ABSTRACT

Title of dissertation: POSITIVE RATIONAL STRONG SHIFT EQUIVALENCE AND THE MAPPING CLASS GROUP OF A SHIFT OF FINITE TYPE

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This thesis studies two independent topics in symbolic dynamics, the positive rational strong shift equivalence and the mapping class group of a shift of finite type.

In the first chapter, we give several results involving strong shift equivalence of positive matrices over the rational or real numbers, within the path component framework of Kim and Roush. Given a real matrix B with spectral radius less than 1, we consider the number of connected components of the space $\mathcal{T}_+(B)$ of positive invariant tetrahedra of B. We show that $\mathcal{T}_+(B)$ has finitely many components. For many cases of B, we show that $\mathcal{T}_+(B)$ is path connected. We also give examples of B for which $\mathcal{T}_+(B)$ has 2 components. If $\mathbb{S}$ is a subring of $\mathbb{R}$ containing $\mathbb{Q}$ we show that every primitive matrix over $\mathbb{S}$ with positive trace is strong shift equivalent to a positive doubly stochastic matrix over $\mathbb{S}_+$ (and consequently the nonzero spectra of primitive stochastic positive trace matrices are all achieved by positive doubly stochastic matrices). We also exhibit a family of 2×2 similar positive stochastic

matrices which are strong shift equivalent over $\mathbb{R}_+$, but for which there is no uniform bound on the lag and matrix sizes of the strong shift equivalences required.

For an SFT (X_A, σ_A), let $\mathcal{M}_A$ denote the mapping class group of σ_A. $\mathcal{M}_A$ is the group of flow equivalences of the mapping torus Y_A, (i.e., self homeomorphisms of Y_A which respect the direction of the suspension flow) modulo the subgroup of flow equivalences of Y_A isotopic to the identity. In the second chapter, we prove several results for the mapping class group $\mathcal{M}_A$ of a nontrivial irreducible SFT (X_A, σ_A) as follows. For every $n \in \mathbb{N}$, $\mathcal{M}_A$ acts n-transitively on the set of circles in the mapping torus Y_A of (X_A, σ_A). The center of $\mathcal{M}_A$ is trivial. $\mathcal{M}_A$ contains an embedded copy of $\mathrm{Aut}(\sigma_B)/ < \sigma_B >$ for any SFT (X_B, σ_B) flow equivalent to (X_A, σ_A). A flow equivalence $F : Y_A \to Y_A$ has an invariant cross section if and only if F is induced by an automorphism of the first return map to some cross section of Y_A (such a return map is an irreducible SFT flow equivalent to σ_A). However, there exist elements of $\mathcal{M}_A$ containing no flow equivalence with an invariant cross section. Finally, we define the groupoid $PE_{\mathbb{Z}}(A)$ of positive equivalences from A. There is an associated surjective group homomorphism $\pi_A : PE_{\mathbb{Z}}(A) \to \mathcal{M}_A/\mathcal{S}_A$ (where $\mathcal{S}_A$ is the normal subgroup of $\mathcal{M}_A$ generated by Nasu's simple automorphisms of return maps to cross sections). In the case of trivial Bowen-Franks group, there is another group homomorphism, $\rho_A : PE_{\mathbb{Z}}(A) \to \mathrm{SL}(\mathbb{Z})$. We show that for every $[F] \in \mathcal{M}_A/\mathcal{S}_A$ and V in $\mathrm{SL}(\mathbb{Z})$ there exists g in $PE_{\mathbb{Z}}(A)$ such that $\pi_A(g) = [F]$ and $\rho_A(g) = V$.

POSITIVE RATIONAL STRONG SHIFT EQUIVALENCE AND THE MAPPING CLASS GROUP OF A SHIFT OF FINITE TYPE

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2011

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Dedication

To my parents, Prapai and Jumrat Chuysurichay.

Acknowledgments

I am greatly indebted to Professor Michael M. Boyle, my thesis advisor, for his thoughtful and helpful advice and for his patience. Sincere thanks and deep appreciation are also extended to Professors Michael Brin, Brian Hunt, Michael Jakobson, James Purtilo, and Jonathan M. Rosenberg, who have served on my preliminary oral and final defense committees.

I would like to thank all of the teachers who have taught me, for their valuable knowledge. I am also grateful to the Development and Promotion of Science and Technology Talents Project (DPST), Thailand, for providing me support throughout my graduate study. I would also like to express my great gratitude to my family for their love and encouragement during my study.

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List of Abbreviations

$\mathbb{N}$	The set of positive integers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}_+$	The set of nonnegative integers
$\mathbb{Q}_+$	The set of nonnegative rational numbers
$\mathbb{R}_+$	The set of nonnegative real numbers
$\mathrm{Conv}(T)$	The convex hull of T
$\mathrm{Fix}(\sigma_A)$	The set of all fixed points of σ_A
$\mathrm{Per}(\sigma_A)$	The set of all periodic points of σ_A
$\mathcal{T}_+(B)$	The set of positive invariant tetrahedra of B
$\mathcal{S}_+(B)$	The set of positive stochastic matrices similar to $B \oplus 1$
$\mathrm{GL}_n(\mathbb{R})$	The set of $n \times n$ invertible matrices over $\mathbb{R}$
$\mathrm{S}_n(\mathbb{R})$	The set of invertible $n \times n$ matrices over $\mathbb{R}$ with equal row sum
$\mathrm{SL}(\mathbb{Z})$	The stable special linear group over $\mathbb{Z}$
S^1	The unit circle
0_n	The $n \times n$ zero matrix
I_n	The $n \times n$ identity matrix
Δ^n	The standard n-simplex, $\{(l_1, ..., l_{n+1}) \in \mathbb{R}_+^{n+1} : l_1 + \cdots + l_{n+1} = 1\}$
SFT	Shift of finite type
ISFT	Irreducible shift of finite type
MSFT	Mixing shift of finite type

Chapter 1

Strong Shift Equivalence of Positive Matrices

1.1 Introduction

Symbolic dynamics has roots in the study of geodesic flows and general dynamical systems by the discretization of space and time. Applications of symbolic dynamics can be found in hyperbolic dynamics [Bow73], data storage and transmission [ACH83], and linear algebra [BoH91]. The fundamental objects we study in symbolic dynamics are shifts of finite type (SFTs). Shifts of finite type can be represented by nonnegative matrices. Let A be an $n \times n$ nonnegative matrix. We consider A as an adjacency matrix of a finite directed graph $\mathcal{G}_A$ with n ordered vertices and a finite edge set E and A_{ij} = the number of edges from vertex i to vertex j. Let E be the set of all edges in $\mathcal{G}_A$ and X_A be the set of bi-infinite sequences (x_i) such that for all $i \in \mathbb{Z}$, the terminal vertex of x_i is the initial vertex of x_{i+1}, i.e. $X_A = \{(x_i)_{i\in\mathbb{Z}} |\text{ each } x_i x_{i+1} \text{ is a path in } \mathcal{G}_A\}$. Define the *shift map* $\sigma : X_A \to X_A$ by the rule $(\sigma x)_i = x_{i+1}$. Then (X_A, σ) is called an *edge shift of finite type defined by* A. Given two matrices A and B, one naturally ask: do they present topologically conjugate SFTs?

The conjugacy problem for shifts of finite type gives rise to strong shift equivalence theory. In 1973, R.F. Williams introduced strong shift equivalence and showed that two shifts of finite type are topologically conjugate if and only if their presenting matrices are strong shift equivalent over $\mathbb{Z}_+$. Let A and B be nonnegative integral matrices. A and B are *elementary strong shift equivalent over* $\mathbb{Z}_+$ if there are nonnegative integral matrices U, V such that $A = UV$ and $B = VU$. A and B are *strong shift equivalent over* $\mathbb{Z}_+$ if there is a chain of nonnegative integral matrices $A = A_0, A_1, ..., A_l = B$ such that A_i and A_{i+1} are elementary strong shift equivalent over $\mathbb{Z}_+$ for all $i = 0, 1, ..., l-1$. The number l is the *lag* of the given strong shift equivalence. Despite its good-looking definition, strong shift equivalence is still very difficult to fully understand. Williams also introduced a more tractable equivalence relation called shift equivalence and conjectured that shift equivalence and strong shift equivalence over $\mathbb{Z}_+$ are the same. A and B are *shift equivalent over* $\mathbb{Z}_+$ if there are nonnegative integral matrices U, V and a positive integer l such that

$$A^l = UV, B^l = VU, AU = UB, BV = VA.$$

The conjecture was proved false by K.H. Kim and F.W. Roush in 1992 (reducible case) and 1997 (irreducible case). Although Williams' Conjecture is false in general, the gap between shift equivalence and strong shift equivalence over $\mathbb{Z}_+$ remains mysterious.

