}$	0.029143622	0.02914306636412
$\frac{\lvert g_5^{-1}(\Delta_{177})\rvert}{H_5(g_5^{-1}\Delta_{177}))}$	0.067586680	0.06758579038737
$\frac{\lvert g_5^{-1}(\Delta_{171})\rvert}{H_5(g_5^{-1}\Delta_{171}))}$	0.084751133	0.08474976160574
$\frac{\lvert g_5^{-1}(\Delta_{137})\rvert}{H_5(g_5^{-1}\Delta_{137}))}$	0.041765194	0.04175972055139
$\frac{\lvert g_5^{-1}(\Delta_{131})\rvert}{H_5(g_5^{-1}\Delta_{131}))}$	0.043392238	0.04339049233279
$\frac{\lvert g_5^{-1}(\Delta_{127})\rvert}{H_5(g_5^{-1}\Delta_{127}))}$	0.093488773	0.09348621361920
$\frac{\lvert g_5^{-1}(\Delta_{117})\rvert}{H_5(g_5^{-1}\Delta_{117}))}$	0.33683767	0.3368342882525
$\frac{\lvert g_5^{-1}(\Delta_{7777})\rvert}{H_5(g_5^{-1}\Delta_{7777}))}$	0.0039980041	0.003998004132746
$\frac{\lvert g_5^{-1}(\Delta_{1777})\rvert}{H_5(g_5^{-1}\Delta_{1777}))}$	0.0161658656	0.01616586569946

B.4 Admissible domains

We take pullbacks of ξ_0 into domains of ξ_0 according to the algorithm in 2.6.2. This forms ξ_0'. Then we divide ξ_0' into sections which will improve the estimate for distorted relative measure of holes in ξ_0'. Upper bounds are taken for t over $\mathcal{T}_0$

Table B.20: Upper bounds for the distorted relative measure of holes for any domain refined by ξ_0' divided into appropriate sections

ξ_0'			
Domains of each section	D=Upper bound for distortion on the section	R=Upper bound for the relative measure of holes in the section	$\frac{D*R}{1-R+R*D}$
1 through 7	1.022	0.11	0.112149
7 through 14	1.057	0.162	0.169667
15 through 20	1.024	0.095	0.0970587
21 through 25	1.021	0.214	0.217516
26 through 56	1.169	0.283	0.315727

57 through 60	1.029	0.288	0.293897
61 through 64	1.011	0.192	0.193703
65 through 71	1.030	0.138	0.141554
72 through 83	1.063	0.291	0.303764
84 through 108	1.112	0.15	0.164044
109 through 292	2.142	0.291	0.467846
293 through 383	1.351	0.27	0.333193
384 through 429	1.103	0.318	0.33963

Here we do the same for partition ξ_5. The last row(section), shaded in gray, is the region of δ_0 where we use separately to get (2.243).

Table B.21: Upper bounds for the distorted relative measure of holes for any domain refined by ξ_5' divided into appropriate sections

Section	D=Distortion on the section	R=Relative measure of holes in the section	$\frac{D*R}{1-R+R*D}$
1 through 64	1.016	0.026	0.026405
65 through 130	1.057	0.029	0.0306024
131 through 194	1.024	0.017	0.0174009
195 through 257	1.021	0.038	0.0387671
258 through 578	1.169	0.051	0.0591095
579 through 640	1.029	0.051	0.0524015
641 through 702	1.011	0.035	0.0353714
703 through 767	1.030	0.025	0.0257307
768 through 895	1.063	0.052	0.0550955
896 through 2100	1.73	0.057	0.0946708
1153 through 2100	1.56	0.064	0.0963855
2101 through 3076	1.38	0.038	0.0516935
3077 through 4037	1.36	0.048	0.0641711
4038 through 4547	1.12	0.055	0.0611961
4548 through 9214	1.3036	0.2	0.245795

B.5 Final calculations

The following table lists the figures we use to obtain inequality (2.260).

