

ABSTRACT

Title of dissertation: COMPUTATIONAL METHODS FOR
GAME OPTIONS

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Game options are American-type options with the additional property that the seller of the option has the right to cancel the option at any time prior to the buyer exercise or the expiration date of the option. The cancellation by the seller can be achieved through a payment of an additional penalty to the exercise payoff or using a payoff process greater than or equal to the exercise value.

The main contribution of this thesis is a numerical framework for computing the value of such options with finite maturity time as well as in the perpetual setting. This framework employs the theory of weak solutions of parabolic and elliptic variational inequalities. These solutions will be computed using finite element methods.

The computational advantage of this framework is that it allows the user to go from one type of process to another by changing the stiffness matrix in the algorithm. Several types of Lévy processes will be used to show the functionality

of this method. The processes considered are of pure diffusion type (Black-Scholes model), the CGMY process as a pure jump model and a combination of the two for the case of jump diffusion.

Computational results of the option prices as well as exercise, hold and cancellation regions are shown together with numerical estimates of the error convergence rates with respect to the L_2 norm and the energy norm.

COMPUTATIONAL METHODS FOR
GAME OPTIONS

by

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Acknowledgments

to Professor Tobias von Petersdorff and Professor Dilip Madan,

Thank you!

Table of Contents

List of Tables	vi
List of Figures	vi
1 Introduction	1
List of Abbreviations	1
2 Introduction to Game Options	4
2.1 Background	4
2.1.1 Definition and Properties	4
2.1.2 δ -penalty Game Put Option	5
2.2 Stochastic Formulation	6
2.2.1 Discrete Case	6
2.2.2 Continuous Case	9
3 Lévy Processes	11
3.1 Introduction	11
3.1.1 Definition and properties	11
3.1.2 Infinitesimal Generators	13
3.2 Lévy models in finance	13
3.2.1 Black Scholes Model	13
3.2.2 CGMY Model	14
3.2.3 Jump-diffusion model	15
3.3 Option Pricing with Lévy models	15
3.3.1 European options	15
3.3.2 American options	17
4 Elliptic Equations and Inequalities	19
4.1 Elliptic Equations	19
4.1.1 Introduction	19
4.1.2 Variational Formulation for Elliptic Equations	22
4.1.3 Localization to a Bounded Domain	23
4.1.4 Discretization of the Space Domain	23
4.1.5 Finite Elements for Elliptic Equations	24
4.2 Elliptic Inequalities	25
4.2.1 Introduction	25
4.2.2 Variational Formulation for Elliptic Inequalities	26
4.2.3 Localization to a Bounded Domain	32
4.2.4 Discretization of the Space Domain	33
4.2.5 Finite Elements for Elliptic Inequalities	33

5	Parabolic Equations and Inequalities	36
5.1	Parabolic Equations	36
5.1.1	Introduction	36
5.1.2	Variational Formulation for Parabolic Equations	38
5.1.3	Localization of the Space Domain	39
5.1.4	Discretization of the Space Domain	40
5.1.5	Finite Elements for Parabolic Equations	40
5.1.6	Discretization of the Time Domain	41
5.2	Parabolic Inequalities	42
5.2.1	Introduction	42
5.2.2	Variational Formulation for Parabolic Inequalities	43
5.2.3	Localization of the Space Domain	44
5.2.4	Discretization of the Space Domain	45
5.2.5	Finite Elements for Parabolic Inequalities	46
5.2.6	Discretization of the Time Domain	46
6	Finite Element Method for Game Options	49
6.1	Game Option Value as Variational Solution of Parabolic Inequalities	49
6.1.1	From Stochastic Formulation to Parabolic Inequalities	49
6.1.2	Localization of the Space Domain	51
6.1.3	Fully Discrete Galerkin Method for Game Options	61
6.2	Perpetual Game Options	62
6.2.1	Elliptic Inequalities for Perpetual Game Options	62
6.2.2	Localization to a Bounded Domain	63
6.2.3	Galerkin Method for Perpetual Game Options	64
6.2.4	Convergence of the Perpetual Game and American Options	65
7	Implementation and Numerical Results	81
7.1	Implementation of the Finite Elements Method for Game Options	81
7.1.1	The Stiffness Matrix for P.W.L. Finite Elements under CGMY	81
7.1.2	The Load Vector for P.W.L. Finite Elements under CGMY	90
7.1.3	Solving the Linear Complementarity Problem for Game Options	99
7.2	Numerical Results for δ -penalty Game Options	105
7.2.1	Perpetual δ -penalty Game Put Value Functions	105
7.2.2	Perpetual δ -penalty Game Put Error Convergence	108
7.2.3	Perpetual δ -penalty Game Call Value Functions	111
7.2.4	Finite Maturity δ -penalty Game Put Value Functions and Free Regions	114
7.2.5	Finite Maturity δ -penalty Game Put Error Convergence Rates	126
7.2.6	Finite Maturity δ -penalty Game Call Value Functions and Free Regions	128
7.2.7	Finite Maturity δ -penalty butterfly Game Option Value Functions and Free Regions	139
8	Concluding Remarks and Future Research	149

List of Tables

7.1	Perpetual δ -penalty Game Put energy norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$	109
7.2	Perpetual δ -penalty Game Put L_2 norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$	109
7.3	Perpetual δ -penalty Game Put energy norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$	110
7.4	Perpetual δ -penalty Game Put L_2 norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$	110
7.5	Perpetual δ -penalty Game Put energy norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$	111
7.6	Perpetual δ -penalty Game Put L_2 norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$	111
7.7	δ -penalty Game Put energy norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	128
7.8	δ -penalty Game Put L_2 norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	129
7.9	δ -penalty Game Put energy norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	130
7.10	δ -penalty Game Put L_2 norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	131
7.11	δ -penalty Game Put energy norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	132
7.12	δ -penalty Game Put L_2 norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	133

List of Figures

7.1	Perpetual δ -penalty Game Put Value Function for pure a diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$	106
7.2	Perpetual δ -penalty Game Put Value Function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$	107
7.3	Perpetual δ -penalty Game Put Value Function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$	108
7.4	Perpetual δ -penalty Game Call Value Function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$	112
7.5	Perpetual δ -penalty Game Call Value Function for pure a jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$	113
7.6	Perpetual δ -penalty Game Call Value Function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$	114
7.7	δ -penalty Game Put value function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	115

7.8	δ -penalty Game Put value functions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $t=0.9$ and $t=0.6$	116
7.9	δ -penalty Game Put regions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	117
7.10	δ -penalty Game Put value function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	118
7.11	δ -penalty Game Put value functions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $t=.97$ and $t=.94$	119
7.12	δ -penalty Game Put regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	120
7.13	δ -penalty Game Put value function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	121
7.14	δ -penalty Game Put value functions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.975$ and $t=0.95$	122
7.15	δ -penalty Game Put regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	123
7.16	δ -penalty Game Put regions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.03$ $T=1$	124
7.17	δ -penalty Game Put regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.1$ $T=1$	125
7.18	δ -penalty Game Put regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.1$ $T=1$	126
7.19	δ -penalty Game Call value function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	129
7.20	δ -penalty Game Call value functions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $t=0.95$ and $t=0.8$	130
7.21	δ -penalty Game Call regions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	132
7.22	δ -penalty Game Call value function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	134
7.23	δ -penalty Game Call value functions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.97$ and $t=0.94$	135
7.24	δ -penalty Game Call regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	136
7.25	δ -penalty Game Call value function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	137
7.26	δ -penalty Game Call value functions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.97$ and $t=0.94$	138
7.27	δ -penalty Game Call regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	139
7.28	δ -penalty Butterfly Game option value function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	140
7.29	δ -penalty Butterfly Game option value functions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $t=0.9$ and $t=0.6$	141

7.30	δ -penalty Butterfly Game option regions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$	142
7.31	δ -penalty Butterfly Game option value function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	143
7.32	δ -penalty Butterfly Game option value functions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.95$ and $t=0.9$	144
7.33	δ -penalty Butterfly Game option regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	145
7.34	δ -penalty Butterfly Game option value function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	146
7.35	δ -penalty Butterfly Game option value functions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.95$ and $t=0.9$	147
7.36	δ -penalty Butterfly Game option regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$	148

Chapter 1

Introduction

This paper introduces a framework for computing the values of game options with finite maturity time as well as in the perpetual setting. Game options are American-type options with the additional property that the seller of the option has the right to cancel the option at any time prior to the buyer exercise and the expiration date of the option. The cancellation by the seller can be achieved through a payment of an additional penalty to the exercise payoff or more generally using a payoff process greater than or equal to the exercise value.

Game options were introduced to diversify the available financial instruments and to offer a less expensive alternative to American options while still preserving some of the properties. The purpose of this paper is to introduce a framework which allows the user to compute the values of game options in the finite maturity setting as well as in the perpetual case while using pure diffusion, pure jump CGMY or jump diffusion CGMY Lévy processes for the underlying asset.

We are motivated by the limited numerical methods available to price game options, specifically in the case of a jump processes. Another motive is limited understanding of the shape and behavior of cancellation, hold and exercise regions under certain jump processes. Our contribution consists of theoretical results on localization error estimates, existence and uniqueness of perpetual option values and convergence

rates for the perpetual options. We are able to analyze and understand the behavior of solutions for various payoffs and processes. We can analyze the dependence of the solutions on several parameters such as the cancelation penalty or the type of the underlying Lévy process.

Game options, also called Israeli options, were introduced by Kifer in [37]. Stochastic properties of game options have been analyzed in [45], [4] and [20], and explicit formulas for the value of the option are derived for some specific examples of game options in the perpetual setting. Perpetual game options are game options with no expiration date. In the same infinite horizon setting, properties and calculations for convertible perpetual bonds are shown in [68] in the Black-Scholes model and also in a more general model with jumps in [26]. Convertible bonds, a subclass of game options, are bonds which can be recalled by the issuer, and at the same time the holder has the choice to convert the bond to stock or continue receiving coupon payments, hence the game nature of this type of contract and finding its value is reduced to an optimal stopping problem [69], [31].

Another approach for solving the stochastic differential game problem is taken in [28] where this problem is solved by finding a local solution of backward stochastic differential equations in a pure diffusion model. Also in the Black-Scholes model the value of the game option is formulated as an obstacle problem and shown in [53] as a viscosity solution to a reflected forward-backward stochastic differential equation. In a more general setting, including complete and incomplete markets, properties of game options are shown through maximization of utility functions [30] and [41]. Another general approach, based on Monte-Carlo methods, is described in [44] and

uses pricing based on numerical simulation of the possible underlying paths. This method is found useful in cases of game options with more complicated structure. The values of game options in the finite horizon case have been shown to be equivalent to a mixture of exotic options in [46] and [42]. This has been done in a very limited setting of the δ -penalty puts and calls under the Black-Scholes model only. This method breaks down for jump processes. One of the insights of this method is the shape of the cancelation region for jump processes.

The paper is structured as follows:

In chapters 2-5 we present some existing theory on option pricing and variational inequalities. In Chapter 2 we formally define the game options and present some of their properties. Chapter 3 introduces Lévy process and options pricing in the European and American setting. In Chapter 4 we will discuss some existing theory on elliptic equations and inequalities followed by parabolic equations and inequalities in Chapter 5.

Chapter 6 and 7 contain the most significant contribution of this paper. In chapter 6 we will present the finite element method for solving game options and present the implementation details in Chapter 7. The value functions and error convergence rates will also be shown in Chapter 7.

Chapter 2

Introduction to Game Options

2.1 Background

2.1.1 Definition and Properties

Game options are financial contracts of American type with the additional feature that the seller has the right to cancel the contract at some additional cost to the exercise value at any time before exercise or expiration. In this situation, the formal setting will consist of the usual set up, a probability space, $(\Omega, \mathcal{F}, \mathcal{P})$, together with a stochastic process X_t describing the log of the price of the underlying stock. Here $t \in [0 \dots T]$ is a non-negative integer in the discrete model and non-negative real in the continuous case and T is the expiration time. The process X_u , with $0 \leq u \leq t$, generates a filtration \mathcal{F}_t which family of complete σ -algebras $\mathcal{F}_t \subset \mathcal{F}$. In addition we have two non-negative, right continuous with left limits, \mathcal{F}_t adapted stochastic payoff processes, $0 \leq F_t \leq G_t$ with $t \in [0, T]$ as above. In this setting, the game option is a contract between a seller and a buyer which allows the buyer to exercise and the seller to cancel up to the maturity time T . If the buyer decides to exercise at a time $t \leq T$, he/she will receive the payment F_t from the seller. If the seller wants to cancel at $t \leq T$, he/she will have to pay the buyer the value G_t . If both of them decide to exercise at the same time $t \leq T$ then the

payment that the buyer receives from the seller is F_t . If none of the parties exercises their right before expiration then at expiration time the value of the option is F_t . If we denote the seller stopping time (cancelation time) by σ and the buyer stopping time (exercise time) by τ then the payment that the buyer receives from the seller at time $\sigma \wedge \tau$ is $R(\sigma, \tau)$ where

$$R(\sigma, \tau) = G_\sigma \mathbf{1}_{\sigma < \tau} + F_\tau \mathbf{1}_{\tau \leq \sigma}$$

where $\mathbf{1}_q = 1$ in the event that q is true and $\mathbf{1}_q = 0$ otherwise. Here \wedge is the minimum operator.

Similarly to the value of an American option, finding the value of a game option is equivalent an optimal stopping problem [19]. In the American case the optimal stopping time is found by maximizing the payoff over all possible exercise times [32]. In the game option case the optimal stopping time is a saddle point obtained by minimizing the payoff over all possible cancelation times and by maximizing over all possible exercise times [36], see section 2.2.

2.1.2 δ -penalty Game Put Option

The δ -penalty game put option is an example of a game option which is an American put option that give the seller the right to cancel the option by paying a fixed non-negative penalty δ . Therefore the exercise payment will be the standard put value $F_t = (K - S_t)^+$, with $t \in [0, T]$ and K the strike. The cancelation payment will be $G_t = F_t + \delta = (K - S_t)^+ + \delta$. Notice if the penalty δ exceeds

some critical value δ^* the seller will not exercise his/her right to cancel because the penalty is too high and the game option will be just an American put option. This so called critical value δ^* is exactly the value of an "at the money" (initial stock price $S_0 = K$) American put option with the same expiration date. If the penalty is below the critical δ^* the seller will have the opportunity to cancel early in the life of the the option before some critical time t^* . This critical time t^* is such that the value of an "at the money" American put option with expiration $T - t^*$ is δ [46]. Following the critical time the game option behaves just like an American option. In the perpetual case, the game put option values are determined in a simple diffusion model with finite activity in [74] and in a more general setting of spectrally negative Lévy processes in [5]. Game puts with finite maturity are briefly discussed in [42], which focusses on the similar game call option, showing its properties and exercise regions, under the Black-Scholes model.

Also in the finite horizon case and the Black-Scholes model, the game put is described as combination of exotic options [46]. For more general game options same idea is presented in [43] using martingale arguments.

2.2 Stochastic Formulation

2.2.1 Discrete Case

In the discrete setting we have the usual set up, a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, together with a discrete stochastic process S_t describing the price of the underlying stock. Here $t = \{0, \dots, N\}$, S_0 is the initial price of the stock and S_N is the stock

value at expiration time T . We also have a filtration $\mathcal{F}_t \subset \mathcal{F}$ generated by S_u , with $0 \leq u \leq t$ and the two non-negative, right continuous with left limits, \mathcal{F}_t adapted stochastic payoff processes, $0 \leq F_t \leq G_t$ with $t = \{0, 1, \dots, N\}$. We consider a constant risk free interest rate r and some risk neutral measure \mathcal{Q} , equivalent to \mathcal{P} , under which the process $M_t = (1+r)^{-t}S_t$ is a martingale for any $t = \{0, \dots, N\}$.

For $t_1 < t_2$ let us denote the collection of stopping times between t_1 and t_2 by

$$\mathcal{T}_{t_1, t_2} = \{\tau \text{ stopping time}, t_1 \leq \tau \leq t_2\}$$

Under this setting, the value of the game option with exercise process F_t and cancellation process G_t is given by $V(x, t)$, as proved in [37], where $V_N = F_N$ and for any $t = \{0, \dots, N-1\}$

$$\begin{aligned} V(x, t) &= \min_{\sigma \in \mathcal{T}_{t, N}} \max_{\tau \in \mathcal{T}_{t, N}} E^{\mathcal{Q}} \left((1+r)^{t-(\sigma \wedge \tau)} R(\sigma, \tau) \mid X_t = x \right) \\ &= \max_{\tau \in \mathcal{T}_{t, N}} \min_{\sigma \in \mathcal{T}_{t, N}} E^{\mathcal{Q}} \left((1+r)^{t-(\sigma \wedge \tau)} R(\sigma, \tau) \mid X_t = x \right) \end{aligned}$$

Just as in the American option case, this means that there exists an optimal stopping strategy consisting of a saddle point (σ^*, τ^*) , $\sigma^*, \tau^* \geq t$, which represents the optimal stopping points for the seller and the buyer such that the value of the game option $V(x, t)$ the expected value of the payoff R at the minimum of the two optimal stopping points

$$V(x, t) = E^{\mathcal{Q}} \left((1+r)^{t-(\sigma^* \wedge \tau^*)} R(\sigma^*, \tau^*) \mid X_t = x \right)$$

For each $t = \{0, \dots, N\}$ such a pair of optimal stopping points (σ_t^*, τ_t^*) is the first time such that the payoff is equal to the continuation value

$$\sigma_t^* = \min_{s \in \mathcal{T}_{t, N}} \{s \mid V_s \geq X_s\}$$

$$\tau_t^* = \min_{s \in \mathcal{T}_{t,N}} \{s \mid V_s \leq Y_s \text{ or } s = T\}$$

Additionally, the optimal strategy (σ_t^*, τ_t^*) satisfies the following for any $\tau \in \mathcal{T}_{t,N}$

$$V(x, t) \geq E^{\mathcal{Q}} \left((1+r)^{t-(\sigma_t^* \wedge \tau)} R(\sigma_t^*, \tau) \mid X_t = x \right)$$

and for any $\sigma \in \mathcal{T}_{t,N}$

$$V(x, t) \leq E^{\mathcal{Q}} \left((1+r)^{t-(\sigma \wedge \tau_t^*)} R(\sigma, \tau_t^*) \mid X_t = x \right)$$

Such value of the game option can be computed through dynamic programming using the following recursive relations $V_N = F_N$ and for any $t = \{0, \dots, N-1\}$

$$V(x, t) = \min \left(\max \left(E^{\mathcal{Q}} \left((1+r)^{-1} V_{t+1} \mid X_t = x \right), F_t \right), G_t \right)$$

which is equivalent to

$$V(x, t) = \max \left(\min \left(E^{\mathcal{Q}} \left((1+r)^{-1} V_{t+1} \mid X_t = x \right), G_t \right), F_t \right)$$

This method can be used to compute the value of the game option under the Binomial Model or other discrete models. For more details on Binomial Models and other discrete models see [17],[67],[47]. An analysis of this method under the Binomial model is discussed in [38] with estimation errors in the Black-Scholes market. In a similar setting of the Black-Scholes Model, a Binomial approximation of the risk associated with game options is presented in [40] and [39]. We will not pursue this method any further, but will focus instead on continuous models in the general setting of Lévy processes modeling the log on the underlying asset.

2.2.2 Continuous Case

The setting for the continuous time consists of a continuous probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a continuous stochastic process for the log of the stock price S_t , where t is now a real value in $[0, T]$, with T being the expiration date. We will define specific Lévy processes in the next chapter: diffusion, pure jump, jump diffusion. The stock price S_u , with $0 \leq u \leq t$ generates a filtration $\mathcal{F}_t \subset \mathcal{F}$. We consider a constant risk free interest rate r and a risk neutral measure \mathcal{Q} , equivalent to \mathcal{P} , under which the process $M_t = e^{-rt}S_t$ is a martingale for any $t \in [0, T]$. In addition we have two non-negative, CADLAG, \mathcal{F}_t - adapted stochastic payoff processes, $0 \leq F_t \leq G_t$, $t \in [0, T]$, F_t as exercise process and G_t as cancelation process. The value of such game option, as shown in [37], is given by $V(x, t)$ where $V(x, t) = F_T$ and for $t \in [0, T)$

$$V(x, t) = \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} E^{\mathcal{Q}} \left(e^{(t-(\sigma \wedge \tau))r} R(\sigma, \tau) \mid X_t = x \right) \quad (2.1)$$

$$= \sup_{\tau \in \mathcal{T}_{t,T}} \inf_{\sigma \in \mathcal{T}_{t,T}} E^{\mathcal{Q}} \left(e^{(t-(\sigma \wedge \tau))r} R(\sigma, \tau) \mid X_t = x \right) \quad (2.2)$$

This is equivalent to the value of this option being given by the expected value of the payoff process R at the optimal saddle point (σ^*, τ^*)

$$V(x, t) = E^{\mathcal{Q}} \left(e^{(t-(\sigma^* \wedge \tau^*))r} R(\sigma^*, \tau^*) \mid X_t = x \right) \quad (2.3)$$

Similarly to the discrete case, the set of optimal stopping times, (σ_t^*, τ_t^*) , are the first times that the value of the game option hits or crosses the exercise or cancelation boundaries

$$\sigma_t^* = \min_{s \in \mathcal{T}_{t,T}} \{s \mid V_s \geq G_s\}$$

$$\tau_t^* = \min_{s \in \mathcal{T}_{t,T}} \{s \mid V_s \leq F_s \text{ or } s = T\}$$

Any other stopping points $\tau \in \mathcal{T}_{t,T}$ and $\sigma \in \mathcal{T}_{t,T}$ will not be optimal for either the seller and the buyer, when the other party acts optimally,

$$V(x, t) \geq E^{\mathcal{Q}} \left(e^{(t - (\sigma_t^* \wedge \tau))r} R(\sigma_t^*, \tau) \mid X_t = x \right)$$

$$V(x, t) \leq E^{\mathcal{Q}} \left(e^{(t - (\sigma \wedge \tau_t^*))r} R(\sigma, \tau_t^*) \mid X_t = x \right)$$

Chapter 3

Lévy Processes

3.1 Introduction

3.1.1 Definition and properties

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a filtered probability space together with a right continuous filtration $(\mathcal{F}_t)_{t \in [0, T]}$. We will denote the time horizon by $T \in [0, \infty]$. A Lévy process, $X = (X_t)_{0 \leq t \leq T}$ is a real valued, cadlag, and \mathcal{F}_t adapted stochastic process which satisfies the following conditions:

1. $X_0 = 0$ a.s.
2. X has independent increments, i.e. $X_t - X_s$ is independent of \mathcal{F}_s for any $0 \leq s \leq t \leq T$.
3. X has stationary increments, i.e. the distribution of $X_{t+s} - X_t$ does not depend on t , for any $0 \leq s, t \leq T$.
4. X is stochastically continuous, i.e. $\lim_{s \rightarrow t} \mathcal{P}(|X_t - X_s| > \epsilon) = 0$ for every $0 \leq t \leq T$ and $\epsilon > 0$.

Lévy processes are processes with infinitely divisible distributions. One way to describe a Lévy process, X_t , represented by the Lévy triplet (μ, σ, ν) is through its

characteristic exponent, $\psi(u)$, given by the famous Lévy-Khintchine formula

$$\psi(u) = iu\mu - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|<1\}})\nu(dx)$$

where μ represents the drift coefficient, σ is the diffusion coefficient, $1_{\{\cdot\}}$ represents the indicator function, and ν is the Lévy jump measure, which satisfies the following assumption

$$\int_{-\infty}^{\infty} (|z|^2 \wedge 1)\nu(dz) < \infty$$

Intuitively the Lévy measure represents the expected number of jumps of a certain height per unit time interval.

If the Lévy density, $K(x)$ exists, the characteristic exponent, $\psi(u)$, can be rewritten as

$$\psi(u) = iu\mu - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|<1\}})K(x)dx$$

If $\nu(\mathbb{R}) < \infty$, then almost all paths of X have finite number of jumps on every compact interval, i.e. the Lévy process has finite activity.

If $\nu(\mathbb{R}) = \infty$, then almost all paths of X have infinite number of jumps on every compact interval, i.e. the Lévy process has infinite activity.

The variation of the Lévy process depends of the presence of the diffusion part but also on the Lévy measure again.

If $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, then almost all the paths of X have finite variation, i.e. the Lévy process has finite variation.

If $\sigma \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, then almost all the paths of X have infinite variation, i.e. the Lévy process has infinite variation.

If $\nu = 0$ then we have a pure diffusion process with mean μ and variance σ^2 . On

the other hand if the $\sigma = 0$ then we have a pure jump Lévy process.

The only Lévy process with continuous paths is the Brownian motion with drift.

3.1.2 Infinitesimal Generators

The infinitesimal generator \mathcal{L} for a Lévy process is given by

$$\mathcal{L}u(x) = \mu \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \int_{\mathbb{R}} \left(u(x+y) - u(x) - y 1_{\{|y|<1\}} \frac{\partial u}{\partial x}(x) \right) \nu(dy)$$

For any $u(X_t, t)$ differentiable with respect to t and twice differentiable with respect to x , the stochastic process M_t defined as

$$M_t = e^{-rt} u(X_t, t) - \int_0^t e^{-rs} \left(u_t(X_s, s) + \mathcal{L}u(X_s, s) - ru(X_s, s) \right) ds$$

is a martingale. Here r is the risk free interest rate, considered constant.

For more details on properties of Lévy processes we refer to [8], [3], [65].

3.2 Lévy models in finance

3.2.1 Black Scholes Model

In the case of the Black Scholes model, introduced in [9], the dynamics of the log of the stock price, $S_t = e^{X_t}$, is given by

$$dX_t = \mu dt + \sigma dW_t$$

here W_t represents the standard Brownian Motion with respect to the risk neutral measure \mathcal{Q} equivalent to \mathcal{P} . To ensure the martingale condition on the discounted value of the stock price, $e^{-rt} S_t = e^{-rt+X_t}$, the drift has to be equal to the risk free

interest rate $\mu = r - \frac{\sigma^2}{2}$.

W_t has all the properties of a Lévy process listed above and in addition the increments $W_t - W_s$ are normally distributed with zero mean and variance $t - s$, $N(0, t - s)$ for $t \geq s$.

The Lévy measure for the diffusion process $\nu = 0$, and the Lévy triplet is $(\mu, \sigma, \nu = 0)$. The infinitesimal generator is given by

$$\mathcal{L}_{BS}u(x) = \mu \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} \quad (3.1)$$

where $\mu = r - \frac{\sigma^2}{2}$ to ensure the martingale condition with respect to the risk neutral measure \mathcal{Q} which in the case of the Black-Scholes model is uniquely determined.

3.2.2 CGMY Model

We have a jump process under the real world measure, and therefore the risk neutral measure \mathcal{Q} is not unique. The process under the risk neutral measure \mathcal{Q} is a jump process and we assume that it is CGMY.

Under the CGMY model, introduced in [13], the log of the underlying asset, X_t , is a pure jump Lévy process with the Lévy jump density K_{CGMY} given by

$$K_{CGMY}(x) = \begin{cases} C \frac{e^{-G|x|}}{|x|^{1+Y}} \text{ for } x < 0 \\ C \frac{e^{-M|x|}}{|x|^{1+Y}} \text{ for } x > 0 \end{cases} \quad (3.2)$$

where $C > 0, G, M \geq 1$ and $Y < 2$.

If $1 \leq Y < 2$ the CGMY process has infinite variation, and if $0 \leq Y < 1$ the CGMY process has infinite activity.

The Lévy triplet for the CGMY process is $(\mu, 0, K_{CGMY})$ and the infinitesimal gen-

erator is given by

$$\mathcal{L}_{CGMY}u(x) = \mu \frac{\partial u}{\partial x} + \int_{\mathbb{R}} \left(u(x+y) - u(x) - (e^y - 1) \frac{\partial u}{\partial x}(x) \right) K_{CGMY}(y) dy \quad (3.3)$$

where $\mu = r$ to ensure the martingale condition with respect to some risk neutral measure \mathcal{Q} .

3.2.3 Jump-diffusion model

In the jump-diffusion model the log of the underlying asset price is a Lévy process with both the diffusion term, σ and the Lévy measure, ν are non-zero. We will consider a combination of a diffusion process and CGMY with the Lévy triplet (μ, σ, K_{CGMY}) . The infinitesimal generator is given by

$$\mathcal{L}_{JD}u(x) = \mu \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \int_{\mathbb{R}} \left(u(x+y) - u(x) - (e^y - 1) \frac{\partial u}{\partial x}(x) \right) K_{CGMY}(y) dy \quad (3.4)$$

where $\mu = r - \frac{\sigma^2}{2}$ to ensure the martingale condition with respect to some risk neutral measure \mathcal{Q} which in the presence of jumps is not unique.

3.3 Option Pricing with Lévy models

3.3.1 European options

A European option is a financial contract which gives the holder the option to receive a certain payoff, h from the seller only upon the expiration of the option called the maturity time T .

When the log of the price of the underlying asset is modeled by the stochastic process

X_t , the value of such option $V_E(x, t)$ is given by the discounted expected value under a risk neutral measure, \mathcal{Q} , of the terminal payoff, $h(x)$, [58]:

$$V_E(x, t) = e^{r(t-T)} E^{\mathcal{Q}} \left(h(X_T) \mid X_t = x \right)$$

The value $V_E(X, t)$ of a European option with a payoff $h(x)$, satisfies the following partial differential equation, (partial integro-differential equation in the case of process with jumps) [14]:

$$\frac{\partial V_E(x, t)}{\partial t} + \mathcal{L}V_E(x, t) - rV_E(x, t) = 0 \quad (3.5)$$

with the terminal condition $V_E(x, T) = h(x)$.

Here \mathcal{L} is the infinitesimal generator of the Lévy process driving the log of the underlying asset presented in the previous section.

With respect to the time to maturity, $\tau = T - t$, the equation (3.5) above becomes the following initial value problem:

$$\frac{\partial v(x, \tau)}{\partial \tau} - \mathcal{L}v(x, \tau) + rv(x, \tau) = 0 \quad (3.6)$$

with the initial condition $v(x, 0) = h(x)$, where $V_E(x, t) = v(x, T - t)$.

A European put has the payoff defined as $h(x) = (K - e^x)^+$ and for the European call the payoff $h(x) = (e^x - K)^+$. Here $(\cdot)^+ = \max(\cdot, 0)$ and K is the strike price.

Under the Black-Scholes model, the values of the European put and call options can be expressed in closed forms, known as the famous Black-Scholes formula, introduced in [9]. For other type of European options and processes the value of the option can be computed using stochastic methods or numerical methods for solving partial (integro) differential equations [14].

3.3.2 American options

An American option is a financial contract which gives the holder the option to receive a certain payoff, h from the seller at any time before the expiration of the option called the maturity time, T . Pricing of such option becomes an optimal stopping problem as shown in [32], and the value of such option $V_A(x, t)$ is given by the maximum over all possible stopping times of the discounted expected value under a risk neutral measure, \mathcal{Q} , of the payoff at the optimal stopping time, $h(x)$

$$V_A(x, t) = \max_{\sigma \in \mathcal{T}_{t, T}} E^{\mathcal{Q}} \left(e^{(t-\sigma)r} h(X_\sigma) \mid X_t = x \right)$$

A closed form solution is not available for this optimal stopping problem and therefore to find the values of American options, one must rely on numerical methods. There are many works in the literature which propose different numerical solutions to this problem. We will mention the one which relates the optimal stopping problems to the solutions of parabolic partial (integro) differential inequalities. This has been presented in detail for the pure diffusion case in [48], [29] as well as many others. The case when jumps are present is discussed in [14].

The value $V_A(x, t)$ of a American option with a payoff $h(x)$, satisfies the following partial differential inequality, (partial integro-differential inequality in the case of process with jumps)

$$\begin{cases} \frac{\partial V_A(x, t)}{\partial t} + \mathcal{L}V_A(x, t) - rV_A(x, t) \leq 0 \\ V_A(x, t) \geq h(x) \\ \left(\frac{\partial V_A(x, t)}{\partial t} + \mathcal{L}V_A(x, t) - rV_A(x, t) \right) (V_A(x, t) - h(x)) = 0 \end{cases} \quad (3.7)$$

with the terminal condition $V_A(x, T) = h(x)$. The last equation is called the complementarity condition. Here \mathcal{L} is the infinitesimal generator of the Lévy process driving the log of the underlying asset.

With respect to the time to maturity, $\tau = T - t$, the inequality (3.7) above becomes the following initial value problem:

$$\begin{cases} \frac{\partial v(x, \tau)}{\partial \tau} - \mathcal{L}v(x, \tau) + rv(x, \tau) \geq 0 \\ v(x, \tau) \geq h(x) \\ \left(\frac{\partial v(x, \tau)}{\partial \tau} - \mathcal{L}v(x, \tau) - rv(x, \tau) \right) (V_A(x, \tau) - h(x)) = 0 \end{cases} \quad (3.8)$$

with the initial condition $v(x, 0) = h(x)$. Here $V_A(x, t) = v(x, T - t)$.

An American put has the payoff defined as $h(x) = (K - e^x)^+$ and for the American call the payoff $h(x) = (e^x - K)^+$. Here $(\cdot)^+ = \max(\cdot, 0)$ and K is the strike price. If we let $\tau \rightarrow \infty$, the value of the American option converges to the value of a perpetual American put. In the case of the call, since we have no dividends, the American call equals to the European call. As $\tau \rightarrow \infty$ the American call does not converge and the value of the perpetual American call does not exist.

The partial differential inequality, (partial integro-differential inequality in the case of process with jumps) together with the linear complementarity condition form a linear complementarity problem. Solving this type of problem will play an important part in getting numerical values for the game options in chapter 6. This bridge established between the values of American options and solutions to LCP type of problems will have a key motivational role in establishing the connection between values of game options and parabolic partial (integro) differential inequalities which is the core of this paper.

Chapter 4

Elliptic Equations and Inequalities

4.1 Elliptic Equations

4.1.1 Introduction

We shall begin by introducing some basic but necessary notions which will be used as building blocks in our work later. Consider a real Hilbert space \mathcal{V} together with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ with the norm on \mathcal{V} induced by this inner product defined as follows

$$\|v\|_{\mathcal{V}}^2 = \langle v, v \rangle_{\mathcal{V}} \quad \forall v \in \mathcal{V}$$

We identify the dual space of \mathcal{V} by \mathcal{V}^* , with the induced norm $\|\cdot\|_{\mathcal{V}^*}$ defined as

$$\|f\|_{\mathcal{V}^*} = \sup_{v \in \mathcal{V}} \frac{|\langle f, v \rangle|}{\|v\|_{\mathcal{V}}}$$

We have an operator $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ given by the infinitesimal generator of the Lévy process

$$\mathcal{A}(u) = -\mathcal{L}u + ru \tag{4.1}$$

The infinitesimal generator \mathcal{L} is defined in the pure diffusion case in (3.1), for the pure jump process in (3.3) and for jump diffusion in (3.4).

The order of the operator \mathcal{A} depends of the infinitesimal generator of the Lévy process

- in the pure diffusion the differential operator \mathcal{A} is of second order
- when we have pure jump CGMY process, the integro-differential operator \mathcal{A} is of order $Y < 2$ (see [59], [60], or [65] for more details)
- in the jump-diffusion case the operator \mathcal{A} is again of second order

The order of the operator will determine our choice of the energy space \mathcal{V} .

Associated with the operator \mathcal{A} is the bilinear form $a(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ via

$$a(u, v) = \langle \mathcal{A}u, v \rangle \quad \forall u, v \in \mathcal{V} \quad (4.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{V}^* \times \mathcal{V}$ duality pairing. We use $\langle f, v \rangle$ both for $f, v \in L_2(\mathbb{R})$ and for $f \in \mathcal{V}^*, v \in \mathcal{V}$.