In this chapter, we study Williams' Conjecture by relaxing the problem to the level of positive rational and real matrices. The definition of elementary strong shift equivalence, strong shift equivalence, and shift equivalence over $\mathbb{Q}_+$ and $\mathbb{R}_+$ can be defined analogously. We expect that the Williams' conjecture is true for positive rational (or real) matrices. This is the conjecture posed by Mike Boyle in [Bo02a]. The key ingredients we use are geometric objects called *positive invariant tetrahedra* within *the path component method* introduced by Kim and Roush. We summarize the essential features of their method now (providing more detail later).

For the summary we need some definitions. If A is an irreducible matrix, then its *stochasticization* $P(A)$ is the stochastic matrix defined as $P(A) = \frac{1}{\lambda}D^{-1}AD$ where $\lambda > 0$ is the Perron eigenvalue of A and D is the diagonal matrix whose vector of diagonal entries is the stochastic right eigenvector of A. Given an $(n-1)\times(n-1)$ real matrix B with spectral radius < 1, a *positive invariant ordered tetrahedron* for B is an n-tuple $(v_1, ..., v_n)$ of vectors in $\mathbb{R}^{n-1}$ such that the convex hull of $\{v_1, ..., v_n\}$ is an $(n-1)-$dimensional simplex and the convex hull of $\{v_1, ..., v_n\}$ is sent to its interior under B. Let $\mathcal{T}_+^{ord}(B)$ denote the space of positive invariant ordered tetrahedra of B.

Now we can summarize essential features of the path component method of Kim and Roush for positive matrices A and C.

(1) A, C are SSE-$\mathbb{R}_+$ to positive matrices A', C' respectively, which in addition are similar matrices.

(2) If there is a path $A_t, 0 \le t \le 1$, of positive similar matrices from $A = A_0$ to $C = A_1$, then A and C are SSE-$\mathbb{R}_+$. If A and C have rational entries, then they are SSE-$\mathbb{Q}_+$.

(3) For $T = (v_1, ..., v_n) \in \mathcal{T}_+^{ord}(B)$ let P_T denote the stochastic matrix P such that $v_i B = \sum_{i=1}^n p_{ij} v_j$. Then P_T is similar to $B \oplus 1$. A path T_t in $\mathcal{T}^{ord}(B), 0 \le t \le 1$, produces a path of positive similar stochastic matrices $P_{T_t}, 0 \le t \le 1$.

The main point is that conditions $(1) - (3)$ provide sufficient conditions for strong shift equivalence over $\mathbb{R}_+$ (or $\mathbb{Q}_+$). In this framework, Kim and Roush proved that matrices over $\mathbb{R}_+(\mathbb{Q}_+)$ with equal spectral radius, a simple root of the characteristic polynomial, and with no other nonzero eigenvalue, are SSE-$\mathbb{R}_+(\mathbb{Q}_+)$. This is the unique general sufficient condition for SSE-$\mathbb{R}_+(\mathbb{Q}_+)$. The corresponding problem over $\mathbb{Z}_+$ is open. They did this in the end by proving $\mathcal{T}_+^{ord}(B)$ is path connected when B is nilpotent. Consequently we are motivated to study the structure of connected components of $\mathcal{T}_+^{ord}(B)$ for more general B.

In section 1.2, we give general background. In section 1.3, we develop basic ideas about (ordered) tetrahedra, (ordered) positive tetrahedra, and (ordered) positive invariant tetrahedra. In section 1.4, we show that $\mathcal{T}_+^{ord}(B)$ has only finitely many connected components (and therefore there are only finitely many SSE-$\mathbb{R}_+$ classes for positive matrices of a given size).

Section 1.5 gives some basic moves to produce positive invariant ordered tetrahedra which stay in the same connected component. In section 1.6, we give a class of examples for which the space of positive invariant tetrahedra is disconnected: if

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}, \alpha, \beta \in (0,1)$$

and $\alpha + \beta \geq 1$ then $\mathcal{T}_+(B)$ is disconnected (an element of $\mathcal{T}_+(B)$ is a set $\{v_1, ..., v_n\}$ such that $(v_1, ..., v_n) \in \mathcal{T}_+^{ord}(B)$). Unfortunately, we have no example for which we can prove the space of positive stochastic matrices in the same similarity class is disconnected.

In section 1.7, we focus on the space of positive invariant tetrahedra for 1×1 and 2×2 matrices. We show the following

1. $\mathcal{T}_+(B)$ is path connected if B has one of the Jordan forms

(a) $(\lambda), \lambda \in (-1,1)$,

(b) $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha, \beta \in [0,1)$,

(c) $\begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$, $\alpha, \beta \in [0,1)$ and $\alpha + \beta < 1$,

(d) $\begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in [0, \frac{1}{2})$,

(e) $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, $\alpha \in [0,1)$.

2. $\mathcal{T}_+(B)$ has exactly 2 connected components when B has the Jordan form

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}, \alpha, \beta \in (0,1) \text{ and } \alpha + \beta \geq 1.$$

Whether $\mathcal{T}_+(B)$ is path connected is still unknown when B has one of the remaining Jordan forms which are compatible with $B \oplus 1$ being similar to a positive stochastic matrix:

(a) $\begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}$, $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$,

(b) $\begin{pmatrix} -\alpha & 1 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in (0, \frac{1}{2})$,

(c) $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $(\alpha, \beta) \in \text{int}(\text{Conv}(T))$ where $T = \{(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$.

The failure in understanding the number of components of $\mathcal{T}_+(B)$ in the above

3 unknown cases is that we still do not know the geometry of $\mathcal{T}_+(B)$ when B has no nonnegative eigenvalue.

In section 1.8, we show that every positive stochastic matrix over any subsemiring of $\mathbb{R}_+$ containing $\mathbb{Q}_+$ is strong shift equivalent to a positive doubly stochastic matrix. As a consequence, we show that the set of nonzero spectra of doubly stochastic matrices and positive-trace primitive stochastic matrices are the same. In section 1.9, we give an example of a class of 2×2 positive, similar, SSE-$\mathbb{R}_+$ matrices for which there is no uniform bound on the lag and matrix size required for a SSE-$\mathbb{R}_+$. The examples are the stochastic matrices

$$P_t = \frac{1}{4}\begin{pmatrix} 3+t & 1-t \\ 1+t & 3-t \end{pmatrix}, 0 \leq t < 1.$$

Finally, in section 1.10 we collect some miscellaneous results involving the space $\mathcal{T}_+(B)$. For any $n \in \mathbb{N}$, we show that $\mathcal{T}_+(B)$ is path connected if B has the following Jordan form:

(a) $B = 0_n$.

(b) $B = \lambda I_n$ where $-\frac{1}{n} < \lambda < 1$.

(c) B is nilpotent (this is a reproof of the Kim-Roush result).

(d) $B = \lambda I_n + N$ where N is nilpotent.

(e) $B = (\lambda) \oplus 0_n$ where $-1 < \lambda < 1$.

1.2 Definitions and Background

1.2.1 Nonnegative Matrices

Let $A = (a_{ij})$ be a real $n \times n$ matrix. A is *nonnegative* if $a_{ij} \geq 0$ for all i, j. A is *positive* if $a_{ij} > 0$ for all i, j. A is *irreducible* if A is nonnegative, square, and for any (i, j) there is some n such that $(A^n)_{ij} > 0$. A is *primitive* if A is nonnegative, square, and there is some $n \in \mathbb{N}$ such that A^n is positive. The period $\mathrm{per}(i)$ of a state i is the greatest common divisor of all integers $n \in \mathbb{N}$ for which $(A^n)_{ii} > 0$. We define $\mathrm{per}(i) = \infty$ if no such integers exist. The *period* of A, denoted by $\mathrm{per}(A)$, is the greatest common divisor of $\mathrm{per}(i)$ that are finite, or is ∞ if $\mathrm{per}(i) = \infty$ for all $i = 1, ..., n$. A is *aperiodic* if $\mathrm{per}(A) = 1$. A is primitive if and only if it is irreducible and aperiodic. A is *quasi-stochastic* if every row sum of A is 1. A is *stochastic* if it is nonnegative and quasi-stochastic. A is *doubly stochastic* if it is stochastic and every column sum of A is 1.

We will use the following properties of nonnegative matrices.

Theorem 1.2.1. (Perron) Let A be a primitive matrix. Then there exists an eigenvalue λ of A, called the Perron eigenvalue, with the following properties:

(a) $\lambda > 0$,

(b) λ is a simple root of the characteristic polynomial of A,

(c) λ has a positive eigenvector v,

(d) If α is any other eigenvalue of A then $|\alpha| < \lambda$,

(e) any nonnegative eigenvector of A is a positive multiple of v.

