ABSTRACT

Title of dissertation: The Cohomological Equation for Horocycle Maps and Quantitative Equidistribution

James Tanis, Doctor of Philosophy, 2011

Dissertation directed by: Giovanni Forni
Department of Mathematics

There are infinitely many distributional obstructions to the existence of smooth solutions for the cohomological equation $u \circ \phi_1 - u = f$ in each irreducible component of $L^2(\Gamma\backslash PSL(2, R))$, where ϕ_1 is the time-one map of the horocycle flow. We study the regularity of these obstructions, determine which ones also obstruct the existence of L^2 solutions and prove a Sobolev estimate of the solution in terms of f. As an application, we estimate the rate of equidistribution of horocycle maps on compact, finite volume manifolds $\Gamma\backslash PSL(2, R)$ using an auxiliary result from Flaminio-Forni(2003) and one from Venkatesh(2010) concerning the horocycle flow and the twisted horocycle flow, respectively.

The Cohomological Equation for Horocycle Maps and Quantitative Equidistribution

by

James Tanis

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree
of Doctor of Philosophy
2011

Advisory Committee:
Professor Giovanni Forni, Chair/Advisor
Professor Mike Boyle
Professor Michael Jakobson
Professor Lawrence Washington
Professor Thomas Cohen

Acknowledgments

I am indebted to my advisor, Giovanni Forni, for very many helpful discussions, and I am grateful to my parents for their constant support.

Table of Contents

1 Introduction 1

2 Bases and Distributions 14
- 2.1 Orthogonal Bases 14
- 2.2 Relevant Distributions 20

3 Cohomological equation 25
- 3.1 Cohomological equation for the principal and complementary series . 25
- 3.2 Cohomological Equation for the Discrete Series 37

4 Obstructions to L^2 Solutions 54

5 Equidistribution of Horocycle Maps 63

A Appendix A 73

B Appendix B 81

Bibliography 105

Chapter 1

Introduction

We say that f is a *coboundary* for a *flow* $\{\phi_t\}_{t\in R}$ if there is a *transfer function* u such that

$$\frac{d}{dt} u \circ \phi_t\big|_{t=0} = f,$$

and for $T > 0$, f is a coboundary for the *map* ϕ_T if

$$u \circ \phi_T - u = f.$$

In this paper we study the discrete analogue of the classical horocycle flow $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t\in R}$ called the classical horocycle map $\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$, each acting by right multiplication on (compact) homogeneous spaces of the form $\Gamma\backslash PSL(2, R)$. Motivated by the success of using cohomological equations to prove quantitative equidistribution of horocycle flows and nilflows [2], [3], we study the cohomological equation for horocycle maps and quantitative equidistribution.

Representation theory is a natural tool for cohomological equations on homogeneous spaces [2], [3], [10]. Flaminio-Forni's (2003, [2]) detailed analysis of the cohomological equation for the horocycle flow was carried out through its representations in the irreducible, unitary components of $L^2(\Gamma\backslash PSL(2, R))$. We take this approach for the cohomological equation of horocycle maps, but the equation for maps is different from that of flows in an important sense. Bargman's well-known ladder argument allows one to construct a basis in each irreducible component, and [2] shows U represents as an off diagonal matrix in this basis, so that the cohomological equation $Uu = f$ can be solved algebraically by a two-step difference equation. In contrast, the matrix ϕ_T^U is very complicated in this basis, so we instead solve using standard representation models, where vector fields and flows appear as explicit formulas in explicit Sobolev spaces.

Solutions to cohomological equations on homogeneous spaces may or may not have distributional obstructions. Results so far indicate cocycles over higher rank abelian hyperbolic actions of R^d or Z^d are typically either rigid or the cohomology classes are finite in number and easy to describe [4], [8], [10], [12], [16], [5].

In contrast, previous results on cohomological equations for homogeneous R or Z actions show there are infinitely many independent distributional obstructions to the existence of L^2 solutions. Consistent with this picture, we find there are infinitely many distributional obstructions for the horocycle map. Lastly, previous results also show some finite loss of regularity between the Sobolev estimates of the transfer function and its coboundary for horocycle maps (see for example [2], [3], [6]), and we prove this is also true for horocycle maps.

In the second part, we use our analysis of the cohomological equation of horocycle maps to study the equdistribution of the horocycle flow. Horocycle flows are known to have zero entropy, and precise mixing rates for geodesic and horocycle flows were obtained by Ratner [17] and Moore [15], and Ratner proved horocycle flows have polynomial decay of correlations. Concerning ergodicity, Furstenberg [7] proved the horocycle flow is uniquely ergodic (i.e. every orbit equidistributes) in 1970. M. Burger [18] estimated the rate of unique ergodicity for sufficiently smooth functions along oribits of horocycle flows on compact surfaces and on open complete surfaces of positive injectivity radius. P. Sarnak [22] obtained asymptotics for the rate of unique ergodicity of cuspidal horocycles on noncompact surfaces of finite area using a method based on Eisenstein series. For sufficiently regular functions, Flaminio-Forni [2] improved on Burger's estimate for compact surfaces by establishing precise asymptotics in this setting, and in the case of noncompact, finite area surfaces, they generalize the result of P. Sarnak to arbitrary horocycle arcs.

Quantitative equidistribution results for horocycle maps are very recent. Shah's conjecture states that for all $\delta > 0$, the horocycle map $\{\phi^U_{n^\delta}\}_{n\in Z^+}$ equidistributes in $\Gamma\backslash PSL(2, R)$. Venkatesh [23] used a method to upgrade quantitative equidistribution and quantitative mixing to prove upper bounds on the equidistribution rate of the twisted horocycle flow $\{\phi^U_t \times e^{2\pi i t}\}_{t\in R}$ on $SM \times S^1$. By an ingenious argument, he used this to prove $\{\phi^U_{n^{1+\delta}}\}_{n\in Z^+}$ equidistributes in $\Gamma\backslash PSL(2, R)$ whenever $0 \leq \delta < \delta_\Gamma$ for some explicit threshold $1 < \delta_\Gamma << 2$.

We estimate the rate of equidistribution for the horocycle map. As in [2], we use our estimate of the cohomological equation for the map to obtain a rate of equidistribution for coboundaries, and we use the analysis of the flow invariant distributions for the horocycle flow in [2] to estimate the rate of decay for the flow invariant distributions of the map. We use Venkatesh's estimate of the equidistribution of the twisted horocycle flow in [23] to estimate the invariant distributions of the map that are not flow invariant. Then because the ergodic sum of every regular enough function is either controlled by the cohomological equation or is one of the invariant distributions, we conclude our quantitative equidistribution result.

Preliminary definitions

The Poincaré upper half plane is

$$H = \{z \in C | \Im(z) > 0, |dz|^2/(\Im z)^2\}.$$

If $\Gamma \subset PSL(2,R)$ is a discrete subgroup acting without fixed points, then $M := \Gamma \backslash H$ is a Riemannian manifold of constant negative curvature. Let SH be the unit tangent bundle of H. Then fixing $(i,i) \in SH$, the map

$$PSL(2,R) : (i,i) \to SH$$

gives the identification $PSL(2,R) \approx SH$. The elements of the Lie algebra $sl(2,R)$ of $PSL(2,R)$ generate some flows on $\Gamma \backslash SH \approx \Gamma \backslash PSL(2,R) := SM$.

The matrices

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} and V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.1}$$

in $sl(2,R)$ are the stable and unstable "horocycle vector fields" in the sense that they generate flows

$$\phi_t^U := e^{tU} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} and \phi_t^V := e^{tV} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

that correspond to the stable and unstable horocycle flows on SH, respectively. These flows act by right multiplication

$$\phi_t^U(\Gamma x) = \Gamma(x\phi_t^U),$$

where $x \in PSL(2, R)$. Let $T > 0$. and define

$$L_T u := u \circ \phi_T^U - u.$$

The main result in this paper is to find coboundaries for the cohomological equation

$$L_T u = f \tag{1.2}$$

and obtain a Sobolev estimate of the transfer function u in terms of the coboundary f.

Harmonic analysis

Elements of $sl(2, R)$ generate some area preserving flows on SM, and we choose a basis for $sl(2, R)$ to be

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.3}$$

which are generators for the geodesic, orthogonal geodesic and circle vector fields respectively. These generators satisfy the commutation rules

$$[X, Y] = 2\Theta, \quad [\Theta, X] = -2Y, \quad [\Theta, Y] = 2X,$$

and note

$$U = \frac{Y + \Theta}{2} and V = \frac{Y - \Theta}{2}.$$

Let $\mathcal{H}$ be a unitary representation space of $PSL(2, R)$. Each of these vector fields is an essentially skew-adjoint operator on $\mathcal{H}$, so their square is essentially self-adjoint on $\mathcal{H}$. The *Laplacian* $\triangle$ is an essentially self-adjoint operator and the elliptic element of the enveloping algebra of $sl(2,R)$ is defined by

$$I + \triangle := I - (X^2 + Y^2 + \Theta^2).$$

The *Casimir* operator

$$\square := \triangle + 2\Theta^2$$

generates the center of the enveloping algebra for $sl(2,R)$. As such, it acts as a constant $\mu \in R$ on each irreducible, unitary representation space $\mathcal{K}_\mu$, and its value classifies them into three classes. The representation $\mathcal{K}_\mu$ belongs to the *principal series* if $\mu \geq 1$, the *complementary series* if $0 < \mu < 1$ and the *discrete series* if $\mu \in \{-n^2 + n | n \in Z^+\}$. We note that some authors scale the vector fields so that $\mathcal{K}_\mu$ with $\mu \geq 1/4$ are in the principal series [2]. We do not use this convention. In fact, our geodesic flow has speed 2 with respect to the hyperbolic metric of constant curvature -1.

When SM is compact, standard elliptic theory shows $spec(\triangle)$ is pure point and discrete, with eigenvalues of finite multiplicity. When SM is not compact, $spec(\triangle)$ is Lebesgue on $[1, \infty)$ with multiplicity equal to the number of cusps, has possibly embedded eigenvalues of finite multiplicity in $[1, \infty)$, and has at most finitely many eigenvalues of finite multiplicity in (0, 1) (see [22]).

There is a standard unitary representation of $PSL(2,R)$ on the separable Hilbert space $L^2(SM)$ of square integrable functions with respect to the $PSL(2,R)$ invariant volume form on SM. As in Flaminio-Forni (2003), the Laplacian gives unitary representation spaces a natural Sobolev structure. The *Sobolev space of order* $r >$

0 is the Hilbert space $W^r(SM) \subset L^2(SM)$ that is the maximal domain determined by the inner product

$$\langle f, g\rangle_{W^r(SM)} := \langle (1+\triangle)^r f, g\rangle_{L^2(M)}$$

for $f, g \in L^2(SM)$.

The space of *infinitely differentiable functions* is

$$C^\infty(SM) := \cap_{r\geq 0} W^r(SM).$$

For $r > 0$, the distributional dual to $W^r(SM)$ is the Sobolev space $W^{-r}(SM) = (W^{-r}(SM))^*$. The distributional dual to $C^\infty(SM)$ is

$$\mathcal{E}'(SM) := (C^\infty(SM))^*.$$

Because the Casimir operator is the center of the enveloping algebra and acts as an essentially self-adjoint operator, any non-trivial unitary representation $\mathcal{H}$ for $PSL(2,R)$ has a $PSL(2,R)$-invariant direct integral decomposition

$$\mathcal{H} = \int_{\mu \in spec(\Box)} \mathcal{K}_\mu d\beta(\mu), \tag{1.4}$$

where $d\beta(\mu)$ is a Stiltjes measure over the spectrum $spec(\Box)$ (see[13], [22]). The space $\mathcal{K}_\mu$ does not need to be irreducible but is generally a direct sum of an at most countable number of irreducible components given by the spectral multiplicity of $\mu \in spec(\Box)$.

Additionally, all operators in the enveloping algebra are decomposable with respect to the direct integral decomposition (1.4). Then in particular,

$$L^2(SM) = \int_{\oplus_\mu} \mathcal{K}_\mu,$$

and for $r \in R$,

$$W^r(SM) = \int_{\oplus_\mu} W^r(\mathcal{K}_\mu). \tag{1.5}$$

When SM is compact, $L^2(SM)$ decomposes into a direct sum.

Statement of results

$\underline{The Cohomological Equation}$

Let

$$\mu_0 = \inf\{\mu \in spec(\Box) | \mu > 0\}.$$

We consider manifolds SM with a *spectral gap* $\mu_0 > 0$. Let $(x, T) \in SM \times N$ and $r \geq 0$. Let

$$\mathcal{I}(SM) := \{\mathcal{D} \in \mathcal{E}'(SM) : L_T\mathcal{D} = 0\}$$

be the space of invariant distributions for L_T, and let

$$\mathcal{I}_\mu := \mathcal{I}_{SM} := \mathcal{I}(SM) \cap \mathcal{E}^\infty(\mathcal{K}_\mu).$$

By (1.5),

$$\mathcal{I}(SM) = \int_\oplus \mathcal{I}_\mu(SM).$$

Let $\epsilon > 0$ and $\mu_0 > 0$. For all $\mu \in spec(\Box)$, the space $\mathcal{I}_\mu$ has infinite countable dimension.

For $\mu > 0$, $\mathcal{I}_\mu \subset W^{-((1+\Re\sqrt{1-\mu})/2+\epsilon)}(\mathcal{K}_\mu)$.

When $\mu \leq 0$, there is an infinite basis $\{\mathcal{D}_n\}_{n\in N} \cup \{\mathcal{D}^0\} \subset \mathcal{I}_\mu$ of T-invariant distributions such that $\mathcal{D}^0 \in W^{-((1+\Re\sqrt{1-\mu})/2+\epsilon)}(\mathcal{K}_\mu)$ is the flow invariant distribution studied in [2] and $\langle\{\mathcal{D}_n\}_{n\in N}\rangle \subset W^{-(1/2+\epsilon)}(\mathcal{K}_\mu)$.

It will follow from Theorem 1 that the invariant distributions classify the space of coboundaries that have smooth solutions.

Let $\mu_0 > 0, T > 0, r \geq 0$ and $f \in W^{3r+4}(SM) \cap Ann(\mathcal{I}(SM))$. Then there is a unique $L^2(SM)$ solution u to

$$L_T u = f$$

and a constant $C_{r,T} > 0$ such that

$$\|u\|_{W^r(SM)} \leq C_{r,T,SM}\|f\|_{W^{3r+4}(SM)}. \tag{1.6}$$

If $\mathcal{D}$ is an invariant distribution and $u \in C^\infty(SM)$, then from definitions we conclude

$$\mathcal{D}(f) = \mathcal{D}(u \circ \phi^U_T) - \mathcal{D}(u) = 0.$$

In this sense, $\mathcal{D}$ *obstructs* the existence of smooth solutions. Theorem 1 gives the invariant distributions that obstruct the existence of $\mathcal{K}_\mu$ solutions for regular enough coboundaries f.

Let $\mu_0 > 0$, $f \in W^9(SM)$ and $\mathcal{D} \in \int_{\oplus\mu\geq 0}\mathcal{I}_\mu$. If there exists $u \in L^2(SM)$ such that $L_T u = f$, then $\mathcal{D}(f) = 0$. Moreover, this is not true for $\mu < 0$.

Theorem 1 was also proven in [2] for the horocycle flow. We prove estimate (1.6) on every irreducible component, and then glue the solutions together. Explicitly, suppose we are given $0 \leq r < t$, $\{u_\mu\}_\mu, \{f_\mu\}_\mu \in \int_{\oplus_\mu}\mathcal{K}_\mu$ and a constant $C_{r,t} > 0$

such that for all $\mu \in spec(\Box)$,

$$\|u_\mu\|_{W^r(\mathcal{K}_\mu)} \le C_{r,t}\|f_\mu\|_{W^t(\mathcal{K}_\mu)}. \tag{1.7}$$

Write

$$u = \int_{\oplus_\mu} u_\mu \qquad f = \int_{\oplus_\mu} f_\mu,$$

and observe

$$\begin{aligned}\|u\|^2_{W^r(SM)} = \|\int_{\oplus_\mu} u_\mu\|^2_{W^r(\mathcal{K}_\mu)} &= \int_{\oplus_\mu} \|u_\mu\|^2_{W^r(\mathcal{K}_\mu)} \\ \le C^2_{r,t}\int_{\oplus_\mu} \|f_\mu\|^2_{W^t(\mathcal{K}_\mu)} &\le C^2_{r,t}\|f\|^2_{W^t(SM)}.\end{aligned}$$

It therefore suffices to establish (1.7). We do not rely soley on the algebraic properties of $PSL(2,R)$, as in [2], [17]. We instead do all calculations in certain unitarily equivalent, standard models $\mathcal{H}_\mu \simeq \mathcal{K}_\mu$, where vector fields and flows are given by explicit formulas in explicit Sobolev spaces. The unitary equivalence

$$Q_\mu : \mathcal{K}_\mu \to \mathcal{H}_\mu$$

intertwines vector fields on each space, so that Q_μ preserves Sobolev norms. Therefore, the Sobolev norms we calculate in $\mathcal{H}_\mu$ pass back to $\mathcal{K}_\mu$.

The *key idea* in our calculation is to introduce a finite dimensional space Y of additional distributions with the property that for all functions in $Ann(Y)$, the estimate (1.7) is substantially easier to prove. Then we remove these distributions using dual basis to Y consisting of explicit coboundaries and obtain (1.7) for each dual basis element. Combining gives the estimate.

Horocycle maps and the horocycle flow are heuristically related by statement that when $\mu \in spec(\Box)$ and $f \in C^\infty(\mathcal{K}_\mu)$, there exists $u \in \mathcal{K}_\mu$ such that

$$Uu = f if and only if u \circ \phi_T^U - u = A_T(f), \tag{1.8}$$

where

$$A_T(f) = \int_0^T f \circ \phi_t^U dt.$$

In [2], Flaminio-Forni showed that the space of flow invariant distributions is at most two dimensional in any irreducible component in contrast to Theorem 1, which states that the space of T- invariant distributions in each irreducible component is infinite dimensional. Formula (1.8) already suggests this. Roughly speaking, if h and $\hat{h}$ are smooth, then the spectral theorem gives

$$\mathcal{D}(\int_0^T h \circ \phi_t dt) = \hat{\mathcal{D}}(\int_0^T e^{-2\pi it\xi} d\lambda(t)\hat{h}) = 0, \tag{1.9}$$

for some spectral measure λ. So $\hat{\mathcal{D}} = \delta_{n/T}$ is an invariant distribution for all $n \in Z$.

The statement (1.8) suggests that one can obtain (1.6) from Flaminio-Forni's result on the flow. We instead derive (1.6) directly from horocycle maps, because the approach suggested by (1.8) did not give an advantage in either the length or simplicity of the argument, nor did it improve the sharpness of (1.6).

$\underline{Rate of Equidistribution}$:

As an application to the above analysis of the cohomological equation, we prove a rate of equidistribution for horocycle maps. Let

$$\alpha(\mu_0) = \frac{(1-\sqrt{1-\mu_0})^2}{4(3-\sqrt{1-\mu_0})},$$

where μ_0 is the spectral gap. For all $\mu > 0$ and $\mathcal{D} \in \{\mathcal{D}_k\} \cup \{\mathcal{D}^0\} \subset \mathcal{I}_\mu$, define

$$\mathcal{S}_{\mathcal{D}} := \begin{cases} \alpha(\mu_0) if \mathcal{D} = \mathcal{D}_k,\ k \neq 0 \\ \frac{1-\sqrt{1-\mu}}{2} if \mathcal{D} = \mathcal{D}_0 \\ \frac{1+\sqrt{1-\mu}}{2} if \mathcal{D} = \mathcal{D}^0. \end{cases}$$

Let ϕ_1^U be the horocycle map on the unit tangent bundle SM of a compact hyperbolic Riemann surface M with spectral gap $\mu_0 > 0$, and let $s \geq 6$. Then there is a constant $C_s > 0$ such that for all $(x_0, N) \in SM \times Z^+$ and $f \in W^{s+3}(SM)$, we have

$$|\frac{1}{N}\sum_{k=0}^{N-1} f(\phi_k^U x_0)| \leq |\bigoplus_{\mu \in spec(\Box)} \sum_{\mathcal{D} \in \mathcal{I}_\mu(SM)} c_{\mathcal{D}}(x_0, N, s)\mathcal{D}(f) N^{-\mathcal{S}_{\mathcal{D}}} \oplus \mathcal{R}(x, N, s)(f)|, \tag{1.10}$$

where the remainder distribution $\mathcal{R}(x, N, s)$ satisfies

$$\|\mathcal{R}(x, N, s)\|_{W^{-s}(\mathcal{K}_\mu)} \leq \frac{C_s}{N},$$

and for all $\mathcal{D} \in \mathcal{I}(SM)$,

$$|c_{\mathcal{D}}| \leq C_s.$$

We control the remainder using the Sobolev estimate for the cohomological equation of horocycle maps in Theorem 1. For $\tau \in Z$, the decay rate of the invariant distributions is obtained from the identity

$$\frac{1}{N}\int_0^N e^{-2\pi i\tau t}(\phi_t^U x)^* dt = \int_0^1 e^{-2\pi i \tau t}\frac{1}{N}\sum_{n=0}^{N-1} \delta_{\phi_{n+t}^U x} dt$$

together with Theorem 1.5 of [2] and Lemma 3.1 of [23].

Chapter 2

Bases and Distributions

2.1 Orthogonal Bases

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, R).$$

$\underline{Principal and complementary series}$

For Casimir parameter $\mu > 0$, let $\mathcal{H}_\mu$ be a model principal and complementary series representation space. The group action is defined by

$$\pi_\nu : PSL(2, R) \to \mathcal{B}(\mathcal{H}_\mu)$$

$$\pi_\nu(A)f(x) = |cx + a|^{-(\nu+1)} f(\frac{dx - b}{-cx + a}),$$

where $x \in R$ and ν is a representation parameter. When $\mu \geq 1$, then $\nu = \pm\sqrt{1 - \mu}$ and

$$\|f\|_{\mathcal{H}_\mu} = \|f\|_{L^2(R)}.$$

When $0 < \mu < 1$, then $\nu = \sqrt{1 - \mu}$ and

$$\|f\|_{\mathcal{H}_\mu} = \left(\int_{R^2} \frac{f(x)\overline{f(y)}}{|x - y|^{1-\nu}} dxdy \right)^{1/2}.$$

By the change of variable $x = \tan(\theta)$, we have the circle models

$$\mathcal{H}_\mu = L^2([\frac{-\pi}{2}, \frac{\pi}{2}], \frac{d\theta}{\cos^2(\theta)}])$$

for the principal series, and

$$\|f\|_{\mathcal{H}_\mu} = \left(\int_{[-\pi/2,\pi/2]^2} \frac{f(\tan\theta) f(\tan\theta')}{|\tan\theta - \tan\theta'|^{1-\nu}} \frac{d\theta d\theta'}{\cos^2(\theta)\cos^2(\theta')} \right)^{1/2}.$$

Computing derived representations, we get Let $\mu > 0$. The vector fields for the $\mathcal{H}_\mu$ model on R are

$$X = d\pi_\nu(X) = -(1+\nu) - 2x\frac{\partial}{\partial x}$$

$$\Theta = d\pi_\nu(\Theta) = -(1+\nu)x - (1+x^2)\frac{\partial}{\partial x}$$

$$Y = d\pi_\nu(Y) = (1+\nu)x - (1-x^2)\frac{\partial}{\partial x}$$

$$U = d\pi_\nu(U) = -\frac{\partial}{\partial x}$$

$$V = d\pi_\nu(V) = (1+\nu)x + x^2\frac{\partial}{\partial x}.$$

By the change of variable $x = \tan(\theta)$, the vector fields in the circle model are

$$X = d\pi_\nu(X) = -(1+\nu) - \sin(2\theta)\tfrac{\partial}{\partial\theta}$$

$$\Theta = d\pi_\nu(\Theta) = -(1+\nu)\tan(\theta) - \tfrac{\partial}{\partial\theta}$$

$$Y = d\pi_\nu(Y) = (1+\nu)\tan(\theta) - \cos(2\theta)\tfrac{\partial}{\partial\theta}$$

$$U = d\pi_\nu(U) = -\cos^2(\theta)\tfrac{\partial}{\partial\theta}$$

$$V = d\pi_\nu(V) = (1+\nu)\tan(\theta) + \sin^2(\theta)\tfrac{\partial}{\partial\theta}.$$

Remark : We denote both the R-model and the circle model by $\mathcal{H}_\mu$.

Discreteseries

For $\mu \leq 0$, we denote $L^2_{hol}(H, d\lambda_\nu)$ to be the upper half-plane model for the holomorphic discrete series, where $d\lambda_\nu := y^{\nu-1}dxdy$ and $\nu = \sqrt{1-\mu} \in \{2n-1\}_{n\in Z^+}$ is the representation parameter. This model has the group action

$$\pi_\nu : PSL(2,R) \to \mathcal{B}\left(L^2_{hol}(H, d\lambda_\nu)\right)$$

$$\pi_\nu(A) : f(z) \to (-cz+a)^{-(\nu+1)} f\left(\frac{dz-b}{-cz+a}\right). \tag{2.1}$$

The anti-holomorphic discrete series case occurs when $\nu = -\sqrt{1-\mu} < 0$, but we safely restrict ourselves to the holomorphic discrete series case because there is a complex anti-linear isomorphism between the two series of the same Casimir parameter. The space $L^2_{hol}(H, d\lambda_\nu)$ is said to be of *lowest weight* $n := \frac{1+\nu}{2}$.