We assume that the bilinear form $a(\cdot, \cdot)$ has the following two properties for $\forall u, v \in \mathcal{V}$ continuity

$$|a(u, v)| \leq C_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad (4.3)$$

and coercivity

$$a(v, v) \geq C_2 \|u\|_{\mathcal{V}}^2 \quad (4.4)$$

There is a natural norm associated with the bilinear operator $a(\cdot, \cdot)$ which is called the energy norm denoted by

$$\|u\|_a^2 = a(v, v) \quad (4.5)$$

From the coercivity and the continuity condition we can see that the energy norm $\|\cdot\|_a$ is equivalent to $\|\cdot\|_{\mathcal{V}}$.

Consider Ω to be some domain in \mathbb{R} . If the derivative of u doesn't exist in the

classical sense, which may be the case when $u \in L_2(\Omega)$, we can define the derivative in the weak sense (see [76] or other reference on Sobolev spaces for the definition of the weak derivatives).

We shall define $H^k(\Omega)$ to be the space of all functions whose weak partial derivatives of order at most k belong to $L_2(\Omega)$

$$H^k(\Omega) = \{u \in L_2(\Omega) \mid d^\alpha u \in L_2(\Omega) \text{ for } \alpha \leq k\}$$

which will be equipped with the following inner product

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{\alpha \leq k} \int_{\Omega} (d^\alpha u)(d^\alpha v) dx,$$

and the corresponding norm

$$\|u\|_{H^k(\Omega)}^2 = \langle u, u \rangle_{H^k(\Omega)} = \left(\sum_{\alpha \leq k} \int_{\Omega} (d^\alpha u)^2 dx \right)$$

Since $L_2(\Omega)$ is complete, $H^k(\Omega)$ is in turn complete and therefore a Hilbert space.

An alternative definition for Sobolev spaces H^s can be formulated in term of the Fourier Transform, (see [21]). A function $f \in L_2(\mathbb{R})$ is in $H^s(\mathbb{R})$ if the following holds

$$(1 + |x|^s) \hat{f} \in L_2(\mathbb{R})$$

where \hat{f} is the Fourier transform of f .

The $\|\cdot\|_{H^s(\mathbb{R})}$ can be defined as

$$\|u\|_{H^s(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + |\xi|^s)^2 \hat{f}^2 d\xi$$

We will define $\tilde{H}^s(\Omega)$ as follows

$$\tilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}), u(\mathbb{R} \setminus \Omega) = 0\}$$

For the remaining of the paper the energy space that will be considered will depend on the Lévy process used for model the underlying asset. We will use the following Sobolev spaces for \mathcal{V}

$$\mathcal{V} = \tilde{H}^s(\Omega) = \begin{cases} \tilde{H}^1(\Omega) & \text{i.e. } s = 1 \text{ for pure diffusion} \\ \tilde{H}^{\frac{\gamma}{2}}(\Omega) & \text{i.e. } s = \frac{\gamma}{2} \text{ for pure jump CGMY} \\ \tilde{H}^1(\Omega) & \text{i.e. } s = 1 \text{ for jump diffusion} \end{cases} \quad (4.6)$$

The dual space of \mathcal{V} will be identified by $\mathcal{V}^* = H^{-s}(\Omega)$.

4.1.2 Variational Formulation for Elliptic Equations

Consider the Hilbert space \mathcal{V} and its dual \mathcal{V}^* , as defined in (4.6) with $\Omega = \mathbb{R}$.

Let $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ be the continuous and coercive operator defined in (4.1).

Consider the elliptic equation

$$\mathcal{A}u = f \text{ in } \Omega \quad (4.7)$$

with $u \in \mathcal{V}$, and $f \in \mathcal{V}^*$.

We will solve this problem in a more general abstract setting. This is called the variational setting and will allow for more general solutions u . We also have available the theory for existence and uniqueness of solutions with less restrictions on f .

Using test functions $v \in \mathcal{V}$ we obtain the variational form of the equation (4.7)

$$\langle \mathcal{A}u, v \rangle = \langle f, v \rangle \quad \forall v \in \mathcal{V}$$

Using the notations introduced at the beginning of the chapter, the variational form can be rewritten as: find $u \in \mathcal{V}$ such that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V} \quad (4.8)$$

Theorem 4.1. (*Lax-Milgram*)

If $f \in \mathcal{V}^*$, and $a(\cdot, \cdot)$ continuous, (4.3), and coercive, (4.4), then the equation (4.8) admits a unique solution $u \in \mathcal{V}$. u is called the weak or variational solution of equation (4.8). Also there exists a constant C such that the following holds

$$\|u\|_{\mathcal{V}} \leq C \|f\|_{\mathcal{V}^*}$$

where $C = \frac{1}{c_2}$.

Proof see section 2 in [50] for details.

4.1.3 Localization to a Bounded Domain

We localize the space domain to a finite interval $\Omega = (-R, R)$. The error induced by localization decreases exponentially in R .

The variational equality (4.8) can be rewritten as: find $u \in \mathcal{V}$ such that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V} \tag{4.9}$$

By the theorem 4.1 the localized problem (4.9) admits a unique solution in \mathcal{V} .

4.1.4 Discretization of the Space Domain

We discretize the space domain $\Omega = (-R, R)$ using a uniform mesh $-R = x_0 < x_1 < \dots < x_{N+1} = R$.

We replace the infinite dimensional space \mathcal{V} with a finite dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$ of continuous piecewise linear functions on the mesh $\{x_0, x_1, \dots, x_{N+1}\}$

which are zero outside the domain Ω .

$$\mathcal{V}_h = \tilde{H}^s(\Omega) = \begin{cases} \tilde{H}^1(\Omega) & \text{i.e. } s = 1 \text{ for pure diffusion} \\ \tilde{H}^{\frac{Y}{2}}(\Omega) & \text{i.e. } s = \frac{Y}{2} \text{ for pure jump CGMY} \\ \tilde{H}^1(\Omega) & \text{i.e. } s = 1 \text{ for jump diffusion} \end{cases} \quad (4.10)$$

The variational problem (4.9) becomes: find $u \in \mathcal{V}_h$ such that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V}_h \quad (4.11)$$

By the theorem 4.1 the problem (4.11) admits a unique solution in \mathcal{V}_h .

4.1.5 Finite Elements for Elliptic Equations

The finite dimensional space \mathcal{V}_h has a basis of hat functions $\mathcal{V}_h = \text{span}\{\phi_1, \dots, \phi_N\}$.

We define the basis functions $\phi_i(x)$ with $i = 1, \dots, N$

$$\phi_i(x) = \begin{cases} 1 + \frac{x-x_i}{x_i-x_{i-1}} & \text{if } x_{i-1} \leq x < x_i \\ 1 + \frac{x_i-x}{x_{i+1}-x_i} & \text{if } x_i \leq x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (4.12)$$

Any $v \in \mathcal{V}_h$ can be expressed in terms of the hat functions ϕ_i defined in (4.12) since they form a basis for \mathcal{V}_h

$$v(x) = \sum_{i=1}^N v_i \phi_i(x)$$

Let us denote the coefficients v_i , $i = 1, \dots, N$, using the column vector $\vec{v} \in \mathbb{R}^N$,

$$\vec{v}^\top = [v_1, \dots, v_N].$$

The variational problem (4.11) can be written in the matrix form: find $\vec{u} \in \mathbb{R}^N$ such

that:

$$A\vec{u} = \vec{f} \quad (4.13)$$

Here A is called the stiffness matrix defined as

$$A_{ij} = a(\phi_j, \phi_i) = \langle \mathcal{A}\phi_j, \phi_i \rangle \quad (4.14)$$

and \vec{f} is called the load vector

$$f_i = \langle f, \phi_i \rangle \quad (4.15)$$

Notice that A is positive definite due to coerciveness of $a(\cdot, \cdot)$ and therefore the equation above will admit a unique solution which is called the finite element solution of (4.11).

4.2 Elliptic Inequalities

4.2.1 Introduction

Consider the Hilbert space \mathcal{V} and its dual \mathcal{V}^* , as defined in (4.6) with $\Omega = \mathbb{R}$.

Let $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ be the continuous and coercive operator defined in (4.1).

Consider the following system of inequalities

$$\left\{ \begin{array}{ll} \mathcal{A}u \geq f & \text{in } \Omega \\ u \geq g & \text{in } \Omega \\ (\mathcal{A}u - f)(u - g) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R} \setminus \Omega \end{array} \right. \quad (4.16)$$

We will solve this problem in the variational setting which allows for more general solutions u . We also have available the theory for existence and uniqueness of

solutions with less restrictions on f .

One of our motivations for the problem above is the perpetual American put option, which is an American put option for which the maturity time is ∞ . If the elliptic operator \mathcal{A} is the infinitesimal generator of Lévy process for the underlying asset, then the value of a perpetual American put option on such asset can be found by solving the problem (4.16) above. For more details on perpetual American options the reader should consult [10]. Another example where a similar problem is solved is that of perpetual game put and perpetual game call. In this case the solution is bounded from above and below. The second condition of the problem (4.16) becomes $F \leq u \leq G$. The perpetual game call value exists because of the presence of the upper obstacle. The existence and uniqueness of solutions of the perpetual game puts, perpetual game call and perpetual American puts will be discussed in detail in section 6.2.

4.2.2 Variational Formulation for Elliptic Inequalities

Consider $\mathcal{K} \subset \mathcal{V}$ a convex subset of \mathcal{V} defined as

$$\mathcal{K} = \left\{ v \mid v \in \mathcal{V}, v \geq g \right\}$$

Such \mathcal{K} is used in the case of one sided problems like the perpetual American put.

When we have two sided problems, like in the case of game options, the subspace

\mathcal{K} is defined as

$$\mathcal{K} = \left\{ v \mid v \in \mathcal{V}, F \leq v \leq G \right\}$$

Using the notation introduced in the previous section, the problem (4.16) can be reformulated in the variational form, called variational inequality, as follows: find $u \in \mathcal{K}$ such that

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K} \quad (4.17)$$

Theorem 4.2. (*Lions-Stampacchia*)

If $a(\cdot, \cdot)$ is continuous (4.3), coercive (4.4), $f \in \mathcal{V}^*$, and \mathcal{K} is a closed convex subset of \mathcal{V} , then there exists a unique solution $u \in \mathcal{K}$ of the variational inequality (4.17).

Proof. We will show a constructive proof for existence of solutions [52]. This will be used in chapter 7 to construct an iterative method for solving the matrix form of the variational inequality above.

We will first consider the case when $a(\cdot, \cdot)$ is symmetric. In this setting we can solve the variational inequality (4.17) by finding a solution to the following energy minimization problem: find $u \in \mathcal{K}$ that minimizes the following

$$E(u) = \frac{1}{2}a(u, u) - \langle f, u \rangle$$

Since $f \in \mathcal{V}^*$ we have

$$|\langle f, v \rangle| \leq C_f \|v\|_{\mathcal{V}} \quad \forall v \in \mathcal{V}$$

Using the continuity of $a(\cdot, \cdot)$ we get

$$\begin{aligned} E(u) &= \frac{1}{2}a(u, u) - \langle f, u \rangle \\ &\geq \frac{C_2}{2} \|u\|_{\mathcal{V}}^2 - C_f \|u\|_{\mathcal{V}} \\ &\geq \frac{C_2}{2} \left(\|u\|_{\mathcal{V}}^2 - \frac{2C_f}{C_2} \|u\|_{\mathcal{V}} \right) \\ &\geq \frac{C_2}{2} \left(\|u\|_{\mathcal{V}} - \frac{C_f}{C_2} \right)^2 - \frac{C_f^2}{2C_2} \\ &\geq -\frac{C_f^2}{2C_2} \end{aligned}$$

Consider $E_* = \inf_{u \in \mathcal{K}} E(u)$. Then there exists a sequence $u_n \in \mathcal{K}$ with $E_n = E(u_n) \rightarrow E_*$ as $n \rightarrow \infty$.

We want to show next that u_n is a Cauchy sequence. For $\forall u_m, u_n \in \mathcal{K}$ we have that $\frac{u_m + u_n}{2} \in \mathcal{K}$ and therefore we obtain

$$\begin{aligned} E\left(\frac{u_m + u_n}{2}\right) &= \frac{1}{2}a\left(\frac{u_m + u_n}{2}, \frac{u_m + u_n}{2}\right) - \langle f, u_m + u_n \rangle \\ &= \frac{1}{8}a(u_m, u_m) + \frac{1}{8}a(u_n, u_n) + \frac{1}{4}a(u_m, u_n) - \langle f, u_m + u_n \rangle \\ &\geq E_* \end{aligned}$$

This implies that

$$-2a(u_m, u_n) \leq a(u_m, u_m) + a(u_n, u_n) - 4\langle f, u_m + u_n \rangle - 8E_* \quad (4.18)$$

Since $E_n = E(u_n)$ converges to E_* we have that $\forall \epsilon > 0 \exists N(\epsilon)$ such that $|E_n - E_*| \leq \frac{\epsilon}{8}$ for all $n > N(\epsilon)$, which implies that $E_n \leq E_* + \frac{\epsilon}{8}$ for all $n > N(\epsilon)$.

From the definition of the energy norm we get

$$\begin{aligned} \|u_m - u_n\|_{\mathcal{V}}^2 &= a(u_m - u_n, u_m - u_n) \\ &= a(u_m, u_m) + a(u_n, u_n) - 2a(u_m, u_n) \end{aligned}$$

Using (4.18) we obtain for $\forall m, n \geq N(\epsilon)$

$$\begin{aligned} \|u_m - u_n\|_{\mathcal{V}}^2 &\leq a(u_m, u_m) + a(u_n, u_n) + a(u_m, u_m) + a(u_n, u_n) \\ &\quad - 4\langle f, u_m + u_n \rangle - 8E_* \\ &\leq 4\left(\frac{1}{2}a(u_m, u_m) - \langle f, u_m \rangle\right) \\ &\quad + 4\left(\frac{1}{2}a(u_n, u_n) - \langle f, u_n \rangle\right) - 8E_* \\ &\leq 4E_m + 4E_n - 8E_* \leq 4\left(E_* + \frac{\epsilon}{8}\right) + 4\left(E_* + \frac{\epsilon}{8}\right) - 8E_* \\ &\leq \epsilon \end{aligned}$$

Since u_n is a Cauchy sequence in the Hilbert space \mathcal{K} then $\exists u \in \mathcal{K}$ such that $u_n \rightarrow u$ with respect to the energy norm $\|\cdot\|_{\mathcal{V}}$ as $n \rightarrow \infty$.

Next we want to show that $E(u) = E_*$. The continuity of $a(\cdot, \cdot)$ gives us that $a(u_n, u_n) \rightarrow a(u, u)$. From the continuity of f we get that $\langle f, u_n \rangle \rightarrow \langle f, u \rangle$. We have $E(u_n) \rightarrow E(u)$ as $n \rightarrow \infty$ which implies that $E(u) = E_*$.

We need to show that u solves the variational inequality (4.17). Let $v \in \mathcal{K}$ and $w = u + \lambda(v - u)$ with $\lambda \in (0, 1)$.

$$\begin{aligned} E(w) &= E(u + \lambda(v - u)) \\ &= \frac{1}{2}a(u + \lambda(v - u), u + \lambda(v - u)) - \langle f, u + \lambda(v - u) \rangle \\ &= \frac{1}{2}a(u, u) - \langle f, u + \lambda(v - u) \rangle + \\ &\quad + \lambda[a(u, v - u) - \langle f, v - u \rangle] + \frac{1}{2}\lambda^2 a(v - u, v - u) \end{aligned}$$

Using that $E(u) \leq E(v)$ for $\forall v \in \mathcal{K}$ we obtain

$$\lambda[a(u, v - u) - \langle f, v - u \rangle] + \frac{1}{2}\lambda^2 a(v - u, v - u) \geq 0$$

Since $f \in \mathcal{V}^*$ then $\langle f, v \rangle < \infty \forall v \in \mathcal{V}$. Therefore we have that the inequality above is true if the coefficient of λ is positive and therefore we have that (4.17) holds

$$a(u, v - u) - \langle f, v - u \rangle \geq 0$$

We have shown existence of solutions when $a(\cdot, \cdot)$ is symmetric. To prove existence in the case when $a(\cdot, \cdot)$ is non-symmetric we reduced the non-symmetric problem to a symmetric case at each step of a fixed-point iteration problem.

Consider the symmetric bilinear $q(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$

$$q(u, v) = \langle u, v \rangle_{\mathcal{V}} \quad \forall u, v \in \mathcal{V}$$

and the symmetric operator $Q : \mathcal{V} \rightarrow \mathcal{V}^*$ such that

$$q(u, v) = \langle Qu, v \rangle \quad \forall u, v \in \mathcal{V}$$

Let us define the projection $\mathcal{P}_{\mathcal{K}} : \mathcal{V} \rightarrow \mathcal{K}$ such that for $u \in \mathcal{V}$, $u_{\mathcal{K}} = \mathcal{P}_{\mathcal{K}}u$ satisfies

$$\|u - u_{\mathcal{K}}\|_{\mathcal{V}} = \min_{v \in \mathcal{K}} \|u - v\|_{\mathcal{V}}$$

Note that for the projection $\mathcal{P}_{\mathcal{K}}$ defined above, the following inequality is satisfied

$$q(u - \mathcal{P}_{\mathcal{K}}u, v - u) \geq 0 \quad \forall v \in \mathcal{K}$$

We can solve the variational inequality (4.17) iteratively using the projection defined above, given a starting value $u_0 \in \mathcal{V}$

$$u_{n+1} = \mathcal{P}_{\mathcal{K}}(u_n + \alpha Q^{-1}r_n) \quad \forall n \geq 0$$

where r_n is the residual term of the n th iteration defined as

$$r_n = f - \mathcal{A}u_n$$

Each step of the iteration is equivalent to the following variational problem in terms of $q(\cdot, \cdot)$: find $u_{n+1} \in \mathcal{K}$ such that $\forall v \in \mathcal{K}$ we have the following

$$q(u_{n+1} - u_n, v - u_{n+1}) \geq \alpha(\langle f, v - u_{n+1} \rangle - a(u_n, v - u_{n+1}))$$

We can rewrite this as a symmetric problem: find $u_{n+1} \in \mathcal{K}$ such that $\forall v \in \mathcal{K}$

$$q(u_{n+1} - u_n, v - u_{n+1}) \geq \langle \tilde{f}, v - u_{n+1} \rangle$$

where $\tilde{f} = \alpha(f - \mathcal{A}u_n) + Qu_n$.

Notice that $\tilde{f} \in \mathcal{V}^*$ which implies by the first half of our proof that each iteration

step will have a unique solution u_{n+1} . Denote the operator associated with this fixed problem $\mathcal{S}_{\mathcal{K}}$, where

$$\mathcal{S}_{\mathcal{K}}(u) = \mathcal{P}_{\mathcal{K}}(u + \alpha Q^{-1}(f - \mathcal{A}u))$$

In order for our fixed point iteration to converge we need $\mathcal{S}_{\mathcal{K}}$ to be a contraction with respect to the norm $\|\cdot\|_q$ defined as

$$\|u\|_q^2 = q(u, u)$$

Recall the continuity constant C_1 from (4.3) and the coercivity constant C_2 from (4.4). Using this notation we obtain the following estimate for the q-norm of the mapping $\mathcal{S}_{\mathcal{K}}$

$$\begin{aligned} \|\mathcal{S}_{\mathcal{K}}(u) - \mathcal{S}_{\mathcal{K}}(v)\|_q^2 &= \|u + \alpha Q^{-1}(f - \mathcal{A}u) - v - \alpha Q^{-1}(f - \mathcal{A}v)\|_q^2 \\ &= \|u - v - \alpha Q^{-1}\mathcal{A}(u - v)\|_q^2 \\ &= \langle Q(u - v) - \alpha\mathcal{A}(u - v), u - v - \alpha Q^{-1}\mathcal{A}(u - v) \rangle_{\mathcal{V}} \\ &= \|u - v\|_q - 2\alpha a(u - v, u - v) + \\ &\quad + \alpha^2 a(u - v, Q^{-1}\mathcal{A}(u - v)) \\ &= (1 - 2\alpha C_2 + \alpha^2 C_1^2) \|u - v\|_q \\ &= C \|u - v\|_q \end{aligned}$$

with $C = 1 - 2\alpha C_2 + \alpha^2 C_1^2$.

For $0 < \alpha < \frac{2C_2}{C_1^2}$, $C < 1$ which implies that $\mathcal{S}_{\mathcal{K}}$ is a contraction.

Since \mathcal{V} is a Hilbert space we have by the Contraction Mapping Theorem that for $\forall u_0 \in \mathcal{V}$ the fixed point iteration will converge to a unique solution u^* which satisfies $\forall v \in \mathcal{K}$

$$q(u^* - u^*, v - u^*) \geq \alpha (\langle f, v - u^* \rangle - a(u^*, v - u^*))$$

This implies

$$a(u^*, v - u^*) \geq \langle f, v - u^* \rangle$$

which means that u^* satisfies (4.17). This completes our proof of existence of solutions.

To show uniqueness we will consider two solutions u_1 and u_2 which satisfy the variational inequality (4.17)

$$a(u_1, v - u_1) \geq \langle f, v - u_1 \rangle$$

and

$$a(u_2, v - u_2) \geq \langle f, v - u_2 \rangle$$

Let $v = u_2$ in the first inequality, $v = u_1$ in the second inequality and add to obtain

$$a(u_2 - u_1, u_2 - u_1) \leq 0$$

Uniqueness $u_1 = u_2$ follows.

□

4.2.3 Localization to a Bounded Domain

We localize the space domain to a finite interval $\Omega = (-R, R)$. The error induced by localization decreases exponentially in R .

The variational inequality (4.17) can be rewritten as: find $u \in \mathcal{K}$ such that

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K} \tag{4.19}$$

By the theorem 4.2 the localized problem (4.19) admits a unique solution in \mathcal{K} .

4.2.4 Discretization of the Space Domain

We discretize the space domain $\Omega = (-R, R)$ using a uniform mesh $-R = x_0 < x_1 < \dots < x_{N+1} = R$.

We replace the infinite dimensional space \mathcal{V} with a finite dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$ of continuous piecewise linear functions defined in (4.10). Let $\mathcal{K}_h \subset \mathcal{V}_h$ be the set of continuous piecewise linear functions in \mathcal{V}_h defined as

$$\mathcal{K}_h = \left\{ v \mid v \in \mathcal{V}_h, v(x_i) \geq g(x_i), i = 1, \dots, N \right\}$$

In the case of game options \mathcal{K}_h is defined as

$$\mathcal{K}_h = \left\{ v \mid v \in \mathcal{V}_h, F(x_i) \leq v(x_i) \leq G(x_i), i = 1, \dots, N \right\}$$

The variational problem (4.19) becomes: find $u \in \mathcal{K}_h$ such that

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}_h \quad (4.20)$$

By the theorem 4.2 the problem (4.20) admits a unique solution in \mathcal{K}_h .

4.2.5 Finite Elements for Elliptic Inequalities

The finite dimensional space \mathcal{V}_h has a basis of hat functions $\mathcal{V}_h = \text{span}\{\phi_1, \dots, \phi_N\}$.

The basis functions $\phi_i(x)$ are defined in (4.12).

Any $v \in \mathcal{V}_h$ can be expressed in terms of the hat functions ϕ_i defined in (4.12) since they form a basis for \mathcal{V}_h

$$v(x) = \sum_{i=1}^N v_i \phi_i(x)$$

Let us denote the coefficients v_i , $i = 1, \dots, N$, using the column vector $\vec{v} \in \mathbb{R}^N$, $\vec{v}^\top = [v_1, \dots, v_N]$.

Let $\mathcal{K}_N \subset \mathbb{R}_N$ be defined as

$$\mathcal{K}_N = \left\{ v \mid v \in \mathbb{R}_N, v_i \geq g(x_i), i = 1, \dots, N \right\}$$

In the case of game options \mathcal{K}_N is defined as

$$\mathcal{K}_N = \left\{ v \mid v \in \mathbb{R}_N, F(x_i) \leq v_i \leq G(x_i), i = 1, \dots, N \right\}$$

The variational inequality (4.20) can be written in the matrix form as the following linear complementarity problem (LCP): find $\vec{u} \in \mathcal{K}_N$ such that

$$A\vec{u} \geq \vec{f} \tag{4.21}$$

Here A is the stiffness matrix defined in (4.14) and \vec{f} is the load vector defined in (4.15).

By the theorem 4.2 the problem (4.21) admits a unique solution in \mathcal{K}_N . This LCP problem can be solved using the iterative method which was introduced at the beginning of this section and will be presented in detail in chapter 7.

The error bounds for the discrete solution with respect to the energy norm $\|\cdot\|_{\mathcal{V}}$, are given by the following expressions: in general

$$\|u - u_h\|_{\mathcal{V}}^2 \leq C \inf_{v_h \in \mathcal{K}_h} \left(\|u - v_h\|_{\mathcal{V}}^2 + \|\mathcal{A}u - f\|_{\mathcal{V}^*} \|u - v_h\|_{\mathcal{V}} \right) \tag{4.22}$$

If we have the regularity assumption $\mathcal{A}u - f \in L_2$ we get

$$\|u - u_h\|_{\mathcal{V}}^2 \leq C \inf_{v_h \in \mathcal{K}_h} \left(\|u - v_h\|_{\mathcal{V}}^2 + \|\mathcal{A}u - f\|_{L_2} \|u - v_h\|_{L_2} \right) \tag{4.23}$$

The results above hold when the condition $\mathcal{K}_h \subset \mathcal{K}$ is satisfied.

We have the following error bounds in the energy norm based on interpolation by piecewise linear polynomials, I_h [11]

$$\|u - I_h u\|_{H^s} = ch^{m-s} \|u\|_{H^m} \quad (4.24)$$

for $0 \leq s < m \leq 2$.

In the case the δ penalty game options we will use excess to payoff functions and therefore the bounds of \mathcal{K} are zero and constant functions. Since constant functions are piecewise linear functions we have that $\mathcal{K}_h \subset \mathcal{K}$.

We use (4.23) together with (4.24) to get the energy norm error estimates below.

In the Black Scholes model (pure diffusion process) the energy space is $\mathcal{V} = \tilde{H}^1(\Omega)$.

In this case $\mathcal{A}u - f \in L_2$ and the solution $u \in H^2$ [49]. We get the following error bound in the energy norm

$$\|u - u_h\|_{\mathcal{V}} \leq c_1 h \|u\|_{H^2} + c_2 h \|u\|_{H^2} \leq ch \|u\|_{H^2}$$

In the pure jump model (CGMY process) the energy space is $\mathcal{V} = \tilde{H}^{\frac{Y}{2}}(\Omega)$. The solution $u \in H^{1+\frac{Y}{2}-\epsilon}(\Omega)$ due to endpoint barrier singularities and $\mathcal{A}u - f \in L_2$ [60]. We have the following estimate for the energy norm of the error of the discrete solution

$$\begin{aligned} \|u - u_h\|_{\mathcal{V}} &\leq c_1 h \|u\|_{H^{1+\frac{Y}{2}-\epsilon}} + c_2 h^{\frac{1}{2}+\frac{Y}{4}-\epsilon} \|u\|_{H^{1+\frac{Y}{2}-\epsilon}} \\ &\leq ch^{\frac{1}{2}+\frac{Y}{4}-\epsilon} \|u\|_{H^{1+\frac{Y}{2}-\epsilon}} \end{aligned}$$

Note that if u satisfies $u \in H^2(\Omega)$ we will get the better rate

$$\begin{aligned} \|u - u_h\|_{\mathcal{V}} &\leq c_1 h^{2-\frac{Y}{2}} \|u\|_{H^2} + c_2 h \|u\|_{H^2} \\ &\leq ch \|u\|_{H^2} \end{aligned}$$

Chapter 5

Parabolic Equations and Inequalities

5.1 Parabolic Equations

5.1.1 Introduction

In the previous chapter we have introduced elliptic operators and the corresponding elliptic equations and inequalities. We will take the elliptic equations and add the time component by introducing a derivative with respect to time into the equation.

In the elliptic case we have considered one Hilbert space \mathcal{V} , in the parabolic problems we will consider two Hilbert spaces, $\mathcal{V} \subset \mathcal{H}$, such that \mathcal{V} is dense in \mathcal{H} and $\mathcal{V} \hookrightarrow \mathcal{H}$ is an injection.

The inner products in these spaces will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ with the respective induced norms $\| \cdot \|_{\mathcal{V}}$ and $\| \cdot \|_{\mathcal{H}}$ such that the following holds

$$\|v\|_{\mathcal{H}} \geq \|v\|_{\mathcal{V}} \quad \forall v \in \mathcal{H}$$

\mathcal{H} will be identified with its dual, $\mathcal{H} \cong \mathcal{H}^*$, and denoting the dual of \mathcal{V} by \mathcal{V}^* , we have the following nested injection, called the Gelfand triple

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

The dual space \mathcal{V}^* will be equipped with the following dual norm

$$\|w\|_{\mathcal{V}^*} = \sup_{v \in \mathcal{H}, \|v\|_{\mathcal{V}}=1} \langle w, v \rangle_{\mathcal{H}}$$

With $\mathcal{J} = [0, T]$, let the space \mathcal{W} be defined as

$$\mathcal{W} = \left\{ v \mid v \in L_2(\mathcal{J}, \mathcal{V}), v' = \frac{\partial v}{\partial t} \in L_2(\mathcal{J}, \mathcal{V}^*) \right\}$$

equipped with the inner product

$$\langle u, v \rangle_{\mathcal{W}} = \langle u, v \rangle_{L_2(\mathcal{J}, \mathcal{V})} + \langle u', v' \rangle_{L_2(\mathcal{J}, \mathcal{V}^*)}$$

with the associated norm

$$\|u\|_{\mathcal{W}}^2 = \langle u, u \rangle_{\mathcal{W}}$$

Consider the elliptic operator $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ defined in (4.1) and the domain Ω . The time domain \mathcal{J} will be considered to be finite, $\mathcal{J} = [0, T], T < \infty$. In this setting we will introduce the following initial boundary value problem

$$\begin{aligned} u_t - \mathcal{A}u &= f && \text{in } \Omega \times \mathcal{J} \\ u &= 0 && \text{in } (\mathbb{R} \setminus \Omega) \times \mathcal{J} \\ u(\cdot, 0) &= g && \text{in } \Omega \end{aligned} \tag{5.1}$$

We will use the variational setting to solve the problem. This will allow for more general solutions u as well as existence and uniqueness of solutions with less restrictions on f and more general initial data g .

The energy space \mathcal{V} considered will depend on the Lévy process driving the underlying asset, as defined in (4.6). The space \mathcal{H} used here will be $\mathcal{H} = L_2(\Omega)$.

Our the motivation for the problem above is pricing of European options for assets

driven by Lévy process. The values of the options can be obtained as solutions to the parabolic equations above, when the elliptic operator is given by the infinitesimal generator of the Lévy process. Recall from the previous chapter that, in the case of the Black-Scholes model, the infinitesimal generator is of second order. In this situation, one can use fundamental solutions of the equations above to compute the closed forms for the values of European options, such as for example the European call and European put.

5.1.2 Variational Formulation for Parabolic Equations

To get the variational formulation of the parabolic equations we will start with the same technique we have used for the elliptic case. We have the domain $\Omega = \mathbb{R}$ and the energy space \mathcal{V} defined in (4.6). The intermediate space \mathcal{H} considered, which is identified with its dual, will be $\mathcal{H} = L_2(\Omega)$ such that we have the Gelfand triple with dense embeddings introduced in the previous section. The first step to get the variational form of (5.1) is multiplying by the test function $v \in \mathcal{V}$ and integrating over Ω to get

$$\langle u_t, v \rangle + a(u, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V} \quad t \in (0, T] \quad (5.2)$$

with the initial condition

$$u(\cdot, 0) = g \quad \text{in } \Omega$$

where $a(\cdot, \cdot)$ is the continuous (4.3) and coercive (4.4) bilinear associated with the operator \mathcal{A} defined in (4.2). Here u_t is the weak derivative of u with respect to time.

To get the variational formulation of the problem (5.1), the next step is to multiply by a test function and integrate over the time domain. The variational problem (5.1) can now be formulated as follows: find $u = u(x, t) \in L_2(\mathcal{J}, \mathcal{V}) \cap H^1(\mathcal{J}, \mathcal{V}^*)$, such that the following holds for all $v \in \mathcal{V}$ and $\phi \in C_0^\infty$

$$-\int_{\mathcal{J}} \langle u(t), v \rangle \phi'(t) dt + \int_{\mathcal{J}} a(u, v) \phi(t) dt = \int_{\mathcal{J}} \langle f(t), v \rangle \phi(t) dt \quad (5.3)$$

with the initial condition

$$u(\cdot, 0) = g \quad \text{in } \Omega$$

Additionally we also need that $u_t \in L_2(\mathcal{J}, \mathcal{V}^*)$, i.e. $u \in \mathcal{W}$, in order for u to be the weak solution of (5.2) on \mathcal{J} .

Under the assumption

$$g \in \mathcal{H}, \quad f \in L_2(\mathcal{J}, \mathcal{H})$$

variational formulation (5.2) admits a unique solution $u \in \mathcal{W}$, [51],[73], [57], and the following estimate holds [72]

$$\|u\|_{\mathcal{W}} \leq C(\|f\|_{L_2(\mathcal{J}, \mathcal{H})} + \|g\|_{\mathcal{H}})$$

5.1.3 Localization of the Space Domain

We localize the space domain to a finite interval $\Omega = (-R, R)$. We will show in the next chapter using a stochastic proof that the error induced by localization decreases exponentially in R .

The variational equality (5.2) can be rewritten as: find $u \in \mathcal{W}$ such that

$$\langle u_t, v \rangle + a(u, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V} \quad t \in (0, T] \quad (5.4)$$

with the initial condition

$$u(\cdot, 0) = g \quad \text{in } \Omega$$

The localized problem (5.4) admits a unique solution in \mathcal{W} .

5.1.4 Discretization of the Space Domain

We discretize the space domain $\Omega = (-R, R)$ using a uniform mesh $-R = x_0 < x_1 < \dots < x_{N+1} = R$.

We replace the infinite dimensional space \mathcal{V} with a finite dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$ of continuous piecewise linear functions on the mesh $\{x_0, x_1, \dots, x_{N+1}\}$ defined in (4.10). The variational problem (5.4) becomes: find $u \in L_2(\mathcal{J}, \mathcal{V}_h) \cap H^1(\mathcal{J}, \mathcal{V}_h^*)$ such that

$$\langle u_t, v \rangle + a(u, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V}_h \quad t \in (0, T] \quad (5.5)$$

with the initial condition

$$u(\cdot, 0) = g \quad \text{in } \Omega$$

The discretized problem (5.5) has a unique solution in $L_2(\mathcal{J}, \mathcal{V}_h) \cap H^1(\mathcal{J}, \mathcal{V}_h^*)$.

5.1.5 Finite Elements for Parabolic Equations

The finite dimensional space \mathcal{V}_h has a basis of hat functions $\mathcal{V}_h = \text{span}\{\phi_1, \dots, \phi_N\}$.

The basis functions $\phi_i(x)$ are defined in (4.12).

Any $v \in (\mathcal{J}, \mathcal{V}_h)$ can be expressed in terms of the hat functions ϕ_i defined in (4.12)

since they form a basis for \mathcal{V}_h

$$v(x, t) = \sum_{i=1}^N v_i(t) \phi_i(x)$$

Let us denote the coefficients $v_i(t)$, $i = 1, \dots, N$, using the column vector $\vec{v}(t) \in \mathbb{R}^N$, $\vec{v}^\top(t) = [v_1(t), \dots, v_N(t)]$.