A vector $l = (l_1, ..., l_n)$ is called *the left Perron eigenvector* of an $n \times n$ stochastic matrix P if l is positive, $l_1 + \cdots + l_n = 1$, and $lP = l$. For any square matrix A, *the Jordan form away from zero of* A, $J^{\times}(A)$, is the matrix obtained by removing from the Jordan form of A all rows and columns with zeros on the main diagonal.

1.2.2 Shift Spaces and Shifts of Finite Type

Let $\mathcal{A}$ be a finite set of symbols, called the alphabet, and let $\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in \mathcal{A}$ for all $i \in \mathbb{Z}\}$ denote the set of all bi-infinite sequences of elements in $\mathcal{A}$. $\mathcal{A}^{\mathbb{Z}}$ is called the *full $\mathcal{A}$-shift*. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ is given by the rule $(\sigma(x))_i = x_{i+1}$. We topologize $\mathcal{A}$ with the discrete topology. Then the topology of $\mathcal{A}^{\mathbb{Z}}$ is given by the product topology. The metric defined by $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = \frac{1}{k+1}$ where $k = \min\{|i| : x_i \neq y_i\}$ induces the product topology on $\mathcal{A}^{\mathbb{Z}}$. A *word* in the full shift $\mathcal{A}^{\mathbb{Z}}$ is a finite sequence $a_1 a_2 \cdots a_n$ where $a_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$. A *subshift* or a *shift space* of $\mathcal{A}^{\mathbb{Z}}$ is a compact, shift invariant subspace of the full shift $\mathcal{A}^{\mathbb{Z}}$ together with the restriction of the shift map. A *shift of finite type* is a shift space X with the property that there is a finite list of words such that X consists of precisely the sequences in the full shift that do not contain any of these words. For a word w of length n and $k \in \mathbb{N}$, we define the *cylinder set* X_w^k as $X_w^k = \{x \in X : x[k, k+n-1] = w\}$. For $k = 0$, we denote X_w for the cylinder set X_w^0.

Suppose (X, σ_X) and (Y, σ_Y) are shift spaces. A map $f : X \to Y$ is called a *code* if it is continuous and $f \circ \sigma_X = \sigma_Y \circ f$. f is a *block code* if there is a number

n and a function F from the set of words of length $2n+1$ in X to a finite set of alphabets in Y such that $(f(x))_i = F(x_{i-n} \cdots x_{i+n})$. The Curtis - Hedlund - Lyndon Theorem asserts that every code is a block code. If f is surjective, it is called a factor map. If f is injective, then it is called an embedding. If it is bijective then it is called a conjugacy of subshifts. We say that (X, σ_X) and (Y, σ_Y) are *topologically conjugate* if there is a conjugacy $f : X \to Y$.

Let A be an $n \times n$ nonnegative integral matrix. A can be viewed as an adjacency matrix of a finite directed graph G with n ordered vertices and a finite edge set E and A_{ij} = the number of edges from vertex i to vertex j. Let E be the set of alphabet and X_A be the set of bi-infinite sequences (x_i) such that for all $i \in \mathbb{Z}$, the terminal vertex of x_i is the initial vertex of x_{i+1}. Then X_A as a subset of the full E shift with the restriction of the shift map σ_A on X_A is a shift of finite type, called *the edge shift defined by* A. Let (X_A, σ_A) denote the edge shift defined by A. Every shift of finite type is topologically conjugate to an edge shift (X_A, σ_A) for some nonnegative integral matrix A. A shift space (X, σ_X) is *irreducible* if for every ordered paired of words u, v there is a word w such that uwv is also a word in X. (X, σ_X) is *mixing* if for every ordered pair of words u, v there is an N such that for each $n \geq N$ there is a word w of length n such that uwv is also a word in X. An edge shift of finite type defined by A (X_A, σ_A) is irreducible if and only if A is irreducible and it is mixing if and only if A is primitive. The class of mixing shifts of finite type are the basic class of SFTs. Often, problems involving SFTs can be reduced to MSFTs.

1.2.3 Strong Shift Equivalence and Shift Equivalence

Let A and B be square matrices over a semiring $\mathcal{R}$ containing 0 and 1 as the additive and multiplicative identities.

1. A is *elementary strong shift equivalent over* $\mathcal{R}$(ESSE-$\mathcal{R}$) to B if there exist matrices U, V over $\mathcal{R}$ with $A = UV, B = VU$.

2. A is *strong shift equivalent over* $\mathcal{R}$(SSE-$\mathcal{R}$) to B if there exists a finite sequence of matrices over $\mathcal{R}$ $A = A_0, A_1, ..., A_l = B$ such that A_i is ESSE-$\mathcal{R}$ to A_{i+1} for all $i = 0, ..., l-1$. Such a finite sequence is a *strong shift equivalence* over $\mathcal{R}$. The number l is the *lag* of the strong shift equivalence. By the *size* of the strong shift equivalence, we mean $\max\{n_i : 0 \leq i \leq l, A_i \text{ is } n_i \times n_i\}$.

3. A is *shift equivalent over* $\mathcal{R}$(SE-$\mathcal{R}$) to B if there exist matrices U, V over $\mathcal{R}$ and $l \in \mathbb{N}$ such that $A^l = UV, B^l = VU$ and $AU = UB, VA = BV$

For any semiring $\mathcal{R}$, SSE-$\mathcal{R}$ and SE-$\mathcal{R}$ are equivalence relations whereas ESSE-$\mathcal{R}$ is not transitive. In fact, SSE-$\mathcal{R}$ is the transitive closure of ESSE-$\mathcal{R}$. It is obvious that ESSE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ for any semiring $\mathcal{R}$. For all the semiring $\mathcal{R}$ under our consideration the implication cannot be reversed. It is not difficult to show that SSE-$\mathcal{R}$ implies SE-$\mathcal{R}$. For example, suppose that

$$A = U_0V_0,$$

$$A_1 = V_0U_0 = U_1V_1,$$

$$A_2 = V_1U_1 = U_2V_2$$

$$B = V_2U_2.$$

Then $A^3 = U_0U_1U_2V_2V_1V_0$ and $B^3 = V_2V_1V_0U_0U_1U_2$. Thus we choose $U = U_0U_1U_2, V =$

$V_2V_1V_0$ and $l = 3$. It is known that if the semiring $\mathcal{R}$ has nice algebraic structure then SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$. For example, if $\mathcal{R}$ is a Dedekind domain then SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ [BoH93]. Thus SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ for $\mathcal{R} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. The main interest in SSE-$\mathcal{R}$ and SE-$\mathcal{R}$ is when $\mathcal{R} = \mathbb{Z}_+$ and $\mathcal{R} = \mathbb{Q}_+$. Strong shift equivalence and shift equivalence were introduced in a seminal paper of R. F. Williams in [Wi73]. The following theorem of Williams gives the meaning of strong shift equivalence over $\mathbb{Z}_+$ for symbolic dynamics.

Theorem 1.2.2. [Wi73] (X_A, σ_A) and (X_B, σ_B) are topologically conjugate if and only if A is SSE-$\mathbb{Z}_+$ to B.

Shift equivalence over $\mathbb{Z}_+$ also has a meaning in symbolic dynamics. We say that (X_A, σ_A) and (X_B, σ_B) are *eventually conjugate* if there is an $N \in \mathbb{N}$ such that (X_A, σ_A^n) and (X_B, σ_B^n) are topologically conjugate for all $n \geq N$.

Theorem 1.2.3. [LM95, Theorem 7.5.15] (X_A, σ_A) and (X_B, σ_B) are eventually conjugate if and only if A is SE-$\mathbb{Z}_+$ to B.

The advantages of using SE-$\mathbb{Z}_+$ rather than SSE-$\mathbb{Z}_+$ is that SE-$\mathbb{Z}_+$ deals with equations of 4 matrices (not an unknown chain as SSE-$\mathbb{Z}_+$ does). SE-$\mathbb{Z}_+$ is decidable [KR88] whereas it is still unknown if SSE-$\mathbb{Z}_+$ is decidable. In 1974, Williams conjectured that SE-$\mathbb{Z}_+$ implies SSE-$\mathbb{Z}_+$. The conjecture was refuted by Kim and Roush in the reducible case [KR92a] and then the irreducible case [KR99].