Define

$$\alpha : D \to H : \xi \to -i\frac{\xi+1}{\xi-1}$$

to be a conformal map between D and H. For each $\nu \geq 1$, the unit disk model for the holomorphic discrete series is denoted $L^2_{hol}(D, d\sigma_\nu)$ and has the measure $d\sigma_\nu := 4^{-\nu}(1-|\xi|^2)^\nu dudv$. By [21],

$$T_\nu : L^2_{hol}(H, d\lambda_\nu) \to L^2_{hol}(D, d\sigma_\nu) : f(z) \to f\circ\alpha(\xi)\left(\frac{-2i}{\xi-1}\right)^{\nu+1}$$

is an isometry between the two models.

Let $\mu \leq 0$. Then the vector fields in $L^2_{hol}(H, d\lambda_\nu)$ are:

$$X = d\pi_\nu(X) = -(1+\nu) - 2z\tfrac{\partial}{\partial z}$$

$$\Theta = d\pi_\nu(\Theta) = -(1+\nu)z - (1+z^2)\tfrac{\partial}{\partial z}$$

$$Y = d\pi_\nu(Y) = (1+\nu)z - (1-z^2)\tfrac{\partial}{\partial z}$$

$$U = d\pi_\nu(U) = -\tfrac{\partial}{\partial z}$$

$$V = d\pi_\nu(V) = (1+\nu)z + z^2\tfrac{\partial}{\partial z}.$$

By changing variables via the linear fractional transformation α, the vector fields for $L^2_{hol}(D, d\sigma_\nu)$ are:

$$X = d\pi_\nu(X) = -(1+\nu) + (\xi^2 - 1)\tfrac{d}{d\xi}$$

$$\Theta = d\pi_\nu(\Theta) = (1+\nu)i(\tfrac{\xi+1}{\xi-1}) - 2i\xi\tfrac{d}{d\xi}$$

$$Y = d\pi_\nu(Y) = -(1+\nu)i(\tfrac{\xi+1}{\xi-1}) + i(\xi^2+1)\tfrac{d}{d\xi}$$

$$U = d\pi_\nu(U) = i\tfrac{(\xi-1)^2}{2}\tfrac{d}{d\xi}$$

$$V = d\pi_\nu(V) = -(1+\nu)i(\tfrac{\xi+1}{\xi-1}) + i\tfrac{(\xi+1)^2}{2}\tfrac{d}{d\xi}$$

The vectors fields in Claim 2.1 and Claim 2.1 yield the commutation relations

$$[X, Y] = 2\Theta,\ [Y, \Theta] = -2X,\ [X, \Theta] = 2Y,$$

which agree with the commutation relations we get by matrix multiplication.

$\underline{Orthogonalbasis}$

Given $u_0 \in Ker(\Theta)$, we generate the rest of the basis elements by applying the annihilation and creation operators $X \pm iY = \eta_\pm$. Here,

$$[-i\Theta, \eta_+] = i[\eta_+, \Theta] = i[X + iY, \Theta]$$

$$= i([X, \Theta] + i[Y, \Theta]) = i(2Y - i2X) = 2(X + iY) = 2\eta_+.$$

The corresponding statement also holds for $[-i\Theta, \eta_-]$. Suppose $-i\Theta f = kf$. Then

$$-i\Theta(\eta_+ f) = \eta_+(-i\Theta f) + [-i\Theta, \eta_+]f$$

$$= \eta_+(-i\Theta f) + 2\eta_+ f(\theta) = (k+2)\eta_+ f(\theta). \tag{2.2}$$

This procedure generates a family $\{(\eta_\pm)^n f\}$ of orthogonal eigenfunctions for $-i\Theta$ that is a basis for $\mathcal{H}_\mu$ or $L^2_{hol}(H, d\lambda_\nu)$ when $\mu > 0$ or $\mu \leq 0$, respectively.

We calculate concrete formulas for the orthogonal basis vectors $\{u_k\}$ in Appendix A, and we present them here.

(i) Let $\mu > 0$. Then the set $\{u_k = e^{-2ik\theta} \cos^{1+\nu}(\theta)\}_{k \in Z}$ is an orthogonal basis for $\mathcal{H}_\mu$. Moreover, If $\mu \geq 1$, then for all k,

$$\|u_k\|^2 = 1.$$

For $0 < \mu < 1$, there is a constant $C_{SM} > 0$ such that

$$C_{SM}^{-1} \left(\frac{1-\nu}{1+\nu}\right) (1+|k|)^{-\nu} \leq \|u_k\|^2 \leq C_{SM} \left(\frac{1-\nu}{1+\nu}\right) (1+|k|)^{-\nu}.$$

(ii) Let $\mu \leq 0$ and $n = \frac{1+\nu}{2}$ be the lowest weight. Then $\{u_k = \left(\frac{z-i}{z+i}\right)^{k-n} \frac{1}{(z+i)^{\nu+1}}\}_{k=n}^{\infty}$ is an orthogonal basis, and for all $k \geq n$,

$$\|u_k\|^2_{L^2(H, d\sigma_\nu)} = \frac{\pi}{(\nu+1) \cdot 4^\nu} \left(\frac{(k-n)!\nu!}{(k+n-1)!}\right).$$

Statement (i) regarding $\|u_k\|$ can be shown from the calculations in Appendix A together with Lemma 2.1 from [2]. The statement (ii) regarding $\|u_k\|$ is proven in Claim .

2.2 Relevant Distributions

Invariant distributions in our model

$\underline{Principal and complementary series}$

Let $\mu > 0$ and write $f = \Phi \cdot \cos^{1+\nu} \in \mathcal{H}_\mu$ in circle coordinates, where

$$\Phi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-2in\theta}.$$

Then define

$$\delta^{(0)}(f) := \Phi(\frac{\pi}{2}),$$

and now using formula (40) and Theorem 1.1 of [2] one shows that $\delta^{(0)} \in W^{-((1+\nu)/2+\epsilon)}(\mathcal{H}_\mu)$.

For $k \in Z$ and $f \in C_c^\infty(R)$, there are invariant distributions given by

$$\hat{\delta}_{k/T}(f) = \int_R f(x) e^{-2\pi ik/Tx} dx.$$

Observe that when $\mu \geq 1$, basis vectors are not in $L^1(R)$, so the Fourier transform is not immediately defined on $C^\infty(\mathcal{H}_\mu)$. Note that $\delta^{(0}$ is continuous on $C^\infty(\mathcal{H}_\mu)$ and for all basis vectors $\{u_n\} \in Z$, $\delta^{(0)}(u_n) \in \{-1, 1\}$. On the other hand $\delta^{(0)}(f) = 0$ for all $f \in C^\infty(R)$, so $C_c^\infty(R)$ is not dense in $C^\infty(\mathcal{H}_\mu)$. Therefore, we cannot extend $\hat{\delta}_{k/T}$ from $C_c^\infty(R)$ to $C^\infty(\mathcal{H}_\mu)$ by density. By Proposition 2.2, functions in

$Ker(\delta^{(0)})$ are in $L^1(R)$, so we extend the definition of the Fourier transform $\mathcal{F}$ to any $f \in W^{(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)$ by

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(f - \delta^{(0)}(f)\cos^{1+\nu} \circ \arctan). \tag{2.3}$$

With this definition, Proposition 2.2 proves

Let $\mu > 0, \epsilon > 0, T > 0$ and $n \in Z$. Then $\hat{\delta}_{k/T} \in W^{-((1+\Re\nu)/2+\epsilon)}(\mathcal{H}_\mu)$.

<u>*discreteseries*</u>

The discrete series case is similar to the principal and complementary series cases. Let $\mu \leq 0$ and $n = \frac{\nu+1}{2}$ be the lowest weight. By the change of variable $\xi = \left(\frac{z-i}{z+i}\right)$, the basis for $L^2(H, d\lambda_\nu)$ written in the unit disk model $L^2(D, d\sigma_\nu)$ is $\{\xi^{k-n}(\xi-1)^{\nu+1}\}_{k=n}^{\infty}$. Then $f = \Phi \cdot u_0 \in \mathcal{H}_\mu$, where

$$\Phi(\xi) = \sum_{k=n}^{\infty} c_k \xi^{k-n}.$$

Then define $\delta^{(0)}(f) := \Phi(1)$, so $\delta^{(0)} \in W^{-((1+\nu)/2+\epsilon)}(\mathcal{H}_\mu)$, again by formula (40) and Theorem 1.1 of [2].

For $k \in Z$, there are distributions given by Fourier transforms of delta distributions along $R \times \{iy\}$. For $f \in W^{1/2+\epsilon}(\mathcal{H}_\mu)$, $k \in Z$ and $y \in R^+$, define

$$\hat{\delta}_{k,y}(f) = \int_R f(x+iy)e^{-2\pi ik(x+iy)}dx.$$

Let $\mu \leq 0$, $k \in Z$, $T > 0$ and $\epsilon > 0$. Then

$$\hat{\delta}_{k/T,y} \in W^{-(1/2+\epsilon)}(H, d\lambda_\nu)$$

is a T-invariant distribution.

Lemma 2.2 will follow immediately from Lemma B, which proves some decay for functions in $W^{1/2+\epsilon}(\mathcal{H}_\mu)$. Moreover,

Let $\mu \leq 0$, $k \in Z$ and $y_1, y_2 > 0$. Then $\hat{\delta}_{k/T,y_1} = \hat{\delta}_{k/T,y_2}$, and if $k \leq 0$, then $\hat{\delta}_{k/T,y_1} = 0$.

Lemma 2.2 follows from Lemma B and Cauchy's theorem, and the proof is given in Lemma B. We therefore drop the subscript y and declare

$$\hat{\delta}_{k/T} := \hat{\delta}_{k/T,y},$$

for *any* $y > 0$.

Additional distributions at infinity

These distributions are introduced only as a technical tool for making calculations.

$\underline{Principal and complementary series}$

Let $\mu > 0$, and for all $r \in N$, define

$$\delta^{(r)} := (\Theta^r \delta^{(0)}).$$

Then Lemma 6.3 of Nelson, Analytic Vectors ([11]) together with (**??**) shows

$$|\delta^{(r)}(f)| = |\delta^{(0)}(\Theta^r f)| \leq C\|\Theta^r f\|_{1+\Re\nu)/2+\epsilon} \leq C_r\|f\|_{r+(1+\Re\nu)/2+\epsilon}.$$

Hence,

$$\delta^{(r)} \in W^{-(r+(1+\Re\nu)/2+\epsilon)}(\mathcal{H}_\mu).$$

The *Key Point* in proving our estimate for the cohomological equation is that functions and their derivatives that annihilate these additional distributions have additional decay.

Let $\mu > 0, \epsilon > 0$ and $s \geq 0$. Then there is a constant $C_{s,\epsilon} > 0$ such that for all $f \in W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{s})$, $x \in R$ and integers $0 \leq r \leq s$, we have

$$|f^{(r)}(x)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}(1+|\nu|)^r(1+|x|)^{-(s+r+1+\Re\nu)}\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}.$$

We should expect this. If $f(\theta) = \Phi(\theta)\cos^{1+\nu}(\theta)$ is smooth, then Φ has a Taylor series about $\frac{\pi}{2}$, and all of its derivatives vanish up to some order. So Φ decays, which forces f to decay as well. That said, the proof is somewhat long, and we defer it to the appendix.

$\underline{Discreteseries}$

Let $\mu \leq 0$. Then in the same way, we have

$$\delta^{(r)} := (\Theta^r\delta^{(0)}) \in W^{-(r+(1+\nu)/2+\epsilon)}(H, d\lambda_\nu). \tag{2.4}$$

Recall that the parameter ν tends to infinity. For fixed regularity $f \in W^s(\mathcal{H}_\mu)$, when $\nu < s$ the proof of our estimate follows the proof for the principal and complementary series estimate. In particular, we use the following proposition to obtain our estimate: Let $\mu \leq 0$, and let $r, s \in N_0$ satisfy $0 \leq r < (s-1)/2$ and $s \geq 2$. Also let $f \in W^s(H, d\lambda_\nu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{\tilde{s}})$. Then there is a constant $C_s > 0$ such that for all $z \in H$,

$$|f^{(r)}(z)| \le C_s \|f\|_s (1 + |z|)^{-(s/2+2\nu+r+3/2}.$$

When $s \ge \nu$, the proof is different, and we do not use the additional distributions.

Chapter 3

Cohomological equation

3.1 Cohomological equation for the principal and complementary series

The main theorem of this section is the following: Let $\mu > 0, T > 0$ and $r \geq 0$. Then for all $f \in W^{2r+3/2}(\mathcal{H}_\mu) \cap \mathcal{I}_\mu(\mathcal{H}_\mu)$, there exists a unique $\mathcal{H}_\mu$ solution u to the cohomological equation

$$u \circ \phi_T - u = f. \tag{3.1}$$

Additionally, there is a constant $C_r > 0$ such that

$$\|u\|_{W^r(\mathcal{H}_\mu)} \leq \frac{C_r}{1 - \sqrt{1-\mu}} \|f\|_{W^{2r+3/2}(\mathcal{H}_\mu)}. \tag{3.2}$$

Remark : We actually prove the tame estimate

$$\|u\|_{W^r(\mathcal{H}_\mu)} \leq \frac{C_r}{\sqrt{1-\nu}} (1 + |\nu|)^r \|f\|_{W^{r+3/2}(\mathcal{H}_\mu)}$$

in each irreducible component. Because there exists infinitely many irreducible components, the representation parameters ν may tend to infinity, so we absorb ν using the Casimir operator $\square$ and obtain (3.2).

Estimate using additional distributions at infinity

The following theorem mostly proves Theorem 3.1.

Let $\mu > 0, r \geq 0$ and $f \in W^{2r+3/2}(\mathcal{H}_\mu) \cap \{\hat{\delta}_n\}_{n=-\infty}^{\infty} \cup \{\delta^{(k)}\}_{k=0}^{r+1}$. Then there exists a unique $\mathcal{H}_\mu$ solution u to the cohomological equation

$$u(x-T) - u(x) = f(x), \tag{3.3}$$

and there is a constant $C_r > 0$ such that

$$\|u\|_{W^r(\mathcal{H}_\mu)} \leq \frac{C_r}{\sqrt{1-\nu}} \|f\|_{W^{2r+3/2}(\mathcal{H}_\mu)}. \tag{3.4}$$

To ease notation, define

$$s(\nu, \epsilon) := s + (1 + \Re\nu)/2 + \epsilon.$$

Let $\mu > 0, \epsilon > 0,\ s \geq 0$ and $f \in W^{s(\nu,\epsilon)}(\mathcal{H}_\mu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{s-1} \cap \{\hat{\delta}_{n/T}\}_{n=-\infty}^{\infty})$. Let u be defined by

$$u(x) = \sum_{n=1}^{\infty} f(x + nT).$$

Then u is a solution to (3.3) and there is a constant $C_{s,\epsilon} > 0$ such that for all $x \in R$ and $0 \leq r \leq s$,

$$|u^{(r)}(x)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} (1 + |\nu|)^r (1 + |x|)^{-(s+r+\Re\nu)} \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)}.$$

Proof:

Proceeding formally at first, define

$$u(x) := \sum_{n=1}^{\infty} f(x + nT).$$

Then

$$u(x-T) - u(x) = \sum_{n=0}^{\infty} f(x + nT) - \sum_{n=1}^{\infty} f(x + nT) = f(x).$$

So u is formally a solution. We will now show u converges absolutely and uniformly on compact sets. Suppose $x \geq 0$. Using that $T > 0$ and $s \geq 1$, Proposition 2.2 shows

$$|u(x)| \leq \sum_{n=1}^{\infty} |f(x+nT)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} \sum_{n=1}^{\infty} (|x+nT|+1)^{-(s+1+\Re\nu)}$$
$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} \sum_{n=1}^{\infty} (|nT|+1)^{-2} < \infty. \tag{3.5}$$

For $x < 0$, note that

$$\|f\|_{C^0(R)} \leq \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} < \infty.$$

There exists $m \in N$ such that $x + mT \geq 0$, so

$$|u(x)| \leq C(s)\|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} + m\|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} < \infty.$$

Hence, the series defining u converges absolutely.

Moreover, calculation (3.5) shows that the series defining u converges uniformly on compact sets, and as $f \in W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)$, we may differentiate under the sum. Then

$$|u^{(r)}(x) \cdot T| = |\frac{d^r}{dx^r} \sum_{n=1}^{\infty} f(x+nT) \cdot T| \leq \sum_{n=1}^{\infty} |f^{(r)}(x+nT)| \cdot T$$
$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} (1+|\nu|)^r \|f\|_{s(\nu,\epsilon)} \sum_{n=1}^{\infty} (|x+nT|+1)^{-(s+r+1+\Re\nu)} \cdot T. \tag{3.6}$$

So when $x \geq 0$,

$$|u^{(r)}(x)| \leq \frac{C_{s,\epsilon}}{T \cdot \sqrt{1-\nu}} (1+|\nu|)^r \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} (|x|+1)^{-(s+r+\Re\nu)},$$

by the integral estimate.

Now we show that u has the same decay for $x \leq 0$. Proposition 2.2 shows that $f \in L^1(R)$. So the Poisson summation formula applies, and using that $f \in Ann(\{\hat{\delta}_{\frac{n}{T}}\}_{n=\infty}^{\infty})$, we have

$$\sum_{n\in Z} f(x+nT) = \frac{1}{T}\sum_{n\in Z}\hat{f}(\frac{n}{T})e^{2\pi ixn/T} = 0.$$

Therefore,

$$u(x) = \sum_{n=1}^{\infty} f(x+nT) = -\sum_{n=0}^{\infty} f(x-nT).$$

When $x \leq -T$, the integral estimate again shows

$$|u^{(r)}(x)| \leq \sum_{n=0}^{\infty} |f^{(r)}(x-nT)|$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}\cdot T}(1+|\nu|)^r \|f\|_{s(\nu,\epsilon)}(|x|+1)^{-(s+r+\Re\nu)}.$$

Finally, for $-T \leq x \leq 0$, note that

$$\|f^{(r)}\|_{C^0(R)} \leq \|f\|_{s(\nu,\epsilon)},$$

so

$$|u^{(r)}(x)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}\cdot T}(1+|\nu|)^r \|f\|_{s(\nu,\epsilon)}(|x|+1)^{-(s+r+\Re\nu)}.$$

□

Proof of Theorem 3.1 :

First let $r \in N_0$, $f \in W^{2r+3/2}(\mathcal{H}_\mu)$ and recall $\triangle = -(X^2+Y^2+\Theta^2)$, and each vector field in $\{X, Y, \Theta\}$ is of the form

$$c_1 x^l + c_2 x^k \frac{d^j}{dx^j},$$

where $l \in \{0,1\}$ and $k-j \leq 1$, and $c_1, c_2 \in C$. Therefore, $\triangle^r$ consists of terms of the form $x^k \frac{d^j}{dx^j}$, where $0 \leq k-j \leq r$.

Then

$$\|u\|_{W^r(\mathcal{H}_\mu)} \leq \sum_{\substack{0 \leq k \leq 2r \\ 0 \leq j \leq r \\ 0 \leq k-j \leq r}} \|x^k u^{(j)}\|_{\mathcal{H}_\mu} \leq \sum_{\substack{0 \leq k \leq 2r \\ 0 \leq j \leq r \\ 0 \leq k-j \leq r}} \|(|x|+1)^k u^{(j)}\|_{\mathcal{H}_\mu}$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu} \cdot T}(1+|\nu|)^r \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} \sum_{\substack{0 \leq k \leq 2r \\ 0 \leq j \leq r \\ 0 \leq k-j \leq r}} \|(|x|+1)^{k-(s+j+\Re\nu)}\|_{\mathcal{H}_\mu}$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu} \cdot T}(1+|\nu|)^r \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} \|(|x|+1)^{r-(s+\Re\nu)}\|_{\mathcal{H}_\mu}.$$

Note that if $\mu \geq 1$, then $\|\cdot\|_{\mathcal{H}_\mu} = \|\cdot\|_{L^2(R)}$ takes the $L^2(R)$ norm and $\Re\nu = 0$. So for all $0 \leq r < s - 1/2$,

$$\|(|x|+1)^{r-s}\|_{\mathcal{H}_\mu} < \infty,$$

which means

$$\|u\|_{W^r(\mathcal{H}_\mu)} \leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu} \cdot T}(1+|\nu|)^r \|f\|_{W^{s+1/2+\epsilon}(\mathcal{H}_\mu)}.$$

In particular, this holds for $r = s - 1/2 - \epsilon$. Then replacing s with $r + 1/2 + \epsilon$ proves Theorem 1 for $\mu \geq 1$ and $r \in N_0$.

When $0 < \mu < 1$, we have

$$\|u\|_{W^r(\mathcal{H}_\mu)} \leq \sum_{\substack{0 \leq k \leq 2r \\ 0 \leq j \leq r \\ 0 \leq k-j \leq r}} \|x^k u^{(j)}\|_{\mathcal{H}_\mu}$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu} \cdot T}(1+|\nu|)^r \|f\|_{W^{s(\nu,\epsilon)}(\mathcal{H}_\mu)} \|(|x|+1)^{r-(s+\nu)}\|_{\mathcal{H}_\mu}.$$

Here, $\mathcal{H}_\mu$ is defined by the norm

$$\|g\|^2_{\mathcal{H}_\mu} := \int_{R \times R} \frac{g(x)g(y)}{|x-y|^{1-\nu}} dxdy. \tag{3.7}$$

Let $\eta = x - y$, so

$$(3.7) \leq \int_R |g(y)| \int_R \frac{|g(\eta+y)|}{|\eta|^{1-\nu}} d\eta dy$$

$$\leq \int_R |g(y)| \left(\int_{|\eta| \leq 1} \frac{|g(\eta+y)|}{|\eta|^{1-\nu}} d\eta + \int_{|\eta| \geq 1} \frac{|g(\eta+y)|}{|\eta|^{1-\nu}} d\eta \right) dy. \tag{3.8}$$

Notice

$$\int_{|\eta| \leq 1} \frac{|g(\eta+y)|}{|\eta|^{1-\nu}} d\eta \leq C_\nu \|g\|_{L^\infty(R)},$$

and

$$\int_{|\eta| \geq 1} |g(\eta+y)| d\eta = \|g\|_{L^1(R)},$$

There are only finitely many values in $spec(\Box) \subset (0,1)$, so $C_\nu \leq C$ for some absolute constant C, and therefore,

$$(3.8) \leq C \|g\|_{L^1(R)} \left(\|g\|_{L^\infty(R)} + \|g\|_{L^1(R)} \right).$$

.

Observe that for all $0 \leq r < s + \nu - 1$,

$$(|x| + 1)^{r-(s+\nu)} \in L^1(R) \cap L^\infty(R).$$

As before, this holds for $r = s + \nu - 1 - \epsilon$. Setting $s = r + 1 + \epsilon - \Re\nu$, we see

$$s(\nu, \epsilon) = r + 1 - \nu + (1 + \nu)/2 + 2\epsilon \leq r + 3/2.$$

This proves the estimate in Theorem 1 for $r \in N_0$.

Finally, observe that for all $0 \leq r \leq s$, $W^s(\mathcal{H}_\mu)$ is a dense subset of $W^r(\mathcal{H}_\mu)$. Additionally, $\triangle$ is an essentially self-adjoint operator. Then the family $\{W^r(\mathcal{H}_\mu)\}_{r\geq 0}$ is an interpolation family in the sense that for $\alpha \in [0, 1]$, the interpolation space

$$[W^s, W^r]_\alpha \simeq W^{r+(s-r)\alpha}. \tag{3.9}$$

Because the estimate in Theorem 1 holds for all integers r, Theorem 5.1 from [9] completes the proof. □

Estimate of coboundaries

Theorem 3.1 holds under the restricted hypothesis that

$$f \in W^{r+3/2}(\mathcal{H}_\mu) \cap Ann\left(\{\hat{\delta}_n\}_{n=-\infty}^{\infty} \cup \{\delta^{(0)}\}\right).$$

To begin, set $\chi_0 := u_0 (= \cos^{1+\nu}(\theta))$, and recursively define $\{\chi_k\}_{k=1}^r$ by

$$\chi_{k+1} := (\chi_k \circ \phi_T^U - \chi_k). \tag{3.10}$$

Then χ_k is a coboundary for all $k \geq 1$. We show $\{\chi_k\}_{k=1}^r$ is a basis in the dual space to $\langle\{\delta^{(k)}\}_{k=0}^r\rangle$ and obtain a bound for each $\|\chi_k\|_{W^r(\mathcal{H}_\mu)}$. For this, we study the distributions $\phi_T^U\delta^{(k)}$.

Let $\mu > 0$, $r \geq 0$ and $f \in C^\infty(\mathcal{H}_\mu)$. If r is even then

$$\sum_{j=0}^{[\frac{r}{2}]-1} \Big(2(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\Big)\, \delta^{(r-2j-1)},$$

and if r is odd, then

$$\mathcal{L}_U\delta^{(r)}(f) = i(\nu+1)(-2i)^r\delta^{(0)}$$
$$+ \sum_{j=0}^{[\frac{r}{2}]-1} \Big(2(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\Big)\, \delta^{(r-2j-1)}.$$

We defer the proof of Lemma 3.1 to the appendix (see Corollary B). This gives coefficients $\{c_{j,k}\}_{0\leq j,k\leq r} \subset C$ such that

$$\mathcal{L}_U|_{\langle\{\delta^{(k)}\}_{k=0}^r\rangle} = \begin{pmatrix} 0 & c_{0,1} & c_{0,2} & \cdots & c_{0,r} \\ 0 & 0 & c_{1,2} & \cdots & c_{1,r} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & c_{r-1,r} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Exponentiating, we get coefficients $\{e_{j,k}\}_{0\leq j,k\leq r} \subset C$ such that

$$\phi_T^U|_{\langle\{\delta^{(k)}\}_{k=0}^r\rangle} = \begin{pmatrix} 1 & e_{0,1} & e_{0,2} & \cdots & e_{0,r} \\ 0 & 1 & e_{1,2} & \cdots & e_{1,r} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & e_{r-1,r} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where the dependence on T is given by $e_{j,k} = \frac{T^{k-j}}{(k-j)} a_{j,k}$ for all j, k and some coefficients $\{a_{j,k}\} \subset C$.

