

## ABSTRACT

Title of dissertation: SOLVING PRACTICAL DYNAMIC  
PRICING PROBLEMS WITH  
LIMITED DEMAND INFORMATION

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Dynamic pricing problems have received considerable attention in the operations management literature in the last two decades. Most of the work has focused on structural results and managerial insights using stylized models without considering business rules and issues commonly encountered in practice. While these models do provide general, high-level guidelines for managers in practice, they may not be able to generate satisfactory solutions to practical problems in which business norms and constraints have to be incorporated. In addition, most of the existing models assume full knowledge about the underlying demand distribution. However, demand information can be very limited for many products in practice, particularly, for products with short life-cycles (e.g., fashion products). In this dissertation, we focus on dynamic pricing models that involve selling a fixed amount of initial inventory over a fixed time horizon without inventory replenishment. This class of dynamic pricing models have a wide application in a variety of industries. Within this class, we study two specific dynamic pricing problems with commonly-encountered business rules

and issues where there is limited demand information. Our objective is to develop satisfactory solution approaches for solving practically sized problems and derive managerial insights.

This dissertation consists of three parts. We first present a survey of existing pricing models that involve one or multiple sellers selling one or multiple products, each with a given initial inventory, over a fixed time horizon without inventory replenishment. This particular class of dynamic pricing problems have received substantial attention in the operations management literature in recent years. We classify existing models into several different classes, present a detailed review on the problems in each class, and identify possible directions for future research.

We then study a markdown pricing problem that involves a single product and multiple stores. Joint inventory allocation and pricing decisions have to be made over time subject to a set of business rules. We discretize the demand distribution and employ a scenario tree to model demand correlation across time periods and among the stores. The problem is formulated as a MIP and a Lagrangian relaxation approach is proposed to solve it. Extensive numerical experiments demonstrate that the solution approach is capable of generating close-to-optimal solutions in a short computational time.

Finally, we study a general dynamic pricing problem for a single store that involves two substitutable products. We consider both the price-driven substitution and inventory-driven substitution of the two products, and investigate their impacts on the optimal pricing decisions. We assume that little demand information is known and propose a robust optimization model to formulate the problem. We

develop a dynamic programming solution approach. Due to the complexity of the DP formulation, a fully polynomial time approximation scheme is developed that guarantees a proven near optimal solution in a manageable computational time for practically sized problems. A variety of managerial insights are discussed.

SOLVING PRACTICAL DYNAMIC PRICING PROBLEMS  
WITH LIMITED DEMAND INFORMATION

by

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# Chapter 1

## Introduction

Dynamic pricing and related problems have attracted significant attention by researchers in the operations management area in the last two decades (see survey papers by McGill and van Ryzin 1999, Bitran and Caldentey 2003, Elmaghraby and Keskinocak 2003, and McAfee and te Velde 2007, and books by Talluri and van Ryzin 2005, and Phillips 2005). These problems arise from various industries including, for example, airline companies (selling seats), hotel companies (selling rooms), cruise companies (selling cabins), rental car companies (renting cars), or retailers (selling seasonal products or non-seasonal products). The dynamic pricing literature can be generally classified into research on models without inventory replenishment and research on models with inventory replenishment. Models without inventory replenishment deal with problems where there is an initial amount of inventory to be sold over a finite planning horizon by adjusting prices, where the initial inventory is either given in advance as a problem input parameter or to be determined as a decision variable.

This dissertation consists of three essays. First, in Chapter 2, we present a comprehensive survey of literature on dynamic pricing models with a fixed amount of initial inventory over a fixed time horizon without inventory replenishment, and then in Chapters 3 and 4, we study two specific practical dynamic pricing problems within this category of problems, respectively. Both of our problems involve a number of commonly-encountered business rules and practical issues that have received little attention in the literature. Furthermore, there is limited demand information in both problems. Our objective is to develop satisfactory solution approaches for solving practically sized problems and derive managerial insights. Our first problem is a markdown pricing problem that involves a single product and multiple stores.

Joint inventory allocation and pricing decisions have to be made over time subject to a set of business rules. Our second problem is a general dynamic pricing problem for a single store that involves two substitutable products.

Most existing dynamic pricing literature studies structural results and managerial insights using stylized models. While these models, in general, may provide useful and high-level guidelines for managers in practice, they oversimplify practical situations in order to derive neat theoretical results. Consequently, the solutions generated by these models may not be satisfactory in many practical situations and the insights derived may not hold if the underlying assumptions fail. None of the papers in the existing literature have considered all the commonly-used business rules that we consider in our problems. Solutions generated by ignoring these rules may cause significant revenue loss and may not even be feasible in many circumstances.

Next, we present a summary of each of the three parts of the dissertation in the following subsections.

## 1.1 A Survey on Dynamic Pricing Models

Dynamic pricing is a commonly-used tool in revenue management and it enables a firm to increase revenue by compensating for statistical fluctuation of uncertain demand, better matching supply with demand, responding to shift of reservation price, and achieving customer segmentation. Since its early success in the airline industry, dynamic pricing has now been commonly adopted in many other industries as well. Numerous success stories of dynamic pricing applications in practice have motivated a substantial amount of research in the revenue management literature. In the past two decades particularly in the past a few years, we have witnessed a rapidly growing body of literature that addresses a wide variety of dynamic pricing problems. In Chapter 2, we survey existing dynamic pricing models that involve one or multiple sellers selling one or multiple products, each with a given initial inventory, over a fixed time horizon without inventory replenishment. Four key features distinguish this particular problem class from other classes of pricing prob-

lems. First, the initial inventory of each of the products involved is given, and in most models to be reviewed, is not a decision variable. In some models, the amount of initial inventory is a decision variable to be determined at the very beginning of the selling horizon. Second, inventory replenishment is not allowed during the selling horizon. Third, the selling horizon is prespecified and finite. At the end of the selling horizon, any unsold items will perish and can only be salvaged. This is also referred to as “perishability” of the product(s). Fourth, pricing decision in general involves determining a sequence of prices over time rather than specifying a single static price for the entire horizon. Thus, the pricing models surveyed in Chapter 2 all involve multiple time periods rather than a single time period. This particular class of dynamic pricing problems arise in various industries including, e.g., airlines, hotels, car rentals, cruise lines, long-distance bus service, broadcast advertising, sports and entertainment, medical service, freight transportation, and retail industries. We classify existing models into several classes and sub-classes. For each class and sub-class, we present detailed review of existing models. We classify and compare existing models according to some common features studied and summarize their solution approaches and results. We also identify topics and issues that have been ignored in the existing literature and propose them for future research.

## 1.2 Markdown Pricing with Multiple Stores

Most of the existing pricing models have focused on problems with a single product and a single store. Examples include, Gallego and van Ryzin (1994), Bitran and Mondschein (1997), Feng and Xiao (1999, 2000a, 2000b), Zhao and Zheng (2000), and Smith and Achabal (1998). To the best of our knowledge, Bitran et al. (1998) is the only paper that considers a model with multiple stores. Their model, however, does not incorporate many practical constraints that are commonly seen in retail industries. In addition, when modeling demand, they assume that the demand is independent across time which may not be true in many practical situations. Fur-

thermore, their solution procedure can only handle problems with a small number of stores. In contrast, we propose a model in Chapter 3 that is more practical than the model studied by Bitran et al. (1998) and propose a solution approach capable of generating near-optimal solutions for large-scale problem instances. More specifically, in Chapter 3 we study a real-world problem faced by a retailer that involves joint decisions of inventory allocation and markdown pricing of a single product at multiple stores subject to a number of business rules. At the beginning of the markdown planning horizon, there is a certain amount of inventory of a product at a warehouse that needs to be allocated to many retail stores served by the warehouse over the planning horizon. In the same time, a markdown pricing scheme needs to be determined for each store over the planning horizon. The complete demand distribution information is assumed to be unknown. We use demand scenario tree to approximate the demand distribution, which enables us to model demand correlation across time periods and among stores. We formulate the problem as a MIP and develop a Lagrangian relaxation based approach which is implemented on a rolling horizon basis. Extensive computational tests demonstrate that our approach is efficient for solving practically sized problems (50-100 stores) and also the performance of our approach is significantly better under all circumstances compared to a number of benchmark approaches commonly used in practice. A number of interesting managerial insights are also discussed.

### 1.3 Dynamic Pricing with Two Substitutable Products

A number of existing papers consider pricing of multiple products, including Kuyumcu and Popescu (2006), Tang and Yin (2007), Karakul and Chan (2008), Thiele (2009), Adida and Perakis (2006). But only a few of them explicitly model the demand interdependency among products. These papers include, Gallego and van Ryzin (1997), Bitran et al. (2006), Maglaras and Meissner (2006), Dong et al. (2009), Zhang and Cooper (2009), Akcay et al. (2010), and Suh and Aydin (2011). All these papers assume that price can be reset at any time point and none of them

explicitly take into consideration of the inventory-driven substitution when one of the products runs out of stock but the price of this product has to stay unchanged. In addition, most of the existing dynamic pricing models assume the precise knowledge of the underlying probability distribution of a random demand and assume risk neutrality of the decision maker. However, this may not be the case when little sales data is available especially in the case of products with a short-selling season. For this reason, recently there is an increasing research interest in the operations management area to use models that require limited demand information only. Examples include Lan et al. (2008) and Ball and Queyranne (2009) for airline revenue management problems, Perakis and Roels (2008) for the newsvendor model, and Thiele (2006a, 2006b, 2009), and Eren and Maglaras (2009) for pricing problems. In Chapter 4 we consider a problem involving all these issues discussed above. Specifically, we consider a dynamic pricing problem with two substitutable products which involves a number of business rules and issues commonly seen in practice. A given amount of inventory of each product has to be sold over a short selling season without inventory replenishment. Prices of the products can be re-set periodically according to some business rules. There are both price-driven substitution effect and inventory-driven substitution effect between the two products. Demand correlation exists between the two products in each time period and across time periods. However, there is not enough information to precisely estimate the underlying probability distributions of the demand functions. We use a number of lower and upper bounds (instead of a point estimate or a probability distribution function) to characterize the demand of each individual product, the aggregate demand of the two products in each period, and the aggregate demand of the two products across multiple time periods. A robust optimization model is developed in which we maximize the worst-case performance. We develop a dynamic programming algorithm to solve the max-min problem. To speed up the DP algorithm, we further develop a fully polynomial-time approximation scheme (FPTAS) which guarantees a proven near optimal solution. Our extensive computational experiments demonstrate the effectiveness and robustness of the proposed approaches. We also generate a set



of interesting managerial insights on how the price elasticities, demand uncertainty level, and some other problem parameters impact on the optimal price paths of the products. These insights can help store managers make better pricing decisions when facing high demand uncertainty due to lack of information.

## Chapter 2

### A Survey of Dynamic Pricing Models

#### 2.1 Introduction

Dynamic pricing is one of the most fundamental and commonly used revenue management tools. It enables a firm to increase revenue by better matching supply with demand, responding to shift of demand pattern, and achieving customer segmentation. Since its early success in the airline industry, dynamic pricing has now gained popularity in many other industries as well including, e.g., hotel, car rental, cruise lines, long-distance bus service, broadcast advertising, sports and entertainment, medical service, freight transportation, and retail industries. Over the past two decades, numerous success stories of dynamic pricing applications have motivated various dynamic pricing models and a rapidly growing body of research on these models in the operations management literature.

We classify dynamic pricing models into two classes: models with inventory replenishment (denoted as **WR**) and models with no inventory replenishment (denoted as **NR**). Models with inventory replenishment deals with problems where inventory can be replenished (via production or ordering) periodically over the selling horizon. Models without inventory replenishment deal with problems where there is an initial amount of inventory to be sold over a finite planning horizon without adding new inventory, and the initial inventory is either given exogenously or to be determined as a decision variable. Our focus in this survey is on **NR** problems, and hence we are not going to delve into **WR** problems. For **NR** problems, we further classify them based on the nature of competition and the type of customers involved as follows. In terms of the nature of competition, a majority of existing papers study problems with a single firm with no competition (denoted as **NC**), whereas only a handful of papers consider problems with multiple firms with competition (denoted

as **WC**). In terms of the type of customers, early research is mainly focused on problems with myopic customers (denoted as **MC**), whereas most recent research is more focused on problems with strategic customers (denoted as **SC**). Myopic customers are those who always make a “buy-or-leave” decision at the time of arrival. More specifically, at the time of arrival, if their valuation is higher than the selling price, they buy the product immediately; and otherwise they leave the store (in this case, the demand is lost for the firm). Strategic customers are those who always make a “buy-or-wait” decision at the time of arrival by considering possible future price and product availability in order to maximize their expected utility or surplus. Even if their valuation is higher than the current selling price, they may still wait and buy at a later time if they expect to have a higher utility or surplus at a later time. We classify **NR** problems into the following three sub-classes: (1) problems with no competition and myopic customers (**NR-NC-MC**); (2) problems with no competition and strategic customers (**NR-NC-SC**); (3) problems with competition (**NR-WC**), which includes problems with myopic customers and problems with strategic customers.

Gallego and van Ryzin (1994) and Bitran and Mondschein (1997) are two representative early studies in the operations management literature that address **NR** problems. They both consider a **NR-NC-MC** problem with a single product and a single store. Since then, **NR** problems have attracted a rapidly increasing research interest in the literature especially in the last five years. Not only **NR-NC-MC** problems with a single product and a single store, but also their extensions with multiple products or multiple stores, and new classes of problems, including **NR-NC-SC** and **NR-WC**, have received considerable attention. Practical issues that were largely ignored before, such as business rules and demand learning, are also investigated more recently. Given that many new models and results have appeared in the recent literature, we believe that a comprehensive survey on this important class of problems is necessary.

In this chapter, we present a comprehensive review on problems with a finite amount of initial inventory and a fixed time horizon in each of the three problem

classes, i.e., **NR-NC-MC**, **NR-NC-SC**, and **NR-WC**. We will also point out possible directions for future research. More specifically, we survey dynamic pricing problems that involve one or multiple firms selling one or multiple products, each with a finite amount of initial inventory (or capacity), over a fixed time horizon without inventory replenishment. Four key features distinguish this particular problem class from other types of pricing problems. First, the initial inventory of each of the products involved is finite, and in most studies to be reviewed, is given and not a decision variable. In some studies, there might be a one-time opportunity for inventory procurement in the very beginning. In this case, the amount of initial inventory is a decision variable. Second, once the selling season starts, inventory replenishment is not allowed. Third, a given deadline is present for the selling season (i.e., finite time horizon). In the end of the time horizon, any unsold items will perish and can only be salvaged, which is also referred to as “perishability” of the product(s). Fourth, pricing decision is dynamic in general, i.e., it involves determining a sequence of prices over time rather than specifying a single static price for the entire horizon. Thus, the pricing models surveyed in this chapter all involve multiple time periods rather than a single time period. It is possible that in some extreme cases, the optimal price in each period can be identical.

Numerous dynamic pricing problems involving selling a fixed amount of inventory over a fixed time horizon without inventory replenishment can be found in a variety of industries, including travel industry (e.g., airlines selling seats, hotels selling rooms, rental car agencies renting cars, cruise line companies selling cabins), entertainment industry (e.g., theaters selling tickets), and retail industry (e.g., retailers selling seasonal and fashion products in their selling seasons, or clearing the inventory of consumer electronic products at the end of their life-cycles). Clearly in all these examples, the initial inventory (or capacity) is fixed and there is a deadline for the selling season. In the airline example, airlines typically commit a particular type of aircraft to a particular flight, and thus the number of seats on each flight is fixed. Most airlines start to sell seats three to eleven months ahead of departure time. After the departure time (deadline), any unsold seats have no value. In the

retail example, as most of the fashion apparel are made overseas and then shipped to the US, often times the production lead time can be more than six months. The actually selling season, on the other hand, may only last six to eight weeks. Therefore, retailers typically make a one-time order long before the beginning of the selling season. Once the selling season starts, there is no opportunity to replenish the inventory if the demand turns out to be higher than expected. On the other hand, any unsold items at the end of the season can either be donated to charity or liquidated via a discount sales channel. In either case, the retailer can only receive a substantially lower salvage value compared to the product's regular selling price.

There are several existing reviews on some classes of dynamic pricing problems that overlap with the problem classes we review in this survey. Elmaghraby and Keskinocak (2003) review pricing problem classes **WR** and **NR**. Within problem class **NR**, they mainly focus on **NR-NC-MC** problems with a single product and a single store. Bitran and Caldentey (2003) provide a limited review on pricing problem class **NR-NC-MC**. They mainly focus on the case with a single product and a single store. McAfee and te Velde (2007) give a limited review on problem class **NR-NC-MC** which is exclusively focused on the airline industry. Shen and Su (2007) present a review of literature on customer behavior within the areas of revenue management and auction. They cover problem class **NR-NC-SC** and also problem class **NR-NC-MC** with multiple products. In contrast, we provide a detailed review on each of the problem classes **NR-NC-MC**, **NR-NC-SC**, and **NR-WC**. Within problem class **NR-NC-MC**, we review problems with a single product, problems with multiple products, as well as problems with multiple stores. We note that there are no existing papers that consider problems with both multiple products and multiple stores. For problem classes **NR-NC-SC** and **NR-WC**, all existing papers consider single-product-single-store problems. In our survey, we review 58 papers in detail, of which 16 also appear in Shen and Su (2007), 9 appear in Elmaghraby and Keskinocak (2003), 12 appear in Bitran and Caldentey (2003), 6 appear in McAfee and te Velde (2007), and 29 have not been reviewed by any of these four existing survey papers.

The following types of dynamic pricing problems are out of the scope of this survey and hence not reviewed: pricing problems involving replenishment of inventory (e.g., Chen and Simchi-Levi, 2004a, 2004b, Federgruen and Heching, 1999, 2002), pricing problems involving production decisions (e.g., Yano and Gilbert 2004, Deng and Yano 2006, Geunes et al. 2006, Ahn et al. 2007), pricing problems with infinite time horizon (e.g., Chintagunta and Rao, 1996, Raman and Chatterjee, 1995), and pricing problems with a single period where a single price is to be determined (e.g., Kuyumcu and Popescu, 2006, Thiele, 2009, Karakul and Chan, 2008, and Tang and Yin, 2007). Note that in the revenue management literature, although capacity control problems (e.g., Talluri and van Ryzin, 2004, Netessine and Shumsky, 2005) are closely related to dynamic pricing problems, they are not the focus of this survey and hence not reviewed.

The remainder of this chapter is organized as follows. We first in Section 2.2 review papers that consider dynamic pricing problems **NR-NC-MC**. We review papers with single-product-single-store, multi-product, and multi-store in Sections 2.2.1, 2.2.2, 2.2.3, respectively. We then in Section 2.3 review dynamic pricing problems **NR-NC-SC**. Next, in Section 2.4 we review papers that study the dynamic pricing problems **NR-WC**. Finally we conclude this survey in Section 2.5 and identify possible directions for future research.

## 2.2 Models with No Competition and Myopic Customers

Existing literature can be categorized into three types in terms of the way they model the time horizon. Type (i): continuous time horizon where the price changes can occur at any point in time. Type (i) assumption is suitable for most internet stores where prices can be easily adjusted at any time point at a small (or no) cost. Type (ii): discrete time horizon where the price changes can only occur at the beginning of each time period. Type (ii) assumption is appropriate for most physical stores where price changes may incur a significant cost and the prices are usually adjusted according to a fixed schedule, e.g., weekly. Type (iii):

discretized continuous time horizon where the time horizon consists of many discrete time intervals. The length of each interval is sufficiently small such that there is at most one customer arrival within each time interval. Type (iii) assumption is essentially a discrete approximation of type (i) assumption.

In terms of the allowable prices, existing models can be generally categorized into two types. Type (i): continuous allowable prices, i.e., any price can be used (in some cases, there might be a lower and an upper bounds for the allowed prices). Type (i) assumption represents an ideal situation where firms have complete flexibility in setting up the prices. Type (ii): discrete allowable prices, i.e., the price can only be chosen from a pre-determined discrete set of price points. Type (ii) assumption represents a typical situation where firms tend to follow a sound pricing strategy, e.g., customers are much more willing to buy at certain price points (e.g., Allen 2011). Therefore, only a small number of price points are used. For example, in the retail industry, the price for a particular camera model may only be chosen from the set  $\{\$249, \$299, \$349, \$399, \$449, \$499\}$ .

### 2.2.1 Single Product and Single Store

In this section, we review papers that consider the dynamic pricing problem with single product and single store. We first describe the general problem using some common notations. We consider the problem where a firm holds  $N$  units of initial inventory for a single product at the beginning of the selling horizon. The length of the selling horizon  $T$  is fixed. Inventory replenishment during the selling horizon is not allowed. Unsold items will perish at the end of the selling horizon. In that case, the firm can only receive a small salvage value  $s$  for each unit of unsold item. Most papers assume that  $s = 0$  except Smith et al. 1998, Smith and Achabal 1998, Chatwin 2000, Gupta et al. 2006, and Chun 2003. The problem in which  $s > 0$  can be converted to an equivalent problem with  $s = 0$  if  $s$  is a constant regardless of the number of unsold items (see Gallego and van Ryzin, 1994). Due to the relative short selling season, the majority of the papers ignore inventory holding costs and time discounting except Smith et al. (1998) who explicitly incorporate inventory

holding costs in their model. The objective is to determine the optimal price over time (possibly a dynamic pricing scheme) in order to maximize the total expected revenue collected over the entire time horizon, denoted by the corresponding value function  $V(N, T)$ . For any intermediate state where the remaining inventory is  $n$  and remaining time is  $t$ , we denote the value function as  $V(n, t)$ .

To the best of our knowledge, in the operations management literature, the first paper that investigates the problem described above is Gallego and van Ryzin (1994). Since then, the pricing problem for a single product and a single store has received extensive attention in the literature. Other papers include Feng and Gallego (1995, 2000), Bitran and Mondschein (1997), Feng and Xiao (1999, 2000a, 2000b, 2006), Zhao and Zheng (2000), Chatwin (2000), Smith and Achabal (1998), Smith et al. (1998), Anjos et al. (2005), Gupta et al. (2006), McAfee and te Velde (2008), Monahan et al. (2004), Chun (2003), Aydin and Ziya (2009), Neelakantan et al. (2007), Besbes and Zeevi (2009), Lin (2006), Sen and Zhang (2009), and Levin et al. (2007). In what follows, we first (in Section 2.2.1.1) describe the demand models commonly used in the literature and categorize the literature according to some common model assumptions, and we then (in Section 2.2.1.2) review each paper in details and summarize their solution approaches and major results.

### 2.2.1.1 Demand Models and Categorization of Existing Literature

Demand models can be generally categorized into two types, i.e., deterministic model and stochastic model. For deterministic models, the aggregate demand  $D$  within a period of time (for discrete time case) or the demand rate  $\lambda$  at certain point in time (for continuous time case) can be represented as a deterministic function of price  $p$ , time  $t$  and possibly other variables. Other variables may include on-hand inventory level, sales to date, advertisement, etc. We will specify these variables when we review each paper in details in Section 2.2.1.2.

Within the group of stochastic models, we further categorize them into three types. Stochastic model (i) (referred to as general Poisson model hereinafter): demand arrives following a Poisson process with demand intensity  $\lambda$ , which is a func-



tion of price  $p$  and time  $t$ ,  $\lambda = \mathbf{G}(p, t)$ . Stochastic model (ii) (referred to as Poisson reservation price model hereinafter): customers visit the store following a Poisson process with given intensity  $\Lambda_t$  at time point  $t$ . Each customer has a reservation price upon arriving at the store, if the posted price  $p$  is below her reservation price, she will make a purchase, otherwise not. The distribution of reservation price at any given time  $t$  is known with cumulative distribution function  $\mathbf{F}(p, t)$ . Therefore the purchasing probability is  $\bar{\mathbf{F}}(p, t) = 1 - \mathbf{F}(p, t)$ . Poisson reservation price model explicitly models customers' purchasing behavior. Clearly, we can view Poisson reservation price model as a special case of general Poisson model if we make the following transformation,  $\lambda = \Lambda_t \bar{\mathbf{F}}(p, t)$ . Stochastic model (iii) (referred to as general aggregate model hereinafter): aggregate demand  $D$  within a certain period of time can be represented as a stochastic function of price  $p$  and time  $t$ ,  $D = \mathbf{H}(p, t)$ . Note that general Poisson model and Poisson reservation price model are for continuous time case only while general aggregate model is for discrete time case only.

For each type of demand model, we further categorize them into two types, i.e., time-invariant model and time-varying model (also referred to as homogeneous/time-independent and nonhomogeneous/time-dependent demand in some papers). In the following, we use a two-field notation ( $\alpha|\beta$ ) to specify the type of demand models.  $\alpha = \{\mathbb{D}, \mathbb{P}, \mathbb{R}, \mathbb{A}\}$  indicates whether demand model is a deterministic model ( $\alpha = \mathbb{D}$ ), general Poisson model ( $\alpha = \mathbb{P}$ ), Poisson reservation price model ( $\alpha = \mathbb{R}$ ), or general aggregate model ( $\alpha = \mathbb{A}$ ).  $\beta = \{\mathbb{I}, \mathbb{V}\}$  indicates whether demand model is a time-invariant model ( $\beta = \mathbb{I}$ ) or time-varying model ( $\beta = \mathbb{V}$ ). For example, we use  $\mathbb{P}|\mathbb{I}$  to represent a stochastic demand model, i.e., general Poisson model with time-invariant demand rate.

Most papers assume that given a price, one can either perfectly predict the demand (deterministic demand model) or knows exactly the demand distribution (stochastic demand model). In practice, often times this may not be case due to lack of demand information before the season starts and fast-changing market conditions. Several papers (Besbes and Zeevi 2009, Sen and Zhang 2009, and Lin 2006) investigate the dynamic problem assuming that firms do not have precise

knowledge about the underlying demand. So either they do not know the demand function, or they know the demand function without knowing the exact parameter values. In such cases, to improve the performance firms have to update their demand forecast according to the real-time sales information. We treat these models as a separate group and call them demand learning ( $\mathbb{DL}$ ) model. Table 2.1 categorizes the existing models based on their model assumptions, i.e., time horizon, allowable prices, and demand model. Detailed review of each paper is provided in the next section, Section 2.2.1.2

### 2.2.1.2 Review of Existing Literature

In the following, we review the literature based on the type of problems studied. We review problems with continuous time horizon & continuous allowable prices, continuous time horizon & discrete allowable prices, discrete time horizon & continuous allowable prices, and discrete time horizon & discrete allowable prices in Sections 2.2.1.2.1, 2.2.1.2.2, 2.2.1.2.3, and 2.2.1.2.4, respectively. In a separate section (Section 2.2.1.2.5), we review all papers that incorporate demand learning regardless of other model assumptions. Some papers may appear in multiple sections if they involve multiple problems. Within each type of problems, we first consider the case with time-invariant demand function and then with time-varying demand function. Within each type of demand function, we first review deterministic problems, followed by stochastic problems. Note that each type of problem may not cover a complete combination of all possible demand models. For example, for problems with discrete time horizon, none of the papers explicitly consider a time-invariant demand model since it can be treated as a special case of time-varying model.

#### 2.2.1.2.1 Continuous Time Horizon & Continuous Allowable Prices

We first consider the case where the demand intensity is time-invariant and a function of price  $p$  only, i.e.,  $\lambda = \mathbf{G}(p)$ . For the deterministic case, Gallego and van Ryzin (1994) show that a single price is optimal. For the corresponding stochastic model (general Poisson model), they derive the following monotonicity properties

Table 2.1: Overview of Existing Models with Single Product and Single Store

Time horizon	Allowable prices	Demand model	Paper(s)
Continuous	Continuous	D I	Gallego and van Ryzin, 1994
		D V	Gallego and van Ryzin, 1994 Anjos et al., 2005 Smith and Achabal, 1998
		P I	Gallego and van Ryzin, 1994 McAfee and te Velde, 2008
		P V	Gallego and van Ryzin, 1994 Levin et al. 2007
		R I	Aydin and Ziya, 2009
		R V	Bitran and Mondschein, 1997 Zhao and Zheng, 2000
		DL	Besbes and Zeevi, 2009 Lin, 2006
Continuous	Discrete	D I	Gallego and van Ryzin, 1994
		D V	Gallego and van Ryzin, 1994
		P I	Gallego and van Ryzin, 1994 Feng and Gallego, 1995 Feng and Xiao, 1999, 2000a, 2000b Chatwin, 2000
		P V	Gallego and van Ryzin, 1994 Feng and Gallego, 2000 Feng and Xiao, 2006 Chatwin, 2000
		R I	Aydin and Ziya, 2009
		R V	Zhao and Zheng, 2000
		DL	Lin, 2006
Discrete	Continuous	D V	Gupta et al., 2006
		R V	Bitran and Mondschein, 1997
		A V	Gupta et al., 2006 Chun, 2003 Monahan et al., 2004
Discrete	Discrete	A V	Smith et al., 1998 Neelakantan et al., 2007
		DL	Sen and Zhang, 2009

for a particular state  $(n, t)$ .

- Property (i):  $V(n, t)$  is strictly increasing and concave in  $n$ ;
- Property (ii):  $V(n, t)$  is strictly increasing and concave in  $t$ ;
- Property (iii): the optimal price  $p^*(n, t)$  is strictly decreasing in  $n$ ;
- Property (iv): the optimal price  $p^*(n, t)$  is strictly increasing in  $t$ .

For a special case where the demand intensity is an exponential function of price, i.e.,  $\lambda = ae^{-\alpha p}$  where  $a, \alpha$  are given parameters, they obtain a closed form optimal solution. For the general case, they demonstrate that a single price policy is asymptotically optimal when the volume of the expected sales goes to infinity. McAfee and te Velde (2008) investigate the same problem with constant demand elasticity, i.e.,  $\lambda = ap^{-\varepsilon}$  where  $\varepsilon$  is the demand elasticity. They derive the closed form optimal solution for this particular demand function and show that their solution is even simpler compared to the one obtained by Gallego and van Ryzin (1994) with exponential demand function. They also show that with constant demand elasticity, social efficient solution that maximizes the total gain from selling all products, which is believed to be the consequence of competitive market, is also achieved in a monopoly market. Finally, they show that if initially there are more customers than available number of items, then at the average prevailing capacity, delaying the purchase is unprofitable for the customer.

Aydin and Ziya (2009) consider a case where the firm makes a personalized pricing based on individual customer's signal (the firm can infer customer's willingness to pay based on this signal), in addition to inventory level  $n$  and remaining time  $t$  as other papers have considered. They formulate the problem as a stochastic dynamic program by discretizing the continuous time horizon. They show that under some strong conditions, the optimal price is increasing in the signal revealed by the customer.

When the demand intensity is time-varying, if the demand rate can be represented as the following multiplicative form,  $\lambda = \mathbf{G}_1(t)\mathbf{G}_2(p)$ , Gallego and van

Ryzin (1994) show that the problem can be transformed into an equivalent problem with time-invariant demand rate by redefining the time horizon. Therefore, all the results derived for the time-invariant case still hold true (for both deterministic and stochastic cases). Anjos et al. (2005) and Smith and Achabal (1998) focus on the deterministic version of this problem. Similar as the Poisson reservation price model for the stochastic case, Anjos et al. (2005) assume that the demand rate at any time  $t$  depends on the instantaneous arrival rate  $\Lambda_t$  and the probability that a customer makes the purchase  $\mathbf{P}(p, t)$ . Therefore, the instantaneous demand rate at time  $t$  is  $\Lambda_t \mathbf{P}(p, t)$ . They show that under certain conditions, the optimal pricing strategy can be characterized by a family of continuous pricing functions. Smith and Achabal (1998) incorporate the on-hand inventory level in their demand function in addition to price  $p$  and time  $t$ . They assume that the inventory effects are one-sided, i.e., low inventory decreases sales while high inventory has no impact on sales. For a multiplicative separable demand function with exponential price sensitivity, i.e.,  $\lambda = \mathbf{K}(t)\mathbf{Y}(I)e^{-\alpha p}$  where  $\mathbf{K}(t)$  capture seasonal effects and  $\mathbf{Y}(I)$  capture the impact of on-hand inventory  $I$ , they obtain closed form solution. They show that the optimal price should be adjusted to compensate exactly for any reduction due to seasonal effects. They also find that when the demand is impacted by inventory, one should use higher initial price early in the season and offer deeper discount in the end. Their model was implemented at three major retail chains and two of them were considered as highly successful.

Bitran and Mondschein (1997) study the same problem as Gallego and van Ryzin (1994) with Poisson reservation price model. As discussed in Section 2.2.1.1, the equivalent demand intensity at time  $t$  for the Poisson reservation price model is  $\Lambda_t \bar{\mathbf{F}}(p, t)$ , where  $\Lambda_t$  is customers' arrival rate and  $\mathbf{F}(p, t)$  is the cumulative distribution function for the reservation price. They show that if the reservation price distribution is time-invariant (the arrival rate  $\Lambda_t$  can still be time-varying), i.e.,  $\mathbf{F}(p, t) = \mathbf{F}(p)$ , *Properties (iii) & (iv)* hold. This is consistent with the above finding by Gallego and van Ryzin (1994) if we view  $\bar{\mathbf{F}}(p)$  as  $\mathbf{G}_2(p)$  and  $\Lambda_t$  as  $\mathbf{G}_1(t)$ . Under the same condition, they also show that a single price is optimal when the

initial inventory  $N$  is large enough (greater than any possible total demand for the entire time horizon).

Zhao and Zheng (2000) use a very similar modeling framework (e.g., Poisson reservation price model) as Bitran and Mondschein (1997) to investigate the same problem. They show that for the general case where the reservation price distribution  $\mathbf{F}(p, t)$  is time-varying, *Properties (i) & (iii)* still hold. However, *Properties (ii) & (iv)* may not always be valid, e.g., the value function  $V(n, t)$  may not be concave in  $t$  in certain cases. They prove that  $\Delta V(n, t) = V(n, t) - V(n - 1, t)$  is increasing in  $t$  and show that under certain sufficient conditions, *Property (iv)* still holds. This condition requires that the probability that a customer is willing to pay a premium does not increase over time. They justify the use of dynamic pricing as a tool to compensate for (i) normal statistical fluctuation of demand and (ii) shifts of reservation price. Their numerical results demonstrate that compared to the single-price policy, the optimal dynamic pricing can improve the revenue by 2.4% to 7.3% due to (i) and up to 100% due to (ii).

Levin et al. (2007) consider a dynamic pricing model by incorporating price guarantee policy. Customers tend to delay their purchase due to future price uncertainty while firms may provide price guarantee policy (by charging a fee) to induce customers to buy earlier. At each point in time, the firm simultaneously determine the optimal price and price guarantee policy. To develop an analytically and computationally practical model, they discretize the continuous time horizon. The sales revenue at each time point depends not only on the remaining inventory and time, but also on the entire sales and policy history. This makes it difficult to formulate the problem as a dynamic program. Instead, they formulate the problem as a non-linear program with the entire problem being optimized together. In their numerical experiments, they were only able to solve problems with very small size. They further develop a myopic lower-bounding heuristic. Their numerical experiments show that price guarantee policy may increase the revenue by either generating extra sales or collecting fees and the improvement is quite significant.

### 2.2.1.2.2 Continuous Time Horizon & Discrete Allowable Prices

We first consider the case where the demand intensity is time-invariant. For the deterministic case, Gallego and van Ryzin (1994) show that the optimal solution is to use two adjacent prices (chosen from multiple predetermined prices) with each price being applied to a portion of the time horizon. For the corresponding stochastic case, they propose a stopping-time heuristic (ST heuristic) based on the deterministic solution, i.e., start from one of the two prices obtained from the deterministic solution and switch to the other one once a certain number of items has been sold or a certain time has elapsed. They demonstrate that the ST heuristic is asymptotically optimal when the volume of expected sales goes to infinity.

Feng and Gallego (1995) develop the optimal policy for the problem where only two prices are used and prespecified. They consider three cases: (i) a markup case in which one can switch from the initial price  $p_1$  to a given higher price  $p_2$  ( $p_1 < p_2$ ) at certain time; (ii) a markdown case in which one can switch from the initial price  $p_1$  to a given lower price  $p_2$  ( $p_1 > p_2$ ) at certain time; or a markup or markdown case in which one can switch from the initial price  $p$  to either a given lower price  $p_1$  or a given higher price  $p_2$  ( $p_1 < p < p_2$ ) at certain time. They show that a threshold policy is optimal, i.e., it is optimal to increase (resp., decrease) the initial price once the remaining time falls above (resp., below) the time threshold which is dependent on the on-hand inventory level. Feng and Xiao (1999) extend this problem to incorporate the risk. In their extended model, they add a term (a linear function of the variance of the revenue) to the objective function (originally it only includes the expected revenue) to reflect decision-makers' risk attitude (either risk-averse or risk-prone). They obtain the exact solution (time thresholds) in closed form for their extended model.

The problem is further extended by Feng and Xiao (2000a) to the case where there are  $K$  ( $K \geq 2$ ) predetermined prices with  $p_1 < p_2 < \dots < p_K$ . The price can only change monotonically, i.e., either from  $p_1$  to  $p_K$  (markup case) or from  $p_K$  to  $p_1$  (markdown case). For this extension, the authors find the exact optimal solution (again a threshold policy). They show that the value function  $V(n, t)$  is increasing

and piece-wise concave in both  $n$  and  $t$ . The optimal time thresholds are also monotonic in the number of remaining inventory  $n$ . Feng and Xiao (2000b) consider another case where reversible price changes are allowed, i.e., both markdown and markup are allowed. They show that only a subset of given prices that form a concave envelop is potentially optimal. They derive the optimal time thresholds for this case. For the same case, Chatwin (2000) prove that *Properties (i), (ii), (iii), and (iv)* all hold.

Aydin and Ziya (2009) investigate the personalized dynamic pricing problem with two given prices  $p_1$  and  $p_2$ . For each individual customer, according to their signal, the firm either charge the customer  $p_1$  or  $p_2$ . They prove that a threshold policy is optimal and show that the optimal threshold is monotonic in both inventory  $n$  and time  $t$ .

The problem with time-varying demand intensity is investigated by Chatwin (2000), Zhao and Zheng (2000), Feng and Gallego (2000), and Feng and Xiao (2006). Chatwin (2000) prove that *Properties (i), (ii), (iii), and (iv)* all hold if the the prices and demand are piecewise constant functions of time  $t$ . Zhao and Zheng (2000) use Poisson reservation price model. For the general case where the optimal price may fail to be monotonic in  $t$  for any given inventory level  $n$ , they propose a stochastic dynamic program by discretizing the continuous time horizon to obtain the optimal solution. For the special case where the optimal policy is time-monotonic, they develop a procedure to obtain the exact solution, i.e., the optimal time thresholds. Feng and Gallego (2000) is an extension of Feng and Xiao (2000b). They consider two problems. In the first problem, both the price and demand intensity are time-dependent. In the second problem, they consider a Markovian case where both the price and demand intensity also depend on the sales to date, i.e., the demand intensity may increase with sales due to word of mouth or decrease with sales due to the finite population effect. For both problems, they develop efficient algorithms for computing the optimal time thresholds. Feng and Xiao (2006) consider an integrated problem in which the inventory allocation (to different micro-markets) and pricing decisions are optimized together. The optimal policy involves a set of time thresholds



that depends on inventory, price and demand intensity.

#### 2.2.1.2.3 Discrete Time Horizon & Continuous Allowable Prices

Gupta et al. (2006) consider a deterministic model with demand function in time period  $t$   $D_t = K_t e^{-\beta_t p} E[\xi_t]$  where  $K_t$  measures the market size,  $\xi_t$  is a random variable and  $E[\xi_t]$  is its expected value,  $\beta_t$  is a parameter and its inverse ( $1/\beta_t$ ) represents the mean reservation price. They derive the closed form solution and demonstrate that the mean reservation price is the one that determines the relative prices in different periods, the higher the mean reservation price, the higher the optimal price in that period. For a special case where the mean reservation price is identical in each period, i.e.,  $\beta_t$  is independent of  $t$ , a single price is optimal for all periods. This finding is consistent with the one by Gallego and van Ryzin (1994) (see Section 2.2.1.2.1) for the continuous time case if we view  $e^{-\beta_t p}$  as  $\mathbf{G}(p)$ . Their numerical results show that implementing a single-price policy to the general problem where the mean reservation price varies across time can be near optimal under the following two situations: (i) the mean reservation price in each period is close; or (ii) the mean reservation price drops dramatically at the end of the season.

Gupta et al. (2006) also consider a stochastic model with similar demand function  $D_t = K_t e^{-\beta_t p} \xi_t$  where  $\xi_t$  is a random variable that allows demand in different periods to be arbitrarily correlated. For this stochastic problem, they obtain several upper and lower bounds on the expected revenue based on the deterministic solution and develop a heuristic solution approach. They also show that markets with smaller variability of demand tend to be more profitable. Chun (2003) uses negative binomial distribution to model uncertain demand in each period and propose a dynamic program to solve the problem.

Bitran and Mondschein (1997) investigate a problem in which the price change can only occur at certain point in time and the price change can only be monotonic, i.e., nonincreasing or nondecreasing. They use Poisson reservation price model to represent uncertain demand. The problem is formulated as a stochastic dynamic program. To solve this DP, they discretize the continuous allowable price into a set

of closely adjacent price points. Their numerical experiments reveal some insights that are consistent with common industrial practice: (i) the impact of periodic price change and monotonic pricing are negligible as long as an appropriate number of price changes are implemented; (ii) higher uncertainty in the reservation price distribution leads to higher initial price, deeper discount thereafter, higher total expected revenue and more unsold items at the end of the time horizon.

Monahan et al. (2004) study the dynamic pricing problem with multiplicative demand function for period  $t$   $D_t = A_t p^{-\varepsilon}$ , where  $\varepsilon$  is the price elasticity of demand and  $A_t$  is an iid random variable. They formulate the problem as a stochastic dynamic program. They show that the dynamic pricing problem can be converted to an equivalent dynamic stocking factor problem that is independent of the inventory level. And thus the original dynamic pricing problem can be solved by iteratively solving  $T$  single-period optimization problem. They further develop several structural results including, for example, *Property (iv)* holds if  $A_t$  is stationary. These structural results also enable them to develop efficient algorithm for solving the problem.

#### 2.2.1.2.4 Discrete Time Horizon & Discrete Allowable Prices

Smith et al. (1998) and Neelakantan et al. (2007) are the only two papers that fall within this category in the existing literature. Smith et al. (1998) use discrete scenarios to model uncertain demand and consider three possible demand scenarios, i.e., most likely, high and low. They use a multiplicatively separable demand function that models the impact of seasonal effects, price and advertisement. The problem is formulated as a mixed integer program and solved by a commercial solver. Unlike most other papers in the literature, they incorporate inventory holding costs and consider two commonly-used business rules for promotion plan: (i) markdown in two consecutive periods are not allowed; (ii) the total number of markdowns allowed for the entire planning horizon cannot exceed an upper limit. Their case study that involves a major department store chain demonstrates that compared to the buyer's original plan, one can increase the profit by roughly more than 20% if

one employs the optimal plan obtained from their model.

Neelakantan et al. (2007) develop two models for a clearance markdown pricing problem, i.e., a risk-neutral model and a risk-sensitive model. For the risk-neutral model in which they maximize the total expected revenue, the problem is formulated as a dynamic program. For the risk-sensitive model in which they maximize the total expected revenue subject to a constraint that the variance of the revenue cannot exceed a given limit, the problem is formulated as an integer program. They demonstrate by a case study that their models can be used in practice.

#### 2.2.1.2.5 Pricing Models with Demand Learning

All the papers we have reviewed in Sections 2.2.1.2.1 through 2.2.1.2.4 assume that given a price at a particular time, one can either perfectly predict the demand (deterministic case) or knows precisely the demand distribution (stochastic case). In practice, this may not always be the case especially in the context we consider for this survey, i.e., relatively short selling season, lack of information, and rapidly-changing demand pattern. In such a situation, implementing the optimal solution from a model with inaccurate demand information may incur significant revenue loss. Therefore, it is important and critical to update the initial estimate of demand according to the real-time sales data. There are several papers (i.e., Besbes and Zeevi 2009, Lin 2006, and Sen and Zhang 2009) that incorporate demand learning in the pricing decision. All three papers assume that demand arrives following a Poisson process. However, the exact way they model other unknown information is different.

Besbes and Zeevi (2009) study two problems with continuous time horizon and continuous allowable prices. In the first problem, they assume that the demand function belongs to a broad functional class and satisfy some regularity conditions, but the exact functional form is unknown. In the second problem they assume that the demand functional form is known with unknown parametric values. For the first problem, they propose a learning and pricing policy that consists of two phases, exploration phase and exploitation phase. In the exploration phase, different

price points are tested and the corresponding demand realizations are observed. In the exploitation phase, a single price is used and this single price corresponds to the “optimal” price based on observations in the exploration phase. For the second problem, a similar approach is proposed in which the number of price points tested is equal to the number of unknown parameters. For both problems, they use regret to measure the performance and derive lower bounds on the regret. They show their approaches is capable of achieving a regret that is close to the lower bounds.

Lin (2006) uses the Poisson reservation price model to characterize the underlying demand. They assume that customer reservation price distribution  $\mathbf{F}(p, t)$  is given while customer arrival rate  $\Lambda$  is unknown. They use gamma distribution to characterize firms’ knowledge about the customer arrival rate  $\Lambda$  before the selling season starts and update the distribution by incorporating real-time sales data. They show that the total number of customers within a period of time follows a negative binomial distribution. Based on their updated knowledge about the arrival rate, they propose a *variable-rate policy* based on the optimal solution from a surrogate model. When determining the price, this approach only considers the number of future customers while ignores their arrival times. Their numerical experiments demonstrate that the variable-rate policy achieves almost optimal solution and is also quite robust even when the initial knowledge about customers’ arrival rate deviates significantly from its true value.

Sen and Zhang (2009) assume discrete time horizon and discrete allowable prices. The demand is assumed to consist of two components, i.e., base demand  $\Lambda$  and a multiplier  $\psi(p)$  that captures how demand changes with price. The exact functional form of  $\psi(p)$  is unknown and assumed to be one of the given  $K$  functions with certain probability. The exact value of base demand  $\Lambda$  is also unknown with a given Gamma distribution. They use Bayes’ rule to update the knowledge about both  $\Lambda$  and  $\psi(p)$ . They show that the demand in each period follows a linear combination of  $K$  negative binomial distribution. Based on the updated distribution, they propose a dynamic program to solve for the optimal price in each period. Their numerical results show that compared to the pricing model without demand learn-

ing, their model is particularly beneficial when the initial estimate of demand rate is inaccurate, the actual demand mismatches supply, and demand is price-sensitive.

## 2.2.2 Multiple Products

Until very recently dynamic pricing problems that involve multiple products or/and multiple stores have received very little attention in the operations management literature. To the best of our knowledge, Gallego and van Ryzin (1997) is the first paper that investigates the dynamic pricing problem involving multiple products. In the last a few years there is an increasing amount of research interest in this problem. Recent papers that consider the dynamic pricing problems with multiple products include Akcay et al. (2010), Dong et al. (2009), Zhang and Cooper (2009), Suh and Aydin(2011), Maglaras and Meissner (2006), Bitran et al. (2006), Liu and Milner (2006), and Chen and Chen (2010). In the following, we first (in Section 2.2.2.1) give an overview of research by categorizing the existing literature according to some common model assumptions. We then (in Section 2.2.2.2) review each paper in details and summarize and compare their results.

### 2.2.2.1 Demand Models

We first describe the general problem using some common notations. Consider a firm that sells  $N$  products, indexed by  $n = 1, 2, \dots, N$ . At the beginning of the selling horizon, the firm is endowed with initial inventory  $C_n$  for product  $n$ . The length of the selling horizon  $T$  is fixed. Due to the long production lead time and relatively short selling horizon, no inventory replenishment is allowed during the selling horizon if any product stocks out. Any unsold items at the end of the horizon have zero salvage value. Problems with non-zero salvage value can be converted to an equivalent zero-salvage-value problem. In line with common assumptions in the pricing literature with short selling horizon, inventory holding cost and time discounting are ignored.

In the case where there are multiple (substitutable or complementary) prod-

ucts, the demand of one product not only depends on its own price and availability, but also the prices and availabilities of other products. Customers make a purchasing decision by comparing the prices and other non-price characteristics (e.g., quality, styles, features) of all products and choose the one that maximizes their utility. Clearly, changing the relative prices of different products may change customers' decisions with regard to which product to purchase (this is called *price-driven substitution*). When their preferred product is out of stock, customers might be willing to choose other products that are still available (this is called *inventory-driven substitution*) or leave the store without purchasing anything. In such a situation, optimizing the price for each individual product independently may result in a sub-optimal solution. In order to achieve the maximum possible total revenue, one needs to jointly optimize the prices of all products by taking into account the inventory levels of all products and the demand interdependency among all products. All papers in the existing literature consider substitutable products. We are unaware of any paper that investigates the dynamic pricing problem with complementary products. This can be an interesting topic for future research. Chen and Chen (2010) is the only paper that explicitly models both price-driven substitution and inventory-driven substitution. All other papers model price-driven substitution only. Chen and Chen (2010) is also the only paper that uses a robust optimization framework in which they maximize the worst-case revenue. All other papers assume risk-neutrality and maximize the expected revenue.

In the existing literature, all papers assume that demand is stochastic. When modeling uncertain demand, a common assumption made is that customers arrive following a Poisson process except one paper, i.e., Chen and Chen (2010). They use a set of lower and upper bounds to model uncertain demand. This requires only limited demand information instead of full knowledge about the probability distribution. We will discuss their demand model in details in the next section when we review each paper in details. For the rest of the papers, in general the demand model can be further categorized into three types according to the specific way they model the arrival rate. Demand model (i): the vector of demand

arrival rates (or intensities)  $\bar{\lambda} = (\lambda_1, \dots, \lambda_N)$  is a function of time  $t$  and the price vector  $\bar{r} = \{r_1, r_2, \dots, r_N\}$  at time  $t$ , i.e.,  $\bar{\lambda} = \Lambda(\bar{p}, t)$ . This includes the special case where the demand arrival rates are time-invariant. The demand function  $\Lambda(\cdot)$  satisfies some regularity conditions. Demand model (ii): the total demand arrival rate  $\lambda_t$  at time  $t$  for all products is given as an exogenous variable, and the prices of all products jointly determine the purchasing probability for a particular product,  $P_n$ ,  $n = 1, \dots, N$ , where  $\sum_{n=0}^N P_n = 1$  and  $P_0$  denotes the non-purchasing probability. The purchasing probability  $P_n$  is modeled differently in each paper and we give a detailed summary and comparison in the next section. Demand model (iii): the total demand arrival rate  $\lambda_t$  at time  $t$  for all products is a function of the common price  $r$  at time  $t$ . The probability that customers choose a particular product is given. Demand model (iii) only applies to a specific setting where all products have to be priced the same. For example, the same shirt with different colors or/and sizes are usually priced the same in most retail stores. Note that demand models (i) and (ii) are essentially equivalent if we let  $\lambda_n = \lambda_t P_n$ ,  $n = 1, \dots, N$ . Similarly if we do the same transformation, demand model (iii) can also be treated as a special case of model (i) under the condition that all products have to use a common price  $r$ .

Table 2.2 categorizes the existing models based on their model assumptions, i.e., time horizon, allowable prices, and demand model. Detailed review of each paper is provided in the next section (Section 2.2.2.2).

### 2.2.2.2 Review of Existing Literature

Gallego and van Ryzin (1997), and Maglaras and Meissner (2006) use demand model (i) to characterize the uncertain demand. The former assumes continuous time horizon and continuous allowable prices. The latter uses discretized time horizon and assumes continuous allowable prices. They both assume that there is a given initial stock of  $M$  ( $M$  is the number of resources and  $M = 1$  in Maglaras and Meissner, 2006) resources that can be used to produce  $N$  products. Due to the intractability of the proposed stochastic problem, they both rely on the solution to the deterministic counterpart of the stochastic problem to construct heuristics. Gallego

Table 2.2: Overview of Existing Models with Multiple Products

Time horizon	Allowable prices	Demand model	Paper(s)
Continuous	Continuous	Poisson model (i) Poisson model (ii) Poisson model (iii)	Gallego and van Ryzin (1997) Bitran et al. (2006) Liu and Milner (2006)
Discrete	Discrete	Bounds-based	Chen and Chen (2010)
Discretized continuous	Continuous	Poisson model (i) Poisson model (ii)	Maglaras and Meissner (2006), Akçay et al. (2010), Dong et al. (2009) Suh and Aydın (2011)
Discretized continuous	Discrete	Poisson model (ii)	Zhang and Cooper (2009)

Poisson model (i): price-dependent individual arrival rate

Poisson model (ii): price-independent total demand rate and price-dependent purchasing probability

Poisson model (iii): price-dependent total demand rate and price-independent purchasing probability

and van Ryzin (1997) show that the deterministic problem provides an upper bound on the stochastic problem. Based on the solution to the deterministic problem, they propose two heuristics. Heuristic 1 (make-to-stock policy): follow the deterministic price path and preassemble a certain amount of units of each product according to the deterministic solution. Heuristic 2 (make-to-order policy): follow the deterministic price path and assemble and sell products in the order they are received. They prove that both heuristics are asymptotically optimal as the expected sales volume approaches infinity. Maglaras and Meissner (2006) show that the dynamic pricing problem introduced by Gallego and van Ryzin (1997) and the capacity control problem introduced by Lee and Hersh (1993) can be reduced to a common formulation in which the firm controls the aggregate capacity consumption rate. Based on the deterministic solution, they propose three heuristics. Heuristic 1: apply the static price obtained from the deterministic problem over the entire planning horizon (corresponding to Heuristic 2 in Gallego and van Ryzin, 1997). Heuristic 2: apply the static price together with capacity control (make certain products unavailable when



the aggregate consumption rate exceeds its nominal value). Heuristic 3: apply the static price and resolve the deterministic problem on a rolling horizon basis. They show that all three heuristics are asymptotically optimal under the same condition as shown by Gallego and van Ryzin (1997).

Akçay et al. (2010), Dong et al. (2009), Zhang and Cooper (2009), Bitran et al. (2006), and Suh and Aydin (2011) use demand model (ii) to characterize the uncertain demand. Bitran et al. (2006) assume continuous time horizon and all other papers use discretized continuous time horizon. Zhang and Cooper (2009) assume discrete allowable prices and all other papers assume continuous allowable prices. They all assume that customers are utility-maximizers. Given a set of products,  $n = 1, \dots, N$ , customers will choose product  $n$  with probability  $P_n = Pr(U_n = \max_{m \in \{1, \dots, N\}} U_m)$ , where  $U_n$  denotes customers' utility for a particular product  $n$ .

Bitran et al. (2006) use two parameters to characterize a customer's purchasing behavior, the non-purchase utility  $U_0$  and her budget  $w$ . These two parameters are unknown to the firm and follow a given distribution  $F(w, U_0)$ . They also assume that the utility for each product  $U_n$ ,  $n = 1, \dots, N$  is known and can be ranked as follows,  $U_1 > U_2 > \dots > U_N$ . Given the prices for all products  $r_n$ ,  $n = 1, \dots, N$ , a customer will choose the product with the highest utility among those with a price below her budget. Therefore, the probability of choosing product  $n$  is  $F(r_{n-1}, U_n) - F(r_n, U_n)$ . To overcome the difficulty in solving the proposed stochastic problem, they investigate two special cases of the original problem. In special case 1, they assume that there is unlimited initial inventory. In special case 2, they assume that the demand is deterministic. They show that the deterministic solution is asymptotically optimal.

In Akçay et al. (2010), Dong et al. (2009), Zhang and Cooper (2009), and Suh and Aydin (2011), a customer's utility for a particular product  $n$  can be represented by a linear function  $U_n = \theta q_n - r_n + \mu \xi$ , where  $q_n$  is a general quality measure for product  $n$ ,  $\theta$  and  $\xi$  are two independent random variables,  $\theta$  measures customers' sensitivity to quality  $q_n$ , and  $\mu$  is a scalar. These three papers differ in how these parameters are specified. Table 2.3 summarizes the parameter specification in each

paper. Note that the specification in the H model of Akcay et al. (2010), Dong et al. (2009), and Suh and Aydin (2011) results in the well-known multinomial logit (MNL) model. All four papers formulate the problem as a stochastic dynamic program.