1.2.4 Rational Strong Shift Equivalence

Our main interest in this chapter is the rational strong shift equivalence of positive matrices. Understanding this relation is a natural step toward understanding SSE-$\mathbb{Z}_+$, and a natural matrix problem independently. SSE-$\mathbb{Q}_+$ can also be given a description in symbolic dynamics. Two shifts of finite type (X_A, σ_A) and (X_B, σ_B) are *rationally isomorphic* if there is some $k \in \mathbb{N}$ such that $(X_{[k]} \times X_A, \sigma_{[k]} \times \sigma_A)$ and $(X_{[k]} \times X_B, \sigma_{[k]} \times \sigma_B)$ are topologically conjugate, or equivalently, if there is some $k \in \mathbb{N}$ such that (X_{kA}, σ_{kA}) and (X_{kB}, σ_{kB}) are topologically conjugate. Then it is easy to see that (X_A, σ_A) and (X_B, σ_B) are rationally isomorphic if and only if A is SSE-$\mathbb{Q}_+$ to B.

The basic elementary strong shift equivalences are conjugations by permutation matrices and state splitting and amalgamations. If A and B are matrices over a semiring $\mathcal{R}$ with $B = PAP^{-1}$ where P is a permutation matrix then A and B are ESSE-$\mathcal{R}$ because $A = UV$ and $B = VU$ where $U = AP^{-1}$ and $V = P$. State splitting and amalgamations are basic elementary strong shift equivalence which connect matrices from different dimensions. They were first introduced in [Wi73] for matrices over $\mathbb{Z}_+$. In this thesis, we extend the same idea to matrices over subsemirings of $\mathbb{R}_+$. Let A be an $n \times n$ nonnegative matrix over a subsemiring of $\mathbb{R}_+$. Let A' be an $(n+1) \times n$ matrix obtained by splitting row i of A into rows i and $i+1$ and the other rows of A and A' are the same. We duplicate column i of A' and form an $(n+1) \times (n+1)$ matrix B. Let U be an $n \times (n+1)$ matrix obtained by duplicating column i of the identity matrix I_n and set $V = A'$. Then $A = UV$ and

$B = VU$. We say that B is obtained from A by a *row splitting* and A is obtained from B by a *row amalgamation*.

Example 1.2.4. Let $A = \begin{pmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{3} & 1 & 3 \\ 1 & 4 & \frac{1}{4} \end{pmatrix}$. We split the second row of A to obtain

$$A' = \begin{pmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{6} & \frac{1}{3} & 2 \\ \frac{1}{6} & \frac{2}{3} & 1 \\ 1 & 4 & \frac{1}{4} \end{pmatrix} = V.$$

Then we duplicate the second column of A' and obtain

$$B = \begin{pmatrix} 2 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 2 \\ \frac{1}{6} & \frac{2}{3} & \frac{2}{3} & 1 \\ 1 & 4 & 4 & \frac{1}{4} \end{pmatrix}.$$

We get the matrix U by duplicating the second column of I_3:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $A = UV$ and $B = VU$.

Column splitting and column amalgamations are defined similarly by switching the role of rows to columns. it is well-known that every strong shift equivalence can be factored as a series of row splitting followed by a conjugacy and then by column amalgamations [KR91a]. The study of rational strong shift equivalence for primitive rational matrices can be reduced to the positive case by the following result.

Theorem 1.2.5. [KR86] Any primitive rational square matrix with a positive trace is strong shift equivalent over $\mathbb{Q}_+$ to a positive matrix.

Mike Boyle also stated the following conjecture in [Bo02a]

Conjecture 1.2.6. (Positive Rational Shift Equivalence Conjecture) Suppose A, B are square positive matrices which are shift equivalent over $\mathbb{Q}_+$. Then A, B are strong shift equivalent over $\mathbb{Q}_+$.

The following is the only known theorem which asserts for some unital subring $\mathcal{R}$ of $\mathbb{R}$, that all matrices in some nontrivial SE-$\mathcal{R}_+$ class are SSE-$\mathcal{R}_+$.

Theorem 1.2.7. [KR90] Let $\mathcal{R}$ be $\mathbb{Q}_+$ or $\mathbb{R}_+$. Suppose A and B are square matrices over $\mathcal{R}$ similar to $(\lambda) \oplus N$ with $\lambda > 0$ and N is nilpotent. If A and B are SE-$\mathcal{R}$, then A and B are SSE-$\mathcal{R}$.

To prove this theorem, Kim and Roush built up a more general structure for approaching the problem geometrically. First, they move the shift equivalence classes to similarity classes by proving the following theorem.

Theorem 1.2.8. [KR90] Let S be $\mathbb{Q}$ or $\mathbb{R}$. Let A, B be positive matrices. If A is SE-S_+ to B then there are positive matrices C, D over S such that A is SSE-S_+ to C, B is SSE-S_+ to D, and C, D are similar over S.

Then they establish *the path component method*.

Theorem 1.2.9. [KR91a] Let A, B be positive real matrices such that there is a path P_t of positive real similar matrices joining $P_0 = A$ and $P_1 = B$. Then A and B are strong shift equivalent over $\mathbb{R}_+$. If in addition A and B are rational matrices, then A and B are SSE-$\mathbb{Q}_+$.

Consequently we are motivated to study the path connected components of positive real matrices in the same similarity class over $\mathbb{R}$. We recall a standard fundamental construction.

Definition 1.2.10. If A is an irreducible matrix with spectral radius λ, then the *stochasticization* of A is the stochastic matrix $P(A) = \frac{1}{\lambda}D^{-1}AD$, where D is the diagonal matrix whose vector of diagonal entries is the stochastic right eigenvector of A.

Example 1.2.11. Let $A = \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}$. Then $\lambda = 11$ and $D = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, so

$$P(A) = \frac{1}{11}\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 10 & 1 \\ 9 & 2 \end{pmatrix}.$$

Given $c > 0$, A_t is a path of positive similar matrices from A_0 to A_1 if and only if cA_t is a path of positive similar matrices from cA_0 to cA_1. So, without loss of generality, we may study the path components of positive real matrices of spectral radius 1 in the same similarity class. For such a matrix A, let $P(A) = D^{-1}AD$ be its stochasticization as above. Now $(tD + (1-t)I)^{-1}A(tD + (1-t)I), 0 \leq t \leq 1$, gives a path of positive similar matrices from A to $P(A)$. On the other hand, if $A_t, 0 \leq t \leq 1$, is a path of positive similar matrices from A to B, and $P_t = D_t^{-1}A_tD_t$ is the stochasticization as above, then $P_t, 0 \leq t \leq 1$, is a path of positive similar stochastic matrices from $P(A)$ to $P(B)$. So, there is a path of positive similar matrices from A to B if and only if there is a path of positive similar stochastic matrices from $P(A)$ to $P(B)$. Paths of positive stochastic matrices can be studied geometrically as paths of positive invariant tetrahedra.

1.3 Positive Invariant Tetrahedra

Positive invariant tetrahedra are basic tools in the path component method developed by Kim and Roush. They play an important role in the proof of Theorem 1.2.7. In this section we study basic properties of tetrahedra, positive tetrahedra, and positive invariant tetrahedra which will be used throughout this chapter.

1.3.1 Tetrahedra and Positive Tetrahedra

Definition 1.3.1. Let T be a set of n vectors in $\mathbb{R}^{n-1}$. T is a *tetrahedron* if the convex hull of T, Conv(T), is an $(n-1)-$ dimensional (geometric) simplex. T is a *positive tetrahedron* when in addition the interior of its convex hull contains the origin. An *ordered (positive) tetrahedron* is a tuple of vectors whose the set of all vectors in the tuple forms a (positive) tetrahedron. We denote $T = \{v_1, ..., v_n\}$ for a tetrahedron and $T = (v_1, ..., v_n)$ for an ordered tetrahedron.

Example 1.3.2. $T_0 = \{(1,0),(0,1),(-1,0)\}$ is a tetrahedron but not a positive tetrahedron. $T_1 = \{(1,0),(0,0),(-1,0)\}$ is not a tetrahedron. $T_2 = \{(1,0),(0,1),(-1,-1)\}$ is a positive tetrahedron.

We recall some basic facts in the next two propositions. The results hold for both tetrahedra and ordered tetrahedra.

Proposition 1.3.3. Let $T = \{v_1, ..., v_n\} \subseteq \mathbb{R}^{n-1}$. The following statements are equivalent

(a) T is a tetrahedron.

(b) For every i, the set $\{v_1 - v_i, ..., v_{i-1} - v_i, v_{i+1} - v_i, ..., v_n - v_i\}$ is a basis of $\mathbb{R}^{n-1}$.

(c) There exists i such that the set $\{v_1 - v_i, ..., v_{i-1} - v_i, v_{i+1} - v_i, ..., v_n - v_i\}$ is a basis of $\mathbb{R}^{n-1}$.

(d) If $\sum_{i=1}^{n} r_i v_i = \sum_{i=1}^{n} s_i v_i$ where $\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i$ then $r_i = s_i$ for all $i = 1, ..., n$.