Let $\chi_0 = \cos^{1+\nu}(\theta)$ and for all $0 \leq k \leq r$, define χ_k as in (3.10). Then for all $1 \leq j < k$ and $0 \leq k \leq r$, we have

$$\delta^{(j)}(\chi_k) = 0.$$

Proof:

For each $0 \leq k \leq r$, let $P(k)$ be the statement

$$forall 1 \leq j < k, \delta^{(j)}(\chi_k) = 0.$$

We will show by induction that $P(k)$ holds for all k. The statement $P(0)$ holds trivially. Now suppose that $P(k)$ holds, and we show $P(k+1)$ holds as well. Let $1 \leq j < k+1$. Then

$$\delta^{(j)}(\chi_{k+1}) = \delta^{(j)}(\chi_k \phi_T^U - \chi_k) = (\phi_T^U \delta^{(j)})(\chi_k) - \delta^{(j)}(\chi_k)$$

$$= \sum_{m=0}^{j} e_{m,j} \delta^{(m)}(\chi_k) - \delta^{(j)}(\chi_k)$$

$$= \sum_{m=0}^{j-1} e_{m,j} \delta^{(m)}(\chi_k) = 0,$$

because $P(k)$ holds by the induction assumption. □

Let $\{\chi_k\}_{k=0}^{r}$ be defined by (3.10). Then for all $1 \leq k \leq r$,

$$\delta^{(j)}(\chi_k) = \begin{cases} \Pi_{j=0}^{k-1} e_{j,j+1} if j = k \\ 0 \quad if j < k \end{cases} \tag{3.11}$$

Proof:

By Lemma 3.1, it remains to examine the case $j = k$, and we again go by induction. Observe

$$\delta^{(1)}(\chi_1) = \delta^{(1)}(\chi_0 \circ \phi_T^U - \chi_0)$$

$$= (c_{0,1}\delta^{(0)} + \delta^{(1)})(\chi_0) - \delta^{(1)}(\chi_0) = c_{0,1}.$$

Now suppose that (3.11) holds for $k \leq r-1$ and we show that it holds for $k+1$. Then

$$\delta^{(k+1)}(\chi_{k+1}) = \delta^{(k+1)}(\chi_k \circ \phi_T^U - \chi_k)$$

$$= \sum_{j=0}^{k+1} e_{j,k+1}\delta^{(j)}(\chi_k) - \delta^{(k+1)}(\chi_k)$$

$$= \sum_{j=0}^{k} e_{j,k+1}\delta^{(j)}(\chi_k). \tag{3.12}$$

By Lemma 3.1 followed by the induction assumption, we conclude

$$(3.12) = e_{k,k+1}\delta^{(k)}(\chi_k) = \Pi_{j=0}^{k} e_{j,j+1}. \quad \square$$

For convenience, we define $P_k = \Pi_{j=0}^{k-1}|e_{j,j+1}|$ for all $k \geq 0$. Let $\mu \in spec(\Box)$, $\epsilon > 0$ and $f \in W^r(\mathcal{H}_\mu) \cap Ann(\{\delta^{(0)}\})$. Then there are coefficients $\{f_k\}_{k=0}^r$ and a constant C_r satisfying

$$|f_k| \leq \begin{cases} \frac{C_r(\mu_0}{T^{k(k+1)/2}}\|f\|_{k+(1+\Re\nu)/2+\epsilon} if T < 1 \\ C_r(\mu_0)\|f\|_{k+(1+\Re\nu)/2+\epsilon} otherwise \end{cases}$$

and

$$f_d := f - \sum_{k=1}^{r} f_k\chi_k \in Ann(\{\delta^{(k)}\}_{k=0}^r.$$

Proof:

From Lemma 3.1, we see by exponentiating the matrix $\mathcal{L}_U|_{\langle\{\delta^{(k)}\}_{k=0}^r\rangle}$ that

$$e_{j,j+1} = Tc_{j,j+1} = \left\{ \begin{array}{r} 2T(\nu+1)ifj=0 \\ \\ 2T(j+1)[2j+(\nu+1)otherwise. \end{array} \right. .$$

In fact, Lemma 3.2 proves the same identity when $\mu \leq 0$. Then for $\mu \in spec(\Box)$, there is a constant $C_r > 0$ such that

$$\Pi_r \geq C_r T^k.$$

Now recursively define $f_1 = \frac{\delta^{(1)}(f)}{\Pi_1}$ and if f_j have been defined for $1 \leq j < k$, define

$$f_k := \frac{\delta^{(k)}(f) - \sum_{j=1}^{k-1} f_j \delta^{(k)}(\chi_k)}{\Pi_k}.$$

We prove by induction that for all $1 \leq k \leq r$,

$$f_k \leq \left\{ \begin{array}{r} \frac{C_r(\mu_0)}{T^{k(k+1)/2}} \|f\|_{k+(1+\Re\nu)/2+\epsilon} if\mathrm{T} ¡ 1 \\ \\ C_r(\mu_0)\|f\|_{k+(1+\Re\nu)/2+\epsilon}. \end{array} \right. \tag{3.13}$$

We consider the case $T < 1$, and the case $T \geq 1$ will then be clear. For the base case notice that

$$f_0 \leq \frac{C_0(\mu_0)}{T}\|f\|_{(1+\Re\nu)/2}.$$

Then assuming (3.13) holds for $1 \leq j < k$, observe

$$\begin{aligned} f_k &\leq \frac{C_k(\mu_0)}{T^k}\left(C_k\|f\|_{k+(1+\Re\nu)/2+\epsilon} + \sum_{j=1}^{k-1} \frac{C_j(\mu_0)\|f\|_{j+(1+\Re\nu)/2}}{T^{j(j+1)/2}}\delta^{(k)}(\chi_j)\right) \\ &\leq \frac{C_k(\mu)}{T^{k(k+1)/2}}\|f\|_{k+(1+\Re\nu)/2+\epsilon}. \end{aligned}$$

Finally, we prove by induction that for all $0 \leq j \leq k$, $\delta^{(j)}(f_d) = 0$. By assumption $\delta^{(0)}(f) = 0$, and by flow invariance, $\delta^{(0)}(\chi_k) = 0$ for $k \geq 1$. So $\delta^{(0)}(f_d) = 0$.

Assume that $\delta^{(m)}(f_d) = 0$ for $0 \leq m < j$. Moreover, by construction and Lemma 3.1, if $0 \leq j \leq k$, then by

$$\delta^{(j)}(f_d) = \delta^{(j)}(f) - \sum_{k=1}^{r} f_k \delta^{(j)}(\chi_k)$$

$$= \delta^{(j)}(f) - \sum_{k=1}^{j} f_k \delta^{(j)}(\chi_k) = \delta^{(j)}(f) - \sum_{k=1}^{j-1} f_k \delta^{(j)}(\chi_k) - f_j \Pi_j = 0,$$

from the definition of f_j. $\square$

Now we *prove Theorem* 3.1. Let f_d be defined as in Lemma 3.1. Because $f_d \in Ann(\{\delta^{(j)}\}_{j=0}^{r})$, Theorem 3.1 shows that f_d has a transfer function u_d and there is a constant $C_r > 0$ such that

$$\|u_d\|_{W^r(\mathcal{H}_\mu)} \leq C_r \|f_d\|_{W^{r+1+\epsilon}(\mathcal{H}_\mu)}.$$

For each $1 \leq k \leq r$, χ_k is a coboundary by construction, and there is a constant $C_{r,\epsilon} > 0$ such that

$$\|\chi_k\|_{W^r(\mathcal{H}_\mu)} \leq C_{r,\epsilon}.$$

Then define

$$u := u_d + \sum_{k=1}^{r} f_k \chi_{k-1}.$$

Then

$$u(x - T) - u(x) = u_d(x - T) - u_d(x) + \sum_{k=1}^{r} f_k[\chi_{k-1}(x - T) - \chi_{k-1}(x)]$$

$$= f_d + \sum_{k=1}^{r} f_k \chi_k(x) = f.$$

Moreover,

$$\|u\|_r \leq \|u_d\|_r + \sum_{k=0}^{r-1} |f_k| \|\chi_k\|_r$$

$$\le \frac{C_r}{(1-\sqrt{1-\mu_0})\cdot T}\left(\|f_d\|_{2r+3/2} + \frac{1+T^{r(r+1)/2}}{T^{r(r+1)/2}}\|f\|_{r+1}\right)$$
$$\le \frac{C_{r,T}}{1-\sqrt{1-\mu_0}}\|f\|_{2r+3/2}.$$

Lastly, u is the unique $\mathcal{H}_\mu$ solution, because if w is any $\mathcal{H}_\mu$ solution to (3.3), then $w-u \in \mathcal{H}_\mu$ and is T-periodic, which means $w = u$ in $\mathcal{H}_\mu$. $\square$

The proof of Theorem 3.1 follows from Theorem 3.1 by showing the space of T-invariant distributions is precisely $S_0 = \langle \{\hat{\delta}_{n/T}\}_{n\in Z} \cup \{\delta^{(0)}\}\rangle$. Section 3 shows the elements of S_0 are T-invariant, and for the other inclusion, suppose there exists $\mathcal{D} \in \mathcal{I}_\mu - S_0$. Then let $f \in Ann(S_0)$ be such that $\mathcal{D}(f) \neq 0$. Note that coboundaries are in the kernel of all invariant distributions, so f is not a coboundary. But by Theorem 3.1, f is a coboundary. Contradiction. $\square$

3.2 Cohomological Equation for the Discrete Series

Our main theorem of this section is Let $\mu \le 0$, $T > 0$, $r \ge 0$, and $f \in W^{3r+4}(\mathcal{H}_\mu) \cap \mathcal{I}_\mu(\mathcal{H}_\mu)$. Then there is a constant $C_{r,T} > 0$ and a unique $\mathcal{H}_\mu$ transfer function u satisfying

$$u \circ \phi^U_T - u = f, \tag{3.14}$$

and

$$\|u\|_{W^r(\mathcal{H}_\mu)} \le C_{r,T}\|f\|_{W^{3r+4}(\mathcal{H}_\mu)}.$$

We remind the reader that we safely restrict our attention to the holomorphic discrete series. Additionally, for fixed $\nu \ge 1$, we have the biholomorphic map

$\alpha : D \to H : \xi \to -i\left(\frac{\xi+1}{\xi-1}\right) := z$ and the isometry

$$T_\nu : L^2_{hol}(H, d\lambda_\nu) \to L^2_{hol}(D, d\sigma_\nu) : f(z) \to f \circ \alpha(\xi) \left(\frac{-2i}{\xi - 1}\right)^{\nu+1} \tag{3.15}$$

linking the holomorphic unit disk model with the holomorphic upper half-plane model.

The argument is divided into two similar cases, when $\nu < s$ and when $\nu \geq s$. When $\nu < s$, the function f does not have enough decay to easily estimate its transfer function, so we use the additional distributions at infinity as we did in our estimate for the principal and complementary series. It turns out that we do not need them when $\nu \geq s$.

Case $\nu <$ s

Our immediate goal is to prove

Let $\mu \leq 0$, $T > 0$, $r \geq 0$, and $f \in W^{3r+2}(H, d\lambda_\nu) \cap Ann(\{\delta_{k/T}\}_{k\in Z^+} \cup \{\delta^{(k)}\}_{k=0}^{r+1})$. Then there is a constant $C_{r,T} > 0$ and a unique $L^2(H, d\lambda_\nu)$ transfer function such that for all $z \in H$,

$$u(z - T) - u(z) = f(z), \tag{3.16}$$

and

$$\|u\|_{W^r(H,d\lambda_\nu)} \leq C_{r,T}\|f\|_{W^{3r+2}(H,d\lambda_\nu)}.$$

Our method of proving this is the same as we did for the principal and complementary series. Throughout, let $\tilde{s} = \lfloor \frac{s-1}{2} \rfloor$. Let $\mu \leq 0$ and r, s be integers that satisfy $0 \leq r < \tilde{s}$ and $s \geq 2$. Then there is a constant $C_s > 0$ such that for all $f \in W^s(H, d\lambda_\nu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{s})$ we have

$$|f^{(r)}(z)| \leq C_{r,s} \cdot \nu^r \|f\|_{W^s(H,d\lambda_\nu)}(1+|z|)^{-(s/2+\nu+r+3/2)}.$$

We show this in Appendix B, and it allows is to prove

Let $\mu \leq 0$, r, s be integers that satisfy $0 \leq r < \tilde{s}$ and $s \geq 2$. Also let $T > 0$, $f \in W^s(H, d\lambda_\nu) \cap Ann(\{\hat{\delta}_{k/T}\}_{k \in Z^+} \cup \{\delta^{(r)}\}_{r \geq 0}^{\tilde{s}})$. Then there is a constant $C_s > 0$ and a unique solution u to the cohomological equation

$$u(z - T) - u(z) = f(z)$$

such that for all $z \in H$,

$$|u^{(r)}(z)| \leq \frac{C_{r,s}}{T} \nu^r \|f\|_{W^s(H,d\lambda_\nu)}(1+|z|)^{-(s/2+\nu+r+1/2)}.$$

Proof:

For all $z \in \mathcal{H}$, define

$$u(z) := \sum_{n=1}^{\infty} f(z + nT).$$

Then formally,

$$u(z - T) - u(z) = \sum_{n=0}^{\infty} f(z + nT) - \sum_{n=1}^{\infty} f(z + nT) = f(z).$$

Sobolev embedding gives that $\|f\|_{C^0(H)} \leq C_\epsilon \|f\|_{(1+\nu)/2+\epsilon} \leq C\|f\|_s$ for some fixed constant $C := C_\epsilon > 0$.Hence, u is an actual solution. Additionally, if $\Re z > 0$,

$$|u(z)| \leq \sum_{n=1}^{\infty} |f(z + nT)|$$

$$\leq C_s \|f\|_s \sum_{n=1}^{\infty} (1+|z+nT|)^{-(s/2+\nu+r+3/2)} < \infty.$$

This also shows that the series defining u converges uniformly on compact sets for $\Re z > 0$, and because $f$is holomorphic, we can differentiate under the sum. Then for $\Re z > 0$, Proposition 3.2 gives

$$\begin{aligned} |u^{(r)}(z) \cdot T| &= \sum_{n=1}^{\infty} |f^{(r)}(z+nT)| \cdot T \\ &\leq \|f\|_s \sum_{n=1}^{\infty} (1+|z+nT|)^{-(s/2+\nu+r+3/2)} \cdot T \\ &\leq C_{r,s} \nu^r \|f\|_s (1+|z|)^{-(s/2+\nu+r+1/2)}, \end{aligned} \tag{3.17}$$

where we get the last inequality by the integral estimate.

Now consider the case $\Re z \leq 0$. Proposition B together with the fact that f is uniformly bounded on H shows that for all $y > 0$, $f(\cdot + iy) = L^1(R)$. By assumption $f \in Ann(\{\hat{\delta}_{k/T}\}_{n \in Z^+}) = Ann(\{\hat{\delta}_{n/T}\}_{n \in Z})$, so for each $y > 0$, the Poisson summation formula gives

$$\sum_{n=-\infty}^{\infty} f((x+nT)+iy) = \sum_{n=-\infty}^{\infty} \hat{\delta}_{n/T,y}(f) e^{2\pi i n/T x} = 0.$$

Therefore,

$$u(z) = \sum_{n=1}^{\infty} f(x+nT+iy) = -\sum_{n=0}^{\infty} f(x-nT+iy).$$

So

$$|u(z)| \leq \sum_{n=0}^{\infty} |f(x-nT+iy)|.$$

The sum $\sum_{n=0}^{\infty} f(z-nT)$ also converges uniformly on compact sets and differentiating under the sum, we get that whenever $\Re z < 0$,

$$|u^{(r)}(z)| \leq \sum_{n=0}^{\infty} |f^{(r)}(z-nT)| \leq \frac{C_{r,s}}{T} \nu^r \|f\|_s (1+|z|)^{-(s/2+\nu+r+1/2)}.$$

Finally, if $w \in L^2(H, d\lambda_\nu)$ is any solution to (3.16), then $u - w \in L^2(H, d\lambda_\nu)$ and is T-periodic, so $u = w$ in $L^2(H, d\lambda_\nu)$ $\square$.

Now we prove Theorem 3.2 :

It remains to estimate $\|u\|_{W^r(H,d\lambda_\nu)}$. Recall $\triangle = -(X^2 + Y^2 + \Theta^2)$, and each vector field in $\{X, Y, \Theta\}$ is of the form

$$z^l + z^k \frac{d^j}{dz^j},$$

where $l \in \{0, 1\}$ and $k - j \leq 1$. Therefore, $\triangle^r$ consists of terms of the form $z^k \frac{d^j}{dz^j}$, where $0 \leq k - j \leq r$. So

$$\begin{aligned}
\|u\|_r &\leq \sum_{\substack{0 \leq k \leq 2r \\ 0 \leq j \leq r \\ 0 \leq k-j \leq r}} \|z^k u^{(j)}\|_{L^2(H,d\lambda_\nu)} \\
&\leq \sum_{\substack{0 \leq k \leq 2r \\ 0 \leq j \leq r \\ 0 \leq k-j \leq r}} \|(|z|+1)^k u^{(j)}\|_{L^2(H,d\lambda_\nu)} \\
&\leq \frac{C_{r,s}}{T} \nu^r \|f\|_s \left\| \frac{(1+|z|)^{2r}}{(1+|z|)^{s/2+\nu+r+1/2}} \right\|_{L^2(H,d\lambda_\nu)} \\
&\leq \frac{C_{r,s}}{T} \nu^r \|f\|_s \left(\int_H (1+|z|)^{2r-s-2\nu-1} \Im(z)^{\nu-1} dx dy \right)^{1/2}. \qquad (3.18)
\end{aligned}$$

Note (3.18) $< \infty$ if $2r - s - 2\nu - 1 + \nu - 1 = 2r - s - \nu - 2 < -1$, which holds whenever $2r - 2 < s$. This always holds given our assumption that $2r + 1 < s$. Then choose $s = 2r + 2$.

Finally, Claim A shows that $\Box f = (1-\nu^2)f$, so that $\nu^2 f = (1-\Box)f$. Then

$$\|\nu^r f\|_{2r+2} = \|(1-\Box)^{r/2} f\|_{2r+2} \leq C_r \|f\|_{3r+2},$$

by Lemma 6.3 of Nelson [16]. Finally, interpolation gives the estimate for all $r \in R^+$.

the solution is unique for the same reason as in Lemma 3.2. $\Box$ $\Box$

Remove additional distributions at infinity

In this subsection, we prove Theorem 3.2 holds under the restricted hypothesis that $f \in$

$W^{3r+2}(H, d\lambda_\nu) \cap Ann(\{\delta_{n/T}\}_{n\in Z} \cap \{\delta^{(0)}\})$.

Our proof goes the same way as the proof of Theorem 3.1 for the principal and complementary series. As a first step, we have

Let $\mu \leq 0$, $r \geq 0$ and $f \in W^{r+2}(\mathcal{H}_\mu)$. If r is even then

$$\mathcal{L}_U \delta^{(r)}(f) = \sum_{j=0}^{[\frac{r}{2}]-1} \left(2(1+\nu)\tbinom{r}{2j+1}(2i)^{2j} - \tbinom{r}{2(j+1)}(2i)^{2(j+1)}\right)\delta^{(r-2j-1)}(f) - (2i)^r\delta^{(1)},$$

and if r is odd, then

$$\mathcal{L}_U \delta^{(r)}(f) = \sum_{j=0}^{[\frac{r}{2}]} \left(2(1+\nu)\tbinom{r}{2j+1}(2i)^{2(j+1)} - (2i)^{2j}\delta^{(r-2j-1)}(f)\right). \quad \Box$$

Lemma 3.2 shows the matrix

$$\mathcal{L}_U|_{\langle\{\delta^{(k)}\}_{k=0}^r\rangle} = \begin{pmatrix} 0 & c_{0,1} & c_{0,2} & \cdots & c_{0,r} \\ 0 & 0 & c_{1,2} & \cdots & c_{1,r} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & c_{r-1,r} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is strictly upper triangular. Exponentiating, we again get coefficients $\{e_{j,k}\}_{0\leq j,k\leq r} \subset C$ such that

$$\phi_T^U|_{\langle\{\delta^{(k)}\}_{k=0}^r\rangle} = \begin{pmatrix} 1 & e_{0,1} & e_{0,2} & \cdots & e_{0,r} \\ 0 & 1 & e_{1,2} & \cdots & e_{1,r} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & e_{r-1,r} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

As before, we recursively define a basis of coboundaries $\{\chi_k\}_{k=0}^r$ in the dual space to $\langle\{\delta^{(k)}\}_{k=0}^r\rangle$. Set $\chi_0 := u_n$ and given χ_k, define

$$\chi_{k+1} := \chi_k \circ \phi_T^U - \chi_k. \tag{3.19}$$

Lemmas 3.1, 3.1 and 3.1 do not depend on the particular representation space, so we have

Let $\mu \leq 0$, $r \in N_0$ and $\{\chi_k\}_{k=0}^r$ be defined by (3.19). Then $\{\chi_k\}_{k=0}^r$ is a linearly independent set of coboundaries, and for all $1 \leq j \leq k \leq r$,

$$\delta^{(j)}(\chi_k) = \Pi_k := \begin{cases} \Pi_{j=0}^{k-1} e_{j,j+1} if j = k \\ 0 \quad if j < k \end{cases}. \quad \square$$

Additionally Let $\mu \leq 0$, $\epsilon > 0$ and $f \in W^r(\mathcal{H}_\mu) \cap Ann(\{\delta^{(0)}\})$. Then there are coefficients $\{f_k\}_{k=0}^r$ and a constant $C_r > 0$ satisfying

$$|f_k| \leq \begin{cases} \frac{C_r(\mu_0}{T^{k(k+1)/2}}\|f\|_{k+(1+\Re\nu)/2+\epsilon} if T < 1 \\ C_r(\mu_0)\|f\|_{k+(1+\Re\nu)/2+\epsilon} otherwise \end{cases}$$

and

$$f_d := f - \sum_{k=1}^{r} f_k \chi_k \in Ann(\{\delta^{(k)}\}_{k=0}^{r}.$$

Proof:

Proof of Theorem 3.2 : This follows in the same way as the proof of Theorem 3.1 for the principal and complementary series. Using Lemma 3.2 and the fact that $\nu \geq 1$, we conclude

$$|c_{j,j+1}| = |-2\nu j + 1| \geq 1.$$

Hence,

$$\Pi_k := \Pi_{j=0}^{k-1} e_{j,j+1} = T^k \Pi_{j=0}^{k-1} c_{j,j+1} \geq T^k.$$

Recall $f \in Ann(\{\hat{\delta}_n\}_{n=-\infty}^{\infty} \cap \{\delta^{(0)}\})$ and let f_d be defined as in Lemma 3.2. By Lemma 3.2, $f_d \in Ann(\{\delta^{(k)}\}_{k=0}^{r})$, and as each χ_k is a coboundary for $k \geq 1$, f_d is also a coboundary, so it is in the kernel of all invariant distributions. Then we apply Theorem 3.2 and conclude there is a constant $C_{r,\epsilon} > 0$ and a transfer function u_d to f_d such that

$$\|u_d\|_r \leq \frac{C_r}{T} \nu^r \|f_d\|_r.$$

Define

$$u := u_d + \sum_{k=1}^{r} f_k \chi_{k-1}$$

and again note that

$$u(z-T) - u(z) = [u_d(z-T) - u_d(z)] + \sum_{k=1}^{r} f_k[\chi_{k-1}(z-T) - \chi_{k-1}(z)] = f.$$

Note there is a constant $C_r > 0$ such that for all $k \geq 0$, $\|\chi_k\|_r \leq C_r$, and the estimate of f_k then proves

$$\|u\|_r \leq C_r(\mu_0)\left(\|f_d\|_{3r+2} + \frac{1+T^{r(r+1)/2}}{T^{r(r+1)/2}}\|f\|_{r+1}\right) \leq C_{r,T}(\mu_0)\|f\|_{3r+2}. \tag{3.20}$$

Case $\nu \geq$ s

We let $H^+ = \{z \in H | \Re z > 0\}$ and $H^- = \{z \in H | \Re z < 0\}$. In this subsection we prove

Let $\mu \leq 0, \nu \geq 5, r \geq 0$ and $T > 0$, and let $f \in W^{3r+4}(H, d\lambda_\nu) \cap Ann(\{\hat{\delta}_{k/T}\}_{k=1}^{\infty})$. Then there is a unique $L^2(H, d\lambda_\nu)$ transfer function u to the cohomological equation (3.16), which satisfies

$$\|u\|_{W^r(H,d\lambda_\nu)} \leq \frac{C_r}{T}\|f\|_{W^{3r+4}(H,d\lambda_\nu)}.$$

Let $\{u_k\}_{k\geq n} \subset L^2(H, d\lambda_\nu)$ be the basis discussed in Section 2, and for $k \geq 0$, write

$$u_k(z) := \left(\frac{z-i}{z+i}\right)^{k-n}\left(\frac{1}{z+i}\right)^{\nu+1}.$$

Let $\mu \leq 0$. Then for all integers $k \geq 0$,

$$\|u_{k+n}\|_{W^s(H,d\lambda_\nu)} = \frac{\sqrt{\pi}}{\sqrt{\nu+1}\cdot 2^\nu}(1+\mu+8(k+n)^2)^{s/2}\left(\frac{k!\nu!}{(k+\nu)!}\right)^{1/2}.$$

This is proved in Claim B in the appendix.