Table 2.3: Parameter Specification of Different Choice Models

$\theta$	$\mu$	$\xi$	Paper
Uniform distribution	0	N.A.	Akcay et al. (2010), V model
Deterministic	Positive	Gumbel distribution	Akcay et al. (2010), H model Dong et al. (2009) Suh and Aydin (2011)
Deterministic	Positive	Truncated normal distribution	Zhang and Cooper (2009)

Akcay et al. (2010) consider two choice models. In the V model, customers' valuations of the product attributes are uniform, e.g., customers always prefer a deluxe room over a standard room if the prices are the same. In the H model, customers' valuations of product attributes are idiosyncratic, e.g., some customers may prefer a blue shirt while others may prefer a white shirt if everything else is the same. They show that the V model and the H model have different pricing policy structure. For the V model, they show that the optimal price for a product is driven by its *aggregate inventory* which equals the total inventory of all products with higher quality plus its own inventory. For the H model, they show that the optimal price for a product is driven by its individual inventory. They also develop a polynomial time exact algorithm for the V model.

Dong et al. (2009) show that the optimal price of a product is the sum of the marginal value of inventory (in later periods) and the profit margin of an immediate sale (in the current period). They demonstrate by numerical examples that the monotonicity properties that hold in the single product case (as shown by Gallego and van Ryzin, 1994) may not hold in the multi-product case, i.e., the optimal price

may not necessarily decrease with time or inventory level. They also show that full-scale dynamic pricing significantly outperforms a unified static pricing or mixed dynamic pricing when the inventory is scarce.

Zhang and Cooper (2009) recognize the computational challenge in solving their model directly due to multi-dimensionality. Hence, they propose five heuristics. These heuristics are based on price pooling, inventory pooling, value approximation and policy approximation. Their numerical experiments reveal that: (i) the revenue loss by using a common price for all products can be quite significant; (ii) revenue loss due to restricting the price changes to be at pre-specified time points is small.

Suh and Aydin (2011) use a similar model framework as Dong et al. (2009). But they focus on the case with two products. They show that the marginal value of a product increases in the remaining time but decreases with its own inventory and the other product's inventory. The optimal price, however, does not hold the monotonicity property, i.e., the optimal price is not monotonic in the remaining time and the other product's inventory level. This is consistent with the finding by Dong et al. (2009) for the multiple-product case.

Liu and Milner (2006) use demand model (iii) to characterize the uncertain demand. They assume continuous time horizon and continuous allowable prices. Their model differs from all other models in an additional constraint that requires all products to have a common price at any time despite the fact that the demand for each product might be different. This problem arises in certain situations where the products only differ in some minor attributes. In this case, firms are reluctant to annoy the customers by pricing those products differently. For example, when a firm sells the same type of shirt with different colors or sizes, they are usually priced identically. They demonstrate that the monotonicity property that holds for the single-product case may not hold in their case, i.e., the optimal price may not necessarily increase with a reduction in inventory. This creates challenge for solving the proposed problem. Therefore, they examine two special cases. For special case 1 where the demand rate is a deterministic function of the price, they show that a  $N$ -segment policy is optimal (recall  $N$  is the number of products). To be

more specific, the planning horizon can be divided into  $N$  segments. Within each segment, a single price is charged to deplete the inventory of a particular product. They show that the price is decreasing over time. The solution to the deterministic problem provides an upper bound for the original stochastic problem. For special case 2 where the demand is stochastic but only a single price markdown is allowed, they show a threshold policy is optimal. Specifically, the higher price is charged first. Whenever the remaining time falls below the threshold (which depends on the current inventory level), it is optimal to switch to the lower price. The solution provides an lower bound for the original problem. For the general problem, they develop four heuristics based on the solution obtained from the two special cases. They show that these heuristics perform well and are asymptotically optimal.

Chen and Chen (2010) assume discrete time horizon and discrete allowable prices. In their paper, they consider two substitutable products. In contrast to all other papers, they assume that there is not enough information to accurately characterize the underlying demand distribution. Instead, they use three types of bounds to model uncertain demand for any given prices: (i) lower and upper bounds for the demand of each individual product; (ii) lower and upper bounds for the total demand of the two products; and (iii) lower and upper bounds for the total demand of the two products from period 1 to period  $t$ . These bounds define the uncertainty space and enable them to model demand substitution between the two products and across time periods. Their model also differs from all other models in that it incorporates commonly-used business rules. These rules are due to established market norms and have been largely ignored in the existing literature. They develop a robust optimization framework in which they maximize the worst-case revenue. They formulate the problem as a dynamic program. They demonstrate that lower demand may not necessarily generate lower revenue, which implies that the entire uncertainty space needs to be searched in order to find the worst-case demand. This creates computational challenge for solving large problems. To expedite the running speed, they further develop a Fully Polynomial Time Approximation Scheme (FP-TAS) that delivers a proven near-optimal solution in a manageable computational

time for practically sized problems. Their numerical experiments show that compared to the risk-neutral solution that maximizes the expected revenue, the robust solution not only increases the worst case revenue, but also significantly reduce the variance of total revenue generated, while the mean revenue loss is small.

### 2.2.3 Multiple Stores

In a case where a single firm operates multiple stores, pricing and inventory decisions need to be coordinated among stores and optimized jointly in order to maximize the total revenue. To the best of our knowledge, in the existing literature there are only two papers that investigate the dynamic pricing problems with multiple stores, i.e., Bitran et al. (1998) and Chen et al. (2011).

Both papers assume discrete time horizon. Bitran et al. (1998) assume continuous allowable prices while Chen et al. (2011) assume discrete allowable prices. In terms of the demand model, Bitran et al. (1998) use Poisson process to model uncertain demand. Although their solution approach does not rely on this assumption, it does require precise knowledge about the underlying demand distribution. Chen et al. (2011), on the other hand, use scenario tree to model uncertain demand. This requires only limited demand information as opposed to the full knowledge about the probability distribution. It also enables them to model demand correlation across time periods and among stores.

Retail chains make pricing and inventory decisions differently. Even for the same retail chain, pricing and inventory decisions are made differently for different products. There are two types of pricing mechanisms commonly seen in practice (Hruschka, 2007, and Shankar and Bolton, 2004). Pricing mechanism (i): all stores on the same retail chain use a common price for the same product. Pricing mechanism (ii): different prices are used for different store locations but stores within the same geographical region have to use similar prices. Pricing mechanism (i) allows firms to maintain a corporate image while pricing mechanism (ii) gives firms more flexibility in setting up the prices according to the demand pattern of each individual store, and thus increase their total revenue. These two mechanisms coexist

in practice even for the same retail chain. For example, at the beginning of the selling season, products are priced the same (regular selling price) but at the end of the season, different discounts are applied at different store locations according to the corresponding sales pattern. There are three types of mechanisms commonly encountered in practice for inventory allocation. Inventory allocation mechanism (i): initial inventory is allocated to each store in the very beginning without further inventory redistribution among stores. Inventory allocation mechanism (ii): inventory is allocated to each store in the very beginning and then it is rebalanced among stores to respond to the sales pattern at each individual store. Inventory allocation mechanism (iii): inventory is mostly kept in the central warehouse and it is delivered to each store period by period according to the sales pattern. In terms of pricing and inventory decisions, these two papers complement each other. Bitran et al. (1998) investigate pricing mechanism (i) and inventory allocation mechanisms (i) and (ii). Chen et al. (2011) investigate pricing mechanism (ii) and inventory allocation mechanism (iii).

In terms of mathematical formulation and solution approach, Bitran et al. (1998) formulate the problem as a stochastic dynamic program. Due to the dimensionality issues, they develop several heuristics based on their stochastic DP formulation. They show that their methodology significantly outperforms the expertise of a product manager of a large retail chain in Chile. Chen et al. (2011) formulate the problem as a mixed integer program. They propose a Lagrangian relaxation approach, which decomposes their original large problem into many small problems. They also demonstrate that their approach performs significantly better than those commonly used in practice. Both papers have found that when the demand uncertainty is high, one should start with higher initial price and then adjust the price according to the realized demand. Chen et al. (2011) show that in the multiple-store case, two price markdowns are in general enough to achieve satisfactory performance as long as one chooses the right price as well as the right time to make a price change. This result extends the finding by Gallego and van Ryzin (1994) who show that in the single-store case, one price change is in general enough.

Chen et al. (2011) also quantify the revenue impact of using pricing mechanism (i) versus mechanism (ii). They demonstrate by numerical experiments that it may incur significant revenue loss if one has to use a common price for all stores in a situation where stores vary significantly in terms of price-sensitivity. Their numerical results also imply that one should offer more price markdowns, deeper discount for stores that are more price-sensitive. Correspondingly, one should allocate more inventory to those stores relative to their expected demand.

### 2.3 Models with No Competition and Strategic Customers

Early work on dynamic pricing as those reviewed in Section 2.2 assumes that customers are myopic in the sense that they make purchasing decisions purely based on the current product price and availability. Customers will purchase a product as long as the product is available and their valuation of that product is higher than the selling price at the time of arrival without considering possible future price and availability. In other words, the demand can be represented as a function of current price only.

Often times, this may not be the case in a real market where customers can easily observe and predict (with certain level of accuracy) the price dynamics according to their own shopping experience or with the aid of advanced information technology. For example, as markdown pricing becomes a common practice for consumer electronics products at the end of their life-cycles or fashion apparel at the end of their selling season, experienced customers who expect a decreasing price pattern may only purchase in the “clearance period” to maximize their surplus. Even in a situation where the price path may not be monotone over time such as in the airline industry, customers may still be able to find an “appropriate” time to purchase by resorting to some online deal forum or prediction websites. For instance, if one searches the price for an airline ticket from the airfare prediction website **Bing Travel** (<http://www.bing.com/travel/>), it not only provides a list of prices from different websites, but also predicts whether the fare will increase, stay

steady, or drop (see Figure 2.1). In such a situation, some customers may behave strategically in the sense that they may delay their purchase in anticipation for a possible price reduction in the future. Such customers are referred to as strategic customers in this chapter, and are also referred to as rational customers or forward-looking customers in other papers. When determining the price schedule over time, ignoring such strategic customer behavior may result in sub-optimal solutions and cause substantial revenue loss.

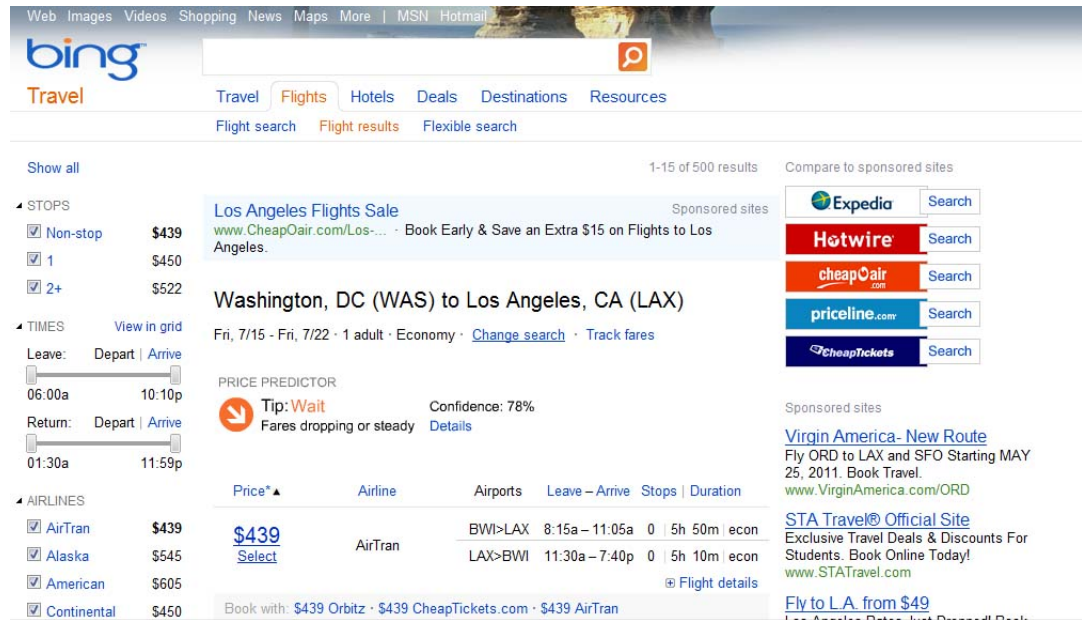


Figure 2.1: Airfare Prediction at Bing Travel

Research on dynamic pricing problems with consideration of strategic customer behavior first appeared in the economics literature. Stokey (1979) is the first paper that looks into this issue. She considers a continuous time model where the price may change continuously over time and shows that price discrimination is not the optimal strategy for a large class of consumer utility function. Other papers include, for example, Stokey (1981) and Harris and Raviv (1981). All these papers either assume infinite initial inventory (Stokey, 1979, 1981) or focus on the design of optimal market mechanism (Harris and Raviv, 1981), so they are out of the scope of this survey and



are not reviewed.

Dynamic pricing by incorporating customer behavior has received little attention in the operations management literature until very recently. In the last several years, we have observed a rapidly increasing interest in this area. Papers that study the dynamic pricing problem with explicit consideration of strategic customer behavior include Zhang and Cooper (2008), Su (2007, 2010), Aviv and Pazgal (2008), Dasu and Tong (2010), Elmaghraby et al. (2008), Elmaghraby et al. (2009), Levin et al. (2010), Levina et al. (2009), Yin et al. (2009), Liu and van Ryzin (2008), Cachon and Swinney (2009), Lai et al. (2010), Bansal and Maglaras (2009), Ovchinnikov and Milner (2011), Gallego et al. (2008), and Cho et al. (2008). In the following, we first (in Section 2.3.1) categorize the existing literature according to common modeling assumptions. We then in Section 2.3.2 summarize some of the major findings and managerial insights from the existing literature. In the end (in Section 2.3.3), we review each paper in detail.

### 2.3.1 Categorization of Existing Literature

In a dynamic pricing problem with strategic customers, the demand can no longer be simply represented as a function of current price only. A commonly-adopted approach is to use a game theoretic framework where the firm acts as a Stackelberg leader and determines the sales policy (e.g., pricing, initial-inventory decisions) first, and then the customers act as followers and choose when to buy to maximize their expected utilities by taking the firm's decision as well as other customers' decisions into account. In such a setting, both the firm's sales policy and the resulting demand shall be viewed as the consequence of the equilibrium of the game between the firm and the customers.

To model such a game, a majority of the existing papers use a two-period model where the first period can be viewed as the regular selling season when a premium price is charged, and the second period can be viewed as a clearance season when a discount price is charged. A two-period model is adequate to capture the main elements of this game if one's objective is to derive general high-level managerial

insights. To better represent a real market and provide a satisfactory solution to a real-world problem, some papers use a multi-period model with a general number of periods (e.g., Dasu and Tong, 2010) or a continuous time horizon model (e.g., Levin et al., 2010).

There are two types of pricing policies studied in the literature, a *preannounced pricing* policy and a *contingent pricing* policy. In a preannounced pricing policy, the firm determines and announces all the prices for the entire selling season at the very beginning without considering possible future sales. In a contingent pricing policy, the firm dynamically adjusts the price over time according to the realized sales history and level of remaining inventory. In a market setting where all customers are myopic, a contingent pricing policy is clearly better than a preannounced pricing policy as the preannounced policy lacks the flexibility in reacting to sales realization. In a market setting where customers may behave strategically, it is not immediately clear which policy is better. For example, under a contingent pricing policy, a large number of customers may decide to wait as they anticipate that the firm may use a much lower price near the end of the season; whereas under a preannounced pricing policy, if the firm commits upfront to only a small price discount in the markdown period, it may discourage strategic waiting and thus increase sales in the regular season. In the existing literature, some papers consider markdown pricing only where the price is non-increasing over time. Whereas some other papers consider a more general pricing setting where the price is allowed to go up or down.

In some situations, a firm's sales policy may also involve rationing decisions (i.e., control the product availability and fulfill only a fraction of demand). By limiting the product availability in the sales period, the firm can discourage strategic waiting and induce more customers to purchase in the regular selling season. Typically there are two ways to achieve this: (i) limit the initial inventory (or capacity); or (ii) fulfill only a fraction of demand in the markdown period even if there is sufficient inventory. We refer to the former as *initial inventory rationing* and the latter as *sales rationing*.

In Table 2.4, we categorize existing literature based on their assumptions on

time horizon as well as the firm’s pricing and rationing policies considered by each paper. In addition to pricing and rationing, firm may also use other tools such as *quick response* (Cachon and Swinney, 2009), *inventory display format* (Yin et al., 2009), *price matching* (Lai et al., 2010), and *reservation regime* (Elmaghraby et al., 2009) to mitigate the negative impact of strategic customer behavior. We will discuss these tools when we review individual papers in Section 2.3.3

Given a firm’s sales policy, the demand is determined by both the customers’ arrival process and their purchasing behavior. A customer’s purchasing behavior is primarily determined by the customer’s type (myopic or strategic), her valuation of the product and risk attitude. In the following, we summarize how these characteristics are modeled in the existing literature.

There are four types of customer arrival process considered in the literature. Type (i) (which we call *static* arrival process): the total number of potential customers is known and all customers are present at the beginning of the selling season. Type (i) arrival process is most commonly assumed in the existing literature. Type (ii) (which we call *Poisson* arrival process): customers arrive following a Poisson process with a given rate. Type (iii) (which we call *random aggregate* arrival process): the aggregate number of potential customers in each period is represented by a random number (or a combination of random and deterministic numbers). Type (iv) (which we call *continuous constant* arrival process): customers arrive continuously according to a deterministic flow with a given constant rate.

In terms of customer composition, most papers assume that all customers are strategic while the remaining assume that there is a mixture of strategic and myopic customers. We note that in the general models of Su (2007), Levin et al. (2010) and Levina et al. (2009), all customers are strategic. However, in a limiting case, the customers in their models can become myopic. In Su (2007) where waiting incurs a waiting cost  $b$  per unit time, when  $b = \infty$ , a customer will purchase the product without waiting if her valuation is higher than the current selling price and hence behave like a myopic customer. In Levin et al. (2010) and Levina et al. (2009), a customer’s utility from a future purchase is discounted by a factor  $\beta \in [0, 1]$  per

Table 2.4: Modeling Firm's Policy

Time Horizon	Pricing Scheme		Rationing	Paper(s)
Two-period	Markdown	Contingent	Initial-inventory	Cachon and Swinney, 2009 Lai et al., 2010
			No	Aviv and Pazgal, 2008
		Preannounced	Initial-inventory	Liu and van Ryzin, 2008 Yin et al., 2009
			Sales	Zhang and Cooper, 2008 Ovchinnikov and Milner, 2011 Gallego et al., 2008
			No	Aviv and Pazgal, 2008 Elmaghraby et al., 2009
	General	Contingent	Initial-inventory	Su, 2010
			No	Su, 2010
	Multi-period	Markdown	Preannounced	Initial-inventory
No				Elmaghraby et al., 2008 Dasu and Tong, 2010
General		Contingent	Initial-inventory	Dasu and Tong, 2010
			No	Dasu and Tong, 2010
		Preannounced	Sales	Bansal and Maglaras, 2009
Continuous		General	Contingent	No
	Preannounced			Initial-inventory
	Preannounced		Sales	Su, 2007
	Preannounced		Sales	Su, 2007

decision period. In a limiting case where  $\beta = 0$ , future purchase has zero value to a customer and hence the customer can be viewed as a myopic customer.

The vast majority of the papers assume that customers know their own valuation of the product over the entire course of the selling season. Levin et al. (2010) and Levina et al. (2009), on the other hand, assume that customers only know their valuation of the product in the current period and the valuations of the product in future periods are unknown and independent of their valuation in the current period.

In terms of the firm's knowledge about the customers' valuations, there are two cases: deterministic and stochastic. In the deterministic case, customers' valuations are known to the firm. In other words, given any price  $p$ , the firm knows exactly the number of customers whose valuation is higher than  $p$ . In this case, if all customers are myopic, the demand can be represented as a deterministic function of the price. It should be noted that even in the deterministic case, the firm does not know each individual customer's valuation. There are a few exceptions (e.g., the base model in Elmaghraby et al., 2009) where all customers are assumed to have an identical valuation. In this case the firm does know each customer's valuation. In the stochastic case, customers' valuations are unknown to the firm and follow a given probability distribution that is known to the firm. Most papers assume that customers' valuations stay constant over time while a few papers assume that the valuations may decline over time. The latter case is prevalent in the sales of electronic products at the end of their life-cycles and fashion or seasonal products at the end of the season. Note that in Levin et al. (2010) and Levina et al. (2009), customers' valuation is independent across time.

Given a firm's pricing policy, a strategic customer chooses when to purchase the product in order to maximize her expected utility. A commonly-used utility function is defined as  $u = (v - p)^\gamma$ , where  $u$  is the customer's utility,  $v$  is the customer's valuation of the product and  $p$  is the selling price. The case with  $\gamma = 1$  models the situation where the customer is risk-neutral and maximizes her expected consumer surplus,  $v - p$ . The case with  $0 \leq \gamma < 1$  models the situation where the

customer is risk-averse.

Table 2.5 classifies existing literature based on the structure of the strategic customer behavior which consists of customer arrival process, customer composition, firm's knowledge about customers' valuation, and customers' risk attitude.

Table 2.5: Modeling Strategic Customer Behavior

Arrival Process	Composition	Firm's Knowledge about Customer Valuation		Risk Attitude	Paper(s)
Static	All strategic	Deterministic	Constant	Risk-averse	Bansal and Maglaras, 2009
				Risk-neutral	Elmaghraby et al., 2008
		Stochastic	Constant	Risk-averse	Liu and van Ryzin, 2008
				Risk-neutral	Dasu and Tong, 2010 Elmaghraby et al., 2008 Gallego et al., 2008
			Declining	Risk-neutral	Gallego et al., 2008
			Independent	N/A	Levin et al., 2010 Levina et al. 2009
	Mixed	Deterministic	Constant	Risk-neutral	Zhang and Cooper, 2008
		Stochastic	Constant	Risk-neutral	Gallego et al., 2008
Poisson	All strategic	Deterministic	Constant	Risk-neutral	Elmaghraby et al., 2009
		Stochastic	Constant	Risk-neutral	Elmaghraby et al., 2009 Yin et al., 2009
			Declining	Risk-neutral	Aviv and Pazgal, 2009
	Mixed	Stochastic	Constant	Risk-neutral	Cho et al., 2008
Random aggregate	All strategic	Stochastic	Constant	Risk-neutral	Gallego et al., 2008
	Mixed	Deterministic	Constant	N/A	Su, 2010 Ovchinnikov and Milner, 2011
			Declining	Risk-neutral	Lai et al., 2010
		Stochastic	Declining	Risk-neutral	Cachon and Swinney, 2009
Continuous constant	All strategic	Deterministic	Constant	Risk-neutral	Su, 2007

## 2.3.2 Main Findings and Managerial Insights

In this section, we summarize the main findings and managerial insights reported in the existing literature, structured around the the following four questions: (i) how does strategic customer behavior impact a firm’s profit? (ii) how does strategic customer behavior affect a firm’s optimal pricing decision? (iii) how effective can a rationing policy be? (iv) are there other strategies that can help a firm mitigate the impact of strategic customer behavior? In Section 2.3.3, we present a detailed discussion of main results in each existing paper.

### 2.3.2.1 Impact of Strategic Customer Behavior on a Firm’s Profit

Dynamic pricing enables a firm to achieve market segmentation and hence increase its profit by exploiting heterogeneity of customer valuations and setting different prices for different time periods. However, when customers anticipate a certain pricing pattern, some of them may delay their purchase if they expect that a future purchase may generate a higher utility. Aviv and Pazgal (2008) demonstrate by a numerical example that more than 60% of customers whose valuation of the product is higher than the selling price at the time of arrival may end up waiting in anticipation for a lower price. This type of strategic behavior will certainly suppress the benefit of customer segmentation and thus substantially hurt a firm’s revenue performance even if the firm properly accounts for it when determining the optimal pricing policy, as demonstrated by many papers, e.g., Aviv and Pazgal (2008), Cachon and Swinney (2009), Levin et al. (2010). One may expect that when the initial supply is very limited, customers become more concerned about the product availability in the clearance period and their behavior is close to the myopic case. However, Aviv and Pazgal (2008) show that, even in this case, the firm cannot avoid the negative impact of strategic customer behavior (more than 20% of revenue loss in their numerical example). It is interesting to note that unlike in the myopic case, when customers behave strategically, holding more initial inventory may actually decrease the revenue that a firm can collect (Levin et al., 2010).

When making optimal pricing decisions, ignoring strategic customer behavior by incorrectly assuming that customers are myopic may result in significant revenue loss. For example, Zhang and Cooper (2008), Dasu and Tong (2010), Aviv and Pazgal (2008), and Cachon and Swinney (2009) show through numerical examples in the context of their specific models that the firm's profit loss due to ignoring strategic customer behavior can reach up to 11%, 13%, 21%, 90%, respectively. Revenue loss due to ignoring strategic customer behavior cannot be avoided even if demand learning can be incorporated (Levina et al., 2009).

Although strategic waiting by high-valuation customers hurts a firm's profit, strategic waiting by low-valuation customers, on the other hand, may benefit the firm as shown by Su (2007) and Cho et al. (2008). The net impact depends on the composition of customer population and other model parameters. Su (2007) argues that as more strategic low-valuation customers choose to wait, it increases the competition for product availability in the clearance period when the initial inventory is limited, and thus induces high-valuation customers to purchase earlier at a higher price. Therefore, it helps the firm achieve market segmentation and extract more revenue. Similarly, Cho et al. (2008) argue that low-valuation customers who find that the product price is higher than their valuation might be able to purchase the product later if they strategically choose to wait instead of leaving immediately. This will increase the sales in the clearance period and thus may increase the total revenue collected by the firm.

### 2.3.2.2 Impact of Strategic Customer Behavior on a Firm's Optimal Pricing Decisions

When strategic customers are present, intuition suggests that the firm should reduce the difference between the regular price and the markdown price, as it may induce more customers to purchase at the regular price. This is formally proved by Cachon and Swinney (2009) in the context of their two-period model where the price for the first period is given. In addition, Liu and van Ryzin (2008) show that when



customers are risk-neutral, a single price policy is optimal; however, when customers are risk-averse, a high-low type of pricing policy (high price in the regular season and low price in the clearance season) is always optimal.

Levin et al. (2010) demonstrate that the monotonicity properties (i) - (iv) described in Section 2.2 for problems with myopic customers only may not hold in general when strategic customers are present except for two special cases: customers have limited rationality or initial inventory exceeds potential demand. Su (2007) shows that a pure markup, a pure markdown, or a mixed pricing policy may be optimal depending on the composition of the customer population and some other problem parameters.

Aviv and Pazgal (2008) show that when customers differ substantially in their valuation and their valuations decline slowly over time, dynamic pricing may not be effective in achieving market segmentation. In other words, a static fixed-price policy in this case can be near-optimal. Su (2010) proves that the presence of speculators may benefit the firm. In a situation where the firm is unable to implement a dynamic pricing policy due to fairness and equity consideration, the presence of speculators in the resale market enables the firm to achieve the benefit of dynamic pricing while actually maintaining a fixed-price over the entire selling season.

It is clear that when customers are all myopic, an optimal contingent pricing policy always performs better than or at least equally well as preannounced pricing policies. Levin, et al. (2010) show through numerical examples that preannounced policies always perform worse than fully dynamic ones by up to 4% when initial inventory is low, and by up to 1% when initial inventory is high. Aviv and Pazgal (2008) find in the context of their model that preannounced policies perform essentially the same as contingent policies. However, when facing strategic customers, a preannounced pricing policy could perform better than a contingent pricing policy depending on market characteristics under which the firm operates. Dasu and Tong (2010) show that neither policy dominates the other and the performance gap between the two is small (up to 1.6%) based on their numerical experiment. Aviv and Pazgal (2008) find that a preannounced pricing policy outperforms a contin-

gent pricing policy under certain conditions, including when the initial inventory is high and the customers are highly heterogeneous. Levin et al. (2010) show that the relative performance of the two policies depend on the initial inventory level. Specifically, when the initial inventory is low, a contingent pricing policy outperforms a preannounced pricing policy; whereas when the initial inventory is high, the reverse is true. Cachon and Swinney (2009) find that although a preannounced pricing policy can be substantially better than a contingent policy in certain cases, for the vast majority of the numerical examples they tested, a contingent pricing policy outperforms a preannounced pricing policy. Under a price matching policy, Lai et al. (2010) show that a contingent pricing policy performs better than a preannounced pricing policy.

### 2.3.2.3 Effectiveness of Rationing

As we have discussed earlier, rationing can be achieved either by limiting the initial inventory (e.g., Liu and van Ryzin, 2008) or by controlling the sales in the clearance period even if inventory is available (e.g., Zhang and Cooper, 2008). Zhang and Cooper (2008) find that rationing policy may only provide secondary benefit for improving a firm's revenue compared to the pricing policy. Rationing policy only works when the prices are given and not optimal, and it never completely compensates for improper pricing decisions. Liu and van Ryzin (2008) show that under their model assumptions, rationing is not profitable when customers are risk neutral. However, when customers are risk averse, they show that when the prices are given, rationing tends to be an optimal strategy when the number of high-valuation customers is large, the level of risk-aversion is high and the price difference between the regular season and markdown season is large. However, when the firm has the flexibility to optimize the prices, rationing is always an optimal strategy. They also show that rationing may not work in a competitive market.

### 2.3.2.4 What Else May Play a Role When Strategic Customers Are Present

In addition to pricing and rationing, firms may adopt other strategies to mitigate the negative impact of strategic customer behavior. Examples of other strategies include quick response (i.e. making an additional order of inventory at the beginning of the first period after observing the first-period demand) (Cachon and Swinney, 2009), hiding inventory information (Yin et al., 2009, and Dasu and Tong, 2010), price matching (Lai et al., 2010), and use of a reservation regime (i.e., the firm allows a customer to reserve the product for purchase in the clearance period and the customer is obligated to buy it if the product remains unsold in the clearance period.) (Elmaghraby et al., 2009). It was found that quick response and price matching strategies are very effective in reducing the negative impact of strategic waiting and improving a firm's profit. Numerical examples (Cachon and Swinney, 2009, Lai et al., 2010) demonstrate that one can increase the profit by up to 25% and 35% if one adopts quick response, and price matching, respectively. On the other hand, hiding inventory information only improves the revenue slightly (up to around 1.2% in Dasu and Tong, 2010, and up to 1.7% in Yin et al., 2009). It is also interesting to note that by allowing the customers to reserve the product in the clearance period, it will always improve the firm's revenue while it may reduce customers' expected surplus when all customers have identical valuation and behave strategically (Elmaghraby et al., 2009).

### 2.3.3 Review of Existing Literature

In this section, we present a detailed review of existing literature. For each paper, we first describe the model considered and then summarize its major findings and managerial insights.

Aviv and Pazgal (2008), Zhang and Cooper (2008), and Liu and van Ryzin (2008) all consider two-period markdown pricing problems with strategic customers. In Aviv and Pazgal (2008), customers arrive to the store following a Poisson pro-

cess with a given rate  $\lambda$ . Each customer  $j$  has a valuation of the product which depends on her base valuation  $V_j$  drawn from a given probability distribution and time of purchase  $t$ ,  $V_j(t) = V_j \cdot e^{-\alpha t}$ , where  $\alpha$  is the rate of decline. All customers behave strategically and choose when to purchase the product in order to maximize their expected consumer surplus. The authors investigate both contingent and preannounced pricing strategies. Their major findings include the following: (i) if the starting time of price markdown can be optimally chosen, when the variance of customers' base valuations is small while the rate of decline is large, price segmentation in the case of strategic customers can be as effective as in the case of myopic customers; (ii) if the starting time of price markdown is fixed, when the variance of customers' base valuations is large while the rate of decline is small, strategic customer behavior may significantly suppress the benefits of price segmentation; (iii) the firm cannot avoid the negative impact of strategic customer behavior even when the initial inventory level is low; (iv) in the case of myopic customers, the preannounced pricing strategy performs equally well as the contingent pricing strategy; (v) in the case of strategic customers, under certain conditions, the preannounced pricing strategy is advantageous to the firm compared to the contingent pricing strategy; (vi) in most cases where the preannounced pricing strategy significantly outperforms the contingent pricing strategy, its performance is similar to a fixed-price strategy; (vii) ignoring strategic customers when determining the optimal pricing policy can be quite costly. In Zhang and Cooper (2008), the firm determines both pricing and (sales) rationing policies. Only a proportion of the customers are strategic while the remaining are myopic. All customers arrive at the beginning of the first period. The authors characterize the demand as a deterministic function of price and specifically focus on two types of demand functions, linear and exponential. They consider both cases of infinite initial inventory and finite initial inventory. For each case, they consider flexible pricing (i.e. prices are decision variables) and fixed pricing (i.e. prices are given). Their primary insights include the following: (i) when the initial inventory is infinite, if the firm can choose the optimal prices, rationing does not improve revenue; (ii) ignoring strategic customer behavior results in lower

prices in both periods, and significant revenue loss especially when the proportion of strategic customers is large; (iii) when the prices are given, rationing may benefit the firm; (iv) rationing can never completely compensate for the revenue loss due to improper pricing decisions. In Liu and van Ryzin (2008), the firm determines both the pricing and (initial-inventory) rationing policies. The prices for the two periods are preannounced. All customers are strategic. A given number of customers are present at the beginning of the selling season with each customer's valuation drawn from a common probability distribution. The firm is risk neutral and maximizes the expected profit. Customers are risk-averse and maximize their expected utility by deciding when to purchase the product. The authors show that: (i) when the prices are given, in general, rationing tends to be an optimal strategy for the case with large number of high-value customers, high level of risk-aversion and large price difference between the two periods; (ii) when the firm can optimally choose the prices, then rationing is always an optimal strategy; (iii) competitive market does not favor rationing as an optimal strategy.

Elmaghraby et al. (2008), Dasu and Tong (2010), and Bansal and Maglaras (2009) all consider multi-period dynamic pricing problem where the firm has  $K$  units of a single product to sell to  $N$  potential customers which are all strategic and present at the beginning of the selling season. In Elmaghraby et al. (2008), each customer has a valuation of the product which is constant over time and each customer may demand multiple units which can be satisfied across multiple time periods. The firm preannounces a markdown pricing scheme and the initial inventory before the season starts. The authors consider two settings, a complete information setting where the firm knows the set of customer valuations, but does not know each individual customer's valuation, and an incomplete information setting where the firm knows that customers' valuations are drawn from given nonoverlapping intervals. Their major findings include: (i) under complete information setting, a two-step pricing is optimal; (ii) under incomplete information setting, if at most one price will occur within each customer's valuation range, then at most three price steps are needed; (iii) under both settings, the buyer will submit all-or-nothing bid

at each price step. In Dasu and Tong (2010), each customer has a valuation of the product drawn from a given probability distribution and her valuation remains constant over time. They examine both preannounced and contingent pricing policies. Their main findings include the following: (i) when initial inventory exceeds demand ( $K \geq N$ ), a single price (static pricing) is optimal; (ii) for both preannounced and contingent pricing schemes, the structure of the Bayesian Nash Equilibrium is a threshold-type; (iii) ignoring strategic customer behavior will result in significant revenue loss; (iv) when customers are strategic, neither preannounced pricing scheme nor contingent pricing scheme dominates the other and their performance gap is small; (v) preannounced pricing scheme with two or three price changes can generate close-to-optimal revenue; (vi) firms benefit slightly from hiding the initial inventory level. In Bansal and Maglaras (2009), before the selling season starts, the firm preannounces its sales policy consisting of the price  $p_t$  and sales rationing fill rate  $r_t$  for each period  $t$ . There are  $N$  customer types and each type  $i$  differs in its valuation  $v_i$  and risk aversion parameter  $\gamma_i$ . Given the firm's policy, customers strategically time their purchase to maximize their expected utility  $(v_i - p_t)^{\gamma_i} r_t$ . All customers within the same type will make the same decision and there is no strategic interaction among customers. They show that their dynamic pricing problem can be formulated as static mechanism design problem. They also demonstrate that when customers' risk aversion is low, a two-price policy is near-optimal.

Su (2007) studies a general deterministic dynamic pricing problem with continuous time horizon. The monotonicity of price change is not explicitly required; instead, it is driven endogenously by customers' purchasing behavior. Customers arrive continuously over time with a deterministic constant rate and differ in two dimensions: valuation (high or low) and waiting cost (patient or impatient). Pricing and rationing policies are preannounced before the selling season starts. Upon arrival, each customer chooses to purchase immediately, exit the market, or wait for a lower price with waiting cost incurred per unit time. His main findings include: (i) the structure of the optimal pricing policies (pure markup, pure markdown, or mixed) is driven by the composition of customer population, which provides al-

ternative explanations for why markups are common in the travel industry while markdowns are common in the fashion industry; (ii) strategic customer behavior may benefit the firm because patient-low-type customers compete with high-type customers for product availability, which discourages high-type customers from waiting; (iii) optimal selling policy is socially efficient in the sense that the sum of the firm’s revenue and total consumer surplus is maximized.

Levin et al. (2010) and Levina et al. (2009) all study stochastic dynamic pricing problems with continuous time horizon and the following main characteristics: (i) all customers are strategic and arrive at the beginning of the horizon, (ii) customers’ valuations at each time point are drawn from a common known time-varying probability distribution and are independent across time, (iii) contingent pricing scheme is used, (iv) no rationing is considered. Levin et al. (2010) consider a monopoly market where the firm has  $K$  units to sell to  $N$  potential customers. Each individual customer controls its shopping intensity and the aggregate demand intensity is the sum of each individual’s shopping intensity. Each customer purchases at most one unit and times their purchase (by controlling the shopping intensity) to maximize their expected present value of utility. To make the problem tractable, the authors discretize the continuous time horizon and formulate the problem as a stochastic dynamic game. Their primary findings include: (i) the monotonicity properties (i) - (iv) described in Section 2.2 for problems with myopic customers do not hold in general; (ii) the monotonicity properties hold when customers have limited rationality or when initial inventory exceeds potential demand ( $K \geq N$ ); (iii) in the presence of strategic customers, revenue may not necessarily increase as the initial inventory increases; (iv) ignoring strategic customer behavior may result in significant revenue loss except for the case when the initial inventory is relatively low; (v) contingent pricing outperforms (underperforms) a preannounced pricing policy when initial inventory is low (high); (vi) when initial inventory is a decision variable, strategic customer behavior results in lower optimal initial inventory; (vii) proper initial inventory decision can partially compensate for the impact of strategic behavior. Levina et al. (2009) extend the work by Levin et al. (2010) by incor-

porating demand learning. In their model, the customer behavior is specified by a constant parameter vector which may include unknown components. The knowledge about the distribution of these unknown components is updated periodically according to the realized sales data by some algorithm. Pricing decisions are re-optimized by using simulation-based optimization. Their approach does not require specific distributional assumptions. Computational results demonstrate that the proposed approach is robust to deviation from actual market if learning is incorporated. They also show that ignoring strategic customer behavior may result in inferior solution even if demand learning is incorporated.

Several papers consider some other strategies jointly with dynamic pricing and rationing decisions in the presence of strategic customers, including Cachon and Swinney (2009) considering quick response, Lai et al. (2010) considering price matching, Elmaghraby et al. (2009) considering reservation regime, and Yin et al. (2009) considering inventory display format. All these papers consider markdown pricing in a selling horizon with two time periods. Cachon and Swinney (2009) consider a problem where the firm has the capability of quick response (i.e. making an additional order of inventory at the beginning of the first period after observing the first-period demand). There are three types of customers: myopic customers who always buy in the first period, bargain-hunting customers who only buy in the second period, and strategic customers who choose when to buy to maximize their expected utility. Strategic customers' valuation is known in the first period and follows a given distribution in the second period. The total population in the first period (myopic and strategic customers together) is a random variable while the composition is known. The first-period price is given. The firm makes the ordering decision before the season starts and determines the second-period price after observing the first-period demand. The firm may have an opportunity to use the quick response strategy (i.e order more after the first-period demand is realized). They show that: (i) when strategic customers are present, firms tend to order less, raise the markdown price, and make less profit compared to the case where all customers are myopic; (ii) ignoring strategic customers may result in



significant revenue loss especially in the case where the firm does not have quick response capability; (iii) quick response is more valuable in the case of strategic customers than myopic customers and the difference can be quite significant; (iv) a contingent pricing policy is in general better than a preannounced pricing policy. Lai et al. (2010) investigate the impact of posterior price matching (PM) policy on a firm’s profit and optimal pricing and initial inventory decisions. Under a PM policy, the firm will refund the difference if the price is marked down in a future period. There are two types of customers by their valuation, high-valuation and low-valuation customers with valuation  $V_H, V_L$ , respectively, where  $V_L$  stays unchanged while  $V_H$  decreases to  $V_h$  in the second period. The number of low-valuation customers is infinite whereas the number of high-valuation customers is a random variable. A certain percentage of the high-valuation customers are strategic. The authors find that: (i) the PM policy induces strategic customers to purchase earlier, and thus makes it possible for the firm to set a higher price in the regular selling season; (ii) PM policy significantly improves the firm’s profit if the fraction of strategic customers and their valuation decline over time are both modest; (iii) when the variance of the number of high-valuation customers is high, the PM policy also increases customer surplus; (iv) with PM policy, a contingent pricing scheme performs better than a preannounced pricing scheme. Elmaghraby et al. (2009) analyze two operating regimes, a “no reservation regime” (NR) and a “reservation regime” (WR), in a preannounced markdown pricing setting. They assume that the firm has a single unit of a product for sale at the regular price  $p_h$  over a continuous time horizon  $[0, T]$ . If the product is not sold by the end of the season, it will be sold at a clearance price  $p_l$  ( $p_h \geq p_l$ ). Customers arrive following a Poisson process with a rate that depends on  $p_h$  and  $p_l$ . They consider a base model where all customers have an identical valuation, and an extended model where there are two customer classes with two distinct valuations. Upon arrival, if the product is still available, customers can choose to buy the product at  $p_h$ , wait until the end of the season and enter a lottery, or reserve the product. In the latter case, if the product is available at the end of the season, the customer who reserved the product is obligated to

buy it. They analyze the firm's revenue and the customers' expected surplus under two regimes and show that under WR (compared to NR): (i) given prices  $p_h, p_l$ , if customers behave strategically, the firm will be better off while the customers will be worse off; (ii) given  $p_h$ , if the firm can optimize  $p_l$ , the firm will be better off, and the customers will be worse (better) off if they are very (not) sensitive towards the markdown price; (iii) when there are two classes of customers, the firm will be better (worse) off while the customers will be worse (better) off if there is a large (small) percentage of high-valuation customers. Yin et al. (2009) compare the impact of two inventory display formats on a firm's optimal profit in a markdown pricing setting with strategic customers: display all (DA) and display one (DO). Under DA, the firm displays all available units; while under DO, the firm only displays one unit at a time if the product is still available. The firm preannounces a regular price  $p_h$  for regular season  $[0, T]$  and markdown price  $p_l$  for end-of-season sale if the product is still available. Customers arrive following a Poisson process with given rate  $\lambda$ . A certain percentage of the customers have high valuation  $v_1$  while the rest have low valuation  $v_0$ . They demonstrate by numerical examples that: (i) DO format creates an increased sense of inventory scarcity and induces high-valuation customers to purchase earlier, and hence improves the firm's profit; (ii) the revenue improvement of DO format compared to DA format is generally small; (iii) changing the display format from DA to DO and simultaneously reoptimizing the prices can improve the firm's revenue significantly more than changing the display format alone; (iv) changing the display format only recovers a small amount of revenue loss due to strategic customer behavior.

Su (2010) considers a two-period pricing problem in the presence of speculators. There are four types of customers: myopic customers, strategic customers, random customers, and low-value customers. Each type of customers has a known valuation. The numbers of myopic customers and strategic customers are known. The number of random customers is unknown and realizes at the beginning of the second period. The number of low-value customers is infinite. Myopic customers always purchase in the first period, random customers and low-value customers only

purchase in the second period, while strategic customers choose when to purchase. In addition, there are speculators who may purchase the product in the first period purely for resale in the second period at a higher price. The author finds that: (i) the presence of speculators can benefit the firm; (ii) speculative resale enables the firm to achieve the benefit of dynamic pricing while maintaining a single price; (iii) speculative behavior leads to lower initial inventory if initial inventory is a decision variable.

Ovchinnikov and Milner (2011), and Gallego et al. (2008) are the only two papers that consider a dynamic pricing problem with multiple seasons. They both assume that in each season there are two periods representing a regular selling period and a markdown period, respectively. The firm determines the prices for both periods and a sales rationing policy for the markdown period. It is assumed that customers' purchasing behavior in a season is impacted by the firm's decisions in the past seasons. In Ovchinnikov and Milner (2011), there are two types of customers with different valuations which are known to the firm. All low-valuation customers and a given fraction of the high-valuation customers are strategic and choose to wait and buy in the second period. The fraction of high-valuation customers who are strategic changes over season as the firm changes the amount of inventory offered in the markdown period in each season. The evolution of the purchasing behavior over time can be characterized by a learning function. The authors focus on two types of function: a self-regulating learning function and a smoothing learning function. They show that: (i) when customer's purchasing behavior follows a self-regulating function, the firm's optimal policy is to allocate some inventory for the markdown period in each season; (ii) when customer's purchasing behavior follows a smoothing learning function, the firm should periodically allocate some inventory to the markdown period over a number of seasons, and then allocate no inventory to the markdown period in the following season; (iii) for the case where there are three types of customer with overbooking allowed, the optimal policy may increase the revenue by 5-15%; (iv) allowing (disallowing) overbooking is beneficial if there are few (many) high-value customers. In Gallego et al. (2008), the total number of

potential customers is given. Customers' valuations follow a known distribution. All customers are strategic and maximize their expected surplus. Customers update their beliefs about inventory availability in the markdown period of a season according to the inventory availabilities in the past seasons. They first study the equilibrium setting and show that a single-price policy for the whole season is optimal. They also show that a markdown pricing can be optimal if (i) there is a mixture of strategic customers and myopic customers; (ii) the total number of potential customers is unknown; or (iii) customers' valuation distribution is unknown. The authors then study the dynamic non-equilibrium setting where the aggregate total demand is a Poisson random variable. Their numerical experiments show that (i) when the total demand is unknown, it is optimal for the firm to limit the second-period inventory and the behavior of this optimal inventory limit depends on how customers update their belief on the inventory availability in the markdown period; (ii) customers' learning behavior from season to season can significantly increase the variability of sales revenue.

Cho et al. (2008) extend the work by Gallego and van Ryzin (1994) to incorporate strategic customer behavior. In their work, they focus on the impact of strategic waiting on the customers' benefits under the assumption that the firm's pricing decisions follow Gallego and van Ryzin (1994) in which strategic customer behavior is not considered. They consider two policies that customers will follow when making a purchasing decision: (1) threshold time policy (i.e., for any  $k$  units of inventory remaining, purchase if and only if the remaining time is below some threshold  $t_k$ ), and (2) threshold price policy (i.e., for any  $k$  units of inventory remaining, purchase if and only if the price is below some threshold  $p_k$ ). They show that these two policies are equivalent. They show through simulation that strategic waiting can benefit both the customers and the firm. Customers who cannot afford the price at the time of arrival may purchase the product later if the price drops below their valuations. Therefore, strategic waiting enables the firm to retain some potential customers and increase the total sales.

## 2.4 Models with Competition

Dynamic pricing problems with competition have received relatively little attention in the literature. We are aware of a handful of papers that consider dynamic pricing decisions facing competition and fall within the framework of this survey, including Dasci and Karakul (2009), Gallego and Hu (2009), Granot et al. (2007), Xu and Hopp (2006), Perakis and Sood (2006), and Levin et al. (2009). All papers consider a general dynamic pricing problem where price change is not required to be monotone over time. Levin et al. (2009) is the only paper that considers strategic customer behavior and all other papers assume that customers are myopic. In Table 2.6, we categorize existing models according to their modeling assumptions including time horizon (two-period, multi-period with a general number of time periods, or continuous), pricing policy (contingent or preannounced), demand (stochastic or deterministic), and market (duopoly or oligopoly). In what follows, we review each of the papers in detail.

Table 2.6: Overview of Existing Models with Competition

Time horizon	Pricing Policy	Demand	Market	Paper(s)
Two-period	Contingent	Deterministic	Duopoly	Dasci and Karakul, 2009
	Preannounced	Deterministic	Duopoly	Dasci and Karakul, 2009
Multi-period	Contingent	Stochastic	Duopoly	Granot et al., 2007
	Preannounced	Stochastic	Oligopoly	Perakis and Sood, 2006
Continuous	Contingent	Stochastic	Oligopoly	Levin et al., 2009 Xu and Hopp, 2006 Gallego and Hu, 2009

### 2.4.1 Review of Existing Literature

Dasci and Karakul (2009) consider a two-period dynamic pricing problem in a duopoly market. The two firms hold an equal amount of initial inventory of an identical product at the beginning of the selling season. The number of customer arrivals in each period is given and all customers in the same period have identical valuation. However, customers arrive in different periods may differ in their valuation. So there are two customer classes. The authors compare two pricing schemes, a dynamic pricing scheme in which the two firms first simultaneously determine the first-period prices and then simultaneously determine the second-period prices according to the first-period sales, and a fixed-ratio pricing scheme in which the two firms simultaneously determine the first-period prices only and the second-period price of each firm is a given fixed ratio of its first-period price. The problem is modeled as a dynamic game between the two firms when the firms use the dynamic pricing scheme, and as a static game when the firms use the fixed-ratio pricing scheme. They find that: (i) for most cases in a duopoly market, the fixed-ratio pricing policy outperforms the dynamic pricing policy in terms of the expected profit for both firms; (ii) under dynamic pricing scheme, due to the fact that there are two customer classes and each firm desires to serve the more lucrative class, the two firms cannot reach a collusive solution (i.e., the two firms will not cooperate to increase their revenue) even if the initial inventory is low; (iii) in equilibrium, one firm assumes the role of a low-cost high-volume alternative while the other assumes the role of a high-cost low-volume alternative; (iv) under dynamic pricing scheme, there is less competition in the second period as the firms become more asymmetric (the low-cost firm that wins in the first period becomes the smaller competitor in the second period).

Gallego and Hu (2009) extend the work by Gallego and van Ryzin (1994) to an oligopoly market. They formulate the problem as a stochastic game on a continuous time horizon. They assume that customers arrive following a Poisson process with the given time-varying total arrival rate  $\lambda(t)$ . The probability that

a customer purchases from a particular firm is jointly determined by the prices set by all firms. Due to the intractability of the proposed stochastic model, they focus on the corresponding deterministic differential game and prove the existence of open-loop and closed-loop Nash equilibria. Based on the equilibrium solutions to the differential game, they further propose pricing heuristics and demonstrate that these heuristics are asymptotic equilibrium for the stochastic game. Their numerical examples show that (i) when a firm's initial inventory is lower, the firm tends to price higher; as a result, other firms respond by increasing their prices too; (ii) in case where irrational firms deviate from their equilibrium strategies, they consistently suffer more than rational firms; (iii) when irrational firms with limited initial inventory maximize their revenue rates, rational firms may suffer or benefit depending on their initial inventory and price-sensitivity relative to other firms; (iv) when irrational firms use market clearing price when they have abundant initial inventory, all rational firms suffer.

Granot et al. (2007) consider a multi-period dynamic pricing problem in a duopoly market. They assume that customers' valuations follow a given distribution which is known to the firms. Customers are assumed to follow a zigzag shopping behavior. That is, a customer visits only one store in each period. If the price she observes in that store is below her valuation, she will buy the product; and otherwise she will visit the other store in the next period. This process will continue until she has found a price below her valuation or the selling season ends. The authors show that: (i) under competition, the prices set by each firm and the corresponding profit are significantly lower than the ones under monopoly market; (ii) price decreases exponentially over time under competition; (iii) these effects of competition increase (decrease) as the total fraction of the market the two firms can satisfy increases (decreases).

Xu and Hopp (2006) consider a continuous-time dynamic pricing problem in a monopoly and oligopoly market, respectively. They assume that customers arrive following a geometric Brownian motion and the demand function is isoelastic. For the monopoly case, they derive a closed-form optimal solution. They show that all

stochasticity in customer arrivals is absorbed into the optimal pricing policy and thus the resulting inventory trajectory is deterministic. They also show that dynamic pricing coupled with optimal initial inventory decision substantially outperforms static pricing coupled with optimal initial inventory decision. For the oligopoly case, their problem is a pricing and inventory decision game. They establish a weak perfect Bayesian equilibrium for the game. They find that cooperative pricing can be achieved even in a non-cooperative setting. However, competition among firms drive each firm to overstock, and thus hurts firms' profits. They show that when competition is not too severe, contingent pricing outperforms preannounced pricing; and when competition becomes very intense, preannounced pricing performs better.

Perakis and Sood (2006) use a robust framework to study a multi-period dynamic pricing problem in an oligopolistic market. They assume that demand for each firm in a given period is a function of the prices set by all firms in that period. The exact values of demand function parameters, however, are unknown and belong to a known uncertainty set. They propose a robust policy that maximizes the revenue for each firm under the most adverse instances of parameters within their uncertainty set. Due to the lack of a concave objective function, they propose a variational inequality reformulation. They prove the existence of equilibrium policies and develop an iterative learning algorithm for computing the market equilibrium policies. Their numerical results show that: (i) typically prices are higher in periods where the demand sensitivity is lower; (ii) by using a robust policy, firms' payoffs are much less sensitive to the uncertain parameters as compared to the policies that ignore uncertainty in parameters and simply use nominal values; (iii) compared to other firms that use a policy based on nominal values of uncertain parameters, firms that use robust policy can obtain a payoff that has much less variation while the mean payoff is slightly lower; (iv) a firm can balance the tradeoff between smaller variation and larger mean payoff by adjusting the budget of robustness in the optimization model.

Levin et al. (2009) extend the work by Levin et al. (2010) to an oligopolistic market. It is assumed that there are  $M$  firms and each firm  $m$  has  $K_m$  units of dif-



ferentiated product  $m$  to sell to a total of  $N$  potential customers. Customers belong to  $S$  segments and all customers within the same segment have a valuation drawn from a common distribution. The problem is formulated as a stochastic dynamic game. At the beginning of each period, all firms first simultaneously choose the prices and then all customers choose their shopping intensities. Their numerical experiments demonstrate that: (i) the impact of strategic customer behavior increases with increased competition among firms; (ii) firms that ignore strategic customer behavior can incur significant revenue loss; (iii) firms that provide better quality products are generally less affected; (iv) strategic customer behavior reduces price variability over time (i.e., leads to a flatter price path over time) but increases the price variability at the end of the season for the stronger firm that provides better quality product.

## 2.5 Future Research Directions

In Sections 2.2 to 2.4, we have provided a state-of-the-art review of dynamic pricing models with finite initial inventory and fixed time horizon without inventory replenishment. Among the various types of problems we have reviewed, problems with strategic customers and problems with multiple products have received the most attention in recent years. Other topics such as pricing under competition and pricing with demand learning have also started to gain popularity.

The following issues are commonly encountered in practice, and yet have received very little or no attention in the dynamic pricing literature. We believe that these issues are worth of investigation in future research.

### 2.5.1 Business Rules and Constraints

In their survey paper, Elmaghraby and Keskinocak (2003) pointed out “Another disconnect between most of the academic literature and practice is the incorporation of business rules into pricing decisions.” Most papers in the existing literature largely ignore commonly-used business rules and practical constraints and build

highly stylized models in order to make their formulation mathematically tractable. This allows them to obtain structural results and derive general managerial insights. As a consequence, the resulting solution might be appealing from a theoretic point of view, but may not be optimal and sometimes may not even be feasible from a practical point of view. For example, most papers allow unlimited price changes and do not set a limit on the magnitude of each price change. This gives the firm more freedom in dynamically adjusting the price according the realizations of uncertain demand. However, frequent price changes and substantial price differences from one period to another period may significantly change customers' purchasing behavior when customers behave strategically. Consequently, it may also change the underlying demand function as assumed to be given exogenously in most papers. In such a case, the solution which is optimal for the original demand function may no longer be optimal for the new demand function. Moreover, in a situation where only limited number of price changes are allowed, the solution obtained from the model that ignores such a constraint is infeasible. Dynamic pricing models that consider business rules have received very little attention in the existing literature. We are only aware of a few papers that incorporate this issue including, e.g., Chen et al. (2011) for a joint inventory allocation and markdown pricing problem with multiple stores and Chen and Chen (2010) for a robust general dynamic pricing problem with two competing products. We believe that incorporating business rules and constraints in a dynamic pricing model will allow us to better understand the problem and generate more practical insights which might be substantially different from the ones generated from previous models that ignore such constraints.