(e) If $\sum_{i=1}^{n} c_i v_i = 0$ and $\sum_{i=1}^{n} c_i = 0$ then $c_i = 0$ for all $i = 1, ..., n$.

(f) For every $v \in \mathbb{R}^{n-1}$, there is a unique representation $v = \sum_{i=1}^{n} c_i v_i$ where $c_i \in \mathbb{R}, i = 1, ..., n$ and $\sum_{i=1}^{n} c_i = 1$.

(g) $\text{Conv}(T)$ has nonempty interior in $\mathbb{R}^{n-1}$.

Proof. (a) $\Rightarrow$ (b) Fix $i \in \{1, ..., n\}$. If T is a tetrahedron, then $\text{Conv}(T)$ is an $(n-1)-$dimensional simplex. Thus $\{v_1 - v_i, ..., v_{i-1} - v_i, v_{i+1} - v_i, ..., v_n - v_i\}$ is a linearly independent set, hence a basis of $\mathbb{R}^{n-1}$.

(b) $\Rightarrow$ (c) This is obvious.

(c) $\Rightarrow$ (d) Without loss of generality, assume that $\{v_2 - v_1, ..., v_n - v_1\}$ is a basis of $\mathbb{R}^{n-1}$. Suppose that $\sum_{i=1}^n r_i v_i = \sum_{i=1}^n s_i v_i$ where $\sum_{i=1}^n r_i = \sum_{i=1}^n s_i$. Then

$$\begin{aligned}\sum_{i=2}^n (r_i - s_i)(v_i - v_1) &= \sum_{i=1}^n (r_i - s_i)(v_i - v_1) \\ &= \sum_{i=1}^n r_i v_i - \sum_{i=1}^n s_i v_i - \Big(\sum_{i=1}^n r_i - \sum_{i=1}^n s_i\Big) v_1 \\ &= 0.\end{aligned}$$

Thus $r_i = s_i$ for all $i = 2, ..., n$. Since $\sum_{i=1}^n r_i = \sum_{i=1}^n s_i$, $r_1 = s_1$. Therefore, $r_i = s_i$ for all $i = 1, ..., n$.

(d) $\Rightarrow$ (e) Suppose that $\sum_{i=1}^n c_i v_i = 0$ and $\sum_{i=1}^n c_i = 0$. Then $\sum_{i=1}^n c_i v_i = \sum_{i=1}^n 0 \cdot v_i$ and $\sum_{i=1}^n c_i = \sum_{i=1}^n 0$. Thus $c_i = 0$ for all $i = 1, ..., n$.

(e) $\Rightarrow$ (a) Suppose that $\sum_{i=1}^n c_i (v_i - v_1) = 0$. Then

$$\begin{aligned}\Big(-\sum_{i=2}^n c_i\Big) v_1 + \sum_{i=2}^n c_i v_i &= \sum_{i=1}^n c_i (v_i - v_1) \\ &= 0.\end{aligned}$$

By assumption, we have $c_i = 0$ for all $i = 2, ..., n$. This shows that $\{v_2 - v_1, ..., v_n - v_1\}$ is a linear independent set of $n-1$ vectors in $\mathbb{R}^{n-1}$, hence a basis. Thus T is a tetrahedron.

(a)-(e) $\Rightarrow$ (f) Suppose $v \in \mathbb{R}^{n-1}$. Then $v - v_1 = \sum_{i=2}^n c_i (v_i - v_1)$ for some scalar c_i, so $v = (1 - \sum_{i=2}^n c_i) v_1 + \sum_{i=2}^n c_i v_i = \sum_{i=1}^n d_i v_i$ where $d_1 = 1 - \sum_{i=2}^n c_i$ and $d_i = c_i$ for all $i = 2, ..., n$. Note that $\sum_{i=1}^n d_i = 1$. Then (d) implies that the representation is unique.

(f) $\Rightarrow$ (e) Suppose that $\sum_{i=1}^n c_i v_i = 0$ and $\sum_{i=1}^n c_i = 0$. Suppose there is an i such that $c_i \neq 0$. Then $c_i = -\sum_{k \neq i} c_i$ and $v_i = -\frac{1}{c_i} \sum_{k \neq i} c_k v_k$. Note that

$\sum_{k \neq i} \frac{-c_k}{c_i} = 1$. But $v_i = 1 \cdot v_i + \sum_{k \neq i} 0 \cdot v_k$. Since the representation is unique, $c_k = 0$ for all $k \neq i$. This implies $c_i = 0$ which is a contradiction. Therefore, $c_i = 0, 1 \leq i \leq n$.

(f) $\Rightarrow$ (g) For any $v \in \mathbb{R}^{n-1}$, define $f(v) = (c_1, ..., c_n)$ where $v = \sum_{i=1}^{n} c_i v_i$ and $\sum_{i=1}^{n} c_i = 1$. Since the representation is unique, f is well-defined and continuous. Let $x_0 = \frac{1}{n} \sum_{i=1}^{n} v_i$. Then $x_0 \in \mathrm{Conv}(T)$. Since $f(x_0) = (\frac{1}{n}, ..., \frac{1}{n})$ has positive coordinates in $\mathbb{R}^n$, there exists an open ball $B(x_0, \epsilon) \subset \mathbb{R}^{n-1}$ for some $\epsilon > 0$ such that $f(x)$ has positive coordinates for all $x \in B(x_0, \epsilon)$ by continuity of f. This shows that $B(x_0, \epsilon) \subset \mathrm{Conv}(T)$ and hence x_0 is in the interior of $\mathrm{Conv}(T)$.

(g) $\Rightarrow$ (f) Let $v \in \mathbb{R}^{n-1}$. Choose $x_0 \in \mathrm{int}(\mathrm{Conv}(T))$. Then the origin is in the interior of the convex hull of $\{v_1 - x_0, ..., v_n - x_0\}$. There is $t > 0$ such that $t(v - x_0)$ is in the convex hull of $\{v_1 - x_0, ..., v_n - x_0\}$. Thus there is a unique representation $0 = \sum_{i=1}^{n} r_i(v_i - x_0)$ and $t(v - x_0) = \sum_{i=1}^{n} s_i(v_i - x_0)$ where $r_i, s_i > 0, 1 \leq i \leq n$ and $\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i = 1$. Let $c_i = r_i + \frac{s_i - r_i}{t}, 1 \leq i \leq n$. Then $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} r_i + \frac{1}{t} \sum_{i=1}^{n} (s_i - r_i) = 1$ and

$$\begin{aligned}
\sum_{i=1}^{n} c_i v_i - x_0 &= \sum_{i=1}^{n} c_i(v_i - x_0) \\
&= \sum_{i=1}^{n} r_i(v_i - x_0) + \frac{1}{t} \sum_{i=1}^{n} s_i(v_i - x_0) - \frac{1}{t} \sum_{i=1}^{n} r_i(v_i - x_0)) \\
&= v - x_0
\end{aligned}$$

Hence $v = \sum_{i=1}^{n} c_i v_i$. □

Proposition 1.3.4. Let $T = \{v_1, ..., v_n\} \subseteq \mathbb{R}^{n-1}$ be a tetrahedron. Then the following statements are equivalent.

(a) T is a positive tetrahedron.

(b) There are positive scalars c_i such that $\sum_{i=1}^n c_i v_i = 0$ and $\sum_{i=1}^n c_i = 1$.

(c) There are positive scalars c_i such that $\sum_{i=1}^n c_i v_i = 0$.

Proof. (a) $\Rightarrow$ (b) Since T is a positive tetrahedron, the origin must be in the interior of Conv(T). Thus $0 = \sum_{i=1}^n c_i v_i$ for some $c_i > 0, 1 \leq i \leq n$ and $\sum_{i=1}^n c_i = 1$.

(b) $\Rightarrow$ (c) This is obvious.

(c) $\Rightarrow$ (a) Suppose that $\sum_{i=1}^n c_i = 0$ for some $c_i > 0, 1 \leq i \leq n$. For each i, let $d_i = \frac{c_i}{c_1 + \cdots + c_n}$. Then $\sum_{i=1}^n c_i = 0$ and $\sum_{i=1}^n d_i = 1$. Thus the origin is in the interior of Conv(T) and hence T is a positive tetrahedron. $\square$

In many situations, it is convenient to represent a (ordered) tetrahedron with a matrix. We may represent an ordered tetrahedron $T = (v_1, ..., v_n)$ by an $n \times (n-1)$ matrix whose row i is the vector v_i,i.e.

$$T = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

For a lighter notation, we will use T for both the ordered tetrahedron and its associated matrix. For a tetrahedron $S = \{w_1, ..., w_n\}$, we may represent S by the matrix of the ordered tetrahedron $(w_1, ..., w_n)$. Also, sometimes we will choose an order for S tacitly.