Consider the mass matrix B defined as

$$B_{ij} = \langle \phi_j, \phi_i \rangle \tag{5.6}$$

The variational problem (5.5) can be written in the semi-discrete form: find $\vec{u}(t) \in \mathbb{R}^N$ such that

$$B\vec{u}'(t) + A\vec{u}(t) = \vec{f}(t) \quad \text{for } t \in (0, T] \tag{5.7}$$

with the initial condition $\vec{u}(0) = \vec{g}$.

Notice that the mass matrix B is a Gram matrix and therefore positive definite and invertible. The stiffness matrix A is positive definite due to coerciveness of $a(\cdot, \cdot)$. This implies that (5.7) above will admit a unique solution for all $t \in (0, T]$, which is called the semi-discrete solution of (5.2).

5.1.6 Discretization of the Time Domain

We will proceed now with the discretization of the time domain $\mathcal{J} = [0, T]$.

We will use a uniform mesh $0 = t_0 < t_1 < \dots < t_M = T$ with the time step $\Delta t = k$, $k = t_m - t_{m-1}$, for $m = 1, \dots, M$. The time derivative at each discrete time $\vec{u}'(t_m)$ will be approximated using finite differences schemes, i.e. θ -scheme, such as Backward Euler and Crank-Nicholson. These methods will result in equations

relating consecutive solutions at each discrete time which can be solved recursively, starting with t_0 . This is the so called time stepping method.

In the θ -scheme, $\theta \in [0, 1]$ and all $m = 1, \dots, M$, the time derivative $\vec{u}'(t_{m-1} + \theta)$ is approximated by

$$\frac{\vec{u}(t_m) - \vec{u}(t_{m-1})}{k}$$

We also approximate $\vec{u}(t_{m-1} + \theta)$ by

$$\theta \vec{u}(t_m) + (1 - \theta) \vec{u}(t_{m-1})$$

and $\vec{f}(t_{m-1} + \theta)$ by

$$\theta \vec{f}(t_m) + (1 - \theta) \vec{f}(t_{m-1})$$

This will result in the following time stepping equation for $\theta \in [0, 1]$ and $m = 1, \dots, M$

$$(B + \theta k A) \vec{u}(t_m) + ((1 - \theta) k A - B) \vec{u}(t_{m-1}) = k \left(\theta \vec{f}(t_m) + (1 - \theta) \vec{f}(t_{m-1}) \right) \quad (5.8)$$

with the initial condition $\vec{u}(t_0) = \vec{g}$.

If we let $\theta = 1$ we have the Backward Euler method, and for $\theta = \frac{1}{2}$ we have the Crank-Nicholson method. For further details on the fully discrete Galerkin method see [54], [56], [70].

5.2 Parabolic Inequalities

5.2.1 Introduction

Consider the Hilbert space \mathcal{V} and its dual \mathcal{V}^* , as defined in (4.6) with $\Omega = \mathbb{R}$.

Let $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ be the continuous and coercive operator defined in (4.1).

Consider the following system of inequalities with boundary conditions

$$\left\{ \begin{array}{ll} u_t + \mathcal{A}u \geq f & \text{in } \Omega \\ u \geq \psi & \text{in } \Omega \\ (u_t + \mathcal{A}u - f)(u - \psi) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R} \setminus \Omega \\ u = g & \text{for } t = 0 \end{array} \right. \quad (5.9)$$

We will use the variational setting to solve the problem. This will allow for more general solutions u as well as existence and uniqueness of solutions with less restrictions on f and more general initial data g .

Our most important motivation for this problem are the time dependent American options introduced in Chapter 3 and the game options introduced in Chapter 2. If the elliptic operator \mathcal{A} is the infinitesimal generator of Lévy process for the underlying asset, then it is shown in [14] that the value of an American option on such asset can be found by finding the solution to the problem (5.9) above.

In the case of game options the solution is bounded from above and below. The second condition of the problem (5.9) becomes $F \leq u \leq G$. Finding values of game options using variational inequalities will be discussed in section 6.1.

5.2.2 Variational Formulation for Parabolic Inequalities

The intermediate space \mathcal{H} considered, which is identified with its dual $\mathcal{H} \cong \mathcal{H}^*$, will be $L_2(\Omega)$ such that we have the Gelfand triple defined at the beginning of this chapter.

Consider $\mathcal{K} \subset \mathcal{V}$ a convex subset of \mathcal{V} defined as

$$\mathcal{K} = \left\{ v \mid v \in \mathcal{V}, v \geq \psi \right\}$$

Such \mathcal{K} is used in the case of one sided problems like the perpetual American put.

When we have two sided problems, like in the case of game options, the subspace \mathcal{K} is defined as

$$\mathcal{K} = \left\{ v \mid v \in \mathcal{V}, F \leq v \leq G \right\}$$

We define the following solution subspace $\mathcal{W}_{\mathcal{K}} \subset \mathcal{W}$

$$\mathcal{W}_{\mathcal{K}} = \left\{ v \mid v \in \mathcal{W}, v(t) \in \mathcal{K} \right\} \quad t \in \mathcal{J}$$

The variational form of the problem (5.9) above can be formulated as follows: find $u \in \mathcal{W}_{\mathcal{K}}$ such that the following holds

$$\langle u_t, v - u \rangle + a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K} \quad t \in (0, T) \quad (5.10)$$

with the initial condition $u(\cdot, 0) = g$.

Here $a(\cdot, \cdot)$ is the continuous (4.3) and coercive (4.4) bilinear associated with the operator \mathcal{A} defined in (4.2). u_t is the weak derivative of u with respect to time.

If $f \in \mathcal{W}$ and $g \in \mathcal{K}$ then the variational problem (5.10) admits a unique solution $u \in \mathcal{W}_{\mathcal{K}}$, see [27], [55] and the following holds

$$u, u' \in L_2(\mathcal{J}, \mathcal{V}) \cap L^\infty(\mathcal{J}, \mathcal{V})$$

5.2.3 Localization of the Space Domain

We localize the space domain to a finite interval $\Omega = (-R, R)$. We will show in the next chapter using a stochastic proof that the error induced by localization

decreases exponentially in R .

The variational inequality (5.10) can be rewritten as: find $u \in \mathcal{W}_{\mathcal{K}}$ such that

$$\langle u_t, v - u \rangle + a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K} \quad t \in (0, T) \quad (5.11)$$

with the initial condition $u(\cdot, 0) = g$.

The localized problem (5.11) admits a unique solution in $\mathcal{W}_{\mathcal{K}}$.

5.2.4 Discretization of the Space Domain

We discretize the space domain $\Omega = (-R, R)$ using a uniform mesh $-R = x_0 < x_1 < \dots < x_{N+1} = R$.

We replace the infinite dimensional space \mathcal{V} with a finite dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$ of continuous piecewise linear functions defined in (4.10). Let $\mathcal{K}_h \subset \mathcal{V}_h$ be the set of continuous piecewise linear functions in \mathcal{V}_h defined as

$$\mathcal{K}_h = \left\{ v \mid v \in \mathcal{V}_h, v(x_i) \geq \psi(x_i), i = 1, \dots, N \right\}$$

In the case of game options \mathcal{K}_h is defined as

$$\mathcal{K}_h = \left\{ v \mid v \in \mathcal{V}_h, F(x_i) \leq v(x_i) \leq G(x_i), i = 1, \dots, N \right\}$$

Let $\mathcal{W}_{\mathcal{K}_h} \subset \mathcal{W}$ be the following solution subspace

$$\mathcal{W}_{\mathcal{K}_h} = \left\{ v \mid v \in \mathcal{W}, v(t) \in \mathcal{K}_h \right\} \quad t \in \mathcal{J}$$

The variational inequality (5.11) becomes: find $u \in \mathcal{W}_{\mathcal{K}_h}$ such that

$$\langle u_t, v - u \rangle + a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}_h \quad t \in (0, T) \quad (5.12)$$

with the initial condition $u(\cdot, 0) = g$.

The discretized problem (5.12) admits a unique solution in $\mathcal{W}_{\mathcal{K}_h}$.

5.2.5 Finite Elements for Parabolic Inequalities

The finite dimensional space \mathcal{V}_h has a basis of hat functions $\mathcal{V}_h = \text{span}\{\phi_1, \dots, \phi_N\}$.

The basis functions $\phi_i(x)$ are defined in (4.12).

Any $v \in (\mathcal{J}, \mathcal{V}_h)$ can be expressed in terms of the hat functions ϕ_i defined in (4.12) since they form a basis for \mathcal{V}_h

$$v(x, t) = \sum_{i=1}^N v_i(t) \phi_i(x)$$

Let us denote the coefficients $v_i(t)$, $i = 1, \dots, N$, using the column vector $\vec{v}(t) \in \mathbb{R}^N$, $\vec{v}^\top(t) = [v_1(t), \dots, v_N(t)]$.

Let $\mathcal{K}_N \subset \mathbb{R}_N$ be defined as

$$\mathcal{K}_N = \left\{ v \mid v \in \mathbb{R}_N, v_i \geq \psi(x_i), i = 1, \dots, N \right\}$$

In the case of game options \mathcal{K}_N is defined as

$$\mathcal{K}_N = \left\{ v \mid v \in \mathbb{R}_N, F(x_i) \leq v_i \leq G(x_i), i = 1, \dots, N \right\}$$

The variational problem (5.12) can be written in the semi-discrete form: find $\vec{u}(t) \in \mathcal{K}^N$ such that for all $t \in (0, T]$ the following holds

$$B\vec{u}'(t) + A\vec{u}(t) \geq \vec{f}(t) \tag{5.13}$$

with the initial condition $\vec{u}(0) = \vec{g}$.

The semi-discrete problem (5.13) admits a unique solution in \mathcal{K}^N .

5.2.6 Discretization of the Time Domain

We will proceed now with the discretization of the time domain $\mathcal{J} = [0, T]$.

We will use a uniform mesh $0 = t_0 < t_1 < \dots < t_M = T$ with the time step $\Delta t = k$,

$k = t_m - t_{m-1}$, for $m = 1, \dots, M$. The time derivative at each discrete time $\vec{u}'(t_m)$ will be approximated using finite differences schemes, i.e. θ -scheme.

In the θ -scheme, $\theta \in [0, 1]$ and all $m = 1, \dots, M$, the time derivative $\vec{u}'(t_{m-1} + \theta)$ is approximated by

$$\frac{\vec{u}(t_m) - \vec{u}(t_{m-1})}{k}$$

We also approximate $\vec{u}(t_{m-1} + \theta)$ by

$$\theta \vec{u}(t_m) + (1 - \theta) \vec{u}(t_{m-1})$$

and $\vec{f}(t_{m-1} + \theta)$ by

$$\theta \vec{f}(t_m) + (1 - \theta) \vec{f}(t_{m-1})$$

This will result in the following variational inequalities for $\theta \in [0, 1]$ and $m = 1, \dots, M$: find $\vec{u}(t_m) \in \mathcal{K}_N$ such that

$$(B + \theta k A) \vec{u}(t_m) + ((1 - \theta) k A - B) \vec{u}(t_{m-1}) \geq k \left(\theta \vec{f}(t_m) + (1 - \theta) \vec{f}(t_{m-1}) \right) \quad (5.14)$$

with the initial condition $\vec{u}(t_0) = \vec{g}$.

If we let $\theta = 1$ we have the Backward Euler method, and for $\theta = \frac{1}{2}$ we have the Crank-Nicholson method. Consider the following notation

$$\tilde{A} = B + \theta k A$$

$$\tilde{B} = (1 - \theta) k A - B$$

$$\vec{\tilde{f}} = \theta \vec{f}(t_m) + (1 - \theta) \vec{f}(t_{m-1})$$

To find a solution to the discrete variational inequality (5.15), we have to solve the following M LCP problems for $m = 1, \dots, M$: find $\vec{u}(t_m) \in \mathcal{K}_N$ such that

$$\tilde{A} \vec{u}(t_m) + \tilde{B} \vec{u}(t_{m-1}) \geq k \vec{\tilde{f}} \quad (5.15)$$

with the initial condition $\vec{u}(t_0) = \vec{g}$.

Each LCP problem admits a unique solution $\vec{u}(t_m) \in \mathcal{K}_N$ for $m = 1, \dots, M$. The LCP problems can be solved using the iterative method which was introduced in the previous chapter and will be presented in detail in chapter 7.

Chapter 6

Finite Element Method for Game Options

6.1 Game Option Value as Variational Solution of Parabolic Inequalities

6.1.1 From Stochastic Formulation to Parabolic Inequalities

In chapter 2, the value of the game option was given as a solution of the following optimal stopping problem

$$\begin{aligned} V^G(x, t) &= \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} E^{\mathcal{Q}} \left(e^{(t-\sigma \wedge \tau)r} R(\sigma, \tau) \mid X_t = x \right) \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} \inf_{\sigma \in \mathcal{T}_{t,T}} E^{\mathcal{Q}} \left(e^{(t-\sigma \wedge \tau)r} R(\sigma, \tau) \mid X_t = x \right) \end{aligned} \quad (6.1)$$

with

$$R(\sigma, \tau) = G_{\sigma} \mathbf{1}_{\sigma < \tau} + F_{\tau} \mathbf{1}_{\tau \leq \sigma}$$

and σ the stopping time for the issuer and τ the stopping time for the holder. Here $F(x, t)$ and $G(x, t)$ are the \mathcal{F}_t adapted stochastic payoff processes corresponding to exercise and cancelation values and \mathcal{Q} is some risk neutral martingale measure, equivalent to \mathcal{P} , which is uniquely determined only in the pure diffusion case.

We want to show the connection between the values of game options and weak solutions of parabolic variational inequalities. We claim that the value of the game option $V^G = u$, where $u(x, t)$ is the variational solution of problem defined below.

Consider the Hilbert space \mathcal{V} and its dual \mathcal{V}^* , as defined in (4.6) with $\Omega = \mathbb{R}$. Let

$\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ be the continuous and coercive operator defined in (4.1).

Consider the following system of inequalities with boundary conditions

$$\left\{ \begin{array}{ll} u_t + \mathcal{A}u \geq f & \text{in } \Omega \\ F \leq u \leq G & \text{in } \Omega \\ (u_t + \mathcal{A}u - f)(u - F)(G - u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R} \setminus \Omega \\ u = g & \text{for } t = 0 \end{array} \right. \quad (6.2)$$

We will use the variational setting to solve the problem. This will allow for more general solutions u as well as existence and uniqueness of solutions with less restrictions on f and more general initial data g .

From section 5.2 we know by [27] that the problem above admits a unique variational solution, $u(x, t)$ as long as $f \in L_2$. To support our claim that $V^G = u$ we have the following evidence

- Game options under a pure diffusion process are equal to the viscosity solutions of the problem (6.2) [53],[18]. This is proved by making the connection through solutions of reflected forward-backward differential equations.
- In the case of Game options under pure diffusion, the value of the stochastic differential games like the game option (6.1) is given by variational solutions of the inequalities such as (6.2) [25], [24], [6], and [7]
- American option values under pure diffusion models are computed using solutions of variational inequalities. For details we refer to [29], [1], [34], [66], [61], [75], [63].

- American option values under CGMY jump models have been found using variational solution of parabolic inequalities [55], [60], [14].

These results establish a solid ground for us to claim that the values of game options can be found through solutions of variational inequalities, such as (6.2), in the case of pure diffusion, pure jump CGMY process and jump diffusion.

6.1.2 Localization of the Space Domain

Consider a barrier game option with the payoff (6.1) when the log price is between some barriers $(-R, R)$ and $F(t)$ when the log price is outside the barriers $(-R, R)$ with $R > 0$. The value of such option will be denoted by $\tilde{V}^G(x, t)$ and will have the following payoff structure

$$\begin{aligned} \tilde{V}^G(x, t) = & \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} E^{\mathcal{Q}} \left(e^{(t-\sigma \wedge \tau)r} R(\sigma, \tau) \mathbf{1}_{\theta > \sigma \wedge \tau} + \right. \\ & \left. + e^{(t-\theta)r} F_{\theta} \mathbf{1}_{\theta \leq \sigma \wedge \tau} \mid X_t = x \right) \end{aligned} \quad (6.3)$$

where σ is the stopping time for the issuer and τ is the stopping time for the holder. Here θ is the the first time that the process X_t leaves the domain $(-R, R)$ defined as

$$\theta = \inf \{s \geq t \mid X_s \notin (-R, R)\}$$

Notice that the value of the barrier option will converge to the value of the option defined in (6.2) as the barriers approach $\pm\infty$ i.e. $R \rightarrow \infty$.

We will show the errors estimates of the localization error using a probabilistic approach, first for the the pure diffusion case and then for cases involving jumps,

like the CGMY and jump diffusion. We will discuss the error estimates first for European options, then continue with American and Game options.

The value of a European option is given by

$$V^E(x, t) = E^{\mathcal{Q}} \left(e^{(t-T)r} F_T \mid X_t = x \right) \quad (6.4)$$

for some given bounded payoff function F . The localized analog of the European option on the bounded domain $\Omega = (-R, R)$ has the following structure

$$\tilde{V}^E(x, t) = E^{\mathcal{Q}} \left(e^{(t-T)r} F_T \mathbf{1}_{\theta > T} + e^{(t-\theta)r} H_{\theta} \mathbf{1}_{\theta \leq T} \mid X_t = x \right) \quad (6.5)$$

where the payoff functions F and H are both bounded. Here θ is the the first time that the process X_t leaves the domain $(-R, R)$ defined as

$$\theta = \inf \{s \geq t \mid X_s \notin (-R, R)\}$$

We denote $V^E(x, t)$ by V_t^E and $\tilde{V}^E(x, t)$ by \tilde{V}_t^E .

Theorem 6.1. *If we have the underlying process X_s given by the Brownian motion $X_s = x + \mu(s - t) + \sigma(W_s - W_t)$ then we get the following error bound*

$$|V_t^E - \tilde{V}_t^E| \leq C \exp \left(-(R - \mu T - |x|)^2 \right) + C \exp \left(-(R - \mu T + |x|)^2 \right)$$

Proof. First we will use the fact that the payoff functions F and H are bounded,

i.e. $\|F\|_\infty < \infty$ and $\|H\|_\infty < \infty$

$$\begin{aligned}
|V_t^E - \tilde{V}_t^E| &= E^{\mathcal{Q}} \left(e^{(t-T)r} F_T \mathbf{1}_{\theta \leq T} + e^{(t-\theta)r} H_\theta \mathbf{1}_{\theta \leq T} \mid X_t = x \right) \\
&\leq E^{\mathcal{Q}} \left(C \|F\|_\infty \mathbf{1}_{\theta \leq T} + C \|H\|_\infty \mathbf{1}_{\theta \leq T} \mid X_t = x \right) \\
&\leq C E^{\mathcal{Q}} \left(\mathbf{1}_{\theta \leq T} \mid \mathcal{F}_t \right) \\
&\leq C \mathcal{Q} \left(\exists t \leq \theta \leq T, X_\theta \notin \Omega \mid X_t = x \right) \\
&\leq C \mathcal{Q} \left(\sup_{t \leq \theta \leq T} |x + \mu(\theta - t) + \sigma(W_\theta - W_t)| \geq R \right) \\
&\leq C \mathcal{Q} \left(\sup_{t \leq \theta \leq T} |x + \sigma(W_\theta - W_t)| \geq R - \mu T \right) \\
&\leq C \mathcal{Q} \left(\sup_{0 \leq \theta \leq T-t} |x + \sigma W_\theta| \geq R - \mu T \right) \\
&\leq C \mathcal{Q} \left(\sup_{0 \leq \theta \leq T} |x + \sigma W_\theta| \geq R - \mu T \right)
\end{aligned}$$

The following bounds are taken from section 5.2.1 in [48]

$$\mathcal{Q} \left(\sup_{s \leq T} W_s \geq a \right) \leq \exp \left(-\frac{a^2}{T} \right)$$

and

$$\mathcal{Q} \left(\inf_{s \leq T} W_s \leq -a \right) \leq \exp \left(-\frac{a^2}{T} \right)$$

Using the results above we can obtain the following bounds

$$\mathcal{Q} \left(\sup_{s \leq T} (x + \sigma W_s) \geq a \right) \leq \exp \left(-\frac{(a - |x|)^2}{\sigma^2 T} \right)$$

and

$$\mathcal{Q} \left(\inf_{s \leq T} (x + \sigma W_s) \leq -a \right) \leq \exp \left(-\frac{(a + |x|)^2}{\sigma^2 T} \right)$$

These two results imply that

$$\mathcal{Q} \left(\sup_{0 \leq \theta \leq T} |x + \sigma W_\theta| \geq a \right) \leq \exp \left(-\frac{(a - |x|)^2}{\sigma^2 T} \right) + \exp \left(-\frac{(a + |x|)^2}{\sigma^2 T} \right)$$

and therefore

$$\begin{aligned}
|V_t^E - \tilde{V}_t^E| \leq & C_{T,\sigma} \exp(-(R - \mu T - |x|)^2) + \\
& + C_{T,\sigma} \exp(-(R - \mu T + |x|)^2)
\end{aligned} \tag{6.6}$$

□

This proves that the convergence is uniform in t , since the constant C doesn't depend on t . This will allow us to replace the fixed t with any random stopping time τ .

Theorem 6.2. *If the underlying process X_s is a Lévy process with the CGMY jump density, i.e. the pure jump CGMY process, CGMY jump-diffusion process, and if $0 < \alpha < \min(G, M)$, then we get the following error estimate*

$$|V_t^E - \tilde{V}_t^E| \leq C \exp(-\alpha R)$$

Proof. Using that F and H are bounded, i.e. $\|F\|_\infty < \infty$ and $\|H\|_\infty < \infty$, we obtain through the same steps as before the following

$$|V_t^E - \tilde{V}_t^E| \leq C \mathcal{Q} \left(\sup_{0 \leq \theta \leq T} |x + X_\theta| \geq R \right)$$

Notice that for $0 < \alpha < \min(G, M)$ we get that

$$\int_{|x|>1} e^{\alpha|x|} \nu(dx) < \infty$$

From this result we can use the following from [64] and section 4.1 in [15]

$$\mathcal{Q} \left(\sup_{s \leq T} (X_s) \geq R \right) \leq C_{T,\alpha} \exp(-\alpha R)$$

and

$$\mathcal{Q} \left(\inf_{s \leq T} (X_s) \leq -R \right) \leq C_{T,\alpha} \exp(-\alpha R)$$

Using these results we obtain the following

$$\mathcal{Q} \left(\sup_{s \leq T} (x + X_s) \geq a \right) \leq C_{T,\alpha} \exp(-\alpha|a - |x||)$$

and

$$\mathcal{Q} \left(\inf_{s \leq T} (x + X_s) \leq -a \right) \leq C_{T,\alpha} \exp(-\alpha|a + |x||)$$

These two results imply that

$$\mathcal{Q} \left(\sup_{0 \leq \theta \leq T} |x + X_\theta| \geq R \right) \leq C_{T,\alpha} \exp(-\alpha|R - |x||) + C_{T,\alpha} \exp(-\alpha|R + |x||)$$

and therefore

$$|V_t^E - \tilde{V}_t^E| \leq C_{T,\alpha} \exp(-\alpha R) \tag{6.7}$$

□

Notice again that that the convergence is uniform in t , since the constant C doesn't depend on t , which will allow us to replace the fixed t with any random stopping time τ for the CGMY process as well.

The exponential decay of the localization error for American options has been discussed before in the case of pure diffusion as well as diffusion with jumps. We mention some of these references [14], [35].

The value of an American option is given by

$$V^A(x, t) = \sup_{\tau \in \mathcal{T}_{0,T}} E^{\mathcal{Q}} \left(e^{(t-\tau)r} F_\tau \mid X_t = x \right) \tag{6.8}$$

for some given bounded payoff function F . The localized analog of the American option on the bounded domain $\Omega = (-R, R)$ has the following structure

$$\tilde{V}^A(x, t) = \sup_{\tau \in \mathcal{T}_{0,T}} E^{\mathcal{Q}} \left(e^{(t-\tau)r} F_T \mathbf{1}_{\theta > \tau} + e^{(t-\theta)r} H_\theta \mathbf{1}_{\theta \leq \tau} \mid X_t = x \right) \tag{6.9}$$

where the payoff functions F and H are both bounded. Here θ is the the first time that the process X_t leaves the domain $(-R, R)$ defined as

$$\theta = \inf \{s \geq t \mid X_s \notin (-R, R)\}$$

We denote $V^A(x, t)$ by V_t^A and $\tilde{V}^A(x, t)$ by \tilde{V}_t^A .

Theorem 6.3. *If we have the underlying process X_s given by the Brownian motion $X_s = x + \mu(s - t) + \sigma(W_s - W_t)$ then we get the following error bound*

$$|V_t^A - \tilde{V}_t^A| \leq C \exp(-(R - \mu T - |x|)^2) + C \exp(-(R - \mu T + |x|)^2)$$

Proof. Let us define the V_t as the European option with the payoff F

$$V(x, t) = E^{\mathcal{Q}} \left(e^{(t-T)r} F_T \mid X_t = x \right) \quad (6.10)$$

and \tilde{V}_t as the European barrier option with the payoff F

$$\tilde{V}(x, t) = E^{\mathcal{Q}} \left(e^{(t-T)r} F_T \mathbf{1}_{\theta > T} + e^{(t-\theta)r} F_{\theta} \mathbf{1}_{\theta \leq T} \mid X_t = x \right) \quad (6.11)$$

We denote $V(x, t)$ by V_t and $\tilde{V}(x, t)$ by \tilde{V}_t . From the uniform bound (6.6) we get that

$$\begin{aligned} V_{\tau} &\leq C_{T,\sigma} \exp(-(R - \mu T - |x|)^2) + \\ &\quad + C_{T,\sigma} \exp(-(R - \mu T + |x|)^2) + \tilde{V}_{\tau} \end{aligned} \quad (6.12)$$

where τ is any random stopping time in $[0, T]$. Notice that taking the supremum over stopping time τ of the expression V_{τ} gives us exactly the American option, V_t^A

$$V_t^A = \sup_{\tau \in \mathcal{T}_{t,T}} V_{\tau} \quad (6.13)$$

and the supremum over stopping time τ of \tilde{V}_{τ} gives us \tilde{V}_t^A

$$\tilde{V}_t^A = \sup_{\tau \in \mathcal{T}_{t,T}} \tilde{V}_{\tau} \quad (6.14)$$

We can use these results in our desired formula to get

$$\begin{aligned}
|V_t^A - \tilde{V}_t^A| &\leq |\sup_{\tau \in \mathcal{T}_{t,T}} V_\tau - \tilde{V}_t^A| \\
&\leq \sup_{\tau \in \mathcal{T}_{t,T}} |(C_{T,\sigma} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\sigma} \exp(-(R - \mu T + |x|)^2) + \tilde{V}_\tau)| - \tilde{V}_\tau^A \\
&\leq C_{T,\sigma} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\sigma} \exp(-(R - \mu T + |x|)^2) + \sup_{t \leq \tau \leq T} \tilde{V}_\tau - \tilde{V}_\tau^A \\
&\leq C_{T,\sigma} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\sigma} \exp(-(R - \mu T + |x|)^2)
\end{aligned}$$

□

Theorem 6.4. *If the underlying process X_s is a Lévy process with the CGMY jump density, i.e. the pure jump CGMY process, CGMY jump-diffusion process, and if $0 < \alpha < \min(G, M)$, then we get the following error estimate*

$$|V_t^A - \tilde{V}_t^A| \leq C \exp(-\alpha R)$$

Proof. Using the uniform convergence in (6.7) we obtain in the jump diffusion case that

$$V_\tau \leq C_{T,\alpha} \exp(-\alpha R) + \tilde{V}_\tau$$

where τ is any random stopping time in $[0, T]$. We can use this result together with (6.13) and (6.14) in our desired expression to obtain

$$\begin{aligned}
|V_t^A - \tilde{V}_t^A| &\leq |\sup_{\tau \in \mathcal{T}_{t,T}} V_\tau - \tilde{V}_t^A| \\
&\leq \sup_{\tau \in \mathcal{T}_{t,T}} |(C_{T,\alpha} \exp(-\alpha R) + \tilde{V}_\tau)| - \tilde{V}_\tau^A \\
&\leq C_{T,\alpha} \exp(-\alpha R) + \sup_{t \leq \tau \leq T} \tilde{V}_\tau - \tilde{V}_\tau^A \\
&\leq C_{T,\alpha} \exp(-\alpha R)
\end{aligned}$$

□

Theorem 6.5. *Let V_t^G be the game option value defined in (6.1) and \tilde{V}_t^G the barrier game option from (6.3). If we have the underlying process X_s given by the Brownian motion $X_s = x + \mu(s - t) + \kappa(W_s - W_t)$ then the following error bound holds*

$$|V_t^G - \tilde{V}_t^G| \leq C \exp(-(R - \mu T - |x|)^2) + C \exp(-(R - \mu T + |x|)^2)$$

Proof. Let us define the \hat{V}_t as the European option with the payoff G

$$\hat{V}(x, t) = E^{\mathcal{Q}} \left(e^{(t-T)r} G_T \mid X_t = x \right) \quad (6.15)$$

and $\hat{\hat{V}}_t$ as the European barrier option with the payoff G and F

$$\hat{\hat{V}}(x, t) = E^{\mathcal{Q}} \left(e^{(t-T)r} G_T \mathbf{1}_{\theta > T} + e^{(t-\theta)r} F_{\theta} \mathbf{1}_{\theta \leq T} \mid X_t = x \right) \quad (6.16)$$

We denote $V^G(x, t)$ by V_t^G and $\tilde{V}^G(x, t)$ by \tilde{V}_t^G . Also denote $\hat{V}(x, t)$ by \hat{V}_t and $\hat{\hat{V}}(x, t)$ by $\hat{\hat{V}}_t$. Using the fact the G and F are bounded and the uniform convergence result (6.6) we obtain the following

$$\begin{aligned} \hat{V}_{\sigma} &\leq C_{T,\kappa} \exp(-(R - \mu T - |x|)^2) + \\ &\quad + C_{T,\kappa} \exp(-(R - \mu T + |x|)^2) + \hat{\hat{V}}_{\sigma} \end{aligned} \quad (6.17)$$

where σ is any random stopping time in $[0, T]$. Notice that taking the infimum and supremum over stopping times σ and τ of the expressions \hat{V}_{σ} and V_{τ} gives us exactly the Game option, V_t^G

$$V_t^G = \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} \left(\hat{V}_{\sigma} \mathbf{1}_{\sigma < \tau} + V_{\tau} \mathbf{1}_{\tau \leq \sigma} \right) \quad (6.18)$$

and the infimum and supremum over stopping times σ and τ of $\hat{\hat{V}}_{\sigma}$ and \tilde{V}_{τ} gives us \tilde{V}_t^G

$$\tilde{V}_t^G = \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} \left(\hat{\hat{V}}_{\sigma} \mathbf{1}_{\sigma < \tau} + \tilde{V}_{\tau} \mathbf{1}_{\tau \leq \sigma} \right) \quad (6.19)$$

We can use these results together with (6.12) in our desired formula to get

$$\begin{aligned}
|V_t^G - \tilde{V}_t^G| &= |\inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} (\hat{V}_\sigma \mathbf{1}_{\sigma < \tau} + V_\tau \mathbf{1}_{\tau \leq \sigma}) - \tilde{V}_t^G| \\
&\leq |\inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} [(C_{T,\kappa} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\kappa} \exp(-(R - \mu T + |x|)^2) + \hat{V}_\sigma) \mathbf{1}_{\sigma < \tau} + \\
&\quad + (C_{T,\sigma} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\sigma} \exp(-(R - \mu T + |x|)^2) + \tilde{V}_\tau) \mathbf{1}_{\tau \leq \sigma}] - \tilde{V}_t^G| \\
&\leq |\inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} [C_{T,\kappa} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\kappa} \exp(-(R - \mu T + |x|)^2) + \hat{V}_\sigma \mathbf{1}_{\sigma < \tau} + \\
&\quad + \tilde{V}_\tau \mathbf{1}_{\tau \leq \sigma}] - \tilde{V}_t^G| \\
&\leq |C_{T,\kappa} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\kappa} \exp(-(R - \mu T + |x|)^2) + \\
&\quad + \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} [\hat{V}_\sigma \mathbf{1}_{\sigma < \tau} + \tilde{V}_\tau \mathbf{1}_{\tau \leq \sigma}] - \tilde{V}_t^G| \\
&\leq C_{T,\kappa} \exp(-(R - \mu T - |x|)^2) + \\
&\quad + C_{T,\kappa} \exp(-(R - \mu T + |x|)^2)
\end{aligned}$$

□

Theorem 6.6. *If the underlying process X_s is a Lévy process with the CGMY jump density, i.e. the pure jump CGMY process, CGMY jump-diffusion process, and if $0 < \alpha < \min(G, M)$, then we get the following error estimate holds*

$$|V_t^G - \tilde{V}_t^G| \leq C \exp(-\alpha R)$$

Proof. Using the uniform convergence in (6.7) we obtain in the jump diffusion case that

$$V_\tau \leq C_{T,\alpha} \exp(-\alpha R) + \tilde{V}_\tau$$

and the analog

$$\hat{V}_\sigma \leq C_{T,\alpha} \exp(-\alpha R) + \hat{\tilde{V}}_\sigma$$

where τ and σ are random stopping times in $[0, T]$. We can use this result together with (6.18) and (6.19) in our desired expression to obtain

$$\begin{aligned} |V_t^G - \tilde{V}_t^G| &= |\inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} (\hat{V}_\sigma \mathbf{1}_{\sigma < \tau} + V_\tau \mathbf{1}_{\tau \leq \sigma}) - \tilde{V}_t^G| \\ &\leq |\inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} \left[(C_{T,\alpha} \exp(-\alpha R) + \hat{\tilde{V}}_\sigma) \mathbf{1}_{\sigma < \tau} + \right. \\ &\quad \left. + (C_{T,\alpha} \exp(-\alpha R) + \tilde{V}_\tau) \mathbf{1}_{\tau \leq \sigma} \right] - \tilde{V}_t^G| \\ &\leq |\inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} [C_{T,\alpha} \exp(-\alpha R) + \\ &\quad + \hat{\tilde{V}}_\sigma \mathbf{1}_{\sigma < \tau} + \tilde{V}_\tau \mathbf{1}_{\tau \leq \sigma}] - \tilde{V}_t^G| \\ &\leq |C_{T,\alpha} \exp(-\alpha R) + \\ &\quad + \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} [\hat{\tilde{V}}_\sigma \mathbf{1}_{\sigma < \tau} + \tilde{V}_\tau \mathbf{1}_{\tau \leq \sigma}] - \tilde{V}_t^G| \\ &\leq C_{T,\alpha} \exp(-\alpha R) \end{aligned}$$

□

We have shown that the localization error decreases exponentially with respect to the barrier R for both cases the pure diffusion and diffusion with jumps given by the CGMY process. To get the variational solution u for the barrier option \tilde{V}^G , we use excess the lower payoff, $\tilde{u} = u - F$. Notice that \tilde{u} will be identically zero outside the barriers $(-R, R)$. We will also define the difference between the cancellation payoff, G and the exercise payoff F , $\psi = G - F$.

Note that the excess payoff should be considered with respect to a function which has the same asymptotic behavior as the solution at $\pm\infty$.