Table B.22: Figures from initial steps of induction

Table

n	U_n = upper bound of $\left\lvert\frac{\frac{\partial^2 \mathcal{G}_{n,i}^{-1}}{\partial t \partial z}}{\frac{\partial \mathcal{G}_{n,i}^{-1}}{\partial z}}\right\rvert$	S_n = upper bound for $\left\lvert\mathcal{T}^{(n-1)}\right\rvert$	C_n = upper bound for $e^{U_n S_n}$	upper bound for the relative measure of holes after five pullbacks of $\hat{\xi}_{[\frac{n}{3}]}$ on the x-axis	upper bound for distortion of $g_{(n-1)}$	upper bound for relative measure of holes $\mathcal{H}_n$ after five pullbacks of $\hat{\xi}_{[\frac{n}{3}]}$ into $\Delta^{(n-1)}$	upper bound for $\left\lvert\frac{\bigcup \mathcal{T}(\delta^{(n)})}{\mathcal{T}^{(n-1)}}\right\rvert$ calculated by $\mathcal{M}_n^c = \frac{1+4\epsilon_0}{1-4\epsilon_0} C_n \mathcal{H}_n$	lower bound for $\left\lvert\frac{\bigcup \mathcal{T}^{(n)}}{\mathcal{T}^{(n-1)}}\right\rvert$ calculated by $\mathcal{M}_n = \frac{1-4\epsilon_0}{1+4\epsilon_0}\frac{1}{C_n}(1-\mathcal{H}_n)$
6	$8.9+1.5*1.5526+200 = 211.23$	$4.64851 * 10^{-6}$	1.001	0.80238453268	$\frac{15}{13} = 1.15385$	Distorted ratio 0.824099 Numerically, 0.80184718 throws away some domains close to y_5	0.84454 , 0.821736, $(1 - 0.821736) = 0.178264$	0.193260, Direct calculations from actual parameters, 0.19814951
7	$211.23 + 12 * 1.5526 + 200 \approx 429.87$	$\left\lvert\mathcal{T}^{(6)}\right\rvert \leq \frac{4}{1-4\epsilon_0}\vartheta_1 \left\lvert\Delta^{(5)}\right\rvert \approx 4.57636 * 10^{-8}$	1.00002	0.773247352	$\left\lvert\frac{g'_{(6)}(x)}{g'_{(6)}(y)}\right\rvert \leq \left\lvert\frac{(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{1,i_1})'(g_{(5)}(x))}{(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{1,i_1})'(g_{(5)}(y))}\right\rvert \left\lvert\frac{g'_{(5)}(x)}{g'_{(5)}(y)}\right\rvert \leq 1.29 * 1.15385 \approx 1.48847$	0.835413	0.855715 $(1 - 0.855715) = 0.144285$	0.160681
8	$429.87 + 12 * 1.5526 + 200 \approx 648.51$	$\left\lvert\mathcal{T}^{(7)}\right\rvert \leq \frac{4}{1-4\epsilon_0}\vartheta_1 \frac{1}{9}\left\lvert\Delta^{(5)}\right\rvert \approx 5.08485 * 10^{-9}$	1.000004	0.773247352	$\left\lvert\frac{g'_{(7)}(x)}{g'_{(7)}(y)}\right\rvert \leq \left\lvert\frac{(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{1,i_1})'(g_{(6)}(x))}{(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{1,i_1})'(g_{(6)}(y))}\right\rvert \left\lvert\frac{g'_{(6)}(x)}{g'_{(6)}(y)}\right\rvert \leq 1.29 * 1.48847 \approx 1.92013$	0.867512	0.888587 $(1 - 0.888587) > 0.111412$	0.129345
9	867.15	$5.64983 * 10^{-10}$	1.0000005 3916215	0.773247352	$\left\lvert\frac{g'_{(8)}(x)}{g'_{(8)}(y)}\right\rvert \leq \left\lvert\frac{(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{1,i_1})'(g_{(7)}(x))}{(\hat{f}_{0,i_s}\circ\cdots\circ\hat{f}_{1,i_1})'(g_{(7)}(y))}\right\rvert \left\lvert\frac{g'_{(7)}(x)}{g'_{(7)}(y)}\right\rvert \leq 1.29 * 1.92013 \approx 2.47696$	0.8941420656	0.9158625645	0.1033474189
10	1085.79	$6.27765 * 10^{-11}$	$1 + 7 * 10^{-8}$	0.773247352	1.29* 2.47696=3.19528	0.9159391459	0.9381887437	0.08206731051
11	1304.43	$6.9752 * 10^{-12}$	$1 + 10^{-8}$	0.773247352	1.29*3.19528= 4.12191	0.9335814085	0.9562595080	0.06484344655
12	1523.07	$7.7502 * 10^{-13}$	$1 + 2 * 10^{-9}$	0.773247352	1.29 * 4.12191 = 5.31727	0.9477322943	0.9707541326	0.05102815531