### 2.5.2 Strategic Customers with Bounded Rationality

Dynamic pricing problems with strategic customers have received substantial attention in the last several years. However, most papers (two exceptions are Levina et al., 2009, and Levin et al., 2010 for a special case of their model) assume that all customers are fully rational and highly sophisticated in the sense that they are always able to obtain necessary information, correctly expect the firm and other

customers' behavior and make their optimal decisions accordingly. In a practical situation, this may not be the case. For example, a customer may never know the firm's inventory information, the potential market size, or other customers' valuation distribution. Given limited information, they may not be able to predict the firm and other customers' behavior. In addition, even if all the information is available, customer's decision making process is in general straightforward and rarely involves solving complicated optimization problems as assumed in the existing models. Moreover, often times, their decision making is also subject to psychological biases and cognitive limitations (Shen and Su, 2007). This is referred to as *bounded rationality* or *limited rationality* in the literature. Therefore, assuming that customers are fully rational and highly sophisticated may not be appropriate in practice. An alternative approach is to build a descriptive model based on sales data, survey or lab experiments to characterize customers' actual purchasing behavior. For more details about modeling bounded rationality, one may refer to Su (2008) for a newsvendor problem and Simon (1982) and Conlisk (1996) for a review of the evolution and development of limited rationality.

### 2.5.3 Non-equilibrium Market Situation

When modeling strategic customer behavior, all papers except Gallego et al. (2008) and Ovchinnikov and Milner (2011), study the equilibrium situation where both the firm and the customers can correctly anticipate the other's optimal behavior and make their decisions accordingly. Often times, it is in the firm's interest to prevent such an equilibrium situation by deviating from its optimal equilibrium behavior and making their strategy unpredictable to the customers. For example, although airfare in general increases as the departure time gets closer, airlines may occasionally offer last minute tickets at a deep discount to attract low-valuation high-flexibility customers to fill up unsold seats. However, whether or not and at what time these last minute deals will be offered never follow any pattern. This prevents high-valuation inflexible customers from taking advantage of such last minute deal and induces them to pay higher price earlier in order to secure a seat. It is natural

to ask how the customers' behavior and the firm's profit will be impacted in such a non-equilibrium situation. Gallego et al. (2008) and Ovchinnikov and Milner (2011) are the only two papers in the existing literature that study such dynamic non-equilibrium behavior. They both consider a setting with multiple seasons in which customers adaptively update their beliefs about the product availability in the markdown period in each season according to the firm's policies in previous seasons. Both papers limit their attention to a two-period markdown pricing problem for each season and make special assumptions on customer valuation distribution. We believe that it might be interesting to investigate a multi-period general dynamic pricing problem (involved in each season) with more general form of customer valuation distribution.

#### 2.5.4 Complementary Products

As we have noted in Section 2.2.2, in the multi-product case, all existing papers consider substitutable products only. We are unaware of any existing literature that considers dynamic pricing problems with complementary products. If a firm sells multiple complementary products, the demand of one product is positively correlated with the demand of other products. Thus, unlike in the substitutable product case, reducing the price of one product in this case will increase the demand of this product and may also increase the demand of other products (Walters, 1991). For example, reducing the price of a particular camera model may also increase the demand of its accessories. In addition, unlike in the substitutable product case where customers generally purchase one of the products, in the complimentary product case, customers may purchase multiple products together according to their total price (Wang, 2006). For example, often times customers purchase a camera, a camera case, and memory cards together. In this case, the commonly-adopted consumer discrete choice model in the substitutable product case may no longer be suitable. Therefore, modeling customers' purchasing behavior may require a different modeling framework. Moreover, compared to the substitutable product case, the availability of one product in this case may have a more significant impact

on the demand of its complementary products. For example, once a particular camera model is sold out, the demand for its accessories may also drop to zero. We believe that dynamic pricing problems with complementary products is a very interesting direction for future research.

### 2.5.5 Empirical Verification and Validation

Most of the existing models that we have reviewed make assumptions on customer purchasing behavior which are not verified by real data. The solutions and insights from most of the existing papers have not been tested in a real market. Although most papers claim that their models and solutions may significantly improve the firm's profit, the actual impact in practice is unclear due to the lack of empirical verification and validation. In particular in the case with strategic customers, different modeling assumptions may lead to conflicting conclusions. In this case, empirical work is especially needed in order to better understand customer purchasing behavior and derive more relevant insights. We believe that empirical work certainly deserves more attention in future research.

## Chapter 3

### Markdown Optimization at Multiple Stores

#### 3.1 Introduction

Markdown pricing is a common technique used by sellers to match supply with demand for time sensitive goods. Consumer electronics products and fashion apparel retailers are two examples of sellers that adopt this technique. According to the National Retail Federation, marked-down goods, which accounted for just 8% of department-store sales three decades ago, now account for over 20% of sales (Merrick 2001). The primary objective of markdown pricing is to manipulate the prices based on the demand in a situation where the demand for the product at a given price drops monotonically over time. Usually for time sensitive goods such as digital cameras, the demand drops steadily with time towards the end of the product life-cycle. Hence the sellers have to lower the prices to stimulate sales. But the extent of this markdown is very critical. If the sellers do not reduce the price sufficiently, there may be excess inventory at the end of the life-cycle that has to be discarded or sold at a very low salvage value. On the other hand, if the price level is set too low, all the items may sell very quickly resulting in a reduction in possible revenue. So a balanced approach is very important in ensuring high revenue for the firm.

In this chapter, we study a markdown pricing problem commonly faced by many large retailers that we have worked with in the last several years. Based on our extensive experience with industry, this problem contains most commonly encountered issues involved in markdown optimization in practice. Below we describe our problem and show that several aspects of the problem are new and have received little attention in the markdown pricing literature. The precise mathematical notation and problem formulation are given in Section 3.2.

We consider a typical large retailer consisting of several warehouses and hundreds of stores in its supply chain. Each warehouse serves many stores. For example, a warehouse in the state of New York may be responsible for allocating inventories to fifty different stores owned by the retail chain in the north-eastern United States. Near the end of a selling season for some products, the retailer has to get rid of the remaining inventory of the products at each warehouse in a limited amount of time before the selling season ends. The retailer needs to allocate the existing inventory of each warehouse to the multiple stores served by the warehouse and in the same time determine a markdown pricing scheme at each store subject to a number of business rules. It is impractical to consider all the stores and all the products together in a single model. A commonly used approach in practice is to decompose the markdown decisions by warehouses and products so that each warehouse of the retailer is independent and each product is dealt with separately in making inventory allocation and markdown decisions. Therefore, we consider the problem of a single warehouse and a single product in this chapter, which is described as follows.

At the beginning of the markdown planning horizon, there is a given amount of inventory of a product in a central warehouse which needs to be allocated to a set of stores over time. The length of the markdown planning horizon is typically short, varying from a couple of weeks to no more than 3 months, because the retailer needs to get rid of the current product as soon as possible in order to (i) minimize the impact of the current product on the sales of one or more new products being introduced; and (ii) have sufficient shelf space for the new products. In addition to the inventory allocation decision, the retailer needs to determine a markdown pricing scheme for each store over the planning horizon. There are a discrete set of allowable prices that can be used for the product at all the stores. For example, if the regular price of a digital camera is \$159.99, then the allowed markdown prices could be \$143.99, \$127.99, \$111.99, \$95.99, and \$79.99, which represent 10% off, 20% off, 30% off, 40% off, and 50% off, respectively, from the regular price. We note that although in theory the price of a product can be set to any number within a certain interval, there are certain price points at which consumers become much

more willing to buy, and hence retailers that follow a sound pricing strategy often use a small set of popular price points for a product (e.g., Allen 2011). At a store, price changes can only occur at the beginning of a time period, where a period typically consists of one to two weeks, and once a change occurs the new price should remain unchanged for the entire time period. For that reason, the planning horizon is assumed to consist of a discrete number of time periods, and the pricing decision at a store is to set a price for each time period. The inventory allocated to the stores and the prices set at the stores over time must follow a set of business rules given below:

- i) Each store must be allocated at least a given minimum amount of inventory in the first period.
- ii) The prices set at each store must be non-increasing over the time.
- iii) The number of markdowns allowed at each store cannot exceed a given upper limit.
- iv) In each period, if there is a markdown from the price used in the previous period, the price change must be within given lower and upper limits (e.g., at least 10% and at most 30% off the current price).
- v) There may exist clusters of stores (e.g., stores in a given geographical area) for which the prices during any period must be within a given range from each other. That is, in each period, the difference between the maximum and minimum prices among the stores within the same cluster must not exceed a given upper limit (e.g., within \$10).

These rules are due to established market norms and the costs associated with implementing markdowns, and may also be desirable from a consumer's standpoint. Rule (i) is not enforced in later periods because normally customers do not expect the availability of the product in the future periods once they realize the product is on sale. Rule (ii) is simply because of the nature of markdown pricing. Rules (iii) and (iv) reflect the fact that frequent price changes and significant price difference



from one period to the next may confuse the consumers. It has been long understood (Hall and Hitch 1939) that frequent price changes can even make a retailer appear unfair or dishonest, as customers try to interpret the retailer's motives behind a price change. Rule (iii) is also reasonable in order to ease markdown implementation and save the associated implementation costs. Rule (v) allows stores to price the same product differently with the constraint that stores within a cluster have to use similar prices. This minimizes the possibility of closely located stores competing for customers. As pointed out by Hruschka (2007) and Shankar and Bolton (2004), many large retailers do allow different prices at different stores in the same chain for the same product. In fact, many retailers, e.g., Walgreen, Target and Radioshack, explicitly specify on their websites that pricing may vary by store location. This gives a retailer flexibility in pricing their products at different stores based on store specific characteristics such as demographics, location, and competition.

Since there are daily shipments of regular products from the warehouse to each store, markdown items can often get a free ride. Therefore, there is no shipping cost for the markdown items from the warehouse to the stores. In each period, only necessary amount of inventory is shipped to the stores to satisfy the demand in that period. Therefore, the inventory is mostly kept at the warehouse and the holding cost over a short planning horizon is negligible and not considered. At the end of the planning horizon, any unsold items are typically either sold at a deep discount through a liquidation channel, or donated to some charities. In the latter case, the company may receive some tax benefit. Hence, there is a salvage value for the unsold items in both cases. The problem is to decide how many units to be allocated to each store in each period and what price to use in each period at each store so that the total expected revenue (sales revenue + salvage value) of all the stores over the planning horizon is maximized subject to the required business rules.

Although the problem we consider does not involve shipping and inventory costs, as justified above, we show in Section 3.7 that for situations where shipping and inventory costs need to be considered, our problem formulation and solution approach (developed in Sections 3.2 and 3.3) can be easily modified to handle such

situations.

### 3.1.1 Related Literature

Three main characteristics distinguish the problem we consider from most of the markdown pricing problems considered in the literature: (1) incorporation of business rules, (2) joint consideration of multiple stores, and (3) practical demand modeling. Most existing models in the dynamic pricing literature (including the markdown pricing literature) are stylized, oversimplify practical issues, and do not consider most of the business rules that we consider in this chapter and are commonly encountered in practice. In reviewing dynamic pricing literature, Elmaghraby and Keskinocak (2003) write that *“another disconnect between most of the academic literature and practice is the incorporation of business rules into pricing decisions”*, and justify the use of commonly accepted rules in practice. Due to the nature of markdown pricing, rule (ii) is considered in all existing markdown pricing problems. However, only few papers have incorporated one or two of the other rules that we consider. Bitran and Mondschein (1997) consider rule (iii), and Perakis and Harsha (2010) consider rules (iii) and (iv). However, both papers consider a single store only.

Elmaghraby and Keskinocak (2003) also note that one of the most important missing links between the academic markdown pricing literature and the real world is the need to consider multiple stores or sales channels, with possibly different demand patterns simultaneously. Multi-store problems are clearly more complex than single-store ones because two levels of decisions, inventory allocation and pricing, have to be made jointly. Furthermore, pricing decisions at different stores are coupled because of business rule (v) and hence the problem cannot be decomposed by stores even after the inventory allocation is done.

Bitran et al. (1998) is the only markdown pricing paper we are aware of that considers a model with multiple stores. As in our model, there are a finite number of discrete time periods in their model so that price changes are periodic and a price once set for a period stays unchanged during that period. However, our model

differs from theirs in a number of dimensions. First of all, the practical constraints on inventory allocation and markdown pricing (i.e., rules (i), (iii), (iv), and (v)) involved in our model do not exist in their model. Secondly, they allow continuous choices of prices such that any price can be used, whereas in our model (as in most retail settings in practice) there is a finite and discrete set of pre-selected price levels that can be used. Thirdly, they require that all the stores use the same price in each period, whereas we allow prices to be different at different stores as long as business rule (v) is satisfied. Fourthly, in their model, all the inventory allocated to a store for the entire planning horizon is delivered to the store at the very beginning of the planning horizon, whereas we deliver a necessary amount of inventory to each store period by period. Fifthly, although in our computational experiments, we use a demand function in order to draw some managerial insights, our model itself does not require an explicit demand function and makes no assumptions about the demand distributions. Their model assumes Poisson arrival and employs the concept of reservation price to model the random demand. Finally, the approach we develop (in Section 3.2) allows demand correlation across stores and across time periods. In their model, demand for a given price in a given period at a given store is independent of the demand in other time periods and independent of the demand at other stores. In addition to those modeling differences, our solution approach is different from theirs. They use dynamic programming. Since the number of possible states is prohibitively large, they propose heuristics based on a state-aggregation technique applied to the DP formulations. In their computational experiment, only problems with a small number of stores are tested. Their DP formulations can generate optimal solutions for problems with 2 stores, and their heuristics are evaluated by comparing to the optimal solution based on 2-store problems and to a policy used in practice by a fashion retail chain based on 8-store problems from the retail chain. We formulate our problem as mixed integer programs and develop a Lagrangian relaxation based decomposition approach which is capable of solving much larger problems (e.g., 50 stores) which are often faced by large retail chains in practice.

Furthermore, the demand modeling approach we use in our problem is more

practical than in most existing markdown pricing problems in the literature. Most papers reviewed in this section do not consider possible demand correlation across time periods, and assume that the demand distribution at each time point is independent and known precisely in advance. However, in practice, demand over time is often correlated, and the probability distribution of the demand is often not known precisely and needs to be re-estimated over time as more market information becomes available over time. Therefore, in our problem, we allow demand correlations across time, do not assume precise demand distribution, and use a rolling horizon approach in which the demand is updated over time. Our demand modeling details are discussed in Section 3.2. In the literature, Feng and Gallego (2000) and Gupta et al. (2006) are the only two papers that consider demand correlation across time. Feng and Gallego (2000) model the demand correlation across time by assuming that the demand intensity at any time point is a function of the total sales up to this time point. Gupta et al. (2006), on the other hand, employs a random component in the demand function that explicitly models the interdependency of demand in different periods.

Next, we review other relevant literature briefly. For a more detailed review, see the survey articles by Elmaghraby and Keskinocak (2003) and Bitran and Caldentey (2003). Existing markdown pricing models can be classified into several different classes in terms of number of stores considered (one or multiple), number of products (one or multiple), nature of price changes (continuous or periodic), nature of allowed prices (discrete or continuous), assumption on demand (deterministic or stochastic), customer behavior (myopic or strategic), and nature of competition (monopoly or competitive). As discussed earlier, all the existing markdown pricing models, except the one considered by Bitran et al. (1998), involve a single store. In terms of the nature of price changes, some papers (e.g., Gallego and van Ryzin 1994, Bitran and Mondschein 1997, Feng and Gallego 1995, 2000, Feng and Xiao 1999, 2000a, 2000b, Smith and Achabal 1998, and Chatwin 2000) study problems with continuous price changes allowed such that the price can be changed at any point in time, whereas some others (e.g., Federgruen and Heching 1999, Smith et

al. 1998, Mantrala and Rao 2001, and Neelakantan et al. 2007) consider problems with periodic price changes such that a price once set for a period stays unchanged during that period. In terms of allowed prices, there are papers (e.g., Gallego and van Ryzin 1994, Bitran and Mondschein 1997, Smith and Achabal 1998, Zhao and Zheng 2000, and Anjos et al. 2005) that allow continuous prices such that any price can be used, and papers (e.g., Chatwin 2000, Smith et al. 1998) that require prices to be chosen from a finite and discrete set of pre-selected price levels. There are a few papers (e.g., Dong et al. 2009, Zhang and Cooper 2009) that study the pricing problem involving multiple products, and most others study the problem with only a single product.

Most of the models in the markdown pricing area including those reviewed above and our model assume that customers are myopic and hence the demand in a period is independent of the prices in the previous periods. When the customers behave strategically, they consider both the current price and possible price changes in the future. This scenario is inherently more difficult to analyze. Aviv and Pazgal (2008), Elmaghraby et al. (2008), and Zhang and Cooper (2008) are a few studies that take into account the strategic behavior of customers. Most of the literature study the dynamic pricing problem in a monopoly setting in which a firm's objective is to maximize its own revenue without considering the impact from other firms that sell similar products. When there is competition in the market, the problem becomes much more complex. Papers that investigate the pricing problem in a competitive market include Gallego and Hu (2007), Perakis and Sood (2006), and Levin et al. (2009).

### 3.1.2 Organization of The Chapter

The remainder of this chapter is organized as follows. In Section 3.2, we introduce necessary notation, model stochastic demand using discrete demand scenarios, and formulate our problem as a mixed integer program (MIP). As we show in Chapter A of the appendix, our problem is NP-hard even if the demand is deterministic and there is only a single store or a single time period. Therefore, the overall prob-

lem is computationally intractable and it is unlikely that one can derive an optimal solution to the MIP formulation of the problem within a reasonable amount of computational time. We thus focus on heuristic solution approaches. In Section 3.3, we propose an optimization based heuristic solution approach implemented on a rolling horizon basis. We develop a Lagrangian relaxation based decomposition approach to solve the problem involved at the beginning of each period under the rolling horizon approach. In Section 3.4, we describe several benchmark markdown approaches that are commonly used in practice. In Section 3.5, we conduct an extensive set of computational experiments under various practical situations, and demonstrate that our approach outperforms the benchmark approaches under various circumstances. In Section 3.6, we discuss a number of managerial insights derived from our computational study. These insights can help managers make better markdown pricing decisions in practice. Finally, we conclude this chapter in Section 3.7 by discussing how some other practical issues can be formulated and solved in a similar way.

## 3.2 Problem Formulation

We define the following notation to be used when we formulate our problem:

$1, \dots, T$ : Markdown time periods, where  $T$  is the length of the planning horizon.

$N = \{1, \dots, n\}$ : Set of  $n$  retail stores.

$I_0$ : Total inventory available at the warehouse to be allocated to the stores at the beginning of the planning horizon.

$I_r^{\min}$ : Minimum amount of inventory that has to be assigned to store  $r \in N$ , according to business rule (i).

$M = \{1, \dots, m\}$ : Set of  $m$  allowable price levels.

$p_1$ : Regular price before markdown.

$p_j$ : Price corresponding to allowed price level  $j$ , for  $j \in M$ . We assume without loss of generality that  $p_1 > p_2 > \dots > p_m$ .

$D_{rjt}$ : The demand corresponding to price level  $j$  for store  $r$  in period  $t$ .

$u_j$ : The minimum price level that can be set for a period according to business

rule (iv) if there is a price drop from the previous period and if the price level  $j$  is used in the previous period.

$v_j$ : The maximum price level that can be set for a period according to business rule (iv) if there is a price drop from the previous period and if the price level  $j$  is used in the previous period.

$R$ : Maximum number of markdowns allowed for any store over the planning horizon, according to business rule (iii).

$Q$ : Number of store clusters. A store cluster includes at least two stores. All the stores within a cluster are required to have prices in each period that satisfy business rule (v).

$C_q$ : The  $q$ th store cluster, which is a subset of  $N$ , for  $q = 1, \dots, Q$ .

$n_I$ : Number of independent stores (stores that do not belong to any store cluster, i.e., stores in  $N \setminus (C_1 \cup \dots \cup C_Q)$ ).

$G$ : Number of price clusters used to model business rule (v). A price cluster is the collection of prices that are within the allowed range from each other by business rule (v). If the rule requires that all the stores in a given store cluster should use prices within a range of 10% from each other, then a price cluster would consist of prices that are at most 10% away from each other. Given the  $m$  allowable prices, all the price clusters can be enumerated.

$E_g$ : The  $g$ th price cluster, for  $g = 1, \dots, G$ .

$s$ : Salvage value per unsold item at the end of the planning horizon.

The demand  $D_{rjt}$  at each store  $r$  in each period  $t$  is a stochastic function of the price  $p_j$  used. However, the probability distributions of the demand functions are not completely known beforehand for the following reasons. First, market dynamics often change over time, and hence it can be very difficult, if not impossible, to generate an accurate distribution for demand in a future period. This is the case especially at the beginning of the planning horizon when little information is known. Second, the product has always been sold at the regular price and hence the retailer has no historical sales data of the product at the markdown prices. Although histor-

ical data from the clearance sales of similar products may be used to construct the probability distribution of the demand for the product of interest, such distribution is only a rough approximation.

Consequently, it is crucial to incorporate up-to-date market information which becomes available over time as markdown sales progress into the decision process. Therefore, we approach our problem using a *rolling horizon* framework where at the beginning of each period  $\tau$ , demand distributions in the remaining periods are re-estimated by utilizing the latest market information, and the problem consisting of the remaining periods  $\tau, \tau + 1, \dots, T$  is re-solved and the solution is implemented for the current period  $\tau$  only. A detailed description of our solution approach is given in Section 3.3.

In the following, we model stochastic demand using discrete scenarios and formulate our problem at the beginning of a particular time period  $\tau$  as a mixed integer program (MIP) with the demand scenarios.

### 3.2.1 Demand Scenario Tree

As we discussed earlier, it is difficult to know the precise probability distributions of stochastic demand functions. In fact, even if accurate demand distributions can be obtained, incorporating them into an optimization model that involves multiple stores and multiple periods can make the model extremely difficult to solve. Therefore, we do not try to precisely characterize stochastic demand functions. Instead, we approximate the stochastic demand over a given planning horizon by a finite number of *demand scenarios*. The demand scenarios can be viewed as a representative set of forecasts on possible demand realizations over the planning horizon. The actual demand realization over the planning horizon may not match exactly any of the scenarios used because the number of possible demand realizations is normally far more than the number of scenarios used.

Using demand scenarios to model uncertain demand is a common technique employed in the literature. Examples include Eppen et al. (1989) and Lucas et al. (2001) for production capacity planning problems, Smith et al. (1998) for a



seasonal product pricing and advertisement planning problem, Bent and van Hentenryck (2004) and Hvattum et al. (2006) for vehicle routing problems, and Chang et al. (2005) for a natural disaster preparedness problem. Estimating a limited number of possible demand scenarios is much easier than characterizing the entire distribution. Furthermore, as we will show later in our computational experiments, approximating the random demand with a discrete set of scenarios can still yield satisfactory solutions.

The demand scenarios together form a tree structure. In the demand scenario tree, the root node represents the beginning of the planning horizon and the terminal nodes represent the end of the planning horizon. Any intermediate nodes represent the end of one period and start of the next period. Each arc represents a possible demand outcome in a particular time period  $t$ . A path from the root node to a node at the end of period  $t$  represents the demand evolution from the beginning of the planning horizon until the end of time period  $t$ . A path from the root node to a terminal node represents a complete demand evolution over the entire planning horizon, which is called a demand scenario. Associated with each scenario, there is an estimated probability that indicates how likely the actual demand realization is represented by this particular scenario. Decisions are made at the root node of the tree (i.e., beginning of the planning horizon) before knowing which demand scenario is going to occur.

One of the advantages of using a demand scenario tree is that it enables us to model demand correlation across time periods. In the problem we consider, there is limited demand information at the beginning of the planning horizon. More information becomes available over time. Realized demand in one period often contains important market information which should be used to forecast possible demand in the future periods. For example, if the realized demand is high in one period, it is very likely that the demand in the next several periods is going to be high as well. Using a demand scenario tree enables us to model such dependency across time periods.

Figure 3.1 illustrates an example of demand scenario tree for the case with a

single store, two time periods, and two allowable price levels,  $p_1 = 50$  and  $p_2 = 40$ . The two numbers in the parenthesis on each arc represent the demand values under the two prices  $p_1$  and  $p_2$ , respectively. There are two possible demand outcomes in time period 1, representing, for example, good or bad market conditions in this period, with the respective demand values  $(60, 120)$  or  $(40, 80)$ . Again there are two possible demand outcomes in time period 2 and these outcomes depend on the demand realization in time period 1. If the demand realization in period 1 is high, as represented by the outcome  $(60, 120)$ , then the two possible outcomes in period 2 are  $(75, 130)$  and  $(55, 110)$ , respectively. In contrast, if the demand realization in period 1 is low, as represented by the outcome  $(40, 80)$ , then the two possible outcomes in period 2 are  $(45, 90)$  and  $(35, 70)$ , respectively. Altogether, there are four different scenarios in this example. As we can see from this simple example, using the tree structure, we are able to model a situation where there are demand correlation across time periods.

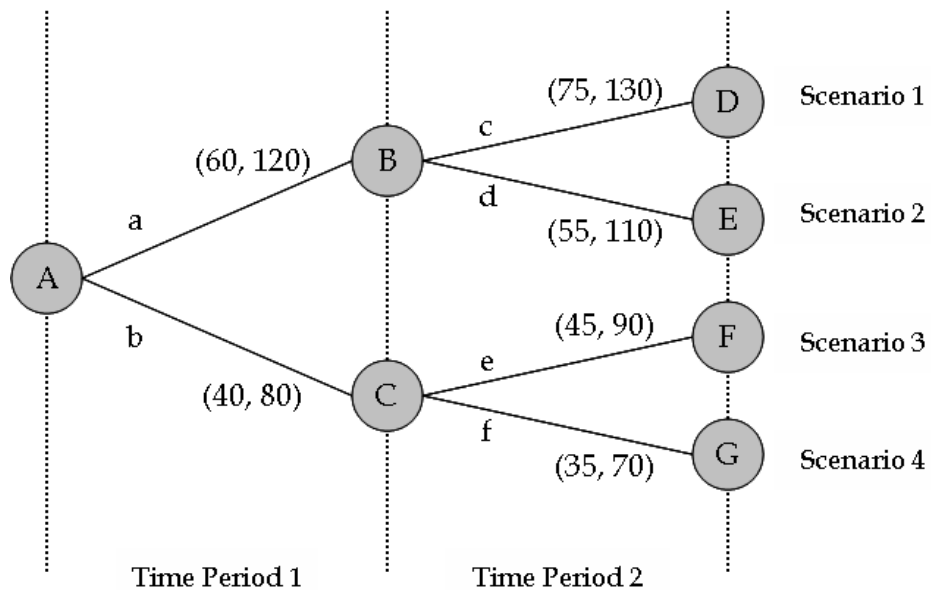


Figure 3.1: A Simple Example of Demand Scenario Tree

In our rolling horizon approach, whenever we roll ahead for one time period,

we re-forecast the demand based on the latest market information and create a new scenario tree for the remaining time periods. Suppose that we are at the beginning of time period  $\tau$ . Using all the information available up to the current time point, a new demand scenario tree is created for the remaining planning horizon consisting of periods  $\tau, \tau + 1, \dots, T$ . We define the following notation to describe this scenario tree. See Section 3.5.1.3 for details on how a demand scenario tree is constructed in our computational experiments.

$\Omega$ : The set of scenarios in the demand scenario tree.

$P_\omega$ : The probability associated with scenario  $\omega$ , with  $0 < P_\omega \leq 1$  and  $\sum_{\omega \in \Omega} P_\omega = 1$ .

$D_{rjt}^\omega$ : The demand at store  $r$  for price level  $j$  in period  $t$  under scenario  $\omega$ .

$\mathcal{A}(t)$ : The set of arcs in period  $t$  in the scenario tree, for  $t = \tau, \dots, T$ .

$\Gamma_t^\alpha$ : The set of scenarios that share a common arc  $\alpha$  in period  $t$  in the scenario tree, for  $\alpha \in \mathcal{A}(t)$  and  $t = \tau, \dots, T$ . We denote  $\Gamma_t^\alpha = \{\omega_1^\alpha, \omega_2^\alpha, \dots, \omega_{|\Gamma_t^\alpha|}^\alpha\}$ , where  $\omega_i^\alpha \in \Omega$ , for  $i = 1, \dots, |\Gamma_t^\alpha|$ , and  $|\Gamma_t^\alpha|$  is the number of scenarios in  $\Gamma_t^\alpha$ .

For the example shown in Figure 1,  $\Omega = \{1, 2, 3, 4\}$ ,  $\mathcal{A}(1) = \{a, b\}$ ,  $\mathcal{A}(2) = \{c, d, e, f\}$ ,  $\Gamma_1^a = \{1, 2\}$ , and  $\Gamma_1^b = \{3, 4\}$ .

### 3.2.2 Formulation

In this section we give an approximate mixed integer programming (MIP) formulation for the problem we face at the beginning of each period. Suppose that we are at the beginning of a particular period  $\tau$  and we have created a new demand scenario tree for the remaining planning horizon consisting of periods  $\tau, \tau + 1, \dots, T$ . Suppose that the price level used in period  $\tau - 1$  for store  $r$  is  $j_{0r}$ , the remaining number of allowable markdowns is  $R_{0r}$  (which is  $R$  minus the number of markdowns already implemented in the first  $\tau - 1$  periods), and the total amount of remaining inventory in the warehouse is  $I'_0$  (which is  $I_0$  minus the total amount of inventory already allocated to the stores in the first  $\tau - 1$  periods). Our problem is to determine how much inventory to allocate to each store and which price level to use in each store in each of the remaining periods  $\tau, \tau + 1, \dots, T$  so that the total expected

revenue is maximized.

Our formulation incorporates the given demand scenarios and treats the involved decisions in the following way: (i) Pricing decisions for all the periods are made at the root node (i.e., at the beginning of period  $\tau$ ) independent of the scenarios; (ii) Inventory allocation decisions across time are made based on the actual demand realizations, and are hence scenario dependent. We note that the pricing decision for the current period  $\tau$  is indeed independent of the scenarios because in reality the prices for period  $\tau$  are set before the actual demand realization in period  $\tau$  is known. However, the pricing decision for future periods  $\tau + 1, \dots, T$  that we make at the beginning of period  $\tau$  should be scenario dependent because in reality the pricing decisions for a future period  $t$  ( $t > \tau$ ) depend on what have happened (i.e., the actual demand realizations) in periods  $\tau, \tau + 1, \dots, t - 1$ .

So our formulation does not formulate our problem precisely; instead it is an approximation. However, it should be noted that in our overall solution approach (described in Section 3.3.1) which is rolling horizon based, after we solve our formulation at the beginning of each period  $\tau$ , only the solution for the current period  $\tau$  is implemented. This means that even though pricing decisions for future periods are also included in our formulation, they are not implemented. Since we solve a new formulation (which incorporates the latest demand information) every time we move one time period forward, the pricing decisions that are implemented in each period are in fact made based on the actual demand realizations up to the beginning of this time period.

Another formulation where the pricing decisions are scenario dependent is given in Chapter B of the appendix. As we show in the appendix, the formulation with scenario-dependent pricing decisions is much larger in scale and much more time consuming to solve than our approximate formulation. Furthermore, in the appendix we show that under the same overall rolling horizon based solution framework, the revenue gain by using the formulation with scenario-dependent pricing decisions (relative to the approximate formulation) is small (varying from 0.6% to 2.3%). For these reasons, we adopt the approximate formulation and use it in our overall

solution approach described in Section 3.3.1. Below, we describe our approximate formulation.

Before giving the complete formulation, we discuss briefly how the various business rules are formulated. We define the following decision variables to formulate business rules (i) - (iv):

$I_r^\omega$ : Non-negative continuous variable denoting the total inventory allocated to store  $r$  across all the periods  $\tau, \dots, T$  under scenario  $\omega$

$X_{rjt}$ : Binary variable indicating whether price level  $j$  is selected at store  $r$  in period  $t$

$H_{rj}$ : Binary variable indicating whether markdown price level  $j(j > j_{0r})$  is ever used at store  $r$  in the periods  $\tau, \dots, T$ . This variable is used to model business rule (iii).

$S_{rjt}^\omega$ : Non-negative continuous variable indicating the quantity sold at store  $r$  for price level  $j$  during period  $t$  under scenario  $\omega$

It is well known that in general, an integer programming formulation is easier to solve when it is tightly formulated (i.e., its LP relaxation is tight) even if the tighter formulation requires a larger number of constraints and variables. Hence, our goal is to formulate each business rule and each constraint as tight as possible. Rules (i) (i.e., minimum inventory allocation in the first period) and (iii) (i.e., number of markdowns allowed) are straightforward. We use the following constraint to formulate rule (ii) (i.e., non-increasing prices over time):

$$\sum_{j=j_{0r}}^h X_{rj(t+1)} \leq \sum_{j=j_{0r}}^h X_{rjt}, \quad \forall r \in N, h \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\} \quad (3.1)$$

Constraint (3.1) ensures that at each store, for each time period, if a particular price is chosen, then either that price or a higher one should have been chosen for the earlier period. Enforcing this for each set of adjacent time periods ensures that price once decreased will not be increased again.

Rule (iv) (i.e., lower and upper bounds on price change) can be formulated as

follows:

$$X_{rjt} \leq X_{rj(t+1)} + \sum_{l=u_j}^{v_j} X_{rl(t+1)}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m-1\}, t \in \{\tau, \dots, T-1\} \quad (3.2)$$

This constraint works as follows. If the left hand side of this constraint is 1 (i.e., the price level for period  $t$  is  $j$ ), one of the variables specified on the right hand side should also be 1 (i.e., either  $X_{rj(t+1)}$  is 1, meaning that the price level remains the same in period  $t+1$ , or one of  $X_{rl(t+1)}$ 's is 1 for some  $l$  within the range prescribed by rule (iv). If the left hand side is zero, then the constraint is redundant.

To formulate rule (v) (i.e., stores in a store cluster use similar prices), we define the following decision variables:

$Y_{qgt}$ : Binary variable that takes a value of 1 if the store cluster  $C_q$  uses the price cluster  $E_g$  in period  $t$  and 0 otherwise

Then rule (v) can be formulated as:

$$\sum_{g=1}^G Y_{qgt} = 1, \quad \forall q \in \{1, \dots, Q\}, t \in \{\tau, \dots, T\} \quad (3.3)$$

$$Y_{qgt} \leq \sum_{j \in E_g} X_{rjt}, \quad \forall q \in \{1, \dots, Q\}, r \in C_q, \\ g \in \{1, \dots, G\}, t \in \{\tau, \dots, T\} \quad (3.4)$$

The first constraint, (3.3), ensures that exactly one price cluster is used for each store cluster in each period. The second constraint, (3.4), guarantees that if a particular price cluster is used by a store cluster then each store within the cluster uses one of the prices in that price cluster.

In addition to the constraints discussed above, we need to add the so-called nonanticipativity constraints for some variables. When two scenarios share the same demand history up to time period  $t$ , all the decisions up to time period  $t$  must be identical for these two scenarios. Since variables  $S_{rjt}^\omega$ 's are scenario and time period dependent, they must satisfy such nonanticipativity constraints as follows.

$$S_{rjt}^{\omega_i^\alpha} = S_{rjt}^{\omega_{i+1}^\alpha}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\}, \\ i = 1, \dots, |\Gamma_t^\alpha| - 1, \alpha \in \mathcal{A}(t) \quad (3.5)$$

Now we are ready to give the complete MIP formulation for the problem we have to solve at the beginning of each rolling horizon consisting of periods  $\tau, \tau + 1, \dots, T$ . We denote this formulation as  $[\text{MIP}_\tau]$ .

$$[\text{MIP}_\tau] \quad \max \quad \sum_{\omega \in \Omega} \left( P_\omega \sum_{r=1}^n \sum_{j=j_{0r}}^m \sum_{t=\tau}^T p_j S_{rjt}^\omega \right) + s \sum_{\omega \in \Omega} \left( P_\omega \left( I'_0 - \sum_{r=1}^n \sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^\omega \right) \right) \quad (3.6)$$

Subject to:

$$\sum_{j=j_{0r}}^m X_{rjt} = 1, \quad \forall r \in N, t \in \{\tau, \dots, T\} \quad (3.7)$$

$$\sum_{j=j_{0r}}^h X_{rj(t+1)} \leq \sum_{j=j_{0r}}^h X_{rjt}, \quad \forall r \in N, h \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\} \quad (3.8)$$

$$X_{rjt} \leq X_{rj(t+1)} + \sum_{l=u_j}^{v_j} X_{rl(t+1)}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m-1\}, t \in \{\tau, \dots, T-1\} \quad (3.9)$$

$$\sum_{l=j_{0r}}^j X_{rlt} \leq \sum_{l=j_{0r}}^{v_j} X_{rl(t+1)}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m-1\}, t \in \{\tau, \dots, T-1\} \quad (3.10)$$

$$H_{rj} \geq X_{rjt}, \quad \forall r \in N, j \in \{j_{0r} + 1, \dots, m\}, t \in \{\tau, \dots, T\} \quad (3.11)$$

$$\sum_{j=j_{0r}+1}^m H_{rj} \leq R_{0r}, \quad \forall r \in N \quad (3.12)$$

$$\sum_{g=1}^G Y_{qgt} = 1, \quad \forall q \in \{1, \dots, Q\}, t \in \{\tau, \dots, T\} \quad (3.13)$$

$$Y_{qgt} \leq \sum_{j \in E_g} X_{rjt}, \quad \forall q \in \{1, \dots, Q\}, r \in C_q, g \in \{1, \dots, G\}, t \in \{\tau, \dots, T\} \quad (3.14)$$

$$S_{rjt}^\omega \leq D_{rjt}^\omega X_{rjt}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T\}, \omega \in \Omega \quad (3.15)$$

$$\sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^\omega \leq I_r^\omega, \quad \forall r \in N, \omega \in \Omega \quad (3.16)$$

$$S_{rjt}^{\omega_i^\alpha} = S_{rjt}^{\omega_{i+1}^\alpha}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\},$$

$$i = 1, \dots, |\Gamma_t^\alpha| - 1, \alpha \in \mathcal{A}(t) \quad (3.17)$$

$$\sum_{r=1}^n I_r^\omega \leq I'_0, \quad \forall \omega \in \Omega \quad (3.18)$$

$$X_{rjt}, H_{rj}, Y_{qgt} \in \{0, 1\}, I_r^\omega, S_{rjt}^\omega \geq 0, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\},$$

$$q \in \{1, \dots, Q\}, g \in \{1, \dots, G\} t \in \{\tau, \dots, T\}, \omega \in \Omega \quad (3.19)$$

In addition, there is one more constraint that we have to add if  $\tau = 1$ , to formulate business rule (i), as follows.

$$I_r^\omega \geq I_r^{\min}, \quad \forall r \in N, \omega \in \Omega \quad (3.20)$$

In the above formulation, the objective function (3.6) maximizes the total expected revenue under all scenarios by taking into account the revenue collected from sales as well as the salvage value. Constraint (3.7) makes sure that each period each store is allotted one and only one price level. Constraints (3.8) and (3.9) enforce rules (ii) and (iv), as explained earlier. Constraint (3.10) ensures that the maximum price drop restriction is not violated. This constraint is redundant for the integer feasible region as the maximum price drop restriction is included in constraint (3.9), but adding this constraint makes the LP-relaxation tighter. We demonstrate this by an example. Consider a problem instance with just one store, where we have four price levels and two time periods  $t \in \{1, 2\}$  ( $\tau = 1$  and  $T = 2$ ). There is no minimum price drop restriction, but the maximum price drop restricts any jumps of more than one level during a price change. That is, price can be marked down from level 1 to level 2, but not to level 3. Now consider a LP-relaxation solution as follows:  $X_{111} = X_{121} = 0.5$ ,  $X_{131} = X_{141} = 0$  for the first period, and  $X_{112} = X_{132} = 0$ ,  $X_{122} = X_{142} = 0.5$  for the second period. It can be easily verified that this particular solution satisfies (3.9), but not (3.10). Thus the additional constraint (3.10), though redundant for the integer feasible region, helps tighten the feasible region of the LP relaxation.

Constraints (3.11) and (3.12) formulate rule (iii). Constraints (3.13) and (3.14) formulate rule (v). Constraint (3.15) makes sure that in a period we sell at a



particular price only if that price has been selected and sales are always no more than the demand. Constraint (3.16) limits the total sales at a store to the quantity that has been allocated to that store under any scenario. Constraint (3.17) is the nonanticipativity constraint as discussed earlier. Constraint (3.18) ensures that the total quantity allocated across all the stores under any scenario does not exceed the inventory available at the beginning of period  $\tau$ . Constraint (3.20) formulates rule (i) which is enforced in the very first period.

We note that in a period, the inventory may not be sufficient to satisfy the demand. Under such cases,  $S_{rjt}^\omega$  will be strictly less than  $D_{rjt}^\omega$ . Also, if there is inventory available, demand should be satisfied. That is, inventory cannot be held back for future demands, while refusing current demand. Although this is not modeled, it is enforced implicitly by the formulation. The reason for this is that since price markups are not allowed, revenue per unit item is non-increasing over time at each store. So, in an optimal solution, demand will not be refused when inventory is available.

### 3.3 Solution Approach

In the appendix we prove that our problem is NP-hard (i.e., computationally intractable) even when the demand is deterministic and when there is only a single store or there is only a single time period. Therefore, it is very unlikely that one can find an optimal solution to our problem within a reasonable computational time even if there is no demand uncertainty. This justifies us to use a heuristic approach. In this section, we propose an optimization based heuristic solution approach implemented on a rolling horizon basis. Below we first describe the overall rolling horizon based solution approach, followed by the description of a Lagrangian relaxation algorithm for solving the mixed integer programming problem  $[MIP_\tau]$  involved at the beginning of each period  $\tau$  in the overall approach.

### 3.3.1 Overall Rolling Horizon Based Approach

Our overall solution approach consists of the following procedures.

Initially, set the current period  $\tau = 1$ .

**Step 1:** At the beginning of period  $\tau$ , use the latest demand information to create a demand scenario tree for the remaining planning horizon consisting of period  $\tau, \tau + 1, \dots, T$  (see Section 3.5.1.3 for details on how demand scenario trees are constructed in our computational experiments). Formulate the problem over this planning horizon as the MIP formulation  $[\text{MIP}_\tau]$  (described in Section 3.2.2). Solve this formulation by the Lagrangian relaxation algorithm (described below in Section 3.3.2). This gives a solution that specifies a price to use at each store in each period (i.e.,  $X_{rjt}$  values).

**Step 2:** Given the solution from Step 1, the part for period  $\tau$  is implemented as follows: (i) the prices given in this solution for period  $\tau$  (i.e.,  $X_{rj\tau}$  values) are set at the stores for period  $\tau$ ; (ii) necessary inventory is shipped daily to each store as sales progress in period  $\tau$ . Since as explained in Section 3.1, there are daily shipments of regular products from the warehouse to each store and markdown items can get a free ride, we can assume that the demand at each store is satisfied daily if there is enough inventory. If the available inventory in the warehouse is enough to satisfy the total demand of all the stores over all the days in period  $\tau$ , then by the end of period  $\tau$ , the total amount of inventory shipped to each store  $r$  is equal to the actual demand of that store in period  $\tau$ . Otherwise, the total amount of inventory shipped to each store  $r$  is equal to the total demand of that store up to the day (before the end of period  $\tau$ ) when the inventory in the warehouse is depleted.

**Step 3:** At the end of period  $\tau$ , update the available inventory at the warehouse. If the available inventory at the warehouse is nonzero, then set  $\tau = \tau + 1$  and go to Step 1.

### 3.3.2 Solving [MIP $_{\tau}$ ]

In Step 1 of the overall rolling horizon based solution approach described in the previous subsection, we need to solve the MIP formulation [MIP $_{\tau}$ ] (given in Section 3.2.2) at the beginning of each period  $\tau$ , for  $\tau = 1, 2, \dots, T$ . This formulation has a very large scale for problems with a practical size. For instance, for a problem with 50 stores, 8 time periods, 8 allowable prices, and 81 scenarios (which is one of the problem configurations we test in our computational experiments described in Section 3.5), this MIP formulation includes more than 4,900 integer variables ( $X_{rjt}$ ,  $H_{rj}$ ,  $Y_{qgt}$ ), 260,000 continuous variables ( $S_{rjt}^{\omega}$ ,  $I_r^{\omega}$ ) and 260,000 constraints. It is impractical to solve such a large scale MIP problem directly. Thus, we propose a Lagrangian relaxation based decomposition approach to get a near optimal solution for this formulation within a reasonable amount of time.

The idea is to relax certain constraints and add them to the objective function so that the original problem can be decomposed into smaller problems. We move constraint (3.18) to the objective function with appropriate penalties (known as Lagrangian multipliers). The relaxed problem has a new objective function as shown below in equation (3.21) where  $\lambda^{\omega} \geq 0$  is the Lagrangian multiplier for constraint (3.18) corresponding to scenario  $\omega \in \Omega$ .

$$\begin{aligned} \max \quad & \sum_{\omega \in \Omega} P_{\omega} \left( \sum_{r=1}^n \sum_{j=j_{0r}}^m \sum_{t=\tau}^T p_j S_{rjt}^{\omega} \right) + s \sum_{\omega \in \Omega} P_{\omega} \left( I'_0 - \sum_{r=1}^n \sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^{\omega} \right) \\ & + \sum_{\omega \in \Omega} \lambda^{\omega} \left( I'_0 - \sum_{r=1}^n I_r^{\omega} \right) \end{aligned} \quad (3.21)$$

The Lagrangian relaxation problem (with objective function (3.21) subject to constraints (3.7) through (3.17), and (3.19), plus (3.20) if  $\tau = 1$ ) is now decomposed into  $Q + n_I$  subproblems. Each subproblem corresponds to a particular store cluster  $q$  or an independent store  $r$ .

The subproblem for store cluster  $q$ , for  $q = 1, \dots, Q$ , can be stated as follows.

$$\begin{aligned} \max \quad & \sum_{\omega \in \Omega} \left( P_{\omega} \sum_{r \in C_q} \sum_{j=j_{0r}}^m \sum_{t=\tau}^T p_j S_{rjt}^{\omega} \right) - s \sum_{\omega \in \Omega} \left( P_{\omega} \sum_{r \in C_q} \sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^{\omega} \right) \\ & - \sum_{\omega \in \Omega} \sum_{r \in C_q} \lambda^{\omega} I_r^{\omega} \end{aligned} \quad (3.22)$$

Subject to: Constraints (3.7) through (3.17), and (3.19), plus (3.20) if  $\tau = 1$ , for store cluster  $q$  only. Remove “ $q \in \{1, \dots, Q\}$ ” in constraints (3.13), (3.14), and (3.19), and change “ $r \in N$ ” to “ $r \in C_q$ ” in all other constraints.

Similarly, the subproblem for an independent store  $r \in N \setminus (C_1 \cup \dots \cup C_Q)$  can be stated as follows.

$$\begin{aligned} \max \quad & \sum_{\omega \in \Omega} \left( P_{\omega} \sum_{j=j_{0r}}^m \sum_{t=\tau}^T p_j S_{rjt}^{\omega} \right) - s \sum_{\omega \in \Omega} \left( P_{\omega} \sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^{\omega} \right) \\ & - \sum_{\omega \in \Omega} \lambda^{\omega} I_r^{\omega} \end{aligned} \quad (3.23)$$

Subject to: Constraints (3.7) through (3.12), (3.15) through (3.17), (3.19), plus (3.20) if  $\tau = 1$ , for store  $r$  only. Remove “ $r \in N$ ” in all these constraints.

For any given set of Lagrangian multipliers  $\lambda^{\omega}$  for  $\omega \in \Omega$ , we solve the Lagrangian relaxation problem by solving the  $Q + n_I$  subproblems as defined above. The objective function of the Lagrangian relaxation problem (3.21) is the summation of the objective functions of all the subproblems plus a constant term  $sI'_0 + \sum_{\omega \in \Omega} \lambda^{\omega} I'_0$ . In practice, typically there are only a small number of stores (no more than 5) in a store cluster. Thus the subproblem for each store cluster is much smaller than the original problem. Clearly the subproblem for each independent store is even smaller.

For any given set of non-negative Lagrangian multipliers  $\lambda^{\omega}$  for  $\omega \in \Omega$ , the optimal objective function value of the Lagrangian relaxation problem provides an upper bound for the problem [MIP $_{\tau}$ ]. The problem of finding the optimal Lagrangian multipliers that generate the minimum upper bound is called the Lagrangian dual. A commonly used approach for solving the Lagrangian dual is the subgradient algorithm. In order to implement the subgradient algorithm, one needs to specify

appropriate initial values and choose a step size in each iteration to iteratively update the values of the Lagrangian multipliers. Appropriate initial values and step sizes are critical in ensuring the convergence of the subgradient algorithm. Unfortunately, these values are often problem specific and very difficult to find.

The Lagrangian multipliers for our problem, however, have a special economic interpretation which can be used to identify a range for the possible values of the optimal Lagrangian multipliers. It can be seen that  $\lambda^\omega$  is the expected marginal revenue of inventory under scenario  $\omega$ . On one hand, an extra unit of inventory can generate an additional revenue of at least  $s$  (e.g., keep this unit in the warehouse and sell it at the end of the planning horizon to receive the salvage value). On the other hand, the maximum additional revenue an extra unit of inventory can generate is no more than  $p_1$ , the maximum possible selling price. Therefore, the possible value of the optimal Lagrangian multiplier  $\lambda^\omega$  must be within the range  $[sP_\omega, p_1P_\omega]$ .

Knowing the range of  $\lambda^\omega$  enables us to develop an efficient algorithm for solving the Lagrangian dual. We search a value within this range for each Lagrangian multiplier  $\lambda^\omega$  in a heuristic way. The value we find for each  $\lambda^\omega$  may not be optimal, but gives a fairly tight Lagrangian upper bound, which enables us to generate near optimal solutions to  $[\text{MIP}_\tau]$ , as shown in our computational experiment described in Section 3.5. Our idea is very similar to the line search algorithms commonly used in the nonlinear programming literature (Bazaraa, et al. 1993). It works as follow. We set the initial value of each Lagrangian multiplier  $\lambda^\omega$  to be the middle point of its possible range, i.e.,  $P_\omega(p_1 + s)/2$ . We then solve the Lagrangian relaxation problem. If the resulting solution violates constraint (3.18) for a specific  $\omega$ , we increase  $\lambda^\omega$  by a step size  $\xi^\omega$ ; otherwise, we decrease  $\lambda^\omega$  by  $\xi^\omega$ . Unlike most commonly used subgradient algorithms for solving a Lagrangian dual problem where step sizes typically depend on how much the constraint(s) moved to the objective function is (are) violated, our algorithm uses pre-determined step sizes which do not depend on the magnitude by which constraint (3.18) is violated. Below we describe our algorithm in detail.

### Lagrangian Relaxation Algorithm (LRA) for Solving [MIP<sub>τ</sub>]

**Step 0: Initialization.** For  $\forall \omega \in \Omega$ , set the initial Lagrangian multiplier  $\lambda^\omega = P_\omega(p_1 + s)/2$  and the initial step size  $\xi^\omega = P_\omega(p_1 - s)/4$ . Set the initial Lagrangian upper bound  $F^0 = \infty$ . Set the iteration counter  $K = 1$ . Set the stopping criterion counter  $L = 0$ .

**Step 1: Solve the Lagrangian Relaxation Problem.** For iteration  $K$ , solve the Lagrangian relaxation problem by solving each subproblem with the given multipliers  $\lambda^\omega$ 's. This gives a new objective function value  $F$  and a new solution. If  $F < F^0$ , make this solution the incumbent solution.

**Step 2: Check the stopping criteria.** If  $(F^0 - F)/F^0 < \alpha$  for some parameter  $\alpha$  (e.g.,  $\alpha = 0.05\%$ ),  $L = L + 1$ ; otherwise,  $L = 0$ . If  $L = 2$  or  $K = 15$ , stop the algorithm and take the incumbent solution; otherwise, update the Lagrangian upper bound,  $F^0 = \min(F^0, F)$ .

**Step 3: Update the Lagrangian multipliers and the step size.** For  $\forall \omega \in \Omega$ , check if the solution obtained in Step 1 violates constraint (3.18). If yes, let  $\lambda^\omega = \lambda^\omega + \xi^\omega$ ; otherwise let  $\lambda^\omega = \lambda^\omega - \xi^\omega$ . Reduce the step size by half, i.e.,  $\xi^\omega = \xi^\omega/2$ . Update the iteration counter,  $K = K + 1$ , and go back to Step 1.

We note that in this algorithm  $L$  is used to keep track of the number of consecutive iterations in which the optimal objective value of the Lagrangian relaxation is not improved by the required minimum percentage value  $\alpha$ . It can be seen from Step 2 that the algorithm is terminated if there is no improvement beyond the minimum requirement in two consecutive iterations (i.e.,  $L = 2$ ), or if the total number of iterations reaches 15.

The solution obtained in this algorithm may not be feasible for the problem [MIP<sub>τ</sub>] since constraint (3.18) might be violated. Therefore, we need to construct a feasible solution based on the solution generated by this algorithm. This can be achieved by the following procedure. We first fix the integer variables  $X_{rjt}$ 's,  $H_{rj}$ 's, and  $Y_{qgt}$ 's in the formulation [MIP<sub>τ</sub>] with the values from the solution of this algorithm. This results in a linear programming formulation with continuous variables  $S_{rjt}^\omega$ 's and  $I_r^\omega$ 's only. We then solve the LP problem by a direct LP solver

to obtain an optimal solution for variables  $S_{rjt}^\omega$ 's and  $I_r^\omega$ 's.

### 3.4 Benchmark Approaches

In this section, we first describe four simple markdown pricing policies commonly used in practice. We then introduce a sequential approach used by several companies we have worked with. Finally, we provide an upper bound on the total revenue one can achieve under a particular demand realization. These benchmark approaches and upper bound are used in Section 3.5 to evaluate the performance of our approach described in Section 3.3.

#### 3.4.1 Simple Markdown Pricing Policies

In practice, most companies use simple markdown pricing policies for ease of implementation. The following four approaches have been widely used (Mantrala and Rao 2001).

- Simple Policy 1 (referred to herein as P1): Under this policy, regular price  $p_1$  is used for the first two periods and then 25% is marked down every two periods thereafter, i.e., 25% off (i.e.,  $0.75p_1$ ) for periods 3 and 4, 50% off (i.e.,  $0.5p_1$ ) for periods 5 and 6, and 75% off (i.e.,  $0.25p_1$ ) for the remaining periods. The fact that the allowable prices are discrete in our problem may not allow us to implement this policy in an exact way. If this happens, we use the nearest allowable price.
- Simple Policy 2 (referred to herein as P2): Under this policy, regular price  $p_1$  is used for the first half planning horizon and then 50% discount is applied to the remaining planning horizon, i.e.,  $p_1$  for periods 1 to  $\lfloor T/2 \rfloor$  and  $p_1/2$  for periods  $\lfloor T/2 \rfloor + 1$  to  $T$ . In the case if  $p_1/2$  is not an allowed price, we use the nearest allowable price.
- Simple Policy 3 (referred to herein as P3): Regular price  $p_1$  is applied to the entire planning horizon.
- Simple Policy 4 (referred to herein as P4): 25% off (i.e.,  $0.75p_1$ ) is applied to the entire planning horizon. If  $0.75p_1$  is not an allowed price, we use the nearest

allowable price.

P1 is adapted from the well-known Filene’s markdown policy (Bitran and Mondschein 1997). P3 and P4 are single price policies. A single price policy is proved to be asymptotically optimal under certain conditions when there is only a single store in the problem (Gallego and van Ryzin 1994). While these simple policies are easy to implement, they do not react to changing market conditions and are not updated over time. These policies can be implemented at once at the beginning of the planning horizon and do not require a rolling horizon approach.

### 3.4.2 Sequential Approach

Several retailers that we have collaborated with use a more sophisticated approach than the simple policies described in the previous subsection. They use a so-called sequential approach (referred to herein as SA). Similar to our approach, this approach is also implemented on a rolling horizon basis. However, it differs from our approach in the following aspects: (i) it does not model stochastic demand by a scenario tree; instead, it uses expected demand only and treats the problem as deterministic; (ii) it makes inventory allocation decision first, followed by pricing decision (and hence is called “sequential”); and (iii) it uses the same price for all stores within the same cluster in each time period.

This approach works as follows. Under the rolling horizon framework, assume that we are at the beginning of time period  $\tau$ .

**Step 1:** Available inventory is allocated to each store cluster and each independent store proportionally to its total expected demand in the remaining horizon (periods  $\tau$  through  $T$ ) under the price used in the previous period  $\tau - 1$ . It should be noted that the determination of such an inventory allocation is solely for determining the pricing decisions in the second step. Only necessary inventory for one period is actually shipped to each store in each period as in our approach discussed in Section 3.3.1.