We consider the solution space $\tilde{\mathcal{K}}$ defined as

$$\tilde{\mathcal{K}} = \{v \mid v \in \mathcal{V}, 0 \leq v \leq \psi\} \quad (6.20)$$

The variational formulation for the barrier option on $\Omega = (-R, R)$ in terms of the excess payoff \tilde{u} becomes the following: find $\tilde{w} \in \tilde{\mathcal{K}}$ such that the following holds for $\forall v \in \tilde{\mathcal{K}}$ and $t \in [0, T]$

$$\langle \tilde{u}_t, v - \tilde{u} \rangle + a(\tilde{u}, v - \tilde{u}) \geq -a(F, v - \tilde{u}) \quad (6.21)$$

with the initial condition $\tilde{u} = 0$ for $t = 0$.

Here $a(\cdot, \cdot)$ is the continuous (4.3) and coercive (4.4) bilinear associated with the operator \mathcal{A} defined in (4.2). Here u_t is the weak derivative of u with respect to time. Notice that t is considered time to maturity and $\tilde{V}^G(x, t) = \tilde{u}(x, T - t)$.

The localized problem (6.21) admits a unique solution \tilde{u} in $(\mathcal{J}, \tilde{\mathcal{K}})$, where $\mathcal{J} = [0, T]$ denotes the time domain.

6.1.3 Fully Discrete Galerkin Method for Game Options

We apply space and time discretization to the localized problem (6.21) (see section 5.2 for details).

In the case of excess to lower exercise payoff value function the $\tilde{\mathcal{K}}_N$ is defined as

$$\tilde{\mathcal{K}}_N = \left\{ v \mid v \in \mathbb{R}_N, 0 \leq v_i \leq \psi(x_i), i = 1, \dots, N \right\} \quad (6.22)$$

To get the fully discrete solutions of the variational inequality (6.21), we have to solve the following M LCP problems for $m = 1, \dots, M$: find $\vec{u}(t_m) \in \tilde{\mathcal{K}}_N$ such that

$$\tilde{A}\vec{u}(t_m) + \tilde{B}\vec{u}(t_{m-1}) \geq k\vec{f} \quad (6.23)$$

with the initial condition $\vec{u}(t_0) = \vec{g}$.

Here we used the following notation

$$\tilde{A} = B + \theta kA \tag{6.24}$$

$$\tilde{B} = (1 - \theta)kA - B$$

$$\vec{f} = \theta \vec{f}(t_m) + (1 - \theta) \vec{f}(t_{m-1})$$

where A is the mass matrix defined in (4.14), B is the mass matrix defined in (5.6)

and the load \vec{f} in this case will be defined as

$$f_i = -\langle \mathcal{A}F, \phi_i \rangle$$

We will show in detail how to compute the special load vector \vec{f} in section 7.2 for pure diffusion, pure jump CGMY and jump diffusion.

Each LCP problem admits a unique solution $\vec{u}(t_m) \in \tilde{\mathcal{K}}_N$ for $m = 1, \dots, M$.

The LCP problems can be solved using the iterative method which was introduced in the previous chapter and will be presented in detail in chapter 7.

6.2 Perpetual Game Options

6.2.1 Elliptic Inequalities for Perpetual Game Options

One important contribution of the time independent problems is qualitative understanding of the finite horizon problems as expiration time $T \rightarrow \infty$. If there exists a solution to the perpetual problem then the time dependent solutions will

converge to it as $T \rightarrow \infty$.

Consider the Hilbert space \mathcal{V} and its dual \mathcal{V}^* , as defined in (4.6) with $\Omega = \mathbb{R}$. Let $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ be the continuous and coercive operator defined in (4.1).

Consider $\mathcal{K} \subset \mathcal{V}$ a convex subset of \mathcal{V} defined as

$$\mathcal{K} = \left\{ v \mid v \in \mathcal{V}, F \leq v \leq G \right\}$$

The variational formulation for the value of perpetual game options is as follows: find $u \in \mathcal{K}$ such that

$$a(u, v - u) \geq 0 \quad \forall v \in \mathcal{K} \tag{6.25}$$

where $a(\cdot, \cdot)$ is the continuous (4.3) and coercive (4.4) bilinear associated with the operator \mathcal{A} defined in (4.2). The variational problem (6.25) has a unique solution $u \in \mathcal{K}$ (see section 4.2 for details).

We use excess the lower payoff, $\tilde{u} = u - F$. The solution subspace will be in this case $\tilde{\mathcal{K}}$ defined in (6.20). The variational form in this case becomes: find $\tilde{u} \in \tilde{\mathcal{K}}$ such that

$$a(\tilde{u}, v - \tilde{u}) \geq a(F, v - \tilde{u}) \quad \forall v \in \tilde{\mathcal{K}} \tag{6.26}$$

If $\mathcal{A}F \in \mathcal{V}^*$ then the variational problem (6.26) has a unique solution $\tilde{u} \in \tilde{\mathcal{K}}$

6.2.2 Localization to a Bounded Domain

We introduce the perpetual barrier option which takes the value of the lower exercise value when the value of the option exits the bounded domain $\Omega = (-R, R)$. We have shown in the previous section that the localization error decays exponentially in R in the finite horizon case. Since the perpetual case is the limit as $T \rightarrow \infty$

of the finite maturity, hence it is reasonable to expect the localization error to decrease exponentially in R .

We consider excess to lower payoff, $\tilde{u} = u - F$. Notice that \tilde{u} will be identically zero outside the barriers $(-R, R)$.

The variational formulation for the barrier option on $\Omega = (-R, R)$ in terms of the excess payoff \tilde{u} becomes the following: find $\tilde{u} \in \tilde{\mathcal{K}}$

$$a(\tilde{u}, v - \tilde{u}) \geq \langle \mathcal{A}F, v - \tilde{u} \rangle \quad \forall v \in \tilde{\mathcal{K}} \quad (6.27)$$

The localized variational problem (6.27) has a unique solution $\tilde{u} \in \tilde{\mathcal{K}}$

6.2.3 Galerkin Method for Perpetual Game Options

We apply Galerkin discretization using p.w.l. to the localized problem (6.27) (see section 4.2 for details).

Consider $\tilde{\mathcal{K}}_N$ defined in (6.22). To get the discrete solutions of the variational inequality (6.27), we have to solve the following the following LCP problem: find $\vec{u} \in \tilde{\mathcal{K}}_N$ such that

$$A\vec{u} \geq \vec{f} \quad (6.28)$$

Here A is the mass matrix defined in (4.14) and the load \vec{f} in this case will be defined as

$$f_i = -\langle \mathcal{A}F, \phi_i \rangle$$

We will show in detail how to compute the special load vector \vec{f} in section 7.2 for pure diffusion, pure jump CGMY and jump diffusion.

The LCP problem admits a unique solution $\vec{u}(t_m) \in \tilde{\mathcal{K}}_N$.

The LCP problems can be solved using the iterative method which was introduced in the previous chapter and will be presented in detail in chapter 7. The discrete solution of the perpetual game option $u_h = \sum_{i=1}^N \vec{u}_i \phi_i + F_h$, where F_h is the p.w.l approximation of the lower exercise payoff F .

Denote the discretization error by $e_h = u - u_h$. Using the results in chapter 4 we get the error estimates below for the discrete solutions of perpetual game options.

In the Black Scholes model, (pure diffusion process) the energy space is $\mathcal{V} = \tilde{H}^1(\Omega)$.

In this case $\mathcal{A}u - f \in L_2$ and the solution $u \in H^2$ we get the following error bound in the energy norm [11]

$$\|u - u_h\|_{\mathcal{V}} \leq ch \|u\|_{H^2}$$

In the pure jump model, (CGMY process), the energy space is $\mathcal{V} = \tilde{H}^{\frac{\gamma}{2}}(\Omega)$. The solution $u \in H^{1+\frac{\gamma}{2}}(\Omega)$ and $\mathcal{A}u - f \in L_2$ [60]. We have the following estimate for the energy norm of the error of the discrete solution

$$\|u - u_h\|_{\mathcal{V}} \leq ch^{\frac{1}{2} + \frac{\gamma}{4}} \|u\|_{H^{1+\frac{\gamma}{2}}}$$

Numerical results on error convergence rates will be shown in Chapter 7. (see Tables 7.1, 7.2, 7.3 and 7.4).

6.2.4 Convergence of the Perpetual Game and American Options

In contradiction to its American counterpart, the perpetual game call option will have a solution on the entire real line. This is due to the presence of the upper obstacle. We will show this through the fixed point iteration.

Formally, to get the value of the perpetual game option, we have to solve the fol-

lowing elliptic inequality over the entire domain $\Omega = \mathbb{R}$

$$\begin{aligned}\mathcal{A}\tilde{u} &\geq f \\ 0 &\leq \tilde{u} \leq \psi\end{aligned}\tag{6.29}$$

$$(\mathcal{A}\tilde{u} - f)(\psi - \tilde{u})\tilde{u} = 0$$

where \tilde{u} is the excess payoff value function, \mathcal{A} the elliptic operator on the energy space \mathcal{V} defined in (4.6). \mathcal{A} is given by the infinitesimal generator of the underlying process. Here F is the exercise value function or the lower obstacle, ψ is the difference between the upper and the lower obstacles, and f is given by $f = -\mathcal{A}F$.

Recall the domain of the solution $\tilde{\mathcal{K}}$ defined in (6.20). The variational formulation for our perpetual game option is find $\tilde{u} \in \tilde{\mathcal{K}}$ such that the following inequality holds for $\forall v \in \tilde{\mathcal{K}}$

$$a(\tilde{u}, v - \tilde{u}) \geq \langle f, v - \tilde{u} \rangle\tag{6.30}$$

where $a(\cdot, \cdot)$ is the bilinear associated with the operator \mathcal{A} , defined in (4.1), which satisfies the continuity condition (4.3) and coercivity condition (4.4). This is well-defined problem (see section 4.2) and if $f \in \mathcal{V}^*$ by the theorem 4.2 there exist a unique solution to the variational problem defined above when $\tilde{\mathcal{K}}$ is a closed convex set.

In general, the payoff functions for most of the financial derivatives lead to $f \notin \mathcal{V}^*$.

We will consider two cases for payoff functions with $f \notin \mathcal{V}^*$:

Case 1 is the put payoff function which has the following structure

$$F_{\text{put}}(x) = \begin{cases} K - e^x & \text{if } x < k \\ 0 & \text{if } x \geq k \end{cases}$$

For case 2 we consider the call payoff function

$$F_{\text{call}}(x) = \begin{cases} 0 & \text{if } x \leq k \\ e^x - K & \text{if } x > k \end{cases}$$

where k is the log of the strike K , $k = \log(K)$.

Remark: In the case of the butterfly option we have that the payoff is bounded and has compact support and therefore f satisfies $f \in \mathcal{V}^*$.

We derive the expressions for the first derivatives of the put and call payoffs

$$F'_{\text{put}}(x) = \begin{cases} -e^x & \text{if } x < k \\ 0 & \text{if } x \geq k \end{cases}$$

and

$$F'_{\text{call}}(x) = \begin{cases} 0 & \text{if } x \leq k \\ e^x & \text{if } x > k \end{cases}$$

and second derivatives

$$F''_{\text{put}}(x) = \begin{cases} -e^x & \text{if } x < k \\ 0 & \text{if } x > k \end{cases}$$

and

$$F''_{\text{call}}(x) = \begin{cases} 0 & \text{if } x < k \\ e^x & \text{if } x > k \end{cases}$$

We will consider three case of Levy processes: pure diffusion, pure jump and jump-diffusion. The pure diffusion (i.e. $\nu = 0$) and pure jump cases (i.e. $\sigma = 0$) are just simplified versions of the jump-diffusion case. In the most general jump-diffusion case \mathcal{A} is given by

$$\mathcal{A}u = \frac{\sigma^2}{2}(u' - u'') + r(u - u') - \int_{\mathbb{R}} (u(x+y) - u(x) - (e^y - 1)u') K_{CGMY}(y) dy$$

where the jump density K_{CGMY} is given by

$$K_{CGMY}(x) = \begin{cases} C \frac{e^{-G|x|}}{|x|^{1+Y}} & \text{for } x < 0 \\ C \frac{e^{-M|x|}}{|x|^{1+Y}} & \text{for } x > 0 \end{cases}$$

with $C > 0, G, M \geq 1$ and $Y < 2$.

Using the payoffs for puts and calls we get the following expressions for $f = -\mathcal{A}F$

$$\begin{aligned} f_{\text{put}} &= f_- + f_{\text{int}} + \frac{\sigma^2}{2} K \delta_k \\ f_{\text{call}} &= f_+ + f_{\text{int}} + \frac{\sigma^2}{2} K \delta_k \end{aligned}$$

where

$$\begin{aligned} f_-(x) &= \begin{cases} -rK & \text{if } x < k \\ 0 & \text{if } x \geq k \end{cases} \\ f_+(x) &= \begin{cases} 0 & \text{if } x \leq k \\ rK & \text{if } x > k \end{cases} \end{aligned}$$

and

$$f_{\text{int}} = \begin{cases} \int_{k-x}^{\infty} (e^{x+y} - K) K_{CGMY}(y) dy & \text{if } x < k \\ \int_{-\infty}^{k-x} (K - e^{x+y}) K_{CGMY}(y) dy & \text{if } x > k \end{cases}$$

We can see that f_{int} is infinitely differentiable everywhere except the point k and it has exponential decay at $\pm\infty$. Also near k the it behaves asymptotically like $|x|^{-Y}$, (see section 7.1 for more details), which implies that $f_{\text{int}} \in H^{1-Y}$. Since $\delta_k \in H^{-1}$ then in the case pure diffusion with $\mathcal{V}^* = H^{-1}$ and $f_{\text{int}} = 0$, we have that $\frac{\sigma^2}{2} K \delta_k + f_{\text{int}} \in \mathcal{V}^*$. When we have pure jump CGMY we have that $\mathcal{V}^* = H^{-\frac{Y}{2}}$ and since $Y < 2$ we get that $1 - Y > -\frac{Y}{2}$. This implies that $H^{1-Y} \subset H^{-\frac{Y}{2}}$ and therefore $f_{\text{int}} \in H^{-\frac{Y}{2}} = \mathcal{V}^*$. Since $\sigma = 0$ we obtain again that $\frac{\sigma^2}{2} K \delta_k + f_{\text{int}} \in \mathcal{V}^*$. When we

have CGMY-jump diffusion $\mathcal{V}^* = H^{-1}$ and $1 - Y > -1$ for $Y < 2$. In this case we obtain again that $f_{\text{int}} \in H^{-1} = \mathcal{V}^*$ and therefore $\frac{\sigma^2}{2}K\delta_k + f_{\text{int}} \in \mathcal{V}^*$. This implies that in all three cases we will have $f_{\text{int}} + \frac{\sigma^2}{2}K\delta_k \in \mathcal{V}^*$ which we can denote by f_1

$$f_1 = f_{\text{int}} + \frac{\sigma^2}{2}K\delta_k \in \mathcal{V}^* \quad (6.31)$$

We also have the following bound for $\forall v \in \mathcal{V}$

$$|\langle f_1, v \rangle| \leq C_f \|v\|_{\mathcal{V}}$$

Notice that the function $f_- \notin \mathcal{V}^*$ and therefore $f_{\text{put}} \notin \mathcal{V}^*$. The function $f_+ \notin \mathcal{V}^*$ and therefore $f_{\text{call}} \notin \mathcal{V}^*$. Let us denote the functions f_- and f_+ by f_0

$$f_0 = \begin{cases} f_- & \text{for put options} \\ f_+ & \text{for call options} \end{cases} \quad (6.32)$$

In the case when $f \notin \mathcal{V}^*$, in order for the problem (6.30) to be well-defined we need the additional condition that $\langle f, u \rangle < \infty$.

Because \tilde{K} is bounded only from below, we know that in the case of the American put and call options, the solution to the perpetual problem exists only in the case of the American put option while the American perpetual call does not have a solution. This can be observed also by considering the time dependent problem and noticing that as we let $T \rightarrow \infty$ the American put converges while the American call does not.

Let I be some unbounded interval in \mathbb{R} .

Remark 1: If $f_0 < 0$ on I , $f_0 = 0$ on $\mathbb{R} \setminus I$ and $v \geq 0$ on I , $\forall v \in \tilde{K}$ then (6.30) has a unique solution. This is the case of the perpetual American put and the perpetual

game put.

Remark 2: If $f_0 > 0$ on I , $f_0 = 0$ on $\mathbb{R} \setminus I$ and $v \leq 0$ on I , $\forall v \in \tilde{K}$ then (6.30) has a unique solution. This is the case of the perpetual game call. The perpetual American call fails to satisfy $v \leq 0$ when $v \in \tilde{K}$.

We will show next that in the case of game options the perpetual problem has a solution for the both call and put game options since \tilde{K} is bounded on both sides. First we will consider the case of the perpetual game put option.

Assumptions: \mathcal{A} is defined in (4.1) and satisfies the continuity and coercivity conditions (4.3) and (4.4). \tilde{K} is closed and convex.

Theorem 6.7. *The perpetual game put option value given by complementarity problem (6.30) for $f = f_{put}$, and f_0, f_1 defined in (6.32), (6.31): find $u \in \tilde{K}$ such that $\int_{-\infty}^k |u(x)| dx < \infty$ and*

$$a(u, v - u) \geq \langle f_0, v - u \rangle + \langle f_1, v - u \rangle \quad \forall v \in \tilde{K} \quad (6.33)$$

has a unique solution $u \in \tilde{K}$.

Proof. We will first consider a symmetric operator \mathcal{A} and the bilinear $a(\cdot, \cdot)$ associated with \mathcal{A} which satisfies the continuity and coercivity conditions (4.3) and (4.4). In this case we can solve the variational inequality (6.33) by finding a solution to the following energy minimization problem: find $v \in \tilde{K}$ that minimizes

$$E(v) = \frac{1}{2}a(v, v) - \langle f_0, v \rangle - \langle f_1, v \rangle$$

Note: if $\int_I |v(x)| dx < \infty$ then $E(u) < \infty$ for some $v \in \tilde{K}$ and if $\int_I |v(x)| dx = \infty$ then $E(u) = \infty$ for some $v \in \tilde{K}$.

Since $f_0 \leq 0$ (see (6.32)) and $v \geq 0 \forall v \in \tilde{\mathcal{K}}$ we get that $-\langle f_0, u \rangle \geq 0$ which implies

$$\begin{aligned}
E(u) &\geq \frac{1}{2}a(u, u) - \langle f_1, v - u \rangle \\
&\geq \frac{C_2}{2}\|u\|_{\mathcal{V}}^2 - C_f\|u\|_{\mathcal{V}} \\
&\geq \frac{C_2}{2} \left(\|u\|_{\mathcal{V}}^2 - \frac{2C_f}{C_2}\|u\|_{\mathcal{V}} \right) \\
&\geq \frac{C_2}{2} \left(\|u\|_{\mathcal{V}} - \frac{C_f}{C_2} \right)^2 - \frac{C_f^2}{C_2^2} \\
&\geq -\frac{C_f^2}{C_2^2}
\end{aligned}$$

Since $\exists v \in \tilde{\mathcal{K}}$ such that $E(u) < \infty$ and $E(u)$ is bounded from below we have that $\inf_{u \in \tilde{\mathcal{K}}} E(u)$ exists.

Consider $E_* = \inf_{u \in \tilde{\mathcal{K}}} E(u)$. Then there exists a sequence $u_n \in \tilde{\mathcal{K}}$ with $E_n = E(u_n) \rightarrow E_*$ as $n \rightarrow \infty$.

We want to show next that u_n is a Cauchy sequence. For $\forall u_m, u_n \in \tilde{\mathcal{K}}$ we have from the convexity of $\tilde{\mathcal{K}}$ that $\frac{u_m+u_n}{2} \in \tilde{\mathcal{K}}$ and therefore we obtain

$$\begin{aligned}
E\left(\frac{u_m+u_n}{2}\right) &= \frac{1}{2}a\left(\frac{u_m+u_n}{2}, \frac{u_m+u_n}{2}\right) - \langle f_0, u_m + u_n \rangle - \langle f_1, u_m + u_n \rangle \\
&= \frac{1}{8}a(u_m, u_m) + \frac{1}{8}a(u_n, u_n) + \frac{1}{4}a(u_m, u_n) \\
&\quad - \langle f_0, u_m + u_n \rangle - \langle f_1, u_m + u_n \rangle \\
&\geq E_*
\end{aligned}$$

This implies that

$$\begin{aligned}
-2a(u_m, u_n) &\leq a(u_m, u_m) + a(u_n, u_n) - 4\langle f_0, u_m + u_n \rangle \\
&\quad - 4\langle f_1, u_m + u_n \rangle - 8E_*
\end{aligned} \tag{6.34}$$

Since $E_n = E(u_n)$ converges to E_* we have that $\forall \epsilon > 0 \exists N(\epsilon)$ such that $|E_n - E_*| \leq \frac{\epsilon}{8}$ for all $n > N(\epsilon)$, which implies that $E_n \leq E_* + \frac{\epsilon}{8}$ for all $n > N(\epsilon)$.

From the definition of the energy norm we get

$$\begin{aligned}\|u_m - u_n\|_{\mathcal{V}}^2 &= a(u_m - u_n, u_m - u_n) \\ &= a(u_m, u_m) + a(u_n, u_n) - 2a(u_m, u_n)\end{aligned}$$

Using (6.34) we obtain for $\forall m, n \geq N(\epsilon)$

$$\begin{aligned}\|u_m - u_n\|_{\mathcal{V}}^2 &\leq a(u_m, u_m) + a(u_n, u_n) + a(u_m, u_m) + a(u_n, u_n) \\ &\quad - 4\langle f_0, u_m + u_n \rangle - 4\langle f_1, u_m + u_n \rangle - 8E_* \\ &\leq 4\left(\frac{1}{2}a(u_m, u_m) - \langle f_0, u_m \rangle - \langle f_1, u_m \rangle\right) \\ &\quad + 4\left(\frac{1}{2}a(u_n, u_n) - \langle f_0, u_n \rangle - \langle f_1, u_n \rangle\right) - 8E_* \\ &\leq 4E_m + 4E_n - 8E_* \leq 4(E_* + \frac{\epsilon}{8}) + 4(E_* + \frac{\epsilon}{8}) - 8E_* \\ &\leq \epsilon\end{aligned}$$

Since u_n is a Cauchy sequence in the Hilbert space \mathcal{V} and $\tilde{\mathcal{K}}$ is closed then $\exists u \in \tilde{\mathcal{K}}$

such that $u_n \rightarrow u$ with respect to the energy norm $\|\cdot\|_{\mathcal{V}}$ as $n \rightarrow \infty$.

Next we want to show that $E(u) = E_*$. Let $w = \mathcal{A}^{-1}f_1$ such that

$$\frac{1}{2}a(v, v) - \langle f_1, v \rangle = \frac{1}{2}\|v - w\|_{\mathcal{V}}^2 - \frac{1}{2}\|w\|_{\mathcal{V}}^2$$

Consider \tilde{E} defined as

$$\tilde{E}(v) = \frac{1}{2}\|v - w\|_{\mathcal{V}}^2 - \langle f_0, v \rangle = E(v) + \frac{1}{2}\|w\|_{\mathcal{V}}^2$$

From the definition of f_0 we have for $v \in \tilde{\mathcal{K}}$

$$-\langle f_0, v \rangle = c \int_{-\infty}^k |v(x)| dx$$

where $c > 0$ and therefore

$$E(v) = \frac{1}{2}\|v - w\|_{\mathcal{V}}^2 + c \int_{-\infty}^k |v(x)| dx + \frac{1}{2}\|w\|_{\mathcal{V}}^2$$

Since $v = 0 \in \tilde{\mathcal{K}}$ we have that

$$\tilde{E}_* = \inf_{v \in \tilde{\mathcal{K}}} \tilde{E}(v) \leq \tilde{E}(0) = \frac{1}{2} \|w\|_{\mathcal{V}}^2 \leq C \|f_1\|_{\mathcal{V}^*}^2$$

With $\tilde{E}_n = \tilde{E}(u_n) \rightarrow \tilde{E}_*$, for $n \geq N$ we obtain that $\tilde{E}(u_n) \leq \tilde{E}(0)$ and therefore

$$\frac{1}{2} \|u_n - w\|_{\mathcal{V}}^2 + c \int_{-\infty}^k |u_n(x)| dx \leq C \|f_1\|_{\mathcal{V}^*}$$

which implies

$$\int_{-\infty}^k |u_n(x)| dx \leq C \|f_1\|_{\mathcal{V}^*} \quad (6.35)$$

For fixed R , $-R < k$ we have

$$\begin{aligned} \int_{-R}^k |u - u_n(x)| dx &\leq \left(\int_{-R}^k |u(x) - u_n(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{-R}^k 1 dx \right)^{\frac{1}{2}} \\ &\leq (R + k)^{\frac{1}{2}} \|u - u_n\|_{L_2} \end{aligned}$$

From $\|u - u_n\|_{\mathcal{V}} \rightarrow 0$ we get $\|u - u_n\|_{L_2} \rightarrow 0$ which together with the inequality above implies

$$\int_{-R}^k |u_n(x)| dx \rightarrow \int_{-R}^k |u(x)| dx \text{ as } n \rightarrow \infty \quad (6.36)$$

Using (6.36) and (6.35) we get

$$\int_{-R}^k |u(x)| dx \leq C \|f_1\|_{\mathcal{V}^*}$$

The bound above doesn't depend on R and therefore it implies

$$\int_{-\infty}^k |u(x)| dx = \lim_{R \rightarrow \infty} \int_{-R}^k |u(x)| dx \leq C \|f_1\|_{\mathcal{V}^*} \quad (6.37)$$

The continuity of $a(\cdot, \cdot)$ give us that $a(u_n, u_n) \rightarrow a(u, u)$. From the continuity of f_1 we get that $\langle f_1, u_n \rangle \rightarrow \langle f_1, u \rangle$. Taking $\lim_{n \rightarrow \infty} \tilde{E}(u_n)$ we obtain

$$\lim_{n \rightarrow \infty} \langle f_0, u_n \rangle = \tilde{E}_* - \frac{1}{2} \|u_n - w\|_{\mathcal{V}}^2 < \infty$$

Using the definition of f_0 and that $\tilde{E}(u) \geq \tilde{E}_*$ we obtain

$$\lim_{n \rightarrow \infty} \int_{-\infty}^k |u_n(x)| dx \leq \int_{-\infty}^k |u(x)| dx \quad (6.38)$$

For any $R > -k$ we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^k |u_n(x)| dx \geq \lim_{n \rightarrow \infty} \int_{-R}^k |u_n(x)| dx$$

Using $\|u_n - u\|_{L_2} \rightarrow 0$ we get

$$\lim_{n \rightarrow \infty} \int_{-\infty}^k |u_n(x)| dx \geq \int_{-R}^k |u(x)| dx = \int_{-\infty}^k |u(x)| dx - \int_{-\infty}^{-R} |u(x)| dx$$

Since $\int_{-\infty}^k |u(x)| dx < \infty$ therefore $\lim_{R \rightarrow \infty} \int_{-\infty}^{-R} |u(x)| dx \rightarrow 0$ which implies

$$\lim_{n \rightarrow \infty} \int_{-\infty}^k |u_n(x)| dx \geq \int_{-\infty}^k |u(x)| dx \quad (6.39)$$

From the inequalities (6.38) and (6.39) we obtain

$$\lim_{n \rightarrow \infty} \int_{-\infty}^k |u_n(x)| dx = \int_{-\infty}^k |u(x)| dx \quad (6.40)$$

From the definition of f_0 and (6.40) we obtain

$$\langle f_0, u_n \rangle \rightarrow \langle f_0, u \rangle \quad n \rightarrow \infty$$

This give us $E(u_n) \rightarrow E(u)$ as $n \rightarrow \infty$ which implies that $E(u) = E_*$.

We need to show that u solves the variational inequality (6.33). Let $v \in \tilde{\mathcal{K}}$ and

$w = u + \lambda(v - u)$ with $\lambda \in (0, 1)$, then by convexity $w \in \tilde{\mathcal{K}}$.

$$\begin{aligned}
E(w) &= E(u + \lambda(v - u)) \\
&= \frac{1}{2}a(u + \lambda(v - u), u + \lambda(v - u)) - \langle f_0, u + \lambda(v - u) \rangle \\
&\quad - \langle f_1, u + \lambda(v - u) \rangle \\
&= \frac{1}{2}a(u, u) - \langle f_0, u \rangle - \langle f_1, u \rangle \\
&\quad \lambda[a(u, v - u) - \langle f_0, v - u \rangle - \langle f_1, v - u \rangle] \\
&\quad + \frac{1}{2}\lambda^2 a(v - u, v - u)
\end{aligned}$$

Using that $E(u) \leq E(w)$ we have $\forall \lambda \in (0, 1)$

$$\lambda[a(u, v - u) - \langle f_0, v - u \rangle - \langle f_1, v - u \rangle] + \frac{1}{2}\lambda^2 a(v - u, v - u) \geq 0 \quad (6.41)$$

If $-\langle f_0, v \rangle < \infty$ then the coefficient of λ is finite. If we have $\alpha > 0$ and a quadratic function $f(x) = \alpha x^2 + \beta x \geq 0 \forall x \in (0, 1)$ then $\beta \geq 0$. Therefore the coefficient of λ in (6.41) is non-negative which implies that (6.33) holds.

If $-\langle f_0, v \rangle = \infty$ and since $\forall v \in \tilde{\mathcal{K}}$ we have $a(u, v) < \infty$, $\langle f_0, u \rangle < \infty$, and $\langle f_1, v \rangle < \infty$ therefore the inequality above is satisfied again which implies that u solves (6.33).

We have shown existence of solutions when $a(\cdot, \cdot)$ is symmetric.

To show uniqueness we will consider two solutions u_1 and u_2 which satisfy the variational inequality (6.30)

$$a(u_1, v - u_1) \geq \langle f, v - u_1 \rangle \quad (6.42)$$

and

$$a(u_2, v - u_2) \geq \langle f, v - u_2 \rangle \quad (6.43)$$

Let $v = u_2$ in (6.42) and $v = u_1$ in (6.43). Since both u_1 and u_2 are solutions of (6.30) they satisfy $\int_{-\infty}^k |u_1(x)| dx < \infty$ and $\int_{-\infty}^k |u_2(x)| dx < \infty$. From the definition

of f we get that $\langle f, u_1 \rangle < \infty$ and $\langle f, u_2 \rangle < \infty$. We can add the inequalities (6.42) and (6.43) to obtain

$$a(u_2 - u_1, u_2 - u_1) \leq 0$$

Uniqueness $u_1 = u_2$ follows.

We have shown existence and uniqueness of solutions when $a(\cdot, \cdot)$ is symmetric. Next we will consider the case when $a(\cdot, \cdot)$ is non-symmetric.

To prove existence in the case when $a(\cdot, \cdot)$ is non-symmetric, continuous (4.3) and symmetric (4.4), we reduced the non-symmetric problem to a symmetric case at each step of a fixed-point iteration problem.

Consider the symmetric bilinear $q(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$

$$q(u, v) = \langle u, v \rangle_{\mathcal{V}} \quad \forall u, v \in \mathcal{V}$$

and the symmetric operator $Q : \mathcal{V} \rightarrow \mathcal{V}^*$ such that

$$q(u, v) = \langle Qu, v \rangle \quad \forall u, v \in \mathcal{V}$$

Consider the operator $\mathcal{S}_{\tilde{\mathcal{K}}} : \tilde{\mathcal{K}} \rightarrow \tilde{\mathcal{K}}$, with $u = \mathcal{S}_{\tilde{\mathcal{K}}}(w)$ where u is the solution of the following variational inequality: find $u \in \tilde{\mathcal{K}}$ such that $\forall v \in \tilde{\mathcal{K}}$ we have the following

$$q(u - w, v - u) \geq \alpha(\langle f_0 + f_1, v - u \rangle - a(w, v - u))$$

We can rewrite this as a symmetric problem: find $u \in \tilde{\mathcal{K}}$ such that $\forall v \in \tilde{\mathcal{K}}$

$$q(u - w, v - u) \geq \langle \tilde{f}_0 + \tilde{f}_1, v - u \rangle$$

where

$$\tilde{f}_0 = \alpha f_0$$

and

$$\tilde{f}_1 = \alpha(f_1 - \mathcal{A}w) + Qw$$

From the definition of f_0 , $q(\cdot, \cdot)$ symmetric and $\tilde{f}_1 \in \mathcal{V}^*$ by the first part of our proof in the symmetric case we have that each iteration step will have a unique solution $u_{n+1} \in \tilde{\mathcal{K}}$. In addition the following holds for $\forall n > 0$

$$\int_{-\infty}^k |u_n(x)| dx \leq C_0 \|\tilde{f}_1\|_{\mathcal{V}^*}$$

We want to show that $\mathcal{S}_{\tilde{\mathcal{K}}}$ is a contraction with respect to the norm $\|\cdot\|_q$ defined as

$$\|u\|_q^2 = q(u, u) = \|u\|_{\mathcal{V}}^2$$

Consider $w_1 \in \tilde{\mathcal{K}}$ and $w_2 \in \tilde{\mathcal{K}}$. Let $u_1 = \mathcal{S}_{\tilde{\mathcal{K}}}(w_1)$ and $u_2 = \mathcal{S}_{\tilde{\mathcal{K}}}(w_2)$ which satisfy the variational inequality (6.30)

$$q(u_1, v - u_1) \geq \langle \alpha f_0 + \alpha(f_1 - \mathcal{A}w_1) + Qw_1, v - u_1 \rangle \quad (6.44)$$

and

$$q(u_2, v - u_2) \geq \langle \alpha f_0 + \alpha(f_1 - \mathcal{A}w_2) + Qw_2, v - u_2 \rangle \quad (6.45)$$

To show that $\mathcal{S}_{\tilde{\mathcal{K}}}$ to be a contraction with respect to the norm $\|\cdot\|_q$ we need $\|u_2 - u_1\|_q \leq C\|w_2 - w_1\|_q$, $0 < C < 1$.

Let $v = u_2$ in (6.44) and $v = u_1$ in (6.45). Since both u_1 and u_2 are solutions of (6.30) they satisfy $\int_{-\infty}^k |u_1(x)| dx < \infty$ and $\int_{-\infty}^k |u_2(x)| dx < \infty$. From the definition of f_0 we get that $\langle f_0, u_1 \rangle < \infty$ and $\langle f_0, u_2 \rangle < \infty$. Since the remaining of the terms are in

\mathcal{V}^* we can add the inequalities (6.44) and (6.45) to obtain using $\langle u, v \rangle = q(Q^{-1}u, v)$

$$\begin{aligned}
\|u_2 - u_1\|_q^2 &= q(u_2 - u_1, u_2 - u_1) \\
&\leq \langle Qw_2 - \alpha \mathcal{A}w_2 - Qw_2 + \alpha \mathcal{A}w_2, u_2 - u_1 \rangle \\
&\leq q(w_2 - w_1 - \alpha Q^{-1} \mathcal{A}(w_2 - w_1), u_2 - u_1) \\
&\leq \|w_2 - w_1 - \alpha Q^{-1} \mathcal{A}(w_2 - w_1)\|_q \|u_2 - u_1\|_q
\end{aligned} \tag{6.46}$$

Recall the continuity constant C_1 from (4.3) and the coercivity constant C_2 from (4.4). Using this notation we obtain the following estimate for the q -norm of the mapping $\mathcal{S}_{\tilde{\mathcal{K}}}$

$$\begin{aligned}
\|u_2 - u_1\|_q^2 &\leq \|w_2 - w_1 - \alpha Q^{-1} \mathcal{A}(w_2 - w_1)\|_q^2 \\
&\leq \|w_2 - w_1\|_q^2 - 2\alpha a(w_2 - w_1, w_2 - w_1) + \\
&\quad + \alpha^2 \|Q^{-1} \mathcal{A}(w_2 - w_1)\|_q^2 \\
&\leq \|w_2 - w_1\|_q^2 - 2\alpha C_2 \|w_2 - w_1\|_q^2 + \\
&\quad + \alpha^2 C_1 \|\mathcal{A}(w_2 - w_1)\|_{\mathcal{V}^*}^2 \\
&\leq (1 - 2\alpha C_2 + \alpha^2 C_1^2) \|w_2 - w_1\|_q^2 \\
&\leq C \|w_2 - w_1\|_q^2
\end{aligned}$$

with $C = 1 - 2\alpha C_2 + \alpha^2 C_1^2$.