Let $\mu \leq 0, T >, r \in N_0, s \in 2N$ and $2r+4 \leq s$. Then

$$u(z) := \sum_{m=1}^{\infty} f(z+mT)$$

is a solution to the cohomological equation

$$u(z-T) - u(z) = f(z), \tag{3.21}$$

and for all $0 \leq r < s$, $u^{(r)}$ is defined *Proof*:

Let $f(z) = \sum_{k=n}^{\infty} c_k u_k(z)$, and define

$$u(z) := \sum_{m=1}^{\infty} f(z + mT).$$

Then we formally have

$$u(z - T) - u(z) = f(z).$$

Lemma B shows in particular that if $s > 1/2$ and $z = x + iy \in H$, then

$$|f(z)| \leq C_{r,\nu,y} \|f\|_s \frac{1}{(1 + |z|)^{2s}}.$$

Hence, the series $-\sum_{m=0}^{\infty} f(z + mT)$ converges uniformly on compact sets, and because f is holomorphic, we may differentiate under the sum. Lemma B also shows that $u^{(r)}(z) = -\sum_{m=0}^{\infty} f^{(r)}(z + mT)$ converges uniformly on compact sets, so u is holomorphic and the decay estimate proves $u^{(r)} \in L^2(H, d\lambda_\nu)$. $\square$

Let $\mu \leq 0$ be such that $r \geq 0$. Then for all $k \geq n$,

$$u_k^{(r)}(z) = \sum_{j=1}^{r+1} \sum_{l=0}^{j} \tilde{c}_{j,r} \frac{k!}{(k - (j - l))!} \frac{(\nu + 1)!}{(\nu + 1 - j)!} u_{k+n+l-j}(z)(z + i)^{-r},$$

where we set

$$\frac{k!}{(k - (j - m))!} := 0 \tag{3.22}$$

if $k < j - m$. *Proof*:

Let $\alpha : D \to H$ be the analytic isomorphism between the unit disk and the upper half-plane given in section 2. For $f \in W^r(H, d\lambda_\nu)$, we switch to unit disk coordinates and get

$$f^{(r)}(z) = U^r(f \circ \alpha)(\xi) = (i \frac{(\xi - 1)^2}{2} \frac{d}{d\xi})^r f \circ \alpha(\xi). \tag{3.23}$$

By formula $(B.16)$, there are constants $\{c_{j,r}\}_{j=0}^{r} \subset C$ such that

$$U^r(f \circ \alpha)(\xi) = \sum_{j=1}^{r} c_{j,r}(\xi - 1)^{r+j}(f \circ \alpha)^{(j)}(\xi). \tag{3.24}$$

In unit disk coordinates, we have that for all $k \geq 0$,

$$u_{k+n} \circ \alpha = \xi^k \left(\frac{1-\xi}{-2i}\right)^{\nu+1}.$$

Using the notation in (3.22), we get

$$u_{k+n}^{(j)}(\xi) = (-2i)^{-(\nu+1)} \sum_{l=0}^{j} \binom{j}{l} \frac{d^{(j-l)}}{d\xi^{(j-l)}} \xi^k \frac{d^l}{d\xi^l}(\xi - 1)^{\nu+1}$$

$$= \sum_{l=0}^{j} \binom{j}{l} \frac{k!}{(k-(j-l))!} \xi^{k-(j-l)} \frac{(\nu+1)!}{(\nu+1-l)!}(\xi - 1)^{\nu+1-l}.$$

Combining this with $(B.16)$ allows us to conclude

$$(3.23) = (-2i)^{-(\nu+1)} \sum_{j=1}^{r+1} \sum_{l=0}^{j} \tilde{c}_{j,r} \binom{j}{l} \frac{k!}{(k-(j-l))!} \frac{(\nu+1)!}{(\nu+1-j)!} \xi^{k-(j-l)}(\xi - 1)^{\nu+1-l+j+r}$$

$$= \sum_{j=1}^{r+1} \sum_{l=0}^{j} \cdot \tilde{c}_{j,r} \binom{j}{l} \frac{k!}{(k-(j-l))!} \frac{(\nu+1)!}{(\nu+1-j)!} \left(\frac{z-i}{z+i}\right)^{k+l-j} (z+i)^{-(\nu+1-l+j+r)}$$

$$= \sum_{j=1}^{r+1} \sum_{l=0}^{j} \tilde{c}_{j,r} \binom{j}{l} \frac{k!}{(k-(j-l))!} \frac{(\nu+1)!}{(\nu+1-j)!} u_{k+n+l-j}(z)(z+i)^{-(j+r-l)}. \quad \square$$

Given $k, q, j \in N_0$, define

$$v_{k+n,q,j,T}(z) := \sum_{m=1}^{\infty} (1 + |z + mT|)^q |u_{k+n}^{(j)}(z + mT)|.$$

Let $\mu \leq 0, T > 0$, $r \in N_0$, and $s \in 2N$ be such that $2r + 4 \leq s$, and let $f \in W^s(\mathcal{H}_\mu)$. Then there is a constant $C_s > 0$ such that

$$\|u\|_{W^r(H^+,d\lambda_\nu)} \leq$$

$$\|f\|_s \sum_{\substack{0 \leq q \leq 2r \\ 0 \leq j \leq r \\ 0 \leq q-j \leq r}} \nu^{r-j} \left(\sum_{k=0}^{\infty} \|u_{k+n}\|^{-2}_{W^s(H,d\lambda_\nu)} \|v_{k+n,q,j,T}\|^2_{L^2(H^+,d\lambda_\nu)} \right)^{1/2}.$$

Proof:

Let $f(z) = \sum_{k=n}^{\infty} c_k u_k(z)$, and let $u \in L^2(H, d\lambda_\nu)$ be a solution to (3.21), given by Claim 3.2. Recall the vector fields X, Y, Θ that make up $(1 + \triangle)$ take the form $c_1(1+\nu)^k z^j + c_2 z^l \frac{d^m}{dz^m}$ for $j, k, m \in \{0, 1\}$ and $l \in \{0, 1, 2\}$. Then one can show

$$\|(1+\triangle)^r u\|_{L^2(H^+,d\lambda_\nu)} \leq \sum_{\substack{0 \leq q \leq 2r \\ 0 \leq j \leq r \\ 0 \leq q-j \leq r}} \nu^{r-j} \|(1+|z|)^q u^{(j)}(z)\|_{L^2(H^+,d\lambda_\nu)}.$$

For $z \in H^+$, we have

$$|(1+|z|)^q u^{(j)}(z)| = |(1+|z|)^q \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} c_{k+n} u^{(j)}_{k+n}(z+mT)|$$

$$\leq \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} |c_{k+n}| (1+|z+mT|)^q |u^{(j)}_{k+n}(z+mT)|$$

$$= \sum_{k=0}^{\infty} (|c_{k+n}| \|u_{k+n}\|_s) \left(\|u_{k+n}\|_s^{-1} \sum_{m=1}^{\infty} (1+|z+mT|)^q |u^{(j)}_{k+n}(z+mT)| \right)$$

$$\leq \|f\|_s \left(\sum_{k=0}^{\infty} \|u_{k+n}\|_s^{-2} \left(\sum_{m=1}^{\infty} (1+|z+mT|)^q |u^{(j)}_{k+n}(z+mT)| \right)^2 \right)^{1/2}.$$

Now observe that

$$\|(1+|z|)^q u^{(j)}(z)\|^2_{L^2(H^+,d\lambda_\nu)}$$

$$\leq \|f\|_s^2 \int_{H^+} \sum_{k=0}^{\infty} \|u_{k+n}\|_s^{-2} \left(\sum_{m=1}^{\infty} (1+|z+mT|)^q |u_{k+n}^{(j)}(z+mT)| \right)^2 y^{\nu-1} dxdy$$

$$\leq \|f\|_s^2 \sum_{k=0}^{\infty} \|u_{k+n}\|_s^{-2} \int_{H^+} \left(\sum_{m=1}^{\infty} (1+|z+mT|)^q |u_{k+n}^{(j)}(z+mT)| \right)^2 y^{\nu-1} dxdy. \quad \square$$

With assumptions as in Lemma 3.2, there exists a constant $C_j > 0$ such that

$$\|v_{k+n,q,j,T}\|_{L^2(H,d\lambda_\nu)} \leq C_j \nu^j (k+\nu)^j \frac{1}{\sqrt{\nu-s+1}\cdot 2^{\nu-s}} \left(\frac{(k-j+s/2)!(\nu-s)!}{(k-j+s/2+\nu-s)!} \right)^{1/2}.$$

Proof:

The triangle inequality gives

$$\|v_{k+n,q,j,T}\|_{L^2(H^+,d\lambda_\nu)} = \| \sum_{m=0}^{\infty} (1+|z+mT|)^q |u_{k+n}^{(j)}(z+mT)| \|_{L^2(H^+,d\lambda_\nu)}$$

$$\leq \sum_{m=0}^{\infty} \|(1+|z+mT|)^q u_{k+n}^{(j)}(z+mT)\|_{L^2(H^+,d\lambda_\nu)}. \tag{3.25}$$

Lemma 3.2 shows

$$\|(1+|z+mT|)^q u_{k+n}^{(j)}(z+mT)\|_{L^2(H^+,d\lambda_\nu)}$$

$$\leq C_r \sum_{w=1}^{j} \sum_{l=0}^{w} \frac{k!}{(k-(w-l))!} \frac{(\nu+1)!}{(\nu+1-w)!} \cdot \|(1+|z+mT|)^q u_{k+n+l-w}(z+mT)(z+mT+i)^{-j}\|_{L^2(H^+,d\lambda_\nu)}.$$

Observe

$$\|(1+|z+mT|)^q u_{k+n+l-w}(z+mT)(z+mT+i)^{-j}\|_{L^2(H^+,d\lambda_\nu)}$$

$$\leq \left(\| \frac{y^s}{|z+mT+i|^{s+4}} \|_{L^\infty(H^+,dxdy)} \right)^{1/2}.$$

$$\left(\int_{H^+}(1+|z+mT|)^{2(q-j)-s+4}\left(|\frac{z+m-i}{z+mT+i}|\right)^{2(k+l-w)}\frac{y^{\nu-s-1}}{|z+mT+i|^{2(\nu-s+1)}}dxdy\right)^{1/2}. \tag{3.26}$$

Moreover, because s is even and $\nu-s\geq 1$, it follows that $L^2(H,d\lambda_{\nu-s})$ is a model for the discrete series representation with parameter $\nu-s$. Define

$$u^s_{k+n+l-w}(z):=\left(\frac{z-i}{z+i}\right)^{k+l-w}\frac{1}{(z+i)^{\nu-s+1}},$$

and notice that $k+n+l-w\geq n+\frac{1-s}{2}=\frac{\nu-s+1}{2}\geq 0$, so $u^s_{k+n+l-w}\in L^2(H,d\lambda_{\nu-s+1})$. Let $n_s=\frac{\nu+1-s}{2}$. Because $2r+4\leq s$ and $q\leq 2r$, we know that $2(q-r)-s+4\leq 0$, and then using Claim 3.2,

$$\left(\int_{H^+}(1+|z+mT|)^{2(q-r)-s+4}\left(|\frac{z+mT-i}{z+mT+i}|\right)^{2(k+l-w)}\frac{y^{\nu-s-1}}{|z+mT+i|^{2(\nu-s+1)}}dxdy\right)^{1/2}$$

$$\begin{aligned}
&\leq \|u^s_{(k+l-w+s/2)+n_s}\|_{L^2(H,d\lambda_{\nu-s})}\\
&=\frac{\sqrt{\pi}}{\sqrt{\nu-s+1}\cdot 2^{\nu-s}}\left(\frac{(k+l-w+s/2)!(\nu-s)!}{(k+l-w+s/2+\nu-s)!}\right)^{1/2}\\
&\leq \frac{\sqrt{\pi}}{\sqrt{\nu-s+1}\cdot 2^{\nu-s}}\left(\frac{(k-j+s/2)!(\nu-s)!}{(k-j-s/2+\nu)!}\right)^{1/2}.
\end{aligned} \tag{3.27}$$

Next,

$$\left(\|\frac{y^s}{|z+mT+i|^{s+3}}\|_{L^\infty(H^+,dxdy)}\right)^{1/2}$$

$$\leq\left(\|\frac{1}{(1+|z+mT|)|^3}\|_{L^\infty(H^+,dxdy)}\right)^{1/2}\leq\frac{1}{(1+mT)^{3/2}}.$$

Combining this with (3.27) and (3.26), we conclude

$$(3.25)\leq C_r\sum_{m=1}^{\infty}\sum_{w=1}^{j}\sum_{l=0}^{w}$$

$$\frac{k!}{(k-(w-l))!}\frac{(\nu+1)!}{(\nu+1-w)!}\frac{1}{(1+mT)^{3/2}}$$

$$\cdot \frac{\sqrt{\pi}}{\sqrt{\nu - s + 1} \cdot 2^{\nu - s}} \left(\frac{(k - j + s/2)!(\nu - s)!}{(k - j - s/2 + \nu)!} \right)^{1/2}$$

$$\leq C_j \nu^j \sum_{m=1}^{\infty} \frac{1}{(1 + mT)^{3/2}}$$

$$\cdot \sum_{w=1}^{j} \frac{(k + \nu)!}{(k + \nu - j)!} \frac{1}{\sqrt{\nu - s + 1} \cdot 2^{\nu - s}} \left(\frac{(k - j + s/2)!(\nu - s)!}{(k - j - s/2 + \nu)!} \right)^{1/2}$$

$$\leq \frac{C_j}{T} \nu^j (k + \nu)^j \frac{1}{\sqrt{\nu - s + 1} \cdot 2^{\nu - s}} \left(\frac{(k - j + s/2)!(\nu - s)!}{(k - j - s/2 + \nu)!} \right)^{1/2} . \quad \square$$

With assumptions as in Lemma 3.2, there is a constant $C_{s,j} > 0$ such that

$$\sum_{k=0}^{\infty} \|u_{k+n}\|_s^{-2} \|v_{k+n,q,j,T}\|^2_{L^2(H^+, d\lambda_\nu)} \leq \frac{C_{j,s}}{T} \nu^j .$$

Proof:

Using Claim 3.2, we have

$$\|u_{k+n}\|_s^{-2} = \frac{(\nu + 1)}{\pi} \cdot 4^\nu (1 + \mu + 8(k + n)^2)^{-s} \left(\frac{(k + \nu)!}{k! \nu!} \right),$$

and by $(B.21)$, we have

$$(1 + \mu + 8(k + n)^2)^{-s} \leq (k + \nu)^{-2s} .$$

Therefore,

$$\|u_{k+n}\|_s^{-2} \|v_{k+n,q,j,T}\|^2_{L^2(H^+, d\lambda_\nu)}$$

$$\leq \frac{C_j}{T} \nu^{2j} \frac{(\nu + 1) \cdot 4^\nu}{(\nu - s + 1) \cdot 4^{\nu - s}} \left(\frac{(k + \nu)!}{k! \nu!} \right) \cdot (k + \nu)^{2(j - s)} \left(\frac{(k - j + s/2)!(\nu - s)!}{(k - j - s/2 + \nu)!} \right)$$

$$\leq \frac{C_j}{T} \nu^{2j} \frac{(\nu + 1) \cdot 4^\nu}{(\nu - s + 1) \cdot 4^{\nu - s}}$$

$$\left(\frac{(k + \nu + 2(j - s))!}{k! \nu!} \right) \cdot \left(\frac{(k - j + s/2)!(\nu - s)!}{(k - j - s/2 + \nu)!} \right) . \tag{3.28}$$

Using $-s \leq -2r - 4$, one can show $2(j - s) + 1 \leq -j - s/2$, and noting that $\nu \geq s$, it follows that

$$(3.28) \leq \frac{C_{j,s}}{T} \nu^{2j} \frac{\nu + 1}{(\nu - s + 1)} \frac{(\nu - s)!}{\nu!}$$

$$\cdot \left(\frac{(k-j+s/2)\cdots(k+1)}{(k+\nu-j-s/2)\cdots(k+\nu-j-2(s-j)+1)} \right) \tag{3.29}$$

$$(3.29) \leq \frac{C_{j,s}}{T}\nu^{2j} \left(\frac{(k-j+s/2)\cdots(k+1)}{(k-j+s/2)\cdots(k+1-s+j)} \right) \leq \frac{C_{j,s}}{T}\nu^{2j}\frac{1}{(1+k)^2}. \quad \square$$

Proof of Theorem 3.2 :

Let $\mathcal{F}_1 f$ be the Fourier transform of f in the real coordinate. Let $y > 0$. Then Proposition B proves in particular that for all $z = x + iy \in H$ and $s > \frac{1}{2}$, then

$$|f(z)| \leq C_{r,\nu,y}\|f\|_s \frac{1}{(1+|z|)^{2s}}.$$

Because $f \in W^4(\mathcal{H}_\mu)$, we know that $f(\cdot + iy) \in L^1(R)$, which means the Poisson summation formula applies. Using the assumption that $f \in Ann(\{\hat{\delta}_{k/T}\}_{k\geq 1}) = Ann(\{\hat{\delta}_{k/T}\}_{k=-\infty}^{\infty}$, we get

$$\sum_{m=-\infty}^{\infty} f(z+mT) = \sum_{m=\infty}^{\infty} \mathcal{F}_1\hat{\delta}_{m/T}(f)e^{2\pi imx} = 0.$$

Therefore,

$$u(z) = \sum_{m=1}^{\infty} f(z+mT) = -\sum_{m=0}^{\infty} f(z-mT).$$

The same argument used to estimate $\|u\|_{W^r(H^+,d\lambda_\nu)}$ proves there is a constant $C_s > 0$ such that

$$\|u\|_{W^r(H^-,d\lambda_\nu)} \leq \frac{C_{r,s}}{T}\nu^r\|f\|_s.$$

Then combining the estimates for H^+ and H^-, and setting $s = 2r+4$ proves

$$\|u\|_r \leq \frac{C_r}{T}\nu^r\|f\|_{2r+4} \leq \frac{C_r}{T}\|f\|_{3r+4}$$

when $r \in N_0$. The estimate for $r \geq 0$ and real follows by interpolation.

Finally, the solution u is unique for the same reason discussed at the end of Lemma 3.2, which completes the proof of Theorem 3.2. $\square$

Remark: Rapidly decreasing functions are not dense in $L^2(\mathcal{H}_\mu)$ (see, for example, the discussion around Lemma 2.2). Therefore, we cannot simply consider a subspace of such functions for which the Poisson summation formula holds, and then estimate and extend by density.

Proof of Theorem 5.1: From Theorems 3.2 and 3.2, it remains to show the space of invariant distributions $\mathcal{I}_\mu$ is modeled by $\langle\{\hat{\delta}_{k/T}\}_{k\in Z^+}\cup\{\delta^{(0)}\}\rangle$, which follows by definitions and Theorems 3.2 and 3.2. $\square$

Chapter 4

Obstructions to L^2 Solutions

We prove Theorem 1, which states that $\langle\{\mathcal{I}_\mu\}_{\mu\geq 0}\rangle$ is the space of distributional obstructions to the existence of $L^2(SM)$ solutions.

Let $\mu \in spec(\Box)$, $T > 0$ and $n \in Z$. If $f \in W^5(\mathcal{H}_\mu)$ has a transfer function $u \in \mathcal{H}_\mu$. Then $\hat{\delta}_{n/T}(f) = 0$. *Proof*:

First suppose that $\mu \geq 1$, and let $u \in \mathcal{H}_\mu$ be such that $f = u \circ \phi_T - u$. By extending the Fourier transform on $W^1(\mathcal{H}_\mu)$ as in definition (2.3), we see that $\hat{f}$ is continuous. Note that $\mathcal{H}_\mu$ takes the $L^2(R)$ norm, so

$$\hat{f} = \mathcal{F}(u \circ \phi_T^U - u) = (e^{-2\pi i T\xi} - 1)\hat{u},$$

in $L^2(R)$. Therefore,

$$\hat{u} = \frac{\hat{f}}{(e^{-2\pi i T\xi} - 1)} \tag{4.1}$$

in $L^2(R)$. Because $\hat{f}$ is continuous and $\hat{u} \in L^2(R)$, we conclude $\hat{f}(\frac{n}{T}) = 0$.

For the case $0 < \mu < 1$, suppose to the contrary that $\hat{\delta}_{n/T}(f) \neq 0$. In circle coordinates, let

$$f(\theta) = \Phi(\theta)\cos^{1+\nu}(\theta),$$

where $\Phi(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{2ik\theta}$. Then

$$\|\Phi\|_{C^0([-\pi/2,\pi/2])} \leq \sum_{k=-\infty}^{\infty} |c_k| \leq \|f\|_1.$$

So back in R-coordinates,

$$|f(x)| \leq \frac{\|f\|_1}{(1+x^2)^{(1+\nu)/2}} \in L^2(R) \cap L^1(R).$$

So $\hat{f}$ is again in $L^2(R)$, which means (4.1) holds, and as $\hat{f}$ is continuous, we conclude that near $\frac{n}{T}$,

$$\hat{u}(\xi) \sim \frac{1}{\xi - n/T}. \tag{4.2}$$

Contradiction

When $\mu \leq 0$, again suppose $\hat{\delta}_{n/T}(f) \neq 0$, and recall the norm for the model $L^2(H, d\lambda_\nu)$ is

$$\|f\|_{L^2(H,d\lambda_\nu)} = \int_R \int_R |f(x+iy)|^2 y^{\nu-1} dxdy.$$

Note that because $f \in W^1(\mathcal{H}_\mu)$, Lemma B shows $f(\cdot + iy) \in L^2(R) \cap L^1(R)$. By Lemma 2.2 we may fix any $y > 0$, and let $\mathcal{F}_1$ be the Fourier transform along $R + iy$. Then

$$\begin{aligned} 0 &= \int_R \left(\int_R |u(x - T + iy) - u(x+iy) - f(x+iy)|^2 dx \right) y^{\nu-1} dy \\ &= \int_R \left(\int_R |(e^{-2\pi i T\zeta} - 1)\mathcal{F}_1 u(\zeta + iy) - \mathcal{F}_1 f(\zeta + iy)|^2 dx \right) y^{\nu-1} dy, \end{aligned}$$

so

$$\mathcal{F}_1 u(\zeta + iy) = \frac{\mathcal{F}_1 f(\zeta + iy)}{(e^{-2\pi i T\zeta} - 1)}$$

in $L^2(R)$.

Note $\mathcal{F}_1 f$ is continuous, so $\mathcal{F}_1 u \sim \frac{1}{\zeta - n/T}$ in an ϵ-neighborhood $(\frac{n}{T} - \epsilon, \frac{n}{T} + \epsilon) \times \{y\}$. Hence, $\|u\|_{\mathcal{H}_\mu} = \infty$, which is a contradiction as before. $\square$

Let $\mu > 0$ and $f \in W^5(\mathcal{H}_\mu)$. The operator A_T defined in (1.8) represents as

$$A_T f(x) = \int_0^{-T} f(x+t)dt$$

in the $\mathcal{H}_\mu$ model, and when $\mu \leq 0$, it similarly represents as

$$A_T f(z) = \int_0^{-T} f(x+t)dt$$

in the $L^2(H, d\lambda_\nu)$ model.

Let $\mu \in spec(\Box)$. If $\mu > 0$, then (distributional) $Ker(A_T) = \langle \{\hat{\delta}_{n/T}\}_{n \in Z - \{0\}} \rangle$ in the $\mathcal{H}_\mu$ model. If $\mu \leq 0$, then (distributional) $Ker(A_T) = \langle \{\hat{\delta}_{n/T}\}_{n \in N} \rangle$. *Proof*:

First let $\mu > 0$. Let $\mathcal{D} \in \mathcal{E}'(\mathcal{H}_\mu)$ and $h \in C^\infty(\mathcal{H}_\mu)$. Then taking Fourier transforms, we see

$$(A_T \mathcal{D})h = -\mathcal{D}(A_T h)$$

$$= -\hat{\mathcal{D}}(\int_0^{-T} e^{2\pi i t \xi} dt \hat{h}(\xi)) = -\hat{\mathcal{D}}(\frac{e^{-2\pi i T \xi} - 1}{2\pi i \xi} \hat{h}(\xi)).$$

So

$$\langle \{\hat{\delta}_{n/T}\}_{n \in Z - \{0\}} \rangle \subset Ker(A_T).$$

For the other direction, calculus shows $\frac{e^{-2\pi i T\xi} - 1}{2\pi i \xi} \in C^\infty(R)$ and is invertible on $\cup_{n \in Z} (\frac{n}{T}, \frac{n+1}{T})$. So if $\mathcal{D} \in Ker(A_T) \cap \mathcal{E}'(\mathcal{H}_\mu)$, $supp(\hat{\mathcal{D}}) \subset \{\frac{n}{T}\}_{n \in Z - \{0\}}$. One checks that $\hat{\mathcal{D}} \neq \delta_{n/T}^{(r)}$ for any $r \geq 1$, so we conclude that $\hat{\mathcal{D}} \in \{\delta_{\frac{n}{T}}\}_{n \in Z - \{0\}}$.

For the case $\mu \leq 0$, recall $\mathcal{F}_1 f(x+iy) = 0$ for all $x \leq 0$. Then the same argument with $h(\cdot)$ replaced by $h(\cdot + i)$ proves

$$Ker(A_T) = \langle \{\hat{\delta}_{n/T}\}_{n \in Z} = \langle \{\hat{\delta}_{n/T}\}_{n \in N} \rangle,$$

where the last equality holds because $\hat{\delta}_{n/T}(f) = 0$ for all $n \leq 0$. $\Box$.