**Step 2:** A single price is determined for each store cluster and each independent store and is implemented for the current period  $\tau$ . The price is determined such



that it is feasible with respect to the business rules and it yields the highest total expected revenue if this price is kept in all the remaining periods for this store cluster or the independent store involved. For example, if there are three allowable prices, i.e.,  $p_1 = 100$ ,  $p_2 = 80$ ,  $p_3 = 60$ , with the total expected demand in the remaining horizon being 100, 200, 300 respectively, and if the inventory allocated to this store cluster (or store) is 180, then we will choose  $p_2 = 80$ .

### 3.4.3 Upper Bound

In reality, the actual demand realization in each period is not known until at the end of this period. However, if we assume that the demand realization across the entire planning horizon is known at the very beginning of the planning horizon, then the problem becomes deterministic and can be formulated as  $[\text{MIP}_1]$  with one demand scenario only which is the actual demand realization. Obviously the optimal solution we get by solving this deterministic problem cannot be implemented in reality because it is impossible to know the actual demand realization in advance. However, the optimal objective value of this problem provides an upper bound of the total revenue one can achieve in reality for that particular demand realization.

## 3.5 Computational Experiments

In this section, we conduct computational experiments to address the following questions:

- (i) How good is the Lagrangian relaxation algorithm (LRA) described in Section 3.3.2 for the MIP formulation  $[\text{MIP}_\tau]$  that we have to solve at the beginning of each period  $\tau$ ? Since  $[\text{MIP}_1]$  is the largest in scale, we focus on  $[\text{MIP}_1]$  and compare the solution generated by LRA and the optimal solution generated by a commercial MIP solver.
- (ii) How good is our overall solution approach proposed in Section 3.3.1 for our entire problem? We compare our approach to the benchmark approaches and the upper bound described in Section 3.4.

These questions are investigated using an extensive set of randomly generated test instances. In the following, we first describe how random test problems are generated in Section 3.5.1, followed by the computational results in 3.5.2 including Section 3.5.2.1 for (i) and Section 3.5.2.2 for (ii), respectively.

### 3.5.1 Design of Experiments

Since our problem is motivated by a real-world situation, we design a set of test problems that capture major characteristics commonly encountered in practice. Due to commercial confidentiality, we are not allowed to report the real data sets from the company that we have worked with. We instead generate our own random data sets that closely follow the structure of the real data. Given that there is a large number of parameters in our problem, it is not possible to report test results from varying each of the parameters independently. Hence based on trial runs, we choose a few key parameters to vary while keep the values of the other parameters fixed.

Although our solution approach does not require any specific demand distributions, for ease of generating test problems, we may use some specific demand distributions that closely mimic what might actually happen in reality.

#### 3.5.1.1 Parameter Configurations

We generate test problem instances based on the parameter configurations as follows.

- Number of time periods,  $T = 8$ . A typical time period in practice is for a duration of one or two weeks, and the entire markdown horizon goes anywhere from a few weeks to about three months.
- Number of stores,  $n = 50$ . The number of retail stores served by a typical warehouse for the company that we have worked with varies from 30 to about 100, but in most cases, no more than 50.
- Number of allowable price levels,  $m = 8$ . The regular price  $p_1 = 100$ , and the

markdown prices are set to be 10%, 20%, ..., 60%, and 70% lower than the regular price, respectively, i.e.,  $p_i = 100 - 10(i - 1)$ , for  $i = 2, \dots, 8$ .

- For business rule (i), the minimum amount of inventory that must be allocated to a store in the first period,  $I_r^{\min} = 10$ , for every store  $r \in N$ .
- For business rule (iii), the number of markdowns allowed at each store,  $R = 5$ .
- For business rule (iv), we require that in each period if there is a price drop, it has to drop at least 10% and no more than 30% from the regular price. Given the allowable prices specified above, this means that the parameters  $u_j$  and  $v_j$  associated with this business rule are:  $u_j = \min\{j + 1, 8\}$  and  $v_j = \min\{j + 3, 8\}$ .
- Number of store clusters  $Q$  for business rule (v). One way to think of store clusters is to group together all the stores in a given geographical area. A good measure for this is the Metropolitan Statistical Area (MSA) as defined by the Census Bureau. We set the cluster size (i.e., the number of stores in each cluster) to be 3 or 4. We also assume that about half of the stores belong to a cluster, while the other half are independent (for example, a store in a small city that has just one store of its kind). Therefore, among the 50 stores, 25 of them belong to 8 different store clusters (i.e.,  $Q = 8$ ) while the other 25 are independent (i.e.,  $n_I = 25$ ). We allow the price difference between the stores in a store cluster to be no more than 10% of the regular price. Given the allowable prices set earlier, we can see that there are 7 price clusters, i.e.,  $G = 7$ , and they are:  $E_1 = \{100, 90\}$ ,  $E_2 = \{90, 80\}$ , ...,  $E_7 = \{40, 30\}$ .
- We assume salvage value per unsold item,  $s = 0$ . Any problem with a positive salvage value can be transformed into an equivalent problem with zero salvage value (Gallego and van Ryzin 1994).
- We employ the following multiplicative demand function to model random demand  $D_{rjt}$  for store  $r$  in period  $t$  under price  $p_j$ .

$$D_{rjt} = \theta_{rt} d_r f(p_j) \phi(t) \tag{3.24}$$

This demand function consists of a random variable  $\theta_{rt}$  and three deterministic terms, i.e.,  $d_r$ ,  $f(p_j)$  and  $\phi(t)$ . They are described as follows.

◦  $\theta_{rt}$  is a random component that represents the overall market condition for store  $r$  in period  $t$ . The overall market condition  $\theta_{rt}$  is determined by all random factors that the retailer has little control of. These random factors include, for example, competition from nearby competitors, regional economy, and even local weather. The detailed description for generating  $\theta_{rt}$  is given in Section 3.5.1.2.

◦  $d_r$  is the base demand for store  $r$ , which can be viewed as the expected demand at store  $r$  in the first period when regular price  $p_1$  is used. Base demand  $d_r$  for each store is drawn uniformly from the interval  $[20, 100]$ .

◦  $f(p_j)$  is a function that captures how demand varies with the price  $p_j$ . We employ a commonly used constant elasticity form  $f(p_j) = (p_j/p_1)^{-\beta_r}$ , where  $\beta_r > 0$  is the price elasticity for store  $r$  that measures how sensitive the demand reacts to a price change. If the demand elasticity is higher, then the demand increases more quickly for a given price drop. The value of  $\beta_r$  should always be greater than 1 because otherwise revenue would increase with price and hence there would be no need for markdowns. We allow demand elasticity  $\beta_r$  to be different for different stores because stores at different locations may face different customer bases. Demand elasticity  $\beta_r$  for store  $r$  is uniformly drawn from an interval and fixed over the entire planning horizon. We use two different intervals in our experiments, i.e.,  $[1.0, 2.0]$  and  $[1.0, 3.0]$ . The first one represents the situation in which stores are more homogeneous while the second one represents a more diversified situation.

◦  $\phi(t)$  is a function that captures how demand changes over time. We use the following particular function, i.e.,  $\phi(t) = 1$ , for  $t = 1, 2, 3, 4$ , and  $\phi(t) = 1 - 0.1(t - 4)$ , for  $t = 5, 6, 7, 8$ . This particular function represents a typical markdown situation in which demand stays constant for the first several periods and then starts to decrease.

- Total available inventory  $I_0$ . We set  $I_0$  at three different levels: low, medium, and high. Each of these levels is set to be the total expected demand of all stores

over the entire planning horizon if a particular price is used for all the stores in all the periods. The low level of  $I_0$  is equal to the total expected demand if  $p_2$  (which is 90% of regular price) is used. The medium level of  $I_0$  is equal to the total expected demand if  $p_4$  (which is 70% of the regular price) is used. The high level of  $I_0$  is equal to the total expected demand if  $p_6$  (which is 50% of the regular price) is used. Here, the total expected demand for a given price  $p_j$  is calculated as  $\sum_{r=1}^n \sum_{t=1}^T d_r(p_j/p_1)^{-\beta_r} \phi(t)$ .

When creating problem instances in our computational tests, we vary two parameters,  $\beta_r$  (from either interval  $[1.0, 2.0]$  or interval  $[1.0, 3.0]$ ) and  $I_0$  (low, medium, or high), while keeping all other parameters fixed, as described earlier.

### 3.5.1.2 Generating Random Test Instances

A test instance specifies the values for all the parameters of the problem including the demand realization over the entire planning horizon. We generate random test instances in a way that mimics what may actually happen in practice over the entire planning horizon. However, when we evaluate the performance of our solution approach, we assume that at each time point we only know the part of the information contained in a test instance up to that particular time point.

Section 3.5.1.1 describes how we generate the value of each parameter except the demand realization. In the following we describe how we generate a particular demand realization over the entire planning horizon (i.e., a particular demand path). By (3.24), where  $d_r$ ,  $\beta_r$  and  $\phi(t)$  are given as described in Section 3.5.1.1, the random demand path is uniquely determined by the random variables  $\theta_{rt}$ , for  $\forall r \in N$ ,  $t \in \{1, \dots, T\}$ . Therefore, generating a demand path is equivalent to generating a particular realization of  $\theta_{rt}$ , for  $\forall r \in N$ ,  $t \in \{1, \dots, T\}$ . In the following, we describe how to generate a particular realization of  $\theta_{rt}$  by considering possible demand correlation both across time and across the stores.

To model demand correlation across stores, we assume that the stores can be classified into two groups  $N_1$ ,  $N_2$ , where  $|N_1| = |N_2|$  and  $N_1 \cup N_2 = N$ , such that the stores within each group are facing a similar market condition (i.e., they have

similar  $\theta_{rt}$  values). This is the case when, for example, a subset of stores are closely located to a major competitor while others are not. We generate the values of  $\theta_{rt}$  for the stores in the same group following the same probability distribution. The probability distributions for the two groups are generally different.

In addition, market conditions in consecutive periods are often closely correlated. For example, if the realized market condition for a store in the current period is good, it is very likely that the market condition for that store in the next several periods is also going to be good. We model this time correlation by updating the distribution of  $\theta_{rt}$  based on its realized value of  $\theta_{r,t-1}$  in the previous period.

Specifically, we generate the values of  $\theta_{rt}$  following a two-step procedure. In the first step, we generate the overall market condition for each group  $i = 1, 2$  in each period  $t$  represented by a value  $c_{it}$  based on the value of overall market condition in the previous period  $c_{i,t-1}$ . In the second step, we generate the market condition  $\theta_{rt}$  for each individual store  $r$  in each period  $t$  based on  $c_{it}$ .

**Step 1:** For each store group  $i = 1, 2$ , let  $c_{i0} = 1$ , and from  $t = 1$  to  $T$ , let  $c_{it}$  be a number uniformly drawn from the interval  $[c_{i,t-1} - l_t, c_{i,t-1} + l_t]$ , where  $l_t = 1/2^t$ .

**Step 2:** For  $\forall r \in N_i, t \in \{1, \dots, T\}, i = 1, 2$ , let  $\theta_{rt}$  be a number uniformly drawn from the interval  $[c_{it} - \delta_t, c_{it} + \delta_t]$ , where  $\delta_t = 0.1/2^t$ .

In the above procedure, the lengths of the intervals for generating  $c_{it}$  and  $\theta_{rt}$  decrease over time. This represents a typical case in which uncertainty about the market condition decreases as time progresses and more information is revealed.

### 3.5.1.3 Generating Demand Scenario Tree

Since our solution approach is implemented on a rolling horizon basis, whenever we move forward for one period, we re-generate a demand scenario tree for the remaining horizon based on the latest market information. Suppose we are at the beginning of period  $\tau$  and we need to re-generate a demand scenario tree for the planning horizon consisting of periods  $\tau, \dots, T$ . We assume that we know in advance that the stores are divided into two groups  $N_1$  and  $N_2$  exactly the same way as discussed in Section 3.5.1.2, and that we know in advance that the random

demand follows the function (3.24) (from which the actual demand realizations are generated in Section 3.5.1.2). In order to generate a demand scenario tree, we need to know the values of  $d_r$ ,  $\beta_r$ ,  $\phi(t)$ , and  $\theta_{rt}$  for  $r \in N$ ,  $t = \tau, \dots, T$ . We assume that  $\beta_r$  and  $\phi(t)$  are known in advance as described in Section 3.5.1.1. However, we do not assume that we know the values of  $d_r$  and  $\theta_{rt}$  exactly. We discuss below how the values of  $d_r$  and  $\theta_{rt}$  are generated.

We consider five different cases of  $d_r$ . In the first case (referred to as E00), we assume that we know the value of  $d_r$  exactly with certainty as described in Section 3.5.1.1. This is a reasonable assumption as in many practical situations, even though one may not know the exact demand distribution, the mean of the distribution can be estimated accurately. Most pricing literature in fact assume that both mean and standard deviation of the demand distribution are known. In the remaining four cases, we assume that we do not know the value of  $d_r$  exactly, and hence we may underestimate or overestimate it. Specifically, we consider two cases where we underestimate  $d_r$  by 25% or 50% (referred to as U25 and U50, respectively), and two cases where we overestimate  $d_r$  by 25% or 50% (referred to as O25 and O50, respectively).

With  $d_r$ ,  $\beta_r$  and  $\phi(t)$  specified as above, creating a demand scenario tree is now equivalent to creating a scenario tree of  $\theta_{rt}$  for  $t = \tau, \dots, T$ . To this end, we first estimate the latest overall market condition facing each store group  $N_i$ , denoted by parameter  $c_{i,\tau-1}$ , based on the demand realization in the last period  $\tau - 1$ . Given the actual demand realization in period  $\tau - 1$ , using the demand function (3.24) and the price used in each store  $r$ , we can get the value of  $\theta_{r,\tau-1}$  for each store  $r \in N$ . For  $i = 1, 2$ , we take the simple average of  $\theta_{r,\tau-1}$  over all the stores  $r \in N_i$  as the value of  $c_{i,\tau-1}$ . We then use  $c_{i,\tau-1}$  to generate scenarios of  $\theta_{rt}$  for  $r \in N_i$  over the future periods  $t = \tau, \dots, T$ . We use the following three methods to generate a scenario tree. This enables us to evaluate the impact of the demand scenario tree on the performance of our solution approach.

**DR, DN (1 scenario):** In this method, we generate a single scenario only. We let  $\theta_{rt} = c_{i,\tau-1}$ , for  $r \in N_i$ ,  $i = 1, 2$ ,  $t = \tau, \dots, T$ . This method gives a deterministic

demand estimate for all the periods. We call this method deterministic rolling horizon and denote it as DR. In addition to DR, we also test another approach, denoted as DN. This approach is similar to DR except that the problem is only solved once at the beginning of the entire planning horizon and the solution is implemented for all the periods at once without using a rolling horizon approach.

**S1 (One period, 9 scenarios):** In this method, scenarios are generated based on the estimated market condition in the first period (i.e., period  $\tau$ ) only. We first define three values based on  $c_{i,\tau-1}$ :  $\theta_{i\tau}^1 = c_{i,\tau-1} + 2l_\tau/3$ ,  $\theta_{i\tau}^2 = c_{i,\tau-1}$ ,  $\theta_{i\tau}^3 = c_{i,\tau-1} - 2l_\tau/3$ . These three values approximate the possible interval  $[c_{i,\tau-1} - l_\tau, c_{i,\tau-1} + l_\tau]$  for the market condition  $\theta_{r\tau}$  in period  $\tau$ . We then create three scenarios with equal probability for the possible market condition of the stores within each group  $N_i$  over the entire remaining horizon. That is, we let  $\theta_{r\tau} = \theta_{r,\tau+1} = \dots = \theta_{rT} \in \{\theta_{i\tau}^1, \theta_{i\tau}^2, \theta_{i\tau}^3\}$ , for  $r \in N_i$ ,  $i = 1, 2$ . Since the two store groups are independent, this results in a total of nine scenarios and each of them is associated with probability of  $1/9$ .

**S2 (Two periods, 81 scenarios):** In this method, scenarios are created based on the estimated market condition in the first two periods (i.e., periods  $\tau$  and  $\tau + 1$ ) only. We first create three scenarios with equal probability for the possible market condition in the current period  $\tau$ . That is,  $\theta_{r\tau} \in \{\theta_{i\tau}^1, \theta_{i\tau}^2, \theta_{i\tau}^3\}$ , for  $r \in N_i$ ,  $i = 1, 2$ , where,  $\theta_{i\tau}^1 = c_{i,\tau-1} + 2l_\tau/3$ ,  $\theta_{i\tau}^2 = c_{i,\tau-1}$ ,  $\theta_{i\tau}^3 = c_{i,\tau-1} - 2l_\tau/3$ . Conditioning on each such scenario in period  $\tau$ , we further create three scenarios with equal probability for the remaining periods  $\tau + 1$  to  $T$ . That is, conditioning on  $\theta_{r\tau} = \theta_{i\tau}^k$ ,  $k = 1, 2, 3$ , we let  $\theta_{r,\tau+1} = \dots = \theta_{rT} \in \{\theta_{i\tau}^{k1}, \theta_{i\tau}^{k2}, \theta_{i\tau}^{k3}\}$ , for  $r \in N_i$ ,  $i = 1, 2$ ,  $t = \tau + 1, \dots, T$ , where,  $\theta_{i\tau}^{k1} = \theta_{i\tau}^k + 2l_{\tau+1}/3$ ,  $\theta_{i\tau}^{k2} = \theta_{i\tau}^k$ ,  $\theta_{i\tau}^{k3} = \theta_{i\tau}^k - 2l_{\tau+1}/3$ . This generates nine scenarios for the stores in each group, and hence a total of 81 scenarios for all the stores together. Each one of the scenarios is associated with probability of  $1/81$ .

We note that if the planning horizon consists of period  $T$  only, then S2 is identical to S1. As can be seen, among these three methods, S2 represents the most accurate estimation of the possible outcomes of the market condition across the planning horizon.



## 3.5.2 Computational Results

In this section, we present the results from our computational experiments. In Section 3.5.2.1, we show the performance of the Lagrangian relaxation algorithm (LRA) given in Section 3.3.2 for solving the problem  $[\text{MIP}_1]$  by comparing it to the commercial CPLEX direct MIP solver. We note that  $[\text{MIP}_1]$  has the largest scale among all the MIP problems that we have to solve in our overall solution approach. In Section 3.5.2.2, we show the performance of our overall solution approach with a rolling horizon implementation for solving the overall integrated inventory allocation and markdown pricing problem described in Section 3.3.1 by comparing it to the benchmark approaches and the upper bound described in Section 3.4. All the programs for the computational tests were written in C++ and all the LP and MIP subproblems involved were solved by calling the LP/MIP solver of CPLEX 9.0. The code was run on a PC with a 2.61-GHz AMD Athlon(tm) 64×2 dual core processor and 3.25-GB memory.

### 3.5.2.1 Performance of LRA for Solving $[\text{MIP}_1]$

The test problem instances are generated following the parameter configurations described in Section 3.5.1.1 with the demand scenario tree generated by approach S2 as described in Section 3.5.1.3. There are six sets of test problem instances, corresponding to the two cases of intervals for  $\beta_r$ , i.e.,  $[1.0, 2.0]$  and  $[1.0, 3.0]$ , and three cases of initial inventory levels, i.e.,  $I_0 \in \{\text{low, medium, high}\}$ . For each such combination of  $\beta_r$  and  $I_0$ , we generate five problem instances, resulting in a total of 30 problem instances.

Table 3.1 shows the solution quality and computational time of LRA compared to the CPLEX direct MIP solver, and the number of iterations of LRA, for problems with 50 stores. The solution quality of LRA is defined as the ratio of the objective value of the solution generated by LRA over the optimal objective value generated by CPLEX. CPLEX was able to solve each problem instance to optimality except one instance for which the program was terminated when the optimality gap falls within

0.02%. Table 3.1 shows the median and the worst case values for each performance measure over the 5 test problems. As can be seen from this table, LRA consistently generates near-optimal solutions. Both the median computational time and the variance of the computational time of LRA, on the other hand, are significantly shorter than those of the CPLEX direct solver. Also the number of iterations used in LRA is consistent for different configurations of problems.

Table 3.1: Comparison of LRA and CPLEX Direct Solver for [MIP<sub>1</sub>] with 50 Stores

		Solution Quality of LRA (LRA/CPLEX, %)		Computational Time (s)				Num of Iterations by LRA		
		(LRA/CPLEX, %)		LRA		CPLEX		by LRA		
		Median	Worst	Median	Worst	Median	Worst	Median	Worst	
$\beta_r$	$I_0$	Low	99.6	99.1	230	315	18670	262783	7	10
		Medium	99.8	99.5	331	341	9408	*591460	8	10
		High	99.1	98.8	363	385	2433	8089	10	11
1.0-2.0	Medium	Low	99.2	99.1	284	291	47343	91176	8	9
		Medium	98.1	97.4	318	346	6204	185518	9	10
		High	99.9	99.8	329	366	796	1200	10	10

\* Stopped after 591460 seconds

To demonstrate that LRA can be applied to much larger problems, we test another set of problem instances with 100 stores that are generated in the same way as the problems with 50 stores. Due to the large sizes of these problems, we are unable to obtain the optimal solution using CPLEX. Thus, we report the solution quality of LRA as a ratio of the objective value of the solution over the Lagrangian upper bound both generated by LRA. Table 3.2 provides a comparison of the solution quality and the computational time for both problem 50-store and 100-store problems. As one can see, on one hand, the computational time for problems with 100 stores is approximately twice that for problems with 50 stores. On the other hand, the problem size has little impact on the solution quality as LRA generates equally good solutions for both sets of problems. Therefore, we can conclude that

LRA is an efficient and robust approach for solving the MIP formulations involved in our overall approach.

Table 3.2: Performance of LRA for  $[MIP_1]$ : 50 Stores versus 100 Stores

$\beta_r$ $I_0$		Solution Quality of LRA (LRA/UB, %)				Computational Time (s)			
		50 stores		100 stores		50 stores		100 stores	
		Median	Worst	Median	Worst	Median	Worst	Median	Worst
1.0-2.0	Low	98.7	98.4	99.5	98.0	230	315	515	539
	Medium	99.4	98.8	99.1	98.7	331	341	600	624
	High	96.2	95.8	95.8	95.7	363	385	695	788
1.0-3.0	Low	98.1	97.7	97.9	96.0	284	291	468	503
	Medium	96.2	95.4	96.2	95.5	318	346	614	669
	High	98.9	97.5	97.8	97.1	329	366	668	745

We note that because it is too time-consuming to generate the upper bound (as described in Section 3.4.3) for the overall problem with more than 50 stores, all the test problems used in the next subsection involve 50 stores only.

### 3.5.2.2 Performance of the Overall Solution Approach for the Overall Problem

In this section, we evaluate the performance of our overall approach described in Section 3.3.1 for the overall integrated inventory allocation and markdown pricing problem. We compare this approach with demand scenario tree generated four different ways (DN, DR, S1 and S2 as described in Section 3.5.1.3) with the five benchmark approaches (SA, P1, P2, P3 and P4 as described in Section 3.4). As described earlier, DR, S1, S2 and SA are implemented on a rolling horizon basis,

whereas DN, P1, P2, P3 and P4 are implemented on a non-rolling horizon basis (i.e., pricing decisions are determined at the beginning of the entire planning horizon without re-optimization in later periods). The test problems are generated following the parameter configurations described in Section 3.5.1. Thus in total we test 30 parameter configurations corresponding to two intervals for  $\beta_r$ , i.e.,  $[1.0, 2.0]$ ,  $[1.0, 3.0]$ , three initial inventory levels, i.e.,  $I_0 \in \{\text{low, medium, high}\}$ , and five cases for  $d_r$ , i.e., E00, U25, U50, O25 and O50. For each parameter configuration tested, we randomly generate 2000 test instances, each corresponding to a different demand path which is generated as described in Section 3.5.1.2. The reason we use such a large number of test instances for a given parameter configuration is that we want to have a fair coverage of possible demand realization.

A test instance specifies the values of all the parameters including the demand realization over the entire planning horizon. As discussed in Section 3.4.3, when generating the upper bound for a particular test instance, we assume that the demand realization for the entire planning horizon is known in advance at the very beginning of the planning horizon. When implementing a rolling horizon approach (i.e., DR, S1, S2 and SA), however, we assume that only the demand realization up to the beginning of the current period is known, and use a demand scenario tree instead of the actual demand realization for the future periods. For the non-rolling horizon approach, DN, pricing decisions for the entire planning horizon are determined at the very beginning of the planning horizon based on the information available at that time, i.e., expected demand for each store in each period under each price which is estimated at the beginning of the planning horizon. For all the simple policies (i.e., P1, P2, P3, and P4), pricing decisions for the entire planning horizon are specified at the very beginning of the horizon regardless of the demand realization.

For each of the 2000 test instances of a given parameter configuration, we run each approach as described above to determine the pricing decisions. Once the pricing decisions are determined, the total revenue achieved by a particular approach is computed based on the actual demand realization in that test instance.

For a given parameter configuration, we measure the performance of each approach relative to the upper bound, defined as the ratio of the average revenue achieved by this approach over the 2000 test instances divided by the average upper bound over the same 2000 test instances.

Table 3.3 provides a summary of the performance of all the nine approaches. Each entry of the table represents the average performance of a particular approach relative to the upper bound for a particular parameter configuration over the 2000 test instances corresponding to this parameter configuration. As can be seen from the table, although the benchmark approaches (SA, P1, P2, P3, P4) may have good performance for a few parameter configurations, the Lagrangian relaxation based approaches (DN, DR, S1, S2) overall significantly outperform the benchmark approaches. By employing a Lagrangian relaxation approach, one can often obtain a solution that is close to the upper bound (at least 90%). In addition, the Lagrangian relaxation approaches are much more stable than that of the benchmark approaches across different parameter configurations.

Among all the Lagrangian relaxation approaches, S2 performs the best, followed by S1, DR and DN. Comparing the performance of the two single-scenario approaches DN and DR, we see that by implementing the Lagrangian relaxation approach on a rolling horizon basis (as in the case of DR), one can obtain a solution that is closer to the ideal solution. On average, the resulting performance measure increases by 4.3%, 5.6%, 8.5%, 8.2%, and 16.4% for the case of E00, U25, U50, O25 and O50, respectively. This implies that when the demand is uncertain, implementing a rolling horizon approach can significantly increase the revenue. In addition, the advantage of a rolling horizon approach over a non-rolling horizon approach is more significant when the demand estimation is less accurate (as in the case of U50 and O50).

We also observe that among all the Lagrangian relaxation approaches with a rolling horizon implementation, S1 and S2 outperform DR in the cases of E00, U25, U50 and when inventory level is low. For all other parameter configurations, S1 and S2 are either slightly better than DR (the difference in performance measure is less

Table 3.3: Comparison of Different Markdown Pricing Approaches

$d_r$	$\beta_r$	$I_0$	Performance Compared to Upper Bound (%)								
			Lagrangian Relaxation Approaches				Benchmarks				
			DN	DR	S1	S2	SA	P1	P2	P3	P4
E00	1.0-2.0	Low	92.7	94.5	96.0	96.3	86.0	81.1	87.5	91.2	88.1
		Medium	92.1	96.9	97.0	97.6	87.2	86.4	89.1	79.4	88.2
		High	92.8	98.5	98.5	98.6	87.0	85.4	76.0	65.9	73.6
	1.0-3.0	Low	91.9	93.9	93.9	95.6	86.4	80.3	86.0	87.0	87.3
		Medium	91.9	96.6	96.8	97.2	88.1	84.5	84.7	66.3	79.9
		High	92.0	98.5	98.3	98.5	87.4	75.3	58.0	44.3	53.3
U25	1.0-2.0	Low	90.6	91.0	94.1	94.9	82.4	79.6	86.7	97.0	83.5
		Medium	86.9	95.1	95.2	95.8	85.4	83.7	89.9	85.9	91.8
		High	88.2	98.2	98.2	98.3	87.1	88.7	81.7	70.9	79.1
	1.0-3.0	Low	90.5	90.8	92.6	93.8	83.3	79.7	86.0	95.1	83.3
		Medium	89.8	96.6	96.3	96.5	87.5	83.5	88.4	73.8	87.9
		High	90.5	98.4	98.3	98.4	87.2	82.4	64.5	49.3	59.3
U50	1.0-2.0	Low	88.5	88.6	92.6	93.0	79.9	81.8	89.3	99.5	80.6
		Medium	79.3	89.9	90.2	90.8	79.6	80.7	87.3	91.8	89.4
		High	81.0	98.0	98.2	98.3	88.1	88.5	86.9	75.4	84.2
	1.0-3.0	Low	88.3	88.3	91.4	91.7	81.0	82.0	88.8	99.2	80.5
		Medium	83.6	93.5	93.1	93.5	84.2	81.9	87.6	80.7	91.2
		High	84.9	98.3	98.1	98.1	86.9	85.9	70.4	53.9	64.8
O25	1.0-2.0	Low	89.6	96.2	96.8	97.0	86.9	84.4	88.1	82.8	88.6
		Medium	90.2	97.0	97.1	97.3	86.4	86.9	83.2	72.2	80.8
		High	91.9	98.8	98.8	98.8	89.5	80.0	70.6	61.1	68.3
	1.0-3.0	Low	87.2	95.4	95.6	96.2	86.6	82.4	85.7	76.0	85.8
		Medium	86.2	96.7	96.5	96.8	86.0	83.8	75.7	57.9	69.8
		High	88.2	98.8	98.6	98.8	89.8	67.4	51.8	39.5	47.6
O50	1.0-2.0	Low	81.0	96.3	96.8	96.7	86.5	85.5	83.4	73.3	81.8
		Medium	82.8	97.3	97.5	97.4	87.5	83.5	75.3	65.0	72.8
		High	88.5	99.1	99.2	99.1	94.7	75.8	66.8	57.7	64.6
	1.0-3.0	Low	75.2	94.5	95.0	95.1	85.1	82.6	79.5	63.5	76.1
		Medium	74.3	96.5	96.3	96.6	86.2	78.6	64.3	48.9	59.0
		High	82.3	98.9	99.0	98.8	93.9	61.8	47.7	36.0	43.5

than 1%) or they have similar performance as DR. This can be explained intuitively as follows.

First, when the inventory is low, few markdowns are needed as the inventory can be cleared at a price close to the regular price  $p_1$ . Thus, the pricing decision in the first period is critical as this price might be applied to the remaining horizon as well. Consequently, the performance over the entire planning horizon is primarily dependent on the solution in the first period. On the other hand, the first period involves the highest uncertainty because no demand realizations have been observed yet. When high uncertainty is present, approaches that incorporate demand scenarios as in S1 and S2 are likely to generate better solutions than approaches that use expected demand as in DR. This explains why S1 and S2 have better performance than DR under low inventory level. When the inventory is high, the inventory can only be cleared at a low price. Due to business rule (iv), this low price can only be reached after multiple markdowns. Therefore, the pricing decision in the first period becomes less critical as decisions in the first period can be corrected in the following periods. This explains why the advantage of S1 and S2 becomes less significant when the inventory level is high.

Second, under a low inventory level, the advantage of S1 and S2 over DR when the base demand  $d_r$  is underestimated is more significant than when it is overestimated. The reason is that, when the base demand is underestimated, or equivalently, when the actual demand is higher than expected, one cannot increase the price due to business rule (ii) if the price in the first period is set too low. When the base demand is overestimated, or equivalently, when the actual demand is lower than expected, however, one can drop the price more quickly in the following periods if the price in the first period is set too high. Therefore, the pricing decision in the first period has little impact when demand is overestimated, whereas the pricing decision in the first period becomes critical when demand is underestimated. Since S1 and S2 can generate better solutions (including first-period solutions) than DR, their advantage over DR is more significant when demand is underestimated.

Comparing the performance of the two approaches involving multiple scenar-

ios, S1 and S2, we see that S2 performs slightly better than S1. This implies that incorporating a more accurate demand scenario tree in the Lagrangian relaxation approach can in general improve the performance. However, the extra benefit might be limited depending on the particular parameter configuration.

Among all the benchmark approaches, although none of them completely dominates the others, SA is the best in terms of consistency and the overall performance for all parameter configurations. However, by comparing SA and DR, we see that DR performs consistently better than SA. This indicates that joint consideration of inventory allocation and pricing (as in DR) can generate considerably more revenue than a sequential approach where the two decisions are made separately (as in SA). We can also observe that the performances of the simple markdown pricing policies are unstable and highly dependent on the parameter configuration. For example, the performance of P3 can be as good as 99.5% for one parameter configuration while as bad as 36.0% for another. This makes sense intuitively since simple markdown pricing policies are predetermined and do not react to changing market conditions. Therefore, each simple policy may be suitable for only a small subset of particular situations.

### 3.6 Managerial Insights

In this section, we investigate the impact of business rules, price sensitivity of individual stores, and demand uncertainty on the total expected revenue and the structure of inventory allocation and pricing decisions. To this end, we test on a variety of parameter configurations. For each parameter configuration tested, we generate 1000 random problem instances following the same way as described in Section 3.5.1. Since our tests in Section 3.5.2 have shown that approach S2 outperforms every other approach, we use S2 to solve all the problem instances in this section.

In the first experiment, we intend to see how the total expected revenue changes with the allowed number of markdowns (i.e., business rule (iii)). The test



problems we use have the same parameter configurations as in Section 3.5.1 except that the number of markdowns allowed (i.e.,  $R$ ) can vary from 1 to 5. Table 3.4 shows the results. The case with  $R = 1$  is used as the base case for which the total expected revenue is normalized as 1 (or 100%). The last column of the table shows the total expected revenue of each other case relative to the base case. The table also shows the average number of markdowns actually used. It is clear that when the number of allowed markdowns increases from 1 to 2, there is a significant increase in revenue (approximately 3% for low inventory level case, 5% for medium inventory level case, and 20% for high inventory level case). Afterwards, the revenue change is very small. This suggests that in practice, implementing a markdown scheme that involves 2 markdowns should, in general, yield satisfactory performance. We note that this is similar to the result derived by Gallego and van Ryzin (1994) for a single-store problem. They show that for a single-store problem with a discrete set of allowable prices and a continuous time horizon, a strategy with only two adjacent prices can be asymptotically optimal under certain limiting conditions.

In the second experiment, we test the impact of the constraint on the prices used at different stores in the same period. We compare the case where all the stores have to use a common price in each period (which is the case in the problem studied by Bitran et al. 1998) and the case where the stores do not have to use a common price, but instead, the prices used by the stores in the same store cluster need to fall within a price cluster (which is the case in our problem as specified in business rule (v)). The test problems we use have the same parameter configurations as in Section 3.5.1 except that here we use three different store configurations with different ranges of price elasticity ( $\beta_r$ ) as shown in the first column of Table 3.5. We note that the first case has a fixed price elasticity  $\beta_r = 2.5$ , which represents the situation where all the stores are equally price sensitive, and the other two cases have a larger range of  $\beta_r$  representing the cases where the stores are more heterogeneous. Table 3.5 shows the average revenue gain by allowing different stores to use different prices following our business rule (v), relative to the case where a common price for all the stores must be used in each period. As clearly demonstrated in this table, one

Table 3.4: Impacts of Allowed Number of Markdowns

$I_0$	Allowed Number of Markdowns	Actual Number of Markdowns	Relative Revenue (%)
Low	1	0.73	100.0
	2	0.93	103.0
	3	1.01	103.3
	4	1.02	103.2
	5	1.02	103.1
Medium	1	0.97	100.0
	2	1.41	105.2
	3	1.59	105.9
	4	1.67	105.6
	5	1.65	105.6
High	1	1.00	100.0
	2	1.89	120.4
	3	2.17	122.7
	4	2.19	122.7
	5	2.22	122.8

will always be better off by allowing stores to use different prices. In addition, when stores become more heterogeneous, there is an increasing revenue gain generated by allowing different prices. The impact is especially significant when the initial inventory level is low. This suggests that when the stores are not homogeneous in terms of price sensitivity (which is likely to be the case, for example, when there is a large number of stores involved), implementing a common price scheme for all stores can incur a significant revenue loss.

In the third experiment, we investigate the impact of price elasticity of individual stores. The test problems we use have the same parameter configurations as

Table 3.5: Impact of Non-Common Prices

$\beta_r$	$I_0$	Relative Revenue Gain
2.5	Low	0.8%
	Medium	0.6%
	High	1.2%
2.0 - 3.0	Low	7.2%
	Medium	1.2%
	High	2.1%
1.0 - 4.0	Low	9.1%
	Medium	3.1%
	High	4.3%

in Section 3.5.1 except that we use a wider range of price elasticity  $\beta_r \in [1, 4]$ . We first examine how pricing decisions should be made for stores with different price elasticities. Figures 3.2, 3.3, 3.4, and 3.5 show, respectively, the average number of markdowns, average first period price, average last period price, and average magnitude of each price drop for stores with different price elasticities. Each figure shows the result under three different levels (low, medium, high) of initial inventory  $I_0$ . As one can see that stores with higher price elasticity, in general, have more price markdowns, and lower first and last period prices. This pattern is especially clear when the initial inventory level is low. This indicates that retailers should offer more frequent and deeper price discount for stores that are more price sensitive. The average magnitude of each price drop, on the other hand, is not significantly impacted by price elasticity, as shown in Figure 3.5. We then examine how inventory should be allocated to stores with different price elasticities. For each store, we compute the ratio of the percentage of the actual inventory allocated to this store

to the percentage of the total base demand of this store. If the ratio is greater than 1, it implies that we should allocate more inventory to this store than its actual proportion based on the base demand. Figure 3.6 shows the results. The figure clearly demonstrates that stores with higher price elasticity also have higher ratio. This suggests that we should allocate a higher proportion of inventory to stores with a higher price elasticity. The observation holds true for all three initial inventory levels. This implies that in allocating inventory to stores, we should consider not only base demands of the stores, but also price sensitivities of the store.

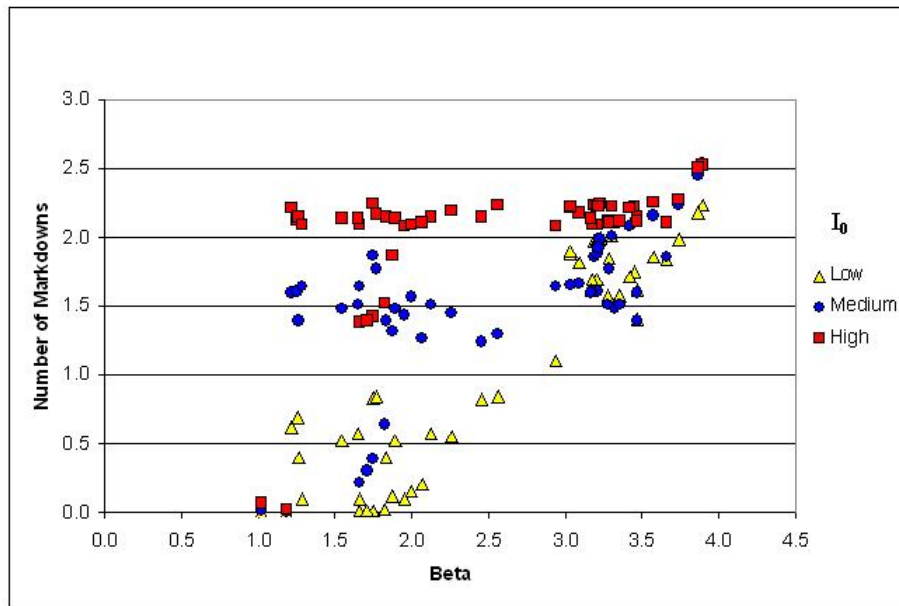


Figure 3.2: How the Number of Markdowns Changes with  $\beta_r$

In the last experiment, we examine the impact of demand uncertainty. We create three sets of 1000 random problem instances. The parameter configurations used to generate test instances are the same as in Section 3.5.1 except for the following differences with respect to the range of  $\theta_{rt}$  (which controls the demand uncertainty). In the first and second set of test instances, the first period market condition  $\theta_{rt}$  is generated from interval  $[0.5, 1.5]$  and  $[0.75, 1.25]$ , respectively. For both sets of problems, the interval length for the market condition  $\theta_{rt}$  in each period after the first period reduces to half of its length in the previous period. The third

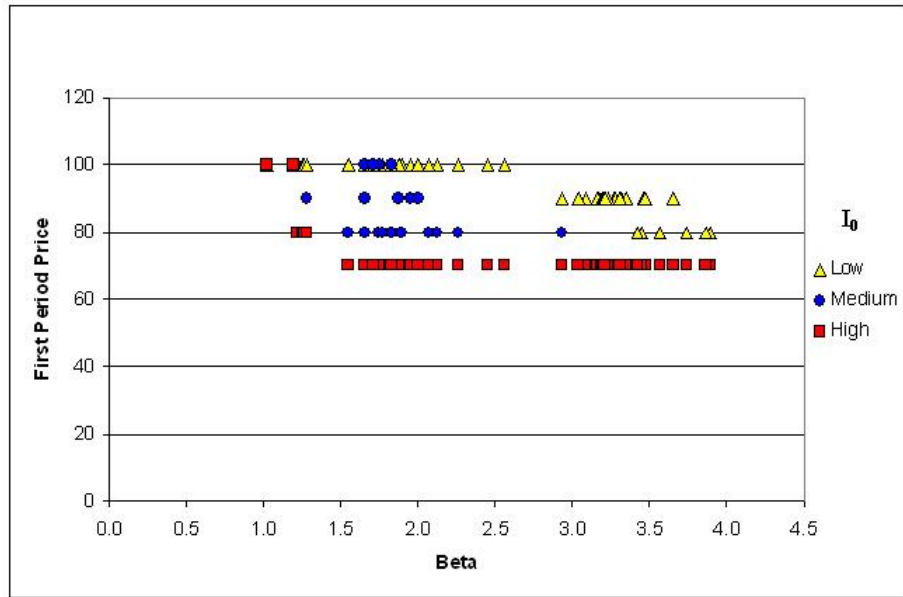


Figure 3.3: How the First Period Price Changes with  $\beta_r$

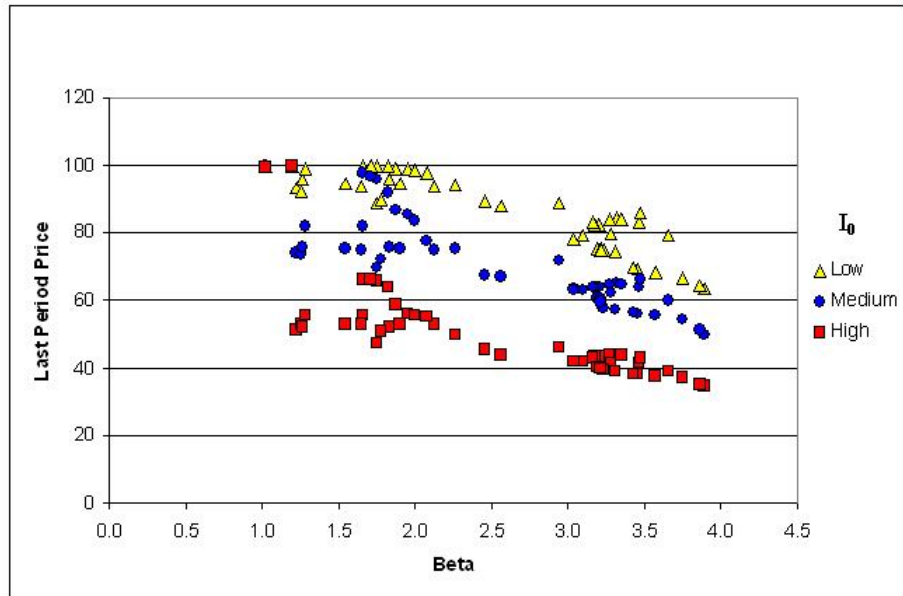


Figure 3.4: How the Last Period Price Changes with  $\beta_r$

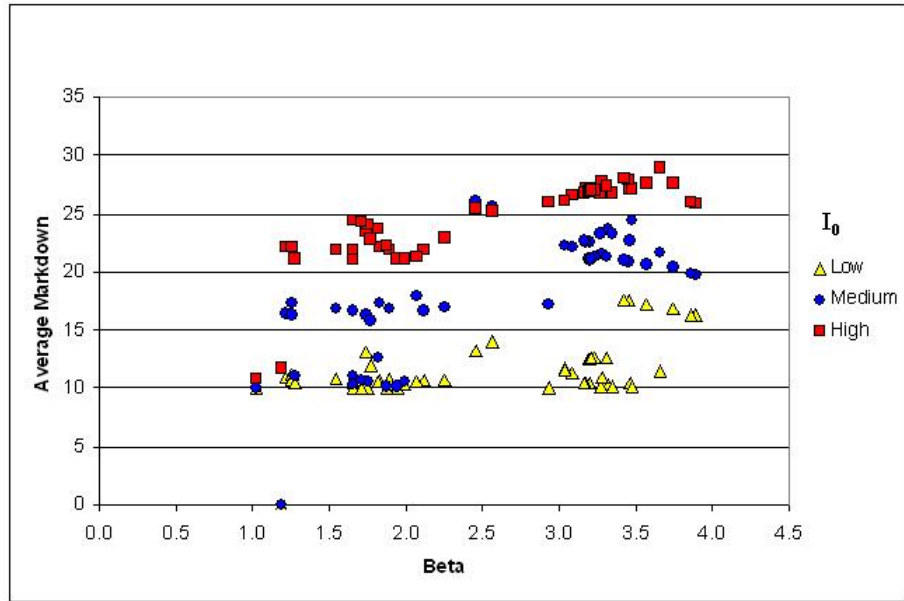


Figure 3.5: How the Average Markdown Changes with  $\beta_r$ .

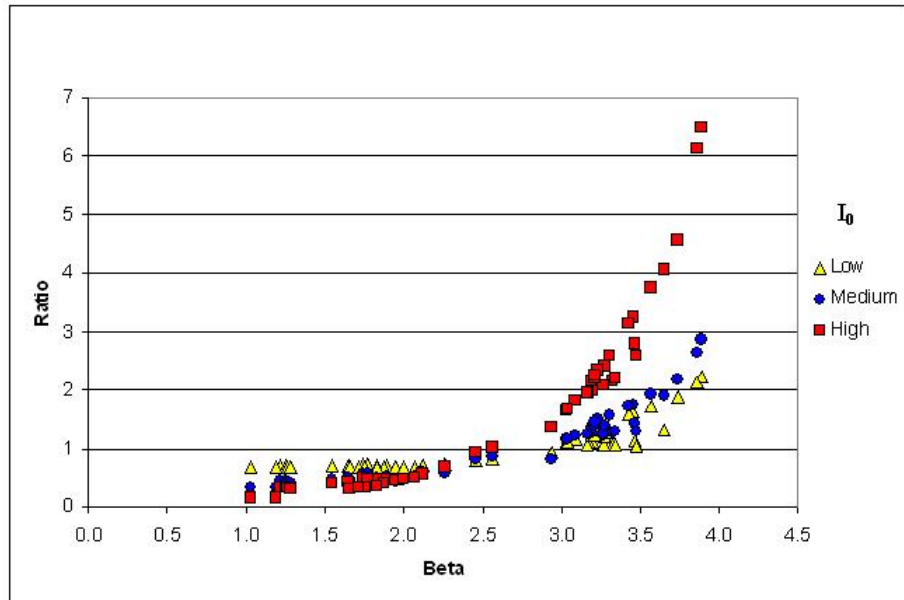


Figure 3.6: How Inventory Allocation Changes with  $\beta_r$ .

set of test instances are deterministic where the demand is given as the expected value. So, in terms of demand uncertainty, these three sets of problems have high, low and no uncertainty. Figures 3.7, 3.8 and 3.9 show the price paths over time for these three sets of problems under low, medium and high initial inventory levels, respectively. Note that the price paths shown in these figures represent the average price path across all stores over all the random test instances. We first look at the low inventory level case. As one can see, when demand is deterministic, the optimal pricing scheme to use is to drop the price to an appropriate level in the very first period and keep this price for the entire planning horizon. In contrast, when the demand is highly uncertain, the price drop in the first period should be more modest and the optimal pricing scheme is to have multiple price drops and each time drop a little. This results in a lower price in the end. For medium and high inventory levels, the price paths for the three sets of problems are very close during the first several periods. And the price tends to be a little lower in the end when demand uncertainty is higher.

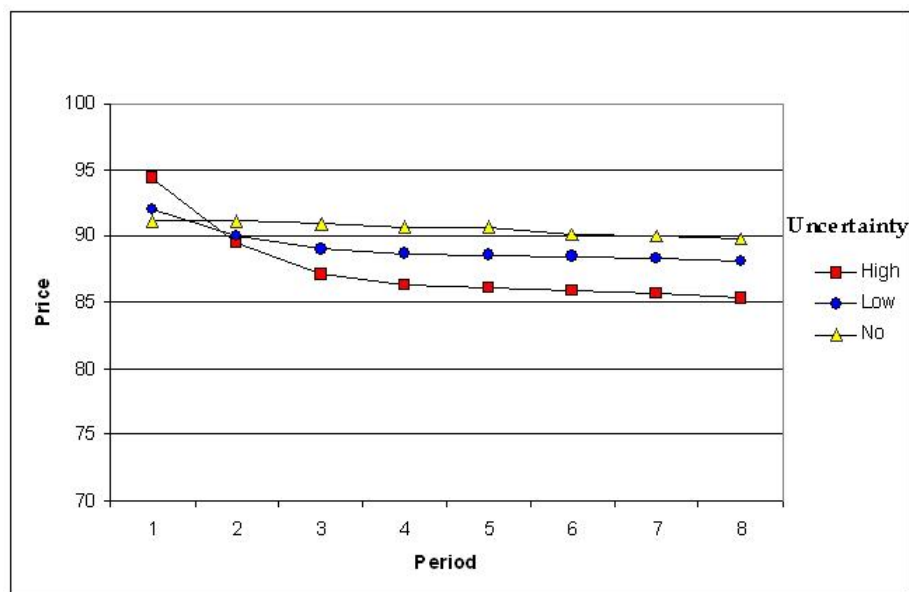


Figure 3.7: Average Price Path under Low Inventory

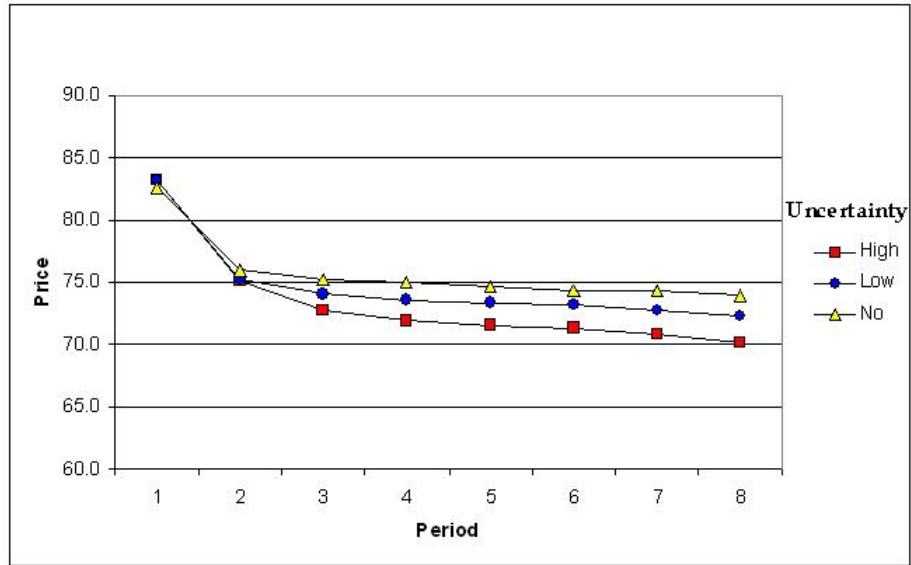


Figure 3.8: Average Price Path under Medium Inventory

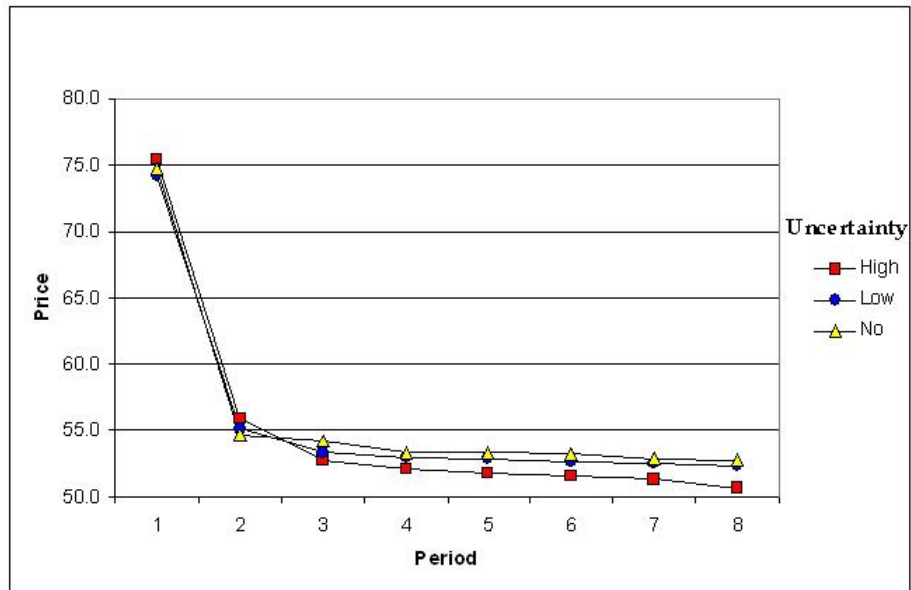


Figure 3.9: Average Price Path under High Inventory



### 3.7 Conclusions and Extensions

In this study, we have considered an integrated inventory allocation and markdown pricing problem faced by many large retailers. The problem involves a large number of stores and a set of business rules which have not been studied in the literature. We have shown that the problem is NP-hard even with a single store or with a single time period and when the demand is deterministic. We have proposed a Lagrangian relaxation based solution approach implemented on a rolling horizon basis for solving our problem. We have developed several different versions of this approach where the demand scenario tree is generated differently. We have evaluated the performance of this approach by comparing it to a set of benchmark approaches commonly used in practice. Our extensive computational experiments have shown that our approach significantly outperforms the benchmark approaches. In addition, we have observed a number of interesting managerial insights that can be used to assist store managers to make better markdown decisions.

We note that our formulation and solution approach can be easily modified to handle a variety of other practical situations that are not explicitly reflected in our problem. In our problem, since markdown items get a free ride with the daily shipments of regular products from the warehouse to each store, shipments of markdown items do not incur additional shipping costs. Thus, there is no incentive for making large but infrequent shipments. Instead, in each period we ship just enough inventory from the warehouse to each store to satisfy the sales of this period only. This results in a large number of shipments from the warehouse to each store, but no inventory redistribution between the stores is needed. In situations where markdown items do not get a free ride and hence associated shipping costs must be considered, it may be optimal to make large but infrequent shipments. This means that some stores may have to keep a large amount of inventory that can be used for multiple periods. Consequently, inventory redistribution between the stores may be necessary in certain time periods. Our formulation  $[MIP_\tau]$  can be modified as follows to accommodate this situation.

For ease of presentation, we use store  $r = 0$  to represent the warehouse. Define variable  $I_{rt}^\omega$  to be the inventory available at store  $r$  at the beginning of period  $t$  before inventory redistribution under scenario  $\omega$ , and variable  $Z_{rst}^\omega$  to be the amount shipped from store  $r$  to store  $s$  at the beginning of period  $t$  under scenario  $\omega \in \Omega$ , for  $r \in N \cup \{0\}$ ,  $s \in N$ ,  $t \in \{\tau, \dots, T\}$  and  $\omega \in \Omega$ . The formulation [MIP $_\tau$ ] is modified by replacing constraint (3.16) with

$$\sum_{j=j_{0r}}^m S_{rjt}^\omega \leq I_{rt}^\omega + \sum_{s \in N \cup \{0\}} Z_{srt}^\omega, \quad \forall r \in N, t \in \{\tau, \dots, T\}, \omega \in \Omega \quad (3.25)$$

and replacing (3.18) with

$$I_{0,t+1}^\omega = I_{0t}^\omega - \sum_{s \in N} Z_{0st}^\omega, \quad \forall t \in \{\tau, \dots, T-1\}, \omega \in \Omega \quad (3.26)$$

$$\begin{aligned} I_{r,t+1}^\omega &= I_{rt}^\omega + \sum_{s \in N \cup \{0\}} Z_{srt}^\omega - \sum_{s \in N} Z_{rst}^\omega - \sum_{j=j_{0r}}^m S_{rjt}^\omega, \\ &\forall r \in N, t \in \{\tau, \dots, T-1\}, \omega \in \Omega \end{aligned} \quad (3.27)$$

where  $I_{r\tau}^\omega$  is the initial inventory at store  $r$  at the beginning of the rolling horizon, which is known and independent of  $\omega$ . Constraint (3.25) ensures that the sales at each store in each period is no more than the available inventory, and constraints (3.26) and (3.27) formulate the relation between the inventory, sales, and redistribution.