For $0 < \alpha < \frac{2C_2}{C_1^2}$ we have that $C < 1$ which implies that $\mathcal{S}_{\tilde{\mathcal{K}}}$ is a contraction.

Since \mathcal{V} is a Hilbert space we have by the Contraction Mapping Theorem that for $\forall u_0 \in \mathcal{V}$ the fixed point iteration will converge to a unique solution u^* which satisfies $\forall v \in \tilde{\mathcal{K}}$

$$q(u^* - u^*, v - u^*) \geq \alpha (\langle f_0 + f_1, v - u^* \rangle - a(u^*, v - u^*))$$

This implies

$$a(u^*, v - u^*) \geq \langle f_0 + f_1, v - u^* \rangle$$

which means that u^* satisfies (6.30). This completes our proof of existence of solutions for the non-symmetric case.

To show uniqueness we will consider two solutions u_1 and u_2 which satisfy the variational inequality (6.30)

$$a(u_1, v - u_1) \geq \langle f_0 + f_1, v - u_1 \rangle$$

and

$$a(u_2, v - u_2) \geq \langle f_0 + f_1, v - u_2 \rangle$$

Let $v = u_2$ in the first inequality, $v = u_1$ in the second inequality. Since both u_1 and u_2 are solutions of (6.30) they satisfy $\int_{-\infty}^k |u_1(x)| dx < \infty$ and $\int_{-\infty}^k |u_2(x)| dx < \infty$.

From the definition of f_0 we get that $\langle f_0, u_1 \rangle < \infty$ and $\langle f_0, u_2 \rangle < \infty$. Since f_1 is in \mathcal{V}^* we can add the inequalities above to obtain

$$a(u_2 - u_1, u_2 - u_1) \leq 0$$

Uniqueness $u_1 = u_2$ follows. This completes the proof of theorem 6.7

□

Remark: The proof for the theorem 6.7 also applies for the perpetual American put case since we only used the fact that $u \in \tilde{K}$ is bounded from below.

In the case of the perpetual game call we will consider the excess to function $\tilde{u} = u - \tilde{F}$ where \tilde{F} is defined as

$$\tilde{F}(x) = \begin{cases} 0 & \text{if } x \leq \log(K - \delta) \\ e^x - K + \delta & \text{if } x > \log(K - \delta) \end{cases}$$

We consider the solution space $\tilde{\mathcal{K}}$ defined as

$$\tilde{\mathcal{K}} = \{v \mid v \in \mathcal{V}, F - \tilde{F} \leq v \leq G - \tilde{F}\} \quad (6.47)$$

Theorem 6.8. *The perpetual game call option value given by complementarity problem (6.30) for $f = f_{call}$, and f_0, f_1 defined in (6.32), (6.31): find $u \in \tilde{\mathcal{K}}$ such that $\int_k^\infty |u(x)|dx < \infty$ and*

$$a(u, v - u) \geq \langle f_0, v - u \rangle + \langle f_1, v - u \rangle \quad \forall v \in \tilde{\mathcal{K}} \quad (6.48)$$

has a unique solution $u \in \tilde{\mathcal{K}}$.

Proof. See proof for theorem 6.7. □

Chapter 7

Implementation and Numerical Results

7.1 Implementation of the Finite Elements Method for Game Options

7.1.1 The Stiffness Matrix for P.W.L. Finite Elements under CGMY

In this section we will present an implementation of the finite element method introduced in chapter 4. We will begin with the methodology used to compute the stiffness matrix and the load vector followed by the Projected Richardson iteration, which is one of the the methods we used to solve the LCP problem numerically. From the definition of the hat functions, ϕ_i , the mass matrix B defined as

$$B_{ij} = \int_{\Omega} \phi_i \phi_j$$

With $h = \Delta x = x_i - x_{i-1}$, the mass matrix is the following tridiagonal matrix

$$B = \frac{h}{6} \begin{pmatrix} 4 & 1 & \cdots & 0 \\ 1 & 4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 4 \end{pmatrix} \quad (7.1)$$

The stiffness matrix A, defined as

$$A_{ij} = a(\phi_j, \phi_i) = \int_{\Omega} (\mathcal{A}\phi_j)\phi_i$$

is tridiagonal only when we the operator \mathcal{A} is local, i.e. the pure diffusion case. From chapter 3 for the jump diffusion case $\mathcal{A}u = \mathcal{L}u - ru$. Denoting $u' = \frac{\partial u}{\partial x}$, $u'' = \frac{\partial^2 u}{\partial x^2}$, and $K = K_{CGMY}$ the jump density, we get

$$\mathcal{A}u = \left(r - \frac{\sigma^2}{2}\right)u' + \frac{\sigma^2}{2}u'' + \int_{\mathbb{R}} (u(x+y) - u(x) - (e^y - 1)u') K(y)dy - ru$$

Consider the following notation $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 - \mathcal{A}_3$

$$\mathcal{A}_0 u = \left(r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - y - 1)K(y)dy\right)u' = \mu u'$$

where

$$\mu = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - y - 1)K(y)dy \quad (7.2)$$

$$\mathcal{A}_1 u = \frac{\sigma^2}{2}u''$$

$$\mathcal{A}_2 u = \int_{\mathbb{R}} (u(x+y) - u(x) - yu') K(y)dy$$

$$\mathcal{A}_3 u = ru$$

with the corresponding matrices

$$A_0 = \int_{\Omega} \phi'_j \phi_i = \frac{1}{2} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & -1 & 0 \end{pmatrix} \quad (7.3)$$

and

$$A_1 = \int_{\Omega} \phi''_j \phi_i = - \int_{\Omega} \phi'_j \phi'_i = \frac{1}{h} \begin{pmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & -2 \end{pmatrix} \quad (7.4)$$

Notice that the operator \mathcal{A}_3 gives us just the mass matrix B defined in (7.1). Next we will work on the integral operator \mathcal{A}_2 .

We will consider the case $u(x) = \phi_i(x)$. This argument also works for any function $u \in C^2(\mathbb{R})$ with $|u|, |u'|, |u''|$ bounded and $|u|, |u'|$ decaying sufficiently fast for $x \rightarrow \pm\infty$, or u with compact support. The hat functions considered $u(x) = \phi_i(x)$ do not have a second derivative in the classical sense, but the derivative ϕ_i'' exists in the distributional sense

$$\phi_i''(x) = \frac{1}{h}\delta_{x_{i-1}} - \frac{2}{h}\delta_{x_i} + \frac{1}{h}\delta_{x_{i+1}} \quad (7.5)$$

We will show how to compute the elements of the stiffness matrix in terms of antiderivatives of the kernel function $K(x)$ and derivatives of u . For simplicity we will assume that $K(x) = 0$ for $x < 0$, since the general case can be obtained by a similar argument on the negative part and then adding them together.

Define the antiderivatives $\tilde{K}_1(x), \tilde{K}_2(x)$ for $x \geq 0$, as follows

$$\tilde{K}_1(x) = \int_0^x K(y)dy \quad (7.6)$$

and

$$\tilde{K}_2(x) = \int_0^x \tilde{K}_1(y)dy \quad (7.7)$$

Consider the following operator

$$(\mathcal{B}_0 u)(x) = \int_0^\infty (u(x+y))K(y)dy \quad (7.8)$$

where $K(y) \in L^1(0, \infty)$. Assumption that $u'(x)$ exists for all $x \in \mathbb{R}$, and that $u(x) \rightarrow 0$ as $x \rightarrow \infty$. Then using using integration by parts and $\tilde{K}_1(x)$ defined in

(7.6), we obtain

$$(\mathcal{B}_0 u)(x) = \int_0^\infty (u'(x+y)) \tilde{K}_1(y) dy \quad (7.9)$$

If we assume additionally that $u''(x)$ exists for all $x \in \mathbb{R}$, and that $|x| \cdot |u'(x)| \rightarrow 0$ as $x \rightarrow \infty$, then, using (7.9) and (7.7), the operator \mathcal{B}_0 defined in (7.8) becomes

$$(\mathcal{B}_0 u)(x) = \int_0^\infty (u''(x+y)) \tilde{K}_2(y) dy \quad (7.10)$$

In the case of $u(x) = \phi_i(x)$, using the distributional derivative of $\phi_i(x)$ defined in (7.5), we get

$$(\mathcal{B}_0 \phi_i)(x) = \frac{1}{h} \tilde{K}_2(x_{i-1} - x) - \frac{2}{h} \tilde{K}_2(x_i - x) + \frac{1}{h} \tilde{K}_2(x_{i+1} - x) \quad (7.11)$$

Define the antiderivatives $K_1(x)$, $K_2(x)$ for $x \geq 0$, as follows

$$K_1(x) = - \int_x^\infty K(y) dy \quad (7.12)$$

and

$$K_2(x) = - \int_x^\infty K_1(y) dy \quad (7.13)$$

Consider the operator \mathcal{B}_1 given by

$$(\mathcal{B}_1 u)(x) = \int_0^\infty (u(x+y) - u(x)) K(y) dy \quad (7.14)$$

with $K(x) \rightarrow 0$ as $x \rightarrow \infty$ and $\min(|x|, 1)K(x) \in L_1(0, \infty)$. Using (7.12) and assuming that $u'(x)$ exists and $|u(x)| \leq c$ for all $x \in \mathbb{R}$, we obtain

$$(\mathcal{B}_1 u)(x) = - \int_0^\infty (u'(x+y)) K_1(y) dy \quad (7.15)$$

Consider the operator \mathcal{B}_2 given by

$$(\mathcal{B}_2 u)(x) = \int_0^\infty (u(x+y) - u(x) - u'(x)y) K(y) dy \quad (7.16)$$

where $\min(|x|^2, |x|)K(x) \in L_1(0, \infty)$. Using (7.13) and (7.15) with the assumption that $u''(x)$ exists and $|u(x)'| \leq c$ for all $x \in \mathbb{R}$, we get the following expression for

$(\mathcal{B}_2)(x)$

$$(\mathcal{B}_2u)(x) = \int_0^\infty (u''(x+y))K_2(y)dy \quad (7.17)$$

In the case of $u(x) = \phi_i(x)$, using the distributional derivative of $\phi_i(x)$ defined in (7.5), we get

$$(\mathcal{B}_2\phi_i)(x) = \frac{1}{h}K_2(x_{i-1} - x) - \frac{2}{h}K_2(x_i - x) + \frac{1}{h}K_2(x_{i+1} - x) \quad (7.18)$$

Define the antiderivatives $\tilde{K}_3(x)$, $\tilde{K}_4(x)$ for $x \geq 0$, with K_1 and K_2 defined in (7.12) and (7.13)

$$\tilde{K}_3(x) = \int_0^x K_2(y)dy \quad (7.19)$$

and

$$\tilde{K}_4(x) = \int_0^x \tilde{K}_3(y)dy \quad (7.20)$$

Assuming that $u''(x)$, $v''(x)$ exists and $|u(x)'| \leq c$, $|v(x)'| \leq c$ for all $x \in \mathbb{R}$, and using the result (7.10) and (7.17), we obtain

$$\int_{-\infty}^\infty (\mathcal{B}_2u)(x)v(x)dx = \int_{-\infty}^\infty v''(x) \int_0^\infty u''(x+y)\tilde{K}_4(y)dydx \quad (7.21)$$

For $u(x) = \phi_i(x)$ and $v(x) = \phi_j(x)$, (7.21) can be interpreted in the distributional sense

$$\int_{-\infty}^\infty (\mathcal{B}_2\phi_j)(x)\phi_i(x)dx = \frac{1}{h}F_j(x_{i-1}) - \frac{2}{h}F_j(x_i) + \frac{1}{h}F_j(x_{i+1}) \quad (7.22)$$

where $F_i(x)$ is defined using the result (7.18)

$$F_i(x) = \frac{1}{h}\tilde{K}_4(x_{i-1} - x) - \frac{2}{h}\tilde{K}_4(x_i - x) + \frac{1}{h}\tilde{K}_4(x_{i+1} - x) \quad (7.23)$$

Using the same argument, similar results can be obtained for $x < 0$. Let us denote the positive and the negative part of the CGMY jump density by

$$K^+(x) = \begin{cases} K(x) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and

$$K^-(x) = \begin{cases} 0 & x \geq 0 \\ K(x) & x < 0 \end{cases}$$

With $K_1^\pm(x), K_2^\pm(x), \tilde{K}_3^\pm(x), \tilde{K}_4^\pm(x)$ defined for $K^\pm(x)$ similarly to (7.12), (7.13), (7.19), (7.20), let

$$F(x) = \begin{cases} \tilde{K}_4^+(x) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ \tilde{K}_4^-(x) & \text{for } x < 0 \end{cases}$$

and

$$p_{ik} = \begin{cases} \frac{1}{h} & \text{for } k = -1 \\ -\frac{2}{h} & \text{for } k = 0 \\ \frac{1}{h} & \text{for } k = 1 \end{cases}$$

Now we have all the ingredients necessary to complete the stiffness matrix. The matrix corresponding to \mathcal{A}_2 is

$$(A_2)_{ij} = \int_{\Omega} (\mathcal{A}_2 \phi_j) \phi_i$$

Using the definition above and the result (7.22) we get the following matrix

$$(A_2)_{ij} = \sum_{k=-1}^1 p_{ik} \sum_{l=-1}^1 p_{jl} F(x_j - x_i) \quad (7.24)$$

To get the fourth antiderivative of the jump density for the CGMY process defined in (3.2) for the case when $1 < Y < 2$, let

$$K^-(x) = \begin{cases} C \frac{e^{-G|x|}}{|x|^{1+Y}} & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

and

$$K^+(x) = \begin{cases} C \frac{e^{-M|x|}}{|x|^{1+Y}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Using the definitions (7.12), (7.13), (7.19), (7.20) we get the following expressions for $K_2^\pm(x)$, $\tilde{K}_3^\pm(x)$, $\tilde{K}_4^\pm(x)$

$$K_1^+(x) = -CM^Y \Gamma(-Y, Mx)$$

$$K_2^+(x) = CM^{Y-1} \left(-Mx\Gamma(-Y, Mx) + \Gamma(1-Y, Mx) \right)$$

$$\tilde{K}_3^+(x) = CM^{Y-2} \left(-\frac{M^2x^2}{2}\Gamma(-Y, Mx) + Mx\Gamma(1-Y, Mx) + \frac{1}{2}\gamma(2-Y, Mx) \right)$$

$$\tilde{K}_4^+(x) = CM^{Y-3} \left(-\frac{M^3x^3}{6}\Gamma(-Y, Mx) + \frac{M^2x^2}{2}\Gamma(1-Y, Mx) + \frac{Mx}{2}\gamma(2-Y, Mx) - \frac{1}{6}\gamma(3-Y, Mx) \right)$$

for $x \geq 0$, and similarly

$$K_1^-(x) = -CG^Y \Gamma(-Y, -Gx)$$

$$K_2^-(x) = CG^{Y-1} \left(Gx\Gamma(-Y, -Gx) + \Gamma(1-Y, -Gx) \right)$$

$$\tilde{K}_3^-(x) = CG^{Y-2} \left(-\frac{G^2x^2}{2}\Gamma(-Y, -Gx) - Gx\Gamma(1-Y, -Gx) + \frac{1}{2}\gamma(2-Y, -Gx) \right)$$

$$\tilde{K}_4^-(x) = CG^{Y-3} \left(\frac{G^2x^3}{6}\Gamma(-Y, -Gx) + \frac{G^2x^2}{2}\Gamma(1-Y, -Gx) - \frac{Gx}{2}\gamma(2-Y, -Gx) - \frac{1}{6}\gamma(3-Y, -Gx) \right)$$

for $x \leq 0$. Here $\Gamma(\alpha, x)$ and $\gamma(\alpha, x)$ are the incomplete gamma functions given by

$$\Gamma(\alpha, x) = \int_x^\infty \frac{e^{-t}}{t^{1-\alpha}} dt$$

and

$$\gamma(\alpha, x) = \int_0^x \frac{e^{-t}}{t^{1-\alpha}} dt$$

We would like to mention that Matlab does not support the incomplete gamma function for the case when $\alpha < 0$. In this situation the following recursive relations need to be used to compute the incomplete gamma functions

$$\Gamma(\alpha, x) = \frac{\Gamma(\alpha + 1, x) - x^\alpha e^{-x}}{\alpha}$$

and

$$\gamma(\alpha, x) = \frac{\gamma(\alpha + 1, x) + x^\alpha e^{-x}}{\alpha}$$

To get the complete form for the stiffness matrix, we also need the expression for μ defined in (7.2), let

$$\mu = r - \frac{\sigma^2}{2} - \mu_- - \mu_+$$

where

$$\mu_- = \int_{-\infty}^0 (e^x - x - 1)K^-(x)dx = \int_{-\infty}^0 C(e^x - x - 1) \frac{e^{-G|x|}}{|x|^{1+Y}} dx$$

and

$$\mu_+ = \int_0^{\infty} (e^x - x - 1)K^+(x)dx = \int_0^{\infty} C(e^x - x - 1) \frac{e^{-M|x|}}{|x|^{1+Y}} dx$$

With the notation above we get the following expressions for μ_- and μ_+

$$\mu_- = \Gamma(-Y) \left((G+1)^Y - YG^{Y-1} - G^Y \right)$$

for $G > 0$, and

$$\mu_+ = \Gamma(-Y) \left((M-1)^Y + YM^{Y-1} - M^Y \right)$$

for $M > 1$. Here $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} \frac{e^{-t}}{t^{1-\alpha}} dt$$

Finally with (7.1), (7.2), (7.3), (7.4), (7.24) we get our expression for the stiffness matrix

$$A = \mu A_0 + \frac{\sigma^2}{2} A_1 + A_2 - rB$$

In the Black Scholes model, (pure diffusion process), since we don't have any jumps, the integral operator corresponding to the jump measure, \mathcal{A}_2 , disappears, and so does the integral part of μ leaving us with the following expression of the stiffness matrix

$$A = \left(r - \frac{\sigma^2}{2}\right) A_0 + \frac{\sigma^2}{2} A_1 - rB$$

When we have the pure jump CGMY process, the diffusion parameter, $\sigma = 0$, reducing the expression of the stiffness matrix to

$$A = (r - \mu_- - \mu_+) A_0 + A_2 - rB$$

In the jump diffusion case, we have both, the jump measure and the diffusion parameter non-zero and therefore the expression for the stiffness matrix will contain all the corresponding matrices, A_0, A_1, A_2, B , and the vector μ , defined above

$$A = \mu A_0 + \frac{\sigma^2}{2} A_1 + A_2 - rB$$

7.1.2 The Load Vector for P.W.L. Finite Elements under CGMY

In this section we will present in detail an algorithm to compute the load vector $\vec{f} \in \mathbb{R}_N$ defined in the previous chapter as

$$f_i = \langle \mathcal{A}f, \phi_i \rangle = \int_{\Omega} (\mathcal{A}f)\phi_i \quad (7.25)$$

with $\Omega = (-R, R)$. The piecewise linear finite element functions $\phi_i \in \mathcal{V}_h$ introduced in (4.12) are defined as

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x_{i-1} \leq x < x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{if } x_i \leq x < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

and the elliptic operator \mathcal{A} defined as $\mathcal{A}u = \mathcal{L}u - ru$, where \mathcal{L} is the infinitesimal generator of the Lévy process introduced in chapter 3. Denoting $u' = \frac{\partial u}{\partial x}$, $u'' = \frac{\partial^2 u}{\partial x^2}$, and K_{CGMY} the jump density, we get

$$\mathcal{A}u = \left(r - \frac{\sigma^2}{2}\right)u' + \frac{\sigma^2}{2}u'' + \int_{\mathbb{R}} (u(x+y) - u(x) - (e^z - 1)u') K_{CGMY}(z)dz - ru$$

Denote the local components of \mathcal{A} by \mathcal{A}_0 and \mathcal{A}_1

$$\mathcal{A}_0 u = \frac{\sigma^2}{2}(u'' - u') \quad (7.26)$$

$$\mathcal{A}_1 u = r(u' - u) \quad (7.27)$$

Denote the non local component of \mathcal{A} by \mathcal{A}_2

$$\mathcal{A}_2 u = \int_{\mathbb{R}} (u(x+y) - u(x) - (e^z - 1)u') K_{CGMY}(z)dz \quad (7.28)$$

The payoff function f is in general defined as

$$f(x) = c_1 + c_2 e^x$$

For the put option we have the following payoff function

$$f(x) = \begin{cases} K - e^x & \text{if } x < k \\ 0 & \text{if } x \geq k \end{cases} \quad (7.29)$$

For the call option the payoff function is defined as

$$f(x) = \begin{cases} 0 & \text{if } x \leq k \\ e^x - K & \text{if } x > k \end{cases} \quad (7.30)$$

Here k is the log of the strike K , $k = \log(K)$.

We will show how to compute the load vector for the case of the payoff of the put option. In this situation the first and the second derivatives of f are defined as

$$f'(x) = \begin{cases} -e^x & \text{if } x < k \\ 0 & \text{if } x > k \end{cases}$$

$$f''(x) = \begin{cases} -e^x & \text{if } x < k \\ 0 & \text{if } x > k \end{cases}$$

First, we will construct the integral of the local operators \mathcal{A}_0 and \mathcal{A}_1 . Using uniform step size $h = \Delta x = x_i - x_{i-1}$, we obtain the following expressions

$$\int_{\Omega} f \phi_i = \int_{x_{i-1}}^{x_{i+1}} f(x) \phi_i(x) dx = \begin{cases} 0 & \text{if } x_{i-1} \geq k \\ K \frac{h}{2} + \frac{K - e^{x_{i-1}}}{h} - K & \text{if } x_i = k \\ Kh + \frac{2K - e^{x_{i+1}} - e^{x_{i-1}}}{h} & \text{if } x_{i+i} \leq k \end{cases} \quad (7.31)$$

$$\int_{\Omega} f' \phi_i = \int_{x_{i-1}}^{x_{i+1}} f'(x) \phi_i(x) dx = \begin{cases} 0 & \text{if } x_{i-1} \geq k \\ \frac{K - e^{x_{i-1}}}{h} - K & \text{if } x_i = k \\ \frac{2K - e^{x_{i+1}} - e^{x_{i-1}}}{h} & \text{if } x_{i+i} \leq k \end{cases} \quad (7.32)$$

$$\int_{\Omega} f'' \phi_i = - \int_{x_{i-1}}^{x_{i+1}} f'(x) \phi_i'(x) dx = \begin{cases} 0 & \text{if } x_{i-1} \geq k \\ \frac{K - e^{x_{i-1}}}{h} & \text{if } x_i = k \\ \frac{2K - e^{x_{i+1}} - e^{x_{i-1}}}{h} & \text{if } x_{i+1} \leq k \end{cases} \quad (7.33)$$

Let \vec{f}^0 be the vector corresponding to the operator \mathcal{A}_0 defined as

$$\vec{f}_i^0 = \int_{\Omega} (\mathcal{A}_0 f) \phi_i \quad (7.34)$$

Using (7.31), (7.32) and (7.33), we obtain the following expression for \vec{f}^0

$$\vec{f}_i^0 = \begin{cases} 0 & \text{if } x_{i-1} \geq k \\ K \frac{\sigma^2}{2} & \text{if } x_i = k \\ 0 & \text{if } x_{i+1} \leq k \end{cases} \quad (7.35)$$

Let \vec{f}^1 be the vector corresponding to the operator \mathcal{A}_1 defined as

$$\vec{f}_i^1 = \int_{\Omega} (\mathcal{A}_1 f) \phi_i \quad (7.36)$$

For \vec{f}^1 we get the following expression using (7.31), (7.32) and (7.33)

$$\vec{f}_i^1 = \begin{cases} 0 & \text{if } x_{i-1} \geq k \\ -rK \frac{h}{2} & \text{if } x_i = k \\ -rKh & \text{if } x_{i+1} \leq k \end{cases} \quad (7.37)$$

Next, we shall continue with the non-local operator \mathcal{A}_2 . The hat functions considered $\phi_i(x)$ do not have a second derivative in the classical sense, but the derivative ϕ_i'' exists in the distributional sense

$$\phi_i''(x) = \frac{1}{h} \delta_{x_{i-1}} - \frac{2}{h} \delta_{x_i} + \frac{1}{h} \delta_{x_{i+1}} \quad (7.38)$$

We will show how to compute the elements of the load vector D in terms of antiderivatives of the kernel function $K(x)$ and derivatives of f . For simplicity we will

assume that $K(x) = 0$ for $x < 0$, since the negative part can be computed using a similar argument. The general case can then be obtained by adding the negative part and the positive part together.

Define the antiderivatives $\tilde{K}_1(x)$, $\tilde{K}_2(x)$ for $x \geq 0$, as follows

$$\tilde{K}_1(x) = \int_0^x K(y)dy \quad (7.39)$$

and

$$\tilde{K}_2(x) = \int_0^x \tilde{K}_1(y)dy \quad (7.40)$$

Consider the following operator

$$(\mathcal{B}_0 u)(x) = \int_0^\infty (u(x+y))K(y)dy \quad (7.41)$$

where $K(y) \in L^1(0, \infty)$. Assumption that $u'(x)$ exists for all $x \in \mathbb{R}$, and that $u(x) \rightarrow 0$ as $x \rightarrow \infty$. Then using using integration by parts and $\tilde{K}_1(x)$ defined in (7.39), we obtain

$$(\mathcal{B}_0 u)(x) = \int_0^\infty (u'(x+y))\tilde{K}_1(y)dy \quad (7.42)$$

If we assume additionally that $u''(x)$ exists for all $x \in \mathbb{R}$, and that $|x| \cdot |u'(x)| \rightarrow 0$ as $x \rightarrow \infty$, then, using (7.42) and (7.40), the operator \mathcal{B}_0 defined in (7.41) becomes

$$(\mathcal{B}_0 u)(x) = \int_0^\infty (u''(x+y))\tilde{K}_2(y)dy \quad (7.43)$$

In the case of $u(x) = \phi_i(x)$, using the distributional derivative of $\phi_i(x)$ defined in (7.38), we get

$$(\mathcal{B}_0 \phi_i)(x) = \frac{1}{h}\tilde{K}_2(x_{i-1} - x) - \frac{2}{h}\tilde{K}_2(x_i - x) + \frac{1}{h}\tilde{K}_2(x_{i+1} - x) \quad (7.44)$$

Define the antiderivative $K_1(x)$ for $x \geq 0$, as follows

$$K_1(x) = - \int_x^\infty K(y) dy \quad (7.45)$$

Consider the operator \mathcal{A}_2 defined in (7.28) given by

$$(\mathcal{A}_2 u)(x) = \int_{\mathbb{R}} (u(x+y) - u(x) - (e^z - 1)u') K_{CGMY}(z) dz$$

with $K(x)_{CGMY} \rightarrow 0$ as $x \rightarrow \infty$ and $\min(|x|, 1)K(x) \in L_1(0, \infty)$. Using (7.45), assuming that $u'(x)$ exists and $|u(x)| \leq c$ for all $x \in \mathbb{R}$, using integration by parts, we obtain

$$(\mathcal{A}_2 u)(x) = - \int_0^\infty (u'(x+y) - u'(x)e^y) K_1(y) dy \quad (7.46)$$

Define the antiderivative $K_2(x)$ for $x \geq 0$, as follows

$$K_2(x) = -e^{-x} \int_x^\infty e^y K_1(y) dy \quad (7.47)$$

Note that $K_2(x)$ satisfies the following differential equation

$$K_2'(x) + K_2(x) - K_1(x) = 0$$

Using (7.47) and integration by parts with the assumption that $u''(x)$ exists and $|u(x)'| \leq c$ for all $x \in \mathbb{R}$, we get the following expression for $(\mathcal{A}_2)(x)$

$$(\mathcal{A}_2 u)(x) = \int_0^\infty (u''(x+y) - u'(x+y)) K_2(y) dy \quad (7.48)$$

In the case of $u(x) = f(x)$, the expression $f''(x) - f'(x)$ for the put option (7.29) is defined in the distributional sense as

$$f''(x) - f'(x) = K \delta_k$$

where K is the strike of the put option and $k = \log(K)$.

Using this result we obtain the following expression for A_2f

$$(\mathcal{A}_2f)(x) = KK_2(k - x) \quad (7.49)$$

Define the antiderivatives $\tilde{K}_3(x)$, $\tilde{K}_4(x)$ for $x \geq 0$, with K_1 and K_2 defined in (7.45)

and (7.47)

$$\tilde{K}_3(x) = \int_0^x K_2(y)dy \quad (7.50)$$

and

$$\tilde{K}_4(x) = \int_0^x \tilde{K}_3(y)dy \quad (7.51)$$

Assuming that $u''(x)$, $v''(x)$ exists and $|u(x)'| \leq c$, $|v(x)'| \leq c$ for all $x \in \mathbb{R}$, and using the result (7.43) and (7.48), we obtain

$$\int_{-\infty}^{\infty} (\mathcal{B}_2u)(x)v(x)dx = \int_{-\infty}^{\infty} v''(x) \int_0^{\infty} (u''(x+y) - u'(x+y))\tilde{K}_4(y)dydx \quad (7.52)$$

For $u(x) = f(x)$ and $v(x) = \phi_i(x)$, (7.52) can be interpreted in the distributional sense

$$\int_{-\infty}^{\infty} (\mathcal{A}_2f)(x)\phi_i(x)dx = \frac{1}{h}F(x_{i-1}) - \frac{2}{h}F(x_i) + \frac{1}{h}F(x_{i+1}) \quad (7.53)$$

where $F(x)$ is defined using the result (7.49)

$$F(x) = K\tilde{K}_4(k - x) \quad (7.54)$$

Using the same argument, similar results can be obtained for $x < 0$. Let us denote the positive and the negative part of the CGMY jump density by

$$K^+(x) = \begin{cases} K_{CGMY}(x) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and

$$K^-(x) = \begin{cases} 0 & x \geq 0 \\ K_{CGMY}(x) & x < 0 \end{cases}$$

With $K_1^\pm(x), K_2^\pm(x), \tilde{K}_3^\pm(x), \tilde{K}_4^\pm(x)$ defined for $K^\pm(x)$ similarly to (7.45), (7.47), (7.50), (7.51), let

$$F(x) = \begin{cases} \tilde{K}_4^+(x) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ \tilde{K}_4^-(x) & \text{for } x < 0 \end{cases}$$

and

$$p_i = \begin{cases} \frac{1}{h} & \text{for } i = -1 \\ -\frac{2}{h} & \text{for } i = 0 \\ \frac{1}{h} & \text{for } i = 1 \end{cases}$$

Let \vec{f}^2 be the vector corresponding to the operator \mathcal{A}_1 defined as

$$\vec{f}_i^2 = \int_{\Omega} (\mathcal{A}_2 f) \phi_i \quad (7.55)$$

Now we have all the ingredients necessary to compute the non-local component of the load vector \vec{f}^2 defined in (7.55) above. Using the result (7.22) we get the following expression for D^2

$$\vec{f}_i^2 = \sum_{i=-1}^1 -p_i K F(k - x_i) \quad (7.56)$$

where K is the strike of the put option and $k = \log(K)$.

To get the fourth antiderivative \tilde{K}_4 of the jump density for the CGMY process defined in (3.2) for the case when $1 < Y < 2$, let

$$K^-(x) = \begin{cases} C \frac{e^{-G|x|}}{|x|^{1+Y}} & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

and

$$K^+(x) = \begin{cases} C \frac{e^{-M|x|}}{|x|^{1+Y}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Using the definitions (7.45), (7.47), (7.50), (7.51) we get the following expressions for $K_2^\pm(x), K_3^\pm(x), \tilde{K}_3^\pm(x), \tilde{K}_4^\pm(x)$

$$K_1^+(x) = -CM^Y \Gamma(-Y, Mx)$$

$$K_2^+(x) = C \left(M^Y \Gamma(-Y, Mx) - e^{-x} (M-1)^Y \Gamma(-Y, (M-1)x) \right)$$

$$\begin{aligned} \tilde{K}_3^+(x) = C & \left((M-1)^Y (e^{-x} \Gamma(-Y, (M-1)x) - \Gamma(-Y)) + \right. \\ & \left. + M^{Y-1} (M\gamma(-Y, Mx) + Mx\Gamma(-Y, Mx) + \gamma(1-Y, Mx)) \right) \end{aligned}$$

$$\begin{aligned} \tilde{K}_4^+(x) = C & \left((M-1)^Y ((1-x)\Gamma(-Y) - e^{-x} \Gamma(-Y, (M-1)x)) + \right. \\ & + M^Y (x-1)\gamma(-Y, Mx) + M^{Y-1} (x-1)\gamma(1-Y, Mx) + \\ & \left. + M^{Y-2} \left(\frac{M^2 x^2}{2} \Gamma(-Y, Mx) - \frac{1}{2} \gamma(2-Y, Mx) \right) \right) \end{aligned}$$

for $x \geq 0$, and similarly

$$K_1^-(x) = -CM^Y \Gamma(-Y, -Mx)$$

$$K_2^-(x) = C \left(M^Y \Gamma(-Y, -Mx) - e^x (M-1)^Y \Gamma(-Y, -(M-1)x) \right)$$

$$\begin{aligned} \tilde{K}_3^-(x) = C & \left((G-1)^Y (e^x \Gamma(-Y, -(G-1)x) - \Gamma(-Y)) + \right. \\ & \left. + G^{Y-1} (G\gamma(-Y, -Gx) - Gx\Gamma(-Y, -Gx) + \gamma(1-Y, -Gx)) \right) \end{aligned}$$

$$\begin{aligned} \tilde{K}_4^-(x) = C & \left((G-1)^Y ((1+x)\Gamma(-Y) - e^x \Gamma(-Y, -(G-1)x)) + \right. \\ & + G^Y (-x-1)\gamma(-Y, -Gx) - G^{Y-1} (x+1)\gamma(1-Y, -Gx) + \\ & \left. + G^{Y-2} \left(\frac{G^2 x^2}{2} \Gamma(-Y, -Gx) - \frac{1}{2} \gamma(2-Y, -Gx) \right) \right) \end{aligned}$$

for $x \leq 0$. Here $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} \frac{e^{-t}}{t^{1-\alpha}} dt$$

Here $\Gamma(\alpha, x)$ and $\gamma(\alpha, x)$ are the incomplete gamma functions given by

$$\Gamma(\alpha, x) = \int_x^{\infty} \frac{e^{-t}}{t^{1-\alpha}} dt$$

and

$$\gamma(\alpha, x) = \int_0^x \frac{e^{-t}}{t^{1-\alpha}} dt$$

We would like to mention that Matlab does not support the incomplete gamma function for the case when $\alpha < 0$. In this situation the following recursive relations need to be used to compute the incomplete gamma functions

$$\Gamma(\alpha, x) = \frac{\Gamma(\alpha + 1, x) - x^\alpha e^{-x}}{\alpha}$$

and

$$\gamma(\alpha, x) = \frac{\gamma(\alpha + 1, x) + x^\alpha e^{-x}}{\alpha}$$

Also to get the values of the gamma function in Matlab for $\alpha < 0$ we can use the following recursive relation

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + 1)}{\alpha}$$

Finally with (7.35), (7.37), (7.56) we get our expression for the load vector (7.25)

$$\vec{f} = \vec{f}^0 + \vec{f}^1 + \vec{f}^2$$

In the Black Scholes model (pure diffusion process), since we don't have any jumps, the integral operator corresponding to the jump measure, \mathcal{A}_2 , disappears leaving us

with the following expression of the load vector

$$\vec{f} = \vec{f}^0 + \vec{f}^1$$

When we have the pure jump CGMY process the diffusion parameter is $\sigma = 0$, reducing the expression of the load vector to

$$\vec{f} = \vec{f}^1 + \vec{f}^2$$

In the jump diffusion case, we have both, the jump measure and the diffusion parameter non-zero and therefore the expression for the load vector will contain all the corresponding vectors, $\vec{f}^0, \vec{f}^1, \vec{f}^2$

$$\vec{f} = \vec{f}^0 + \vec{f}^1 + \vec{f}^2$$

7.1.3 Solving the Linear Complementarity Problem for Game Options

We have to solve the following Linear Complementarity problem, which in the matrix form can be expressed as follows find $\vec{x} \in \mathcal{K}_N$ such that

$$M\vec{x} \geq \vec{b}$$

$$0 \leq \vec{x} \leq \vec{c}$$

with the complementarity condition

$$(M\vec{x} - \vec{b})(\vec{c} - \vec{x})\vec{x} = 0$$

where the expressions for M, \vec{b}, \vec{c} are given by (6.24) for the finite maturity case. In the perpetual case M, \vec{b}, \vec{c} are defined in (4.14). The matrix M is of the form $M = A$

in time independent case and $M = B + dt\theta A$ in the time dependent, where A is the stiffness matrix (4.14) and B is the mass matrix (5.6). From the expression of M we can see that it is positive definite. This implies existence and uniqueness of solutions [16].