Let $\mu \in spec(\Box) - \{0\}, s > 1$ and $f \in W^s(\mathcal{H}_\mu)$. Then there exists $u \in \mathcal{H}_\mu$ such that

$$\mathcal{L}_U u = f if and only if u \circ \phi_T^U - u = A_T f.$$

The left equality implies the right by the fundamental theorem of calculus. For the converse,

$$\int_0^{-T} f \circ \phi_t^U dt = A_T f = u \circ \phi_T^U - u = \int_0^T \mathcal{L}_U u \circ \phi_t^U dt,$$

which implies

$$A_T(\mathcal{L}_U u - f) = 0.$$

By Claim 4, the (distributional) $Ker(A_T) = \langle \{\hat{\delta}_{n/T}\}_{n \in Z - \{0\}} \rangle$. So there exists $\{c_n\}_{n \in Z} \subset C$ such that

$$\mathcal{L}_U u - f = \mathcal{D} = \sum_{n=-\infty}^{\infty} c_n \hat{\delta}_{n/T}. \tag{4.3}$$

When $\mu \geq 1$, we take Fourier transforms and conclude

$$2\pi i \xi \hat{u} - \hat{f} = \sum_{n=-\infty}^{\infty} c_n \delta_{n/T}.$$

Because $u, f \in L^2(R)$, it follows that $\xi \hat{u}, \hat{f} \in L^2_{loc}(R)$.

For $0 < \mu < 1$, let $\hat{g} \in C_c^\infty(R)$ be a bump function supported on $[-1, 1]$ with $\hat{g}(0) = 1$. Additionally, fix $n \in Z$, and for all $m \in Z$, define

$$\hat{g}_{n,m}(\xi) := \hat{g}(n(\xi - m)).$$

Notice that

$$g_{n,m}(x) = \frac{1}{n} e^{2\pi i m x} g(\frac{x}{n}).$$

Then

$$|c_n| = |\left(\sum_{k=-\infty}^{\infty} c_k \hat{\delta}_k\right)(g_{n,m})| = |\langle \mathcal{L}_U u, g_{n,m}\rangle_{\mathcal{H}_\mu}| + |\langle f, g_{n,m}\rangle_{\mathcal{H}_\mu}|$$

$$\leq |\langle u, g'_{n,m}\rangle_{\mathcal{H}_\mu}| + |\langle f, g_{n,m}\rangle_{\mathcal{H}_\mu}| \leq \|u\|_{\mathcal{H}_\mu}\|g'_{n,m}\|_{\mathcal{H}_\mu} + \|f\|_{\mathcal{H}_\mu}\|g_{n,m}\|_{\mathcal{H}_\mu}, \tag{4.4}$$

where the last equality holds because $\mathcal{L}_U$ is skew-adjoint on $L^2(SM)$ and therefore also on $\mathcal{H}_\mu$.

We will estimate the values $\|g'_{n,m}\|_{\mathcal{H}_\mu}, \|g_{n,m}\|_{\mathcal{H}_\mu}$ with the following lemma.

Let $0 < \mu < 1$ and $h \in \mathcal{H}_\mu$. Then there exists $q > 0$ and a constant $C_{q,\nu} > 0$ such that

$$\|h\|_{\mathcal{H}_\mu} \leq C_{q,\nu}\|h\|_{L^1(R)}(\|h\|_{L^q(\{|x|\geq 1\})} + \|h\|_{L^\infty(\{|x|<1\})}).$$

Proof:

For a given function $h \in \mathcal{H}_\mu$,

$$\|h\|^2_{\mathcal{H}_\mu} = \int_{R^2} \frac{h(r+x)h(x)}{|r|^{1-\nu}} dr dx$$

$$= \int_R h(x)\left(\int_R \frac{h(r+x)}{|r|^{1-\nu}} dr\right) dx. \tag{4.5}$$

Observe that

$$\int_R \frac{h(r+x)}{|r|^{1-\nu}} dr \leq \int_{\{|r|\geq 1\}} \frac{|h(r+x)|}{|r|^{1-\nu}} dr + \int_{\{|r|<1\}} \frac{|h(r+x)|}{|r|^{1-\nu}} dr,$$

and notice

$$\int_{\{|r|<1\}} \frac{|h(r+x)|}{|r|^{1-\nu}} dr \leq C_\nu \|h\|_{L^\infty(R)}. \tag{4.6}$$

Additionally, let $p, q > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then Holder's inequality gives

$$\int_{\{|r|\geq 1\}} \frac{|h(r+x)|}{|r|^{1-\nu}} dr \leq \left(\int_{\{|r|\geq 1\}} \frac{1}{|r|^{p(1-\nu)}} dr\right)^{1/p} \left(\int_{\{|r|\geq 1\}} |h(r+x|)|^q dr\right)^{1/q}.$$

Let $p > 0$ such that

$$p(1-\nu) > 1,$$

so that $(1-\nu) > \frac{1}{p} = 1 - \frac{1}{q}$, and therefore $q < \frac{1}{\nu}$. Because $\nu < 1$, we can choose p such that $1 < q < \frac{1}{\nu}$, and conclude

$$\int_{\{|r|\geq 1\}} \frac{|h(r+x)|}{|r|^{1-\nu}} dr \leq C_{q,\nu} \|f\|_{L^q(R}. \tag{4.7}$$

Then

$$(4.5) \leq C_{q,\nu} \|h\|_{L^1(R)} (\|h\|_{L^q(\{|x|\geq 1\})} + \|h\|_{L^\infty(\{|x|<1\})}). \quad \square$$

This means there exists $q > 1$ such that

$$(4.4) \leq C_{\nu,q} \|u\|_{\mathcal{H}_\mu} \|g'_{n,m}\|_{L^1} (\|g'_{n,m}\|_{L^q} + \|g'_{n,m}\|_{L^\infty})$$

$$+ \|f\|_{\mathcal{H}_\mu} \|g_{n,m}\|_{L^1} (\|g_{n,m}\|_{L^q} + \|g_{n,m}\|_{L^\infty}). \tag{4.8}$$

Notice

$$\frac{d}{dx} g_{nm}(x) = \frac{1}{n^2} g'_{n,m}(\frac{x}{n}) e^{2\pi i m x} + \frac{2\pi i m}{n} e^{2\pi i m x} g(\frac{x}{n}).$$

Then there is a constant $C > 0$ such that

$$\|g'_{n,m}\|_{L^1} + \|g_{n,m}\|_{L^1} \leq Cm, \; \|g'_{n,m}\|_{L^{q_1}} + \|g_{n,m}\|_{L^{q_1}} \leq Cmn^{1/q-1},$$

$$\|g'_{n,m}\|_{L^\infty} + \|g_{n,m}\|_{L^\infty} \leq C\frac{m}{n}.$$

Therefore,

$$\lim_{n\to\infty} \|g'_{n,m}\|_{\mathcal{H}_\mu} = 0.$$

From (4.8) and by letting $n \to \infty$ we conclude

$$c_m = 0.$$

Finally, the case $\mu \leq 0$ is clear, because u and f are holomorphic functions, so $\mathcal{L}_U u - f$ is still holomorphic, but given (4.3), this only happens when $\mathcal{L}_U u - f \equiv 0$. $\square$

Let $\mu \geq 0$, $T > 0$, and $f \in W^9(\mathcal{H}_\mu)$, and suppose there exists $u \in \mathcal{H}_\mu$ such that

$$f = u \circ \phi_T^U - u. \tag{4.9}$$

Then $\delta^{(0)}(f) = 0$. *Proof*:

First let $\mu > 0$. By [11] we have $Uf \in W^8(\mathcal{H}_\mu)$, and by flow invariance $U\delta^{(0)} = 0$. Proposition 4 shows $f \in Ann(\{\hat{\delta}_n\}_{n=-\infty}^{\infty})$ so

$$\hat{\delta}_{n/T}(Uf) = 2\pi i \frac{n}{T} \hat{\delta}_{n/T}(f) = 0.$$

Then Theorem 3.1 shows there exists $g \in W^{4/3}(\mathcal{H}_\mu)$ such that

$$Uf(x) = g(x - T) - g(x).$$

Then for every $M \in R^-$,

$$f(x) - f(M) = \int_M^x Uf(t)dt = \int_M^x [g(t - T) - g(t)]dt$$

$$= \int_M^x g(t - T)dt - \int_M^x g(t)dt = \int_x^{x-T} g(t)dt - \int_M^{M-T} g(t)dt.$$

Write $g(\theta) = \Phi(\theta) \cos^{1+\nu}(\theta)$. Then by Sobolev embedding, $\|\Phi\|_{C^0(R)} \leq \|g\|_1$, so that for all $x \in R$,

$$|g(x)| \leq \frac{\|g\|_1}{\sqrt{1 + |x|}}.$$

Hence,

$$\lim_{M\to-\infty}\int_{M-T}^{M} g(t)dt = 0,$$

and for the same reason,

$$\lim_{M\to-\infty} f(M) = 0,$$

which means

$$f(x) = \int_0^{-T} g(x+t)dt.$$

Now Lemma 4 gives a solution u to $\mathcal{L}_U u = g$. By Lemmas 4.7 and 4.8 of [2], we get $\delta^{(0)}(g) = 0$. Then flow invariance of $\delta^{(0)}$, gives that for all $t \in [0,1]$

$$\delta^{(0)}(g \circ \phi_t^U) = \delta^{(0)}(g) = 0.$$

So

$$0 = \int_0^{-T} \delta^{(0}(g(\cdot - t))dt = \delta^{(0)}(\int_0^{-T} g(\cdot - t)dt) = \delta^{(0)}(f).$$

For $\mu = 0$, the same argument gives us a function g such that $f(x+iy) = \int_0^{-T} g(x-t+iy)dt$. Lemma 4 again gives a solution u to $\mathcal{L}_U u = g$. Because $u \in L^2(H, d\lambda_\nu)$, Lemma 4.9 of [2] shows $\delta^{(0)}(g) = 0$, and again flow invariance of $\delta^{(0)}$ proves $\delta^{(0)}(f) = 0$.

Let $\mu < 0$. Then there exists $f \in W^3(\mathcal{H}_\mu)$ with a solution u to the cohomological equation (4.9) such that $\delta^{(0)}(f) \neq 0$. *Proof*:

We safely restrict ourselves to the holomorphic discrete series, so $\mu < 0$ implies $n = \frac{\nu+1}{2} \geq 2$. Let $f \in W^3(\mathcal{H}_\mu)$ be such that $\delta^{(0)}(f) \neq 0$. Then Lemma 4.5 of [2] shows there is a solution $u \in \mathcal{H}_\mu$ such that $\mathcal{L}_U u = f$. Consider $A_T f = \int_0^{-T} f(x+t)dt$, and note Lemma 4 applies, so we conclude that $f(\cdot) = u(\cdot - T) - u(\cdot)$. Finally, notice that $\delta^{(0)}$ is flow invariant, so

$$\delta^{(0)}(f) = \delta^{(0)} \int_0^{-T} f(\cdot + t) dt = \int_0^{-T} \delta^{(0)}(f(\cdot + t)) dt \neq 0. \quad \square$$

Proof of Theorem 1: Combining Lemmas 4, 4 and Proposition 4, we conclude.

$\square$

Chapter 5

Equidistribution of Horocycle Maps

In this section we assume that SM is compact, so the Laplacian has only pure point spectrum. In each irreducible component $\mathcal{K}_\mu$, we correspond to the invariant distributions $\{\mathcal{D}_n\}_{n\in Z} \cup \{\mathcal{D}^0\} \subset \mathcal{I}_\mu$ the invariant distributions $\{\hat{\delta}_n\}_{n\in Z} \cup \{\delta^{(0)}\} \subset \mathcal{I}_\mu(\mathcal{H}_\mu)$ by the formulas

$$\mathcal{D}_n := (Q_\mu)^* \hat{\delta}_n, \quad \mathcal{D}^0 := (Q_\mu)^* \delta^{(0)},$$

for all $n \in Z$.

Let

$$\alpha(\mu_0) = \frac{(1-\sqrt{1-\mu_0})^2}{4(3-\sqrt{1-\mu_0})},$$

where μ_0 is the spectral gap. For all $\mu > 0$ and $\mathcal{D} \in \{\mathcal{D}_k\} \cup \{\mathcal{D}^0\} \subset \mathcal{I}_\mu$, define

$$\mathcal{S}_\mathcal{D} := \begin{cases} \alpha(\mu_0) if \mathcal{D} = \mathcal{D}_k, \ k \neq 0 \\ \frac{1-\sqrt{1-\mu}}{2} if \mathcal{D} = \mathcal{D}_0 \\ \frac{1+\sqrt{1-\mu}}{2} if \mathcal{D} = \mathcal{D}^0. \end{cases}$$

Let ϕ_1^U be the horocycle map on the unit tangent bundle SM of a compact hyperbolic Riemann surface M with spectral gap $\mu_0 > 0$, and let $s \geq 6$. Then there is a constant $C_s > 0$ such that for all $(x_0, N) \in SM \times Z^+$ and $f \in W^{2s+2}(SM)$, we have

$$|\frac{1}{N}\sum_{k=0}^{N-1} f(\phi_k^U x_0)| \leq |\bigoplus_{\mu\in spec(\Box)} \sum_{\mathcal{D}\in\mathcal{I}_\mu(SM)} c_\mathcal{D}(x_0, N, s)\mathcal{D}(f)N^{-\mathcal{S}_\mathcal{D}} \oplus \mathcal{R}(x, N, s)(f)|, \tag{5.1}$$

where the remainder distribution $\mathcal{R}(x, N, s)$ satisfies

$$\|\mathcal{R}(x, N, s)\|_{W^{-s}(\mathcal{K}_\mu)} \leq \frac{C_s}{N},$$

and for all $\mathcal{D} \in \mathcal{I}(SM)$,

$$|c_{\mathcal{D}}| \leq C_s.$$

Remainder distribution

Let $\mu > 0$ and $s \geq 6$. Then there exists a constant $C_s > 0$ such that for all $(x_0, N) \in SM \times N$,

$$\frac{\|\mathcal{R}(x_0, N, 1, s)\|_{W^{-s}(SM)}}{N} \leq \frac{C_s}{N}.$$

Proof:

Let $f \in W^s(SM)$. Because $\mathcal{R}(x_0, N, 1, s) \in \mathcal{I}(SM)^\perp$, we can write $f = f_{\mathcal{I}} \oplus f_{\mathcal{C}}$, where $f_{\mathcal{I}} \in (Ann(\mathcal{I}(SM))^\perp \subset Ker(\mathcal{R}(x_0, N, 1, s))$, and $f_{\mathcal{C}} \in Ann(\mathcal{I}(SM))$ is the coboundary component. Then by the splitting in (5.1),

$$\frac{\mathcal{R}(x_0, N, 1, s)}{N} f = \frac{\mathcal{R}(x, N, 1, s)}{N} f_{\mathcal{C}} = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\phi_n^U(x_0)} f_{\mathcal{C}}. \tag{5.2}$$

Theorem **??** and Sobolev embedding show there exists $\frac{1}{2} < r < s, C_s > 0$ and a (unique) transfer function $u \in W^r(SM)$ to the cohomological equation (1.2) satisfying

$$\|u\|_{C^0(SM)} \leq C_r \|u\|_r \leq C_s \|f_{\mathcal{C}}\|_s.$$

Therefore,

$$(5.2) \leq \frac{1}{N} \sum_{n=0}^{N-1} [u(\phi^U_{(n+1)T} x_0) - u(\phi^U_{nT} x_0)]$$

$$= \frac{1}{N}[u(\phi^U_{NT}x_0) - u(x_0)] \leq \frac{C_s}{N}\|f_C\|_s \leq \frac{C_s}{N}\|f\|_s,$$

where the last inequality holds by orthogonal projection. □

Flow invariant distributions

We remark that in Theorem 1.5 of [6] Flaminio-Forni proved asymptotics for the decay of the invariant distributions of the horocycle flow in each irreducible representation $\mathcal{K}_\mu$ to be

$$\begin{cases} c_0(x_0, N, s)\mathcal{D}_0 \sim N^{-1/2(1-\nu)} \\ d_0(x_0, N, s)\mathcal{D}^0 \sim N^{-1/2(1+\nu)} \end{cases}. \tag{5.3}$$

First we need a Lemma.

Let $\mu > 0$, $s > 1$ and $\sum_{n=-\infty}^{\infty} c_n(x, N, 1, s)\mathcal{D}_n \in W^{-s}(\mathcal{K}_\mu)$. Then there is a constant $C_s > 0$ such that for all n,

$$|c_n(x, N, 1, s)| \leq C_s(1 + |\nu|)^s n^s.$$

Proof:

First suppose that $s \in N$ and let $f_0 \in C_c^\infty(R)$ be supported in $[-1, 1]$ and satisfy $\int_R f_0(x)dx = 1$, and let $\chi_\tau = e^{2\pi i\tau x} f_0 \in C_c^\infty(R)$ be such that $\hat{\delta}_\tau(\chi_\tau) = 1$ and for all $\tau \neq n$, $\hat{\delta}_n(\chi_\tau) = 0$. Note that because $\chi_\tau \in C_c^\infty(R)$, we get for free that $\delta^{(0)}(\chi_\tau) = 0$. Then Sobolev embedding gives

$$|c_n| = |c_n\mathcal{D}_n(Q_\mu^{-1}\chi_n)|$$

$$\leq |\left(\left(\sum_{n=-\infty}^{\infty} c_n\mathcal{D}_n + d_0\mathcal{D}^0\right) \oplus \mathcal{R}\right)(Q_\mu^{-1}\chi_n)| + |\mathcal{R}(Q_\mu^{-1}\chi_n)|$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Q_\mu^{-1}\chi_n(\phi_n^U(x_0)) + C_s\|Q_\mu^{-1}\chi_n\|_s \leq C_s\|\chi_n\|_s.$$

We can estimate $\|\chi_n\|_s$ using our concrete formulas for X, Y, Θ in $(1+\triangle)^{s/2} = (1-(X^2+Y^2+\Theta^2))^{s/2}$. Because $supp(f_0) \subset [-1,1]$ and $n \geq 1$, we have

$$\|\chi_n\|_s \leq C_s(1+|\nu|)^s\|(1+U^s)\chi_n\|_0 \leq C_s n^s\|f_0\|_s.$$

Then the Lemma for real $s > 1$ follows by interpolation. $\square$

Let $\mu > 0$ and $s \geq 6$. Then there exists a constant $C_s > 0$ such that for all $(x_0, N) \in SM \times N$,

$$|c_0(x_0, N, 1, s)| \leq N^{-(1-\Re\sqrt{1-\mu})/2},$$

and

$$|d_0(x_0, N, 1, s)| \leq C_s N^{-(1+\Re\sqrt{1-\mu})/2}.$$

Proof:

We prove the decay estimate for $c_0(x_0, N, 1, s)$. Let $f_0 \in C^\infty(\mathcal{K}_\mu) \cap Ker(\mathcal{D}^{(0)})$ be such that $\mathcal{D}^0(f_0) = 1$ and $\mathcal{D}_0(f_0) = 0$.

We have

$$\int_0^1 \phi_t^U\left(\left(\sum_{n=-\infty}^{\infty} c_n(x_0, N, 1, s)\mathcal{D}_n + d_0(x_0, N, 1, s)\mathcal{D}^0\right) \oplus \frac{\mathcal{R}(x_0, N, 1, s)}{N}\right) dt$$

$$= \int_0^1 \phi_t^U\left(\frac{1}{N}\sum_{n=0}^{N-1}\delta_{\phi_n^U(x_0)}\right) dt = \frac{1}{N}\int_0^N (\phi_t^U(x_0))^* dt.$$

Additionally, using Proposition 5, we get

$$\int_0^1 \phi_t^U\left(d_0(x_0, N, 1, s)\mathcal{D}^{(0)} + \mathcal{R}(x_0, N, 1, s)\right) dt \in W^{-s}(SM),$$

and Sobolev embedding shows

$$\frac{1}{N}\int_0^N (\phi_t^U(x_0))^* dt \in W^{-s}(SM).$$

Hence,

$$\int_0^1 \phi_t^U \left(\sum_{n=-\infty}^{\infty} c_n(x_0, N, 1, s)\mathcal{D}_n \right) dt \in W^{-s}(SM).$$

Then we may separate the integral and conclude

$$\int_0^1 d_0(x_0, N, 1, s)\phi_t^U \mathcal{D}_0 dt$$

$$= \frac{1}{N}\int_0^N (\phi_t^U(x_0))^* dt - \int_0^1 \phi_t^U \left(\sum_{n\in Z} c_n\mathcal{D}_n \right) dt + \int_0^1 \phi_t^U \mathcal{R} dt.$$

In the same way one shows $\sum_{n=-\infty}^{\infty} c_n\mathcal{D}_n \in W^{-s}(\mathcal{K}_\mu)$, so Lemma 5 shows $|c_n| \leq C_s n^s$. Because the unitary equivalence Q_μ intertwines vector fields, it also intertwines the flow ϕ_t^U. Therefore, we get

$$(\phi_t^U \mathcal{D}_n)(f_0) = (Q_\mu^* \hat{\delta}_n)(f_0 \circ \phi_{-t}^U) = \hat{\delta}_n(Q_\mu(f_0 \circ \phi_{-t}^U)) = \hat{\delta}_n((Q_\mu f_0) \circ \phi_{-t}^U)$$

$$= \hat{\delta}_n((Q_\mu f_0)(\cdot + t)) = e^{2\pi int}\hat{\delta}_{2\pi n}(Q_\mu f_0) = e^{2\pi int}\mathcal{D}_n(f_0). \tag{5.4}$$

Notice the Fourier transform $\mathcal{F}(Q_\mu f_0)$ decays faster than the reciprocal of any polynomial, so we get

$$\sum_{n\in Z} c_n\mathcal{D}_n(f_0) = \sum_{n\in Z} c_n\hat{\delta}_n(Q_\mu f_0)$$

converges absolutely. Then using (5.4),

$$\int_0^1 (\phi_t^U)^* \sum_{n\in Z} c_n\mathcal{D}_n(f_n)dt = \sum_{n\in Z} c_n \int_0^1 (\phi_t^U)^*\mathcal{D}_n(f_n)dt = 0. \tag{5.5}$$

Next, because $\mathcal{D}_0$ is horocycle flow invariant, and $\mathcal{D}^0(f_0) = 1$, Proposition 5 together with Theorem 1.5 of [2] give a constant $C_s > 0$ such that for all $(x, N) \in SM \times N$,

$$|d_0(x_0, N, 1, s)| \leq |\frac{1}{N}\int_0^N f_0(\phi_t^U(x_0))dt| + |\int_0^1 \phi_t^U(\mathcal{R})(x_0, N, 1, s)(f_0)dt|$$

$$\leq C_s N^{-(1+\Re\sqrt{1-\mu})/2}.$$

The estimate for the coefficient $c_0(x_0, N, 1, s)$ follows in the same way. $\square$

Invariant Distributions for the Map

Let $\mu \in spec(\Box)$ and $\tau \in Z$. Then

$$|c_\tau(x_0, N, 1, s)| \leq C\tau T^{\alpha(\mu_0)},$$

where $\alpha(\mu_0) = \frac{(1-\sqrt{1-\mu_0})^2}{8(3-\sqrt{1-\mu_0})}$, and $\mu_0 > 0$ is again the spectral gap of the Laplacian $\triangle_{SM}$ on SM. *Proof*:

As in the proof of Proposition 5, we isolate the coefficients c_n by

$$c_\tau(x_0, N, 1, s)\mathcal{D}_\tau = \sum_{n\in Z}\int_0^1 e^{-2\pi i n\tau t} c_n(x, N, 1, s)\phi_t^U \mathcal{D}_n dt$$

$$= \int_0^1 e^{-2\pi i\tau t}\phi_t^U \left(\sum_{n\in Z} c_n(x, N, 1, s)\mathcal{D}_n \oplus \mathcal{R}\right) dt - \int_0^1 e^{-2\pi i\tau t}\phi_t^U \mathcal{R} dt$$

$$= \int_0^1 e^{-2\pi i\tau t}\phi_t^U \frac{1}{N}\sum_{n=0}^{N-1}(\phi_n^U(x_0))^* dt - \int_0^1 e^{-2\pi i\tau t}\phi_t^U \mathcal{R} dt$$

$$= \frac{1}{N}\int_0^N e^{-2\pi i\tau t}(\phi_t^U(x_0))^* dt - \int_0^1 e^{-2\pi i\tau t}\phi_t^U \mathcal{R} dt. \tag{5.6}$$

Define

$$\gamma_{T,\tau} = \frac{1}{N}\int_0^N e^{-2\pi i\tau t}(\phi_t^U(x_0))^* dt.$$

By Proposition 5 and Minkowski's integral inequality,

$$\| \int_0^1 e^{-2\pi i\tau t}\phi_t^U \mathcal{R}(x_0, N, 1, s) dt\|_{-1} \leq \int_0^1 \|\mathcal{R}(x_0, N, 1, s)\|_{-1} dt \leq \frac{C_s}{N}.$$

Therefore, it remains to estimate the integral for the twisted horocycle flow, which is given to us by a recent result of Venkatesh.

[Venkatesh (2010)] Let $\int_{SM} f dvol = 0$. Then there exists a constant $C > 0$ such that for all $(x_0, N) \in SM \times N$ and $\tau \in R$, we have

$$|\gamma_{T,\tau}(f)| \leq C\|f\|_{W^1(SM)} N^{-\alpha(\mu_0)},$$

where $\alpha(\mu_0) = \frac{(1-\sqrt{1-\mu_0})^2}{8(3-\sqrt{1-\mu_0})}$, and $\mu_0 > 0$ is again the spectral gap.

Proof of 916:

Combining this with (5.6) and Proposition 5, we conclude

$$|c_\tau(x, N, 1, s)|\|\mathcal{D}_\tau\|_{-1} \leq CN^{-\alpha(\mu_0)}.$$

Then Proposition 5 follows by showing there is a constant $C > 0$ such that $\|\mathcal{D}_\tau\|_{-1} = \|\hat{\delta}_\tau\|_{-1} \leq C\tau$. $\square$

Venkatesh's proof of Theorem 5 is short, and we reproduce it here for the convenience of the reader.