1.3.2 New Tetrahedra from Old

In this section, we develop several ways to construct a new tetrahedron and positive tetrahedron from the given one. These results will be used in later sections.

Proposition 1.3.5. Let $T = \{v_1, ..., v_n\} \subseteq \mathbb{R}^{n-1}$ be a positive tetrahedron.

(a) If $A \in \mathrm{GL}_{n-1}(\mathbb{R})$, then $TA = \{v_1A, ..., v_nA\}$ is a positive tetrahedron.

(b) If $A \in \mathrm{S}_n(\mathbb{R})$, then AT is a positive tetrahedron.

Proof. (a) Since T is a tetrahedron and A is invertible, the set $\{v_2A - v_1A, ..., v_nA - v_1A\}$ is a basis of $\mathbb{R}^{n-1}$. Thus TA is a tetrahedron by Proposition 1.3.3 (c).

(b) Let $AT = \{w_1, ..., w_n\}$. Suppose that $c_1w_1 + \cdots + c_nw_n = 0$ with $c_1 + \cdots + c_n = 0$. Let $(d_1, ..., d_n) = (c_1, ..., c_n)A$. Then we have $(d_1, ..., d_n)T = (c_1, ..., c_n)AT = 0$. Let r be the row sum of A. Then

$$\begin{aligned} d_1 + \cdots + d_n &= (d_1, ..., d_n)(1, ..., 1)^t \\ &= (c_1, ..., c_n)A(1, ..., 1)^t \\ &= r(c_1, ..., c_n)(1, ..., 1)^t \\ &= r(c_1 + \cdots + c_n) \\ &= 0. \end{aligned}$$

We have that $(d_1, ..., d_n) = 0$ since T is a tetrahedron. This implies $(c_1, ..., c_n) = 0$ since A is invertible. Thus AT is a tetrahedron by Proposition 1.3.3 (e). $\square$

Proposition 1.3.6. Let $T_0 = \{v_1, ..., v_{n-1}, v_n\}, T_1 = \{v_1, ..., v_{n-1}, w_n\}$ be positive tetrahedra. Then $\mathrm{Conv}(T_0) \cap \mathrm{Conv}(T_1)$ is the convex hull of a positive tetrahedron.

Proof. Clearly, Conv$(T_0)\cap$Conv(T_0) contains a neighborhood of the origin. The issue is to show there is a single vector u such that this intersection equals Conv$(v_1, ..., v_{n-1}, u)$. For $i = 1, ..., n-1$, let

G_i be the supporting hyperplane of T_0 containing $T_0 - \{v_i\}$,

H_i be the supporting hyperplane of T_1 containing $T_1 - \{v_i\}$,

H_n be the supporting hyperplane of T_0 and T_1 containing $\{v_1, ..., v_{n-1}\}$,

G_i^+ be the half space containing T_0 and having G_i as its boundary, and

H_i^+ be the half space containing T_1 and having H_i as its boundary.

Then Conv$(T_0) = G_1^+ \cap G_2^+ \cap \cdots \cap H_n^+$ and Conv$(T_1) = H_1^+ \cap H_2^+ \cap \cdots \cap H_n^+$. Let $T_1^{(i)} = G_i^+ \cap G_{i-1}^+ \cap \cdots \cap G_1^+ \cap T_1$. We will show that $T_1^{(i)}$ is a simplex for any $i = 1, ..., n-1$. If $T_1^{(1)} = G_1^+ \cap T_1 = $ Conv(T_1) then for $i = 1$ we are done. Suppose that $T_1^{(1)} \neq$ Conv(T_1). Note that

$$v_1 = H_2 \cap H_3 \cap \cdots \cap H_n,$$

$$v_2 = G_1 \cap H_3 \cap \cdots \cap H_n,$$

$$\vdots$$

$$v_{n-1} = G_1 \cap H_2 \cap \cdots \cap H_{n-2} \cap H_n$$

and $H_2 \cap \cdots \cap H_{n-1}$ is a line passing through v_1 and w_n, say L. Thus $G_1 \cap H_2 \cap \cdots \cap H_{n-1}$ is empty, a point, or the line L. If $G_1 \cap H_2 \cap \cdots \cap H_{n-1}$ is empty then $G_1^+ \cap T_1 = T_1$ which contradicts the assumption. Since $v_1 \notin G_1$, $G_1 \cap H_2 \cap \cdots \cap H_{n-1} \neq L$. Thus $G_1 \cap H_2 \cap \cdots \cap H_{n-1}$ is a point. Let $u_n = G_1 \cap H_2 \cap \cdots \cap H_{n-1}$. We will show that

$$G_1^+ \cap H_2^+ \cap \cdots \cap H_n^+ = \text{Conv}(v_1, ..., v_{n-1}, u_n).$$

Because u_n is the convex combination of v_1 and w_n, both of which are in $G_1^+ \cap H_n^+$, we have $G_1^+ \cap H_2^+ \cap \cdots \cap H_n^+ \supset \mathrm{Conv}(v_1, ..., v_{n-1}, u_n)$. To show the other containment, suppose that $x \in G_1^+ \cap H_2^+ \cap \cdots \cap H_n^+$. Determine a_i in $\mathbb{R}^{n-1}$ by the conditions $G_1 = \{x \in \mathbb{R}^{n-1} : x \cdot a_1 = 1\}$ and $H_i = \{x \in \mathbb{R}^{n-1} : x \cdot a_i = 1\}$ for $i = 2, ..., n$. Then $G_1^+ = \{x \in \mathbb{R}^{n-1} : x \cdot a_1 \leq 1\}$ and $H_i^+ = \{x \in \mathbb{R}^{n-1} : x \cdot a_i \leq 1\}$ for $i = 2, ..., n$. Because $\{v_1, ..., v_{n-1}, u_n\}$ is a tetrahedron, there are scalars c_i such that $x = c_1 v_1 + \cdots + c_{n-1} v_{n-1} + c_n u_n$ and $c_1 + \cdots + c_n = 1$. Then

$$\begin{aligned} x \cdot a_1 &= c_1 v_1 \cdot a_1 + \cdots + c_{n-1} v_{n-1} \cdot a_1 + c_n u_n \cdot a_1 \\ &= c_1 v_1 \cdot a_1 + c_2 + \cdots + c_n \\ &= c_1 v_1 \cdot a_1 + 1 - c_1. \end{aligned}$$

Thus $c_1 = \frac{1 - x \cdot a_1}{1 - v_1 \cdot a_1}$. Similarly, $c_i = \frac{1 - x \cdot a_i}{1 - v_i \cdot a_i}$ for $i = 2, ..., n-1$ and $c_n = \frac{1 - x \cdot a_n}{1 - u_n \cdot a_n}$. Since $x \in G_1^+, x \cdot a_1 \leq 1$ (the denominators are positive; e.g., since $v_1 \in G_1^+ \setminus G_1, v_1 \cdot a_1 < 1$). Since $v_1 \in G_1^+ \setminus G_1$, we have $v_1 \cdot a_1 < 1$, and thus $c_1 = \frac{1 - x \cdot a_1}{1 - v_1 \cdot a_1} \geq 0$. Similarly, $c_i \geq 0$ for all $i = 2, ..., n$. Hence $x \in \mathrm{Conv}(v_1, ..., v_{n-1}, u_n)$. This proves the claim. Consequently, $T_1^{(1)}$ is a tetrahedron. By using the same argument, $T_1^{(i)}$ is a tetrahedron for any $i = 2, ..., n-1$. Thus

$$\begin{aligned} \mathrm{Conv}(T_0) \cap \mathrm{Conv}(T_1) &= H_n^+ \cap T_1^{(n-1)} \\ &= T_1^{(n-1)} \end{aligned}$$

is a tetrahedron since $T_1^{(n-1)} \subseteq \mathrm{Conv}(T_1) \subseteq H_n^+$. This completes the proof. $\square$

Theorem 1.3.7. Let $T_0 = \{v_1, ..., v_n\}$ be a positive tetrahedron. Then the following statements hold.

(a) If $T_1 = \{w_1, ..., w_n\} \subset \mathbb{R}^{n-1}$ and $\text{Conv}(T_0) \subseteq \text{Conv}(T_1)$, then T_1 is a positive tetrahedron.

(b) If $c_1, c_2, ..., c_n$ are positive then $T_1 = \{c_1v_1, c_2v_2, ..., c_nv_n\}$ is a positive tetrahedron.

(c) If $T_1 = \{v_1, ..., v_{n-1}, w_n\}$ is a positive tetrahedron then $T_t = \{v_1, ..., v_{n-1}, (1-t)v_n + tw_n\}, 0 \le t \le 1$ are positive tetrahedra.