We will show a proof of uniqueness and existence based on the fixed point iteration introduced in section 4.2. The energy space considered here is \mathbb{R}^N with the standard dot product

$$\langle u, v \rangle_{\mathcal{V}} = \langle u, v \rangle_q = q(u, v) = \langle u, v \rangle_{\mathbb{R}^N} = \vec{u} \cdot \vec{v}$$

The operator $Q : \mathcal{V} \rightarrow \mathcal{V}^*$ associated with the symmetric bilinear $q(u, v)$ satisfies

$$\langle u, v \rangle_q = q(u, v) = \langle Qu, v \rangle_q$$

Hence $Q = Q^{-1} = I$, where I is the N by N identity matrix. The purpose of the following fixed point iteration method for solving the LCP is a constructive proof of existence and uniqueness and to show an implementation method that we can also use to get an estimate on the necessary work to achieve a desired accuracy and the additional work required to improve the accuracy.

$\mathcal{K}_N \in \mathbb{R}_N$ is defined as

$$\mathcal{K}_N = \left\{ \vec{x} \mid \vec{x} \in \mathbb{R}_N, \quad 0 \leq \vec{x} \leq \vec{c} \right\}$$

Define the projection $P_{\mathcal{K}_N} : \mathbb{R}_N \rightarrow \mathcal{K}_N$

$$\vec{y} = P_{\mathcal{K}_N}(\vec{x}) \quad \text{s.t.} \quad \vec{y}_i = \begin{cases} 0 & \text{if } \vec{x}_i \leq 0 \\ \vec{x}_i & \text{if } 0 < \vec{x}_i < \vec{c}_i \\ \vec{c}_i & \text{if } \vec{x}_i \geq \vec{c}_i \end{cases}$$

Note that the projection $P_{\mathcal{K}_N}$ is a non-expanding with respect to the Euclidean norm

$$\|\cdot\|$$

$$\|P_{\mathcal{K}_N}\vec{x}_1 - P_{\mathcal{K}_N}\vec{x}_2\| \leq \|\vec{x}_1 - \vec{x}_2\|$$

The linear complementarity problem above can be solved numerically using fixed point iteration, also called Projected Richardson iteration

$$\vec{x}_{n+1} = \mathcal{S}_{\mathcal{K}_N}(\vec{x}_n) \quad n > 0, \quad x_0 \in \mathbb{R}_N$$

where $\mathcal{S}_{\mathcal{K}_N}$ is the operator associated with the fixed point problem defined as

$$\mathcal{S}_{\mathcal{K}_N}(\vec{x}) = \mathcal{P}_{\mathcal{K}_N}(\vec{x} + \rho(\vec{b} - A\vec{x})) \quad \rho > 0$$

Let μ be defined in terms of the smallest eigenvalue of $M + M^\top$

$$\mu = \lambda_{\min} \left[\frac{1}{2}(M + M^\top) \right]$$

Therefore we have that $\langle (M + M^\top)\vec{x}, \vec{x} \rangle \geq 2\mu\|\vec{x}\|^2$ and we also consider $C = \|M\|$

hence we have $\|M\vec{x}\| \leq C\|\vec{x}\|$.

Using these we will show that $\mathcal{S}_{\mathcal{K}_N}$ is a contracting map

$$\begin{aligned} \|\mathcal{S}_{\mathcal{K}_N}\vec{x} - \mathcal{S}_{\mathcal{K}_N}\vec{y}\|^2 &= \|P_{\mathcal{K}_N}(\vec{x} + \rho(\vec{b} - A\vec{x})) - P_{\mathcal{K}_N}(\vec{y} + \rho(\vec{b} - A\vec{y}))\|^2 \\ &\leq \|(\vec{x} - \vec{y}) - \rho M(\vec{x} - \vec{y})\|^2 \\ &\leq \|\vec{w} - \rho M\vec{w}\|^2 \\ &\leq \langle \vec{w} - \rho M\vec{w}, \vec{w} - \rho M\vec{w} \rangle \\ &\leq \langle \vec{w}, \vec{w} \rangle - \rho \langle (M + M^\top)\vec{w}, \vec{w} \rangle + \langle M\vec{w}, M\vec{w} \rangle \\ &\leq \|\vec{w}\|^2 - 2\rho\mu\|\vec{w}\|^2 + \rho^2 C^2 \|\vec{w}\|^2 \\ &\leq q^2 \|\vec{x} - \vec{y}\|^2 \end{aligned}$$

where $q^2 = 1 - 2\rho\mu + \rho^2 C^2$.

If $0 < \rho < \frac{2\mu}{C^2}$ then $q < 1$ and therefore M is a contracting map. The optimal choice for ρ is $\rho = \frac{\mu}{C^2}$ for which

$$q = \sqrt{1 - \frac{\mu^2}{C^2}} \quad (7.57)$$

From the expression of the mass matrix B , (7.1), and the definition of M we can get an estimate for μ if we have a uniform mesh with step $h = \Delta x$, which gives us $\mu = \frac{C_2 h}{3}$, where C_2 is the coercivity constant (4.4).

For the estimate of C in terms of h we get $C = C_1 C' h^{-1}$, where C_1 is the continuity constant (4.3), when have pure diffusion and jump-diffusion, using the inverse inequality property $\|u\|_{\mathcal{V}} = \|u\|_{\tilde{\mathcal{H}}^1} \leq C' h^{-1} \|u\|_{L_2} \leq C'' h^{-\frac{1}{2}} \|\vec{u}\|$. In the case of the pure jump CGMY we have $C = C_1 C' h^{1-Y}$ using the inverse inequality property $\|u\|_{\mathcal{V}} = \|u\|_{\tilde{\mathcal{H}}^{\frac{Y}{2}}} \leq C' h^{\frac{-Y}{2}} \|u\|_{L_2} \leq C'' h^{\frac{1-Y}{2}} \|\vec{u}\|$.

The expression for q in the equation (7.57) in the case of pure diffusion and jump diffusion becomes

$$q = \sqrt{1 - ch^4} \quad (7.58)$$

For the case of pure jump we have

$$q = \sqrt{1 - ch^{2+Y}} \quad (7.59)$$

Theorem 7.1. *For $\forall \vec{x}_0 \in \mathbb{R}_N$ there exists a unique solution \vec{x}^* of the LCP given by the projected Richardson iteration which converges to \vec{x}^* i.e. $\vec{x}_n \rightarrow \vec{x}^*$ as $n \rightarrow \infty$. Moreover, if we let \vec{e}^n be the error term associated with the n th iteration, $\vec{e}^n = \vec{x}^* - \vec{x}_n$, then the following bounds on the Euclidean norm of error term \vec{e}^n hold*

a priori error estimate

$$\|\vec{e}_n\| \leq \frac{q^n}{1-q} \|\vec{x}_1 - \vec{x}_0\|$$

a posteriori error estimate

$$\|\vec{e}_n\| \leq \frac{q}{1-q} \|\vec{x}_n - \vec{x}_{n-1}\|$$

Proof. The existence and uniqueness of the solution are given by the Contracting Map Theorem since $\mathcal{S}_{\mathcal{K}_N}$ is a contracting map when $0 < \rho < \frac{2\mu}{C^2}$ (7.57) and therefore in this case the projected Richardson algorithm will converge.

The posteriori error estimate

$$\begin{aligned} \|\vec{e}_n\| &= \|\vec{x}^* - \vec{x}_n\| = \|\vec{x}^* - \vec{x}_{n+1} + \vec{x}_{n+1} - \vec{x}_n\| \\ &\leq \|\vec{x}^* - \vec{x}_{n+1}\| + \|\vec{x}_{n+1} - \vec{x}_n\| \\ &\leq \|\mathcal{S}_{\mathcal{K}_N} \vec{x}^* - \mathcal{S}_{\mathcal{K}_N} \vec{x}_n\| + \|\mathcal{S}_{\mathcal{K}_N} \vec{x}_n - \mathcal{S}_{\mathcal{K}_N} \vec{x}_{n-1}\| \\ &\leq q \|\vec{x}^* - \vec{x}_n\| + q \|\vec{x}_n - \vec{x}_{n-1}\| \\ &\leq q \|\vec{e}_n\| + q \|\vec{x}_n - \vec{x}_{n-1}\| \end{aligned}$$

The priori error estimate

$$\|\vec{x}_n - \vec{x}_{n-1}\| = \|\mathcal{S}_{\mathcal{K}_N} \vec{x}_{n-1} - \mathcal{S}_{\mathcal{K}_N} \vec{x}_{n-2}\| \leq q \|\vec{x}_{n-1} - \vec{x}_{n-2}\| \leq q^{n-1} \|\vec{x}_1 - \vec{x}_0\|$$

Using the previous expression we the desired result

$$\|\vec{e}_n\| \leq \frac{q}{1-q} \|\vec{x}_n - \vec{x}_{n-1}\| \leq \frac{q^n}{1-q} \|\vec{x}_1 - \vec{x}_0\|$$

□

If we denote the residual associated with the nth iteration by $\vec{r}_n = \vec{b} - A\vec{x}_{n-1}$ then we can control the error of the iteration just by checking the norm of the residual. This means that one way to improve our algorithm is a good initial vector \vec{x}_0 .

In the case of the time dependent game option, when we have a sequence of LCPs, the best choice for \vec{x}_0 is the solution from the previous time step.

To estimate the amount of work we use the estimates for $q = 1 - ch^\alpha$ from the expressions (7.58) and (7.59). For the pure diffusion and jump diffusion we have $\alpha = 4$ and for the pure jump we have $\alpha = 2 + Y$. If the desired accuracy after n steps is $\|\vec{e}_n\| = O(h^b)$ then the number of steps is $n = O(h^{-\alpha} \log h)$. This is a somewhat pessimistic estimate and in practice the algorithm shows significantly better performance.

For numerical computations we have also used the PATH solver to solve LCP problem which uses a Newton type method. For more details on the PATH see [22]. Another numerical solver based on the Newton method that we have used to compare our results is Fischer's LCP solver [23].

The above 2 LCP solvers outperformed the Projected Richardson method which was the slowest even for some more relaxed restrictions on the residual used to terminate the iteration. One way to speed up the LCP solver is to relax this termination condition on the residual and make it comparable to errors induced by other sources such as discretization. While a fast LCP solver is crucial to solve our problem, the focus of this paper was to analyze convergence rates of the finite element method for values of game options. We are aware of existing methods that could be used to improve the efficiency of the LCP solvers and we would like to mention that one could employ faster LCP solvers since this was one of the limitations when using this framework.

7.2 Numerical Results for δ -penalty Game Options

7.2.1 Perpetual δ -penalty Game Put Value Functions

One important contribution of the time independent problems is qualitative understanding of the finite horizon problems as expiration time $T \rightarrow \infty$. If there exist a solution to the perpetual problem then the time dependent solutions will converge to it as $T \rightarrow \infty$.

The space domain of the time independent game options will be divided into three regions

1. Exercise region where the buyer exercises to receive the exercise payoff F , and the following holds

$$V(x) = F(x) \quad \text{and} \quad \mathcal{A}V(x) \leq 0$$

2. Cancellation region where the seller cancels by paying the cancellation payoff G , and the following holds

$$V(x) = G(x) \quad \text{and} \quad \mathcal{A}V(x) \leq 0$$

3. Hold region where both buyer and seller wait, and the following holds

$$F(x) < V(x) < G(x) \quad \text{and} \quad \mathcal{A}V(x) = 0$$

For the pure diffusion case the cancellation region is only a single contact point at $S = K$.

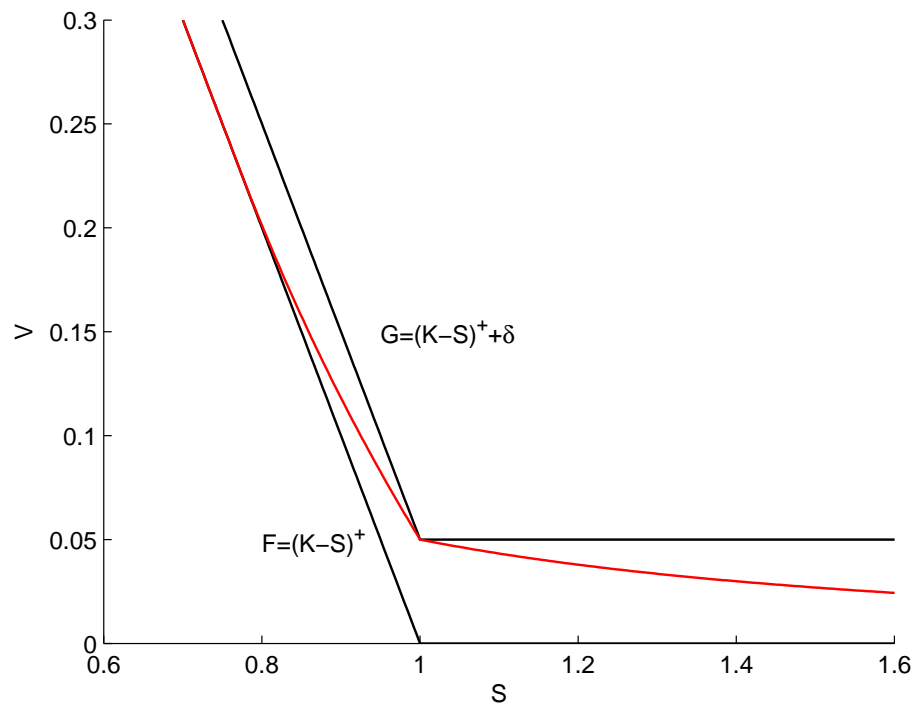


Figure 7.1: Perpetual δ -penalty Game Put Value Function for pure a diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$

In the presence of jumps, which is the case of pure jump CGMY processes and CGMY jump diffusion processes, the contact region is no longer a single point.

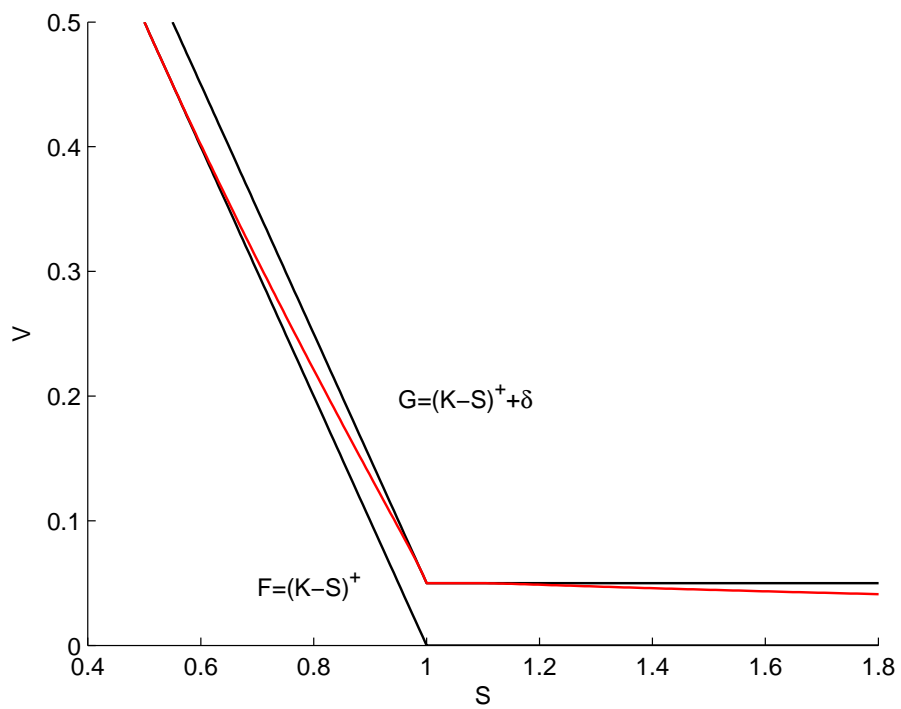


Figure 7.2: Perpetual δ -penalty Game Put Value Function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$

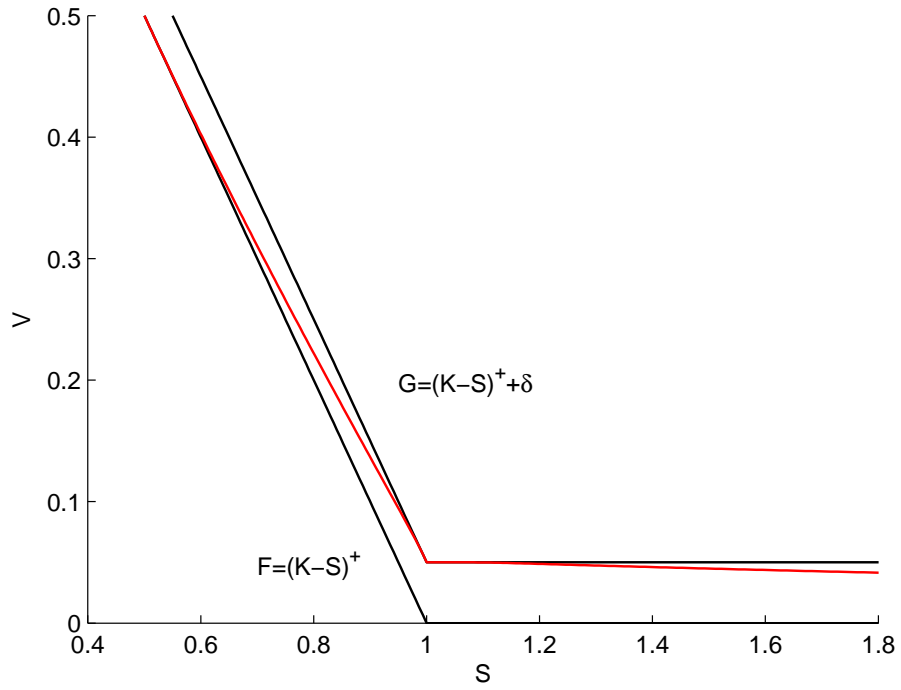


Figure 7.3: Perpetual δ -penalty Game Put Value Function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$

7.2.2 Perpetual δ -penalty Game Put Error Convergence

The convergence rate of the error will be shown with respect to the energy norm $\|\cdot\|_E$ and the L_2 norm $\|\cdot\|_{L_2}$.

The energy norm will be computed using the stiffness matrix A computed in the previous section. For a column vector x we have

$$\|x\|_E^2 = x^\top Ax$$

The L_2 norm will be computed using the mass matrix M computed in the previous section. For a column vector x we have

$$\|x\|_{L_2}^2 = x^\top Mx$$

The number of space points in each uniform grid used to computed each value v_i will be denoted by n_i . The difference d_i will be computed using an overkill solution v_N where $N = 2^{12}$ for all the computations and the missing values of v_i

$$d_i = v_i - v_N$$

The barrier value used in the numerical computation is $R = 2$.

n_i	2^4	2^5	2^6	2^7	2^8	2^9
$e_i = \ d_i\ _E$	0.01039	0.00555	0.00285	0.00144	0.00072	0.00036
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.9047	0.9623	0.9832	0.9926	1.0087

Table 7.1: Perpetual δ -penalty Game Put energy norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$

n_i	2^4	2^5	2^6	2^7	2^8	2^9
$e_i = \ d_i\ _{L_2}$	0.00194	0.00085	0.00036	0.00015	0.00006	0.00002
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.1933	1.2328	1.2702	1.2953	1.3087

Table 7.2: Perpetual δ -penalty Game Put L_2 norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$

n_i	2^4	2^5	2^6	2^7	2^8	2^9
$e_i = \ d_i\ _E$	0.03079	0.01629	0.00842	0.00425	0.00204	0.00101
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.9182	0.9514	0.9875	1.0568	1.0087

Table 7.3: Perpetual δ -penalty Game Put energy norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$

n_i	2^4	2^5	2^6	2^7	2^8	2^9
$e_i = \ d_i\ _{L_2}$	0.00452	0.00204	0.00088	0.00037	0.00015	0.00006
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.1480	1.2014	1.2689	1.2854	1.2917

Table 7.4: Perpetual δ -penalty Game Put L_2 norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$

If we denote the space discretization interval by $\Delta x = cn^{-1}$, we can see from the results above that in all cases the energy norm of the error term $\|e_n\|_E \approx O(\Delta x)$. In the case of the pure CGMY this is higher than the theoretical error estimate in chapter 6. One of the reasons for the better error convergence rate is that the theoretically dominant term is decreased by an exponential factor with respect to the barrier R . Therefore the first order error term becomes dominant.

The L_2 norm of the error term $\|e_n\|_{L_2} \approx O(\Delta x^{1.3})$. Theoretical results are available only for elliptic equations. The theoretical convergence rate for the European options with respect to the L_2 norm is $O(h^2)$. We can only expect the rates for the elliptic inequality case not to exceed the equality case.

n_i	2^4	2^5	2^6	2^7	2^8	2^9
$e_i = \ d_i\ _E$	0.04142	0.02181	0.0114	0.00587	0.00295	0.00143
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.9248	0.9266	0.9671	0.99221	1.0353

Table 7.5: Perpetual δ -penalty Game Put energy norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$

n_i	2^4	2^5	2^6	2^7	2^8	2^9
$e_i = \ d_i\ _{L_2}$	0.00464	0.00210	0.00093	0.00039	0.00016	0.00006
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.1444	1.1736	1.2362	1.2769	1.2861

Table 7.6: Perpetual δ -penalty Game Put L_2 norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$

7.2.3 Perpetual δ -penalty Game Call Value Functions

As opposed to their American counterparts, the perpetual game put options have solutions due to the presence of the upper obstacle. We can see that even in the pure diffusion case the cancelation region is no longer a single point as we have seen in the case of the perpetual game put options but consists of the entire upper obstacle for stock price larger than the strike. Below the strike the contact is still reduced to one single point at K .

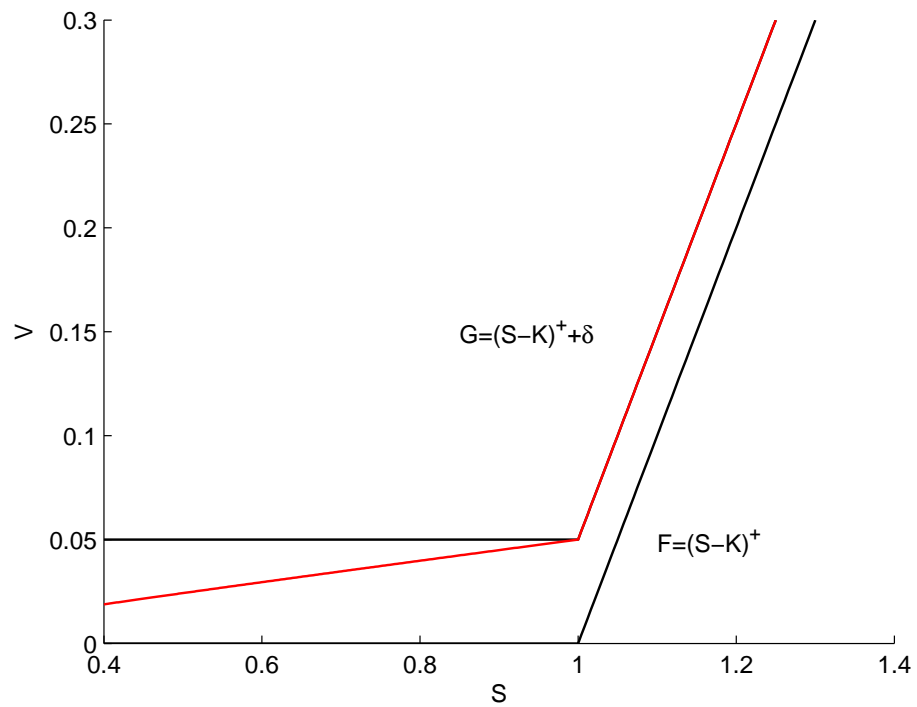


Figure 7.4: Perpetual δ -penalty Game Call Value Function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$

When we have pure jump or jump diffusion processes we can see the cancellation region extending to a small interval to the left of K due to the presence of jumps.

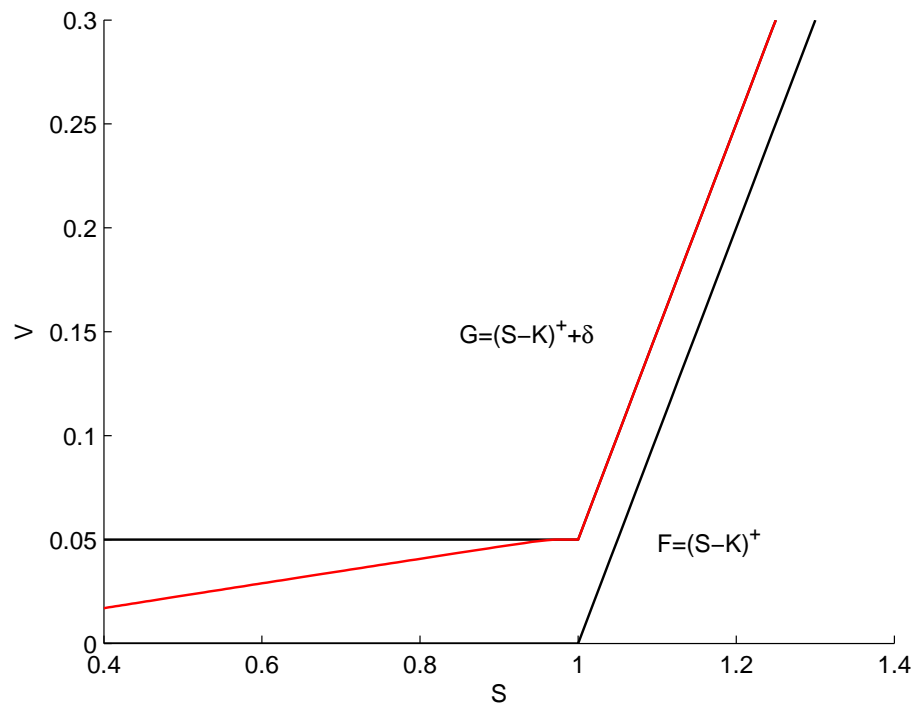


Figure 7.5: Perpetual δ -penalty Game Call Value Function for pure a jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$

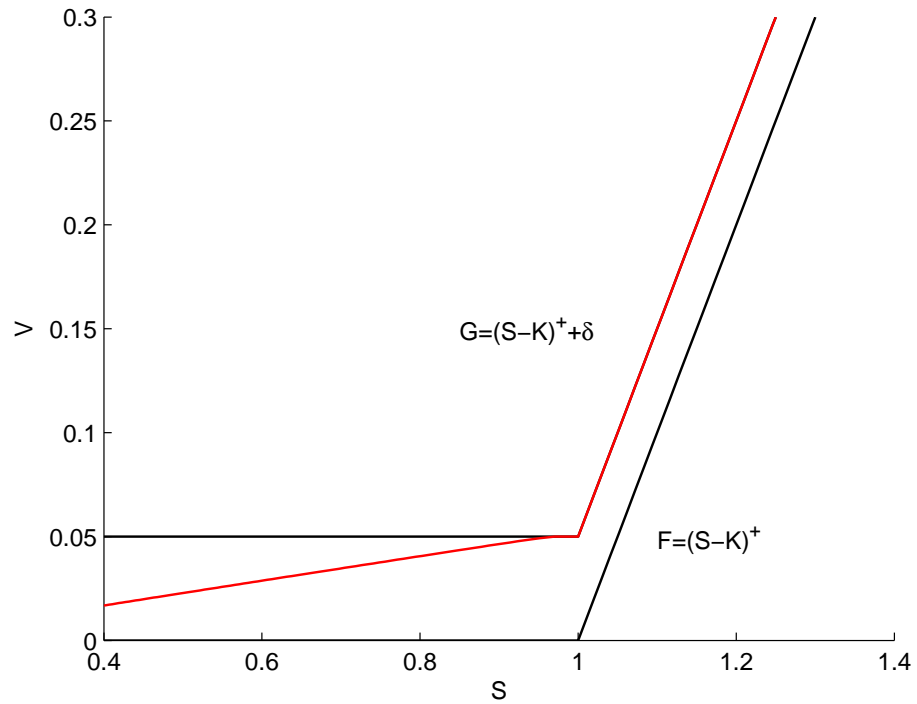


Figure 7.6: Perpetual δ -penalty Game Call Value Function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$

7.2.4 Finite Maturity δ -penalty Game Put Value Functions and Free Regions

The space-time domain of the game options will be divided into three regions

1. Exercise region where the buyer exercises to receive the exercise payoff F , and the following holds

$$V(x, t) = F(x, t) \quad \text{and} \quad V_t(x, t) + \mathcal{A}V(x, t) \leq 0$$

2. Cancellation region where the seller cancels by paying the cancellation payoff

G , and the following holds

$$V(x, t) = G(x, t) \quad \text{and} \quad V_t(x, t) + \mathcal{A}V(x, t) \leq 0$$

3. Hold region where both buyer and seller wait, and the following holds

$$F(x, t) < V(x, t) < G(x, t) \quad \text{and} \quad V_t(x, t) + \mathcal{A}V(x, t) = 0$$

For the pure diffusion case the cancellation region is only a single point, which allows it to be separated into an Exotic option and an option of Exotic type. This has been discussed in [43].

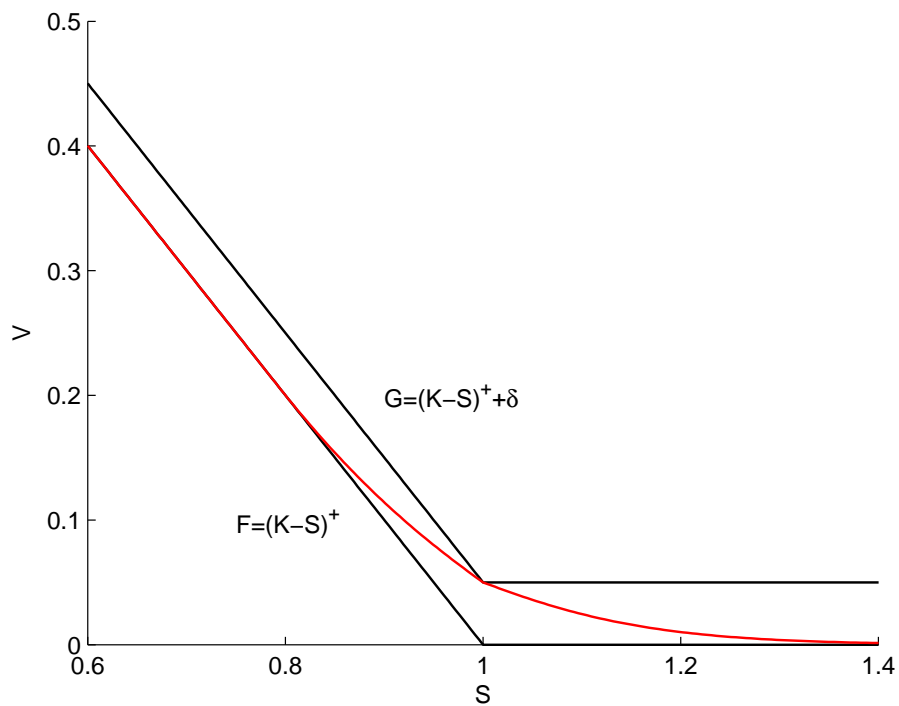


Figure 7.7: δ -penalty Game Put value function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

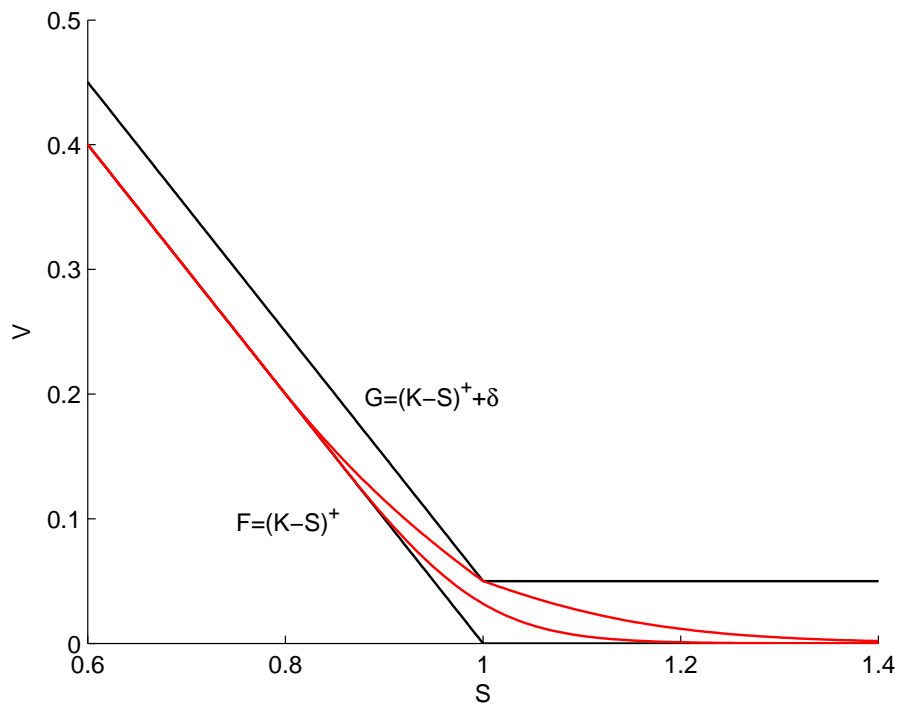


Figure 7.8: δ -penalty Game Put value functions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $t=0.9$ and $t=0.6$

The lower function represents δ -penalty Game Put value at $t = 0.9$ when the game option has not reached the cancelation region and therefore behaves like an American option. The upper function is δ -penalty Game Put value at $t = 0.6$ when the first contact is made with the upper obstacle which will lead to the cancelation region. The lower function can also be interpreted as the value of a δ -penalty Game Put with expiration time $T = 0.1$ and the upper function represents the value of a δ -penalty Game Put with expiration time $T = 0.4$

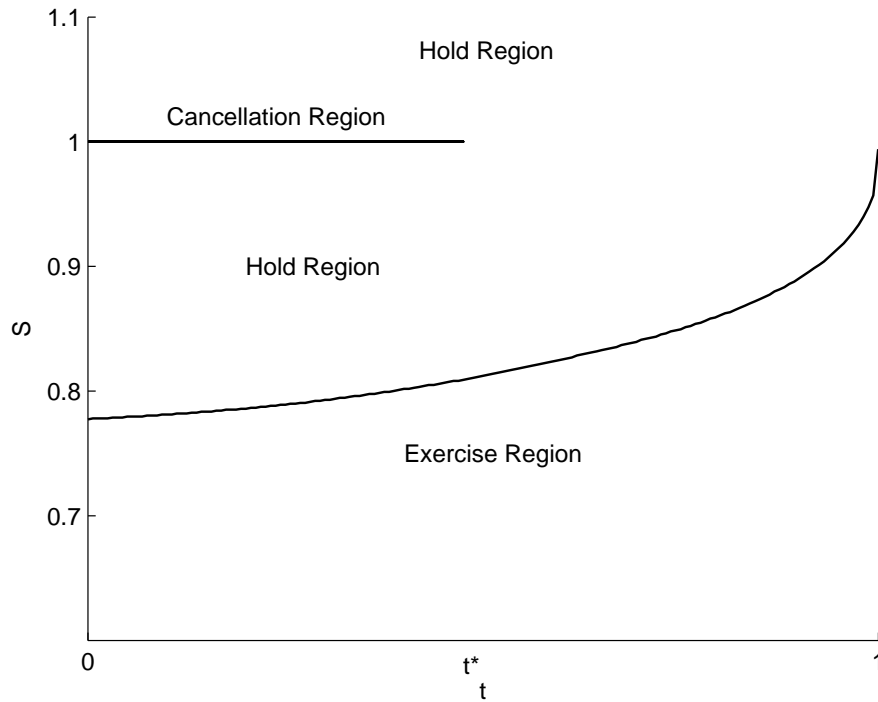


Figure 7.9: δ -penalty Game Put regions for a pure diffusion process with $\sigma=0.2$
 $K=1$ $\delta=0.05$ $T=1$

In the figure above, t^* represents the time such that the value of the corresponding American option with expiration time $T = t^*$ at $S = K$ is δ . Following t^* the behavior of the Game option is that of an American option, in the sense that there will be no cancellation.