Proof of Theorem 5 :

Let

$$\varpi = \begin{cases} 0 if \mu_0 \geq 1 \\ \sqrt{1-\mu_0} if 0 < \mu_0 < 1 \end{cases}.$$

Let $H > 1$. Let ρ_H be a distribution on $W^2(SM)$ defined by

$$\rho_H(f) = \frac{1}{H}\int_0^H e^{2\pi i\tau t} f(\phi_t^U(x_0))dt$$

where $f \in W^2(SM)$. So

$$f * \rho_H(x_0) = \frac{1}{H}\int_0^H e^{-2\pi i\tau h} f(\phi_h^U(x_0))dh,$$

and $f*\rho_H \in W^2(SM)$.

Let $f \in C^\infty(\Gamma\backslash G)$ and denote $f*\sigma_H$ the right convolution of f by ρ_H. One checks there is a constant $C>0$ such that

$$|\gamma_{T,\tau}(f)-\gamma_{T,\tau}(f*\rho_H)| \leq C\frac{H}{T}\|f\|_{W^2(SM)},$$

and by Cauchy Schwartz we also have

$$|\gamma_{T,\tau}(f*\rho_H)|^2 \leq \gamma_{T,0}(|f*\rho_H|^2).$$

Theorem 1.5 of [6] proves

$$\left(\gamma_{T,0}(|f*\rho_H|^2)-vol(|f*\rho_H|^2)|\right) \leq CT^{-1/2(1-\sqrt{1-\mu_0})}\|f*\rho_H\|_{W^4(SM)}, \tag{5.7}$$

where μ_0 is the spectral gap of the laplacian $\triangle_{SM}$.

Notice

$$|f*\rho_H(x_0)|^2 = \frac{1}{H^2}\int_{[0,H]^2} e^{-2\pi i\tau(h_1-h_2)}f(\phi^U_{h_1}(x_0))\overline{f(\phi^U_{h_2}(x_0))}dh_1dh,$$

and applying (5.7), we have

$$\begin{aligned}
|\gamma_{T,\tau}(f)| &\leq C\frac{H}{T}\|f\|_1+\left(\frac{1}{H^2}\int_{h_1,h_2\in[0,H]^2}\gamma_{T,0}(\phi^U_{h_1}f\overline{\phi^U_{h_2}f})dh_1dh_2\right)^{1/2}\\
&\leq C\frac{H}{T}\|f\|_1+\left(\frac{1}{H^2}\int_{h_1,h_2\in[0,H]^2}\langle\phi^U_{h_1}f,\phi^U_{h_2}f\rangle_{L^2(SM)}dh_1dh_2\right)^{1/2}\\
&\quad+\left(T^{-1/2(1-\sqrt{1-\mu_0})}\left(\gamma_{T,0}(|f*\rho_H|^2)-vol(|f*\rho_H|^2)|\right)\right)^{1/2}\\
&\leq C\frac{H}{T}\|f\|_1+\left(\frac{1}{H^2}\int_{h_1,h_2\in[0,H]^2}\langle\phi^U_{h_1-h_2}f,f\rangle dh_1dh_2\right)^{1/2}\\
&\quad+T^{-1/4(1-\sqrt{1-\mu_0})}\left(\sup_{(h_1,h_2)\in[0,H]^2}\|\phi^U_{h_1}f\cdot\overline{\phi^U_{h_2}f}\|_2\right)^{1/2}.
\end{aligned}$$

Then using quantitative mixing of the horocycle flow (see [17]) and basic properties of Sobolev norms, we conclude

$$\langle \phi_h^U f, f \rangle \leq C_\epsilon (1 + |h|)^{2\kappa - 1 + \epsilon} \|f\|_2^2$$

for some $C_\epsilon > 0$ and

$$\sup_{(h_1, h_2) \in [0, H]^2} \|\phi_{h_1}^U f \cdot \overline{\phi_{h_2}^U f}\|_2 \leq (1 + |h_1| + |h_2|)^2 \|f\|_2^2.$$

This implies

$$|\gamma_{T,\tau}(f)| \leq C(\frac{H}{T} + H^{-\varpi - 1/2 + \epsilon} + T^{-1/4(1-\varpi)})\|f\|_2.$$

Then choosing $H^{\alpha - 1/2} = HT^{-1/4(1-\varpi)}$ gives the result. $\square$

We have now given an upper bound for the rate of decay of the remainder distribution and all invariant distributions. To finally *prove Theorem* **??**, we need a Lemma.

Let $\mu > 0, s \geq 2$ and $f \in W^s(\mathcal{H}_\mu)$. Then for all $\xi \in R$,

$$|\hat{f}(\xi)| \leq C_s \|f\|_{W^{s+2}(\mathcal{H}_\mu)} (1 + |\xi|)^{-(s+1)}.$$

Proof:

Observe that

$$\begin{aligned} \|\xi^s \hat{f}\|_{C^0(R)} &\leq \|\xi^s \hat{f}\|_{L^2(R)} + \|\frac{d}{d\xi}(\xi^s \hat{f})\|_{L^2(R)} \\ &\leq \|f^{(s)}\|_{L^2(R)} + \|x f^{(s)}\|_{L^2(R)}. \end{aligned} \tag{5.8}$$

Next, Lemma B shows that under the change of variable $x = \tan(\theta)$,

$$|f^{(s)}(x)| = |U^s f(\theta)| \leq C_s \sum_{j=1}^{s} \cos^{s+j}(\theta) |f^{(j)}(\theta)|,$$

and formula $(B.6)$ proves

$$|f^{(j)}(\theta)| \leq C_s(1+|\nu|)^j \sum_{k=0}^{j} |\Phi^{(k)}(\theta)| \cos^{1-j+\nu}(\theta),$$

where $\Phi(\theta) = \sum_{m=-\infty}^{\infty} c_m e^{-2\pi i m\theta}$. By Sobolev's inequality, each $|\Phi^{(k)}(\theta)| \leq \|f\|_{W^{k+2}(\mathcal{H}_\mu)}$. Combining and switching back to R coordinates, we find a constant $C_s > 0$ such that for all $x \in R$,

$$|f^{(s)}(x)| \leq C_s(1+|\nu|)^j(1+|x|)^{-(s+1)}\|f\|_{s+2}. \tag{5.9}$$

By Lemma 6.3 of [11], we conclude

$$(5.9) \leq C_s(1+|x|)^{-(s+1)}\|f\|_{2s+2}.$$

Therefore,

$$(5.8) \leq C_r\|f\|_{2s+2} \quad \square$$

Now we prove Theorem 5. Recall that $\alpha(\mu_0) = \frac{(1-\sqrt{1-\mu_0})^2}{8(3-\sqrt{1-\mu_0})}$. By Lemma 5, there is a constant $C_s > 0$ such that

$$|\mathcal{D}_n(f)| = |\hat{\delta}_n(Q_\mu f)| \leq C\frac{\|f\|_{2s+2}}{n^s}.$$

Therefore, the series

$$\frac{1}{N}|\sum_{k=0}^{N-1} f(\phi_k^U x_0)| = |\left(\sum_{n\in Z} c_n(x_0,N,s)\mathcal{D}_n(f) + d_0(x,N,s)\mathcal{D}^0(f)\right) \oplus \frac{\mathcal{R}(x,N,s)(f)}{N}|$$

$$\leq \sum_{n\in Z} |n|^{s+1}N^{-\alpha(\mu_0)}\|f\|_{2s+2}|n|^{-s+3} + C_s\|f\|_{2s+2}N^{-(1+\nu)/2} + \frac{\|f\|_s}{N}$$

converges absolutely, and Theorem 5 follows.

Appendix A

Appendix A

Formulas for the Principal and Complementary series

Vector fields

The models for the principal and complementary series are discussed in Section 2.

Let $\mu > 0$. The vector fields for the $\mathcal{H}_\mu$ model on R are

$$X = d\pi_\nu(X) = -(1+\nu) - 2x\tfrac{\partial}{\partial x}$$

$$\Theta = d\pi_\nu(\Theta) = -(1+\nu)x - (1+x^2)\tfrac{\partial}{\partial x}$$

$$Y = d\pi_\nu(Y) = (1+\nu)x - (1-x^2)\tfrac{\partial}{\partial x}$$

$$U = d\pi_\nu(U) = -\tfrac{\partial}{\partial x}$$

$$V = d\pi_\nu(V) = (1+\nu)x + x^2\tfrac{\partial}{\partial x}.$$

By the change of variable $x = \tan(\theta)$, the vector fields in the circle model are

$$X = d\pi_\nu(X) = -(1+\nu) - \sin(2\theta)\tfrac{\partial}{\partial\theta}$$

$$\Theta = d\pi_\nu(\Theta) = -(1+\nu)\tan(\theta) - \tfrac{\partial}{\partial\theta}$$

$$Y = d\pi_\nu(Y) = (1+\nu)\tan(\theta) - \cos(2\theta)\tfrac{\partial}{\partial\theta}$$

$$U = d\pi_\nu(U) = -\cos^2(\theta)\tfrac{\partial}{\partial\theta}$$

$$V = d\pi_\nu(V) = (1+\nu)\tan(\theta) + \sin^2(\theta)\tfrac{\partial}{\partial\theta}.$$

Remark : We denote both the R-model and the circle model by $\mathcal{H}_\mu$.

Discreteseries

The following commutations relations of the formulas in Claim 2.1 hold:

$$[X,Y] = (XY - YX) = 2\Theta,\ [\Theta,Y] = 2X,\ [\Theta,X] = -2Y,\ [X,U] = 2U.$$

These formulas agree with the commutation relations of the matrices.

Construction of basis

The goal of this subsection is to construct an orthogonal basis of vectors $\{u_n\}_{n=-\infty}^{\infty} \subset C^\infty(\mathcal{H}_\mu)$ for the irreducible represention space $\mathcal{H}_\mu$, which we do using the annihilation and creation operators $X \pm iY$.

We have $\cos^{1+\nu}(\theta) \in Ker(\Theta)$.

For all $n \in Z$, define

$$u_n := e^{-2in\theta} \cos^{1+\nu}(\theta).$$

Let $n \in Z^+$. Then

$$\begin{cases} (X+iY)u_n = -(1+\nu+2n)u_{n+1} \\ (X-iY)u_n = [-(1+\nu)+2n]u_{n-1}. \end{cases}$$

We have

$$X+iY = -(\nu+1)(1-i\tan(\theta) - ie^{-2i\theta}\frac{\partial}{\partial\theta},$$

and then note

$$\textbf{-(X + i Y)} \cos^{\nu+1}(\theta)$$

$$= [(\nu+1) - i(\nu+1)\tan(\theta)]\cos^{\nu+1}(\theta) + ie^{-2i\theta}(\cos^{\nu+1}(\theta))'$$

$$= (\nu+1)\cos^{\nu+1}(\theta) - i(\nu+1)\sin(\theta)\cos^{\nu}(\theta) - ie^{-2i\theta}(\nu+1)\cos^{\nu}(\theta)\sin(\theta)$$

$$= (\nu+1)\cos^{\nu+1}(\theta) - i(\nu+1)\sin(\theta)\cos^{\nu}(\theta)(1+e^{2i\theta})$$

$$= (\nu+1)\cos^{\nu+1}(\theta)(1-2i\sin(\theta)e^{i\theta}).$$

Check that

$$(1-2i\sin(\theta)e^{i\theta}) = e^{-2i\theta}.$$

Hence,

$$(X+iY)\cos^{\nu+1}(\theta) = -(1+\nu)\cos^{\nu+1}(\theta)e^{-2i\theta}.$$

Observe

$$(X+iY)u_n = (X+iY)(\cos^{\nu+1}(\theta)e^{-2in\theta})$$

$$= [(X+iY)\cos^{\nu+1}(\theta)]e^{-2in\theta} - ie^{-2i\theta}\cos^{\nu+1}(\theta)(\frac{d}{d\theta}e^{-2in\theta})$$

$$= -(\nu+1)\cos^{\nu+1}(\theta)e^{-2i\theta}e^{-2in\theta} - ie^{-2i\theta}(-2in)\cos^{\nu+1}(\theta)e^{-2in\theta}$$

$$= -(\nu+1)\cos^{\nu+1}(\theta)e^{-2i(n+1)\theta} - 2n\cos^{\nu+1}(\theta)e^{-2i(n+1)\theta} = -(1+\nu+2n)u_{n+1}.$$

We similarly have

$$X - iY = -(\nu+1)(1+i\tan(\theta) + ie^{2i\theta}\frac{\partial}{\partial\theta},$$

and one shows

$$(X-iY)\cos^{\nu+1}(\theta) = -(1+\nu)\cos^{1+\nu}(\theta)e^{2i\theta}.$$

In the same way it follows that

$$(X-iY)u_n = (-(\nu+1)+2n)u_{n-1}. \ \Box$$

Let $\mu > 0$. Recall the Casimir operator $\Box := -X^2 - Y^2 + \Theta^2$ and the Laplacian $\triangle := (-X^2 - Y^2 - \Theta^2)$. Then for all $n \in Z$,

$$\Box u_n = (1-\nu^2)u_n, and \triangle u_n = (1-\nu^2+8n^2)u_n.$$

Let $\mu > 0$. The set $\{u_n\}_{-\infty}^{\infty} \subset C^\infty(\mathcal{H}_\mu)$ is an orthogonal basis for $\mathcal{H}_\mu$. *Proof*:

By construction $\langle\{u_n\}_{-\infty}^{\infty}\rangle \subset \mathcal{H}_\mu$ is irreducible, and therefore $\langle\{u_n\}_{-\infty}^{\infty}\rangle = \mathcal{H}_\mu$. Additionally, general theory shows it is an orthogonal basis. One checks that $u_n \in \mathcal{H}_\mu$ for each n. Finally, Claim A shows that each u_n is an eigenfunction for $\triangle$, so $u_k \in C^\infty(\mathcal{H}_\mu)$. $\Box$

Formulas for the Discrete series

The upper half-plane models are discussed in Section 2.

Vector fields

Let $\mu \leq 0$. The vector field formulas for the upper half-plane are:

$$X = d\pi_\nu(X) = -(1+\nu) - 2z\tfrac{\partial}{\partial z}$$

$$\Theta = d\pi_\nu(\Theta) = -(1+\nu)z - (1+z^2)\tfrac{\partial}{\partial z}$$

$$Y = d\pi_\nu(Y) = (1+\nu)z - (1-z^2)\tfrac{\partial}{\partial z}$$

$$U = d\pi_\nu(U) = -\tfrac{\partial}{\partial z}$$

$$V = d\pi_\nu(V) = (1+\nu)z + z^2\tfrac{\partial}{\partial z}.$$

Our remaining goal is to construct an orthogonal basis $\{u_k\}_{k=n}^{\infty} \subset C^\infty(\mathcal{H}_\mu)$ for the holomorphic irreducible representation space $L^2(H, d\lambda_\nu)$.

Construction of basis

Given the Θ derivative in the upper half-plane representation, the function

$$u_0 = \left(\frac{z-i}{z+i}\right)^{-n}\left(\frac{1}{z+i}\right)^{\nu+1} \in Ker\Theta.$$

Bargmann's well-known ladder argument (see (2.2)) gives For all $k \in N_0$,

$$-i\Theta((X-iY)^k u_0) = -2k\ (X-iY)^k u_0. \quad \square$$

For all integers $k \geq n$, define

$$u_k = \left(\frac{z-i}{z+i}\right)^{k-n} \left(\frac{1}{z+i}\right)^{\nu+1}.$$

Note that

$$u_n = \left(\frac{1}{z+i}\right)^{\nu+1}$$

and each $u_k = \left(\frac{1}{z+i}\right)^{\nu+1} \in L^2(H, d\lambda_\nu)$.

We have

$$\begin{cases} (X-iY)u_n = 2(\nu+1)u_{n+1} \\ \qquad\qquad (X+iY)u_n = 0. \end{cases}$$

Proof:

We have

$$(X-iY)u_n = \left(-(1+\nu) - 2z\frac{\partial}{\partial z}\right) - i\left((1+\nu)z - (1-z^2)\frac{\partial}{\partial z}\right)u_n$$

$$= \left(-(1+\nu)(1+iz) - i(z-i)^2\frac{\partial}{\partial z}\right)u_n$$

$$= -i(1+\nu)(z-i)\left(\frac{1}{z+i}\right)^{\nu+1} + i(1+\nu)(z-i)\left(\frac{z-i}{z+i}\right)\left(\frac{1}{z+i}\right)^{\nu+1}$$

$$= \left(-i(1+\nu)(z-i) + i(z-i)(1+\nu)\right)u_{n+1} = 2(\nu+1)u_n = (1+\nu+2n)u_n.$$

On the other hand,

$$(X+iY)u_n = \left(i(1+\nu)((i+z) + i(z+i)^2\frac{\partial}{\partial z}\right)\left(\frac{1}{z+i}\right)^{\nu+1}$$

$$= i(1+\nu)(i+z)\left(\frac{1}{z+i}\right)^{\nu+1} - i(1+\nu)(z+i)^2\left(\frac{1}{z+i}\right)^{\nu+2} = 0. \quad \Box$$

Let $k \geq n$ be an integer. Then

$$\begin{cases} (X+iY)u_k = (1+\nu-2k)u_{k-1} \\ (X-iY)u_k = (1+\nu+2k)u_{k+1}. \end{cases}$$

Proof:

We have

$$(X-iY)u_k = (X-iY)\left(\left(\frac{z-i}{z+i}\right)^{k-n} u_n\right)$$

$$= \left(\frac{z-i}{z+i}\right)^{k-n}(X-iY)u_n - i(z-i)^2 u_n \frac{\partial}{\partial z}\left(\frac{z-i}{z+i}\right)^{k-n}$$

$$= \left(\frac{z-i}{z+i}\right)^{k-n}(1+\nu+2n)u_n - i(z-i)^2 u_n (k-n)\left(\frac{z-i}{z+i}\right)^{k-n-1}\frac{2i}{(z+i)^2}$$

$$= (1+\nu+2n)u_{k+1} + 2(k-n)u_{k+1} = (1+\nu+2k)u_{k+1}.$$

Next,

$$(X+iY)u_k = \left(\frac{z-i}{z+i}\right)^{k-n}((X+iY)u_n) + i(z+i)^2 u_n \frac{\partial}{\partial z}\left(\frac{z-i}{z+i}\right)^{k-n}$$

$$= i(k-n)(z+i)^2 u_n \left(\frac{z-i}{z+i}\right)^{k-n-1}\frac{2i}{(z+i)^2} = -2(k-n)u_{n-1} = (1+\nu-2k)u_{n-1}. \quad \Box$$

Lastly, Bargmann's ladder argument proves For all integers $k \geq n$,

$$-i\Theta((X-iY)^k u_0) = -2k\ (X-iY)^k u_0. \quad \Box$$

Therefore, the set $\{u_k\}_{k\geq n}$ is a basis for $L^2(H, d\lambda_\nu)$. Moreover, $\{u_k\} \subset C^\infty(\mathcal{H}_\mu)$, which is proven from the following formulas:

Let $\mu \leq 0$. Then for all integers $k \geq n$, we have

$$\Box u_k = (1-\nu^2)u_k,$$

and

$$\triangle u_k = (1 - \nu^2 + 8k^2)u_k. \quad \square$$

Appendix B

Appendix B

In this section, we prove the lemmas we needed in section 3 and Proposition 4.3 from section 4.

Principal and complementary series

Relevantdistributions

Additional distributions at infinity

Recall from section 3 that $\delta^{(0)} \in W^{-(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)$ is the flow invariant distribution. We prove the important proposition listed previously as Proposition 4.3: Let $\mu > 0$ and $s \geq 0$. Then there is a constant $C_{s,\epsilon} > 0$ such that for all $f \in W^{s+1/2+\Re\nu/2+\epsilon}(\mathcal{H}_\mu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{s-1})$, $x \in R$ and $r \geq 0$, we have

$$|f^{(r)}(x)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}}(1+|\nu|)^r(1+|x|)^{-(s+r+1+\Re\nu)}\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}.$$

This will require several steps.

There exists constants $0 < c < C$ such that for all $|\theta| \in [3\frac{\pi}{8}, \frac{\pi}{2}]$ and $\alpha \in (0,1]$,

$$c^\alpha|\theta - \frac{\pi}{2}|^\alpha \leq |\cos^\alpha(\theta - \frac{\pi}{2})| \leq C^\alpha|\theta - \frac{\pi}{2}|^\alpha.$$

Proof:

First let $\alpha = 1$. Expanding the Taylor series of $\cos(\theta)$ about $\frac{\pi}{2}$, we see

$$|\cos(\theta - \frac{\pi}{2})| = |\sum_{n=0}^{\infty} \frac{\cos^{(n)}(\frac{\pi}{2})}{n!}(\theta - \frac{\pi}{2})^n| = |\sum_{n=0}^{\infty} \frac{\cos^{(2n+1)}(\frac{\pi}{2})}{(2n+1)!}(\theta - \frac{\pi}{2})^{2n+1}|$$

$$= |(\theta - \frac{\pi}{2})\sum_{n=0}^{\infty} \frac{\cos^{(2n+1)}(\frac{\pi}{2})}{(2n+1)!}(\theta - \frac{\pi}{2})^{2n}|. \tag{B.1}$$

Notice the coefficients satisfy

$$|\frac{\cos^{(2n+1)}(\frac{\pi}{2})}{(2n+1)!}| \leq 1,$$

so there is a constant $C > 0$ such that

$$(B.1) \leq C|\theta - \frac{\pi}{2}|.$$

Also notice that $|3\frac{\pi}{8} - \frac{\pi}{2}| = \frac{\pi}{8}$ and so

$$|\sum_{n=1}^{\infty} \frac{\cos^{(2n+1)}(\frac{\pi}{2})}{(2n+1)!}(\theta - \frac{\pi}{2})^{2n}| \leq \sum_{n=1}^{\infty}(\frac{\pi}{8})^{2n} \leq \sum_{n=0}^{\infty}(\frac{\pi}{8})^n = \frac{1}{1-\pi/8} - 1 < 1.$$

So there is a constant $c > 0$ such that

$$c|\theta - \frac{\pi}{2}| \leq (B.1).$$

Then the result follows for $\alpha \in (0, 1]$ by taking powers. $\square$

Let $\mu > 0$, $s \geq 0$, $\epsilon > 0$ and $f \in W^{s+1/2+\Re\nu/2+\epsilon}(\mathcal{H}_\mu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{s-1})$. Then there is a constant $C_{s,\epsilon} > 0$ such that for all $|\theta| \in [\frac{3\pi}{8}, \frac{\pi}{2}]$ and integers $0 \leq j \leq s-1$, we have

$$|\Phi^{(j)}(\theta)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}} \cos^{s-j}(\theta)\|f\|_{W^{s+1/2+\Re\nu/2+\epsilon}(\mathcal{H}_\mu)}.$$

Proof:

Let $0 \leq j \leq [s]$ be an integer. Sobolev embedding gives

$$\|\Phi\|_{C^{\lfloor s \rfloor - j}} \leq C_{s,\epsilon}\left(\sum_{n=-\infty}^{\infty}(1+|n|)^{2(s-j)+1+2\epsilon}|c_n|^2\right)^{1/2}. \tag{B.2}$$

Then using Lemma 2.1 of [2], we see that

$$(B.2) \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}\|f\|_{W^{s-j+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)} < \infty. \tag{B.3}$$

Let

$$g(t_1) := \Phi^{(j)}(t_1(\theta - \frac{\pi}{2}) + \frac{\pi}{2}).$$

Then the Fundamental Theorem of Calculus gives

$$\Phi^{(j)}(\theta) = g(1) - g(0) = \int_0^1 g'(t_1)dt_1 = (\theta - \pi/2)\int_0^1 \Phi^{(j+1)}(t_1(\theta-\pi/2)+\pi/2)dt_1.$$

Now for $t \in [0,1]$, let $\theta_t = t(\theta - \pi/2) + \pi/2$, and we see in the same way that

$$\Phi^{(j+1)}(\theta_{t_1}) = (\theta - \pi/2)\int_0^1 (\theta_{t_1} - 1)\int_0^1 \Phi^{(j+2)}(t_2(\theta_{t_1} - \pi/2) + \pi/2)dt_2dt_1.$$

Notice that $\theta_{t_1} - \pi/2 = t_1(\theta - \pi/2)$, and define $\vec{t}_{r-j-1} = t_{r-j-1}\cdots t_1$. Then iterating proves

$$|\Phi^{(j)}(\theta)| \leq |\theta - \pi/2|^{\tilde{s}-r}\int_{[0,1]^{s-r}} |\Phi^{(r)}(\theta_{\vec{t}_{r-j-1}})|d\vec{t}_{r-j-1}. \tag{B.4}$$

Then

$$|\Phi^{(r)}(\theta_{\vec{t}_{r-j-1}})| = |\Phi^{(r)}(\theta_{\vec{t}_{r-j-1}}) - \Phi^{(r)}(\pi/2)|$$

$$\leq (\vec{t}_{r-j-1}(\theta - \pi/2))^\alpha \|\Phi^{(r)}\|_{C^\alpha([-\pi/2,\pi/2])} \leq |\theta - \pi/2|^\alpha \|\Phi\|_{C^s([-\pi/2,\pi/2])}.$$

Combining this with $(B.3)$ and $(B.4)$ gives

$$|\Phi^{(j)}(\theta)| \leq \theta^{r-j+\alpha}\|\Phi\|_{C^s([-\pi/2,\pi/2])} \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}|\theta - \pi/2|^{s-j}\|f\|_{W^{s-j+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}.$$

Finally, Claim B allows us to conclude. $\square$

Let $\mu > 0$ and $j \geq 0$. Then there is a constant $C_j > 0$ such that for all $\theta \in [\frac{3\pi}{8}, \frac{\pi}{2}]$,

$$|\frac{d^j}{d\theta^j}\cos^{1+\nu}(\theta)| \leq C_j(1+|\nu|)^j|\cos^{1+\Re\nu-j}(\theta)|.$$

Proof:

The claim is immediate for $\nu = 0$.