Proof. (a) Since $\text{Conv}(T_0)$ has nonempty interior, $\text{Conv}(T_1)$ has nonempty interior. Clearly, the origin is in the interior of $\text{Conv}(T_1)$. Thus T_1 is a positive tetrahedron.

(b) Let $c = \min\{c_1, ..., c_n\}$. Then $T_2 = \{cv_1, ..., cv_n\}$ is clearly a positive tetrahedron. Moreover, $\text{Conv}(T_2) \subseteq \text{Conv}(T_1)$. Consequently, T_1 is a positive tetrahedron by part (a).

(c) By Proposition 1.3.6, $\text{Conv}(T_0) \cap \text{Conv}(T_1)$ is the convex hull of a tetrahedron. Let $v \in \text{Conv}(T_0) \cap \text{Conv}(T_1)$. Suppose that $v = c_1v_1 + \cdots + c_{n-1}v_{n-1} + c_nv_n$ and $v = d_1v_1 + \cdots + d_{n-1}v_{n-1} + d_nw_n$ for some c_j, d_j. Then $\frac{tv}{c_n} = \frac{c_1t}{c_n}v_1 + \cdots + \frac{c_{n-1}t}{c_n}v_{n-1} + tv_n$ and $\frac{(1-t)v}{d_n} = \frac{d_1(1-t)}{d_n}v_1 + \cdots + \frac{d_{n-1}(1-t)}{d_n}v_{n-1} + (1-t)w_n$. Thus

$$\begin{aligned}\Big[\frac{t}{c_n} + \frac{1-t}{d_n}\Big]v &= \Big[\frac{c_1t}{c_n} + \frac{d_1(1-t)}{d_n}\Big]v_1 + \cdots + \Big[\frac{c_{n-1}t}{c_n} + \frac{d_{n-1}(1-t)}{d_n}\Big]v_{n-1} + v_n(t)\\ v &= \Big[\frac{c_1d_nt + d_1c_n(1-t)}{d_nt + c_n(1-t)}\Big]v_1 + \cdots + \Big[\frac{c_{n-1}d_nt + d_{n-1}c_n(1-t)}{d_nt + c_n(1-t)}\Big]v_{n-1}\\ &+ \Big[\frac{c_nd_n}{d_nt + c_n(1-t)}\Big]v_n(t).\end{aligned}$$

Hence $v \in \text{Conv}(T_t)$ for all $t \in [0,1]$. By part (a), $T_t, 0 \le t \le n$ are positive tetrahedra. □

1.3.3 Invariant Tetrahedra and Positive Invariant Tetrahedra

Let B be an $(n-1) \times (n-1)$ real matrix. For the rest of this chapter, we always assume that B has all eigenvalues less than 1 in absolute value.

Definition 1.3.8. Let $T = \{v_1, ..., v_n\}$ be a (ordered) tetrahedron. T is called an *(ordered) invariant tetrahedron* for B if the convex hull of T is sent to itself under B. T is called a *(ordered) positive invariant tetrahedron* of B if the convex hull of T is sent to its interior under B.

Proposition 1.3.9. Let T be an invariant tetrahedron of a matrix B. Then the origin must be in the convex hull of T. If T is a positive invariant tetrahedron of B then T is also a positive tetrahedron.

Proof. Note that $TB \subseteq \text{Conv}(T)$. Then $TB^n \subseteq \text{Conv}(T))$ for all $n \in \mathbb{N}$. But $\lim_{n\to\infty} TB^n = \{0\}$, so $\{0\} \subseteq \text{Conv}(T)$. If T is a positive invariant tetrahedron of B then the origin must be in the interior of $\text{Conv}(T)$. Thus T is a positive tetrahedron. $\square$

Theorem 1.3.10. Let $T = \{v_1, ..., v_n\}$ be a tetrahedron. Then T is a (positive) invariant tetrahedron of B if and only if there is a (positive) stochastic matrix P such that $TB = PT$.

Proof. Suppose that $T = \{v_1, ..., v_n\}$ is an invariant tetrahedron of B. For each $i \in \{1, ..., n\}$ we have $v_i B = \sum_{j=1}^n p_{ij} v_i$ for some $p_{ij} \geq 0$ such that $\sum_{j=1}^n p_{ij} = 1$ since $v_i B$ is in $\text{Conv}(T)$. Put $P = (p_{ij})$. Then P is stochastic and satisfies the equation $TB = PT$.

Suppose conversely that $TB = PT$ for some (positive)stochastic matrix P. Let $v = \sum_{i=1}^{n} c_i v_i$ where $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$. Then

$$
\begin{aligned}
vB &= \sum_{i=1}^{n} c_i v_i B \\
&= \sum_{i=1}^{n} c_i \Big(\sum_{j=1}^{n} p_{ij} v_j \Big) \\
&= \sum_{i=1}^{n} d_i v_i
\end{aligned}
$$

where $d_i = \sum_{j=1}^{n} c_j p_{ji}$. Note that $d_i \geq 0$ and

$$
\begin{aligned}
\sum_{i=1}^{n} d_i &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_j p_{ji} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} c_j p_{ji} \\
&= \sum_{j=1}^{n} c_j \Big(\sum_{i=1}^{n} p_{ji} \Big) \\
&= \sum_{j=1}^{n} c_j \\
&= 1.
\end{aligned}
$$

Thus vB is in the convex hull of T. Note that T is positive invariant if and only if $p_{ij} > 0$ for all i, j. $\square$

Let $\mathcal{T}_+(B)$ denote the set of all positive invariant tetrahedra of B, $\mathcal{T}_+^{ord}(B)$ denote the space of ordered positive invariant tetrahedra of B, and $\mathcal{S}_+(B)$ denote the space of positive stochastic matrices similar to $B \oplus 1$. We topologize $\mathcal{T}_+(B)$ by using the Hausdorff metric and topologize $\mathcal{T}_+^{ord}(B)$ and $\mathcal{S}_+(B)$ by using the subspace topology of the Euclidean space. The space of positive invariant tetrahedra has proved its worth in the study of rational strong shift equivalence by Kim and Roush.

Remark 1.3.11. There is a natural continuous function $\mathcal{T}_+^{ord}(B) \to \mathcal{T}_+(B)$ defined by $(v_1, ..., v_n) \mapsto \{v_1, ..., v_n\}$. Thus a path in $\mathcal{T}_+^{ord}(B)$ induces a path in $\mathcal{T}_+(B)$.

Theorem 1.3.12. Let T be an ordered positive tetrahedron and denote $\mathbf{1} = (1, 1, ..., 1)^t$. Then $T \in \mathcal{T}_+^{ord}(B)$ if and only if the matrix $(T\ \mathbf{1})(B \oplus 1)(T\ \mathbf{1})^{-1}$ is positive.

Proof. If $T \in \mathcal{T}_+(B)$ then there is a positive stochastic matrix P such that $TB = PT$. This implies that $(T\ \mathbf{1})(B \oplus 1) = P(T\ \mathbf{1})$, so $(T\ \mathbf{1})(B \oplus 1)(T\ \mathbf{1})^{-1} = P$ is positive. Conversely, suppose that $P = (T\ \mathbf{1})(B \oplus 1)(T\ \mathbf{1})^{-1}$ is positive. We will show that P has row sum 1. Note that $(T\ \mathbf{1})(0, 0, ..., 1)^t = (1, 1, ..., 1)^t$. Thus

$$\begin{aligned} P(1, 1, ..., 1)^t &= (T\ \mathbf{1})(B \oplus 1)(T\ \mathbf{1})^{-1}(1, 1, ..., 1)^t \\ &= (T\ \mathbf{1})(B \oplus 1)(0, 0, ..., 1)^t \\ &= (T\ \mathbf{1})(0, 0, ..., 1)^t \\ &= (1, 1, ..., 1)^t. \end{aligned}$$

Hence P is positive and stochastic. Then we have $(T\ \mathbf{1})(B \oplus 1) = P(T\ \mathbf{1})$ and it can be reduced to $TB = PT$. Therefore, $T \in \mathcal{T}_+(B)$. $\square$

Definition 1.3.13. For an ordered tetrahedron $T = (v_1, ..., v_n)$ in $\mathbb{R}^{n-1}$, let P_T be the quasi-stochastic matrix P such that $v_i B = \sum_{i=1}^n p_{ij} v_j, 1 \le i \le n$.

Remark 1.3.14. If T is an ordered invariant tetrahedron of B then P_T is stochastic. If in addition T is positive then P_T is positive and stochastic.

For any $T \in \mathcal{T}_+^{ord}(B)$, define $\pi_B(T) = P_T$. We recall the following theorem in [KR90].