We can see that the cancellation region is just a single line at $S = K$ for $t < T^*$. The holder's perspective on this is the following [43] if $S > K$, because of the positive interest rate $r > 0$, it is more advantageous to wait and pay the cancellation penalty value δ once the stock reaches the strike price K ; on the other hand if $S < K$ the discounted cancellation function stopped at K is a supermartingale and therefore

the holder will expect to pay the least when S reaches K .

In the presence of jumps, which is the case of pure jump CGMY process and CGMY jump diffusion process, the contact region is no longer a single point and therefore the option cannot be priced using American and Exotic type options.

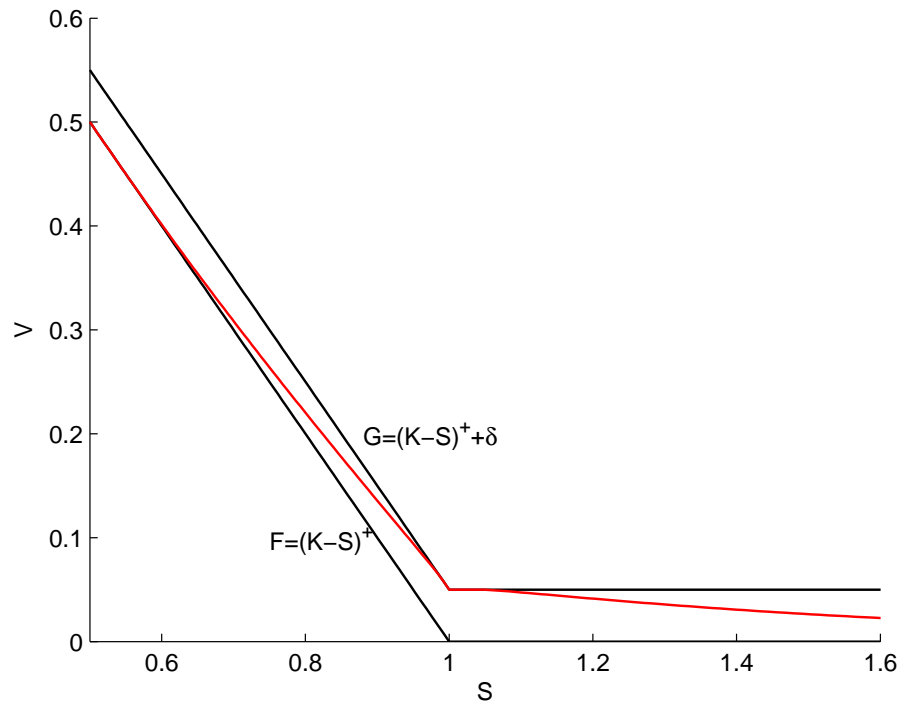


Figure 7.10: δ -penalty Game Put value function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

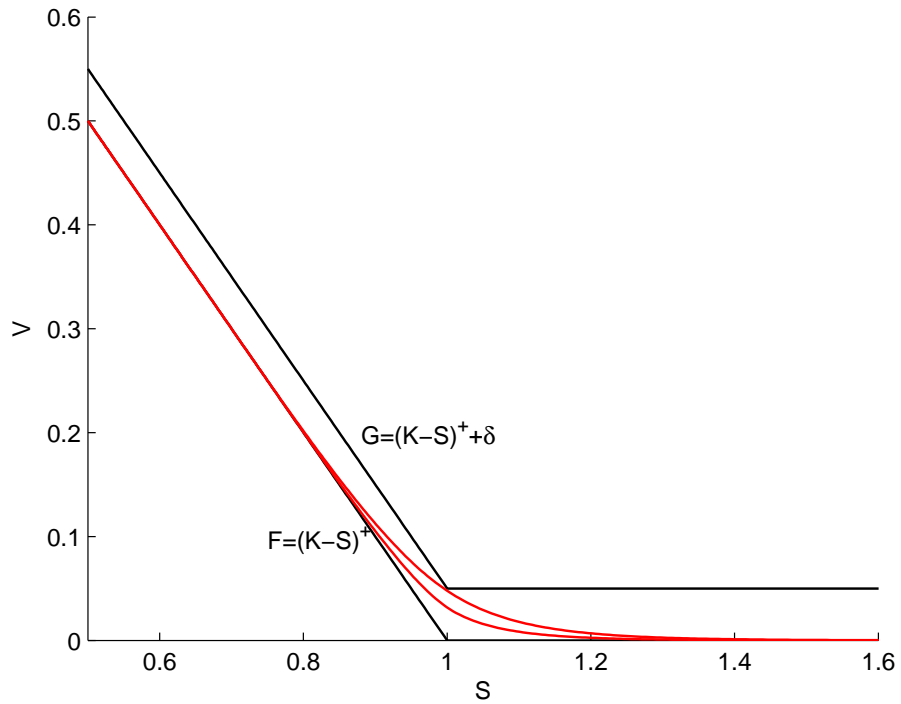


Figure 7.11: δ -penalty Game Put value functions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $t=.97$ and $t=.94$

The lower function represents δ -penalty Game Put value at $t = 0.97$ when the game option has not reached the cancelation region and therefore behaves like an American option. The upper function is δ -penalty Game Put value at $t = 0.94$ when the first contact is made with the upper obstacle which will lead to the cancelation region. The lower function can also be interpreted as the value of a δ -penalty Game Put with expiration time $T = 0.03$ and the upper function represents the value of a δ -penalty Game Put with expiration time $T = 0.06$

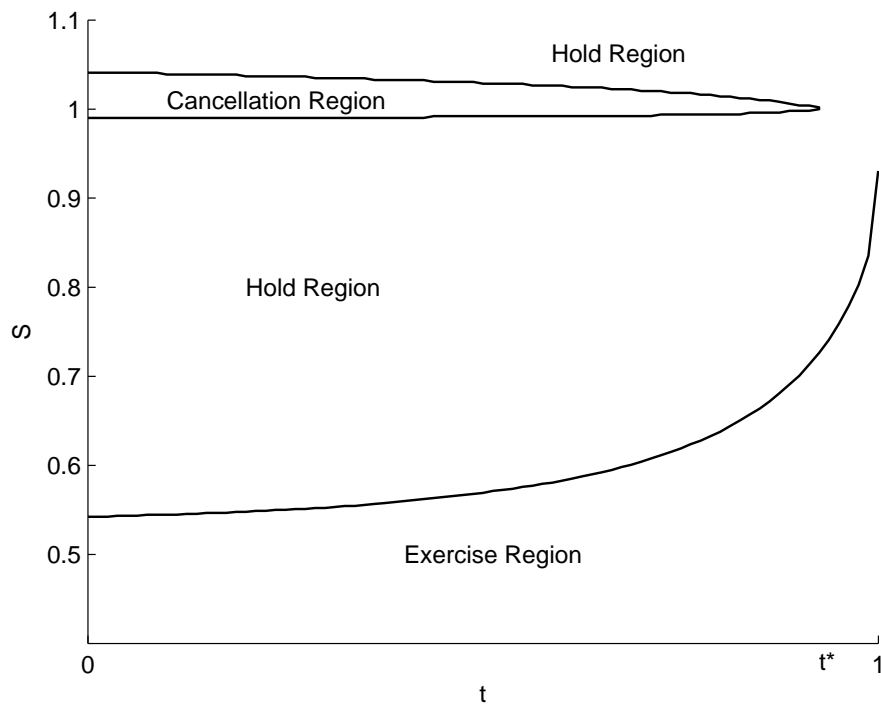


Figure 7.12: δ -penalty Game Put regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

We can see here that the cancellation region is no longer a single line. One explanation is that the holder can no longer wait until the stock price reaches K because of the presence of jumps. This means that there is a positive probability that the stock could jump over the the value K causing the holder to have to pay more and therefore it is optimal to cancel as soon as the stock gets a certain distance close K .

We will see the same results for the value function for the case when we have a jump-diffusion process and similar behavior for the exercise, hold and cancellation regions.

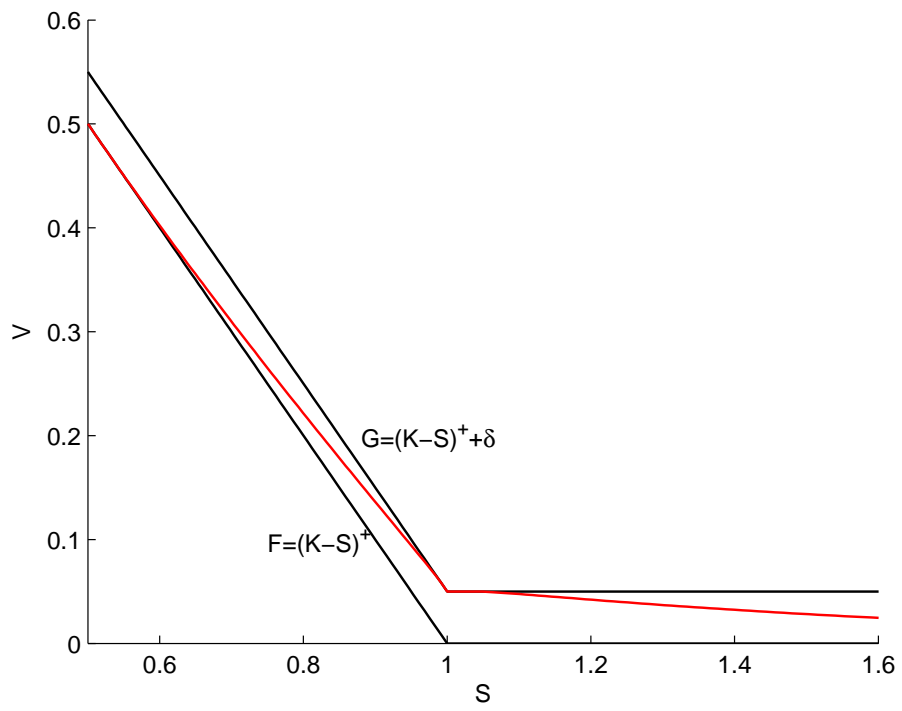


Figure 7.13: δ -penalty Game Put value function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

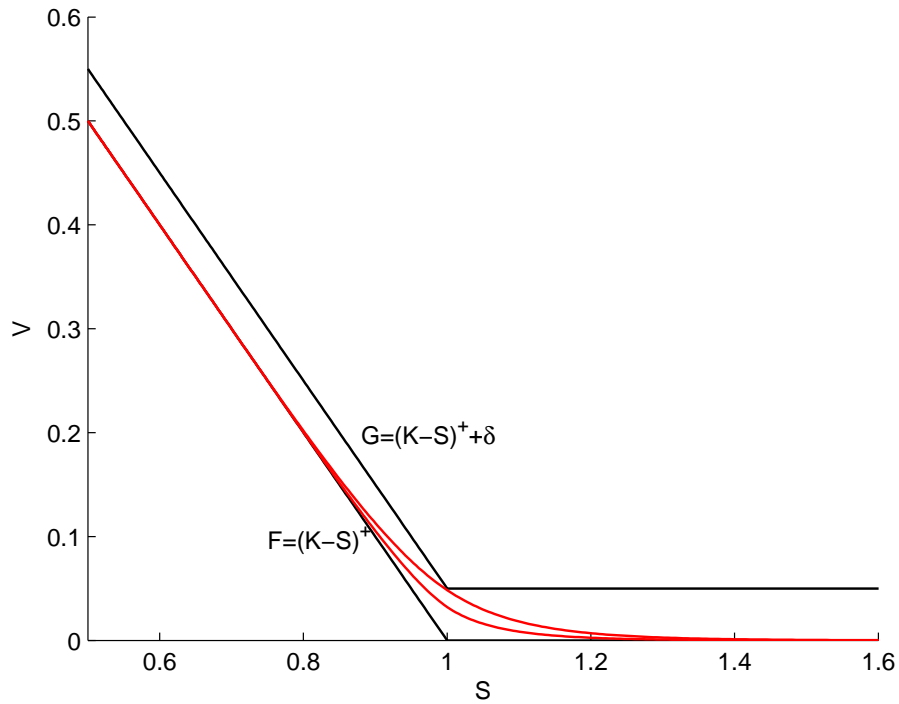


Figure 7.14: δ -penalty Game Put value functions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.975$ and $t=0.95$

The lower function represents δ -penalty Game Put value at $t = 0.975$ when the game option has not reached the cancellation region and therefore behaves like an American option. The upper function is δ -penalty Game Put value at $t = 0.95$ when the first contact is made with the upper obstacle which will lead to the cancellation region. The lower function can also be interpreted as the value of a δ -penalty Game Put with expiration time $T = 0.025$ and the upper function represents the value of a δ -penalty Game Put with expiration time $T = 0.05$

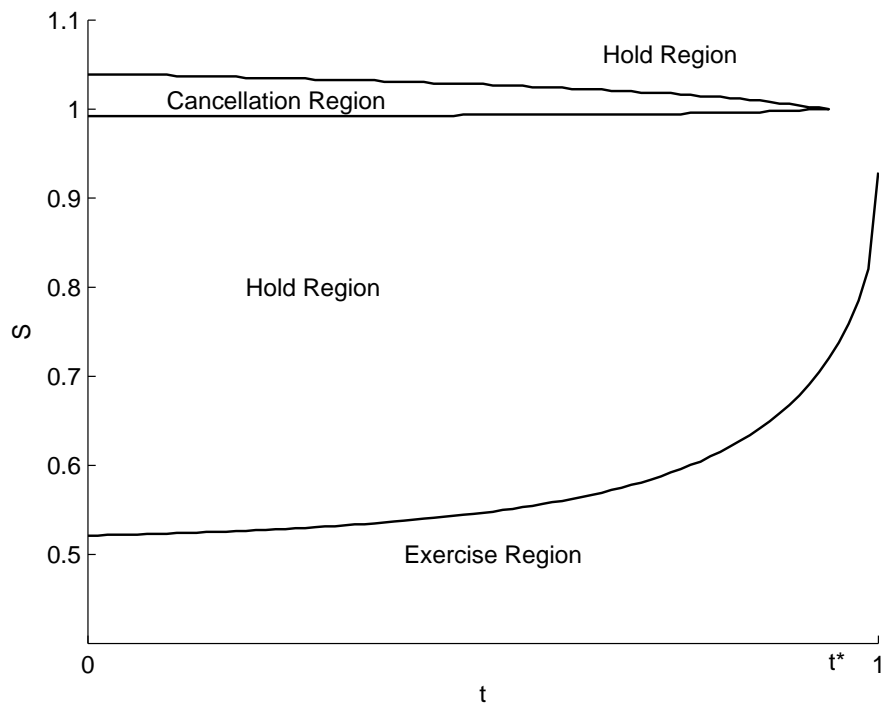


Figure 7.15: δ -penalty Game Put regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

A decrease in δ induces an increase in t^* as illustrated in Figures 7.9 and 7.16 below in the case of pure diffusion, where δ is decreased from 0.03 to 0.05. This means that the American behavior of the Game option is reduced since the value reaches the upper obstacle sooner. If δ is greater than or equal to the corresponding American option with the same expiration at $K = T$ than the Game option will become an American option and the seller will never cancel.

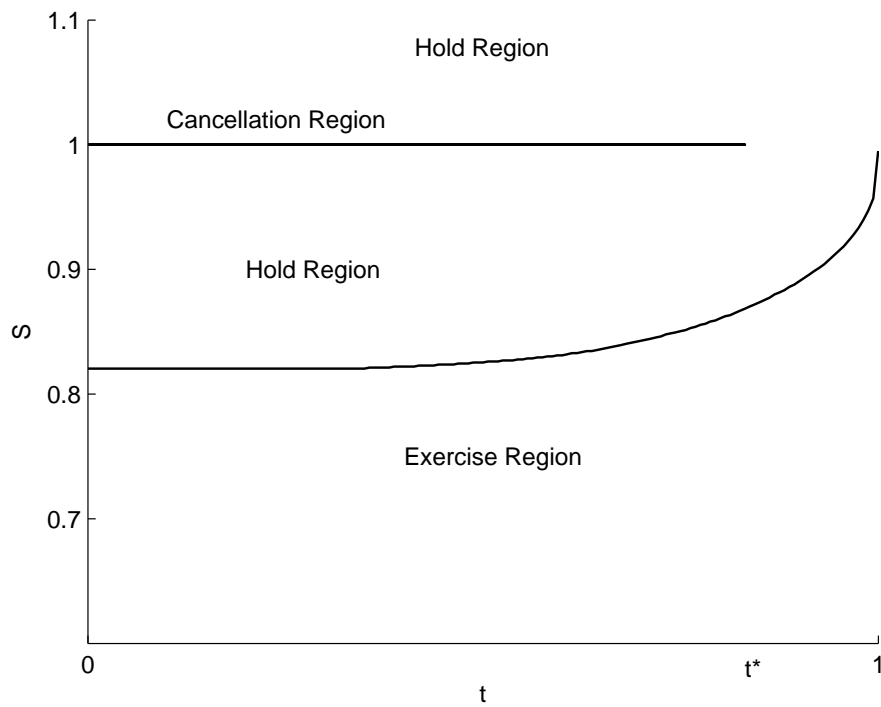


Figure 7.16: δ -penalty Game Put regions for a pure diffusion process with $\sigma=0.2$
 $K=1$ $\delta=0.03$ $T=1$

An increase in δ induces a decrease in t^* as illustrated in Figures 7.12 and 7.17 below in the case of pure jump and Figures 7.15 and 7.18 below for jump diffusion, where δ is increased from 0.05 to 0.1. This means that the Game option will behave like an American option longer since the value reaches the upper obstacle later.

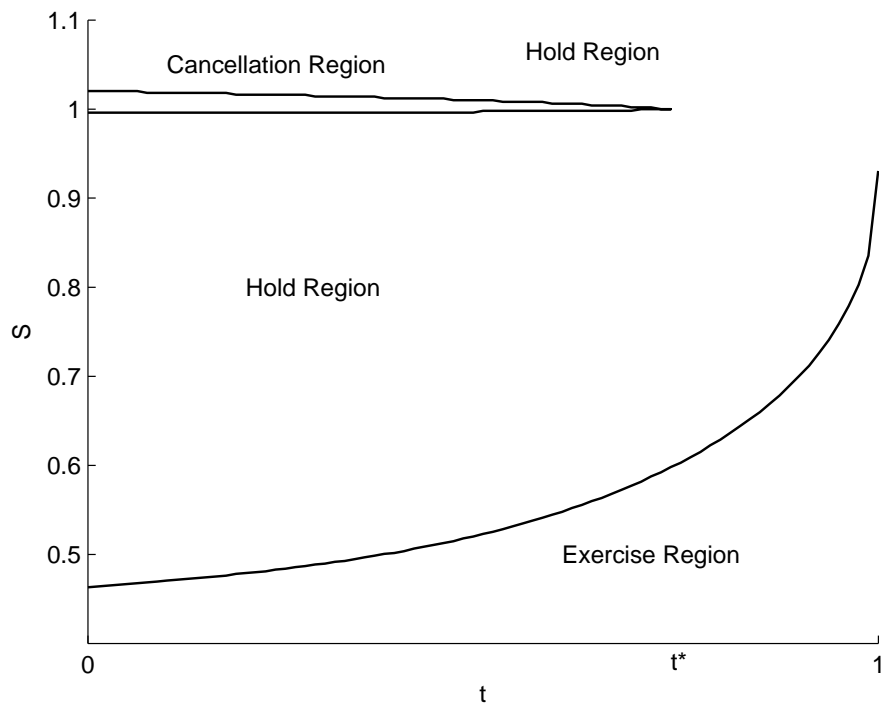


Figure 7.17: δ -penalty Game Put regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.1$ $T=1$

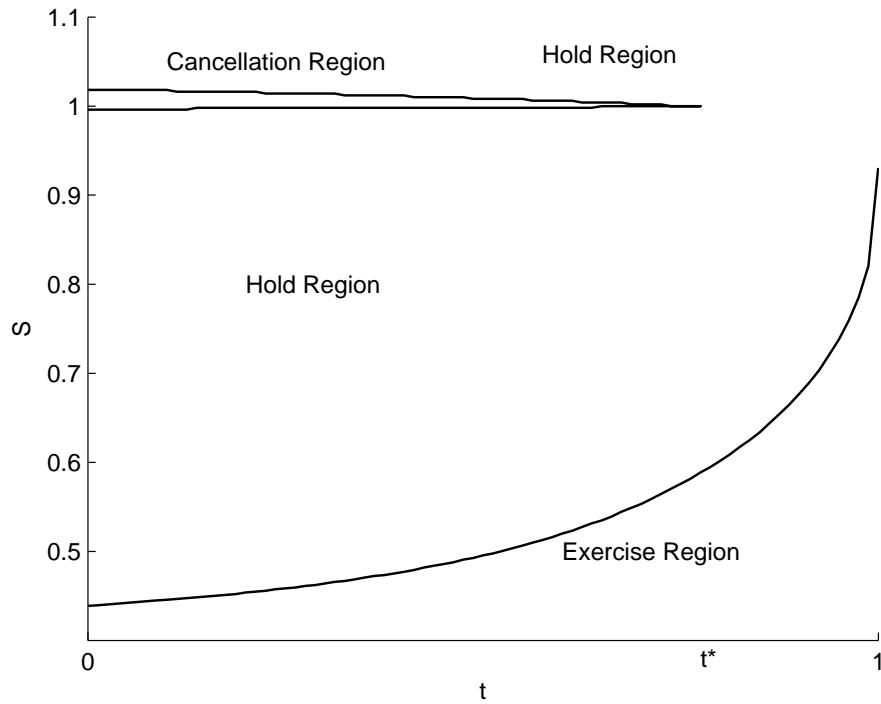


Figure 7.18: δ -penalty Game Put regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.1$ $T=1$

If δ is greater than or equal to the corresponding American option with the same expiration at $K = T$ than the Game option will become an American option and the seller will never cancel.

7.2.5 Finite Maturity δ -penalty Game Put Error Convergence Rates

For finite maturity game options, the convergence rate of the error will be shown with respect to the energy norm $\|\cdot\|_a$ and the L_2 norm $\|\cdot\|_{L_2}$.

The energy norm will be computed using the stiffness matrix A computed in the

previous section. For a column vector x we have

$$\|x\|_E^2 = x^\top Ax$$

The L_2 norm will be computed using the mass matrix M computed in the previous section. For a column vector x we have

$$\|x\|_{L_2}^2 = x^\top Mx$$

The value of the option at each time will be denoted by $v_{n_i, m_i}(x, t)$, where n_i is the number of space points in each uniform space grid used to compute the value at each time and m_i represents the number of time periods. The difference d_{n_i, m_i} will be computed at the final time T using an overkill solution $v_{N, M}$ where $N = 2^{10}$ and $M = 2^{10}$ for all the computations and the missing values of $v_{n_i, m_i}(x, T)$ interpolated

$$d_{n_i, m_i} = v_{n_i, m_i}(x, T) - v_{N, M}(x, T)$$

The barrier value used in the numerical computation is $R = 2$. Time stepping is performed using the Backward Euler method.

If we denote the space discretization interval by $\Delta x = n^{-1}$ and the time interval by $\Delta t = n^{-1}$, we can see from the results above that in all cases the energy norm of the error term $\|e_n\|_E \approx O(\Delta x) + O(\Delta t)$ while the L_2 norm of the error term $\|e_n\|_{L_2} \approx O(\Delta x^{1.2}) + O(\Delta t)$.

n_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{n_i, M}\ _E$	0.00718	0.00359	0.00179	0.00089	0.00043
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.0019	1.0036	1.0093	1.0354
m_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{N, m_i}\ _E$	0.00125	0.00063	0.00031	0.00014	0.00007
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.9938	1.0230	1.0564	1.0780

Table 7.7: δ -penalty Game Put energy norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

7.2.6 Finite Maturity δ -penalty Game Call Value Functions and Free Regions

The Game call option has no exercise region, which is a characteristic inherited from the corresponding American call option.

n_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{n_i, M}\ _{L_2}$	0.00386	0.00158	0.00064	0.00026	0.00011
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.2885	1.2967	1.3047	1.3051
m_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{N, m_i}\ _{L_2}$	0.00070	0.00036	0.00018	0.00008	0.00004
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.9539	1.0031	1.0683	1.0977

Table 7.8: δ -penalty Game Put L_2 norm error rates for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

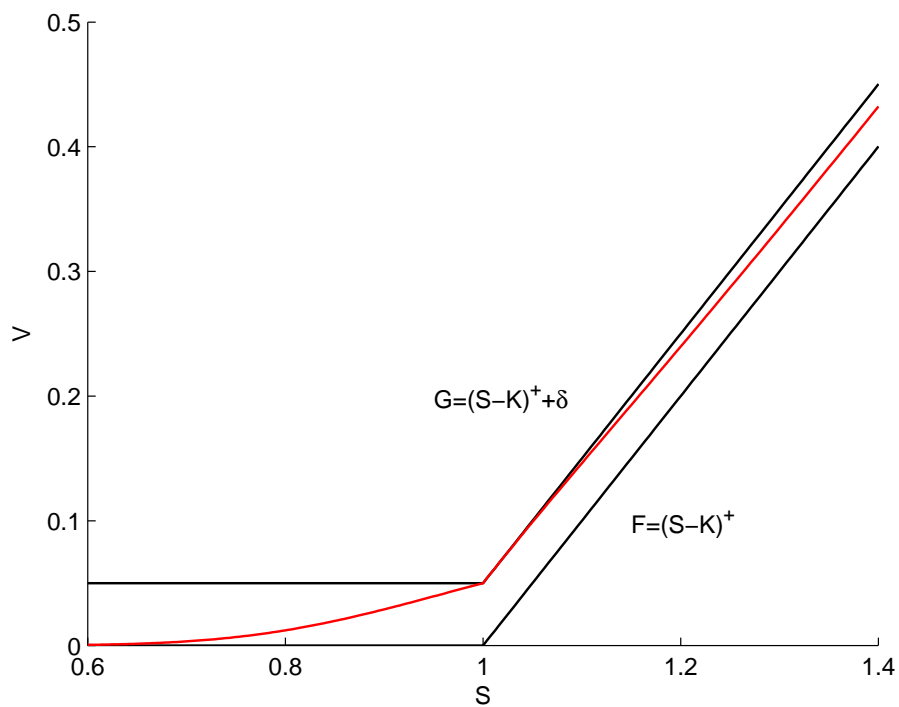


Figure 7.19: δ -penalty Game Call value function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

n_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{n_i, M}\ _E$	0.01808	0.00825	0.00363	0.00157	0.00067
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.1306	1.1869	1.2083	1.2234
m_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{N, m_i}\ _E$	0.00830	0.00403	0.00190	0.00087	0.00041
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.0424	1.0801	1.1075	1.1162

Table 7.9: δ -penalty Game Put energy norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

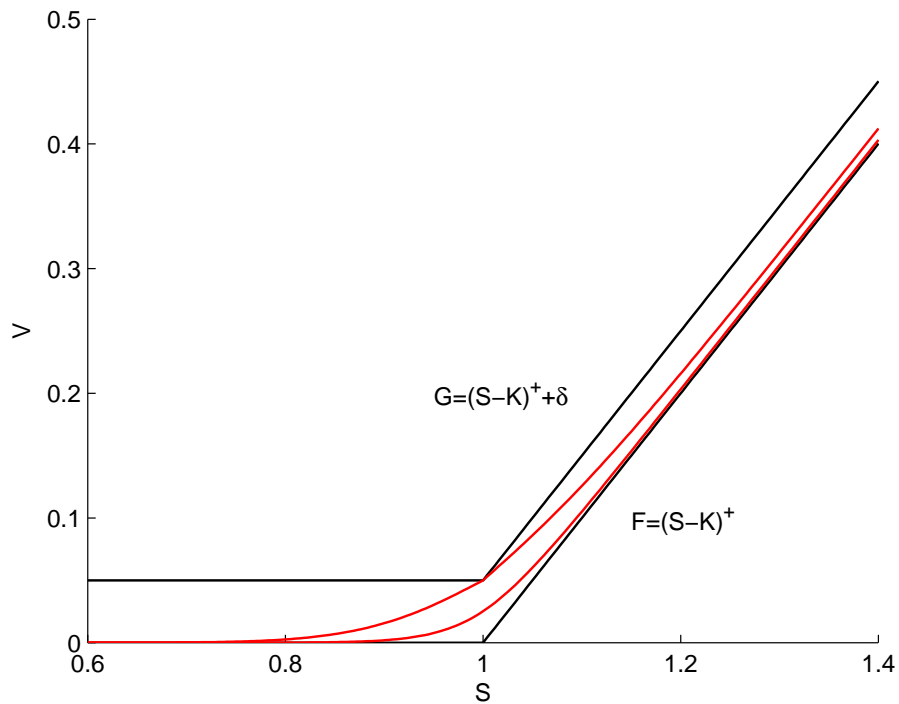


Figure 7.20: δ -penalty Game Call value functions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $t=0.95$ and $t=0.8$

n_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{n_i, M}\ _{L_2}$	0.00387	0.00172	0.00077	0.00034	0.00015
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.1691	1.1560	1.1983	1.1967
m_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{N, m_i}\ _{L_2}$	0.00271	0.00135	0.00061	0.00028	0.00013
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.0062	1.0412	1.0980	1.1083

Table 7.10: δ -penalty Game Put L_2 norm error rates for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

The lower function represents δ -penalty Game Call value at $t = 0.95$ when the game option has not reached the cancelation region and therefore behaves like an American option. The upper function is δ -penalty Game Call value at $t = 0.8$ when the first contact is made with the upper obstacle which will lead to the cancelation region. The lower function can also be interpreted as the value of a δ -penalty Game Call with expiration time $T = 0.05$ and the upper function represents the value of a δ -penalty Game Call with expiration time $T = 0.2$

n_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{n_i, M}\ _E$	0.01660	0.00862	0.00439	0.00223	0.00108
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.9451	0.9725	0.9847	1.0367
m_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{N, m_i}\ _E$	0.00872	0.00431	0.00208	0.00097	0.00046
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.0159	1.0501	1.0961	1.1055

Table 7.11: δ -penalty Game Put energy norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

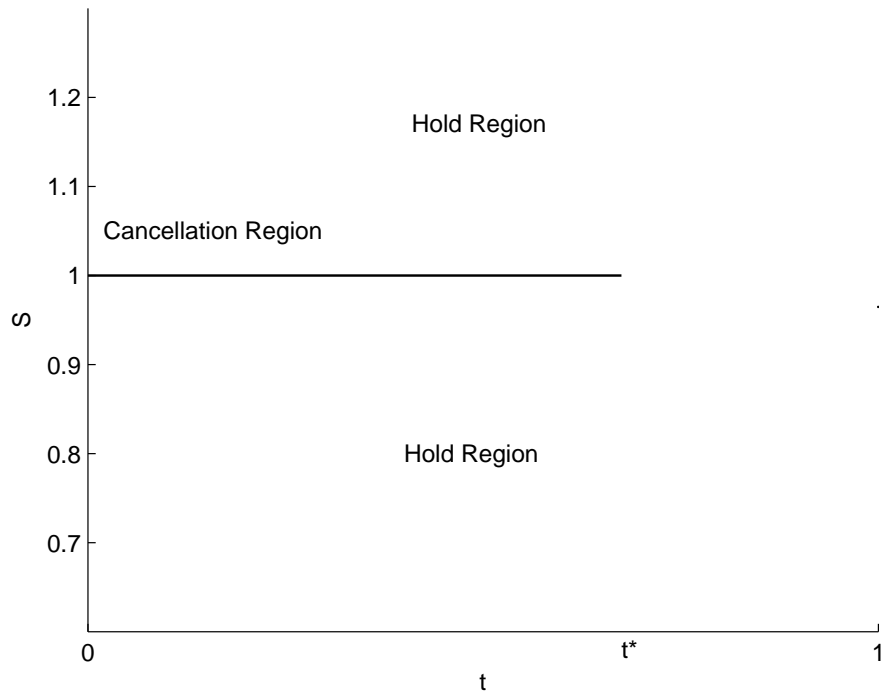


Figure 7.21: δ -penalty Game Call regions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

n_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{n_i, M}\ _{L_2}$	0.00389	0.00185	0.00086	0.00037	0.00016
$\log_2 \frac{e_i}{e_{i+1}}$	-	1.06541	1.0989	1.1994	1.2026
m_i	2^4	2^5	2^6	2^7	2^8
$e_i = \ d_{N, m_i}\ _{L_2}$	0.00292	0.00148	0.00070	0.00033	0.00015
$\log_2 \frac{e_i}{e_{i+1}}$	-	0.97873	1.0826	1.0886	1.0900

Table 7.12: δ -penalty Game Put L_2 norm error rates for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

When have have pure diffusion we observe that the cancelation region is again a line. This has been discussed in [42].