If $\nu \neq 0$, then taking j derivatives gives a term

$$\Pi_{k=0}^{j}(1-k+\nu)\sin(\theta)\cos^{\nu+1-k}(\theta) \tag{B.5}$$

plus terms with a higher power of $\cos(\theta)$. Because $\cos(\frac{\pi}{2}) = 0$, the term $(B.5)$ dominates. Finally, note there is a constant $C_j > 0$ such that

$$|(B.5)| \leq C_j(1+|\nu|)^j \cos^{1+\Re\nu-j}(\theta). \quad \square$$

Let $\mu > 0$ and $s \geq 0$. There is a constant $C_s > 0$ such that for all $\mu > 0$, $0 \leq j \leq s$ and $f \in W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{s-1})$, we have

$$|f^{(j)}(\theta)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}(1+|\nu|)^j \cos^{s-j+1+\Re\nu}(\theta)\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}.$$

Proof:

There are constants $\{c_k\}_{k=0}^{j} \subset Z$ such that

$$\frac{d^j}{d\theta^j}f(\theta) = \sum_{k=0}^{j} c_k \binom{j}{k}\left(\frac{d^{j-k}}{d\theta^{j-k}}\Phi(\theta)\right)\left(\frac{d^k}{d\theta^k}\cos^{1+\nu}(\theta)\right). \tag{B.6}$$

By Lemma B and Claim B, we have

$$\begin{aligned}(B.6) &\leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}\sum_{k=0}^{j} c_k(1+|\nu|)^k \cos^{s-(j-k)+1+\Re\nu-k}(\theta) \\ &\leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}(1+|\nu|)^j\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}\cos^{s-j+1+\Re\nu}(\theta). \quad \square\end{aligned}$$

Let $r \geq 0$ be an integer, $\mu > 0$ and $f \in C^r([-\frac{\pi}{2}, \frac{\pi}{2}])$. Then there is a constant $C_r > 0$ such that for all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$|U^r f(\theta)| \leq C_r \sum_{j=1}^{r} \cos^{r+j}(\theta)|f^{(j)}(\theta)|.$$

Proof:

In circle coordinates, we know

$$U = \cos^2\theta \frac{d}{d\theta}.$$

Proceeding by induction, suppose $r \geq 1$ and there are constants $\{c_{j,k}\} \subset C$ such that

$$U^r f(\theta) = \sum_{\substack{1 \leq j \leq r \\ 0 \leq k \\ j+k \leq r}} c_{j,k} \cos^{r+j+k}(\theta) \sin^{r-(j+k)}(\theta) f^{(j)}(\theta).$$

Notice this holds for $r = 0$ (and $r = 1$). The induction assumption gives

$$U^{r+1} f(\theta) = (\cos^2\theta \frac{d}{d\theta}) \sum_{\substack{1 \leq j \leq r \\ 0 \leq k \\ 0 \leq j+k \leq r}} c_{j,k} \cos^{r+j+k}(\theta) \sin^{r-(j+k)}(\theta) f^{(j)}(\theta)$$

$$= \cos^2(\theta) \sum_{\substack{1 \leq j \leq r \\ 0 \leq k \\ 0 \leq j+k \leq r}} c_{j,k} \cdot (-(r+j+k) \cos^{r+j+k-1}(\theta) \sin^{r-(j+k)+1}(\theta) f^{(j)}(\theta)$$

$$+(r-j+k) \cos^{r+j+k+1}(\theta) \sin^{r-(j+k)-1}(\theta) f^{(j)}(\theta) + \cos^{r+j+k}\theta \sin^{r-(j+k)}\theta f^{(j+1)}(\theta))$$

$$= \sum_{\substack{1 \le j \le r \\ 0 \le k \\ 0 \le j+k \le r}} c_{j,k} \cdot (-(r+j+k)\cos^{r+j+k+1}(\theta)\sin^{r-(j+k)+1}(\theta)f^{(j)}(\theta)$$

$$+(r-j+k)\cos^{r+j+k+3}(\theta)\sin^{r-(j+k)-1}(\theta)f^{(j)}(\theta)$$

$$+\cos^{r+j+k+1}(\theta)\sin^{r-(j+k)}(\theta)f^{(j+1)}(\theta)). \tag{B.7}$$

The only concerning term is

$$\cos^{r+j+k+3}(\theta)\sin^{r-(j+k)-1}(\theta)f^{(j)}(\theta),$$

which occurs when $r-(j+k) \neq 0$. In this case the assumption $j+k \le r$ gives $j+k \le r-1$, so

$$r+j+k+3 \le 2(r+1).$$

This completes the induction.

Finally, for all $k \ge 0$ and $j \ge 1$ such that $j+k \le r$,

$$\cos^{r+j+k}(\theta)\sin^{r-(j+k)}(\theta) \le \cos^{r+j}(\theta). \quad \square$$

Proof of Proposition B :

Let $\mu \ge 1$ and $0 \le r \le s$. Combining Lemma B with Lemma B gives

$$|U^r f(\theta)| \le C(s)(1+|\nu|)^r \sum_{j=1}^{r} \cos^{r+j+s-j+1+\Re\nu}(\theta)\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}$$

$$\le \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}(1+|\nu|)^r \cos^{s+r+1+\Re\nu}(\theta)\|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}. \tag{B.8}$$

Now observe that $\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$, so in R-coordinates, for all $|x| \geq \tan(\frac{3\pi}{8})$,

$$(B.8) \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}(1+|\nu|)^r \frac{1}{(1+|x|)^{s+r+1+\Re\nu}} \|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}.$$

When $|x| \leq \tan(\frac{3\pi}{8})$, there a constant $\frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}} > 0$ such that

$$\|f^{(r)}\|_{C^0(R)} \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}} \|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}.$$

So for all $0 \leq r \leq s$ and $x \in R$,

$$|f^{(r)}(x)| \leq \frac{C_{s,\epsilon}}{\sqrt{1-\Re\nu}}(1+|\nu|)^r \frac{1}{(1+|x|)^{s+r+1+\Re\nu}} \|f\|_{W^{s+(1+\Re\nu)/2+\epsilon}(\mathcal{H}_\mu)}. \quad \square$$

The distributions $\phi^U_T \delta^{(r)}$

Let $r \geq 0$ be an integer, $\mu > 0$, and $f \in C^\infty(\mathcal{H}_\mu)$. If r is even, then

$$\mathcal{L}_Y \delta^{(r)}(f) = \sum_{j=0}^{[\frac{r}{2}]-1} \left((\nu+1)(-2)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\right) \delta^{(r-2j-1)} - \delta^{(r+1)},$$

and if r is odd, then

$$\mathcal{L}_Y \delta^{(r)}(f) = -i(\nu+1)(-2i)^r \delta^{(0)} - \delta^{(r+1)}$$
$$+ \sum_{j=0}^{[\frac{r}{2}]-1} (-2)\left((\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\right) \delta^{(r-2j-1)}.$$

Proof:

Write $f(\theta) = \Phi(\theta)\cos^{\nu+1}(\theta)$, so we have

$$\mathcal{L}_Y \delta^{(r)}(f) = \delta^{(r)}(-Yf) = \delta^{(r)}(-Y \sum_{n \in Z} c_n u_n)$$

$$= \delta^{(r)} \left(\frac{i}{2} \sum_{n \in Z} c_n [(X + iY) - (X - iY)] u_n \right)$$

$$= \delta^{(r)} \left(\frac{i}{2} \sum_{n \in Z} c_n [(-(\nu + 1) - 2n) e^{-2i(n+1)\theta} - (-(\nu + 1) + 2n) e^{-2i(n-1)\theta}] \cos^{\nu+1}(\theta) \right)$$

$$= \frac{i}{2} \sum_{n \in Z} c_n [(-(\nu+1)-2n)(-2i(n+1))^r (-1)^{n+1} - (-(\nu+1)+2n)(-2i(n-1))^r (-1)^{n+1}]$$

$$= \frac{i}{2} (-2i)^r \sum_{n \in Z} c_n (-1)^{n+1} [(-(\nu + 1) - 2n)(n + 1)^r - (-(\nu + 1) + 2n)(n - 1)^r]$$

$$= \frac{i}{2} (-2i)^r \sum_{n \in Z} c_n (-1)^{n+1} [-(\nu + 1)((n + 1)^r - (n - 1)^r) - 2n((n + 1)^r + (n - 1)^r)].$$

The binomial theorem gives

$$(n + 1)^r - (n - 1)^r = \sum_{j=0}^{r} \binom{r}{j} (n^{r-j} - (-1)^j n^{r-j}) = 2 \sum_{j=0}^{[\frac{r-1}{2}]} \binom{r}{2j+1} n^{r-2j-1}, \tag{B.9}$$

and

$$(n + 1)^r + (n - 1)^r = \sum_{s=0}^{r} \binom{r}{j} (n^{r-j} + (-1)^j n^{r-j}) = 2 \sum_{j=0}^{[\frac{r}{2}]} \binom{r}{2j} n^{r-2j}. \tag{B.10}$$

Hence,

$$\mathcal{L}_Y \delta^{(r)}(f)$$

$$= \frac{i}{2} (-2i)^r \sum_{n \in Z} c_n (-1)^{n+1} [-(\nu + 1) 2 \sum_{j=0}^{[\frac{r-1}{2}]} \binom{r}{2j+1} n^{r-2j-1} - 2 \cdot 2 \sum_{j=0}^{[\frac{r}{2}]} \binom{r}{2j} n^{r-2j+1}]$$

$$= i(-2i)^r \sum_{n \in Z} c_n (-1)^n (\nu + 1) \sum_{j=0}^{[\frac{r-1}{2}]} \binom{r}{2j+1} n^{r-2j-1}$$

$$+ i(-2i)^r \sum_{n \in Z} c_n (-1)^n 2 \sum_{j=0}^{[\frac{r}{2}]} \binom{r}{2j} n^{r-2j+1}]$$

$$= i \sum_{j=0}^{[\frac{r-1}{2}]} (\nu + 1)(-2i)^{2j+1} \binom{r}{2j+1} \sum_{n \in Z} c_n (-1)^n (-2in)^{r-2j-1}$$

$$+ 2i \sum_{j=0}^{[\frac{r}{2}]} (-2i)^{2j-1} \binom{r}{2j} \sum_{n \in Z} c_n (-1)^n (-2in)^{r-2j+1}]. \tag{B.11}$$

By definition,

$$\delta^{(r)}(f) = \sum_{n\in Z}(-1)^n c_n(-2in)^r.$$

So

$$\begin{aligned}(B.11) &= i\sum_{j=0}^{[\frac{r-1}{2}]}(\nu+1)(-2i)^{2j+1}\tbinom{r}{2j+1}\delta^{(r-2j-1)}(f)\\ &\quad +2i\sum_{j=0}^{[\frac{r}{2}]}(-2i)^{2j-1}\tbinom{r}{2j}\delta^{(r-2j+1)}(f)\\ &= 2\sum_{j=0}^{[\frac{r-1}{2}]}(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1}\delta^{(r-2j-1)}(f)\\ &\quad -\sum_{j=0}^{[\frac{r}{2}]}(-2i)^{2j}\tbinom{r}{2j}\delta^{(r-2j+1)}(f).\end{aligned} \tag{B.12}$$

If r is even, then $[\frac{r-1}{2}] = [\frac{r}{2}-\frac{1}{2}] = [\frac{r}{2}]-1$. So

$$(B.12) = \sum_{j=0}^{[\frac{r}{2}]-1}\left(2(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\right)\delta^{(r-2j-1)} - \delta^{(r+1)}.$$

If r is odd, then $[\frac{r-1}{2}] = [\frac{r}{2}]$. So

$$(B.12) = i(\nu+1)(-2i)^r\delta^{(0)} - \delta^{(r+1)}$$

$$+\sum_{j=0}^{[\frac{r}{2}]-1}\left(2(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\right)\delta^{(r-2j-1)}. \ \Box$$

Let $\mu > 0$, $r \geq 0$ and $f \in C^\infty(\mathcal{H}_\mu)$. If r is even then

$$\sum_{j=0}^{[\frac{r}{2}]-1}\left(2(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\right)\delta^{(r-2j-1)},$$

and if r is odd, then

$$\mathcal{L}_U\delta^{(r)}(f) = i(\nu+1)(-2i)^r\delta^{(0)}$$

$$+\sum_{j=0}^{[\frac{r}{2}]-1}\left(2(\nu+1)(-2i)^{2j}\tbinom{r}{2j+1} - (-2i)^{2(j+1)}\tbinom{r}{2(j+1)}\right)\delta^{(r-2j-1)}.$$

Proof:

Let $\mu \geq 1$ and $f \in W^{r+2}(\mathcal{H}_\mu)$. Notice that $U = \frac{1}{2}(Y + \Theta)$, so

$$\mathcal{L}_U \delta^{(r)}(f) = \delta^{(r)}(-Uf) = \frac{1}{2}\delta^{(r)}(-Yf - \Theta f) = \frac{1}{2}(\mathcal{L}_Y \delta^{(r)}(f) + \mathcal{L}_\Theta \delta^{(r)}(f))$$

Finally, observe that by definition of $\delta^{(r+1)}$, we have

$$\mathcal{L}_\Theta \delta^{(r)}(f) = \delta^{(r)}(-\Theta f) = \delta^{(r+1)}(f). \quad \square$$

Invariant distributions

Let $\mu > 0$, $n \in Z$, $T > 0$ and $\epsilon > 0$. Then $\hat{\delta}_{n/T} \in W^{-((1+\nu)/2+\epsilon)}(\mathcal{H}_\mu)$ is a T-invariant distribution. *Proof*:

First let $\tilde{f} := f - \delta^{(0)}(f)\cos^{1+\nu}(\arctan x)$. Proposition B gives

$$|\hat{\delta}_{n/T}(f)| = |\hat{\delta}_{n/T}(\tilde{f})| \leq \int_R |\tilde{f}(x)| dx$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} \int_R (1+|x|)^{-(1+\epsilon)} dx \|\tilde{f}\|_{(1-\Re\nu)/2+\epsilon}$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} \|f - \delta^{(0)}(f)\cos^{1+\nu} \circ \arctan\|_{(1-\Re\nu)/2+\epsilon}$$

$$\leq \frac{C_{s,\epsilon}}{\sqrt{1-\nu}} \|f\|_{(1+\Re\nu)/2+\epsilon},$$

where the last step holds because $|\delta^{(0)}(f)| \leq C_\epsilon \|f\|_{(1+\Re\nu)/2+\epsilon}$, by Section 3.

Clearly, $\hat{\delta}_{n/T}$ is linear, and it is T-invariant, because

$$\hat{\delta}_{n/T} f(\cdot + T) = \int_R \tilde{f}(x+T) e^{2\pi i x n/T} dx$$

$$= \int_R \tilde{f}(x) e^{2\pi i (x-T)n/T} dx = \int_R \tilde{f}(x) e^{2\pi i x n/T} dx = \hat{\delta}_{n/T}(f). \quad \square$$

Let $\mu \leq 0$, $n \in Z$ and $T > 0$. Then $\hat{\delta}_{n/T,y} \in W^{-(1/2+\epsilon)}(\mathcal{H}_\mu)$ is a T-invariant distribution.

In the same way, one shows that $\hat{\delta}_{n/T,y}$ is a T-invariant distribution.

<u>*Discreteseries*</u>

For $1 \geq 2$, define

$$\tilde{s} := \lfloor \frac{s-1}{2} \rfloor,$$

and let $\{u_k\}_{k \geq n} \subset L^2(H, d\lambda_\nu)$ be the basis discussed in section 2 and discussed further in Appendix A. Let

$$\alpha : D \to H : \xi \to -i \left(\frac{\xi + 1}{\xi - 1} \right) := z$$

be the conformal map between D and H.

Let $\mu \leq 0$, and let $r, s \in N_0$ satisfy $0 \leq r < (s-1)/2$ and $s \geq 2$. Also let $f \in W^s(H, d\lambda_\nu) \cap Ann(\{\delta^{(r)}\}_{r=0}^{\tilde{s}})$. Then there is a constant $C_s > 0$ such that for all $z \in H$,

$$|f^{(r)}(z)| \leq C_s \|f\|_s (1 + |z|)^{-(s/2+2\nu+r+3/2}.$$

Define the space

$$\mathcal{P}_\nu(D) = \{\sum_{k=n}^{M} c_k u_k | M \in N, \{c_k\}_{k=n}^{M} \subset C, \xi \in D\}.$$

Let $\mu \leq 0$ and $r, s \in N_0, s \geq 2$ and $0 \leq r < \tilde{s}$. Also let $f \in \mathcal{P}_\nu(D) \cap Ann(\{\delta^{(r)}\}_{r=0}^{\tilde{s}-1})$. Then for all integers $0 \leq r \leq \tilde{s}$,

$$\Phi^{(r)}(1) = 0.$$

Proof:

Because $f \in \mathcal{P}_\nu(D)$, we may differentiate Φ term by term. Then

$$\frac{d^r}{d\theta^r}\Phi(e^{2\pi i\theta}) = \frac{d^r}{d\theta^r}\sum_{n\geq\nu} c_n e^{2\pi in\theta} = \sum_{n\geq\nu} c_n(2\pi in)^r e^{2\pi in\theta}.$$

With this, we have

$$\frac{d^r}{d\theta^r}\Phi(e^{2\pi i\theta})|_{\theta=0} = \sum_{n\geq\nu} c_n(2\pi in)^r = (-1)^r\delta^{(r)}(f) = 0. \tag{B.13}$$

Note that Φ has a Taylor series expansion about $\xi = 1$ given by

$$\Phi(\xi) = \sum_{n=0}^{\infty} k_n(\xi-1)^n,$$

where $\{k_n\}_{n=0}^{\infty} \subset C$. Hence,

$$\Phi(e^{2\pi i\theta}) = \sum_{n=0}^{\infty} k_n(e^{2\pi i\theta}-1)^n.$$

Observe

$$k_0 = \Phi(1) = \delta^{(0)}(f) = 0.$$

Then for $0 \leq r \leq \tilde{s}-1$,

$$(2\pi i)^r k_r \frac{n!}{(n-r)!} = \frac{d^r}{d\theta^r}\Phi(e^{2\pi i\theta})|_{\theta=0} = (-1)^r\delta^{(r)}(f) = 0. \ \square$$

Let $j \in N$ and given $t_1, \ldots, t_j \in R$, let

$$\vec{t_j} := t_1 \cdots t_j.$$

Let $\mu \leq 0$ and $r, s \in N_0, s \geq 2$ and $0 \leq r < \tilde{s}$. Also let $T_\nu f \in \mathcal{P}_\nu(D) \cap Ann(\{\delta^{(r)}\}_{r\geq 0}^{\tilde{s}-1})$. Then for all $\xi \in \mathcal{D}$, we have

$$|\Phi^{(r)}(\xi)| \leq |\xi-1|^{\tilde{s}-r}\int_{[0,1]^{s-r}} |\Phi^{(\tilde{s})}(\vec{t}_{\tilde{s}-r}(\xi-1)+1)|d\,\overrightarrow{t}_{\tilde{s}-r}.$$

Proof:

Let

$$g_1(t) := \Re\Phi^{(r)}(t(\xi-1)+1)$$

and

$$g_2(t) := \Im\Phi^{(r)}(t(\xi-1)+1)$$

and let $g(t) = g_1(t) + g_2(t)$. Then Lemma B and the Fundamental Theorem of Calculus show

$$\Re\Phi^{(r)}(\xi) = g_1(1) - g_1(0) = \int_0^1 g_1'(t_1)dt_1,$$

and

$$\Im\Phi^{(r)}(\xi) = g_2(1) - g_2(0) = \int_0^1 g_2'(t_1)dt_1.$$

So

$$\Phi^{(r)}(\xi) = \int_0^1 g'(t_1)dt_1 = (\xi-1)\int_0^1 \Phi^{(r+1)}(t_1(\xi-1)+1)dt_1.$$

Now let $\xi_{t_1} = t_1(\xi-1)+1$, and we see in the same way that

$$\Phi^{(r+1)}(\xi_{t_1}) = (\xi-1)\int_0^1 (\xi_{t_1}-1)\int_0^1 \Phi'(t_2(\xi_{t_1}-1)+1)dt_2dt_1.$$

Notice that $\xi_{t_1} - 1 = t_1(\xi-1)$. Then iterating gives the lemma. $\square$

Define

$$\mathcal{P}_\nu(H) := \{f \in L^2(H, d\lambda_\nu) | f \circ \alpha \in \mathcal{P}_\nu(D)\}.$$

Let $\mu \leq 0$, $r \in N_0, s \geq 2$ and $0 \leq r < \tilde{s}$. Also let $f \in \mathcal{P}_\nu(H) \cap Ann(\{\delta^{(r)}\}_{r\geq 0}^{\tilde{s}-1})$. Then there is a constant $C_{r,s} > 0$ such that for all $z \in H$,

$$|f^{(r)}(z)| \leq C_{r,s}\|f\|_s(1+|z|)^{-((s-1)/2+\nu+r+2)}.$$

Proof

Notice $f \circ \alpha \in \mathcal{P}_\nu(D)$, so

$$f \circ \alpha(\xi) = \Phi(\xi)(\xi - 1)^{\nu+1}.$$

Then

$$(f \circ \alpha)^{(r)}(\xi) = \sum_{j=0}^{r} \tbinom{r}{j} \Phi^{(r-j)}(\xi) \frac{d^j}{d\xi^j}(\xi - 1)^{\nu+1}$$

$$\leq \sum_{j=0}^{r} \frac{((\nu+1))!}{(\nu+1-j)!} \int_{[0,1]^{\tilde{s}-j}} |\Phi^{(\tilde{s})}(\xi_{\vec{t}_{\tilde{s}-j}})| d\overrightarrow{t}_{\tilde{s}-j} |\xi - 1|^{\nu+1-j+\tilde{s}-r+j}$$

$$\leq C_s \nu^r \sum_{j=0}^{r} \int_{[0,1]^{\tilde{s}-j}} |\Phi^{(\tilde{s})}(\xi_{\vec{t}_{\tilde{s}-j}})| d\overrightarrow{t}_{\tilde{s}-j} |\xi - 1|^{\tilde{s}+\nu+1-r}. \tag{B.14}$$

Because D is convex, we know that $\xi_{\vec{t}_{\tilde{s}-j}} \in D$ for all $\vec{t}_{\tilde{s}-j}$. Recall that $\nu < s$ and let $0 < \epsilon = \frac{s-\nu}{2}$. For all $\xi_{\vec{t}_{\tilde{s}-j}} \in D$,

$$|\Phi^{(\tilde{s})}(\xi_{\vec{t}_{\tilde{s}-j}})| \leq \sum_{n=-\infty}^{\infty} |c_n| n^{\tilde{s}}$$

$$\leq C_\epsilon \left(\sum_{n=-\infty}^{\infty} |c_n|^2 n^{2\tilde{s}+1+\epsilon} \right)^{1/2} \leq C_\epsilon \|f\|_{\tilde{s}+\frac{\nu+1}{2}+\epsilon} \tag{B.15}$$

Because ν is odd, we have

$$\tilde{s} + \frac{\nu+1}{2} + \epsilon = \lfloor \frac{s-1+\nu+1}{2} \rfloor + \epsilon \leq s.$$

Then

$$(B.14) \leq C_s \nu^r \|f\|_s (\xi - 1)^{\tilde{s}+\nu+1-r},$$

By way of induction, we prove that for $r \geq 1$, there are constants $\{c_j\} \subset C$ such that

$$U^r(f \circ \alpha)(\xi) = \sum_{j=1}^{r} c_j (\xi - 1)^{r+j} (f \circ \alpha)^{(j)}(\xi). \tag{B.16}$$

By Claim A,

$$U(f\circ\alpha)(\xi) = i\frac{(\xi-1)^2}{2}\frac{d}{d\xi}(f\circ\alpha)(\xi).$$

Then

$$U^{r+1}(f\circ\alpha)(\xi) = (i\frac{(\xi-1)^2}{2}\frac{d}{d\xi})\sum_{j=1}^{r} c_j(\xi-1)^{r+j}(f\circ\alpha)^{(j)}$$

$$= i\frac{(\xi-1)^2}{2}\sum_{j=1}^{r} c_j\left((r+j)\xi-1)^{r+j-1}f^{(j)} + (\xi-1)^{r+j}f^{(j+1)}\right)$$

$$= \sum_{j=1}^{r+1}\tilde{c}_j(\xi-1)^{r+j}(f\circ\alpha)^{(j)},$$

for some constants $\{\tilde{c}_j\}\subset C$, which proves formula $(B.16)$ holds for all $0\leq r\leq\tilde{s}$.

In particular,

$$|U^r f(\xi)| \leq C_r\sum_{j=0}^{r}|\xi-1|^{r+j}|(f\circ\alpha)^{(j)}(\xi)|$$

$$\leq C_{r,s}\nu^r\|f\|_s\sum_{j=0}^{r}|\xi-1|^{\tilde{s}+\nu+1+r} \leq C_{r,s}\nu^r\|f\|_s\sum_{j=0}^{r}|\xi-1|^{(s-1)/2+\nu+r+1}. \tag{B.17}$$

Then the Lemma follows from the change of variable given by the linear fractional transformation $\alpha:\xi\to -i\frac{\xi+1}{\xi-1}$ (see $(B.23)$). $\square$

Proof of Proposition B :

Clearly, $\mathcal{P}_\nu(H)$ is dense in $W^s(H, d\lambda_\nu)$. Then let $\eta>0$ and $f_\eta\in(\mathcal{P}(H,d\lambda_\nu))\cap Ann(\{\delta^{(r)}\}_{r=0}^{s})$ satisfy

$$\|f-f_\eta\|_{W^s(H,d\lambda_\nu)} < \eta.$$

As $r<\frac{s-1}{2}$ and $\nu<s$, take $0<\epsilon=\frac{s-\nu}{2}$ and conclude that for all $z\in H$,

$$|\frac{d^r}{dz^r}(f-f_\eta)(z)| \leq \|\frac{d^r}{dz^r}(f-f_\eta)(z)\|_{(1+\nu)/2+\epsilon} \leq \|f-f_\eta\|_s < \eta,$$

where we use Nelson [9] in the second inequality.