Table B.24: More figures from initial steps of induction

Table (part 2)

n	U_n = upper bound of $\left\|\frac{\frac{\partial^2 \mathcal{G}_{n,i}^{-1}}{\partial t \partial z}}{\frac{\partial \mathcal{G}_{n,i}^{-1}}{\partial z}}\right\|$	S_n = upper bound for $\left\|\mathcal{T}^{(n-1)}\right\|$	C_n = upper bound for $e^{U_n S_n}$	upper bound for the relative measure of holes after five pullbacks of $\hat{\xi}_{[\frac{n}{3}]}$ on the x-axis	upper bound for distortion of $g_{(n-1)}$	upper bound for relative measure of holes $\mathcal{H}_n$ after five pullbacks of $\hat{\xi}_{[\frac{n}{3}]}$ into $\Delta^{(n-1)}$	upper bound for $\left\|\frac{\bigcup \mathcal{T}(\delta^{(n)})}{\mathcal{T}^{(n-1)}}\right\|$ calculated by $\mathcal{M}_n^c = \frac{1+4\epsilon_0}{1-4\epsilon_0} C_n \mathcal{H}_n$	lower bound for $\left\|\frac{\bigcup \mathcal{T}^{(n)}}{\mathcal{T}^{(n-1)}}\right\|$ calculated by $\mathcal{M}_n = \frac{1-4\epsilon_0}{1+4\epsilon_0} \frac{1}{C_n} (1 - \mathcal{H}_n)$
13	1741.71	$8.61123 * 10^{-14}$	$1 + 2 * 10^{-10}$	0.773247352	$1.29 * 5.31727 = 6.85928$	0.9590006521	0.9822962146	0.04002703138
14	1960.35	$9.56804 * 10^{-15}$	$1 + 2 * 10^{-11}$	0.773247352	$1.29 * 6.85928 = 8.84847$	0.9679218988	0.9914341717	0.03131735567
15	2178.99	$1.06312 * 10^{-15}$	$1 + 3 * 10^{-12}$	0.3520572748	$1.29 * 8.84847 = 11.4145$	$a + (1-a) * b = 0.453 + (1-0.453) * 0.5 = 0.7265$	0.7439429150	0.2672090909
16	2397.63	$1.18125 * 10^{-16}$	$1 + 3 * 10^{-13}$	0.3520572748	$1.29 * 111.4145 = 14.727247 < 15.5$	0.7265	0.7439429150	0.2672090909
17	2616.27	$1.3125 * 10^{-17}$	$1 + 4 * 10^{-14}$	0.3520572748	$1.29 * 14.7247 = 18.9949 > 15.5$ so we use 15.5	0.7265	0.7439429150	0.2672090909
18	2834.91	$1.45833 * 10^{-18}$	$1 + 5 * 10^{-15}$	0.3520572748	15.5	0.7265	0.7439429150	0.2672090909
19	3053.55	$1.62037 * 10^{-19}$	$1 + 5 * 10^{-16}$	0.3520572748	15.5	0.7265	0.7439429150	0.2672090909
20	3272.19	$1.80041 * 10^{-20}$	$1 + 6 * 10^{-17}$	0.3520572748	15.5	0.7265	0.7439429150	0.2672090909
21	3490.83	$2.00046 * 10^{-21}$	$1 + 7 * 10^{-18}$	0.3520572748	15.5	0.7265	0.7439429150	0.2672090909
22	3709.47	$2.22273 * 10^{-22}$	$1 + 9 * 10^{-19}$	0.3520572748	15.5	0.7265	0.7439429150	0.2672090909
23	3928.11	$2.4697 * 10^{-23}$	$1 + 10^{-19}$	0.3520572748	15.5	0.7265	0.7439429150	0.2672090909

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