In the presence of jumps, i.e. the pure jump and the jump diffusion cases, the cancelation region is no longer a line, as illustrated in the figures below

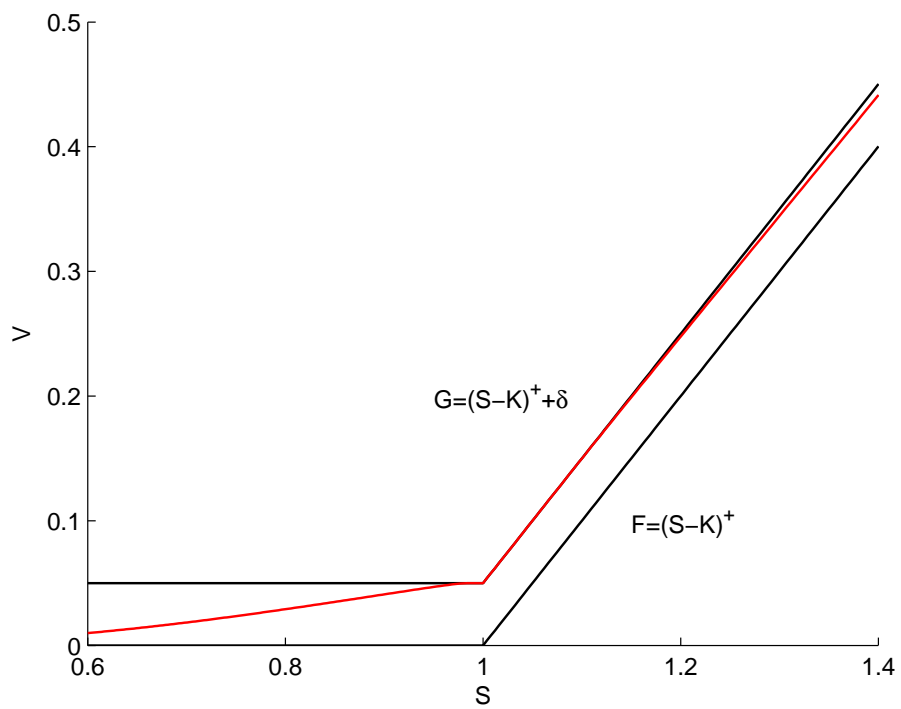


Figure 7.22: δ -penalty Game Call value function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

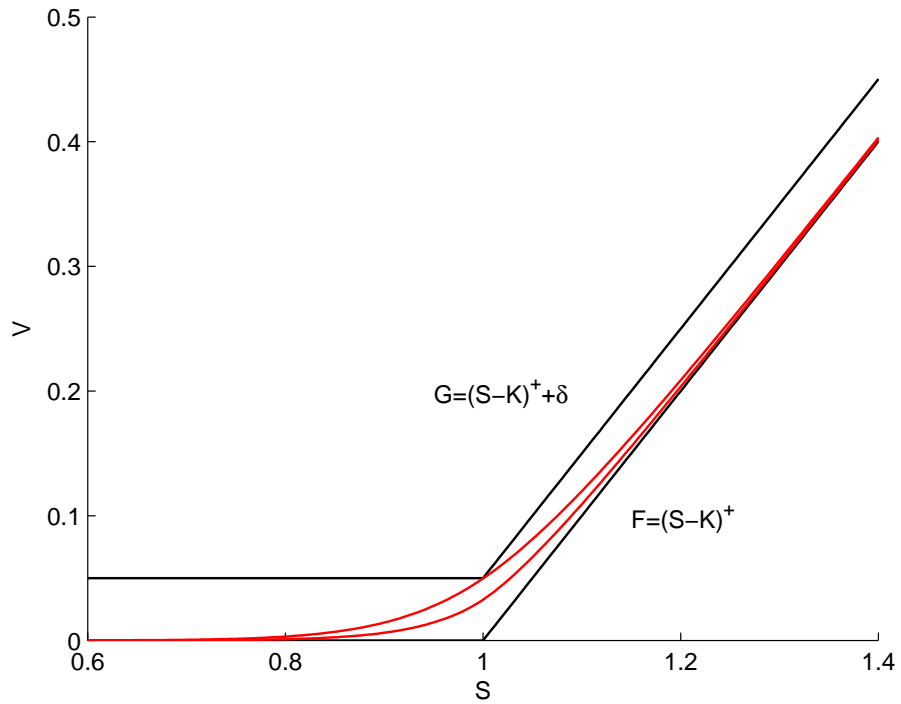


Figure 7.23: δ -penalty Game Call value functions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.97$ and $t=0.94$

The lower function represents δ -penalty Game Call value at $t = 0.97$ when the game option has not reached the cancellation region and therefore behaves like an American option. The upper function is δ -penalty Game Call value at $t = 0.94$ when the first contact is made with the upper obstacle which will lead to the cancellation region. The lower function can also be interpreted as the value of a δ -penalty Game Call with expiration time $T = 0.03$ and the upper function represents the value of a δ -penalty Game Call with expiration time $T = 0.06$

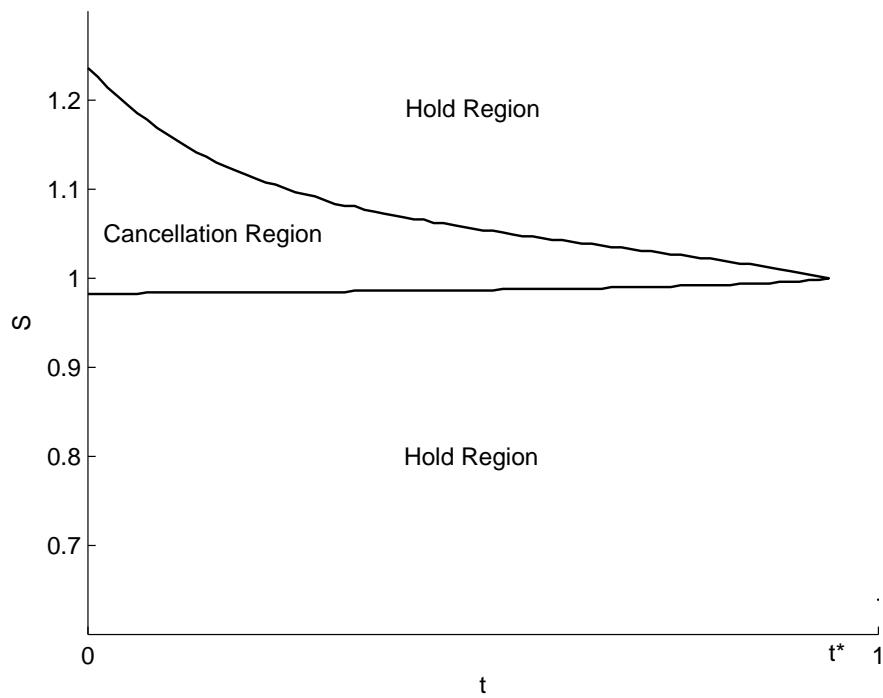


Figure 7.24: δ -penalty Game Call regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

The case of jumps diffusion has similar results to the pure jump case, as illustrated in the figure to follow

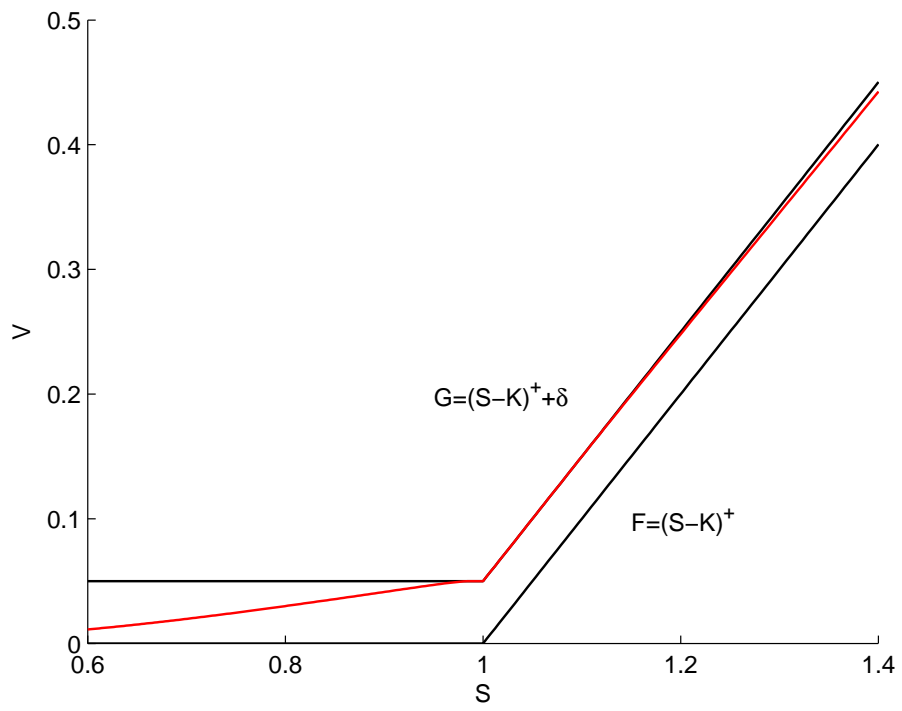


Figure 7.25: δ -penalty Game Call value function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

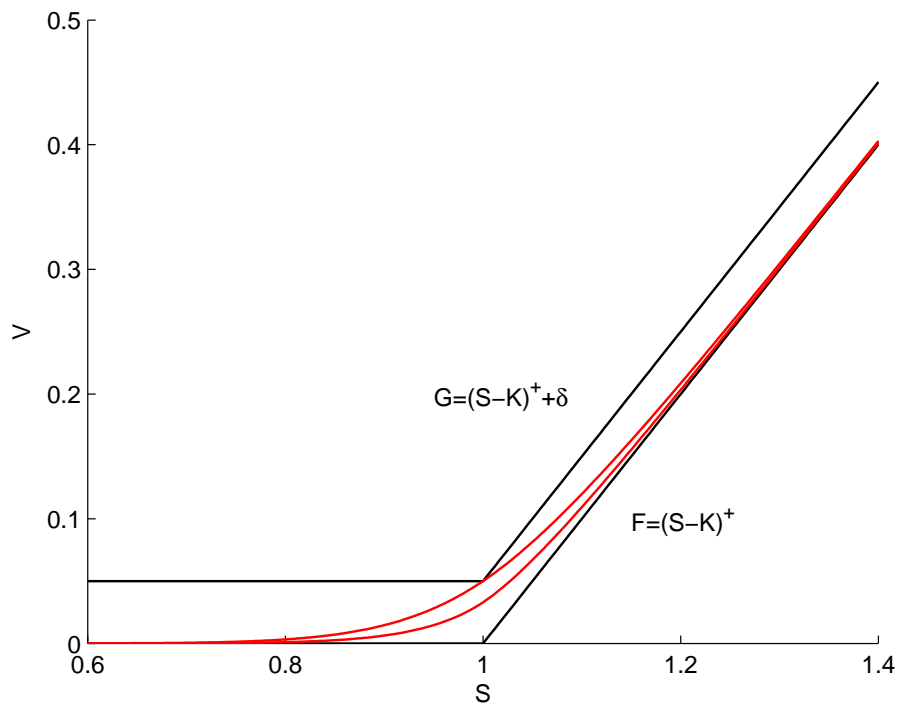


Figure 7.26: δ -penalty Game Call value functions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.97$ and $t=0.94$

The lower function represents δ -penalty Game Call value at $t = 0.97$ when the game option has not reached the cancellation region and therefore behaves like an American option. The upper function is δ -penalty Game Call value at $t = 0.94$ when the first contact is made with the upper obstacle which will lead to the cancellation region. The lower function can also be interpreted as the value of a δ -penalty Game Call with expiration time $T = 0.03$ and the upper function represents the value of a δ -penalty Game Call with expiration time $T = 0.06$

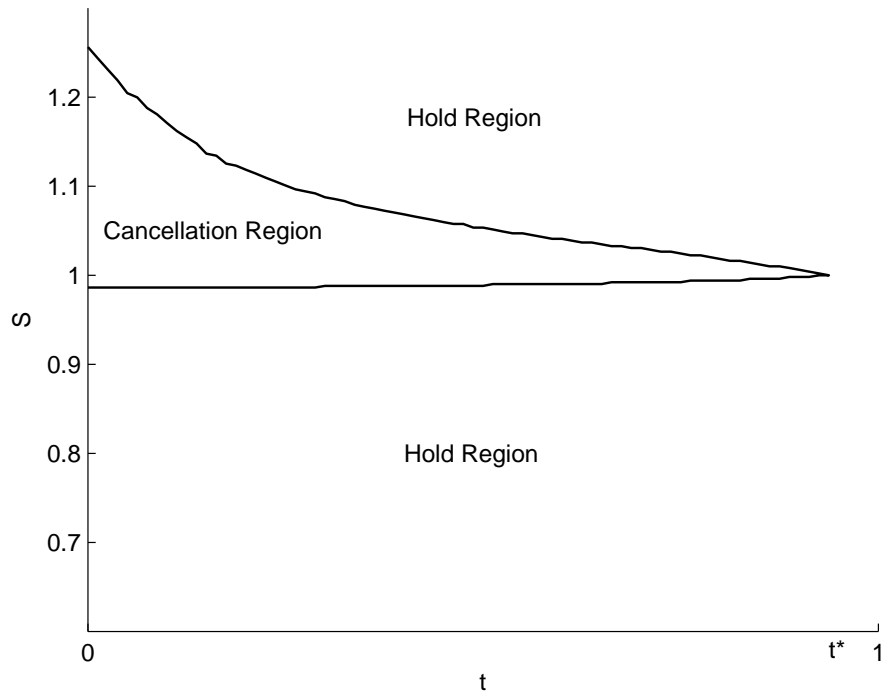


Figure 7.27: δ -penalty Game Call regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

7.2.7 Finite Maturity δ -penalty butterfly Game Option Value Functions and Free Regions

To show the flexibility of the numerical framework introduced in this paper we apply it to the δ -penalty butterfly Game options, which have the following payoffs exercise payoff

$$F = (K - .2 - S)^+ - 2(K - S)^+ + (K + .2 - S)^+$$

and cancelation payoff

$$G = F + \delta = (K - .2 - S)^+ - 2(K - S)^+ + (K + .2 - S)^+ + \delta$$

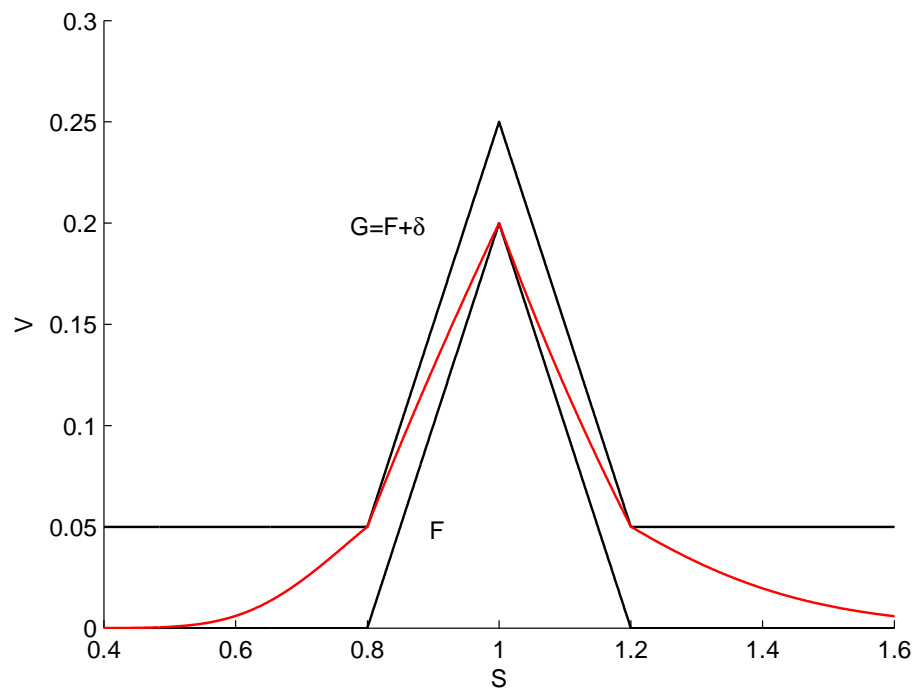


Figure 7.28: δ -penalty Butterfly Game option value function for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

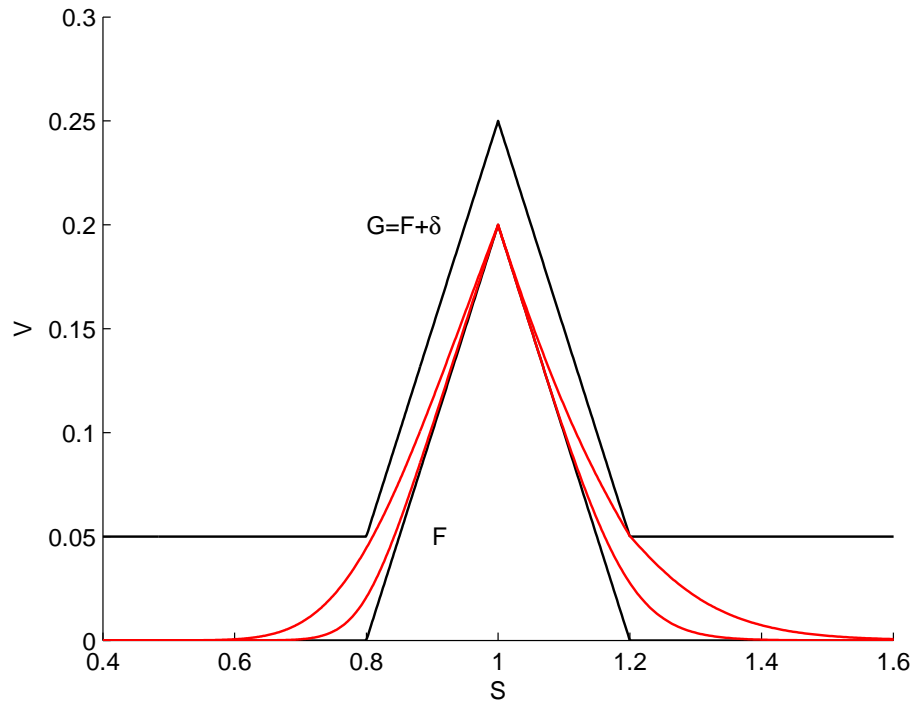


Figure 7.29: δ -penalty Butterfly Game option value functions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $t=0.9$ and $t=0.6$

The lower function represents δ -penalty butterfly Game option value at $t = 0.9$ when the game option has not reached the cancellation region and therefore behaves like an American option. The upper function is δ -penalty butterfly Game option value at $t = 0.6$ when the first contact is made with the upper obstacle which will lead to the cancellation region. The lower function can also be interpreted as the value of a δ -penalty butterfly Game option with expiration time $T = 0.1$ and the upper function represents the value of a δ -penalty butterfly Game option with expiration time $T = 0.4$

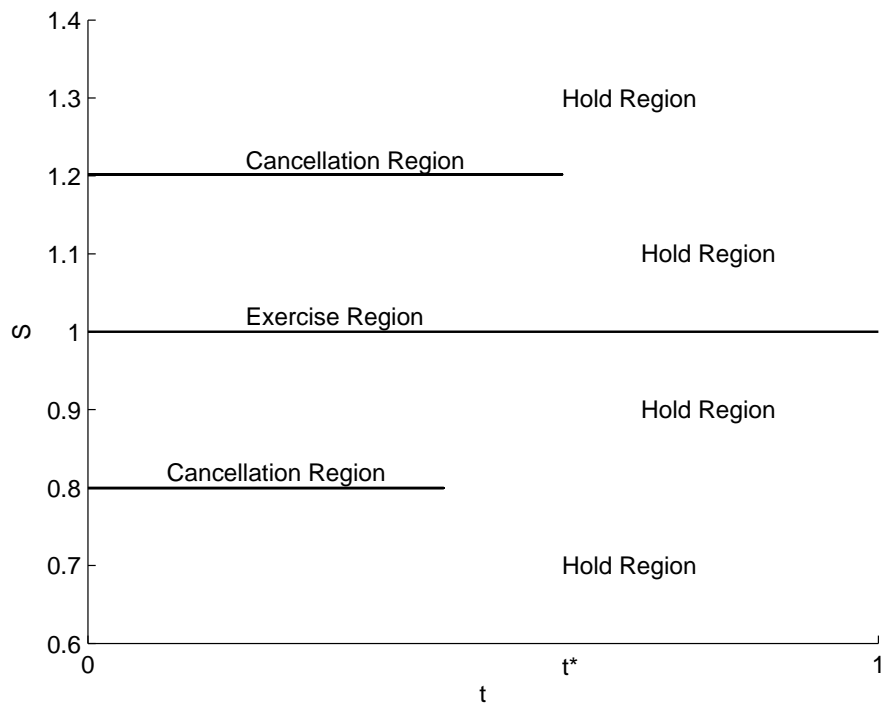


Figure 7.30: δ -penalty Butterfly Game option regions for a pure diffusion process with $\sigma=0.2$ $K=1$ $\delta=0.05$ $T=1$

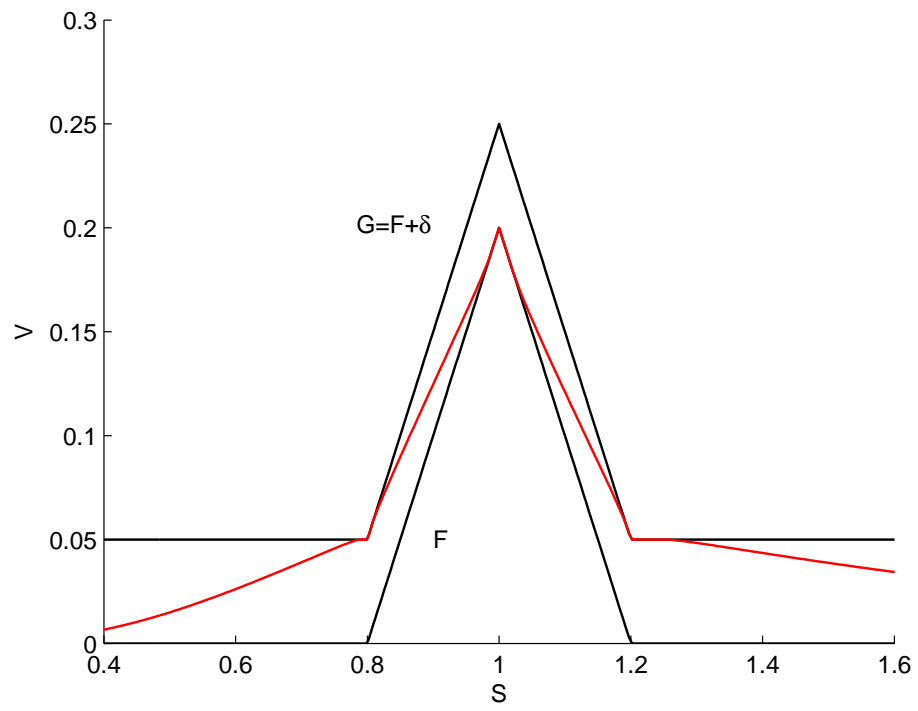


Figure 7.31: δ -penalty Butterfly Game option value function for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

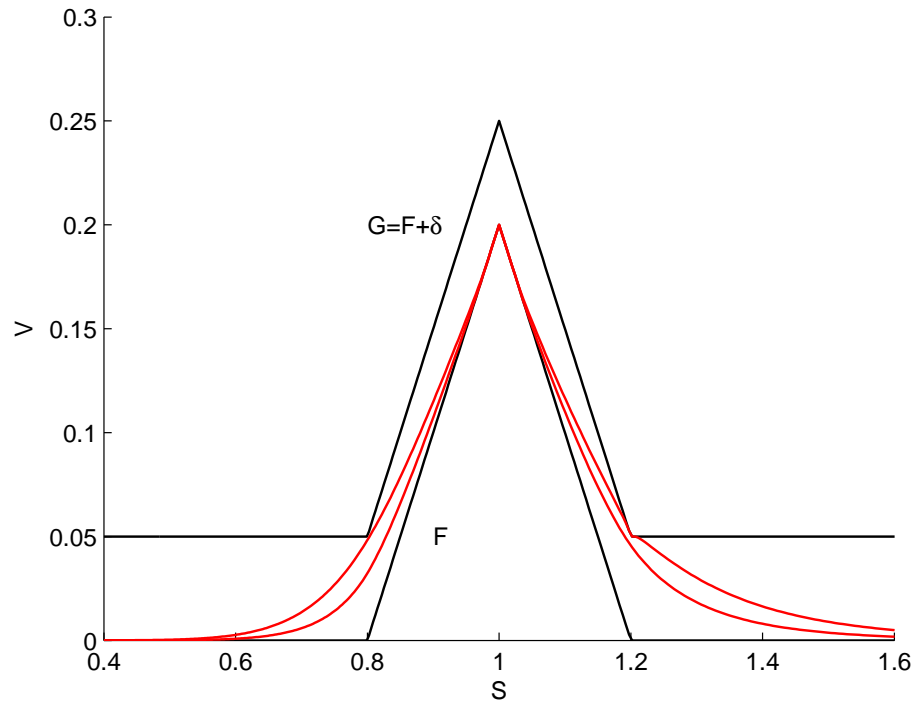


Figure 7.32: δ -penalty Butterfly Game option value functions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.95$ and $t=0.9$

The lower function represents δ -penalty butterfly Game option value at $t = 0.95$ when the game option has not reached the cancellation region and therefore behaves like an American option. The upper function is δ -penalty butterfly Game option value at $t = 0.9$ when some contact has been made with the upper obstacle which will lead to the cancellation region. The lower function can also be interpreted as the value of a δ -penalty butterfly Game option with expiration time $T = 0.05$ and the upper function represents the value of a δ -penalty butterfly Game option with expiration time $T = 0.1$

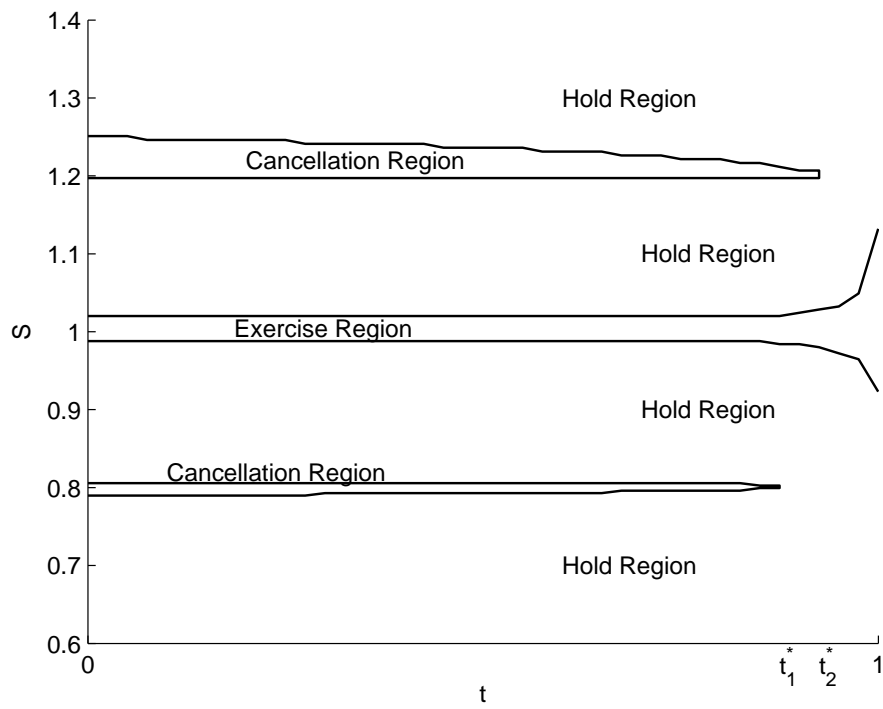


Figure 7.33: δ -penalty Butterfly Game option regions for a pure jump CGMY process with $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

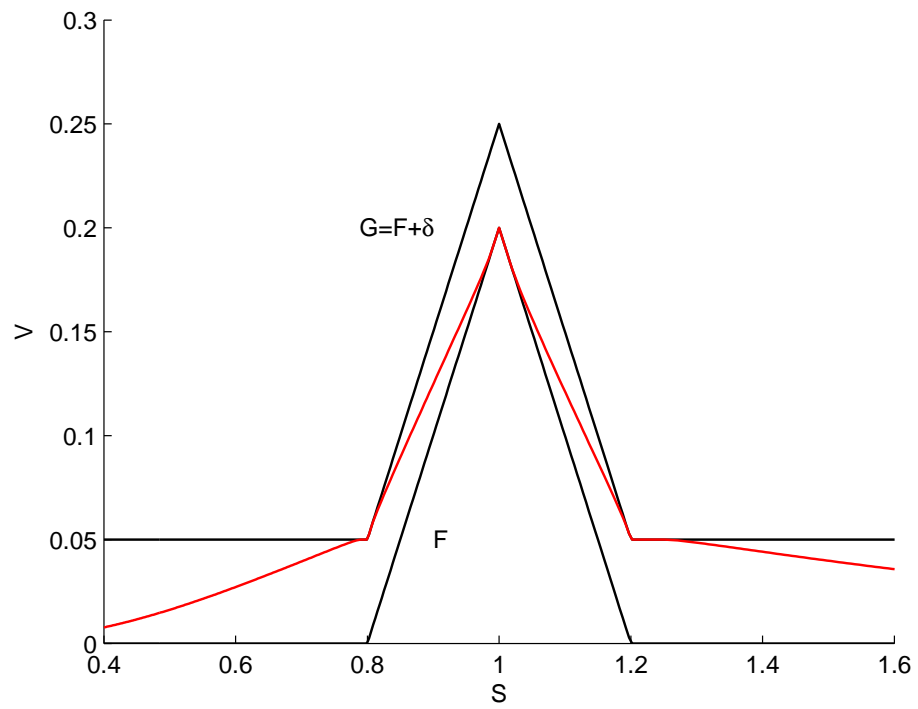


Figure 7.34: δ -penalty Butterfly Game option value function for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

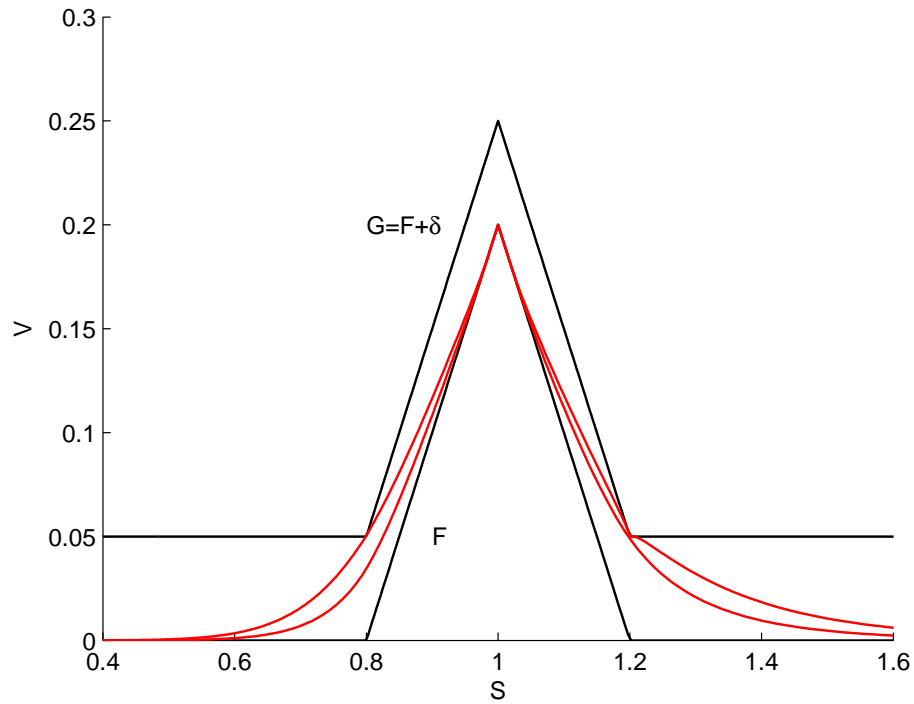


Figure 7.35: δ -penalty Butterfly Game option value functions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $t=0.95$ and $t=0.9$

The lower function represents δ -penalty butterfly Game option value at $t = 0.95$ when the game option has not reached the cancelation region and therefore behaves like an American option. The upper function is δ -penalty butterfly Game option value at $t = 0.9$ when some contact has been made with the upper obstacle which will lead to the cancelation region. The lower function can also be interpreted as the value of a δ -penalty butterfly Game option with expiration time $T = 0.05$ and the upper function represents the value of a δ -penalty butterfly Game option with expiration time $T = 0.1$

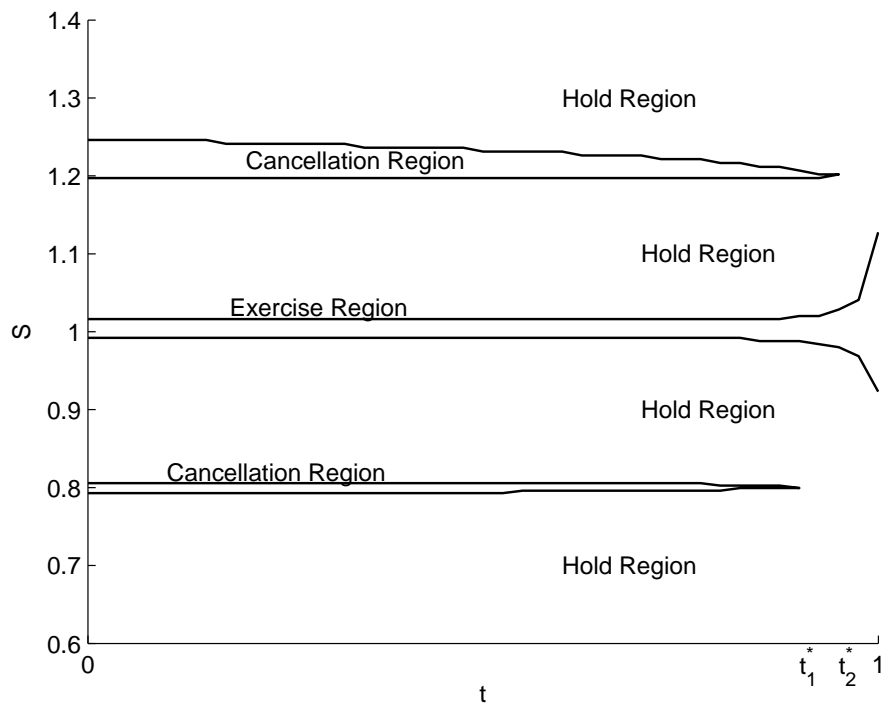


Figure 7.36: δ -penalty Butterfly Game option regions for a jump diffusion CGMY process with $\sigma=0.2$ $Y=1.25$ $K=1$ $\delta=0.05$ $T=1$

Chapter 8

Concluding Remarks and Future Research

- The framework presented in this paper offers a robust method for computing values of game options. The purpose of this framework was to offer a method for computing values of game options and to justify the theoretical results by analyzing the error convergence rates.
- The numerical results confirm the theory for the time independent problem, while in the time dependent case the available theoretical results for the error convergence are more limited.
- The results on cancelation regions for pure jump processes and jump-diffusion helps to get a better understanding of the shape of these regions as well as of the financial interpretation of the short position
- The part that was most exhaustive computationally was the LCP solver which we were able to notice by comparing it to the European options where we don't need to solve an LCP problem. Improvements on the speed of the LCP solver can be achieved by relaxing this termination condition on the residual and making it comparable to errors induced by other sources. Another way to improve the LCP performance is using a faster LCP solver based on interior point methods.

- In the case of pure jump and jump diffusion the stiffness matrix was a full matrix. Storing a full matrix also put a limitation on the number of degrees of freedom we could use based on the available memory. One way to improve the computation time is by using fast methods for handling full matrices. Some of these methods include wavelets, [72], multipole, and clustering.
- We have shown results for the standard put, call and butterfly payoff function, but a great advantage of this framework is that it can handle a wide variety of payoff functions.
- While we have implemented the one dimensional version, this methods can be carried over into multidimensional game options. It can also be applied to baskets of game options or other game derivatives with multiple underlying assets.

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