Building on the above newly defined variables, we can also incorporate inventory and shipping costs into the problem by simply subtracting the total inventory cost (a linear function of  $I_{rt}^\omega$ ) and total shipping cost (a function of  $Z_{rst}^\omega$ ) from the current objective function (3.6). We can also easily incorporate the costs associated with the number of markdowns by subtracting a linear function of  $H_{rj}$  from the current objective function (3.6).

Our Lagrangian relaxation solution approach can also be modified to solve the revised formulation. We can relax the formulation by moving (3.25), (3.26) and (3.27) to the objective function with appropriate Lagrangian multipliers. This will result in a relaxed formulation which can be decomposed by store clusters in a similar manner as discussed in Section 3.3.2.

## Chapter 4

### Robust Dynamic Pricing with Two Substitutable Products

#### 4.1 Introduction

##### 4.1.1 Motivation and Related Literature

Retail stores typically offer a variety of products for customers to choose from. The demand for one product is influenced not only by its own price, but also by the prices of the other products sold in the same time. There is a *substitution effect* (or *complimentary effect*) between two products if increasing the price of one product can increase (or decrease) the demand of another product. Pricing decisions for multiple products should be made jointly and adjusted over time by exploiting the possible substitution or complimentary relationships among the products over time.

Pricing and related problems have attracted significant attention by researchers in the operations management area in the last fifteen years (see survey papers by McGill and van Ryzin 1999, Bitran and Caldentey 2003, and Elmaghraby and Keskinocak 2003). Most of the work to date so far has focused on single-product problems which obviously do not model substitution or complimentary effect between products. Several papers (e.g. Kuyumcu and Popescu 2006, Tang and Yin 2007, Karakul and Chan 2008, Thiele 2009) consider *static pricing* of multiple products in which pricing decisions are made once at the beginning of the planning horizon and not adjusted over time. Several papers (e.g. Adida and Perakis 2006) study *dynamic pricing* of multiple products, in which product prices are adjusted over time, but assume that the demand of a product is not influenced by the prices of the other products.

We are aware of a handful of papers only (Gallego and van Ryzin 1997, Bitran et al. 2006, Maglaras and Meissner 2006, Dong et al. 2009, Zhang and Cooper

2009, and Akcay et al. 2010) that study multi-product dynamic pricing with the explicit consideration of demand inter-dependency among the products. They all consider substitutable products and assume that (i) given a price vector, demand probability distributions are known and independent across time; and (ii) price change is allowed at any time during the planning horizon. Demand substitution is also modeled similarly in all these papers. In Bitran et al. (2006), Dong et al. (2009), Zhang and Cooper (2009) and Akcay et al. (2010), the total demand rate (for all the products together) is given exogenously while the product prices jointly determine the probability of a customer choosing a particular product (or leaving the store without purchasing any product). In Gallego and van Ryzin (1997) and Maglaras and Meissner (2006), the demand rate for each individual product is determined jointly by the product prices through a demand function. In either case, changing the price of one product may affect the demand of other products, referred to as *price-driven* substitution. In addition, they all assume that at the moment when the inventory of a product becomes zero, the price for this product is set to be large enough so that the probability that a customer will buy this product is zero. This modeling approach only works for the case where price change is allowed at any time. As we discuss below, in most practical situations, price changes can only occur periodically. In such a case, when the inventory of a product stocks out, there may exist unmet demand for this product before the price for this product can be re-set. A fraction of the unmet demand may turn to other products. This type of substitution, referred to as *inventory-driven* substitution, is not considered in these papers. In formulating their proposed model, Maglaras and Meissner (2006), Zhang and Cooper (2009), Dong et al. (2009) and Akcay et al. (2010) use stochastic dynamic programming whereas Gallego and van Ryzin (1997) and Bitran et al. (2006) treat their model as a stochastic control problem. Gallego and van Ryzin (1997), Maglaras and Meissner (2006), and Bitran et al. (2007) develop heuristic procedures whereas Dong et al. (2009) and Akcay et al. (2010) are able to solve small problem instances using their exact DP formulation.

Most of the existing pricing research including the papers reviewed above stud-

ies structural properties and managerial insights using stylized models. These insights may provide useful *general, high-level* guidelines for managers in practice. However, most of the stylized models oversimplify practical situations and do not closely reflect business norms and rules commonly encountered in practice. Elmaghraby and Keskinocak (2003) write that “*another disconnect between most of the academic literature and practice is the incorporation of business rules into pricing decisions*”, and justify the use of commonly accepted rules in practice. Consequently, the insights generated based on a stylized model that ignores common business norms and rules do not provide a satisfactory solution to a practical problem. In fact, such insights may no longer hold when some of the underlying assumptions change. For example, one of the best-known insights in the single-product dynamic pricing case (Gallego and van Ryzin 1994) is that when the demand follows a certain probability distribution and is *independent across time*, it is optimal to markup the price after a unit is sold and markdown the price over time if no item is sold. However, when the demand is not independent across time (as in many practical situations), this strategy may not work (Su 2007). To deal with a diverse set of problems in practice, one must use a model which incorporates the business rules and issues commonly seen in the real world.

Furthermore, most of the existing dynamic pricing models assume the precise knowledge of the underlying probability distribution of a random demand and assume risk neutrality of the decision maker. However, in practice, it is often difficult to completely characterize the demand, especially with little sales data available in the case of products with a short selling season. Even for products long established in the market, estimating cross elasticities between products can be a daunting challenge. For this reason, there is an increasing interest in recent years in the operations management area to use models that require limited demand information only. Examples include Lan et al. (2008) and Ball and Queyranne (2009) for airline revenue management problems, Perakis and Roels (2008) for the newsvendor model, and Thiele (2006a, 2006b, 2009), Lim and Shanthikumar (2007), Lim et al. (2008), and Eren and Maglaras (2009) for pricing problems.

When modeling demand, Ball and Queyranne (2009) do not require any demand information while Lan et al. (2008) require only lower and upper bounds of the uncertain demand. Thiele (2006a, 2006b, 2009) requires both the lower and upper bounds and expected value of the uncertain demand. Eren and Maglaras (2009) assume that only the support of the distribution of customers' willingness to pay is known. Lim and Shanthikumar (2007) and Lim et al. (2008) assume that the unknown probability distribution of the demand is known to lie within a neighborhood of a given nominal distribution. Perakis and Roels (2008), on the other hand, require the knowledge of some other parameters of the underlying demand distribution (e.g., mean, variance, symmetry, unimodality). All these papers develop robust optimization approaches. The objective functions considered in these papers include both relative performance measures such as maximizing competitive ratio (Ball and Queyranne 2009, Lan et al. 2008, Eren and Maglaras 2009), and absolute performance measures such as maximizing worst-case revenue (Thiele 2006a, 2006b, 2009, Lim and Shanthikumar 2007, Lim et al. 2008) or minimizing maximum regret (Perakis and Roels 2008). There are also papers (e.g., Besbes and Zeevi 2009) that study pricing problems with limited demand information, but use a framework different from robust optimization. Besbes and Zeevi (2009) consider a single-product problem under the assumption that the demand function belongs to a known parametric family with unknown parameters or to a class of functions without knowing the exact functional form. The sales data up to a time period is used to "learn the demand" and optimize the pricing policies thereafter.

#### 4.1.2 Our Model

In this chapter, we study a dynamic pricing problem with two substitutable products involving a set of business rules and practical issues and use a robust optimization modeling approach to formulate the problem. Because of the short selling season (e.g. 8 weeks) and long supply lead time (as in the case of fashion products, some toys, and some high tech products), no inventory replenishment is possible. The decision maker needs to sell a given amount of inventory of each

product by periodically adjusting the prices of the products over the selling season. In practice, price changes can only occur at the beginning of a time period, where a period typically consists of one to two weeks. Once a price change occurs the new price should remain unchanged for the entire period. For that reason, the selling season is assumed to consist of a discrete number of time periods and the decision is to set a price for each product in each period. In addition, we consider the following business rules which are widely used in practice:

- i) There are a discrete set of allowable prices that can be used for each product over the selling season (e.g., for a particular digital camera, the allowable price set is  $\{\$269, \$249, \$229, \$199, \$179, \$159\}$ ).
- ii) The number of price changes allowed for each product cannot exceed a given upper limit (e.g., there are no more than 4 price changes over the entire planning horizon).
- iii) In each period, if there is a price change for a product from the price used in the previous period, the price change must be within a given lower and upper limits (e.g., at least 10% but no more than 30% markup or markdown).

These rules are due to established market norms and are also desirable from a consumer's standpoint. There are certain price points at which consumers become much more willing to buy, and hence retailers that follow a sound pricing strategy often use a small set of popular price points for a product (e.g., Allen 2011). Frequent price changes and significant price differences from one period to the next may confuse the consumers, who are searching for appropriate prices, not necessarily the lowest ones. It has been long understood (Hall and Hitch 1939) that frequent price changes can even make a retailer appear unfair or dishonest, as customers try to interpret the firms' motives behind a price change. To the best of our knowledge, no existing research has considered all these rules in a single pricing model.

In our model, the substitution effect between the two products is driven by price and availability in the following way. If both products have inventory, the

demand for each product is driven by the prices of both products (i.e., price-driven substitution). However, when one of the products is out of stock, a fraction of the unsatisfied demand for this product will buy the other product (i.e., inventory-driven substitution), which results in a larger demand for the other product. This is commonly used in the inventory literature (e.g., Smith and Agrawal 2000, Netessine and Rudi 2003, Kraiselburd et al. 2004, and Nagarajan and Rajagopalan 2008). We also consider demand substitution across time periods by assuming that the total demand of the two products across multiple time periods falls within a certain interval. None of the above reviewed multi-product pricing literature considers inter-temporal demand substitution. However, today's customers are increasingly sophisticated and extremely adept at finding the best deals. As a result, there often exists inter-temporal demand substitution which needs to be taken into consideration when a retailer makes pricing decisions (Su 2007, Shen and Su 2007).

Since as discussed above, demand information is often limited in practice, in our model we do not assume the full knowledge of the demand distribution. Instead, we assume that in each period  $t$ , given a price pair, we know the expected demand of each product in the current period and the following bounds on the uncertain demand: (i) a lower bound and upper bound of the demand for each product in the current period; (ii) a lower bound and upper bound of the total demand of the two products in the current period; (iii) a lower bound and upper bound of the total demand of the two products from the first period through the current period. Type (ii) bounds model uncertainty associated with demand substitution between the two products in a period. Type (iii) bounds model uncertainty associated with inter-temporal substitution of the total demand of the two products. From a forecasting point of view, it is also important to use type (ii) and (iii) bounds. A well-known principle in demand forecasting is that aggregate forecast is more accurate than individual ones. Thus, type (iii) bounds are generally tighter and easier to obtain than type (ii) bounds which are generally tighter and easier to obtain than type (i) bounds.



We formulate our model as a robust optimization problem in which we maximize the worst-case total sales revenue over all possible demand realizations subject to a number of constraints induced by the given business rules and the nature of demand substitution described earlier.

### 4.1.3 Contributions and Organization

Our study contributes to the existing literature in the following three ways. First, as discussed above, our model is more practical than most existing multi-product dynamic pricing models. We explicitly consider various business rules, whereas the existing literature has largely ignored such practical issues and relied on stylized models. We explicitly model demand substitution between the two products (both price-driven and inventory-driven substitutions) and across time periods, whereas inventory-driven substitution between the two products and inter-temporal demand substitution are not considered in most existing dynamic pricing models. We assume that there is limited demand information and use intervals to characterize demand, whereas demand distributions are precisely known in most existing models.

Second, we develop a dynamic programming algorithm to solve our problem. To speed up the DP algorithm, we further develop a fully polynomial-time approximation scheme which guarantees a proven near optimal solution in a manageable computational time for practically sized problems. For a special case of the problem where only price markdowns are allowed, we show that the search spaces in the DP algorithm can be reduced greatly so that the algorithm is capable of generating optimal solutions in a reasonable amount of computational time. Extensive numerical experiments are conducted to show the effectiveness and robustness of the proposed solution approaches.

Third, we generate a set of interesting managerial insights on how the price elasticities, demand uncertainty level, number of allowed price changes, and some other problem parameters impact on the optimal price paths of the products. We also compare the optimal prices obtained by our max-min approach and a risk-

neutral approach, and evaluate the value of dynamic pricing. These insights can help store managers make better pricing decisions when facing high demand uncertainty due to lack of information.

The rest of this chapter is organized as follows. In Section 4.2, we define the problem precisely and present a dynamic programming formulation. In Section 4.3, we develop a fully polynomial time approximation scheme (FPTAS) for solving the DP formulation of the problem. In Section 4.4, we study a special case of the problem where only price markdowns are allowed and develop structural properties which are used to reduce the computational time of the DP algorithm. In Section 4.5, computational tests are conducted and the corresponding managerial insights are discussed. We conclude this chapter in Section 4.6. The proofs for all the lemmas and theorems are presented in Chapter D of the Appendix.

## 4.2 Problem Definition and Formulation

### 4.2.1 Problem Definition

We consider a dynamic pricing problem in which a firm sells two products (indexed as  $k = 1, 2$ ) with initial inventory level  $I_1, I_2$ , respectively, over a short selling season consisting of a finite number of time periods,  $1, 2, \dots, T$ . Typically in practice, each time period represents one to two weeks. Due to long supply lead time and short selling horizon, inventory replenishment is not allowed. In line with most of the existing literature, we do not consider inventory holding cost or time discounting factor. Any unsold items after the selling horizon have zero value. The case in which unsold items have non-zero salvage value can be easily converted to the case with zero-salvage value (see, e.g., Gallego and van Ryzin 1994). Price change can only occur at the beginning of each period and the price for a product, once set, has to stay unchanged for the entire time period.

The three business rules (i), (ii) and (iii) discussed in Section 4.1 must be followed. Following business rule (i) (i.e., discrete set of allowable prices), we denote the set of allowable prices for product  $k \in \{1, 2\}$  as  $\{p_k^1, \dots, p_k^{l_k}, \dots, p_k^{m_k}\}$  with

$p_k^1 > \dots > p_k^{l_k} > \dots > p_k^{m_k}$ , where  $m_k$  denotes the number of allowed price levels for product  $k$ ,  $p_k^1$  denotes the regular selling price of product  $k$ , and  $l_k \in \{1, 2, \dots, m_k\}$  is a general index that denotes a particular price level for product  $k$ , where larger  $l_k$  corresponds to a lower price. These prices are predetermined and given as part of the model input. In the remaining of this chapter, we will use price  $p_k^{l_k}$  and price level  $l_k$  interchangeably. Following business rule (ii) (i.e., a limited number of price changes allowed), we define  $R_k$  to be the maximum number of price changes allowed for product  $k$ . To model business rule (iii) (i.e., allowed magnitude for a price change), for each price level  $l_k \in \{1, 2, \dots, m_k\}$ , we define a corresponding set of price levels  $F_k^{l_k}$  to be the set of allowable price levels that can be used for product  $k$  in a period if the price level used in the previous period is  $l_k$ . We give a simple example to illustrate the concept. Let us consider the problem where product 1 has 6 possible price levels,  $\{\$269, \$249, \$229, \$199, \$179, \$159\}$ . Suppose that the price level used in the previous period for product 1 is  $l_1 = 3$  that corresponds to  $\$229$ . Suppose that business rule (iii) requires that each time if there is a price change, the change has to be at least 10% but no more than 30% of the original price ( $\$229$  in this case). With some simple calculation, one can easily see that the set of the allowed price levels for the current period is  $F_1^3 = \{1, 3, 4, 5\}$ .

Next we describe how uncertain demand is modeled. Given a price pair  $(l_1, l_2)$  of the two products in time period  $t$ , we denote the uncertain demand of each product  $k$  to be  $D_{kt}^{l_1 l_2}$ , for  $k = 1, 2$ . We assume that we know the expected value of  $D_{kt}^{l_1 l_2}$ , denoted as  $\bar{D}_{kt}^{l_1 l_2}$ . As discussed in Section 4.1, we use three types of bounds to characterize our estimates of the underlying uncertainty space of  $D_{1t}^{l_1 l_2}$  and  $D_{2t}^{l_1 l_2}$ . First, the actual demand  $D_{kt}^{l_1 l_2}$  is within a known lower bound  $L_{kt}^{l_1 l_2}$  and upper bound  $U_{kt}^{l_1 l_2}$ , i.e.

$$L_{kt}^{l_1 l_2} \leq D_{kt}^{l_1 l_2} \leq U_{kt}^{l_1 l_2}, \quad (4.1)$$

for  $k = 1, 2, t \in \{1, \dots, T\}, l_1 \in \{1, \dots, m_1\}, l_2 \in \{1, \dots, m_2\}$ .

Secondly, the total demand of the two products in period  $t$  is also within a known

lower bound  $L_t^{l_1 l_2}$  and upper bound  $U_t^{l_1 l_2}$ , i.e.

$$L_t^{l_1 l_2} \leq D_{1t}^{l_1 l_2} + D_{2t}^{l_1 l_2} \leq U_t^{l_1 l_2}, \quad (4.2)$$

for  $t \in \{1, \dots, T\}$ ,  $l_1 \in \{1, \dots, m_1\}$ ,  $l_2 \in \{1, \dots, m_2\}$ .

which models the uncertainty of demand substitution between the two products in each period. Since in general estimate of aggregate demand is more accurate and the corresponding bounds are tighter, we assume that  $L_t^{l_1 l_2} \geq L_{1t}^{l_1 l_2} + L_{2t}^{l_1 l_2}$  and  $U_t^{l_1 l_2} \leq U_{1t}^{l_1 l_2} + U_{2t}^{l_1 l_2}$ , although the results derived in this chapter still work without this assumption. We also note that we do not require the bounds defined in (4.1) and (4.2) to be symmetric about the expected values  $\bar{D}_{kt}^{l_1 l_2}$  or  $\bar{D}_{1t}^{l_1 l_2} + \bar{D}_{2t}^{l_1 l_2}$ .

Finally, to model the uncertainty of inter-temporal demand substitution, we define  $\gamma_t$  as a price path from time period 1 to time period  $t$ , which specifies a particular price pair  $(l_1, l_2)$  used in each period  $\tau$ , for  $\tau \in \{1, 2, \dots, t\}$ . If  $(l_1, l_2)$  is the price pair used in period  $\tau$  in price path  $\gamma_t$ , we denote  $(l_1, l_2, \tau) \in \gamma_t$ . If price path  $\gamma_t$  is used over the periods  $1, 2, \dots, t$ , then the total expected demand of the two products over these periods is known to be  $\sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} (\bar{D}_{1\tau}^{l_1 l_2} + \bar{D}_{2\tau}^{l_1 l_2})$ , which is denoted as  $\bar{D}_t(\gamma_t)$  with a slight abuse of notation. The actual total demand of the two products over these periods,  $\sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} (D_{1\tau}^{l_1 l_2} + D_{2\tau}^{l_1 l_2})$ , which is unknown, must be in some neighborhood of the expected value  $\bar{D}_t(\gamma_t)$ . We assume that this neighborhood is bounded by a known lower bound and upper bound, each at most  $B_t$  units away from  $\bar{D}_t(\gamma_t)$ , where  $B_t$  is known and measures the cumulative uncertainty from period 1 to  $t$ . In summary,

$$\bar{D}_t(\gamma_t) - B_t \leq \sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} (D_{1\tau}^{l_1 l_2} + D_{2\tau}^{l_1 l_2}) \leq \bar{D}_t(\gamma_t) + B_t, \quad (4.3)$$

for  $t \in \{1, \dots, T\}$  and any price path  $\gamma_t$

Due to inter-temporal substitution effect, it is expected that the bounds involved in constraint (4.3) are in general tighter than those involved in constraint (4.2), i.e.  $\bar{D}_t(\gamma_t) - B_t \geq \sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} L_{\tau}^{l_1 l_2}$  and  $\bar{D}_t(\gamma_t) + B_t \leq \sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} U_{\tau}^{l_1 l_2}$ , although we do not make such assumptions in our model.

The set of possible demand realizations (which we call demand uncertainty space) in time period  $t$  given price path  $\gamma_t$  is determined by constraints (4.1), (4.2), and (4.3). Clearly, this set not only depends on the price pair in the current period, it also depends on the price path up to the current period and the actual demand realizations in the previous periods. In Section 4.2.2, we will discuss how the demand uncertainty spaces are represented in our dynamic programming formulation.

In case one of the products stocks out before the end of the planning horizon, we assume that a known fraction of the unsatisfied customers for this product will buy the other product (this is the result of inventory driven substitution). We denote  $\alpha_t^{l_1 l_2}$  ( $\beta_t^{l_1 l_2}$ ) to be the fraction of unsatisfied demand who will switch from product 2 to product 1 (from product 1 to product 2) in period  $t$  if product 2 (product 1) is out of stock when the prices for the products are  $l_1, l_2$ . For ease of presentation, we call  $\alpha_t^{l_1 l_2}, \beta_t^{l_1 l_2}$  *demand conversion rates*. Therefore, if product 1 stocks out in period  $t$  under price pair  $(l_1, l_2)$ , the actual demand for product 2 will be  $D_{2t}^{l_1 l_2} + \lfloor \beta_t^{l_1 l_2} (D_{1t}^{l_1 l_2} - i_1) \rfloor$ , where  $D_{1t}^{l_1 l_2}, D_{2t}^{l_1 l_2}$  are the initial demands before the inventory-driven substitution effect and  $i_1$  is the available inventory of product 1 at the beginning of period  $t$  (hence  $D_{1t}^{l_1 l_2} - i_1$  is the unmet demand for product 1). Throughout this chapter, we only deal with integer-valued demand and inventory. We also assume that  $\alpha_t^{l_1 l_2} p_1^{l_1} \leq p_2^{l_2}$  and  $\beta_t^{l_1 l_2} p_2^{l_2} \leq p_1^{l_1}, \forall t \in \{1, 2, \dots, T\}, l_k \in \{1, 2, \dots, m_k\}$ , which simply states that one cannot make more profits by intentionally making the inventory of one product “unavailable” and forcing customers to buy the other product.

We now describe the decision process in our problem. For ease of presentation, we assume that there are a *decision maker* who does not know the actual demand realization in advance and seeks to maximize the total revenue over the planning horizon in the worst case, and an *adversary* who controls the actual realization of demand. At the beginning of each period  $t \in \{1, 2, \dots, T\}$  starting from period 1, the decision maker first chooses an allowable price pair for the two products for the current period. Then, given this price pair, the adversary picks a particular realization of demand for the current period from the set of possible demand realizations defined by constraints (4.1), (4.2), and (4.3). Next, at the beginning of period  $t + 1$ ,

after having known the actual demand in period  $t$ , the decision maker chooses an allowable price pair for period  $t + 1$ , and this process continues until the end of period  $T$ .

We note that our problem is *NP*-hard even if there is a single product only and there is no demand uncertainty. The *NP*-hardness proof given in Chen et al. (2011) for a single-product deterministic markdown pricing problem can be applied to this special case of our problem.

## 4.2.2 Problem Formulation

We formulate our problem as a dynamic programming problem. Constraint (4.3) which, together with (4.1) and (4.2), defines the possible demand realizations in a period  $t$ , involves the price path  $\gamma_t$  from period 1 to  $t$ . However, it would make the DP extremely inefficient if we keep track of the price path in the DP. To avoid this, we introduce a state variable called *cumulative demand deviation*, and use this to convert constraint (4.3) into an equivalent constraint such that representing this equivalent constraint in the DP will not involve the price path explicitly. Given a price path  $\gamma_\tau$  over the first  $\tau$  time periods, we define the cumulative demand deviation over the first  $\tau$  time periods, denoted as  $d_\tau(\gamma_\tau)$ , to be the cumulative difference between the actual demand realization and the expected demand over these time periods, i.e.,  $d_\tau(\gamma_\tau) = \sum_{j=1}^{\tau} \sum_{(l_1, l_2, j) \in \gamma_\tau} (D_{1j}^{l_1 l_2} + D_{2j}^{l_1 l_2}) - \bar{D}_\tau(\gamma_\tau)$ . Constraint (4.3) is equivalent to the constraint that the total demand deviation in the first  $t$  time periods is between  $-B_t$  and  $B_t$ , i.e.,

$$-B_t \leq d_t(\gamma_t) \leq B_t \quad (4.4)$$

Suppose that the cumulative demand deviation in the first  $t - 1$  periods is known and its value is denoted as  $d$  ( $d$  can be negative). Then, we have  $d_t(\gamma_t) = d + (D_{1t}^{l_1 l_2} + D_{2t}^{l_1 l_2}) - (\bar{D}_{1t}^{l_1 l_2} + \bar{D}_{2t}^{l_1 l_2})$ . Thus constraint (4.4) is further equivalent to

$$(\bar{D}_{1t}^{l_1 l_2} + \bar{D}_{2t}^{l_1 l_2}) - B_t - d \leq D_{1t}^{l_1 l_2} + D_{2t}^{l_1 l_2} \leq (\bar{D}_{1t}^{l_1 l_2} + \bar{D}_{2t}^{l_1 l_2}) + B_t - d \quad (4.5)$$

Constraints (4.3) and (4.5) are equivalent. However, they require a different set of parameters. Constraint (4.3) requires to know the price path used, whereas

constraint (4.5) requires to know the cumulative demand deviation  $d$  in the first  $t - 1$  periods and the price pair  $(l_1, l_2)$  in period  $t$ . We use constraint (4.5) to replace (4.3) when we formulate our problem below. This way we do not need to keep track of the price path; instead we keep track of cumulative demand deviation, which makes our DP formulation more efficient. We note that the lower bound (upper bound) in constraint (4.5) can be larger than, equal to, or smaller than that in (4.2), depending on the values of  $B_t$  and  $d$ .

Given that price pair  $(l_1, l_2)$  is used in time period  $t$  and that the cumulative demand deviation from period 1 through period  $t - 1$  is  $d$ , we use  $\Omega_{t,d}^{l_1 l_2}$  to denote the set of possible demand realizations, also called *demand uncertainty space* for ease of presentation, in period  $t$ . By (4.1), (4.2) and (4.5), we have ,

$$\begin{aligned} \Omega_{t,d}^{l_1 l_2} = \{ & (D_{1t}^{l_1 l_2}, D_{2t}^{l_1 l_2}) \mid \\ & L_{1t}^{l_1 l_2} \leq D_{1t}^{l_1 l_2} \leq U_{1t}^{l_1 l_2}, \\ & L_{2t}^{l_1 l_2} \leq D_{2t}^{l_1 l_2} \leq U_{2t}^{l_1 l_2}, \\ & L_t^{l_1 l_2} \leq D_{1t}^{l_1 l_2} + D_{2t}^{l_1 l_2} \leq U_t^{l_1 l_2}, \\ & (\bar{D}_{1t}^{l_1 l_2} + \bar{D}_{2t}^{l_1 l_2}) - B_t - d \leq D_{1t}^{l_1 l_2} + D_{2t}^{l_1 l_2} \leq (\bar{D}_{1t}^{l_1 l_2} + \bar{D}_{2t}^{l_1 l_2}) \\ & \quad \quad \quad + B_t - d \} \end{aligned}$$

We note that when  $t = 1$ , since the cumulative deviation before the first period must be 0,  $\Omega_{1,d}^{l_1 l_2}$  is well defined for  $d = 0$ , but not defined for  $d \neq 0$ . Figure 4.1 shows a typical shape of a demand uncertainty space  $\Omega_{t,d}^{l_1 l_2}$  for given  $t, d, l_1, l_2$  (which is the collection of the integer points in the shaded area).

In the following, we present our DP formulation. For ease of presentation, in the DP formulation below and the corresponding proofs later, we skip some superscripts and subscripts and simply denote demand for product 1 and 2 as  $D_1, D_2$ , and cumulative deviation up to time period  $t - 1$  as  $d$ . Our DP goes backward from time period  $T$  to 1. We define the value function  $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  as the maximum revenue one can achieve from time period  $t$  to  $T$  under the following conditions: (1) initial inventories of the two products at the beginning of time period  $t$  are  $i_1, i_2$ , respectively; (2) the price levels used for the two products in period  $t - 1$  are  $l_1, l_2$ , respectively; (3) the remaining number of price changes that are allowed for

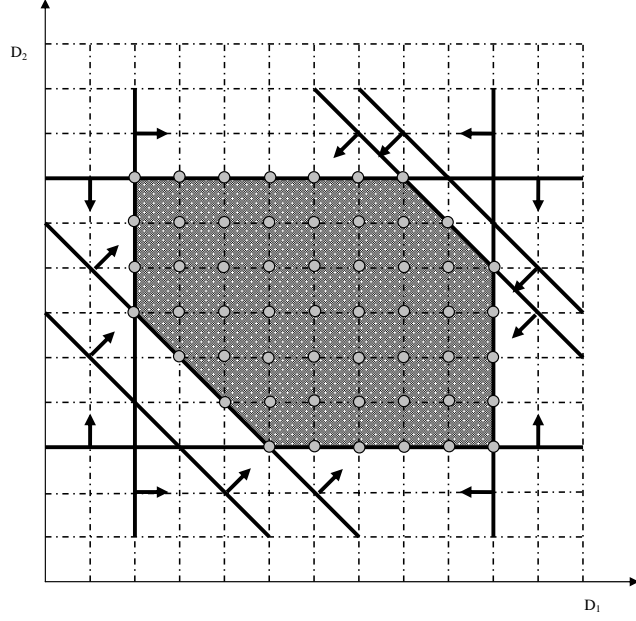


Figure 4.1: Typical shape of a demand uncertainty space,  $\Omega_{t,d}^{l_1 l_2}$

the two products from time period  $t$  to  $T$  is  $r_1, r_2$ , respectively; (4) the cumulative demand deviation in the first  $t-1$  periods is  $d$ . As one can see, the number of states is in the order of  $TI_1 I_2 m_1 m_2 R_1 R_2 B_{\max}$ , where  $B_{\max} = \max\{B_t | t = 1, \dots, T\}$ .

The problem can be formulated as a max-min problem with the following recursive equation.

$$V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) = \max_{(l'_1, l'_2)} \min_{(D_1, D_2)} \{p_1^{l'_1} S_1 + p_2^{l'_2} S_2 + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d')\} \quad (4.6)$$

subject to:

$$l'_1 \in F_1^{l_1} \text{ if } r_1 \geq 1, \text{ or } l'_1 = l_1 \text{ if } r_1 = 0, \quad (4.7)$$

$$l'_2 \in F_2^{l_2} \text{ if } r_2 \geq 1, \text{ or } l'_2 = l_2 \text{ if } r_2 = 0, \quad (4.8)$$

$$(D_1, D_2) \in \Omega_{t,d}^{l'_1 l'_2}, \quad (4.9)$$

$$S_1 = \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor), \quad (4.10)$$

$$S_2 = \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor), \quad (4.11)$$

$$i'_1 = i_1 - S_1, \quad (4.12)$$

$$i'_2 = i_2 - S_2, \quad (4.13)$$

$$r'_1 = r_1 \text{ if } l'_1 = l_1, \text{ or } r'_1 = r_1 - 1 \text{ otherwise}, \quad (4.14)$$

$$r'_2 = r_2 \text{ if } l'_2 = l_2, \text{ or } r'_2 = r_2 - 1 \text{ otherwise}, \quad (4.15)$$



$$d' = d + (D_1 + D_2) - (\bar{D}_{1t}^{l'_1 l'_2} + \bar{D}_{2t}^{l'_1 l'_2}) \quad (4.16)$$

In the above formulation, (4.6) defines the value function  $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$ . The decisions to make include first setting the price pair  $(l'_1, l'_2)$  for period  $t$  by the decision maker following constraints (4.7) and (4.8), and then giving the demand realization  $(D_1, D_2)$  for period  $t$  by the adversary following constraint (4.9). All the other variables are uniquely defined by the remaining constraints (4.10) - (4.16). The variables  $S_1, S_2$  are the sales of the two products in period  $t$ , and  $(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d')$  together form a state in period  $t + 1$ .

The boundary conditions are as follows:

$$V_t(0, 0, l_1, l_2, r_1, r_2, d) = 0, \forall t \in \{1, \dots, T\}, l_k \in \{1, \dots, m_k\}, r_k \in \{0, \dots, R_k\}, d \in \{-B_{t-1}, \dots, 0, \dots, B_{t-1}\}$$

$$V_{T+1}(i_1, i_2, l_1, l_2, r_1, r_2, d) = 0, \forall i_k \in \{0, \dots, I_k\}, l_k \in \{1, \dots, m_k\}, r_k \in \{0, \dots, R_k\}, d \in \{-B_T, \dots, 0, \dots, B_T\}.$$

The optimal solution is found by calculating  $V_1(I_1, I_2, 1, 1, R_1, R_2, 0)$ .

The above formulation is fairly straightforward. However, we demonstrate in the following that solving this DP to optimality can be extremely time consuming.

Since the adversary's goal is to minimize the decision maker's total revenue, one may think that the adversary should choose the demand realization  $(D_1, D_2)$  as low as possible or at least one of the corner points in the demand uncertainty space  $\Omega_{t,d}^{l'_1 l'_2}$ . If this is true, then we can eliminate most of the demand uncertainty space and look for values of  $(D_1, D_2)$  on the left or/and lower boundary or corner points of the demand uncertainty space. Unfortunately, this intuition is not valid. We give an example in Chapter C of the Appendix to show that the worst-case demand can be in the middle of a demand uncertainty space. This implies that to solve the DP formulation, we may have to search all possible demand points in a demand uncertainty space in order to find the worst one. Therefore, in the worst case, the computational time of the proposed DP is  $O(TI_1I_2m_1^2m_2^2R_1R_2D_{1,\max}D_{2,\max}B_{\max})$ , where  $D_{1,\max}$  and  $D_{2,\max}$  are the largest possible demand for product 1 and 2 in any period at any prices. As one can see, the computational time increases with the amount of initial inventory as well as the magnitude of the demand. Therefore,

for practical problems with high initial inventory and large demand, it can take an excessive amount of computational time to solve this DP to optimality.

In the next section, we propose a fully polynomial time approximation scheme (FPTAS). This approximation scheme guarantees a proven near optimal solution in a manageable computational time for practically sized problems.

### 4.3 Approximation Algorithm

In this section, we develop a fully polynomial time approximation scheme (FPTAS) for solving the DP formulation given in Section 4.2. For any  $\epsilon > 0$ , a FPTAS generates a feasible solution  $A(I)$  for any problem instance  $I$  such that its objective value  $Z(A(I))$  is at most  $\epsilon$  away from the optimal objective value  $Z^*(I)$ , i.e.,  $|Z(A(I)) - Z^*(I)| \leq \epsilon Z^*(I)$ , and the computational time used to generate  $A(I)$  is polynomial in the problem input size and  $1/\epsilon$ . A FPTAS is the strongest possible result one can achieve for a  $NP$ -hard problem unless  $P = NP$  (e.g. Vazirani 2001, page 68).

Instead of checking all feasible solutions in the solution spaces of the DP (which we may have to do in order to find an optimal solution), in our approximation scheme we only check the solutions in the approximate solution spaces which are much smaller than the original solution spaces. This will lose some precision but save the computational time. More specifically, we partition the state spaces and the demand uncertainty spaces in the DP into smaller spaces and consider a single solution only in each smaller space. We describe below how this is done.

We partition the space of  $i_1$  into intervals of length  $\Delta_1$ , the space of  $i_2$  into intervals of length  $\Delta_2$ , and the space of each of  $D_1, D_2, d$  into intervals of length  $\Delta$ , where  $\Delta_1, \Delta_2, \Delta$  are positive integers. We choose the smallest integer point in each interval to represent all the points in this interval. The space formed by these representative points alone form an approximate space. Therefore, for any value of  $i_1, i_2, d$  in their original state space, there is a corresponding value, denoted as  $\tilde{i}_1, \tilde{i}_2, \tilde{d}$ , in the corresponding approximate space which is the smallest integer point

in the interval containing  $i_1, i_2, d$ . Similarly, for each possible demand  $(D_1, D_2)$  in an original demand uncertainty space, there is a corresponding demand, denoted as  $(\tilde{D}_1, \tilde{D}_2)$ , in the corresponding approximate demand uncertainty space. For example, if  $\Delta_1 = 5$ , then we divide the space of  $i_1$  into intervals  $[0, 5)$ ,  $[5, 10)$ , ..., and use a single point  $\tilde{i}_1 = 0$  in the approximate space to represent any value of  $i_1 \in [0, 5)$  in the original space, a single point  $\tilde{i}_1 = 5$  in the approximate space to represent any value of  $i_1 \in [5, 10)$  in the original space, and so on. It can be seen that  $\tilde{i}_1 \geq i_1 - (\Delta_1 - 1)$ ,  $\tilde{i}_2 \geq i_2 - (\Delta_2 - 1)$ ,  $\tilde{D}_1 \geq D_1 - (\Delta - 1)$ ,  $\tilde{D}_2 \geq D_2 - (\Delta - 1)$ , and  $\tilde{d} \geq d - (\Delta - 1)$ , for any integer valued  $i_1, i_2, D_1, D_2, d$ , respectively. It can also be seen that  $\tilde{i}_1, \tilde{i}_2$  are always within the original space of  $i_1, i_2$ , but  $\tilde{d}, \tilde{D}_1, \tilde{D}_2$  may be out of their original space because  $\tilde{d}, \tilde{D}_1, \tilde{D}_2$  can be smaller than the lower bound of  $d, D_1, D_2$ , respectively.

We note that in the remainder of this chapter, the tilde notation applied to a number  $x$  (resulting in  $\tilde{x}$ ) always represents the above described relationship between  $x$  and  $\tilde{x}$ , i.e.  $\tilde{x}$  is the smallest integer point in the approximate space corresponding to the original space where  $x$  belongs. For ease of presentation, we may use symbols  $\Phi_1, \Phi_2, \Phi$  to express such relationship between a point in the original space and the corresponding point in the approximate space as follows:  $\tilde{i}_1 = \Phi_1(i_1), \tilde{i}_2 = \Phi_2(i_2), \tilde{D}_1 = \Phi(D_1), \tilde{D}_2 = \Phi(D_2), \tilde{d} = \Phi(d)$ . For example, if  $\Delta_1 = 10, i_1 = 22$ , then  $\tilde{i}_1 = \Phi_1(i_1) = \Phi_1(22) = 20$ .

In an approximate state space, no approximation is applied to the other state variables  $l_1, l_2, r_1, r_2$  (i.e., all possible values of these variables are considered). We use  $\tilde{\Omega}_{t,d}^{l_1 l_2}$  to denote the resulting approximate demand uncertainty space of an original demand uncertainty space  $\Omega_{t,d}^{l_1 l_2}$ .

Our approximation algorithm uses the same DP as formulated in Section 4.2 except that it is implemented in the approximate state spaces and approximate demand spaces. For ease of presentation, we refer to our approximation algorithm as AS (which stands for Approximation Scheme) hereinafter. We use  $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  to denote the value function with the following recursive re-

lation.

$$\begin{aligned}
A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) &= \max_{(l'_1, l'_2)} \min_{(\tilde{D}_1, \tilde{D}_2)} \{p_1^{l'_1} S_1 + p_2^{l'_2} S_2 + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}')\} \\
\text{subject to:} \quad & l'_1 \in F_1^{l_1} \text{ if } r_1 \geq 1 \text{ or } l'_1 = l_1 \text{ if } r_1 = 0 \\
& l'_2 \in F_2^{l_2} \text{ if } r_2 \geq 1 \text{ or } l'_2 = l_2 \text{ if } r_2 = 0 \\
& (\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1, l'_2} \\
& S_1 = \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1, l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& S_2 = \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1, l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& \tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - S_1\} \\
& \tilde{i}'_2 = \Phi_2\{\tilde{i}_2 - S_2\} \\
& r'_1 = r_1 \text{ if } l'_1 = l_1, \text{ or } r'_1 = r_1 - 1 \text{ otherwise,} \\
& r'_2 = r_2 \text{ if } l'_2 = l_2, \text{ or } r'_2 = r_2 - 1 \text{ otherwise,} \\
& \tilde{d}' = \Phi\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_{1t}^{l'_1, l'_2} + \bar{D}_{2t}^{l'_1, l'_2})\}
\end{aligned}$$

The boundary conditions are as follows:

$$\begin{aligned}
A_t(0, 0, l_1, l_2, r_1, r_2, \tilde{d}) &= 0, \quad \forall t, l_1, l_2, r_1, r_2, \tilde{d} \\
A_{T+1}(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) &= 0, \quad \forall \tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}.
\end{aligned}$$

An approximate solution is found by calculating  $A_1(\tilde{I}_1, \tilde{I}_2, 1, 1, R_1, R_2, 0)$ .

Since in the AS the revenue calculation is based on approximated values of demand and inventory, the value  $A_1(\tilde{I}_1, \tilde{I}_2, 1, 1, R_1, R_2, 0)$  does not represent the actual revenue one can achieve by implementing the solution from the AS. Thus, we need to recalculate the revenue obtained by implementing the solution from the AS. To this end, we define value function  $R_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  to be the actual revenue obtained by implementing the solution from the AS for the state  $(i_1, i_2, l_1, l_2, r_1, r_2, d)$  in the original state space. The values  $R_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  are computed backward from period  $T$  to 1. In period  $t$ , for any state  $(i_1, i_2, l_1, l_2, r_1, r_2, d)$ , its value  $R_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  is calculated by solving the following optimality equation, where  $(l'_1, l'_2)$  is the given optimal price pair for period  $t$  obtained from the AS for the corresponding approximate state  $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ .

$$R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) = \min_{(D_1, D_2)} \{p_1^{l'_1} S_1 + p_2^{l'_2} S_2 + R_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d')\}$$

subject to:

$$\begin{aligned}
(D_1, D_2) &\in \Omega_{t,d}^{l'_1 l'_2} \\
S_1 &= \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
S_2 &= \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
i'_1 &= i_1 - S_1 \\
i'_2 &= i_2 - S_2 \\
r'_1 &= r_1 \text{ if } l'_1 = l_1, \text{ or } r'_1 = r_1 - 1 \text{ otherwise,} \\
r'_2 &= r_2 \text{ if } l'_2 = l_2, \text{ or } r'_2 = r_2 - 1 \text{ otherwise,} \\
d' &= d + (D_1 + D_2) - (\bar{D}_{1t}^{l'_1 l'_2} + \bar{D}_{2t}^{l'_1 l'_2})
\end{aligned}$$

The value  $R_1(I_1, I_2, 1, 1, R_1, R_2, 0)$  is then the actual total revenue of the solution from the AS.

In the following, we first make some observations about the demand uncertainty spaces. Since these properties are straightforward, we omit the proofs. These observations will be used later when we prove some properties of the value functions of the DPs.

**Observation 1** For any  $\delta \geq 0$  and  $d$  such that  $\Omega_{t,d}^{l_1 l_2}$  and  $\Omega_{t,d-\delta}^{l_1 l_2}$  are nonempty, if  $(D_1, D_2) \in \Omega_{t,d-\delta}^{l_1 l_2}$ , but  $(D_1, D_2) \notin \Omega_{t,d}^{l_1 l_2}$ , then there exists  $(D'_1, D'_2) \in \Omega_{t,d}^{l_1 l_2}$  such that  $D'_1 \leq D_1 \leq D'_1 + \delta, D'_2 \leq D_2 \leq D'_2 + \delta$  and  $(D_1 + D_2) - (D'_1 + D'_2) \leq \delta$ .

**Observation 2** For any  $\delta \geq 0$  and  $d$  such that  $\tilde{\Omega}_{t,d}^{l_1 l_2}$  and  $\tilde{\Omega}_{t,d+\delta}^{l_1 l_2}$  are nonempty, if  $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t,d+\delta}^{l_1 l_2}$ , but  $(\tilde{D}_1, \tilde{D}_2) \notin \tilde{\Omega}_{t,d}^{l_1 l_2}$ , then there exists  $(\tilde{D}'_1, \tilde{D}'_2) \in \tilde{\Omega}_{t,d}^{l_1 l_2}$  such that  $\tilde{D}_1 \leq \tilde{D}'_1 \leq \tilde{D}_1 + \delta, \tilde{D}_2 \leq \tilde{D}'_2 \leq \tilde{D}_2 + \delta$  and  $(\tilde{D}_1 + \tilde{D}_2) - (\tilde{D}'_1 + \tilde{D}'_2) \leq \delta$ .

**Observation 3** For any  $d$  such that  $\Omega_{t,d}^{l_1 l_2}$  and  $\tilde{\Omega}_{t,d}^{l_1 l_2}$  are nonempty, and  $\Delta \geq 1$ , if  $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t,d}^{l_1 l_2}$ , then there exists  $(D'_1, D'_2) \in \Omega_{t,d}^{l_1 l_2}$  such that  $D'_1 - (\Delta - 1) \leq \tilde{D}_1 \leq D'_1 + (\Delta - 1), D'_2 - (\Delta - 1) \leq \tilde{D}_2 \leq D'_2 + (\Delta - 1)$  and  $(\tilde{D}_1 + \tilde{D}_2) - (D'_1 + D'_2) \leq d - \tilde{d}$ .

**Observation 4** For any  $d$  such that  $\Omega_{t,d}^{l_1 l_2}$  and  $\tilde{\Omega}_{t,d}^{l_1 l_2}$  are nonempty, and  $\Delta \geq 1$ , if  $(D_1, D_2) \in \Omega_{t,d}^{l_1 l_2}$ , then there exists  $(\tilde{D}'_1, \tilde{D}'_2) \in \tilde{\Omega}_{t,d}^{l_1 l_2}$  such that  $\tilde{D}'_1 - (\Delta - 1) \leq D_1 \leq \tilde{D}'_1 + (\Delta - 1), \tilde{D}'_2 - (\Delta - 1) \leq D_2 \leq \tilde{D}'_2 + (\Delta - 1)$  and  $(\tilde{D}'_1 + \tilde{D}'_2) - (D_1 + D_2) \leq d - \tilde{d}$ .

Next we present four lemmas to show some properties of the value functions  $V_t(\cdot)$ ,  $A_t(\cdot)$ ,  $R_t(\cdot)$  and their relationships. Lemma 1 gives properties of the value functions  $V_t(\cdot)$  of the exact DP while Lemma 2 gives properties of the value functions  $A_t(\cdot)$  of the approximate DP. Each result in each lemma provides an upper bound on the change of the value function caused by a change of one or two particular state variables. Based on these results, we then characterize the gap between  $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  and  $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  in Lemma 3 and the gap between  $R_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  and  $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  in Lemma 4. Lemmas 3 and 4 are the building blocks for the proof of our main result, Theorem 1, given after these lemmas.

**Lemma 1** *In any period  $t$ , for any  $\delta_1, \delta_2, \delta \geq 0$ , the following inequalities hold as long as the value of each state variable involved is within its domain.*

- (i)  $V_t(i_1 + \delta_1, 0, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, 0, l_1, l_2, r_1, r_2, d)$
- (ii)  $V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1$
- (iii)  $V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_2^1$
- (iv)  $V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1$
- (v)  $V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1(\delta_1 + i_2) - p_2^1 i_2$
- (vi)  $V_t(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1 - p_2^1 \delta_1$
- (vii)  $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d - \delta) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta - p_2^1 \delta$

**Lemma 2** *In any period  $t$ , for any  $\delta_1, \delta_2, \delta \geq 0$ , the following inequalities hold as long as the value of each state variable involved is within its domain in the approximate state space.*

- (i)  $A_t(\tilde{i}_1 + \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, 0, l_1, l_2, r_1, r_2, \tilde{d})$
- (ii)  $A_t(\tilde{i}_1 + \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1$
- (iii)  $A_t(\tilde{i}_1 + \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_2^1$
- (iv)  $A_t(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1$
- (v)  $A_t(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1(\tilde{\delta}_1 + \tilde{i}_2 + (T - t)(\Delta_1 - 1)) - p_2^1 \tilde{i}_2$

$$\begin{aligned}
(vi) \quad & A_t(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1 - p_2^1 (\tilde{\delta}_1 + (T-t)(\Delta_2 - 1)) \\
(vii) \quad & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \tilde{\delta}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 (\tilde{\delta} + 1) - p_2^1 (\tilde{\delta} + 1)
\end{aligned}$$

**Lemma 3** *In any period  $t$ , for any  $\Delta_1, \Delta_2, \Delta \geq 1$ , the following inequality holds as long as the value of each state variable involved is within its domain in the corresponding state space,*

$$\begin{aligned}
& A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
\geq & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \\
& -(T-t+1)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& -(T-t+1)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \}
\end{aligned}$$

**Lemma 4** *In any period  $t$ , for any  $\Delta_1, \Delta_2, \Delta \geq 1$ , the following inequality holds as long as the value of each state variable involved is within its domain in the corresponding state space,*

$$\begin{aligned}
& R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \\
\geq & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
& -(T-t+1)p_1^1 \{ 10(\Delta - 1) + (T-t+1)(\Delta_1 - 1) + 2(\Delta_2 - 1) \} \\
& -(T-t+1)p_2^1 \{ 10(\Delta - 1) + 2(\Delta_1 - 1) + (T-t+1)(\Delta_2 - 1) \}
\end{aligned}$$

Next we show that under some mild conditions on problem parameters and when  $\Delta_1$ ,  $\Delta_2$  and  $\Delta$  are defined properly, the approximation algorithm (AS) constructed earlier gives a fully polynomial-time approximation scheme (FPTAS). Define  $D_j^{max}$  to be the maximum possible demand of product  $j$  in any period under any price level, respectively, for  $j = 1, 2$ . Define  $D_j^{total}$  to be the minimum possible total demand of product  $j$  over all the time periods if product  $j$  is priced at level  $m_j$  whereas the other product is priced at level 1 in every time period, for  $j = 1, 2$ . Define  $B_{max} = \max\{B_t \mid t = 1, \dots, T\}$ .

We assume that there exists a positive finite integer  $C_0$  independent of the problem parameters such that the following conditions hold.

- (i)  $\max(p_1^1, p_2^1) / \min(p_1^{m_1}, p_2^{m_2}) \leq C_0$ ;
- (ii)  $I_j / D_j^{total} \leq C_0$ , for  $j = 1, 2$ ;
- (iii)  $D_j^{max} / D_j^{total} \leq C_0$ , for  $j = 1, 2$ ;
- (iv)  $D_j^{max} / I_j \leq C_0$ , for  $j = 1, 2$ ;
- (v)  $B_{max} / D_j^{total} \leq C_0$ , for  $j = 1, 2$ .
- (vi)  $B_{max} / I_j \leq C_0$ , for  $j = 1, 2$ .

We allow the value of  $C_0$  to be any positive finite integer as long as it is not problem parameter dependent. We can set  $C_0$  sufficiently large (e.g. 100) such that the above assumptions are easily justified from a practical point of view. Since the two products are substitutable, one product should not be priced, for example, 100 times, higher than the other product. Similarly, the highest price used for a product should not be too much (for example, 100 times) higher than the lowest price used for the product. This justifies (i). To justify the other assumptions, it is reasonable to assume that under the same price pair, the maximum possible demand for a product in a period should not be, for example, 100 times, higher than the minimum possible demand (i.e.,  $U_{kt}^{l_1 l_2} \leq 100L_{kt}^{l_1 l_2}$ ). By definitions of  $D_j^{total}$  and  $D_j^{max}$ ,  $D_1^{total} \geq \sum_{t=1}^T L_{1t}^{m_1, 1}$ ,  $D_2^{total} \geq \sum_{t=1}^T L_{2t}^{1, m_2}$ ,  $D_1^{max} \leq \max\{U_{1t}^{m_1, 1} \mid t = 1, \dots, T\}$ , and  $D_2^{max} \leq \max\{U_{2t}^{1, m_2} \mid t = 1, \dots, T\}$ . Thus,  $D_1^{max} \leq U_{1\tau}^{m_1, 1} \leq 100L_{1\tau}^{m_1, 1} \leq 100D_1^{total}$ , for some  $\tau \in \{1, \dots, T\}$ . Similarly, it can be shown that  $D_2^{max} \leq 100D_2^{total}$ . This justifies (iii). We can justify (v) similarly. In reality, the available inventory at the beginning  $I_j$  should not be too large relative to  $D_j^{total}$  because otherwise there will be inventory remaining in the end and hence it can be reduced to a lower level without affecting the solution. Similarly,  $I_j$  should not be too small relative to  $D_j^{max}$  or allowed cumulative demand deviation  $B_t$  because otherwise the problem becomes trivial. This justifies (ii), (iv) and (vi).

For any given  $\epsilon > 0$ , we define the lengths of the intervals  $\Delta_1, \Delta_2, \Delta$  used in our approximation scheme as follows:

$$\Delta_j = \lceil \frac{I_j \epsilon}{102C_0^2 T^2} \rceil, \text{ for } j = 1, 2, \text{ and}$$

$$\Delta = \max\{\theta_1, \theta_2, \theta_3\}, \text{ where}$$

$$\theta_j = \lceil \frac{D_j^{max} \epsilon}{102C_0^2 T} \rceil, \text{ for } j = 1, 2, \text{ and}$$



$$\theta_3 = \lceil \frac{B_{max}\epsilon}{102C_0^2T} \rceil.$$

**Theorem 1** *For any  $\epsilon > 0$ , the approximation algorithm AS with the values of  $\Delta_1, \Delta_2, \Delta$  defined as above generates a solution that is within a relative error  $\epsilon$  from the optimality with running time  $O(T^8 m_1^2 m_2^2 R_1 R_2 / \epsilon^5)$ .*

This theorem means that AS is a fully polynomial time approximation scheme (FP-TAS).

#### 4.4 A Special Case

In this section, we consider a special case of the general problem defined in Section 4.2. We derive some optimality properties for this special case. These properties enable us to significantly reduce the search spaces in the DP and solve practically-sized problems to optimality within a reasonable computational time. In this special case, we assume that (i) only price markdown is allowed for both products, i.e., the price used for each product must be non-increasing over the time; and (ii) the demand for each product is independent across time periods.

Both assumptions are reasonable in many practical settings and are commonly made in the markdown pricing literature. Markdown pricing is a common technique used by retailers to sell the remaining inventory of a product in the end of the product life cycle. According to the National Retail Federation, marked-down goods, which accounted for just 8% of department store sales three decades ago, now account for over 20% of sales (Merrick 2001). The primary objective of markdown pricing is to stimulate sales by lowering the price over time. Marking down the price over time is proven to be effective for time sensitive goods such as digital cameras for which the demand drops steadily with time towards the end of the product life cycle.

Assumption (ii) is made in most existing dynamic pricing literature including the handful of papers that study multi-product dynamic pricing problems (Gallego and van Ryzin 1997, Bitran et al. 2006, Maglaras and Meissner 2004, Dong et al. 2009, Zhang and Cooper 2009, and Akcay et al. 2010, which are all reviewed in Section 4.1). Problems with inter-temporal demand correlation are clearly more

difficult to solve than problems without such correlation. Thus, it is likely that Assumption (ii) is made in the literature to make the problem more tractable. However, even if there is no tractability issue, this assumption may have to be made in situations where there is little information about the inter-temporal correlation so that the decision maker is not able to characterize such correlation in a meaningful way. For example, following our demand uncertainty modeling approach described in Section 4.2.1, due to lack of any meaningful information, the parameter  $B_t$  that measures the cumulative demand uncertainty in the first  $t$  time periods is estimated to be so large that the bounds involved in (4.3) are too loose to be useful (i.e.,  $\bar{D}_t(\gamma_t) - B_t \leq \sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} L_{\tau}^{l_1 l_2}$  and  $\bar{D}_t(\gamma_t) + B_t \geq \sum_{\tau=1}^t \sum_{(l_1, l_2, \tau) \in \gamma_t} U_{\tau}^{l_1 l_2}$  and hence constraint (4.3) is dominated by constraint (4.2)).

Assumption (ii) means that there is no inter-temporal demand substitution. Therefore, when defining demand uncertainty spaces, we no longer consider constraint (4.3) (or equivalently constraint (4.5)). The demand uncertainty spaces are defined by constraints (4.1) and (4.2) only. Given the price pair  $(l_1, l_2)$  for time period  $t$ , we denote the demand uncertainty space in period  $t$  as  $\Omega_t^{l_1 l_2}$ . By (4.1) and (4.2), we have,

$$\Omega_t^{l_1 l_2} = \{ (D_{1t}^{l_1 l_2}, D_{2t}^{l_1 l_2}) \mid \begin{aligned} &L_{1t}^{l_1 l_2} \leq D_{1t}^{l_1 l_2} \leq U_{1t}^{l_1 l_2}, \\ &L_{2t}^{l_1 l_2} \leq D_{2t}^{l_1 l_2} \leq U_{2t}^{l_1 l_2}, \\ &L_t^{l_1 l_2} \leq D_{1t}^{l_1 l_2} + D_{2t}^{l_1 l_2} \leq U_t^{l_1 l_2} \} \end{aligned}$$

The shape of a demand uncertainty space is similar to that in the case of the general problem, shown in Figure 4.1.