Hence,

$$|f^{(r)}(z)| \leq |\frac{d^r}{dz^r}(f - f_\eta)(z)| + |f_\eta^{(r)}(z)|$$

$$\leq \eta + C_{r,s} \cdot \nu^r(|z|+1)^{-((s-1)/2+\nu+r+1)} \|f_\eta\|_{W^s(\mathcal{H}_\mu)}. \quad \square$$

The case $\nu \geq$ s

Let $\mu \leq 0$. Then for all integers $k \geq 0$,

$$\|u_{k+n}\|_{W^s(H,d\lambda_\nu)} = \frac{\sqrt{\pi}}{\sqrt{\nu+1} \cdot 2^\nu}(1+\mu+8(k+n)^2)^{s/2}\left(\frac{k!\nu!}{(k+\nu)!}\right)^{1/2}.$$

Proof:

The unit disc model $L^2(D, d\sigma_\nu)$ has the measure $d\sigma_\nu := 4^{-\nu}(1-|\xi|^2)^\nu dudv$, and there is an isometry

$$T_\nu : L^2(H, d\lambda_\nu) \to L^2(D, d\sigma_\nu) : u_n \to u_n \circ \alpha(\xi)\left(\frac{-2i}{\xi-1}\right)^{\nu+1},$$

where $\alpha(\xi) = -i\left(\frac{\xi+1}{\xi-1}\right)$. One checks that $u_n \circ \alpha(\xi) = \left(\frac{\xi-1}{-2i}\right)^{\nu+1}$, so $T_\nu u_n = 1$, and therefore

$$\|u_n\|^2_{L^2(H,d\lambda_\nu)} = \|T_\nu u_n\|^2_{L^2(D,d\sigma_\nu)} = 4^{-\nu}\int_D (1-|\xi|^2)^\nu d\xi$$

$$= 4^{-\nu}\int_0^{2\pi}\int_0^1 (1-r^2)^\nu r dr d\theta = \pi 4^{-\nu}\int_0^1 t^\nu dt = \frac{\pi}{(\nu+1)4^\nu}.$$

Let $\{\tilde{u}_k\}$ be the basis studied in [2]. Lemma 2.1 of [2] gives that for all $k \geq n$,

$$\|\tilde{u}_k\|^2 = \tilde{\Pi}_{\nu,k} = \frac{(k-n)!\nu!}{(k+n-1)!}.$$

In particular,

$$\|\tilde{u}_n\|^2 = \tilde{\Pi}_{\nu,n} = 1.$$

Hence,

$$\|u_n\| = \frac{\sqrt{\pi}}{(\nu+1)2^{\nu}}\|\tilde{u}_n\|.$$

We generate the other basis vectors from the creation operator. For all $k > n$,

$$u_k = \frac{1}{\nu+2k-1}\eta_+ u_{k-1},$$

and

$$\tilde{u}_k = \frac{1}{\nu+2k-1}\eta_+ \tilde{u}_{k-1}.$$

By iterating we conclude

$$\|u_k\| = \sqrt{\frac{\pi}{\nu+1}}2^{-\nu}\|\tilde{u}_k\| = \sqrt{\frac{\pi}{\nu+1}}2^{-\nu}\left(\frac{(k-n)!\nu!}{(k+n-1)!}\right)^{1/2}.$$

The Claim follows by Claim A. $\square$

Let $\mu \leq 0$ and r, s be integers such that $0 \leq r < s \leq \nu$. If $f \in W^s(H, d\lambda_\nu)$, then

$$f^{(r)}(\cdot + iy) \in L^1(R).$$

The first step is to prove Let $\mu \leq 0$ and r, s be integers such that $0 \leq r \leq s \leq \nu$. If $f \in W^s(H, d\lambda_\nu)$, then for all $\xi \in D$,

$$|(f \circ \alpha)^{(r)}(\xi)|$$
$$\leq C_r \nu^{r+1/4}\|f\|_s \sum_{j=0}^{r}\left(\sum_{k=r-j}^{\infty}\frac{(k+n-1)!}{(k-n)!\nu!}(k+\nu)^{-2s+2r}|\xi|^{2k}\right)^{1/2}|\xi|^{-r}|1-\xi|^{(\nu+1)-r} \tag{B.18}$$

Proof:

We have

$$f \circ \alpha(\xi) = (-2i)^{-(\nu+1)} \sum_{k=0}^{\infty} c_{k+n} \xi^k (1-\xi)^{\nu+1}.$$

Therefore

$$(f \circ \alpha)^{(r)}(\xi) = (-2i)^{-(\nu+1)} \sum_{j=0}^{r} \left(\sum_{k=r-j}^{\infty} c_{k+n} \frac{d^{r-j}}{d\xi^{r-j}} \xi^k \right) \frac{d^j}{d\xi^j} (1-\xi)^{\nu+1}$$

$$= \sum_{j=0}^{r} \left(\sum_{k=r-j}^{\infty} c_{k+n} \frac{k!}{(k-r+j)!} \xi^{k-r+j} \right) \frac{(\nu+1)!}{((\nu+1)-j)!} (-1)^j (1-\xi)^{\nu+1-j}$$

$$\leq \frac{C_r}{2^\nu} \nu^r \sum_{j=0}^{r} \left(\sum_{k=r-j}^{\infty} |c_{k+n}| \frac{k!}{(k-r+j)!} |\xi|^k \right) |\xi|^{-r+j} |1-\xi|^{\nu+1-j}. \tag{B.19}$$

Using Lemma 3.2 we multiply and divide by $\|u_k\|_s = \frac{\sqrt{\pi}}{\sqrt{\nu+1} \cdot 2^\nu} (1+\mu+8(k+n)^2)^{s/2} \left(\frac{k!\nu!}{(k+\nu)!} \right)^{1/2}$ to get

$$(B.19) = \frac{C_r}{2^\nu} \nu^r \sum_{j=0}^{r}$$

$$\left(\sum_{k=r-j}^{\infty} |c_{k+n}| \|u_{k+n}\|_s \|u_{k+n}\|_s^{-1} \frac{k!}{(k-r+j)!} |\xi|^k \right) |\xi|^{-r+j} |1-\xi|^{\nu+1-j}$$

$$\leq C_r \sqrt{\nu+1} \nu^r \|f\|_s$$

$$\sum_{j=0}^{r} \left(\sum_{k=r-j}^{\infty} \frac{(k+\nu)!}{k!\nu!} (1+\mu+8(k+n)^2)^{-s} \left(\frac{k!}{(k-r+j)!} \right)^2 |\xi|^{2k} \right)^{1/2}$$

$$\cdot |\xi|^{-r+j} |1-\xi|^{\nu+1-j}. \tag{B.20}$$

Notice that

$$1+\mu+8(k+n)^2 = 1+1-\nu^2+8(k+n)^2$$

$$= 1+1-(2n-1)^2+8(k+n)^2$$

$$= 1+4n^2+4n+16kn+8k^2 \geq k^2+4kn+4n^2$$

$$= (k+(\nu+1))^2 \geq (k+\nu)^2. \tag{B.21}$$

Additionally, $0 \leq j \leq r < \nu$, so

$$\left(\frac{k!}{(k-r+j)!}\right)^2 \leq \left(\frac{(k+\nu)!}{(k+\nu-r)!}\right)^2 \leq (k+\nu)^{2r}.$$

Combing this with $(B.20)$ and $(B.21)$, we conclude. $\square$

Now let

$$B_T := \{z \in H : |z-i| < T/3\} and B_T^{c,0} := int(H - B_T).$$

Let $\mu \leq 0$ and r, s be integers such that $0 \leq r \leq s \leq \nu$. Also let $z \in B_T^{c,0}$ and $f \in W^s(H, d\lambda_\nu)$. If $\nu/2 + r < s$, the

$$|f^{(r)}(z)| \leq C_r \left(\frac{1+T}{T}\right)^r \frac{\nu^{r+1/2}}{\sqrt{\nu!}} \|f\|_s \left(\frac{1}{\Im z}\right)^{1/2} |\frac{1}{1+|z|^2+2\Im(z)}|^{(\nu+r+1)/2},$$

and if $\nu/2 + r \geq s$, then

$$|f^{(r)}(z)| \leq C_r \nu^{r+1/4} \left(\frac{1+T}{T}\right)^r \|f\|_s \left(\frac{1}{\Im z}\right)^{\nu/2-s+r+1/2} \left(\frac{1}{1+|z|^2+2\Im(z)}\right)^{s-r/2}.$$

In particular, if $r = 0$, $s > 1/2$ and $z = x + iy \in H$, then

$$|f(z)| \leq C_{r,\nu,y} \|f\|_s \frac{1}{(1+|z|)^{2s}}.$$

Proof:

Then by $(B.16)$ and Lemma B we have

$$|f^{(r)}(z)| = U^r(f \circ \alpha)(\xi) \leq C \sum_{j=1}^{r} |\xi - 1|^{r+j} |f^{(j)}(\xi)|$$

$$\leq C_r \frac{\nu^{r+1/2}}{\sqrt{\nu!}} \|f\|_s \left(\sum_{k=r-j}^{\infty} \frac{(k+\nu)!}{k!} (k+\nu)^{-2s+2r} |\xi|^{2k}\right)^{1/2} |\xi|^{-r} |1-\xi|^{\nu+1+r} \tag{B.22}$$

For the following case 1) and case 2), let $q = |\xi|^2$.

Case 1 : $\nu/2 + r < s$. Then $\nu - 2s + 2r < 0$, which means

$$\frac{(k+\nu)!}{k!} (k+\nu)^{-2s+2r} \leq (k+\nu)^{\nu-2s+2r} \leq 1,$$

and therefore

$$\left(\sum_{k=0}^{\infty} \frac{(k+\nu)!}{k!}(k+\nu)^{-2s+2r}|\xi|^{2k}\right)^{1/2}$$
$$\leq \left(\sum_{k=0}^{\infty} q^k\right)^{1/2} = \left(\frac{1}{1-q}\right)^{1/2}.$$

Observe that

$$q = |\frac{x+i(y-1)}{x+i(y+1)}|^2 = 1 - \frac{4\Im z}{1+|z|^2+2\Im z},$$

so

$$\frac{1}{1-q} = \frac{1+|z|^2+2\Im z}{4\Im z}.$$

Next, because $z \in B_T^c$, we may write it as $z = i + \alpha\omega T$ where $\alpha \geq 1$ and $\omega \in S^1$, which means

$$|\xi|^{-1} = |\frac{z+i}{z-i}| = |\frac{2+\alpha\omega T}{\alpha\omega T}| \leq C\frac{1+T}{T},$$

for some constant $C > 0$ Therefore,

$$|\xi|^{-r} \leq C_r \left(\frac{1+T}{T}\right)^r.$$

Lastly,

$$|1-\xi| = |\frac{(z-i)-(z+i)}{z+i}| = |\frac{2}{z+i}|$$
$$= \left(\frac{4}{1+|z|^2+2\Im(z)}\right)^{1/2}. \tag{B.23}$$

Combining these facts gives

$$(B.22) \leq C_r \left(\frac{1+T}{T}\right)^r \frac{\nu^{r+1/2}}{\sqrt{\nu!}}\|f\|_s \left(\frac{1+|z|^2+2\Im(z)}{4\Im z}\right)^{1/2} \left(\frac{4}{1+|z|^2+2\Im(z)}\right)^{(\nu+1+r)/2}$$
$$\leq C_r \left(\frac{1+T}{T}\right)^r \frac{\nu^{r+1/2}}{\sqrt{\nu!}}\|f\|_s \left(\frac{1}{\Im z}\right)^{1/2} |\frac{1}{1+|z|^2+2\Im(z)}|^{(\nu+r+1)/2}.$$

Case 2 : $\nu/2 + r \geq s$. Then $\nu - 2s + 2r \geq 0$, so that

$$(B.22) \leq \left(\sum_{k=0}^{\infty} \frac{(k+\nu-2s+2r)!}{k!} q^k\right) = \left(\sum_{k=\nu-2s+2r}^{\infty} \frac{k!}{(k-\nu+2s-2r)!} q^{k-\nu+2s-2r)}\right)$$

$$= \frac{d^{\nu-2s+r}}{dq^{\nu-2s+2r}} \left(\sum_{k=0}^{\infty} q^k\right) = \frac{d^{\nu-2s+2r}}{dq^{\nu-2s+2r}} \left(\frac{1}{1-q}\right)$$

$$= (\nu - 2s + 2r)! \left(\frac{1}{1-q}\right)^{\nu-2s+2r+1}.$$

Hence,

$$(B.22) = C_r \nu^{r+1/4} \left(\frac{1+T}{T}\right)^r \frac{\sqrt{(\nu-2s+2r)!}}{\sqrt{\nu!}} \|f\|_s$$

$$\left(\frac{1+|z|^2+2\Im(z)}{4\Im z}\right)^{(\nu-2s+2r+1)/2} \left(\frac{4}{1+|z|^2+2\Im(z)}\right)^{(\nu+1+r)/2}$$

$$\leq C_r \nu^{r+1/4} \left(\frac{1+T}{T}\right)^r \|f\|_s \left(\frac{1}{\Im z}\right)^{\nu/2-s+r+1/2} \left(\frac{1}{1+|z|^2+2\Im(z)}\right)^{s-r/2}. \quad \square$$

Now we prove *Proposition B*. Because f is holomorphic, it is bounded compact sets, so it is bounded on $B_T \cap (-\infty, \infty) \times \{y\}$. Then the proposition follows from Lemma B. $\square$

Now we prove

Let $\mu \leq 0$, $k \in Z$ and $y_1, y_2 > 0$. Then $\hat{\delta}_{k/T,y_1} = \hat{\delta}_{k/T,y_2}$, and if $k \leq 0$, then $\hat{\delta}_{k/T,y_1} = 0$. *Proof* :

Say $y_1 > y_2$, and let $s > 1/2$. Additionally, or all $n \in N$, let Γ_n be the closed curve with sides

$$\Gamma_n = \{[-n+iy_1, n+iy_1]\} \cup \{[n+iy_1, n+iy_2]\} \cup \{[n+iy_2, -n+iy_2]\} \cup \{[-n+iy_2, -n+iy_1]\}.$$

Let $f \in W^s(\mathcal{H}_\mu)$, and note that f is holomorphic. Then by Cauchy's theorem,

$$0 = \int_{\Gamma_n} f(z) e^{-2\pi i k/T z} dz.$$

By Lemma B, there is a constant $C_{\nu,y_1,y_2} > 0$ such that

$$|\int_{[-n+iy_2,-n+iy_1]} f(z)e^{-2\pi ik/Tz}dz| + |\int_{[n+iy_2,n+iy_1]} f(z)e^{-2\pi ik/Tz}dz|$$

$$\leq C_{\nu,y_1,y_2} \|f\|_s (1+|n|)^{-2s}.$$

Letting $n \to \infty$, we conclude $\hat{\delta}_{k/T,y_1} = \hat{\delta}_{k/T,y_2}$.

The second statement is proved in the same way. $\square$

The matrix of $\phi_T^U(\delta^{(k)})_{k=0}^r$ is determined by calculating $\phi_T^U \delta^{(k)}$.

Let $\mu \leq 0$, and $r \geq 0$. Also, let $f = \sum_{n=\nu}^\infty c_n v_{-n} \in W^{r+2}(\mathcal{H}_\mu)$. If r is even then

$$\mathcal{L}_Y \delta^{(r)}(f) = -(2i)^r \delta^{(1)}(f)$$
$$- \sum_{j=0}^{[\frac{r}{2}]-1} \left(i(2i)^{2j+1}\nu \cdot \tbinom{r}{2j+1} \delta^{(r-2j-1)} + (2i)^{2j} \tbinom{r}{2j} \delta^{(r-2j+1)} \right)(f).$$

If r is odd, then

$$\mathcal{L}_Y \delta^{(r)}(f) = -\sum_{j=0}^{[\frac{r}{2}]} \left(i(2i)^{2j+1}\nu \cdot \tbinom{r}{2j+1} \delta^{(r-2j-1)} + (2i)^{2j} \tbinom{r}{2j} \delta^{(r-2j+1)} \right)(f).$$

Proof:

Let $f \in W^{r+2}(\mathcal{H}_\mu)$. Then Lemma 6.3 of [11] implies

$$\|Yf\|_{W^{r+1}(\mathcal{H}_\mu)} \leq \|\triangle f\|_{W^{r+1}(\mathcal{H}_\mu)} \leq \|f\|_{W^{r+2}(\mathcal{H}_\mu)}.$$

Hence,

$$Y : W^{r+2}(\mathcal{H}_\mu) \to W^{r+1}(\mathcal{H}_\mu)$$

is a bounded operator. Additionally, observe that

$$|Yf| \leq \|f\|_{W^1(\mathcal{H}_\mu)} \leq C \sum_{k=n}^{\infty} k^2 |c_k|^2.$$

So the series defining Yf converges absolutely. So we may move Y under the sum.

Therefore, we have

$$\mathcal{L}_Y \delta^{(r)}(f) = \delta^{(r)}(-Y \sum_{k=n}^{\infty} c_k u_k)$$

$$= \frac{i}{2} \delta^{(r)}(\sum_{k=n}^{\infty} c_k [(X+iY) - (X-iY)] u_k$$

$$= \frac{-i}{2} \delta^{(r)}(\sum_{k=n}^{\infty} c_k (1+\nu-2k) u_{k-1} - (1+\nu+2k) u_{k+1}. \tag{B.24}$$

By (2.4), $\delta^{(r)} \in W^{-(r+1)}(\mathcal{H}_\mu)$ so we may move $\delta^{(r)}$ under the sum. So

$$(B.24) = (2i)^r \frac{i}{2} \sum_{k=n}^{\infty} c_k [(1+\nu-2k)(k-1)^r - (1+\nu+2k)(k+1)^r]$$

$$= (2i)^r \frac{i}{2} \sum_{k=n}^{\infty} c_k \left(-(1+\nu)[(k+1)^r - (k-1)^r] - 2k[(k-1)^r + (k+1)^r] \right). \tag{B.25}$$

The Binomial Theorem gives

$$(k+1)^r - (k-1)^r = 2 \sum_{j=0}^{[\frac{r-1}{2}]} \tbinom{r}{2j+1} k^{r-2j-1},$$

and

$$(k+1)^r + (k-1)^r = 2 \sum_{j=0}^{[\frac{r}{2}]} \tbinom{r}{2j} k^{r-2j}.$$

So

$$(B.25) = (2i)^r \frac{i}{2} \cdot 2 \sum_{k=n}^{\infty} c_k \left(-(1+\nu) \sum_{j=0}^{[\frac{r-1}{2}]} \tbinom{r}{2j+1} k^{r-2j-1} - 2k \sum_{j=0}^{[\frac{r}{2}]} \tbinom{r}{2j} k^{r-2j} \right)$$

$$= -i(2i)^r (1+\nu) \sum_{j=0}^{[\frac{r-1}{2}]} \tbinom{r}{2j+1} \sum_{k=n}^{\infty} k^{r-2j-1} c_k - (2i)^{r+1} \sum_{j=0}^{[\frac{r}{2}]} \tbinom{r}{2j} \sum_{k=n}^{\infty} k^{r-2j+1} c_k$$

$$-i(1+\nu)\sum_{j=0}^{[\frac{r-1}{2}]}\binom{r}{2j+1}(2i)^{2j+1}\sum_{k=n}^{\infty}(2ik)^{r-2j-1}c_k-\sum_{j=0}^{[\frac{r}{2}]}\binom{r}{2j}(2i)^{2j}\sum_{k=n}^{\infty}(2ik)^{r-2j+1}c_k$$

$$=2(1+\nu)\sum_{j=0}^{[\frac{r-1}{2}]}\binom{r}{2j+1}(2i)^{2j}\delta^{(r-2j-1)}(f)-\sum_{j=0}^{[\frac{r}{2}]}\binom{r}{2j}(2i)^{2j}\delta^{(r-2j+1)}(f). \tag{B.26}$$

If r is even, then $[\frac{r-1}{2}]=[\frac{r}{2}]-1$, so

$$(B.26)=\sum_{j=0}^{[\frac{r}{2}]-1}\left(2(1+\nu)\binom{r}{2j+1}(2i)^{2j}-\binom{r}{2(j+1)}(2i)^{2(j+1)}\right)\delta^{(r-2j-1)}(f)-(2i)^r\delta^{(1)}-\delta^{(r+1)}.$$

If r is odd, then $[\frac{r-1}{2}]=[\frac{r}{2}]$, so

$$(B.26)=\sum_{j=0}^{[\frac{r}{2}]}\left(2(1+\nu)\binom{r}{2j+1}(2i)^{2j}-(2i)^{2(j+1)}\delta^{(r-2j-1)}(f)\right)-\delta^{(r+1)}. \quad \square$$

Let $\mu\le 0$, $r\ge 0$ and $f\in W^{r+2}(\mathcal{H}_\mu)$. If r is even then

$$\mathcal{L}_U\delta^{(r)}(f)=\sum_{j=0}^{[\frac{r}{2}]-1}\left(2(1+\nu)\binom{r}{2j+1}(2i)^{2j}-\binom{r}{2(j+1)}(2i)^{2(j+1)}\right)\delta^{(r-2j-1)}(f)-(2i)^r\delta^{(1)},$$

and if r is odd, then

$$\mathcal{L}_U\delta^{(r)}(f)=\sum_{j=0}^{[\frac{r}{2}]}\left(2(1+\nu)\binom{r}{2j+1}(2i)^{2(j+1)}-(2i)^{2j}\delta^{(r-2j-1)}(f)\right). \quad \square$$

Proof:

Notice that $U=\frac{1}{2}(Y+\Theta)$, so

$$\mathcal{L}_U\delta^{(r)}(f)=\frac{1}{2}(\mathcal{L}_Y\delta^{(r)}(f)+\mathcal{L}_\Theta\delta^{(r)}(f)). \tag{B.27}$$

Observe

$$\mathcal{L}_\Theta\delta^{(r)}(f)=\delta^{(r+1)}(f),$$

by definition of $\delta^{(r+1)}$, so the Corollary follows by combining this with Lemma B. $\square$

Bibliography

[1] M. Burger. *Horocycle flow on geometrically finite surfaces.* Duke Math. J. **6** (1990), 779-803.

[2] L. Flaminio, G. Forni. *Invariant distributions and time averages for horocycle flows.* Duke Math J. **119** No. 3 (2003) 465 - 526.

[3] ———. *Equidistribution of nilflows with application to theta sums* Ergodic theory of dynamical systems **26**(2006) 409-433

[4] S. Ferleger, A. Katok. *First cohomology rigidity of the Weyl chamber flows.* unpublished.

[5] A. Katok, R. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity.* Inst. Hautes É Sci. Publ. Math. **79** (1994) 131-156.

[6] G. Forni. *Solutions of the cohomological equation for area preserving flows on surfaces of higher genus.* Ann. of Math. **146** (1997) 295 - 344.

[7] H. Furstenberg. *Unique ergodicity of the horocycle flow.* Recent advances in topological dynamics. Springer lecture notes 318 (1973), 95 - 115.

[8] A. Katok V. Nitica, A. Torok. *Nonabelian cohomology of abelian Anosov actions.* Ergodic Theory and Dynamical Systems **20** (2001) 259 - 288.

[9] J. Lions *Non-homogeneous boundary value problems and applications. Vol 1.* Springer-Verlag New York, New York, 1972.

[10] D. Mieczkowski; *The First cohomology of parabolic actions for some higher-rank abelian groups and representation theory*; Journal of Modern Dynamics Volume 1, No. 1, 2007, 61 - 92.

[11] E. Nelson. *Analytic vectors.* Ann. of Math. (2) **70** (1959) 572 - 615.

[12] V. Nitica, A. Torok. $Cocycles over TNS actions$. Geometriae Dedicata **102** (2003) 65 - 90.

[13] F. Mautner, *Unitary representations of locally compact groups. I,* Ann. of Math. (2) **51** (1950), 1-25.

[14] F. Mautner. *Unitary representations of locally compact groups. II,* Ann. of Math. (2) **52** (1950), 528 - 556.

[15] C. Moore. *Exponential decay of correlation coefficients for geodesic flows.* Group Representations, Ergodic Theory, Operator Algebras and Mathematical Physics, Math. Sci. Res.Inst. Publ. **6**, Springer, New York. 1987, 163-181.

[16] R. Ramirez. *Cocycles over higher-rank abelian actions of semisimple Lie groups.* Journal of Modern Dynamics **3** No. 1 (2009) 1 - 23.

[17] M. Ratner. *The rate of mixing for geodesic and horocycle flows.* Ergodic theory and dynamical systems **7** (1987), 267-288.

[18] M. Ratner. *Horocycle flows, joinings and rigidity of products.* Ann. of Math. (2) 118(2): 277-313, 1983.

[19] M. Ratner. *Factors of horocycle flows.* Ergodic theory Dynam. Systems, 2(3-4):465-489 (1983), 1982.

[20] R. Berndt. *Representations of linear groups.* Vieweg Sohn Verlag, Wiesbaden 2007.

[21] S. Lang. $SL(2, R)$. Springer-Verlag New York, New York, 1985.

[22] P. Sarnak. *Selberg's eigenvalue conjecture*, Notices Amer. Math. Soc. **42** (1995), no. 11, 1272 - 1277.

[23] A. Venktesh. *Sparse equidistribution problems, period bounds and subconvexity.* Ann. of Math. 172 : 989-1094, 2010.