This special case can be formulated as a DP in a similar way to the DP formulated in Section 4.2.2 for the general problem. We define the value function  $V_t(i_1, i_2, l_1, l_2, r_1, r_2)$  as the maximum revenue one can achieve from time period  $t$  to time period  $T$  under the following conditions: (1) initial inventories of the products at the beginning of time period  $t$  are  $i_1, i_2$ , respectively; (2) the price levels used in period  $t - 1$  are  $l_1, l_2$ , respectively; (3) the remaining number of markdowns that are allowed for the products from time period  $t$  to  $T$  is  $r_1, r_2$ , respectively. The recursive

equation is described as follows.

$$V_t(i_1, i_2, l_1, l_2, r_1, r_2) = \max_{(l'_1, l'_2)} \min_{(D_1, D_2)} \{p_1^{l'_1} S_1 + p_2^{l'_2} S_2 + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2)\} \quad (4.17)$$

$$\text{subject to: } l'_1 \in F_1^{l_1} \cap \{l_1, \dots, m_1\} \text{ if } r_1 \geq 1, \text{ or } l'_1 = l_1 \text{ if } r_1 = 0, \quad (4.18)$$

$$l'_2 \in F_2^{l_2} \cap \{l_2, \dots, m_2\} \text{ if } r_2 \geq 1, \text{ or } l'_2 = l_2 \text{ if } r_2 = 0, \quad (4.19)$$

$$(D_1, D_2) \in \Omega_t^{l'_1 l'_2}, \quad (4.20)$$

$$S_1 = \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor), \quad (4.21)$$

$$S_2 = \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor), \quad (4.22)$$

$$i'_1 = i_1 - S_1, \quad (4.23)$$

$$i'_2 = i_2 - S_2, \quad (4.24)$$

$$r'_1 = r_1 \text{ if } l'_1 = l_1, \text{ or } r'_1 = r_1 - 1 \text{ otherwise,} \quad (4.25)$$

$$r'_2 = r_2 \text{ if } l'_2 = l_2, \text{ or } r'_2 = r_2 - 1 \text{ otherwise.} \quad (4.26)$$

The boundary conditions are as follows.

$$V_t(0, 0, l_1, l_2, r_1, r_2) = 0, \forall t \in \{1, \dots, T\}, l_k \in \{1, \dots, m_k\}, r_k \in \{0, \dots, R_k\}$$

$$V_{T+1}(i_1, i_2, l_1, l_2, r_1, r_2) = 0, \forall i_k \in \{0, \dots, I_k\}, l_k \in \{1, \dots, m_k\}, r_k \in \{0, \dots, R_k\}.$$

The optimal solution is found by calculating  $V_1(I_1, I_2, 1, 1, R_1, R_1)$ .

The above formulation is slightly simpler than the formulation given in Section 4.2.2 for the general problem. Because of the markdown pricing requirement, the set of allowable prices in period  $t$  is  $F_1^{l_1} \cap \{l_1, \dots, m_1\}$  for product 1 and  $F_2^{l_2} \cap \{l_2, \dots, m_2\}$  for product 2. This is reflected in (4.18) and (4.19). In the worst case, the computational time for the proposed DP is  $O(TI_1I_2m_1^2m_2^2R_1R_2D_{1,\max}D_{2,\max})$ .

In the following, we show two results, Lemma 5 and Theorem 2, for the above DP formulation. Lemma 5 is used in the proof of Theorem 2. The properties given in Theorem 2 enable us to consider only a small part of the demand uncertainty spaces in the DP. This, in turn, significantly reduces the computational time.

Lemma 5, says that if we lower the available inventory of a product by one unit at the beginning of a time period, the revenue loss is no more than an amount equal to the price used in the previous period (or the maximum price that can be used in the current period).

**Lemma 5** *In any period  $t$ , the following two inequalities hold as long as the value of each state variable involved is within its domain:*

$$V_t(i_1 - 1, i_2, l_1, l_2, r_1, r_2) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_1^1, \text{ and}$$

$$V_t(i_1, i_2 - 1, l_1, l_2, r_1, r_2) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_2^2.$$

Theorem 2 contains two fairly complex results which are elaborated below after the statement of the theorem. These results essentially mean that given a price pair in any period, lower demand will always result in a lower revenue.

**Theorem 2** *Given any state  $(i_1, i_2, l_1, l_2, r_1, r_2)$  in the beginning of any period  $t$ , for any feasible price pair  $(l_1^l, l_2^l)$  chosen for period  $t$  (i.e.,  $l_1^l \in F_1^{l_1} \cap \{l_1, \dots, m_1\}$ , and  $l_2^l \in F_2^{l_2} \cap \{l_2, \dots, m_2\}$ ), and any demand realization  $(D_1, D_2), (D_1 - 1, D_2), (D_1, D_2 - 1) \in \Omega_t^{l_1^l l_2^l}$ , the following results hold:*

$$\begin{aligned} & p_1^l \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - \min(i_2, D_2)) \rfloor) + p_2^l \min(i_2, D_2 \\ & + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) + V_{t+1}(i_1'', i_2'', l_1^l, l_2^l, r_1', r_2') \\ & \leq p_1^l \min(i_1, D_1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - \min(i_2, D_2)) \rfloor) + p_2^l \min(i_2, D_2 \\ & + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - \min(i_1, D_1)) \rfloor) + V_{t+1}(i_1', i_2', l_1^l, l_2^l, r_1', r_2'), \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} & p_1^l \min(i_1, D_1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - 1 - \min(i_2, D_2 - 1)) \rfloor) + p_2^l \min(i_2, D_2 - 1 \\ & + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - \min(i_1, D_1)) \rfloor) + V_{t+1}(i_1''', i_2''', l_1^l, l_2^l, r_1', r_2') \\ & \leq p_1^l \min(i_1, D_1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - \min(i_2, D_2)) \rfloor) + p_2^l \min(i_2, D_2 \\ & + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - \min(i_1, D_1)) \rfloor) + V_{t+1}(i_1', i_2', l_1^l, l_2^l, r_1', r_2'), \end{aligned} \quad (4.28)$$

where  $i_1'' = i_1 - \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1''' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1^l l_2^l} (D_2 - 1 - \min(i_2, D_2 - 1)) \rfloor)$ , and  $i_2''' = i_2 - \min(i_2, D_2 - 1 + \lfloor \beta_t^{l_1^l l_2^l} (D_1 - \min(i_1, D_1)) \rfloor)$ .

The left side of (4.27) or (4.28) represents the total revenue from period  $t$  to  $T$  given that in period  $t$  the price pair used is  $(l_1^l, l_2^l)$  and the demand realization is

$(D_1 - 1, D_2)$  or  $(D_1, D_2 - 1)$ . The right side of these inequalities represents the total revenue from period  $t$  to  $T$  given that in period  $t$  the price pair used is  $(l'_1, l'_2)$  and the demand realization is  $(D_1, D_2)$ .

The results of (4.27) and (4.28) imply that given any price pair, for any two demand realizations  $(D_1, D_2)$  and  $(D'_1, D'_2)$  in the demand uncertainty space of the given price pair, if  $D'_1 \leq D_1$ ,  $D'_2 \leq D_2$ , then the realization  $(D'_1, D'_2)$  will result in a revenue that is not greater than that generated by the realization  $(D_1, D_2)$ . This observation means that in solving the DP formulation, in any time period  $t$ , for any price pair  $(l_1, l_2)$  chosen by the decision maker, the adversary only needs to consider the demand realizations which are on the left bottom boundary of the demand uncertainty space  $\Omega_t^{l_1 l_2}$ , i.e., integer points on the line segment between point  $A$  and  $B$  shown in Figure 4.2. This reduces the worst-case computational time of the DP from  $O(TI_1I_2m_1^2m_2^2R_1R_2D_{1,\max}D_{2,\max})$  to  $O(TI_1I_2m_1^2m_2^2R_1R_2 \min(D_{1,\max}, D_{2,\max}))$ . Our computational experiment given later (see Section 4.5.3) also shows that the average computational time for a problem instance is significantly reduced.

Intuitively, one may further expect the worst case demand to be one of the two end points of the line segment ( $A$  or  $B$ ). However, we can construct an example to demonstrate that this is not the case. This implies that in the DP algorithm, one has to search the whole line segment in order to find the worst case demand.

## 4.5 Computational Results and Managerial Insights

In this section, we conduct four sets of computational experiments. Unless otherwise stated, all the test problem instances are generated as described in Section 4.5.1. In the first set of experiments, described in Section 4.5.2, we evaluate the performance of the approximation algorithm AS given in Section 4.3. Our computational results show that AS is capable of generating near-optimal solutions in significantly shorter time compared to the exact DP for practically-sized problems. In the second set of experiments, described in Section 4.5.3, we demonstrate that the optimality properties derived in Section 4.4 for the special case enable us to solve

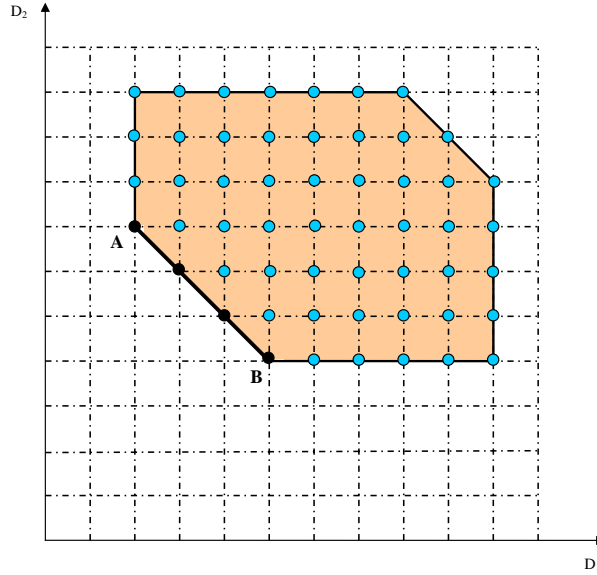


Figure 4.2: Demand uncertainty space  $\Omega_t^{l_1 l_2}$  for the special case, where only demand realizations on the line segment from  $A$  to  $B$  need to be considered by the adversary

much larger problems to optimality in shorter computational time. In the third set of experiments, described in Section 4.5.4, we assess the robustness of the max-min approach that we use by comparing it to a risk-neutral approach which maximizes the total expected revenue. We show that by employing the max-min approach, one can significantly increase the worst case revenue and decrease the variance of total revenue at the expense of average revenue generated. However, this average revenue loss, in general, is very small. In the last set of experiments, described in Section 4.5.5, we derive a set of interesting managerial insights. We show how the optimal pricing strategies change with problem parameters including price elasticities, demand uncertainty level, number of price changes allowed, and demand conversion rates. We also compare the optimal prices obtained by our max-min approach and a risk-neutral approach, and evaluate the value of dynamic pricing. For each computational experiment, the code was run on a PC with a 2.61-GHz AMD Athlon(tm) 642 dual core processor and 3.25-GB memory.

### 4.5.1 Test Problems

We generate test problems that closely follow real-world situations. We use the following configurations.

- Number of time periods,  $T = 6$ . A typical time period in practice is for a duration of one or two weeks, and the entire selling horizon goes anywhere from a few weeks to about several months.
- Number of allowable price levels,  $m_1 = m_2 = 6$ .
- Prices of the products. We consider two cases: (1) the two products have *comparable* prices where the two products have the same price range; (2) the two products have *incomparable* prices where the price of one product is significantly higher than the other product. Case (1) represents situations where the two substitutable products, for example, Canon and Nikon, have similar prices. Case (2) represents situations where the two substitutable products, for example, Dell and Apple, have significantly different prices. The regular prices for the two products are set as follows: for case 1,  $p_1^1 = p_2^1 = 100$ ; for case 2,  $p_1^1 = 100$ ,  $p_2^1 = 50$ . The other allowable prices are set to be 10%, 20%, ..., and 50% lower than the regular price, respectively, i.e.,  $p_1^2 = p_2^2 = 90, \dots, p_1^6 = p_2^6 = 50$  for case 1, and  $p_1^2 = 90, \dots, p_1^6 = 50$ ,  $p_2^2 = 45, \dots, p_2^6 = 25$  for case 2.
- For business rule (ii), the number of price changes allowed,  $R = 4$ .
- For business rule (iii), we require that in each period if there is a price change, it has to be at least 10% but no more than 30% from the regular price. Given the allowable prices specified above, this means that  $F_k^j = \{\max(j-3, 1), \max(j-3, 1)+1, \dots, \min(j+3, 6)-1, \min(j+3, 6)\}$ .

Although our model does not require any explicit form of demand function, we use the following functional form to generate the expected demand for a given price pair  $(l_1, l_2)$  in time period  $t$ :  $\bar{D}_{1t}^{l_1 l_2} = \lfloor f(t)e^{a_{10}-a_{11}\ln(p_1^{l_1})+a_{12}\ln(p_2^{l_2})} \rfloor$  and  $\bar{D}_{2t}^{l_1 l_2} = \lfloor f(t)e^{a_{20}+a_{21}\ln(p_1^{l_1})-a_{22}\ln(p_2^{l_2})} \rfloor$ . This type of demand function is commonly used in the marketing literature (e.g., Reibstein and Gatignon, 1984). In this function, we set  $f(t) = 1.0 - 0.1(t-1)$ , for  $t = 1, 2, \dots, T$ , which captures how demand changes over

time. This represents a typical situation for many time-sensitive products where demand decreases steadily over time. The parameters  $a_{11}, a_{22}$  are *elasticities* for which a larger value indicates that the demand of the product responds more (negatively) to a price increase of this product. The parameters  $a_{12}, a_{21}$  are *cross-elasticities* for which a larger value indicates that the demand of this product responds more (positively) to a price increase of the other product. The parameters  $a_{10}, a_{20}$  are simply constants which are used to control the magnitude of the demand generated. When the two products have comparable prices, we test two scenarios of price elasticities: scenario 1:  $a_{11} = a_{22} = 2.5, a_{12} = a_{21} = 1.5$ ; and scenario 2:  $a_{11} = a_{22} = 1.5, a_{12} = a_{21} = 0.5$ . These two scenarios represent situations in which the two products are equally price-sensitive. However, the demand is more sensitive to a price change in scenario 1 than in scenario 2. When the two products have incomparable prices, we also test two scenarios: scenario 3:  $a_{11} = 2.5, a_{12} = 1.5, a_{22} = 1.5, a_{21} = 0.5$ ; and scenario 4:  $a_{11} = 1.5, a_{12} = 0.5, a_{22} = 2.5, a_{21} = 1.5$ . These two scenarios represent situations in which the demand for one product is more sensitive to a price change than the other product.

The lower and upper bounds for the demand intervals are generated as follows:  $L_{kt}^{l_1 l_2} = \lfloor (1 - \psi + \epsilon) \bar{D}_{kt}^{l_1 l_2} \rfloor$ ,  $U_{kt}^{l_1 l_2} = \lfloor (1 + \psi + \epsilon) \bar{D}_{kt}^{l_1 l_2} \rfloor$ , where  $\psi$  is a parameter between 0 and 1 that measures the uncertainty level and  $\epsilon$  is a random perturbation. Larger  $\psi$  makes the demand interval wider and hence indicates higher demand uncertainty. We let  $\psi = 0.5$ , which implies that the actual demand for each product can be 50% lower or higher than the expected demand. To eliminate the possibility of generating redundant bounds, we have to make sure that the following two conditions hold: (i) the lower bound of the total demand of the two products  $L_t^{l_1 l_2}$  should not be smaller than the sum of the lower bounds of the demand of the two products  $L_{1t}^{l_1 l_2} + L_{2t}^{l_1 l_2}$ , because otherwise  $L_t^{l_1 l_2}$  becomes redundant; (ii)  $L_t^{l_1 l_2}$  should not be greater than either  $L_{1t}^{l_1 l_2} + U_{2t}^{l_1 l_2}$  or  $U_{1t}^{l_1 l_2} + L_{2t}^{l_1 l_2}$ , because otherwise either  $L_{1t}^{l_1 l_2}$  or  $L_{2t}^{l_1 l_2}$  will become redundant. We also have similar conditions for upper bound  $U_t^{l_1 l_2}$ . To this end, we generate the lower and upper bounds for the total demand of the two products in each period  $t$  in the following way:  $L_t^{l_1 l_2} = \lfloor \chi(L_{1t}^{l_1 l_2} + L_{2t}^{l_1 l_2}) + (1 - \chi) \min(L_{1t}^{l_1 l_2} +$



$U_{2t}^{l_1 l_2}, U_{1t}^{l_1 l_2} + L_{2t}^{l_1 l_2}]$ , and  $U_t^{l_1 l_2} = \lfloor \chi(U_{1t}^{l_1 l_2} + U_{2t}^{l_1 l_2}) + (1 - \chi) \max(L_{1t}^{l_1 l_2} + U_{2t}^{l_1 l_2}, U_{1t}^{l_1 l_2} + L_{2t}^{l_1 l_2}) \rfloor$ , where  $\chi$  is a parameter that is randomly generated from the interval between 0 and 1.

We test two initial inventory levels for each product, high and low. The high initial inventory level for a product is set to be the total expected demand when this product takes the second lowest price while the other product takes the second highest price in each period, i.e.,  $I_1 = \sum_{t=1}^T \bar{D}_{1t}^{52}$  and  $I_2 = \sum_{t=1}^T \bar{D}_{2t}^{25}$ . The low initial inventory level is simply 60% of the high initial inventory level. In order to control the problem size so that the computational time of the exact DP is manageable, we choose the value of  $a_{10}, a_{20}$  such that the high initial inventory for both products is 100, i.e.,  $I_1 = I_2 = 100$ . For the problem instances we test, the magnitude of total expected demand of the two products in a single period varies from 5 to 85 depending on the price pair chosen and the time period. The average (over all price pairs and all periods) total expected demand of the two products in a single period is around 30. We set the total allowed cumulative demand deviation to be 30, i.e.,  $B_t = 30, t = 1, 2, \dots, T$ . This implies that, on average over all price pairs, the total cumulative demand up to time period  $t$  can deviate from its expected value by at most 100%, 50%, 33%, 25%, 20%, 17% for  $t = 1, 2, 3, 4, 5, 6$ , respectively. Clearly, this models the situation where the aggregate demand is more accurate. The demand conversion rates are set as  $\alpha_t^{l_1 l_2} = \min(0.25 + 0.05 * (l_1 - l_2), p_2^{l_2} / p_1^{l_1})$  and  $\beta_t^{l_1 l_2} = \min(0.25 + 0.05 * (l_2 - l_1), p_1^{l_1} / p_2^{l_2})$ .

## 4.5.2 Performance of the Approximation Scheme

In this set of experiments, we evaluate the performance of the approximation scheme (AS) given in Section 4.3. We test all the four scenarios of price comparability and price elasticities described in Section 4.5.1. For scenario 1 (where the two products have comparable prices with  $a_{11} = a_{22} = 2.5, a_{12} = a_{21} = 1.5$ ) and scenario 2 (where the two products have comparable prices with  $a_{11} = a_{22} = 1.5, a_{12} = a_{21} = 0.5$ ), we test three different initial inventory levels of the two products, (high, high), (high, low), (low, low). We omit the initial inventory level (low, high) because it has

similar results to the case (high, low). For scenario 3 (where the two products have incomparable prices with  $a_{11} = 2.5, a_{12} = 1.5, a_{22} = 1.5, a_{21} = 0.5$ ) and scenario 4 (where the two products have incomparable prices with  $a_{11} = 1.5, a_{12} = 0.5, a_{22} = 2.5, a_{21} = 1.5$ ), we test four different initial inventory levels, (high, high), (high, low), (low, high) and (low, low). Therefore, we test a total of 14 combinations of parameter values. We test AS with two sets of approximation interval lengths. In the first set, under which the approximation scheme is denoted as AS1, we use the approximation interval lengths  $\Delta = 2$ , and  $\Delta_j = 10$  if the initial inventory level of product  $j$  is high, and  $\Delta_j = 5$  if the initial inventory level of product  $j$  is low, for  $j = 1, 2$ . In the second set, under which the approximation scheme is denoted as AS2, we use the approximation interval lengths  $\Delta = 2$ , and  $\Delta_j = 5$  if the initial inventory level of product  $j$  is high, and  $\Delta_j = 2$  if the initial inventory level of product  $j$  is low, for  $j = 1, 2$ . For each combination of parameter values, we test 5 randomly generated problem instances and report the median and worst case performance measure. In order to make the total computational time manageable for this set of experiments, we set the uncertainty interval parameter  $\psi = 0.25$  instead of  $\psi = 0.5$  as described in Section 4.5.1. We believe that the results should be similar for other values of  $\psi$ .

The results are reported in Table 4.1. In this table, we report the optimality gaps of AS1 and AS2, which are defined as the relative difference (in percent) between the solution obtained by AS1 or AS2 and the optimal solution obtained by solving the exact DP to optimality. We also report the computational time of the exact DP, AS1 and AS2. Clearly, both approximation schemes, AS1 and AS2, generate near-optimal solutions in a significantly shorter time than the exact DP. For example, for initial inventory level (high, high) in scenario 1, if we apply AS1, in the worst case of the five problem instances we test, we can obtain a solution that is only 5.5% lower than the real optimal solution. The computational time, however, can be reduced from 33815 seconds (more than 9 hours) to 146 seconds (less than 3 minutes). Furthermore, by comparing the performance of AS1 and AS2, we found that while the approximation scheme with smaller interval lengths

(AS2) does not always guarantee a better solution, it does generate a solution with smaller optimality gap for the vast majority (more than 90%) of the problem instances. The magnitude of this improvement is problem specific with an average value of 2.5%. This is achieved at the expense of computational time. As one can see, the computational time of AS2 is approximately 4 to 6 times that of AS1. This table demonstrates that overall the approximation scheme performs well for all the problem instances we test.

### 4.5.3 Performance of the DP for the Special Case

For the special case, we test a set of problem instances that are generated similarly as those tested in Section 4.5.2 except for the fact that (i) there is no constraint on the cumulative demand deviation (hence, state variable  $d$  is not needed); and (ii) only price markdowns are allowed. Results show that without employing the optimality properties presented in Section 4.4, it takes about 7 to 15 minutes to solve a test problem instance. On the other hand, if we employ the optimality properties, each test problem instance can be solved to optimality in no more than 3 minutes. Note that the initial inventory levels for these test instances are  $I_1 = I_2 = 100$ . To further demonstrate that the optimality properties are useful for larger problems, we also test another set of problem instances with higher initial inventory levels, i.e.,  $I_1 = I_2 = 500$ . Without employing the optimality properties, it takes 2 to 3 days to solve each problem instance to optimality. However, the computational time is reduced to about 2 hours if one uses the optimality properties. These results clearly demonstrate that for the special case we consider in this study, the optimality properties derived in Section 4.4 enable us to solve much larger problems to optimality within a reasonable amount of computational time.

### 4.5.4 Robustness of the Max-Min Approach

In this set of experiments, we compare the performance of our max-min approach with the risk-neutral approach which maximizes the total expected revenue.

Table 4.1: Performance of the Approximation Scheme AS

Scenario	Initial Inventory	Optimality Gap (%)						Computational Time (in seconds)					
		AS1			AS2			Exact DP		AS1		AS2	
		Worst	Median	AS2 Worst	AS2 Median	AS2 Worst	AS2 Median	Worst	Median	Worst	Median	Worst	Median
Comparable Prices 1: $a_{11} = 2.5, a_{12} = 1.5$ $a_{22} = 2.5, a_{21} = 1.5$	(high,high)	5.5	3.3	3.2	1.9	3.2	1.9	33815	31288	146	139	533	509
	(high,low)	6.4	4.8	8.0	4.1	8.0	4.1	15006	13549	137	131	627	598
	(low,low)	5.1	4.0	6.0	3.4	6.0	3.4	7137	6393	127	123	727	705
Comparable Prices 2: $a_{11} = 1.5, a_{12} = 0.5$ $a_{22} = 1.5, a_{21} = 0.5$	(high,high)	7.2	6.5	7.2	5.5	7.2	5.5	33064	31973	151	146	558	540
	(high,low)	7.3	5.8	7.4	6.5	7.4	6.5	13838	13216	143	138	658	635
	(low,low)	3.8	2.9	6.0	2.7	6.0	2.7	6534	6180	132	129	762	744
Incomparable Prices 3: $a_{11} = 2.5, a_{12} = 1.5$ $a_{22} = 1.5, a_{21} = 0.5$	(high,high)	8.8	6.8	7.9	4.4	7.9	4.4	31844	30166	163	159	596	581
	(high,low)	8.7	7.9	4.9	3.9	4.9	3.9	19760	18446	195	190	890	870
	(low,high)	2.7	2.0	3.1	0.7	3.1	0.7	13806	12513	152	147	695	673
(low,low)	4.9	4.4	3.1	2.3	3.1	2.3	8988	8044	183	177	1051	1011	
Incomparable Prices 4: $a_{11} = 1.5, a_{12} = 0.5$ $a_{22} = 2.5, a_{21} = 1.5$	(high,high)	8.6	5.9	6.4	0.4	6.4	0.4	24896	23946	149	141	544	516
	(high,low)	9.1	6.5	4.1	1.3	4.1	1.3	15056	14528	177	171	809	780
	(low,high)	3.1	2.6	1.8	0.5	1.8	0.5	10554	9996	137	132	628	603
(low,low)	4.5	3.1	3.6	0.7	3.6	0.7	6894	6539	164	159	937	910	

Because the probability distribution of the demand is explicitly used in calculating the total expected revenue, the risk-neutral approach requires to know the demand distribution which is not required in the max-min approach. The risk-neutral approach can be formulated the same way as (4.6) to (4.16) for the max-min approach except that the objective function (4.6) should be replaced by the following.

$$V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) = \max_{(l'_1, l'_2)} \sum_{(D_1, D_2) \in \Omega_{t,d}^{l'_1, l'_2}} \text{Prob}(D_1, D_2) \{p_1^{l'_1} S_1 + p_2^{l'_2} S_2 + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d')\}$$

where  $\text{Prob}(D_1, D_2)$  denotes the probability that the realized demand is  $(D_1, D_2)$ . We test two versions of risk-neutral approaches: (i) one assuming that the underlying demand distribution is uniform (denoted as RN-U); (ii) and the other assuming that the underlying demand distribution is bivariate normal (denoted as RN-N). These three approaches, i.e., max-min approach (denoted as MM), RN-U, and RN-N, are tested under a large number (1,000,000) of demand instances that are randomly generated following two distributions, i.e., uniform and normal. It should be noted that the demand distribution (either uniform or normal) assumed in the risk-neutral approaches (i.e., RN-U or RN-N) represents the decision maker's belief about the underlying demand distribution before the actual demand is revealed while the distribution (again either uniform or normal) used to generate random demand instances represents the actual demand distribution. When these two coincide, it represents the situation where the information about the underlying demand distribution is accurate. Same as in Section 4.5.2, we test 14 combinations of parameter values. Our purpose here is to demonstrate how the max-min approach MM compares to the risk-neutral approaches, RN-U and RN-N, in terms of the worst case, mean, standard deviation and range (max - min) of the revenue generated by each approach.

The results are summarized in Table 4.2. This table shows the percentage increase for each performance measure (worst case revenue, mean revenue, standard deviation and range) if one applies the MM approach as opposed to the RN-U or RN-N approach. For instance, for initial inventory level (high, high) in scenario 1, when

the actual demand distribution is uniform, by applying the MM approach as opposed to the RN-U approach, the mean revenue will decrease by 4.5%. However, the worst case revenue will increase by 6.8%, and the standard deviation and range will decrease by 28.9% and 24.6%, respectively. While the actual performance change might be problem specific, the improvement of the worst case revenue and reduction of variance are fairly significant for most of the problem instances tested. The mean revenue loss, on the other hand, is relatively small in general. This table clearly demonstrates that in most cases by applying the max-min approach, one can increase the worst case revenue and significantly decrease the variance (measured by the standard deviation and the range) of the revenue.

One may note that for scenarios 2 and 4 when the initial inventory level is low for both products, applying the max-min approach increases the standard deviation of the revenue generated, which is different from the results for other cases. This can be explained as follows. When the initial inventory level is low for both products, the risk-neutral approaches (RN-U and RN-N) may simply pick the regular price as the optimal price in each period for the majority of the demand realizations. In this case, even under different demand realizations, as long as the total realized demand is higher than the initial inventory, the revenue is equal to a fixed value, which is the regular price times the initial inventory. This results in a revenue distribution that is highly concentrated on this fixed value, which is also the right end of the corresponding distribution. Clearly, the variance (or standard deviation) of such a revenue distribution can be very small even if the range is large. The max-min approach, on the other hand, is more likely to adjust the price over time in order to protect the worst case scenario. This can lead to different revenue values under different demand realizations even if the total demand of all these demand realizations are all higher than the initial inventory level. The resulting revenue distribution is closer to a normal distribution. Thus, the variance of revenue distribution generated by the max-min approach is likely to be larger than that generated by the risk-neutral approach.

To better illustrate the difference between our max-min approach and the

Table 4.2: Comparison of Max-Min Approach v.s. Risk-neutral Approach

Scenario	Initial Inventory	Actual Demand Distribution	Gap, (MM/RN-U - 1)*100%				Gap, (MM/RN-N - 1)*100%			
			Worst	Mean	STDV	Range	Worst	Mean	STDV	Range
1: Comparable Prices $a_{11} = 2.5, a_{12} = 1.5$ $a_{22} = 2.5, a_{21} = 1.5$	(high,high)  (high,low)  (low,low)	Uniform	6.8	-4.5	-28.9	-24.6	6.8	-4.4	-28.5	-23.6
		Normal	6.8	-5.9	-33.1	-23.7	6.8	-5.9	-32.3	-24.3
		Uniform	6.8	-4.2	-21.5	-18.1	6.8	-4.1	-21.7	-19.5
		Normal	6.8	-5.8	-28.3	-15.4	6.8	-5.8	-27.5	-15.4
		Uniform	8.1	-5.4	-26.9	-27.1	5.4	-5.3	-25.6	-23.1
		Normal	8.1	-5.9	-15.5	-17.7	5.4	-5.9	-15.3	-17.7
2: Comparable Prices $a_{11} = 1.5, a_{12} = 0.5$ $a_{22} = 1.5, a_{21} = 0.5$	(high,high)  (high,low)  (low,low)	Uniform	9.7	-3.4	-51.8	-48.5	9.5	-3.5	-52.6	-48.5
		Normal	9.7	-3.4	-37.2	-41.6	9.5	-3.4	-38.2	-42.9
		Uniform	6.5	-2.4	-34.4	-29.9	7.5	-2.3	-32.2	-29.4
		Normal	6.5	-2.3	-22.2	-27.8	7.5	-2.4	-20.0	-23.1
		Uniform	5.3	-2.6	7.9	-11.8	5.3	-2.6	7.9	-11.8
		Normal	5.3	-1.9	104.3	-11.3	5.3	-1.9	108.9	-11.3
3: Incomparable Prices $a_{11} = 2.5, a_{12} = 1.5$ $a_{22} = 1.5, a_{21} = 0.5$	(high,high)  (high,low)  (low,high)  (low,low)	Uniform	6.2	-2.3	-31.0	-25.5	5.3	-2.3	-28.0	-20.1
		Normal	6.2	-2.4	-31.5	-25.1	5.3	-2.5	-27.8	-29.0
		Uniform	6.2	-2.7	-42.3	-34.6	5.3	-2.7	-39.6	-29.7
		Normal	6.2	-2.6	-46.6	-40.5	5.3	-2.7	-44.5	-37.0
		Uniform	8.3	-2.0	-42.1	-37.4	10.0	-2.0	-32.1	-37.9
		Normal	8.3	-2.0	-35.5	-34.2	10.0	-2.3	-21.4	-27.0
4: Incomparable Prices $a_{11} = 1.5, a_{12} = 0.5$ $a_{22} = 2.5, a_{21} = 1.5$	(high,high)  (high,low)  (low,high)  (low,low)	Uniform	7.7	-2.7	-34.8	-23.7	10.4	-2.6	-34.2	-22.3
		Normal	7.7	-2.6	-33.7	-39.3	10.4	-2.7	-32.9	-41.6
		Uniform	6.1	-0.9	-16.4	-17.6	5.4	-0.9	-16.4	-16.0
		Normal	6.1	-0.9	-12.9	-14.3	5.4	-0.9	-12.5	-9.9
		Uniform	12.9	-4.0	-19.1	-22.0	13.6	-4.0	-18.9	-22.0
		Normal	12.9	-3.2	-16.2	-21.3	13.6	-3.2	-16.2	-21.3
4: Incomparable Prices $a_{11} = 1.5, a_{12} = 0.5$ $a_{22} = 2.5, a_{21} = 1.5$	(high,high)  (high,low)  (low,high)  (low,low)	Uniform	4.1	-3.9	-36.0	-23.9	3.1	-3.9	-36.2	-20.2
		Normal	4.1	-4.2	-32.0	-34.7	3.1	-4.2	-32.3	-29.9
		Uniform	7.2	-5.0	2.6	-23.4	6.7	-5.0	4.5	-21.0
		Normal	7.2	-5.1	38.4	-18.0	6.7	-5.1	39.6	-12.2
		Uniform	6.1	-0.9	-16.4	-17.6	5.4	-0.9	-16.4	-16.0
		Normal	6.1	-0.9	-12.9	-14.3	5.4	-0.9	-12.5	-9.9

risk-neutral approach, Figures 4.3 and 4.4 compare the total revenue generated by two different approaches, MM and RN-U, for initial inventory level (high, high) in scenario 2 when the actual demand distribution is uniform. Specifically, Figure 4.3 shows the distribution of total revenue generated by these two approaches while Figure 4.4 shows the corresponding cumulative distribution. Clearly, the revenue generated by RN-U is distributed over a wider interval while the revenue generated by MM is more concentrated around its mean. The cumulative distributions of these approaches show that using the max-min approach lowers the probability (or risk) of having low total revenue. These observations hold true for other initial inventory levels in other scenarios of price comparability and price elasticities.

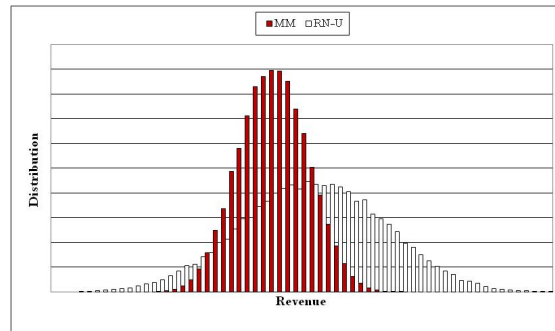


Figure 4.3: Distribution of total revenue generated by MM and RN-U for initial inventory level (high, high) in scenario 2 when the actual demand distribution is uniform

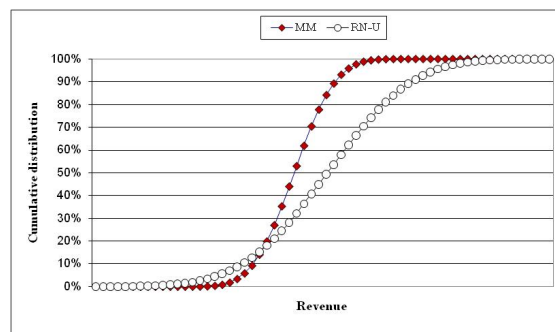


Figure 4.4: Cumulative distribution of total revenue generated by MM and RN-U for initial inventory level (high, high) in scenario 2 when the actual demand distribution is uniform



Table 4.3 shows the results for larger problem instances under scenario 1, where the max-min approach is solved by our approximation scheme AS, and the risk-neutral approaches (RN-U, RN-R) are solved by a similar approximation scheme. Results for other scenarios are similar, and hence not presented in the table. In these problem instances, we set the high initial inventory level  $I_1 = I_2 = 500$ . Low initial inventory level is again 60% of the high initial inventory level, which is 300. In all the approximation schemes, we use the approximation interval lengths  $\Delta = 10$ , and  $\Delta_j = 50$  if the initial inventory level of product  $j$  is high, and  $\Delta_j = 25$  if the initial inventory level of product  $j$  is low, for  $j = 1, 2$ . Other parameters are the same as described in Section 4.5.1. As can be seen, we have similar results for large problem instances for which approximation scheme is applied. That is, implementing the max-min approach can significantly increase the worst case revenue and reduce the variance of the revenue generated, while the mean revenue loss is small.

#### 4.5.5 Managerial Insights

In this set of experiments, we generate a number of managerial insights by investigating the following three questions:

- Q1: How the optimal price paths of the products and the revenue are impacted by the problem parameters such as price elasticities, demand uncertainty level, number of allowed price changes and demand conversion rates?
- Q2: How different are the optimal price paths generated by our max-min approach compared to a risk-neutral approach?
- Q3: What is the value of dynamic pricing?

These questions are answered in Sections 4.5.5.1, 4.5.5.2 and 4.5.5.3, respectively. Unless otherwise stated, we focus on the case with initial inventory level (high, high). We found that when one of the products has low initial inventory, a good strategy is to use the regular price for this product and dynamically adjust the price of the other product according to the realized demand history. Thus the problem reduces

Table 4.3: Comparison of Max-min Approach v.s. Risk-neutral Approach for Larger Problem Instances

Scenario	Initial Inventory	Actual Demand Distribution	Gap, (MM/RN-U - 1)*100%				Gap, (MM/RN-N - 1)*100%			
			Worst	Mean	STDV	Range	Worst	Mean	STDV	Range
Comparable Prices 1: $a_{11} = 2.5, a_{12} = 1.5$ $a_{22} = 2.5, a_{21} = 1.5$	(high,high)	Uniform	9.9	-1.9	-31.8	-19.9	9.5	-2.1	-33.1	-17.8
		Normal	9.9	-1.8	-20.2	-19.1	9.5	-2.0	-21.6	-18.9
	(high,low)	Uniform	9.3	-2.8	-36.6	-30.6	5.9	-3.4	-35.8	-28.7
		Normal	9.3	-4.1	-28.1	-29.1	5.9	-4.8	-23.6	-30.2
	(low,low)	Uniform	8.3	-4.3	-33.6	-22.0	6.3	-4.3	-36.1	-23.4
		Normal	8.3	-4.1	-22.7	-18.2	6.3	-4.8	-25.7	-19.7

to a single-product pricing problem. When both products have low initial inventory, using the regular price for both products will result in a near optimal solution.

#### 4.5.5.1 Impact of Problem Parameters on Optimal Price Paths and Revenue

We first examine the impact of price elasticities on the optimal price paths of the two products. For each of the four scenarios of price comparability and price elasticities described in Section 4.5.1, we conduct simulations to test the optimal prices obtained by our max-min approach. Figures 4.5 to 4.8 show a typical price path for each product under each scenario. We can make the following observations.

- For scenario 1 where the two products have comparable prices and both products are sensitive to price changes, it appears that the best strategy is to use very different prices (one high and one low) for the two products in each period and then reverse their prices at certain point of the selling season (see Figure 4.5). This suggests that one should always promote one product by using a low price while maintaining a regular price or a price close to it for the other product in each period. This maximizes the demand of the product being promoted, and thus achieves the highest possible revenue. When the inventory of the product being promoted drops to a certain level, one should start to promote the other product.
- For scenario 2 where the two products have comparable prices and both products are much less sensitive to price changes compared to scenario 1, it appears that the optimal strategy is to maintain a relatively constant price that is close to the regular price for both products and mildly adjust the price according to the realized demand history (see Figure 4.6). This suggests that a static pricing strategy, e.g., fixing the prices of both products at the regular price, may result in a satisfactory performance.
- For scenario 3 where the two products have incomparable prices (the price range of product 1 is from 50 to 100 and the price range of product 2 is from 25 to 50), and product 1 is much more sensitive to price changes than product 2, the optimal

strategy is to maintain a regular price or a price close to it for the less price-sensitive product, i.e., product 2, and dynamically adjust the price for the more price-sensitive product, i.e., product 1, according to the realized demand history (see Figure 4.7).

- For scenario 4 which is similar to scenario 3 except that product 2 is much more sensitive to price changes than product 1, we observe similar results, i.e., the optimal strategy is to maintain a regular price or a price close to it for the less price-sensitive product, i.e., product 1, and dynamically adjust the price for the more price-sensitive product, i.e., product 2, according to the realized demand history (see Figure 4.8). The results for scenarios 3 and 4 imply that the relative magnitude of price elasticities (as opposed to the relative magnitude of price range) is the primary factor that determines the optimal pricing strategy.

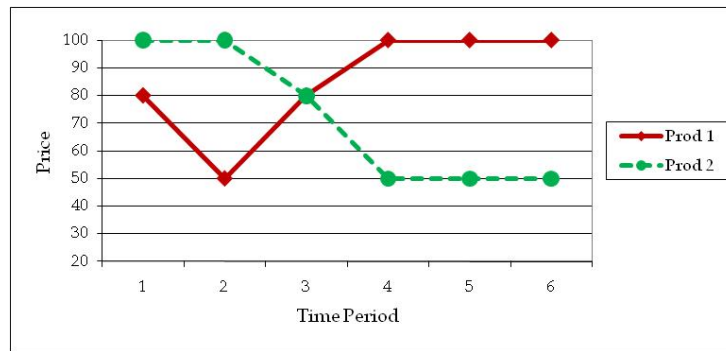


Figure 4.5: Sample Price Path for Scenario 1

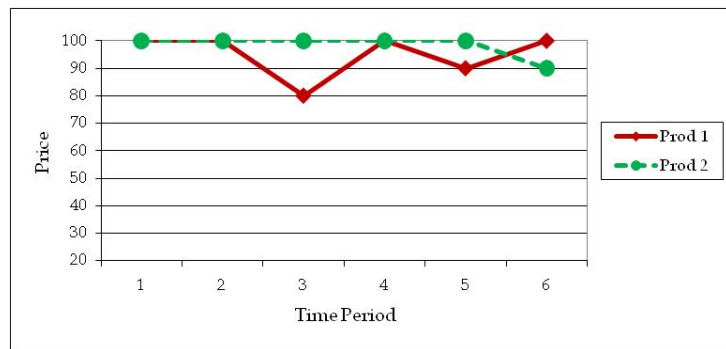


Figure 4.6: Sample Price Path for Scenario 2

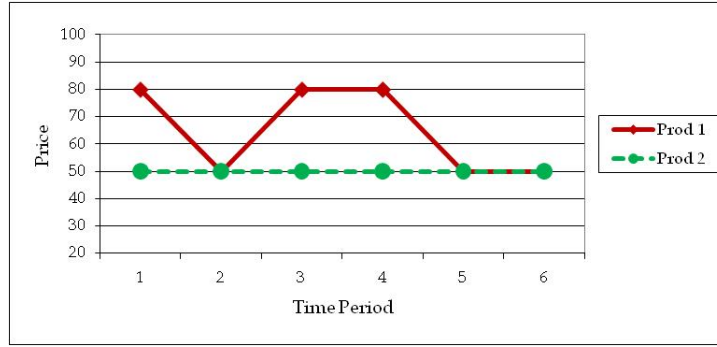


Figure 4.7: Sample Price Path for Scenario 3

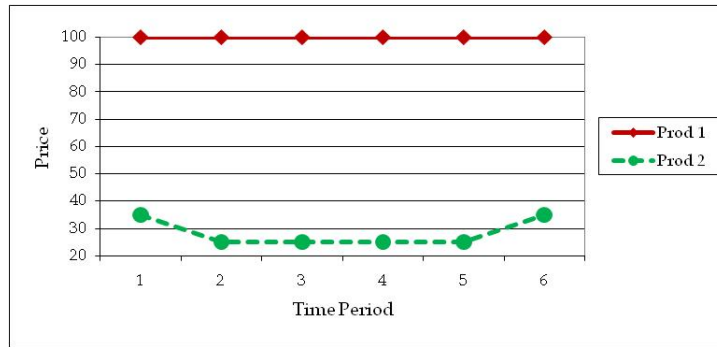


Figure 4.8: Sample Price Path for Scenario 4

We also test another scenario which is similar to scenario 3 except that the price elasticities of both products are higher, i.e.,  $a_{11} = 3.5, a_{12} = 2.0, a_{22} = 2.5, a_{21} = 2.0$ . For this scenario, we have similar findings as in the case of scenario 3, i.e., the optimal strategy is to maintain a regular price or a price close to it for the less price-sensitive product while dynamically adjusting the price of the other product according to the realized demand history. This indicates that when one product is significantly more price sensitive than the other product, the relative magnitude of the price elasticities of the products plays a more important role in determining the optimal pricing strategy than the absolute magnitude of the price elasticities.

We then investigate how the uncertainty level affects the optimal price paths and the revenue. We test two sets of problem instances, one with uncertainty parameter  $\psi = 0.5$  as described in Section 4.5.1, and the other one with uncertainty

parameter  $\psi = 0.25$ . We found that when the uncertainty level is lower, price paths of both products are less responsive to the realized demand history. In this case, different demand scenarios lead to very similar price paths, and in fact, for the majority of the demand scenarios, the price paths are identical. In contrast, when the uncertainty level is higher, optimal price path is fairly sensitive to the realized demand history. Therefore, for low uncertainty case, a deterministic optimization model based on point estimates of demand may perform well, but for high uncertainty case (such as in the problem setting we consider in this study), a model (such as ours) that generates optimal price path dynamically according to the realized demand history is always preferred. We also found that the worst case revenue loss due to overestimating the uncertainty level is relatively small, but the worst case revenue loss due to underestimating the uncertainty level can be fairly significant. Take scenario 1 for instance, when we set the uncertainty level parameter  $\psi = 0.5$  in our max-min model but the actual uncertainty level parameter is 0.25, the worst-case revenue obtained is 2.9% lower than that obtained by using the actual uncertainty level in the model. On the other hand, if we set the uncertainty level parameter  $\psi = 0.25$  in our model but the actual uncertainty level parameter is 0.5, the worst-case revenue obtained is 12.8% lower than that obtained by using the actual uncertainty level in the model. This implies that using an optimization model with small uncertainty intervals may cause significant revenue loss when the actual uncertainty level is high. This also suggests that using wide uncertainty intervals is preferred when it is difficult to accurately estimate these intervals.

We now discuss how the number of allowed price changes affects the revenue. For each of the four scenarios of price comparability and price elasticities described in Section 4.5.1, we test four different cases of the number of allowed price changes ( $R_1 = R_2 = 1, 2, 3$  or  $4$ ). All other parameters are the same as the test instances used in Section 4.5.4. We observe that when the allowed number of price changes increases from 1 to 2, there is a significant increase (about 15%) in the worst-case revenue for scenarios 1, 3 and 4. For scenario 2, however, the revenue increase is much smaller (less than 2%). When the number of allowed price changes increases

from 2 to 4, we observe that the increase in the worse-case revenue for scenario 2 is again insignificant (less than 1%). For scenarios 3 and 4, the increase is moderate, around 2 to 4%, but for scenario 1 the increase is larger, about 8%. Based on this, we can make the following observations which are consistent with what we have observed earlier. When both products are little price-sensitive as in scenario 2, the optimal strategy is to maintain a relatively constant price for each product, and hence allowing 1 or 2 price changes should, in general, be enough to achieve near-optimal revenue. On the other hand, when one product (or both products) is price-sensitive such as in scenarios 3 and 4 (or in scenario 1), one needs to dynamically adjust the price for that product (or both products) in each period according to the realized demand history. In this case, limiting the number of price changes to a small number will likely result in an inferior solution. As we discussed earlier, the optimal pricing strategy for scenario 1 is to use very different prices for the two products in each period and switch their prices at certain time. It requires at least 4 price changes to drop the price of one product to the lowest and then increase it to the highest due to business rule (iii) involved (i.e., price change must be between 10% and 30%). This explains why in the case of scenario 1 there is a significant increase in the worst-case revenue when the number of allowed price changes increases from 2 to 4. We also observe that when the number of allowed price changes increases, the increase in mean revenue is much smaller than the increase in the worst-case revenue. This indicates that for price-sensitive products, when there is high demand uncertainty involved, allowing enough number of price changes is necessary especially when one tries to maximize the worst-case revenue.

Next we investigate how the demand conversion rates  $\alpha_t^{l_1 l_2}, \beta_t^{l_1 l_2}$  impact on the optimal prices. In addition to testing the same problem instances used in all other tests where the demand conversion rates are set as  $\alpha_t^{l_1 l_2} = \min(0.25 + 0.05(l_1 - l_2), p_2^l / p_1^l)$  and  $\beta_t^{l_1 l_2} = \min(0.25 + 0.05(l_2 - l_1), p_1^l / p_2^l)$ , we also test another set of problem instances where the demand conversion rates are set as  $\alpha_t^{l_1 l_2} = \min(0.5 + 0.05(l_1 - l_2), p_2^l / p_1^l)$  and  $\beta_t^{l_1 l_2} = \min(0.5 + 0.05(l_2 - l_1), p_1^l / p_2^l)$ . Results show that demand conversion occurs when one product has a sufficient initial inventory to

satisfy demand in all periods, whereas the other product has a low initial inventory such that it is insufficient to satisfy the demand. In such a case, the conversion rates have no impact on the optimal price of the product with low initial inventory. In fact, using the regular price for this product is optimal. However, the conversion rates have a fairly significant impact on the optimal price of the product with high initial inventory. When the conversion rates are higher, the price of the product with higher initial inventory is also higher compared to the case where the conversion rates are lower. This can be explained as follows. When the conversion rates are higher, the “effective demand” (i.e., original demand plus the converted demand) for the product with high initial inventory is also higher.

#### 4.5.5.2 Max-min versus Risk-neutral

Next, we examine the difference between the optimal prices obtained by our max-min approach and the risk-neutral approach. It appears that under the risk-neutral approach, the price for each product is kept at a relatively constant value with small variation in each period, e.g., 10% price increase or decrease if there is a price change. In addition, the price path is less responsive to the realized demand history, i.e., price paths are very similar especially in the first several periods no matter what the realized demand history in the previous periods is. In contrast, under the max-min approach, more significant price changes are observed. For instance, price can increase by 30% in one period and decrease by 30% in the following period. Price path is more dependent on the realized demand history, e.g., two different demand realizations may lead to very different price paths.

#### 4.5.5.3 Value of Dynamic Pricing

Finally, we compare the dynamic pricing strategy with a static pricing strategy in which a fixed price is used for all periods for each product. Fixed pricing strategy is commonly used in practice because it is easy to implement. For each of the four scenarios of price comparability and price elasticities described in Section 4.5.1, we



test all possible fixed pricing strategies (for the problem instances we test where there are 6 allowable prices for each product, there are 36 possible fixed pricing strategies) and pick the best one in terms of the worst-case performance. We compare our dynamic pricing strategy with this best fixed pricing strategy. Our results show that for scenarios 1, 2, 3 and 4, there is a 10.5%, 5.5%, 3.5%, 2.5%, respectively, increase in the worst case revenue by using the dynamic pricing strategy compared to the best fixed pricing strategy. In reality, even a small percentage increase in revenue can translate into a fairly significant increase in profit. This implies that a dynamic pricing strategy such as ours is necessary especially when both products are price-sensitive (as in scenario 1).

## 4.6 Conclusions

In this chapter, we have studied a dynamic pricing problem involving two substitutable products subject to a set of business rules. Due to the lack of information, it is impossible to precisely characterize the demand distributions. Therefore, we have modeled the demand of each product, aggregate demand of the two products in a period, and aggregate demand of the two products over a number of periods using a set of intervals instead of point estimates or probability distributions. We have proposed a robust optimization approach that maximizes the worst case total revenue. We have developed a fully polynomial time approximation scheme (FPTAS) based on a dynamic program (DP) that generates a proven near optimal solution for practically-sized problems within a reasonable computational time. We have also studied a special case of the general problem where only price markdowns are allowed. We have shown some optimality properties for this special case which enable us to consider only a small subset of the solutions in a demand uncertainty space in the DP, and hence make it possible to solve large problems to optimality within a reasonable computational time.

We have demonstrated through computational experiments that for a variety of combinations of model parameter values, our approximation scheme is capable

of generating near-optimal solutions in a significantly shorter time than the exact DP algorithm. We have also shown that our robust approach increases the worst case total revenue and reduces the variance of the total revenue, compared to a risk-neutral approach. Such an approach is desirable in many practical circumstances where an important goal is to achieve a certain level of revenue or minimize the risk of having a total revenue that falls below a given level. In addition, we have derived a set of interesting managerial insights that can help store managers make better pricing decisions when facing high demand uncertainty due to lack of information.

## Chapter 5

### Conclusion

In this dissertation, we have studied dynamic pricing problems with finite initial inventory and fixed time horizon without inventory replenishment, which we refer to as NR problems. This type of dynamic pricing problems have a wide application in many industries including, e.g., airlines, hotels, cruise lines, rental car companies, long-distance bus companies, and retail industries. In the past two decades particularly in the past a few years, we have witnessed a rapidly-growing body of literature in this area.

This dissertation consists of three essays. In the first essay, we have surveyed existing NR models. We have classified, compared and summarized existing models according to the nature of competition, types of customers, number of products, number of stores, time horizon, allowable prices, and demand models. Then we have studied two specific NR problems in the second and third essays, respectively. In the second essay, we have studied a markdown pricing problem with a single product and multiple stores that are served by a central warehouse. In this problem, joint inventory allocation and markdown pricing decisions need to be made. In the third essay, we have studied a general dynamic pricing problem with two substitutable products and a single store. We have modeled substitution effects between the two products (both price-driven and inventory-driven) and across time periods. In both problems, we have assumed that there is limited information available so that it is not possible to accurately estimate the demand values or demand distributions. To model uncertain demand, we have used scenario-trees in the second essay, and used a set of lower and upper bounds in the third essay. In both problems, we have incorporated commonly-used business rules that have been largely ignored in the existing literature. We have formulated the first problem (the markdown pricing

problem presented in Chapter 3) as a mixed integer program in which we maximize the expected revenue. We have solved it by Lagrangian relaxation and proposed it be implemented on a rolling horizon basis. We have formulated the second problem (the general dynamic pricing problem presented in Chapter 4) as a dynamic program in which we maximize the worst-case revenue. We have developed a full polynomial time approximation scheme that generates a proven near-optimal solution. These algorithms allow us to solve problems with practical sizes in a reasonable amount of time. A number of interesting managerial insights have also been discussed for both problems.

We have found that in the existing literature, the following problems and issues have received no or little interest and may deserve more attention in future work: (i) dynamic pricing problems that incorporates business rules; (ii) strategic customers with bounded rationality; (iii) non-equilibrium market situations; (iv) dynamic pricing problems with complementary products; and (v) empirical research.

## Appendix A

### Problem Complexity

In this section, we prove that our problem is NP-hard (i.e., computationally intractable) even when the demand is deterministic and when there is only a single store or there is only a single time period. In the single-store case, business rule (v) is not applicable. In the single-period case, only one price needs to be determined for each store, and hence business rules (ii) and (iii) are not applicable. The NP-hardness of these special cases means that the general case of our problem is also NP-hard. Thus, there is no simple (i.e., polynomial-time) algorithm that can find an optimal solution to our problem. This justifies the use of heuristic solution approaches for our problem. For concepts such as NP-hardness and polynomial-time algorithms, see the excellent book by Garey and Johnson (1979).

In the following, we first show that the single-store problem with deterministic demand is NP-hard. In this problem, the demand at each price level in each period is known exactly, and the objective is to determine how many units of inventory to allocate to each period and what price level to use in each period such that the total revenue over the planning horizon is maximized subject to the relevant business rules. Since there is only one store, for ease of presentation, we omit the store-related symbol  $r$  from the relevant parameters defined earlier, i.e. we will use  $D_{jt}$  to represent the deterministic demand at price level  $j$  in period  $t$ .

**Theorem 3** *The problem even with a single store and deterministic demand is NP-hard.*

**Proof** We prove this by a reduction from the subset sum problem (SS), a known NP-hard problem (Garey and Johnson 1979).

*SS:* Given a set of  $k$  elements,  $K = \{1, \dots, k\}$ , a positive integer  $a_i$  associated with each element  $i \in K$  and a positive integer  $H$ , does there exist a subset  $Q$  of  $K$  such

that  $\sum_{i \in Q} a_i = H$ ?

Define  $A = \sum_{j \in K} a_j$ . We construct the following instance for the single-store problem based on the instance of SS:

- Number of time periods  $T = k$ .
- Number of allowable price levels  $m = 2k$ , with prices  $p_{2i-1} = 1/M^i + 1$  and  $p_{2i} = 1/(M^i + a_i) + 1$ , for  $i = 1, \dots, k$ , where  $M$  is a sufficiently large positive integer such that  $M > kA$ .
- Demand corresponding to each price level is assumed to be time invariant and denoted as  $D_i$  for price level  $i$ . Let  $D_{2i-1} = M^i$  and  $D_{2i} = M^i + a_i$ , for  $i = 1, \dots, k$ .
- Available inventory  $I_0 = H + \sum_{i=1}^k M^i$ .
- Unit salvage value  $s = 0$ .
- Maximum number of markdowns allowed  $R = k$ .
- For business rule (iv), set  $u_j = 2k$  and  $v_j = j + 1$  such that any price level between  $j + 1$  and  $m$  can be set for the next period if the price level for the current period is  $j$  and if there is a price change in the next period, for  $j = 1, \dots, 2k$ .
- Threshold value for the total revenue of the problem  $F = k + I_0 = k + H + \sum_{i=1}^k M^i$ .

We note that in the above constructed instance, the following hold: (i)  $p_j D_j = 1 + D_j$ , for every  $j = 1, \dots, m$ ; (ii) demand decreases when the price increases; and (iii)  $M^i > A + \sum_{j=1}^{i-1} M^j$ , for  $i = 1, \dots, k$ . Clearly the above instance can be constructed in polynomial time. In the following, we prove that there is a solution to the above instance of our problem with the total revenue greater than or equal to  $F$  if and only if there is a solution to the instance of SS.

*(If part)* If there is a subset  $Q$  of  $K$  such that  $\sum_{i \in Q} a_i = H$ , we construct a solution to the instance of our problem as follows. Define a set  $R = \{2i \mid i \in Q\} \cup \{2i-1 \mid i \in K \setminus Q\}$ . Clearly there are exactly  $k$  elements in  $R$ . Rewrite set  $R = \{[1], \dots, [k]\}$ , where the symbol  $[j]$ , for  $j = 1, \dots, k$ , represents the  $j$ th smallest element of  $R$  (in other words,  $([1], \dots, [k])$  is a permutation of the elements of  $R$  with  $[1] < \dots < [k]$ ).

Use price level  $[j]$  for period  $j$ , for  $j = 1, \dots, T$ . We have

$$\begin{aligned} \sum_{j=1}^T D_{[j]} &= \sum_{j \in Q} D_{2j} + \sum_{j \in K \setminus Q} D_{2j-1} = \sum_{j \in Q} (M^j + a_j) + \sum_{j \in K \setminus Q} M^j \\ &= H + \sum_{j \in K} M^j = I_0 \end{aligned} \quad (\text{A.1})$$

This means that with the chosen price levels for the  $k$  time periods and with the given total inventory  $I_0$ , the demand in each period can be satisfied fully, and that there is no inventory remaining in the end of period  $T$ . Thus the total revenue is equal to

$$\begin{aligned} \sum_{j=1}^T p_{[j]} D_{[j]} &= \sum_{j \in Q} p_{2j} D_{2j} + \sum_{j \in K \setminus Q} p_{2j-1} D_{2j-1} = k + \left( \sum_{j \in Q} D_{2j} + \sum_{j \in K \setminus Q} D_{2j-1} \right) \\ &= k + I_0 = F, \end{aligned}$$

where the last equality is due to (A.1). This means that the constructed solution for the instance of our problem has the total revenue equal to the threshold  $F$ .

*(Only If part)* Given a solution to the instance of our pricing problem with the total revenue greater than or equal to  $F$ , let  $[j]$ ,  $D_{[j]}$ , and  $S_j$  denote the price level used in period  $j$ , the demand in period  $j$ , and the sales volume in period  $j$ , respectively, for  $j = 1, \dots, k$ , where  $[j] \in \{1, \dots, 2k\}$ . Clearly,  $S_j \leq D_{[j]}$ , for  $j = 1, \dots, k$ . There are three possible cases to consider as follows.

Case (i) If  $\sum_{j=1}^k S_j < I_0$ , then the total revenue is

$$\sum_{j=1}^k p_{[j]} S_j = \sum_{j=1}^k (1/D_{[j]} + 1) S_j \leq k + \sum_{j=1}^k S_j < k + I_0 = F$$

This means that the total revenue of the given solution is less than  $F$ . So this case will not happen. This implies that  $\sum_{j=1}^k S_j = I_0$ .

Case (ii) If  $\sum_{j=1}^k S_j = I_0$  and if there is some period  $e \in \{1, \dots, k\}$  such that  $S_e < D_{[e]}$ , then the total revenue is

$$\sum_{j=1}^k p_{[j]} S_j = \sum_{j=1}^k (1/D_{[j]} + 1) S_j < k + \sum_{j=1}^k S_j = k + I_0 = F$$

which again means that the total revenue of the given solution is less than  $F$ . So this case will not happen either. This means that the third case discussed next must happen.

Case (iii) If  $\sum_{j=1}^k S_j = I_0$  and if  $S_j = D_{[j]}$  for each  $j = 1, \dots, k$ , then the demand in each period is fully fulfilled and  $\sum_{j=1}^k D_{[j]} = I_0$ . Next we first prove that for each  $i = 1, \dots, k$ , in the given solution there is exactly one period where either price level  $2i - 1$  or  $2i$  is used.

We show this by induction. We prove by contradiction that there is exactly one period where either price level  $2k - 1$  or  $2k$  is used. Suppose that there is no period where one of these two price levels is used, then the total demand fulfilled is at most  $W = k(A + M^{k-1})$  because the demand at any price level other than  $2k - 1$  and  $2k$  is no more than  $A + M^{k-1}$ . Clearly,  $W < M^k < I_0$ , which contradicts with the fact that  $\sum_{j=1}^k D_{[j]} = I_0$ . Now suppose that there are at least two periods where one of these price levels is used, then the total demand fulfilled is at least  $2M^k > I_0$ , which again contradicts with the fact that  $\sum_{j=1}^k D_{[j]} = I_0$ . This means that there is exactly one period where either price level  $2k - 1$  or  $2k$  is used. Now suppose that for some  $u$  with  $1 \leq u \leq k - 1$ , there is exactly one period where either price level  $2i - 1$  or  $2i$  is used, for each  $i = k, \dots, u + 1$ . We need to prove that the same result holds for  $i = u$ . Given the induction assumption, we can see that the total demand of the periods where a price level  $2u + 1$  or higher is used is at least  $\sum_{j=u+1}^k M^j$  and at most  $\sum_{j=u+1}^k (A + M^j)$ . Suppose that there is no period where one of the two price levels  $2u - 1$  and  $2u$  is used, then the total demand fulfilled is at most  $W = \sum_{j=u+1}^k (A + M^j) + u(A + M^{u-1})$ , where the first summation is an upper bound on the total demand of the periods with a price level  $2u + 1$  or higher, and the second summation is an upper bound on the total demand of the periods with a price level  $2u - 2$  or lower (the number of such periods is  $u$ ). It can be shown that  $W < I_0$ , which contradicts with the fact that  $\sum_{j=1}^k D_{[j]} = I_0$ . Now suppose that there are at least two time periods where one of the two price levels  $2u - 1$  and  $2u$  is used, then the total demand fulfilled is at least  $W = \sum_{j=u+1}^k M^j + 2M^u > I_0$ , which again results in a contradiction. This shows that there is exactly one period where



either price level  $2u - 1$  or  $2u$  is used. Therefore, by induction, we have proved that in the given solution there is exactly one period where either price level  $2i - 1$  or  $2i$  is used, for  $i = 1, \dots, k$ .

Now define set  $U = \{i \in K \mid \text{price level } 2i \text{ is used in the given solution}\}$  and  $V = \{i \in K \mid \text{price level } 2i - 1 \text{ is used in the given solution}\}$ . By the result proved above, we can see that  $U$  and  $V$  are disjoint and  $U \cup V = K$ . Then we have

$$\sum_{j=1}^k D_{[j]} = \sum_{j \in U} (M^j + a_j) + \sum_{j \in V} M^j = \sum_{j \in K} M^j + \sum_{j \in U} a_j \quad (\text{A.2})$$

By the fact that  $\sum_{j=1}^k D_{[j]} = I_0$ , Eq. (A.2) implies that  $\sum_{j \in U} a_j = H$ , which means that the subset  $U$  is a solution to the subset sum instance. This shows the ‘‘Only If part’’. ■

Next we show that the single-period problem is NP-hard. If we view each time period in the single-store problem as a store in the single-period problem, then the proof of Theorem 3, after it is slightly modified, can be used to prove the NP-hardness of the single-period problem. Below we show how this can be done.

**Theorem 4** *The problem even with a single period and deterministic demand is NP-hard.*

**Proof** We prove this by a reduction from the subset sum problem (SS). Most part of the proof of Theorem 3 can be used after we redefine some parameters as follows. Given the instance of SS described in the proof of Theorem 3, we construct an instance for the single-period problem exactly the same as the instance of the single-store problem constructed in the proof of Theorem 3, except that (i) number of time periods  $T = k$  is now replaced by number of stores  $n = k$ ; (ii) demand functions are store-independent and the same notation  $D_i$  represents the demand for price level  $i$  at each store; (iii) for business rule (iv) in the first (and only) time period, set  $u_1 = 2k$  such that any price level can be set at each store. Both the ‘‘If’’ part and ‘‘Only If’’ part can be proved exactly the same way as in the proof of Theorem 3 except that each period  $j$  is now replaced by each store  $j$ , for  $j = 1, \dots, k$ , and  $T$  is replaced by  $n$  ( $n = T = k$ ). ■

## Appendix B

### Formulation with Scenario-Dependent Pricing Decisions

In Section 3.2.2 of this dissertation, we give an approximate MIP formulation  $[\text{MIP}_\tau]$  for the problem that we need to solve at the beginning of each period  $\tau$ . In that formulation, the pricing decisions are modeled as scenario independent. Below in Section B.1 we give another formulation for the same problem where the pricing decisions are formulated as scenario dependent (so this is a precise formulation), and in Section B.2 we compare the computational performance of the two formulations.

#### B.1 Formulation

In this section we give a formulation, where the pricing decisions are formulated as scenario dependent, for the problem that we need to solve at the beginning of each period  $\tau$ . We use the same notation and decision variables as in Section 3.2 except that since we now allow pricing decisions to be scenario dependent, we denote the corresponding binary variables as scenario  $\omega$  dependent, i.e., we use  $X_{rjt}^\omega, Y_{qgt}^\omega, H_{rt}^\omega$  to replace  $X_{rjt}, Y_{qgt}, H_{rt}$ , respectively. We denote this formulation as  $[\text{MIP}_\tau^s]$ , as opposed to  $[\text{MIP}_\tau]$  introduced in Section 3.2.

$$[\text{MIP}_\tau^s] \quad \max \quad \sum_{\omega \in \Omega} \left( P_\omega \sum_{r=1}^n \sum_{j=j_{0r}}^m \sum_{t=\tau}^T p_j S_{rjt}^\omega \right) + s \sum_{\omega \in \Omega} \left( P_\omega \left( I_0' - \sum_{r=1}^n \sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^\omega \right) \right)$$

Subject to:

$$\sum_{j=j_{0r}}^m X_{rjt}^\omega = 1, \quad \forall r \in N, t \in \{\tau, \dots, T\}, \omega \in \Omega$$

$$\sum_{j=j_{0r}}^h X_{rj(t+1)}^\omega \leq \sum_{j=j_{0r}}^h X_{rjt}^\omega, \quad \forall r \in N, h \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\}, \omega \in \Omega$$

$$X_{rjt}^\omega \leq X_{rj(t+1)}^\omega + \sum_{l=u_j}^{v_j} X_{rl(t+1)}^\omega, \quad \forall r \in N, j \in \{j_{0r}, \dots, m-1\},$$

$$\begin{aligned}
& t \in \{\tau, \dots, T-1\}, \omega \in \Omega \\
\sum_{l=j_{0r}}^j X_{rlt}^\omega & \leq \sum_{l=j_{0r}}^{v_j} X_{rl(t+1)}^\omega, \quad \forall r \in N, j \in \{j_{0r}, \dots, m-1\}, \\
& t \in \{\tau, \dots, T-1\}, \omega \in \Omega \\
H_{rjt}^\omega & \geq X_{rjt}^\omega, \quad \forall r \in N, j \in \{j_{0r}+1, \dots, m\}, t \in \{\tau, \dots, T\}, \omega \in \Omega \\
\sum_{j=j_{0r}+1}^m H_{rj}^\omega & \leq R_{0r}, \quad \forall r \in N, \omega \in \Omega \\
\sum_{g=1}^G Y_{qgt}^\omega & = 1, \quad \forall q \in \{1, \dots, Q\}, t \in \{\tau, \dots, T\}, \omega \in \Omega \\
Y_{qgt}^\omega & \leq \sum_{j \in E_g} X_{rjt}^\omega, \quad \forall q \in \{1, \dots, Q\}, r \in C_q, g \in \{1, \dots, G\}, \\
& t \in \{\tau, \dots, T\}, \omega \in \Omega \\
S_{rjt}^\omega & \leq D_{rjt}^\omega X_{rjt}^\omega, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T\}, \omega \in \Omega \\
\sum_{j=j_{0r}}^m \sum_{t=\tau}^T S_{rjt}^\omega & \leq I_r, \quad \forall r \in N, \omega \in \Omega \\
S_{rjt}^{\omega_i^\alpha} & = S_{rjt}^{\omega_{i+1}^\alpha}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\}, \\
& i = 1, \dots, |\Gamma_t^\alpha| - 1, \alpha \in \mathcal{A}(t) \\
X_{rj(t+1)}^{\omega_i^\alpha} & = X_{rj(t+1)}^{\omega_{i+1}^\alpha}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, t \in \{\tau, \dots, T-1\}, \\
& i = 1, \dots, |\Gamma_t^\alpha| - 1, \alpha \in \mathcal{A}(t) \tag{B.1} \\
X_{rj\tau}^{\omega'} & = X_{rj\tau}^{\omega''}, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, \omega', \omega'' \in \Omega \text{ and } \omega' \neq \omega'' \tag{B.2} \\
\sum_{r=1}^n I_r^\omega & \leq I_0', \quad \forall \omega \in \Omega \\
X_{rjt}^\omega, H_{rj}^\omega, Y_{qgt}^\omega & \in \{0, 1\}, I_r^\omega, S_{rjt}^\omega \geq 0, \quad \forall r \in N, j \in \{j_{0r}, \dots, m\}, q \in \{1, \dots, Q\}, \\
& g \in \{1, \dots, G\}, t \in \{\tau, \dots, T\}, \omega \in \Omega
\end{aligned}$$

There is one more constraint that one has to add if  $\tau = 1$ , to formulate business rule (i), as follows.

$$I_r^\omega \geq I_r^{\min}, \quad \forall r \in N, \omega \in \Omega$$

In the above formulation, constraints (B.1) and (B.2) are the non-anticipativity constraints for  $X$  variables. Since pricing decisions for each period are set before a particular demand scenario for this period is realized, if two scenarios share a

common arc in one period, the same prices should be used in the following period. All other constraints are similar to those in  $[\text{MIP}_\tau]$  given in Section 3.2.2.

We note that this formulation is much larger than  $[\text{MIP}_\tau]$  given in Section 3.2.2. The number of integer variables in this formulation is  $|\Omega|$  times that in  $[\text{MIP}_\tau]$  whereas the number of continuous variables remains the same as in  $[\text{MIP}_\tau]$ . The number of constraints is slightly less than  $|\Omega|$  times that in  $[\text{MIP}_\tau]$ .

## B.2 Computational Results

The Lagrangian Relaxation Algorithm (LRA) described in Section 3.3.2 (for solving  $[\text{MIP}_\tau]$  given in Section 3.2.2) can also be applied to solving  $[\text{MIP}_\tau^s]$  given in Section B.1 above. The overall rolling horizon based approach described in Section 3.3.1 can also be easily modified accordingly by replacing  $[\text{MIP}_\tau]$  by  $[\text{MIP}_\tau^s]$  and using the corresponding LRA algorithm. In this section, we conduct computational experiments to compare (i) the performance of LRA algorithm applied to the formulation  $[\text{MIP}_1]$  and  $[\text{MIP}_1^s]$ , respectively, for the problem at the beginning of the first period (which is the largest in scale among all the problems that we need to solve); (ii) the performance of the overall rolling horizon approach based on the formulation  $[\text{MIP}_\tau]$  and  $[\text{MIP}_\tau^s]$ , respectively, for the overall integrated inventory allocation and markdown pricing problem.

Unless otherwise specified, all test problem instances are created similarly as in Section 3.5. We first compare the performance of LRA for solving  $[\text{MIP}_1]$  and  $[\text{MIP}_1^s]$ . Table B.1 summarizes the computational results for problems with 50 stores and problems with 100 stores, respectively. The solution quality of LRA is defined as the ratio of the objective value obtained by LRA for  $[\text{MIP}_1]$  or  $[\text{MIP}_1^s]$  over the corresponding Lagrangian upper bound, represented as a percentage in the table. For each parameter configuration, we test 10 randomly generated problem instances. Both the median and the worst case performance are reported in the table.

As clearly indicated in Table B.1, the solutions obtained by LRA for both formulations (i.e.,  $[\text{MIP}_1]$  and  $[\text{MIP}_1^s]$ ) are near optimal. The computational time for

$[\text{MIP}_1]$  is consistent across all problem instances and is no more than 30 minutes even in the worst case for problems with 100 stores. On the other hand, the computational time for  $[\text{MIP}_1^s]$  is many times that for  $[\text{MIP}_1]$  and has much larger variation. In some cases, LRA may take 20 to 30 hours to solve  $[\text{MIP}_1^s]$  with 100 stores. This may cause implementation difficulty for practically sized problems. This is one of the reasons that we use formulation  $[\text{MIP}_\tau]$  instead of  $[\text{MIP}_\tau^s]$  in our solution approach (given in Section 3.3).

We next compare the solution quality of the overall rolling horizon approach based on  $[\text{MIP}_\tau]$  and  $[\text{MIP}_\tau^s]$ , respectively. To make the computational experiment manageable, we use test problem instances with 10 stores only because there are a large number of test problems involved, and for each test problem it requires a large computation time to solve the formulation  $[\text{MIP}_\tau^s]$  many times across the planning horizon. For each parameter configuration, we test the average performance over 1000 randomly generated problem instances. We focus on the basic case E00 and use approach S2 (81 scenarios). Table B.2 shows the relative revenue one can achieve by using  $[\text{MIP}_\tau^s]$  compared to that by using  $[\text{MIP}_\tau]$ . This table shows that compared to the formulation  $[\text{MIP}_\tau]$  that we use in our approach, using the more complex formulation  $[\text{MIP}_\tau^s]$  may increase the revenue by up to 2.3%, 1.6%, and 0.7% for problems with low, medium and high initial inventory level, respectively. However, as we have seen in Table B.1, the approach with  $[\text{MIP}_\tau^s]$  is much more time consuming than that with  $[\text{MIP}_\tau]$ . In fact, for a large percentage of test problems with 50 stores, the approach with  $[\text{MIP}_\tau^s]$  is not capable of generating a solution in a day (24 hours). In contrast, the approach with  $[\text{MIP}_\tau]$  is capable of solving every test problem with 50 stores in less than an hour. Therefore, we use the formulation  $[\text{MIP}_\tau]$  in our approach.

Table B.1: Comparison of LRA for  $[\text{MIP}_1]$  and  $[\text{MIP}_1^s]$

		Solution Quality of LRA (LRA/UB, %)				Computational Time (s)				
$n$	$\beta_r$	$I_0$	$[\text{MIP}_1]$		$[\text{MIP}_1^s]$		$[\text{MIP}_1]$		$[\text{MIP}_1^s]$	
			Median	Worst	Median	Worst	Median	Worst	Median	Worst
50	1.0-2.0	Low	98.6	97.5	99.5	98.9	281	310	1173	2303
		Medium	98.9	97.5	98.9	98.7	311	419	2452	12691
		High	96.0	95.5	98.9	98.6	356	471	4597	45730
	1.0-3.0	Low	98.2	97.7	99.1	98.4	283	318	1086	9329
		Medium	96.1	95.2	98.8	98.5	338	393	1789	54177
		High	97.4	96.8	97.9	95.3	332	366	1671	26426
100	1.0-2.0	Low	98.9	97.8	99.6	99.5	572	702	2524	10510
		Medium	99.0	97.8	99.0	98.8	730	1254	9660	107598
		High	96.1	95.8	99.0	98.7	749	895	29095	68132
	1.0-3.0	Low	98.0	97.6	99.2	98.6	565	654	2819	18310
		Medium	96.2	96.0	98.8	97.9	692	882	13476	97508
		High	97.2	96.6	98.5	95.9	714	818	10349	74173

Table B.2: Comparison of the Overall Approach with  $[\text{MIP}_\tau]$  v.s. the Overall Approach with  $[\text{MIP}_\tau^s]$

$\beta_r$	$I_0$	Relative Revenue ( $[\text{MIP}_\tau^s]$ -based/ $[\text{MIP}_\tau]$ -based, %)
1.0-2.0	Low	102.0
	Medium	101.6
	High	100.6
1.0-3.0	Low	102.3
	Medium	101.6
	High	100.7

## Appendix C

### Counter Example

As we have discussed in Section 4.2, intuitively, one would expect the worst-case demand to be always on the left or/and lower boundary of the demand uncertainty space or one of the corner points. If this is true, we can eliminate a large portion of the demand uncertainty space and hence significantly reduce the computational time for the proposed DP. We now show, in the following example, that this is not the case. In this example, there are two time periods ( $T = 2$ ), one allowable price for product 1 ( $p_1^1 = 50$ ) and two allowable prices for product 2 ( $p_2^1 = 50, p_2^2 = 45$ ). The initial inventory levels  $I_1 = I_2 = 10$ . The allowed cumulative demand deviations  $B_1 = B_2 = 1$ . We consider a special case in which  $\alpha_t^{l_1 l_2} = 0, \beta_t^{l_1 l_2} = 0, \forall t, l_1, l_2$ . We will show that if the decision maker chooses price pair (50,50) in the first period, the adversary will choose the middle point of the corresponding demand uncertainty space to achieve the lowest total revenue. The expected demand for price pair (50,50) in the first period is (6,6) and the demand intervals are as follows, i.e.,  $D_{11}^{11} \in [5, 7], D_{21}^{11} \in [5, 7], D_{11}^{11} + D_{21}^{11} \in [11, 13]$ . The expected demand for price pair (50,50) in the second period is (6,4) and the demand intervals are  $D_{12}^{11} \in [5, 7], D_{22}^{11} \in [2, 6], D_{12}^{11} + D_{22}^{11} \in [8, 12]$ . Correspondingly, the expected demand for price pair (50,45) in the second period is (4,6) and the demand intervals are  $D_{12}^{12} \in [2, 6], D_{22}^{12} \in [5, 7], D_{12}^{12} + D_{22}^{12} \in [8, 12]$ .

Corresponding to price pair (50,50) selected in the first period, the demand uncertainty space is  $\Omega_{1,0}^{11}$  (note, at the beginning of time period 1,  $d$  must be 0), which is shown in Figure C.1. Demand uncertainty spaces for period 2 corresponding to two price pairs (i.e., (50,50),(50,45)) and three cumulative demand deviations (i.e.,  $d = 1, 0, -1$ ) are shown in Figures C.2 to C.7.

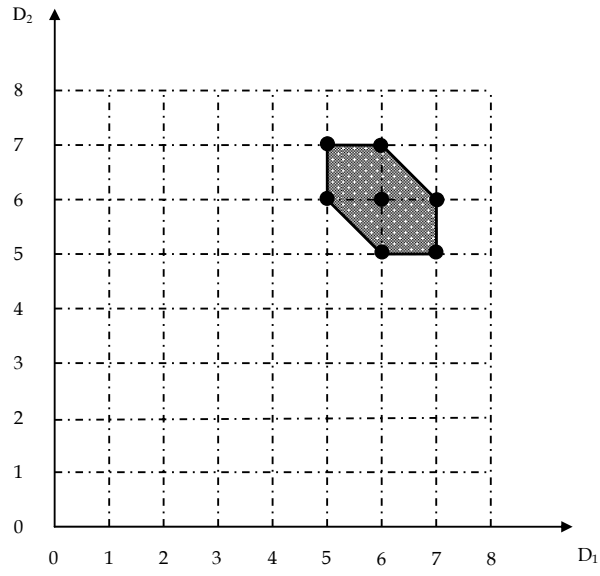


Figure C.1:  $\Omega_{1,0}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 5 \leq D_2 \leq 7, 11 \leq D_1 + D_2 \leq 13\}$

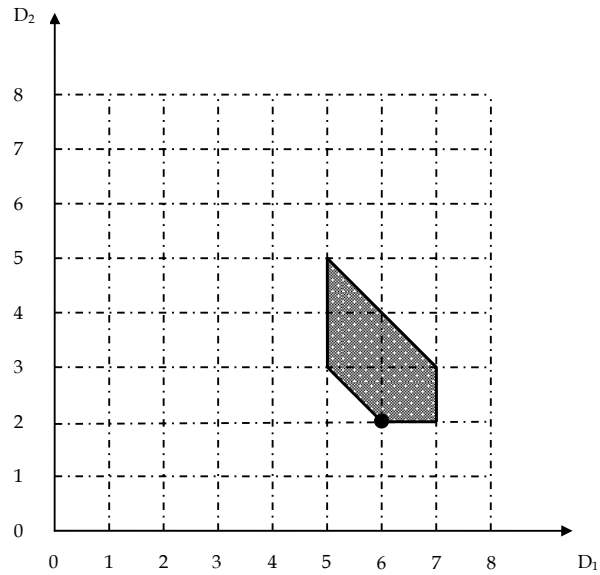


Figure C.2:  $\Omega_{2,1}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 2 \leq D_2 \leq 6, 8 \leq D_1 + D_2 \leq 10\}$



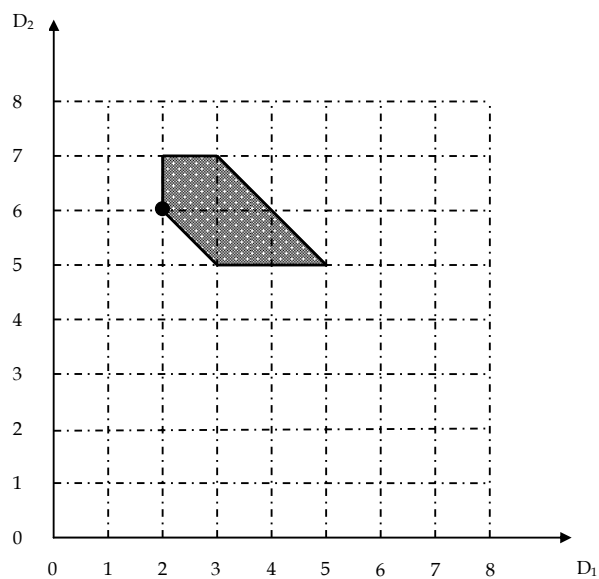


Figure C.3:  $\Omega_{2,1}^{12} = \{(D_1, D_2) \mid 2 \leq D_1 \leq 6, 5 \leq D_2 \leq 7, 8 \leq D_1 + D_2 \leq 10\}$

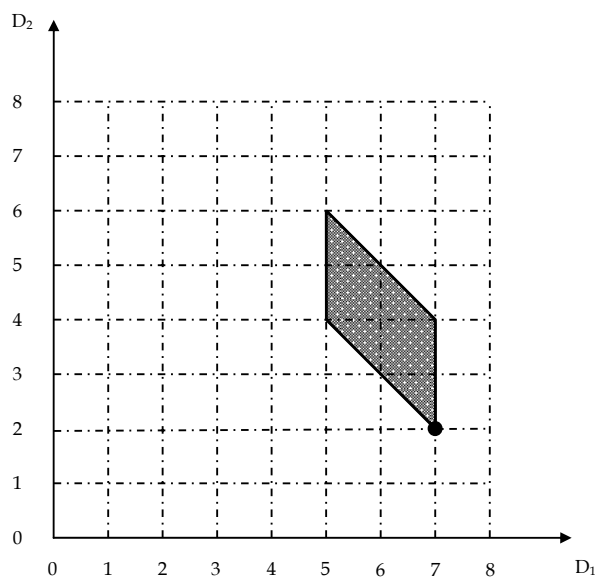


Figure C.4:  $\Omega_{2,0}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 2 \leq D_2 \leq 6, 9 \leq D_1 + D_2 \leq 11\}$

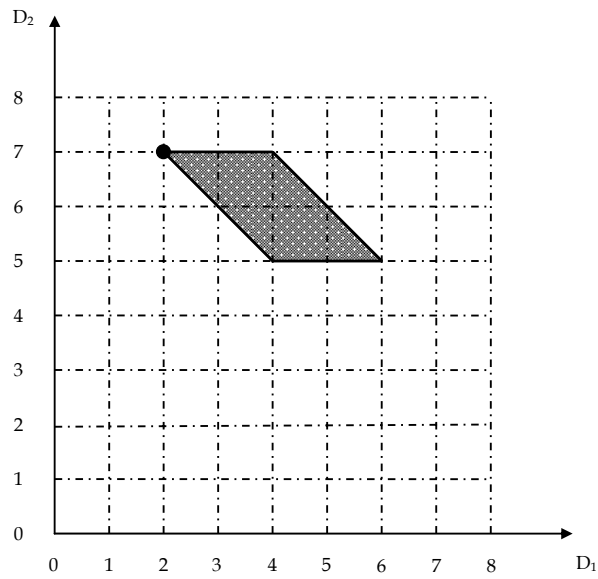


Figure C.5:  $\Omega_{2,0}^{12} = \{(D_1, D_2) \mid 2 \leq D_1 \leq 6, 5 \leq D_2 \leq 7, 9 \leq D_1 + D_2 \leq 11\}$

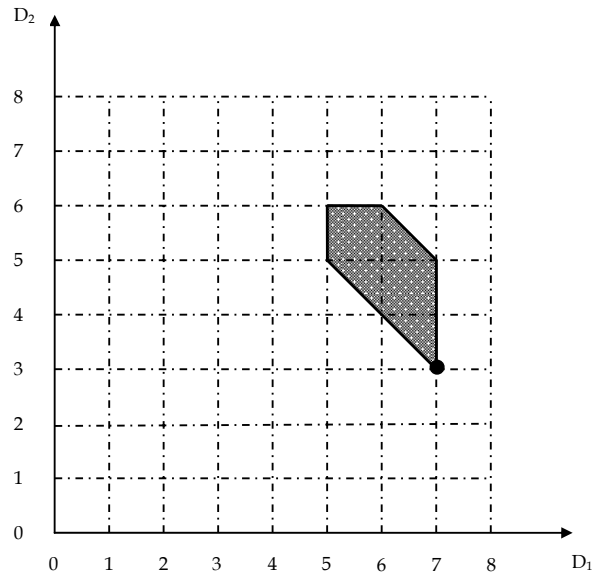


Figure C.6:  $\Omega_{2,-1}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 2 \leq D_2 \leq 6, 10 \leq D_1 + D_2 \leq 12\}$

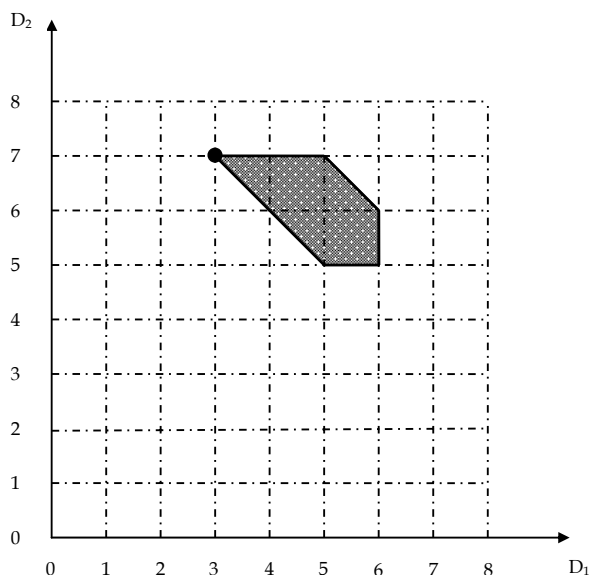


Figure C.7:  $\Omega_{2,-1}^{12} = \{(D_1, D_2) \mid 2 \leq D_1 \leq 6, 5 \leq D_2 \leq 7, 10 \leq D_1 + D_2 \leq 12\}$

Given that the decision maker chooses price pair (50, 50) in the first period, Table C.1 demonstrates why choosing demand (6, 6) will lead to the lowest total revenue. For example, if the adversary chooses demand (6, 6) in the first period, it leads to demand deviation of 0, first period revenue of 600 and remaining inventory of 4 for both products. In the second period, if the decision maker chooses price pair (50, 50), which results in the demand uncertainty space  $\Omega_{2,0}^{11}$ , then the adversary will choose demand realization (7, 2) to minimize the second period revenue (300 in this case). Correspondingly, if the decision maker chooses price pair (50, 45) in the second period, which results in the demand uncertainty space  $\Omega_{2,0}^{12}$ , then the adversary will choose demand realization (2, 7) to minimize the second period revenue (280 in this case). The decision maker knows exactly what the adversary will choose for each possible price pair. After comparing these two possible price pairs, the decision maker will choose (50, 50) since  $300 > 280$ , which results in a total revenue of 900 for two periods (marked with an asterisk in the table). For each possible demand in the first period, we can do the same analysis and compute the corresponding total revenue (marked with an asterisk in the table). Since 900 is the smallest among all these total revenue numbers, it means that demand (6, 6) is

the worst-case demand if the decision maker chooses price pair (50, 50) in the first period.

Table C.1: Worst-case Demand Analysis

First Period				Second Period					Total Revenue
Demand	Demand Deviation	Revenue	Remaining Inventory	Prices	Uncertainty Space	Worst-case Demand	Sales	Revenue	
(6,6)	0	600	(4,4)	(50,50)	$\Omega_{2,0}^{11}$	(7,2)	(4,2)	300	900*
				(50,45)	$\Omega_{2,0}^{12}$	(2,7)	(2,4)	280	880
(6,7)	1	650	(4,3)	(50,50)	$\Omega_{2,1}^{11}$	(6,2)	(4,2)	300	950*
				(50,45)	$\Omega_{2,1}^{12}$	(2,6)	(2,3)	235	885
(6,5)	-1	550	(4,5)	(50,50)	$\Omega_{2,-1}^{11}$	(7,3)	(4,3)	350	900
				(50,45)	$\Omega_{2,-1}^{12}$	(3,7)	(3,5)	375	925*
(5,6)	-1	550	(5,4)	(50,50)	$\Omega_{2,-1}^{11}$	(7,3)	(5,3)	400	950*
				(50,45)	$\Omega_{2,-1}^{12}$	(3,7)	(3,4)	330	880
(7,6)	1	650	(3,4)	(50,50)	$\Omega_{2,1}^{11}$	(6,2)	(3,2)	250	900
				(50,45)	$\Omega_{2,1}^{12}$	(2,6)	(2,4)	280	930*
(7,5)	0	600	(3,5)	(50,50)	$\Omega_{2,0}^{11}$	(7,2)	(3,2)	250	850
				(50,45)	$\Omega_{2,0}^{12}$	(2,7)	(2,5)	325	925*
(5,7)	0	600	(5,3)	(50,50)	$\Omega_{2,0}^{11}$	(7,2)	(5,2)	350	950*
				(50,45)	$\Omega_{2,0}^{12}$	(2,7)	(2,3)	235	835

## Appendix D

### Proofs for Lemmas and Theorems

In this section, we prove all the lemmas and theorems given in Sections 4.3 and 4.4. We prove each result contained in each lemma by backward induction. We show that if the result holds for time period  $t + 1$ , it also holds for time period  $t$ . The result for time period  $T$  can be proved similarly as a special case, and hence is not proved here.

**Lemma 1** *In any period  $t$ , for any  $\delta_1, \delta_2, \delta \geq 0$ , the following inequalities hold as long as the value of each state variable involved is within its domain.*

- (i)  $V_t(i_1 + \delta_1, 0, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, 0, l_1, l_2, r_1, r_2, d)$
- (ii)  $V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1$
- (iii)  $V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_2^1$
- (iv)  $V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1$
- (v)  $V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1(\delta_1 + i_2) - p_2^1 i_2$
- (vi)  $V_t(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1 - p_2^1 \delta_1$
- (vii)  $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d - \delta) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta - p_2^1 \delta$

**Proof** We first prove result (i) by backward induction. We show that if result (i) holds for time period  $t + 1$ , it also holds for time period  $t$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(i_1, 0, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. To simplify notation, in all the proofs hereinafter, we denote  $p_1^* = p_1^{l'_1}, p_2^* = p_2^{l'_2}$  and the expected demand corresponding to price levels  $l'_1, l'_2$  in period  $t$  as  $\bar{D}_1, \bar{D}_2$ . We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 + \delta_1, 0, l_1, l_2, r_1, r_2, d)$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(D_1, D_2), (D_1^*, D_2^*) \in \Omega_{t,d}^{l'_1, l'_2}$ . We also denote

$$i_1''' = i_1 + \delta_1 - \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} D_2 \rfloor), \quad i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} D_2 \rfloor),$$

$$i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} D_2^* \rfloor), \quad d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2), \quad \text{and } d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2).$$

$$\begin{aligned} & V_t(i_1 + \delta_1, 0, l_1, l_2, r_1, r_2, d) \\ & \geq p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} D_2 \rfloor) + V_{t+1}(i_1''', 0, l_1', l_2', r_1', r_2', d'') \\ & \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} D_2 \rfloor) + V_{t+1}(i_1'', 0, l_1', l_2', r_1', r_2', d'') \\ & \quad (\text{by induction and the fact that } i_1''' \geq i_1'') \\ & \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} D_2^* \rfloor) + V_{t+1}(i_1', 0, l_1', l_2', r_1', r_2', d') \quad (\text{by definition of } D_1^*, D_2^*) \\ & = V_t(i_1, 0, l_1, l_2, r_1, r_2, d) \end{aligned}$$

Next we prove result (ii) by backward induction. We show in the following that if result (ii) holds for time period  $t+1$ , it also holds for time period  $t$ . In the following proof, we denote  $l_1', l_2'$  as the optimal price levels for state  $(i_1, 0, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d)$  if price levels  $l_1', l_2'$  are used in that period. Note that  $(D_1, D_2), (D_1^*, D_2^*) \in \Omega_{t,d}^{l_1' l_2'}$ . We also denote  $i_1''' = i_1 + \delta_1 - \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2''' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor)$ ,  $i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} D_2 \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} D_2^* \rfloor)$ ,  $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$ , and  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$ .

Case 1:  $i_2 \leq D_2$

$$\begin{aligned} & V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\ & \geq p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\ & \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor) \\ & \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\ & \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(0, D_2)) \rfloor) - p_1^* (\alpha_t^{l_1' l_2'} i_2 + 1) \\ & \quad + p_2^* \min(0, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1, D_1)) \rfloor) + p_2^* i_2 \\ & \quad + V_{t+1}(i_1'', 0, l_1', l_2', r_1', r_2', d'') \quad (\text{by result (i) and the fact that } i_1''' \geq i_1'', i_2''' = 0) \\ & \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(0, D_2)) \rfloor) \end{aligned}$$

$$\begin{aligned}
& + p_2^* \min(0, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i''_1, 0, l'_1, l'_2, r'_1, r'_2, d'') - p_1^* \text{ (by assumption that } \alpha_t^{l'_1 l'_2} p_1^{l'_1} \leq p_2^{l'_2} \text{)} \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} (D_2^* - \min(0, D_2^*)) \rfloor) \\
& + p_2^* \min(0, D_2^* + \lfloor \beta_t^{l'_1 l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i'_1, 0, l'_1, l'_2, r'_1, r'_2, d') - p_1^1 \text{ (by definition of } D_1^*, D_2^* \text{)} \\
= & V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1
\end{aligned}$$

Case 2:  $i_2 > D_2$

$$\begin{aligned}
& V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
\geq & p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor) \\
& + V_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, d'') \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(0, D_2)) \rfloor) - p_1^* \alpha_t^{l'_1 l'_2} D_2 \\
& + p_2^* \min(0, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) + p_2^* D_2 \\
& + V_{t+1}(i''_1, 0, l'_1, l'_2, r'_1, r'_2, d'') - p_1^1 \text{ (by induction and the fact that } i'''_1 \geq i''_1 \text{)} \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(0, D_2)) \rfloor) \\
& + p_2^* \min(0, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i''_1, 0, l'_1, l'_2, r'_1, r'_2, d'') - p_1^1 \text{ (by assumption that } \alpha_t^{l'_1 l'_2} p_1^{l'_1} \leq p_2^{l'_2} \text{)} \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} (D_2^* - \min(0, D_2^*)) \rfloor) \\
& + p_2^* \min(0, D_2^* + \lfloor \beta_t^{l'_1 l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i'_1, 0, l'_1, l'_2, r'_1, r'_2, d') - p_1^1 \text{ (by definition of } D_1^*, D_2^* \text{)} \\
= & V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1
\end{aligned}$$

We then prove result (iii) by backward induction. We prove that result (iii) holds for time period  $t$  if it holds for time period  $t + 1$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(i_1, i_2, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d)$  if price levels  $l'_1, l'_2$

are used in that period. Note that  $(D_1, D_2), (D_1^*, D_2^*) \in \Omega_{t,d}^{l_1 l_2}$ . We also denote  $i_1''' = i_1 + \delta_1 - \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1 l_2} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2''' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor)$ ,  $i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1 l_2} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2} (D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l_1 l_2} (D_2^* - \min(i_2, D_2^*)) \rfloor)$ ,  $i_2' = i_2 - \min(i_2, D_2^* + \lfloor \beta_t^{l_1 l_2} (D_1^* - \min(i_1, D_1^*)) \rfloor)$ ,  $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$ , and  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$ .

Case 1:  $i_2 \leq D_2$

$$\begin{aligned}
& V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
& \geq p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1 l_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1 l_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') \\
& \quad \text{(by result (i) and the fact that } i_1''' \geq i_1'', i_2''' = i_2'' = 0) \\
& \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1 l_2} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1 l_2} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& \quad + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') \\
& = V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)
\end{aligned}$$

Case 2:  $i_1 + \delta_1 < D_1, i_2 > D_2$

$$\begin{aligned}
& V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
& \geq p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1 l_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1 l_2} (D_2 - \min(i_2, D_2)) \rfloor) + p_1^* \delta_1 \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2} (D_1 - \min(i_1, D_1)) \rfloor) - p_2^* (\beta_t^{l_1 l_2} \delta_1 + 1) \\
& \quad + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'')
\end{aligned}$$



$$\begin{aligned}
& \text{(by result (i) and the fact that } i_1''' = i_1'' = 0, i_2''' \geq i_2'') \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_2^* \text{ (by assumption that } \beta_t^{l_1' l_2'} p_2^{l_2'} \leq p_1^{l_1'}) \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_2^1 \text{ (by definition of } D_1^*, D_2^*) \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_2^1
\end{aligned}$$

Case 3:  $i_1 \leq D_1 \leq i_1 + \delta_1, i_2 > D_2$

$$\begin{aligned}
& V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
\geq & p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) + p_1^* (D_1 - i_1) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1, D_1)) \rfloor) - p_2^* \beta_t^{l_1' l_2'} (D_1 - i_1) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_2^1 \\
& \text{(by result (ii) and the fact that } i_1'' = 0, i_2'' \geq i_2'') \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_2^1 \text{ (by assumption that } \beta_t^{l_1' l_2'} p_2^{l_2'} \leq p_1^{l_1'}) \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_2^1 \text{ (by definition of } D_1^*, D_2^*) \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_2^1
\end{aligned}$$

Case 4:  $i_1 > D_1, i_2 > D_2$

$$V_t(i_1 + \delta_1, i_2, l_1, l_2, r_1, r_2, d)$$

$$\begin{aligned}
&\geq p_1^* \min(i_1 + \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
&\quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1 + \delta_1, D_1)) \rfloor) \\
&\quad + V_{t+1}(i_1''', i_2''', l'_1, l'_2, r'_1, r'_2, d'') \\
&\geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
&\quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
&\quad + V_{t+1}(i_1'', i_2'', l'_1, l'_2, r'_1, r'_2, d'') - p_2^1 \\
&\quad \text{(by induction and the fact that } i_1''' \geq i_1'', i_2''' = i_2'') \\
&\geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
&\quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l'_1 l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
&\quad + V_{t+1}(i_1', i_2', l'_1, l'_2, r'_1, r'_2, d') - p_2^1 \text{ (by definition of } D_1^*, D_2^*) \\
&= V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_2^1
\end{aligned}$$

We next prove result (iv) by backward induction. We show that if result (iv) holds for time period  $t + 1$ , it also holds for time period  $t$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(i_1, 0, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d)$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(D_1, D_2), (D_1^*, D_2^*) \in \Omega_{t,d}^{l'_1 l'_2}$ . We also denote  $i_1''' = i_1 - \delta_1 - \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor)$ ,  $i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} D_2^* \rfloor)$ ,  $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$ ,  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$  and  $X = \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor)$ . We consider the following two cases.

Case 1:  $i_1 - \delta_1 \leq D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor$ .

$$\begin{aligned}
&V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \\
&\geq p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor) + V_{t+1}(i_1''', 0, l'_1, l'_2, r'_1, r'_2, d'') \\
&\geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor) - p_1^*(X - i_1 + \delta_1) \\
&\quad + V_{t+1}(i_1'', 0, l'_1, l'_2, r'_1, r'_2, d'') - p_1^1(i_1 - X) \\
&\quad \text{(by induction and the fact that } i_1''' = 0, i_1'' = i_1 - X) \\
&\geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} D_2^* \rfloor) + V_{t+1}(i_1', 0, l'_1, l'_2, r'_1, r'_2, d') - p_1^1 \delta_1
\end{aligned}$$

$$= V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1$$

Case 2:  $i_1 - \delta_1 > D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor$

$$\begin{aligned} & V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \\ & \geq p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor) + V_{t+1}(i_1''', 0, l'_1, l'_2, r'_1, r'_2, d'') \\ & \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} D_2 \rfloor) + V_{t+1}(i_1'', 0, l'_1, l'_2, r'_1, r'_2, d'') - p_1^1 \delta_1 \\ & \quad (\text{by induction and the fact that } i_1''' = i_1'' - \delta_1) \\ & \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} D_2^* \rfloor) + V_{t+1}(i_1', 0, l'_1, l'_2, r'_1, r'_2, d') - p_1^1 \delta_1 \\ & = V_t(i_1, 0, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1 \end{aligned}$$

Next we prove result (v) by backward induction. We show that if result (v) holds for time period  $t + 1$ , it also holds for time period  $t$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(i_1, i_2, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d)$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(D_1, D_2), (D_1^*, D_2^*) \in \Omega_{t,d}^{l'_1 l'_2}$ . We also denote  $i_1''' = i_1 - \delta_1 - \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(0, D_2)) \rfloor)$ ,  $i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} (D_2^* - \min(i_2, D_2^*)) \rfloor)$ ,  $i_2' = i_2 - \min(i_2, D_2^* + \lfloor \beta_t^{l'_1 l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor)$ ,  $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$ , and  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$ . We also denote  $X_1 = \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(0, D_2)) \rfloor)$ ,  $X_1' = \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $X_2 = \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor)$ . We consider two cases in the following.

Case 1:  $X_1 \leq X_1'$ , note that  $X_1' - X_1 \leq \delta_1$

$$\begin{aligned} & V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \\ & \geq p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(0, D_2)) \rfloor) \\ & \quad + p_2^* \min(0, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1 - \delta_1, D_1)) \rfloor) \\ & \quad + V_{t+1}(i_1''', 0, l'_1, l'_2, r'_1, r'_2, d'') \\ & \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) - p_1^*(X_1' - X_1) \end{aligned}$$

$$\begin{aligned}
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2}(D_1 - \min(i_1, D_1)) \rfloor) - p_2^* X'_2 \\
& + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') - p_1^1(\delta_1 - (X'_1 - X_1) + i_2 - X'_2) - p_2^1(i_2 - X'_2) \\
& \text{(by induction and the fact that } i'''_1 = i''_1 - \delta_1 + (X'_1 - X_1), i''_2 = i_2 - X'_2) \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l'_1 l'_2}(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') - p_1^1(\delta_1 + i_2) - p_2^1 i_2 \text{ (by definition of } D_1^*, D_2^*) \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1(\delta_1 + i_2) - p_2^1 i_2
\end{aligned}$$

Case 2:  $X_1 > X'_1$ , note that  $X_1 - X'_1 \leq X'_2$ .

$$\begin{aligned}
& V_t(i_1 - \delta_1, 0, l_1, l_2, r_1, r_2, d) \\
\geq & p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2}(D_2 - \min(0, D_2)) \rfloor) \\
& + p_2^* \min(0, D_2 + \lfloor \beta_t^{l'_1 l'_2}(D_1 - \min(i_1 - \delta_1, D_1)) \rfloor) \\
& + V_{t+1}(i'''_1, 0, l'_1, l'_2, r'_1, r'_2, d'') \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2}(D_2 - \min(i_2, D_2)) \rfloor) + p_1^*(X_1 - X'_1) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2}(D_1 - \min(i_1, D_1)) \rfloor) - p_2^* X'_2 \\
& + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') - p_1^1(\delta_1 + (X_1 - X'_1) + i_2 - X'_2) - p_2^1(i_2 - X'_2) \\
& \text{(by induction and the fact that } i'''_1 = i''_1 - \delta_1 - (X_1 - X'_1), i''_2 = i_2 - X'_2) \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l'_1 l'_2}(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') - p_1^1(\delta_1 + i_2) - p_2^1 i_2 \text{ (by definition of } D_1^*, D_2^*) \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1(\delta_1 + i_2) - p_2^1 i_2
\end{aligned}$$

We now prove result (vi) by backward induction. We show that result (vi) holds for time period  $t$  if it holds for time period  $t + 1$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(i_1, i_2, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d)$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(D_1, D_2), (D_1^*, D_2^*) \in \Omega_{t,d}^{l'_1 l'_2}$ . We also denote  $i'''_1 = i_1 - \delta_1 - \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i'''_2 = i_2 - \min(i_2, D_2 +$

$\lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1 - \delta_1, D_1)) \rfloor$ ,  $i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor)$ ,  $i_2' = i_2 - \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor)$ ,  $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$ , and  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$ . We consider the following four cases.  
 Case 1:  $i_2 \leq D_2$ . To simplify notation, we denote  $X_1 = \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $X_1' = \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ . Clearly,  $X_1 \leq X_1' \leq X_1 + \delta_1$ .

$$\begin{aligned}
 & V_t(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
 \geq & p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
 & + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1 - \delta_1, D_1)) \rfloor) \\
 & + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
 \geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) - p_1^*(X_1' - X_1) \\
 & + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
 & + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_1^1(\delta_1 + X_1 - X_1') \\
 & \text{(by result (iv) and the fact that } i_1''' = i_1'' - \delta_1 - X_1 + X_1', i_2''' = i_2'' = 0) \\
 \geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
 & + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
 & + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_1^1 \delta_1 \\
 = & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1
 \end{aligned}$$

Case 2:  $i_1 < D_1, i_2 > D_2$

$$\begin{aligned}
 & V_t(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
 \geq & p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
 & + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1 - \delta_1, D_1)) \rfloor) \\
 & + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
 \geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) - p_1^* \delta_1 \\
 & + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
 & + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_2^1 \delta_1
 \end{aligned}$$

$$\begin{aligned}
& \text{(by result (iv) and the fact that } i_1''' = i_1'' = 0, i_2'' - \delta_1 \leq i_2''' \leq i_2'') \\
& \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& \quad + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_1^1 \delta_1 - p_2^1 \delta_1 \text{ (by definition of } D_1^*, D_2^*) \\
& = V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1 - p_2^1 \delta_1
\end{aligned}$$

Case 3:  $i_1 - \delta_1 \leq D_1 \leq i_1, i_2 > D_2$

$$\begin{aligned}
& V_t(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
& \geq p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1 - \delta_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) - p_1^* (D_1 - i_1 + \delta_1) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_1^1 (i_1 - D_1) - p_2^1 (i_1 - D_1 + D_1 - i_1 + \delta_1) \\
& \text{(by result (v) and the fact that } i_1''' = 0, i_1'' = i_1 - D_1, \\
& \quad i_2''' \leq i_2'' \leq i_2'' + (D_1 - i_1 + \delta_1)) \\
& \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& \quad + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_1^1 \delta_1 - p_2^1 \delta_1 \text{ (by definition of } D_1^*, D_2^*) \\
& = V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1 - p_2^1 \delta_1
\end{aligned}$$

Case 4:  $i_1 - \delta_1 > D_1, i_2 > D_2$

$$\begin{aligned}
& V_t(i_1 - \delta_1, i_2, l_1, l_2, r_1, r_2, d) \\
& \geq p_1^* \min(i_1 - \delta_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1 - \delta_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'} (D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'} (D_1 - \min(i_1, D_1)) \rfloor)
\end{aligned}$$

$$\begin{aligned}
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_1^1 \delta_1 - p_2^1 \delta_1 \\
& \quad (\text{by induction and the fact that } i_1''' = i_1'' - \delta_1, i_2''' = i_2'') \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_1^1 \delta_1 - p_2^1 \delta_1 \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta_1 - p_2^1 \delta_1
\end{aligned}$$

Finally we prove result (vii) by backward induction. We prove that result (vii) holds for time period  $t$  if it holds for time period  $t+1$ . We denote  $l_1', l_2'$  as the optimal price levels for state  $(i_1, i_2, l_1, l_2, r_1, r_2, d)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1, i_2, l_1, l_2, r_1, r_2, d - \delta)$  if price levels  $l_1', l_2'$  are used in that period. Note that  $(D_1^*, D_2^*) \in \Omega_{t,d}^{l_1' l_2'}$  and  $(D_1, D_2) \in \Omega_{t,d-\delta}^{l_1' l_2'}$ . We also denote  $i_1''' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2''' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor)$ ,  $i_2' = i_2 - \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor)$ ,  $d''' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$ , and  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$ . We consider the following two cases.

Case 1:  $(D_1, D_2) \in \Omega_{t,d}^{l_1' l_2'}$

$$\begin{aligned}
& V_t(i_1, i_2, l_1, l_2, r_1, r_2, d - \delta) \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d''' - \delta) \\
\geq & p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d''') - p_1^1 \delta - p_2^1 \delta \quad (\text{by induction}) \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_1^1 \delta - p_2^1 \delta \quad (\text{by definition of } D_1^*, D_2^*)
\end{aligned}$$

$$= V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta - p_2^1 \delta$$

Case 2:  $(D_1, D_2) \notin \Omega_{t,d}^{l_1' l_2'}$ . Based on Observation 1, we can find  $(D'_1, D'_2) \in \Omega_{t,d}^{l_1' l_2'}$  such that  $D'_1 \leq D_1, D'_2 \leq D_2$  and  $(D_1 + D_2) - (D'_1 + D'_2) \leq \delta$ . In the following, we denote  $i_1'' = i_1 - \min(i_1, D'_1 + \lfloor \alpha_t^{l_1' l_2'}(D'_2 - \min(i_2, D'_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D'_2 + \lfloor \beta_t^{l_1' l_2'}(D'_1 - \min(i_1, D'_1)) \rfloor)$ ,  $d'' = d + (D'_1 + D'_2) - (\bar{D}_1 + \bar{D}_2)$ ,  $\delta' = (D_1 + D_2) - (D'_1 + D'_2)$ ,  $X_1 = \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $X_2 = \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor)$ ,  $X'_1 = \min(i_1, D'_1 + \lfloor \alpha_t^{l_1' l_2'}(D'_2 - \min(i_2, D'_2)) \rfloor)$ ,  $X'_2 = \min(i_2, D'_2 + \lfloor \beta_t^{l_1' l_2'}(D'_1 - \min(i_1, D'_1)) \rfloor)$ . It can be easily verified that  $\delta' \leq \delta$ ,  $d'' = d''' - \delta'$ ,  $0 \leq X_1 - X'_1 + X_2 - X'_2 \leq \delta'$ .

$$\begin{aligned}
& V_t(i_1, i_2, l_1, l_2, r_1, r_2, d - \delta) \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d''' - \delta) \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d''') - p_1^1(\delta - \delta') - p_2^1(\delta - \delta') \\
& \quad \text{(by induction and the fact that } d'' = d''' - \delta') \\
& \geq p_1^* \min(i_1, D'_1 + \lfloor \alpha_t^{l_1' l_2'}(D'_2 - \min(i_2, D'_2)) \rfloor) + p_1^*(X_1 - X'_1) \\
& \quad + p_2^* \min(i_2, D'_2 + \lfloor \beta_t^{l_1' l_2'}(D'_1 - \min(i_1, D'_1)) \rfloor) + p_2^*(X_2 - X'_2) \\
& \quad + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_1^1(X_1 - X'_1 + X_2 - X'_2) \\
& \quad - p_2^1(X_2 - X'_2 + X_1 - X'_1) \text{ (by result (vi))} \\
& \quad - p_1^1(\delta - \delta') - p_2^1(\delta - \delta') \\
& \geq p_1^* \min(i_1, D'_1 + \lfloor \alpha_t^{l_1' l_2'}(D'_2 - \min(i_2, D'_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D'_2 + \lfloor \beta_t^{l_1' l_2'}(D'_1 - \min(i_1, D'_1)) \rfloor) \\
& \quad + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_1^1 \delta' - p_2^1 \delta' - p_1^1(\delta - \delta') - p_2^1(\delta - \delta') \\
& \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor)
\end{aligned}$$



$$\begin{aligned}
& +V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') - p_1^1 \delta - p_2^1 \delta \text{ (by definition of } D_1^*, D_2^*) \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^1 \delta - p_2^1 \delta
\end{aligned}$$

■

**Lemma 2** *In any period  $t$ , for any  $\delta_1, \delta_2, \delta \geq 0$ , the following inequalities hold as long as the value of each state variable involved is within its domain in the approximate state space.*

$$\begin{aligned}
(i) \quad & A_t(\tilde{i}_1 + \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \\
(ii) \quad & A_t(\tilde{i}_1 + \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \\
(iii) \quad & A_t(\tilde{i}_1 + \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_2^1 \\
(iv) \quad & A_t(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1 \\
(v) \quad & A_t(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
& \quad - p_1^1 (\tilde{\delta}_1 + \tilde{i}_2 + (T - t)(\Delta_1 - 1)) - p_2^1 \tilde{i}_2 \\
(vi) \quad & A_t(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
& \quad - p_1^1 \tilde{\delta}_1 - p_2^1 (\tilde{\delta}_1 + (T - t)(\Delta_2 - 1)) \\
(vii) \quad & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \tilde{\delta}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
& \quad - p_1^1 (\tilde{\delta} + 1) - p_2^1 (\tilde{\delta} + 1)
\end{aligned}$$

**Proof** Results (i), (ii), (iii) and (iv) can be proved similarly as Lemma 1 (i), (ii), (iii), and (iv), and hence the proofs are omitted.

We now prove result (v) by backward induction. We show that if result (v) holds for time period  $t + 1$ , it also holds for time period  $t$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  in period  $t$  and  $\tilde{D}_1^*, \tilde{D}_2^*$  as the corresponding worst-case demand. We denote  $\tilde{D}_1, \tilde{D}_2$  as the worst-case demand in period  $t$  for state  $(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d})$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(\tilde{D}_1, \tilde{D}_2), (\tilde{D}_1^*, \tilde{D}_2^*) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1, l'_2}$ . We also denote  $\tilde{i}_1''' = \Phi_1\{\tilde{i}_1 - \tilde{\delta}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1, l'_2} (\tilde{D}_2 - \min(0, \tilde{D}_2)) \rfloor)\}$ ,  $\tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1, l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)\}$ ,  $\tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1, l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)\}$ ,

$\tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor)\}$ ,  $\tilde{i}'_2 = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor)\}$ ,  $\tilde{d}'' = \Phi\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$ , and  $\tilde{d}' = \Phi\{\tilde{d} + (\tilde{D}_1^* + \tilde{D}_2^*) - (\bar{D}_1 + \bar{D}_2)\}$ . To simplify notation, we denote  $X_1 = \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(0, \tilde{D}_2)) \rfloor)$ ,  $X'_1 = \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)$ ,  $X'_2 = \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)$ . We consider the following two cases.

Case 1:  $X_1 \leq X'_1$ , note that  $X'_1 - X_1 \leq \tilde{\delta}_1$

$$\begin{aligned}
& A_t(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \\
& \geq p_1^* \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(0, \tilde{D}_2)) \rfloor) \\
& \quad + p_2^* \min(0, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor) \\
& \quad + A_{t+1}(\tilde{i}_1''', 0, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) - p_1^*(X'_1 - X_1) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) - p_2^* X'_2 \\
& \quad + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \\
& - p_1^1(\tilde{\delta}_1 - (X'_1 - X_1) + (\Delta_1 - 1) + \tilde{i}_2 - X'_2 + (T - t - 1)(\Delta_1 - 1)) - p_2^1(\tilde{i}_2 - X'_2) \\
& \quad \text{(by induction and the fact that)} \\
& \quad \tilde{i}_1''' = \Phi_1\{\tilde{i}_1 - \tilde{\delta}_1 - X_1\}, \tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - X'_1\}, \tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - X'_2\}) \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& \quad + A_{t+1}(\tilde{i}_1', \tilde{i}_2', l'_1, l'_2, r'_1, r'_2, \tilde{d}') - p_1^1(\tilde{\delta}_1 + \tilde{i}_2 + (T - t)(\Delta_1 - 1)) - p_2^1 \tilde{i}_2 \\
& \quad \text{(by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
& = A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1(\tilde{\delta}_1 + \tilde{i}_2 + (T - t)(\Delta_1 - 1)) - p_2^1 \tilde{i}_2
\end{aligned}$$

Case 2:  $X_1 > X'_1$ , note that  $X_1 - X'_1 \leq X'_2$ .

$$\begin{aligned}
& A_t(\tilde{i}_1 - \tilde{\delta}_1, 0, l_1, l_2, r_1, r_2, \tilde{d}) \\
& \geq p_1^* \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(0, \tilde{D}_2)) \rfloor) \\
& \quad + p_2^* \min(0, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor) \\
& \quad + A_{t+1}(\tilde{i}_1''', 0, l'_1, l'_2, r'_1, r'_2, \tilde{d}'')
\end{aligned}$$

$$\begin{aligned}
&\geq p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) + p_1^* (X_1 - X'_1) \\
&\quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) - p_2^* X'_2 \\
&\quad + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \\
&\quad - p_1^1(\tilde{\delta}_1 + (X_1 - X'_1) + (\Delta_1 - 1) + \tilde{i}_2 - X'_2 + (T - t - 1)(\Delta_1 - 1)) - p_2^1(\tilde{i}_2 - X'_2) \\
&\quad \text{(by induction and the fact that)} \\
&\quad \tilde{i}_1''' = \Phi_1\{\tilde{i}_1 - \tilde{\delta}_1 - X_1\}, \tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - X'_1\}, \tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - X'_2\}) \\
&\geq p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
&\quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
&\quad + A_{t+1}(\tilde{i}_1', \tilde{i}_2', l'_1, l'_2, r'_1, r'_2, \tilde{d}') \\
&\quad - p_1^1(\tilde{\delta}_1 + \tilde{i}_2 + (T - t)(\Delta_1 - 1)) - p_2^1 \tilde{i}_2 \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
&= A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1(\tilde{\delta}_1 + \tilde{i}_2 + (T - t)(\Delta_1 - 1)) - p_2^1 \tilde{i}_2
\end{aligned}$$

We next prove result (vi) by backward induction. We show that result (vi) holds for time period  $t$  if it holds for time period  $t + 1$ . In the following proof, we denote  $l'_1, l'_2$  as the optimal price levels for state  $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  in period  $t$  and  $\tilde{D}_1^*, \tilde{D}_2^*$  as the corresponding worst-case demand. We denote  $\tilde{D}_1, \tilde{D}_2$  as the worst-case demand in period  $t$  for state  $(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(\tilde{D}_1, \tilde{D}_2), (\tilde{D}_1^*, \tilde{D}_2^*) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}$ . We also denote  $\tilde{i}_1''' = \Phi_1\{\tilde{i}_1 - \tilde{\delta}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)\}$ ,  $\tilde{i}_2''' = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor)\}$ ,  $\tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)\}$ ,  $\tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)\}$ ,  $\tilde{i}_1' = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor)\}$ ,  $\tilde{i}_2' = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor)\}$ ,  $\tilde{d}'' = \Phi\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$ ,  $\tilde{d}' = \Phi\{\tilde{d} + (\tilde{D}_1^* + \tilde{D}_2^*) - (\bar{D}_1 + \bar{D}_2)\}$ ,  $X_1 = \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)$ ,  $X_2 = \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor)$ ,  $X'_1 = \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)$ ,  $X'_2 = \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)$ . Clearly,  $X_1 \leq X'_1 \leq X_1 + \tilde{\delta}_1$ . We consider the following four cases.

Case 1:  $\tilde{i}_2 \leq \tilde{D}_2$ .

$$\begin{aligned}
&A_t(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
&\geq p_1^* \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)
\end{aligned}$$

$$\begin{aligned}
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l_1', l_2', r_1', r_2', \tilde{d}'') \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) - p_1^*(X_1' - X_1) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l_1', l_2', r_1', r_2', \tilde{d}'') - p_1^1(\tilde{\delta}_1 + X_1 - X_1') \\
& \text{(by result (iv) and the fact that)} \\
& \tilde{i}_1''' = \Phi_1\{\tilde{i}_1 - \tilde{\delta}_1 - X_1\}, \tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - X_1'\}, \tilde{i}_2''' = \tilde{i}_2'' = 0) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1', \tilde{i}_2', l_1', l_2', r_1', r_2', \tilde{d}') - p_1^1 \tilde{\delta}_1 \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1
\end{aligned}$$

Case 2:  $\tilde{i}_1 < \tilde{D}_1, \tilde{i}_2 > \tilde{D}_2$

$$\begin{aligned}
& A_t(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
\geq & p_1^* \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l_1', l_2', r_1', r_2', \tilde{d}'') \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) - p_1^1 \tilde{\delta}_1 \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l_1', l_2', r_1', r_2', \tilde{d}'') - p_2^1(\tilde{\delta}_1 + \Delta_2 - 1) \\
& \text{(by result (iv) and the fact that)} \\
& \tilde{i}_1''' = \tilde{i}_1'' = 0, \tilde{i}_2''' = \Phi_2\{\tilde{i}_2 - X_2\}, \tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - X_2'\}) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1', \tilde{i}_2', l_1', l_2', r_1', r_2', \tilde{d}') - p_1^1 \tilde{\delta}_1 - p_2^1(\tilde{\delta}_1 + \Delta_2 - 1) \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1 - p_2^1(\tilde{\delta}_1 + \Delta_2 - 1)
\end{aligned}$$

Case 3:  $\tilde{i}_1 - \tilde{\delta}_1 \leq \tilde{D}_1 \leq \tilde{i}_1, \tilde{i}_2 > \tilde{D}_2$

$$\begin{aligned}
& A_t(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
\geq & p_1^* \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l_1', l_2', r_1', r_2', \tilde{d}'') \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) - p_1^*(\tilde{D}_1 - \tilde{i}_1 + \tilde{\delta}_1) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l_1', l_2', r_1', r_2', \tilde{d}'') \\
& - p_1^1(\tilde{i}_1 - \tilde{D}_1) - p_2^1(\tilde{i}_1 - \tilde{D}_1 + \tilde{D}_1 - \tilde{i}_1 + \tilde{\delta}_1 + (\Delta_2 - 1) + (T - t - 1)(\Delta_2 - 1)) \\
& \text{(by result (v) and the fact that} \\
& \tilde{i}_1''' = 0, \tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - \tilde{D}_1\}, \tilde{i}_2''' = \Phi_2\{\tilde{i}_2 - X_2\}, \tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - X_2'\}) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1', \tilde{i}_2', l_1', l_2', r_1', r_2', \tilde{d}') - p_1^1 \tilde{\delta}_1 - p_2^1(\tilde{\delta}_1 + (T - t)(\Delta_2 - 1)) \\
& \text{(by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1 - p_2^1(\tilde{\delta}_1 + (T - t)(\Delta_2 - 1))
\end{aligned}$$

Case 4:  $\tilde{i}_1 - \tilde{\delta}_1 > \tilde{D}_1, \tilde{i}_2 > \tilde{D}_2$

$$\begin{aligned}
& A_t(\tilde{i}_1 - \tilde{\delta}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
\geq & p_1^* \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1 - \tilde{\delta}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l_1', l_2', r_1', r_2', \tilde{d}'') \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l_1 l_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l_1', l_2', r_1', r_2', \tilde{d}'') - p_1^1 \tilde{\delta}_1 - p_2^1(\tilde{\delta}_1 + (T - t - 1)(\Delta_2 - 1)) \\
& \text{(by induction and the fact that } \tilde{i}_1''' = \tilde{i}_1'' - \tilde{\delta}_1, \tilde{i}_2''' = \tilde{i}_2'') \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l_1 l_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor)
\end{aligned}$$

$$\begin{aligned}
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') - p_1^1 \tilde{\delta}_1 - p_2^1 (\tilde{\delta}_1 + (T-t)(\Delta_2 - 1)) \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 \tilde{\delta}_1 - p_2^1 (\tilde{\delta}_1 + (T-t)(\Delta_2 - 1))
\end{aligned}$$

We now prove result (vii) by backward induction. We show that result (vii) holds for time period  $t$  if it holds for time period  $t+1$ . We denote  $l'_1, l'_2$  as the optimal price levels for state  $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  in period  $t$  and  $\tilde{D}_1^*, \tilde{D}_2^*$  as the corresponding worst-case demand. We denote  $\tilde{D}_1, \tilde{D}_2$  as the worst-case demand in period  $t$  for state  $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \tilde{\delta})$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(\tilde{D}_1^*, \tilde{D}_2^*) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}$  and  $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t, \tilde{d} + \tilde{\delta}}^{l'_1 l'_2}$ . We also denote  $\tilde{i}_1''' = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)\}$ ,  $\tilde{i}_2''' = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)\}$ ,  $\tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor)\}$ ,  $\tilde{i}'_2 = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor)\}$ ,  $\tilde{d}''' = \Phi\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$ , and  $\tilde{d}' = \Phi\{\tilde{d} + (\tilde{D}_1^* + \tilde{D}_2^*) - (\bar{D}_1 + \bar{D}_2)\}$ . We consider the following two cases.

Case 1:  $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}$

$$\begin{aligned}
& A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \tilde{\delta}) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l'_1, l'_2, r'_1, r'_2, \tilde{d}''' + \tilde{\delta}) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l'_1, l'_2, r'_1, r'_2, \tilde{d}''') - p_1^1 (\tilde{\delta} + 1) - p_2^1 (\tilde{\delta} + 1) \text{ (by induction)} \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') - p_1^1 (\tilde{\delta} + 1) - p_2^1 (\tilde{\delta} + 1) \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1 (\tilde{\delta} + 1) - p_2^1 (\tilde{\delta} + 1)
\end{aligned}$$

Case 2:  $(\tilde{D}_1, \tilde{D}_2) \notin \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}$ . Based on Observation 2, we can find  $(\tilde{D}'_1, \tilde{D}'_2) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}$  such that  $\tilde{D}_1 \leq \tilde{D}'_1 \leq \tilde{D}_1 + \tilde{\delta}, \tilde{D}_2 \leq \tilde{D}'_2 \leq \tilde{D}_2 + \tilde{\delta}$  and  $(\tilde{D}'_1 + \tilde{D}'_2) - (\tilde{D}_1 + \tilde{D}_2) = 0$ .

$\tilde{D}_2) \leq \tilde{\delta}$ . In the following, we denote  $\tilde{i}_1'' = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor)\}$ ,  $\tilde{i}_2'' = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor)\}$ ,  $\tilde{d}'' = \Phi\{\tilde{d} + (\tilde{D}'_1 + \tilde{D}'_2) - (\tilde{D}_1 + \tilde{D}_2)\}$ ,  $\tilde{\delta}' = (\tilde{D}'_1 + \tilde{D}'_2) - (\tilde{D}_1 + \tilde{D}_2)$ ,  $X_1 = \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)$ ,  $X_2 = \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)$ ,  $X'_1 = \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor)$ ,  $X'_2 = \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor)$ . It can be easily verified that  $1 \leq \tilde{\delta}' \leq \tilde{\delta}$ ,  $X'_1 \geq X_1$ ,  $X'_2 \geq X_2$ ,  $(X'_1 + X'_2) - (X_1 + X_2) \leq \tilde{\delta}'$ . We consider the following three possible sub-cases, i.e., (2a)  $X'_1 = X_1$ ; (2b)  $X'_2 = X_2$ ; (2c)  $X'_1 - X_1 \geq 1$  and  $X'_2 - X_2 \geq 1$ , which also implies that  $X'_1 - X_1 \leq \tilde{\delta}' - 1$  and  $X'_2 - X_2 \leq \tilde{\delta}' - 1$

Case (2a):  $X'_1 = X_1$

$$\begin{aligned}
& A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \tilde{\delta}) \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& \quad + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l'_1, l'_2, r'_1, r'_2, \tilde{d}''' + \tilde{\delta}) \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) - p_2^* \tilde{\delta}' \\
& \quad + A_{t+1}(\tilde{i}_1''', \tilde{i}_2''', l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
& \quad - p_1^1(\tilde{\delta} - \tilde{\delta}' + 1) - p_2^1(\tilde{\delta} - \tilde{\delta}' + 1) \text{ (by induction and the fact that } \tilde{d}'' = \tilde{d}''' + \tilde{\delta}') \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) - p_2^* \tilde{\delta}' \\
& \quad + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - p_1^1 \text{ (by result (iii) and the fact that } \tilde{i}_1''' = \tilde{i}_1'', \tilde{i}_2''' \geq \tilde{i}_2'') \\
& \quad - p_1^1(\tilde{\delta} - \tilde{\delta}' + 1) - p_2^1(\tilde{\delta} - \tilde{\delta}' + 1) \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) \\
& \quad + A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - p_1^1(\tilde{\delta} + 1) - p_2^1(\tilde{\delta} + 1) \\
& \geq p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& \quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor)
\end{aligned}$$

$$\begin{aligned}
& +A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') - p_1^1(\tilde{\delta} + 1) - p_2^1(\tilde{\delta} + 1) \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1(\tilde{\delta} + 1) - p_2^1(\tilde{\delta} + 1)
\end{aligned}$$

Case (2b):  $X'_2 = X_2$ . This case can be proved similarly as case (2a).

Case (2c):  $X'_1 - X_1 \geq 1, X'_2 - X_2 \geq 1$

$$\begin{aligned}
& A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \tilde{\delta}) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}'''_1, \tilde{i}'''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}''' + \tilde{\delta}) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) - p_1^*(\tilde{\delta}' - 1) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) - p_2^*(\tilde{\delta}' - 1) \\
& + A_{t+1}(\tilde{i}'''_1, \tilde{i}'''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
& - p_1^1(\tilde{\delta} - \tilde{\delta}' + 1) - p_2^1(\tilde{\delta} - \tilde{\delta}' + 1) \text{ (by induction and the fact that } \tilde{d}'' = \tilde{d}''' + \tilde{\delta}') \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) - p_1^*(\tilde{\delta}' - 1) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) - p_2^*(\tilde{\delta}' - 1) \\
& + A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - p_1^1 - p_2^1 \\
& \text{(by result (iii) and the fact that } \tilde{i}'''_1 \geq \tilde{i}''_1, \tilde{i}'''_2 \geq \tilde{i}''_2) \\
& - p_1^1(\tilde{\delta} - \tilde{\delta}' + 1) - p_2^1(\tilde{\delta} - \tilde{\delta}' + 1) \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') - p_1^1(\tilde{\delta} + 1) - p_2^1(\tilde{\delta} + 1) \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^1(\tilde{\delta} + 1) - p_2^1(\tilde{\delta} + 1)
\end{aligned}$$

■

**Lemma 3** *In any period  $t$ , for any  $\Delta_1, \Delta_2, \Delta \geq 1$ , the following inequality holds as long as the value of each state variable involved is within its domain in the corresponding state space,*

$$A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$$



$$\begin{aligned}
&\geq V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \\
&\quad - (T - t + 1)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
&\quad - (T - t + 1)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \}
\end{aligned}$$

**Proof** We prove this lemma by backward induction. We show that if the lemma holds for time period  $t + 1$ , it also holds for time period  $t$ . We denote  $l'_1, l'_2$  as the optimal price levels corresponding to  $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$ , and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $\tilde{D}_1, \tilde{D}_2$  as the worst-case demand corresponding to  $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(D_1^*, D_2^*) \in \Omega_{t,d}^{l'_1, l'_2}$ ,  $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t,\tilde{d}}^{l'_1, l'_2}$ . According to Observation 3, for any  $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}_{t,\tilde{d}}^{l'_1, l'_2}$ , we can find  $(D'_1, D'_2) \in \Omega_{t,d}^{l'_1, l'_2}$  such that  $D'_1 - (\Delta - 1) \leq \tilde{D}_1 \leq D'_1 + (\Delta - 1)$ ,  $D'_2 - (\Delta - 1) \leq \tilde{D}_2 \leq D'_2 + (\Delta - 1)$  and  $(\tilde{D}_1 + \tilde{D}_2) - (D'_1 + D'_2) \leq d - \tilde{d} \leq (\Delta - 1)$ . We also denote  $i'''_1 = \tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1, l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)$ ,  $i'''_2 = \tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1, l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)$ ,  $\tilde{i}'''_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1, l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor)\}$ ,  $\tilde{i}'''_2 = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1, l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor)\}$ ,  $i''_1 = i_1 - \min(i_1, D'_1 + \lfloor \alpha_t^{l'_1, l'_2} (D'_2 - \min(i_2, D'_2)) \rfloor)$ ,  $i''_2 = i_2 - \min(i_2, D'_2 + \lfloor \beta_t^{l'_1, l'_2} (D'_1 - \min(i_1, D'_1)) \rfloor)$ ,  $i'_1 = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1, l'_2} (D_2^* - \min(i_2, D_2^*)) \rfloor)$ ,  $i'_2 = i_2 - \min(i_2, D_2^* + \lfloor \beta_t^{l'_1, l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor)$ ,  $d''' = \tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)$ ,  $\tilde{d}''' = \Phi\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$ ,  $d'' = d + (D'_1 + D'_2) - (\bar{D}_1 + \bar{D}_2)$ , and  $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$ . Since  $(\tilde{D}_1 + \tilde{D}_2) - (D'_1 + D'_2) \leq d - \tilde{d}$ , or  $(\tilde{D}_1 + \tilde{D}_2) + \tilde{d} \leq (D'_1 + D'_2) + d$ , we have  $d''' \leq d''$ . In addition, since  $\tilde{d} \geq d - (\Delta - 1)$  and  $(\tilde{D}_1 + \tilde{D}_2) \geq (D'_1 + D'_2) - 2(\Delta - 1)$ , we have  $d''' \geq d'' - 3(\Delta - 1)$ . It can also be verified that  $i''_1 - i'''_1 \leq (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1)$  and  $i''_2 - i'''_2 \leq (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1)$ .

$$\begin{aligned}
&A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
&\geq p_1^* \min(\tilde{i}_1, \tilde{D}_1 + \lfloor \alpha_t^{l'_1, l'_2} (\tilde{D}_2 - \min(\tilde{i}_2, \tilde{D}_2)) \rfloor) \\
&\quad + p_2^* \min(\tilde{i}_2, \tilde{D}_2 + \lfloor \beta_t^{l'_1, l'_2} (\tilde{D}_1 - \min(\tilde{i}_1, \tilde{D}_1)) \rfloor) \\
&\quad + A_{t+1}(\tilde{i}'''_1, \tilde{i}'''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
&\geq p_1^* \min(i_1, D'_1 + \lfloor \alpha_t^{l'_1, l'_2} (D'_2 - \min(i_2, D'_2)) \rfloor) - p_1^* \max(\Delta_1 - 1, 2(\Delta - 1)) \\
&\quad + p_2^* \min(i_2, D'_2 + \lfloor \beta_t^{l'_1, l'_2} (D'_1 - \min(i_1, D'_1)) \rfloor) - p_2^* \max(\Delta_2 - 1, 2(\Delta - 1)) \\
&\quad + V_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}''')
\end{aligned}$$

$$\begin{aligned}
& -(T-t)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& -(T-t)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& \text{(by induction)} \\
\geq & p_1^* \min(i_1, D'_1 + \lfloor \alpha_t^{l'_1 l'_2} (D'_2 - \min(i_2, D'_2)) \rfloor) - p_1^1 \max(\Delta_1 - 1, 2(\Delta - 1)) \\
& + p_2^* \min(i_2, D'_2 + \lfloor \beta_t^{l'_1 l'_2} (D'_1 - \min(i_1, D'_1)) \rfloor) - p_2^1 \max(\Delta_2 - 1, 2(\Delta - 1)) \\
& + V_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, d'') - 3p_1^1(\Delta - 1) - 3p_2^1(\Delta - 1) \\
& \text{(by Lemma 1 (vii))} \tag{D.1} \\
& -(T-t)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& -(T-t)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
\geq & p_1^* \min(i_1, D'_1 + \lfloor \alpha_t^{l'_1 l'_2} (D'_2 - \min(i_2, D'_2)) \rfloor) - p_1^1 \max(\Delta_1 - 1, 2(\Delta - 1)) \\
& + p_2^* \min(i_2, D'_2 + \lfloor \beta_t^{l'_1 l'_2} (D'_1 - \min(i_1, D'_1)) \rfloor) - p_2^1 \max(\Delta_2 - 1, 2(\Delta - 1)) \\
& + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') - 3p_1^1(\Delta - 1) - 3p_2^1(\Delta - 1) \\
& - p_1^1 \{ 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 2(\Delta - 1) \} \\
& - p_2^1 \{ 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 2(\Delta - 1) \} \tag{D.2} \\
& -(T-t)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& -(T-t)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
= & p_1^* \min(i_1, D'_1 + \lfloor \alpha_t^{l'_1 l'_2} (D'_2 - \min(i_2, D'_2)) \rfloor) \\
& + p_2^* \min(i_2, D'_2 + \lfloor \beta_t^{l'_1 l'_2} (D'_1 - \min(i_1, D'_1)) \rfloor) \\
& + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') \\
& -(T-t+1)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& -(T-t+1)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} (D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l'_1 l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') \text{ (by definition of } D_1^*, D_2^*) \\
& -(T-t+1)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
& -(T-t+1)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \}
\end{aligned}$$

$$\begin{aligned}
&= V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \\
&\quad - (T - t + 1)p_1^1 \{ \max(\Delta_1 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \} \\
&\quad - (T - t + 1)p_2^1 \{ \max(\Delta_2 - 1, 2(\Delta - 1)) + 2(\Delta_1 - 1) + 2(\Delta_2 - 1) + 5(\Delta - 1) \}
\end{aligned}$$

In the above from (D.1) to (D.2), we consider four possible cases. In all four cases, we assume that  $(\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) \geq 1$ . The case when  $(\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) = 0$  is trivial because in this case  $\Delta_1 = \Delta_2 = \Delta = 1$ , which implies that the approximation scheme AS does not lose any accuracy and hence  $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) = V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$  and Lemma 3 holds.

Case 1:  $i_1''' \geq i_1''$  and  $i_2''' \geq i_2''$ , according to Lemma 1 (iii), we have

$$\begin{aligned}
&V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2', d'') \\
&\geq V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') - p_1^1 - p_2^1 \\
&\geq V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') \\
&\quad - p_1^1 \{ (\Delta - 1) + (\Delta_1 - 1) + (\Delta_2 - 1) \} - p_2^1 \{ (\Delta - 1) + (\Delta_1 - 1) + (\Delta_2 - 1) \}
\end{aligned}$$

For case 2:  $i_1'' - \{ (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) \} \leq i_1''' < i_1''$  and  $i_2'' - \{ (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) \} \leq i_2''' < i_2''$ ; case 3:  $i_1''' \geq i_1''$  and  $i_2'' - \{ (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) \} \leq i_2''' < i_2''$ ; and case 4:  $i_1'' - \{ (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) \} \leq i_1''' < i_1''$  and  $i_2''' \geq i_2''$ , one can easily verify that the result also holds by applying Lemma 1 (iii) and (vi). ■

**Lemma 4** *In any period  $t$ , for any  $\Delta_1, \Delta_2, \Delta \geq 1$ , the following inequality holds as long as the value of each state variable involved is within its domain in the corresponding state space,*

$$\begin{aligned}
&R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \\
&\geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
&\quad - (T - t + 1)p_1^1 \{ 10(\Delta - 1) + (T - t + 1)(\Delta_1 - 1) + 2(\Delta_2 - 1) \} \\
&\quad - (T - t + 1)p_2^1 \{ 10(\Delta - 1) + 2(\Delta_1 - 1) + (T - t + 1)(\Delta_2 - 1) \}
\end{aligned}$$

**Proof** We prove this lemma by backward induction. We show in the following that the lemma holds for time period  $t$  if it holds for time period  $t + 1$ . We denote  $l_1', l_2'$

as the optimal price levels corresponding to  $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ , and  $\tilde{D}_1^*, \tilde{D}_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand corresponding to  $R_t(i_1, i_2, r_1, r_2, l_1, l_2, d)$  if price levels  $l'_1, l'_2$  are used in that period. Note that  $(\tilde{D}_1^*, \tilde{D}_2^*) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}, (D_1, D_2) \in \Omega_{t, d}^{l'_1 l'_2}$ . Based on Observation 4, for any  $(D_1, D_2) \in \Omega_{t, d}^{l'_1 l'_2}$ , we have  $(\tilde{D}'_1, \tilde{D}'_2) \in \tilde{\Omega}_{t, \tilde{d}}^{l'_1 l'_2}$  such that  $\tilde{D}'_1 - (\Delta - 1) \leq D_1 \leq \tilde{D}'_1 + (\Delta - 1), \tilde{D}'_2 - (\Delta - 1) \leq D_2 \leq \tilde{D}'_2 + (\Delta - 1)$  and  $(\tilde{D}'_1 + \tilde{D}'_2) - (D_1 + D_2) \leq d - \tilde{d} \leq (\Delta - 1)$ . We also denote  $i'''_1 = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor), i'''_2 = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor), \tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor)\}, \tilde{i}''_2 = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor)\}, \tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor)\}, \tilde{i}''_2 = \Phi_2\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor)\},  $d''' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2), \tilde{d}'' = \Phi\{\tilde{d} + (\tilde{D}'_1 + \tilde{D}'_2) - (\bar{D}_1 + \bar{D}_2)\},$  and  $\tilde{d}' = \Phi\{\tilde{d} + (\tilde{D}_1^* + \tilde{D}_2^*) - (\bar{D}_1 + \bar{D}_2)\}$ . It can also be verified that  $\Phi_1\{i'''_1\} \geq \tilde{i}'_1 - \{2(\Delta - 1) + (\Delta_1 - 1)\}, \Phi_2\{i'''_2\} \geq \tilde{i}''_2 - \{2(\Delta - 1) + (\Delta_2 - 1)\},$  and  $\tilde{d}'' \leq \Phi\{d'''\} \leq \tilde{d}' + 4(\Delta - 1)$ .$

$$\begin{aligned}
& R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \\
&= p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
&\quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
&\quad + R_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, d''') \\
&\geq p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
&\quad + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) \\
&\quad - p_1^1 \{(\Delta_2 - 1) + (\Delta - 1)\} - p_2^1 \{(\Delta_1 - 1) + (\Delta - 1)\} \\
&\quad + A_{t+1}(\Phi_1\{i'''_1\}, \Phi_2\{i'''_2\}, l'_1, l'_2, r'_1, r'_2, \Phi\{d'''\}) \tag{D.3}
\end{aligned}$$

$$\begin{aligned}
& - (T - t) p_1^1 \{10(\Delta - 1) + (T - t)(\Delta_1 - 1) + 2(\Delta_2 - 1)\} \\
& - (T - t) p_2^1 \{10(\Delta - 1) + 2(\Delta_1 - 1) + (T - t)(\Delta_2 - 1)\} \quad (\text{by induction}) \\
&\geq p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2} (\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
&\quad + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2} (\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) \\
&\quad - p_1^1 \{(\Delta_2 - 1) + (\Delta - 1)\} - p_2^1 \{(\Delta_1 - 1) + (\Delta - 1)\} \\
&\quad + A_{t+1}(\Phi_1\{i'''_1\}, \Phi_2\{i'''_2\}, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
&\quad - p_1^1 \{5(\Delta - 1)\} - p_2^1 \{5(\Delta - 1)\} \tag{D.4}
\end{aligned}$$

$$\begin{aligned}
& -(T-t)p_1^1\{10(\Delta-1)+(T-t)(\Delta_1-1)+2(\Delta_2-1)\} \\
& -(T-t)p_2^1\{10(\Delta-1)+2(\Delta_1-1)+(T-t)(\Delta_2-1)\} \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) \\
& - p_1^1\{(\Delta_2-1)+(\Delta-1)\} - p_2^1\{(\Delta_1-1)+(\Delta-1)\} \\
& + A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - p_1^1\{5(\Delta-1)\} - p_2^1\{5(\Delta-1)\} \\
& - p_1^1\{4(\Delta-1)+(T-t+1)(\Delta_1-1)+(\Delta_2-1)\} \\
& - p_2^1\{4(\Delta-1)+(\Delta_1-1)+(T-t+1)(\Delta_2-1)\} \tag{D.5} \\
& -(T-t)p_1^1\{10(\Delta-1)+(T-t)(\Delta_1-1)+2(\Delta_2-1)\} \\
& -(T-t)p_2^1\{10(\Delta-1)+2(\Delta_1-1)+(T-t)(\Delta_2-1)\} \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}'_1 + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}'_2 - \min(\tilde{i}_2, \tilde{D}'_2)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}'_2 + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}'_1 - \min(\tilde{i}_1, \tilde{D}'_1)) \rfloor) \\
& + A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \\
& -(T-t+1)p_1^1\{10(\Delta-1)+(T-t+1)(\Delta_1-1)+2(\Delta_2-1)\} \\
& -(T-t+1)p_2^1\{10(\Delta-1)+2(\Delta_1-1)+(T-t+1)(\Delta_2-1)\} \\
\geq & p_1^* \min(\tilde{i}_1, \tilde{D}_1^* + \lfloor \alpha_t^{l'_1 l'_2}(\tilde{D}_2^* - \min(\tilde{i}_2, \tilde{D}_2^*)) \rfloor) \\
& + p_2^* \min(\tilde{i}_2, \tilde{D}_2^* + \lfloor \beta_t^{l'_1 l'_2}(\tilde{D}_1^* - \min(\tilde{i}_1, \tilde{D}_1^*)) \rfloor) \\
& + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
& -(T-t+1)p_1^1\{10(\Delta-1)+(T-t+1)(\Delta_1-1)+2(\Delta_2-1)\} \\
& -(T-t+1)p_2^1\{10(\Delta-1)+2(\Delta_1-1)+(T-t+1)(\Delta_2-1)\} \\
= & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
& -(T-t+1)p_1^1\{10(\Delta-1)+(T-t+1)(\Delta_1-1)+2(\Delta_2-1)\} \\
& -(T-t+1)p_2^1\{10(\Delta-1)+2(\Delta_1-1)+(T-t+1)(\Delta_2-1)\}
\end{aligned}$$

In the above, from (D.3) to (D.4), we assume that  $\Delta \geq 2$ , and thus we have  $\Delta-1 \geq 1$ . The case when  $\Delta = 1$  is trivial and one can easily verify that the result still holds.

Since  $\tilde{d}'' \leq \Phi\{d'''\} \leq \tilde{d}'' + 4(\Delta - 1)$ , according to Lemma 2 (vii), we have,

$$\begin{aligned}
& A_{t+1}(\Phi_1\{i_1'''\}, \Phi_2\{i_2'''\}, l'_1, l'_2, r'_1, r'_2, \Phi\{d'''\}) \\
& \geq A_{t+1}(\Phi_1\{i_1'''\}, \Phi_2\{i_2'''\}, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') - p_1^1(4(\Delta - 1) + 1) - p_2^1(4(\Delta - 1) + 1) \\
& \geq A_{t+1}(\Phi_1\{i_1'''\}, \Phi_2\{i_2'''\}, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') - p_1^1 5(\Delta - 1) - p_2^1 5(\Delta - 1)
\end{aligned}$$

From (D.4) to (D.5), we consider four possible cases. In all four cases, we assume that  $(\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) \geq 1$ . The case when  $(\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta - 1) = 0$  is trivial because in this case  $\Delta_1 = \Delta_2 = \Delta = 1$ , which implies that the approximation scheme AS does not lose any accuracy and hence  $R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) = A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$  and the lemma holds.

Case 1:  $\Phi_1\{i_1'''\} \geq \tilde{i}_1''$  and  $\Phi_2\{i_2'''\} \geq \tilde{i}_2''$ , according to Lemma 2 (iii), we have,

$$\begin{aligned}
& A_{t+1}(\Phi_1\{i_1'''\}, \Phi_2\{i_2'''\}, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
& \geq A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l'_1, l'_2, r'_1, r'_2, \tilde{d}''') - p_1^1 - p_2^1 \\
& \geq A_{t+1}(\tilde{i}_1'', \tilde{i}_2'', l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
& \quad - p_1^1((\Delta - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)) - p_2^1((\Delta - 1) + (\Delta_1 - 1) + (\Delta_2 - 1))
\end{aligned}$$

For case 2:  $\tilde{i}_1'' - \{2(\Delta - 1) + (\Delta_1 - 1)\} \leq \Phi_1\{i_1'''\} < \tilde{i}_1''$  and  $\tilde{i}_2'' - \{2(\Delta - 1) + (\Delta_2 - 1)\} \leq \Phi_2\{i_2'''\} < \tilde{i}_2''$ ; case 3:  $\Phi_1\{i_1'''\} \geq \tilde{i}_1''$  and  $\tilde{i}_2'' - \{2(\Delta - 1) + (\Delta_2 - 1)\} \leq \Phi_2\{i_2'''\} < \tilde{i}_2''$ ; and case 4:  $\tilde{i}_1'' - \{2(\Delta - 1) + (\Delta_1 - 1)\} \leq \Phi_1\{i_1'''\} < \tilde{i}_1''$  and  $\Phi_2\{i_2'''\} \geq \tilde{i}_2''$ , one can easily verify that the result also holds by applying Lemma 2 (iii) and (vi). ■

**Theorem 1** *For any  $\epsilon > 0$ , the approximation algorithm AS with the values of  $\Delta_1, \Delta_2, \Delta$  defined in Section 4.3 generates a solution that is within a relative error  $\epsilon$  from the optimality with running time  $O(T^8 m_1^2 m_2^2 R_1 R_2 / \epsilon^5)$ .*

**Proof** We first estimate the running time of the approximation algorithm AS. In AS, we partition the  $i_j$  dimension into equal intervals of length  $\Delta_j$  and only one value in each interval is considered, for  $j = 1, 2$ . Thus in the algorithm AS, in each period  $t$ , at most  $\lceil I_j / \Delta_j \rceil$  different values of  $i_j$  are considered, for  $j = 1, 2$ . Similarly, in each period  $t$ , at most  $\lceil B_t / \Delta \rceil$  different values of  $d$  are considered, and at most  $\lceil D_j^{\max} / \Delta \rceil$

different values of  $D_j$  are considered, for  $j = 1, 2$ . Thus, the overall running time of the algorithm AS is bounded by  $O(Tm_1^2m_2^2R_1R_2\lceil I_1/\Delta_1\rceil\lceil I_2/\Delta_2\rceil\lceil D_1^{\max}/\Delta\rceil\lceil D_2^{\max}/\Delta\rceil\lceil B_{\max}/\Delta\rceil)$ . By the way  $\Delta_1, \Delta_2, \Delta$  are defined,  $I_j/\Delta_j = O(T^2/\epsilon)$  and  $D_j^{\max}/\Delta \leq D_j^{\max}/\theta_j \leq O(T/\epsilon)$ , for  $j = 1, 2$ . Similarly,  $B_{\max}/\Delta \leq B_{\max}/\theta_3 \leq O(T/\epsilon)$ . This implies that the overall running time of the algorithm is bounded by  $O(T^8m_1^2m_2^2R_1R_2/\epsilon^5)$ , which is polynomial in the problem input size and  $1/\epsilon$ .

Next we show that AS delivers a solution that is within a relative error  $\epsilon$  from the optimality, i.e.,

$$V_1(I_1, I_2, 1, 1, R_1, R_2, 0) - R_1(I_1, I_2, 1, 1, R_1, R_2, 0) \leq \epsilon V_1(I_1, I_2, 1, 1, R_1, R_2, 0). \quad (\text{D.6})$$

By Lemma 3, we have

$$\begin{aligned} A_1(\tilde{I}_1, \tilde{I}_2, 1, 1, R_1, R_2, 0) &\geq V_1(I_1, I_2, 1, 1, R_1, R_2, 0) - Tp_1^1\{3(\Delta_1 - 1) \\ &\quad + 2(\Delta_2 - 1) + 7(\Delta - 1)\} \\ &\quad - Tp_2^1\{2(\Delta_1 - 1) + 3(\Delta_2 - 1) + 7(\Delta - 1)\} \end{aligned} \quad (\text{D.7})$$

Similarly, by Lemma 4, we have

$$\begin{aligned} R_1(I_1, I_2, 1, 1, R_1, R_2, 0) &\geq A_1(\tilde{I}_1, \tilde{I}_2, 1, 1, R_1, R_2, 0) - Tp_1^1\{T(\Delta_1 - 1) \\ &\quad + 2(\Delta_2 - 1) + 10(\Delta - 1)\} \\ &\quad - Tp_2^1\{2(\Delta_1 - 1) + T(\Delta_2 - 1) + 10(\Delta - 1)\} \end{aligned} \quad (\text{D.8})$$

By (D.7) and (D.8), we have

$$\begin{aligned} &V_1(I_1, I_2, 1, 1, R_1, R_2, 0) - R_1(I_1, I_2, 1, 1, R_1, R_2, 0) \\ &\leq Tp_1^1[(T + 3)(\Delta_1 - 1) + 4(\Delta_2 - 1) + 17(\Delta - 1)] \\ &\quad + Tp_2^1[4(\Delta_1 - 1) + (T + 3)(\Delta_2 - 1) + 17(\Delta - 1)]. \end{aligned} \quad (\text{D.9})$$

Clearly, by definition of  $D_1^{\text{total}}$  and  $D_2^{\text{total}}$ ,

$$V_1(I_1, I_2, 1, 1, R_1, R_2, 0) \geq p_1^{m_1} \min\{I_1, D_1^{\text{total}}\}, \quad (\text{D.10})$$

$$V_1(I_1, I_2, 1, 1, R_1, R_2, 0) \geq p_2^{m_2} \min\{I_2, D_2^{\text{total}}\}. \quad (\text{D.11})$$

By (D.9), (D.10) and (D.11), in order to show (D.6), it is sufficient to show that

$$\frac{Tp_1^1(T+3)(\Delta_1-1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/6, \quad (\text{D.12})$$

$$\frac{4Tp_2^1(\Delta_1-1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/6, \quad (\text{D.13})$$

$$\frac{Tp_1^1 17(\Delta-1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/6, \quad (\text{D.14})$$

$$\frac{4Tp_1^1(\Delta_2-1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \epsilon/6, \quad (\text{D.15})$$

$$\frac{Tp_2^1(T+3)(\Delta_2-1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \epsilon/6, \quad (\text{D.16})$$

$$\frac{Tp_2^1 17(\Delta-1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/6, \quad (\text{D.17})$$

We prove (D.12), (D.13) and (D.14) in the following. The relations (D.15), (D.16) and (D.17) can be proved similarly and hence we omit the proofs for them. To prove (D.12) and (D.13), we consider two cases as follows.

**Case 1:** If  $I_1 \leq D_1^{total}$ , then by the definition of  $\Delta_1$  and assumption (i), we have

$$\begin{aligned} \frac{Tp_1^1(T+3)(\Delta_1-1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} &\leq \frac{T(T+3)C_0(\Delta_1-1)}{I_1} \leq \frac{T(T+3)C_0(I_1\epsilon)/(102C_0^2T^2)}{I_1} \\ &= \frac{T+3}{102C_0T}\epsilon \leq \epsilon/6. \end{aligned}$$

This shows (D.12). The relation (D.13) can be shown exactly the same way by using the fact that  $4 \leq T+3$ .

**Case 2:** If  $I_1 > D_1^{total}$ , then by the definition of  $\Delta_1$  and assumptions (i) and (ii), we have

$$\begin{aligned} \frac{Tp_1^1(T+3)(\Delta_1-1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} &\leq \frac{T(T+3)C_0(\Delta_1-1)}{D_1^{total}} \leq \frac{T(T+3)C_0(I_1\epsilon)/(102C_0^2T^2)}{D_1^{total}} \\ &\leq \frac{T+3}{102T}\epsilon \leq \epsilon/6. \end{aligned}$$

This shows (D.12). The relation (D.13) can be shown exactly the same way by using the fact that  $4 \leq T+3$ .

To prove (D.14), by the definition of  $\Delta$ , it is sufficient to prove the following:

$$\frac{Tp_1^1 17(\theta_j-1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/6, \quad \text{for } j = 1, 2, 3$$



To this end, we prove the following three results:

$$\frac{Tp_1^1 17(\theta_1 - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/6, \quad (\text{D.18})$$

$$\frac{Tp_1^1 17(\theta_2 - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \epsilon/6, \quad (\text{D.19})$$

$$\frac{Tp_1^1 17(\theta_3 - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/6. \quad (\text{D.20})$$

To prove (D.18) and (D.20), we consider the following two cases.

**Case 1:** If  $I_1 \leq D_1^{total}$ , then by the definition of  $\theta_1$  and assumptions (i) and (iv), we have

$$\frac{Tp_1^1 17(\theta_1 - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{17TC_0(\theta_1 - 1)}{I_1} \leq \frac{17TC_0(D_1^{max}\epsilon)/(102C_0^2T)}{I_1} \leq \epsilon/6.$$

This shows (D.18). Similarly, by the definition of  $\theta_3$  and assumptions (i) and (vi),

$$\frac{Tp_1^1 17(\theta_3 - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{17TC_0(\theta_3 - 1)}{I_1} \leq \frac{17TC_0(B_{max}\epsilon)/(102C_0^2T)}{I_1} \leq \epsilon/6.$$

This shows (D.20).

**Case 2:** If  $I_1 > D_1^{total}$ , then by the definition of  $\theta_1$  and assumptions (i) and (iii), we have

$$\frac{Tp_1^1 17(\theta_1 - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{17TC_0(\theta_1 - 1)}{D_1^{total}} \leq \frac{17C_0T(D_1^{max}\epsilon)/(102C_0^2T)}{D_1^{total}} \leq \epsilon/6.$$

This shows (D.18). Similarly, by the definition of  $\theta_3$  and assumptions (i) and (v),

$$\frac{Tp_1^1 17(\theta_3 - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{17TC_0(\theta_3 - 1)}{D_1^{total}} \leq \frac{17C_0T(B_{max}\epsilon)/(102C_0^2T)}{D_1^{total}} \leq \epsilon/6.$$

This shows (D.20).

To prove (D.19), we consider the following two cases.

**Case 1:** If  $I_2 \leq D_2^{total}$ , then by the definition of  $\theta_2$  and assumptions (i) and (iv), we have

$$\frac{Tp_1^1 17(\theta_2 - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} = \frac{17Tp_1^1(\theta_2 - 1)}{p_2^{m_2} I_2} \leq \frac{17TC_0(D_2^{max}\epsilon)/(102C_0^2T)}{I_2} \leq \epsilon/6.$$

**Case 2:** If  $I_2 > D_2^{total}$ , then by the definition of  $\theta_2$  and assumptions (i) and (iii), we have

$$\frac{Tp_1^1 17(\theta_2 - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} = \frac{17Tp_1^1(\theta_2 - 1)}{p_2^{m_2} D_2^{total}} \leq \frac{17TC_0(D_2^{max}\epsilon)/(102C_0^2T)}{D_2^{total}} \leq \epsilon/6.$$

■

**Lemma 5** *In any period  $t$ , the following two inequalities hold as long as the value of each state variable involved is within its domain:*

$$V_t(i_1 - 1, i_2, l_1, l_2, r_1, r_2) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_1^1, \text{ and}$$

$$V_t(i_1, i_2 - 1, l_1, l_2, r_1, r_2) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_2^2.$$

**Proof** We prove this lemma by backward induction. We show in the following that if the result holds for time period  $t + 1$ , it also holds for time period  $t$ . We denote  $l'_1, l'_2$  as the optimal price levels for state  $(i_1, i_2, l_1, l_2, r_1, r_2)$  in period  $t$  and  $D_1^*, D_2^*$  as the corresponding worst-case demand. We denote  $D_1, D_2$  as the worst-case demand in period  $t$  for state  $(i_1 - 1, i_2, l_1, l_2, r_1, r_2)$  if price levels  $l'_1, l'_2$  are used in that period. We also denote  $i_1''' = i_1 - 1 - \min(i_1 - 1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2''' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1 - 1, D_1)) \rfloor)$ ,  $i_1'' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1^* + \lfloor \alpha_t^{l'_1 l'_2} (D_2^* - \min(i_2, D_2^*)) \rfloor)$ , and  $i_2' = i_2 - \min(i_2, D_2^* + \lfloor \beta_t^{l'_1 l'_2} (D_1^* - \min(i_1, D_1^*)) \rfloor)$ . We consider three cases in the following.

Case 1:  $i_1 - 1 \geq D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor$ .

$$\begin{aligned} & V_t(i_1 - 1, i_2, l_1, l_2, r_1, r_2) \\ & \geq p_1^* \min(i_1 - 1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\ & \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1 - 1, D_1)) \rfloor) \\ & \quad + V_{t+1}(i_1''', i_2''', l'_1, l'_2, r'_1, r'_2) \\ & \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\ & \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\ & \quad + V_{t+1}(i_1'', i_2'', l'_1, l'_2, r'_1, r'_2) - p_1^* \end{aligned}$$

$$\begin{aligned}
& \text{(by induction and the fact that } i_1''' = i_1'' - 1, i_2''' = i_2'') \\
& \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1 l_2}'(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1 l_2}'(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& \quad + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2') - p_1^{l_1} \quad \text{(by definition of } D_1^*, D_2^*) \\
& = V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_1^{l_1}
\end{aligned}$$

Case 2:  $i_1 \leq D_1 + \lfloor \alpha_t^{l_1 l_2}'(D_2 - \min(i_2, D_2)) \rfloor$  and  $\min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1 - 1, D_1)) \rfloor) = \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1, D_1)) \rfloor)$ .

$$\begin{aligned}
& V_t(i_1 - 1, i_2, l_1, l_2, r_1, r_2) \\
& \geq p_1^* \min(i_1 - 1, D_1 + \lfloor \alpha_t^{l_1 l_2}'(D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1 - 1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2') \\
& = p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1 l_2}'(D_2 - \min(i_2, D_2)) \rfloor) - p_1^* \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2') \quad \text{(note: } i_1''' = i_1'' = 0, i_2''' = i_2'') \\
& \geq p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1 l_2}'(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1 l_2}'(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& \quad + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2') - p_1^{l_1} \quad \text{(by definition of } D_1^*, D_2^*) \\
& = V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_1^{l_1}
\end{aligned}$$

Case 3: if  $i_1 \leq D_1 + \lfloor \alpha_t^{l_1 l_2}'(D_2 - \min(i_2, D_2)) \rfloor$  and  $\min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1 - 1, D_1)) \rfloor) = \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1, D_1)) \rfloor) + 1$ .

$$\begin{aligned}
& V_t(i_1 - 1, i_2, l_1, l_2, r_1, r_2) \\
& \geq p_1^* \min(i_1 - 1, D_1 + \lfloor \alpha_t^{l_1 l_2}'(D_2 - \min(i_2, D_2)) \rfloor) \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1 - 1, D_1)) \rfloor) \\
& \quad + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2') \\
& \geq p_1^* \min(i_1, D_1 + \lfloor \alpha_t^{l_1 l_2}'(D_2 - \min(i_2, D_2)) \rfloor) - p_1^* \\
& \quad + p_2^* \min(i_2, D_2 + \lfloor \beta_t^{l_1 l_2}'(D_1 - \min(i_1, D_1)) \rfloor) + p_2^*
\end{aligned}$$

$$\begin{aligned}
& +V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2') - p_2^* \\
& \quad \text{(by induction and the fact that } i_1''' = i_1'' = 0, i_2''' = i_2'' - 1) \\
\geq & p_1^* \min(i_1, D_1^* + \lfloor \alpha_t^{l_1' l_2'}(D_2^* - \min(i_2, D_2^*)) \rfloor) \\
& + p_2^* \min(i_2, D_2^* + \lfloor \beta_t^{l_1' l_2'}(D_1^* - \min(i_1, D_1^*)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2') - p_1^{l_1} \quad \text{(by definition of } D_1^*, D_2^*) \\
= & V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_1^{l_1}
\end{aligned}$$

Similarly, we can show that  $V_t(i_1, i_2 - 1, l_1, l_2, r_1, r_2) \geq V_t(i_1, i_2, l_1, l_2, r_1, r_2) - p_2^{l_2}$ .

This completes the proof. ■

**Theorem 2** *Given any state  $(i_1, i_2, l_1, l_2, r_1, r_2)$  in the beginning of any period  $t$ , for any feasible price pair  $(l_1', l_2')$  chosen for period  $t$  (i.e.,  $l_1' \in F_1^{l_1} \cap \{l_1, \dots, m_1\}$ , and  $l_2' \in F_2^{l_2} \cap \{l_2, \dots, m_2\}$ ), and any demand realization  $(D_1, D_2), (D_1 - 1, D_2), (D_1, D_2 - 1) \in \Omega_t^{l_1' l_2'}$ , the following results hold:*

$$\begin{aligned}
& p_1^{l_1'} \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l_2'} \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2') \\
\leq & p_1^{l_1'} \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l_2'} \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2'), \tag{D.21}
\end{aligned}$$

and

$$\begin{aligned}
& p_1^{l_1'} \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - 1 - \min(i_2, D_2 - 1)) \rfloor) \\
& + p_2^{l_2'} \min(i_2, D_2 - 1 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1''', i_2''', l_1', l_2', r_1', r_2') \\
\leq & p_1^{l_1'} \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l_2'} \min(i_2, D_2 + \lfloor \beta_t^{l_1' l_2'}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2'), \tag{D.22}
\end{aligned}$$

where  $i_1'' = i_1 - \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2'' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor)$ ,  $i_1' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$ ,  $i_2' = i_2 - \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - \min(i_1, D_1)) \rfloor)$ ,  $i_1''' = i_1 - \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - 1 - \min(i_2, D_2 - 1)) \rfloor)$ , and  $i_2''' = i_2 - \min(i_2, D_2 - 1 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - \min(i_1, D_1)) \rfloor)$ .

**Proof** We prove this theorem by backward induction. We show that if the theorem holds for time period  $t+1$ , it also holds for time period  $t$ . We consider three possible cases in the following.

Case 1:  $\min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) = \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)$  and  $\min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) = \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - \min(i_1, D_1)) \rfloor)$ , then,

$$\begin{aligned}
& p_1^{l_1'} \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l_2''} \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2') \\
= & p_1^{l_1'} \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l_2''} \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2') \quad (\text{note: } i_1'' = i_1', i_2'' = i_2')
\end{aligned}$$

Case 2:  $\min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) = \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) - 1$  and  $\min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) = \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - \min(i_1, D_1)) \rfloor)$ , then,

$$\begin{aligned}
& p_1^{l_1'} \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l_2''} \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2') \\
\leq & p_1^{l_1'} \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor) - p_1^{l_1'} \\
& + p_2^{l_2''} \min(i_2, D_2 + \lfloor \beta_t^{l_1'' l_2''}(D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2') + p_1^{l_1'} \\
& \quad (\text{by Lemma 5 and the fact that } i_1'' = i_1' + 1, i_2'' = i_2') \\
= & p_1^{l_1'} \min(i_1, D_1 + \lfloor \alpha_t^{l_1' l_2'}(D_2 - \min(i_2, D_2)) \rfloor)
\end{aligned}$$

$$\begin{aligned}
& + p_2^{l'_2} \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2)
\end{aligned}$$

Case 3:  $\min(i_1, D_1 - 1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) = \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor)$  and  $\min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) = \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) - 1$ , then,

$$\begin{aligned}
& p_1^{l'_1} \min(i_1, D_1 - 1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l'_2} \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - 1 - \min(i_1, D_1 - 1)) \rfloor) \\
& + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2) \\
\leq & p_1^{l'_1} \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l'_2} \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) - p_2^{l'_2} \\
& + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2) + p_2^{l'_2} \\
& \quad (\text{by Lemma 5 and the fact that } i''_1 = i'_1, i''_2 = i'_2 + 1) \\
= & p_1^{l'_1} \min(i_1, D_1 + \lfloor \alpha_t^{l'_1 l'_2} (D_2 - \min(i_2, D_2)) \rfloor) \\
& + p_2^{l'_2} \min(i_2, D_2 + \lfloor \beta_t^{l'_1 l'_2} (D_1 - \min(i_1, D_1)) \rfloor) \\
& + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2)
\end{aligned}$$

This completes the proof for (D.21). We can prove (D.22) similarly. ■

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