Theorem 1.3.15. [KR90] $\pi_B : \mathcal{T}_+^{ord}(B) \to \mathcal{S}_+(B)$ is continuous and surjective.

Proof. By Theorem 1.3.12, we have $P_T = (T\ \mathbf{1})(B \oplus 1)(T\ \mathbf{1})^{-1}$. Thus π_B is continuous. Let $P \in \mathcal{S}_+(B)$. Suppose that $P = C(B \oplus 1)C^{-1}$ for some $C \in \mathrm{GL}_n(\mathbb{R})$. Let C_n denote the last column of C. Then $PC = C(B \oplus 1)$ implies $PC_n = C_n$. Thus C_n is a right eigenvector of P corresponding to 1, so $C_n = k(1, 1, ..., 1)^t$ for some $k > 0$. Let R_i be the i^{th} row of C. Then $R_i = (v_i, k)$ for some $v_i \in \mathbb{R}^{n-1}, 1 \le i \le n$. Define $T = (v_1, ..., v_n)$. We will show that T is an ordered tetrahedron. Since C is invertible, $\{R_1, ..., R_n\}$ is a basis of $\mathbb{R}^n$. Suppose that $\sum_{i=2}^n c_i(v_i - v_1) = 0$. Then $\sum_{i=2}^n c_i R_i - (\sum_{i=2}^n c_i)R_1 = \sum_{i=2}^n c_i(R_i - R_1) = \sum_{i=2}^n c_i(v_i - v_1, 0) = 0$. Thus $c_i = 0$ for all $i = 2, ..., n$. By Proposition 1.3.3 (c), T is an ordered tetrahedron as claimed. Note that $PC = (PT, C_n)$ and $C(B \oplus 1) = (TB, C_n)$, so $TB = PT$. By Theorem 1.3.10, $T \in \mathcal{T}_+^{ord}(B)$ and $P_T = P$. Thus π_B is surjective. $\square$

The following proposition suggests us to study the space of positive invariant tetrahedra for some simple matrix B, e.g. B may be chosen as the real Jordan canonical form.

Proposition 1.3.16. Let A and B be similar matrices over $\mathbb{R}$. Then

(a) $\mathcal{T}_+^{ord}(A)$ is homeomorphic to $\mathcal{T}_+^{ord}(B)$.

(b) $\mathcal{T}_+(A)$ is homeomorphic to $\mathcal{T}_+(B)$.

Proof. Suppose that $A = CBC^{-1}$. For any $T \in \mathcal{T}_+^{ord}(A)$, there is a positive stochastic matrix P_T such that $TA = P_T T$. Then we have $(TC)B = TAC = P_T(TC)$. Thus

$TC \in \mathcal{T}_+^{ord}(B)$. The map $T \longmapsto TC$ defines a homeomorphism between $\mathcal{T}_+^{ord}(A)$ and $\mathcal{T}_+^{ord}(B)$. The same argument can be applied for $\mathcal{T}_+(A)$ and $\mathcal{T}_+(B)$. $\square$

Proposition 1.3.17. Let $T = (v_1, ..., v_n) \in \mathcal{T}_+^{ord}(B)$ and $l = (l_1, ..., l_n) \in \text{int}(\Delta^{n-1})$. Then the following statements are equivalent.

(a) $l_1 v_1 + \cdots + l_n v_n = 0$.

(b) l is the left Perron eigenvector of P_T.

Proof. (a) Suppose that $l_1 v_1 + \cdots + l_n v_n = 0$. Then

$$(l_1 v_1 + \cdots + l_n v_n)B = 0$$

$$(l_1 p_{11} + \cdots + l_n p_{n1})v_1 + \cdots + (l_1 p_{1n} + \cdots + l_n p_{nn})v_n = l_1 v_1 + \cdots + l_n v_n.$$

Applying Proposition 1.3.3 (d), we get $\sum_{i=1}^{n} l_i p_{ij} = l_j$ for all $j = 1, 2, ..., n$. Consequently, we have $lP_T = l$ and hence l is the left Perron eigenvector of P_T.

(b) Suppose that $lP_T = l$. Then

$$\begin{aligned}
(l_1 v_1 + \cdots + l_n v_n)B &= l_1 v_1 B + l_2 v_2 B + \cdots + l_n v_n B \\
&= l_1(p_{11} v_1 + \cdots + p_{1n} v_n) + \cdots + l_n(p_{n1} v_1 + \cdots + p_{nn} v_n) \\
&= (l_1 p_{11} + \cdots + l_n p_{n1})v_1 + \cdots + (l_1 p_{1n} + \cdots + l_n p_{nn})v_n \\
&= l_1 v_1 + \cdots + l_n v_n.
\end{aligned}$$

Since B has all eigenvalues less than 1 in absolute value, $l_1 v_1 + \cdots + l_n v_n$ can not be an eigenvector of B corresponding to an eigenvalue 1. Therefore $l_1 v_1 + \cdots + l_n v_n = 0$. $\square$

Proposition 1.3.18. Let $S = (v_1, ..., v_n), T = (w_1, ..., w_n) \in \mathcal{T}_+^{ord}(B)$. Then the following are equivalent.

(a) $P_S = P_T$.

(b) There exists an invertible matrix A such that $w_i = v_i A$ for $1 \leq i \leq n$ and $AB = BA$.

Proof. $(b) \Rightarrow (a)$ Suppose that $v_i B = p_{i1} v_1 + \cdots + p_{in} v_n$ for all $i = 1, ..., n$. Then

$$\begin{aligned} w_i B &= v_i AB \\ &= v_i BA \\ &= (p_{i1} v_1 + \cdots + p_{in} v_n) A \\ &= p_{i1} w_1 + \cdots + p_{in} w_n. \end{aligned}$$

Thus $P_S = P_T$.

$(a) \Rightarrow (b)$ Suppose that $v_i B = p_{i1} v_1 + \cdots + p_{in} v_n$ and $w_i B = p_{i1} w_1 + \cdots + p_{in} w_n$ for all $i = 1, ..., n$. Then $l_1 v_1 + \cdots + l_n v_n = 0$ and $l_1 w_1 + \cdots + l_n w_n = 0$ for some $l = (l_1, ..., l_n) \in \operatorname{int}(\Delta^n)$. By Proposition 1.3.3 (g), $\{v_1, ..., v_{n-1}\}$ and $\{w_1, ..., w_{n-1}\}$ are bases of $\mathbb{R}^{n-1}$. Define a linear transformation $L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $L(v_i) = w_i$ for all $i = 1, ..., n-1$. Then

$$\begin{aligned} L(v_n) &= L\Big(-\frac{1}{l_n}(l_1 v_1 + \cdots + l_{n-1} v_{n-1})\Big) \\ &= -\frac{1}{l_n}\Big(l_1 L(v_1) + \cdots + l_{n-1} L(v_{n-1})\Big) \\ &= -\frac{1}{l_n}(l_1 w_1 + \cdots + l_{n-1} w_{n-1}) \\ &= w_n \end{aligned}$$

Thus there is an invertible matrix A such that $w_i = v_i A$ for all $i = 1, ..., n$. For any

$i \in \{1, ..., n\}$, we have

$$\begin{aligned} v_i AB &= w_i B \\ &= p_{i1} w_1 + \cdots + p_{in} w_n \\ &= (p_{i1} v_1 + \cdots + p_{in} v_n) A \\ &= v_i BA. \end{aligned}$$

Therefore, $AB = BA$. $\square$

1.4 $\mathcal{T}_+(B)$ Has Only Finitely Many Connected Components

Definition 1.4.1. A semialgebraic subset of $\mathbb{R}^n$ is a subset of points in $\mathbb{R}^n$ which is the solution set of a boolean combination of polynomial equations and inequalities with real coefficients.

It is well-known that a semialgebraic set has finitely many connected components. See e.g. [BCR98, Theorem 2.4.4] for more details.

Theorem 1.4.2. $\mathcal{T}_+^{ord}(B)$ has finitely many connected components.

Proof. It suffices to show that $\mathcal{T}_+^{ord}(B)$ is a semialgebraic set. Let $T \in \mathcal{T}_+^{ord}(B)$. Then T satisfies the matrix inequality

$$(T\ \mathbf{1})(B \oplus 1)(T\ \mathbf{1})^{-1} > 0$$

which is equivalent to the system of polynomial inequalities in $n(n+1)$ variables

$$\{\det{(T\ \mathbf{1})} > 0, (T\ \mathbf{1})(B \oplus 1)\mathrm{adj}(T\ \mathbf{1}) > 0\} \text{ or}$$

$$\{\det{(T\ \mathbf{1})} < 0, (T\ \mathbf{1})(B \oplus 1)\mathrm{adj}(T\ \mathbf{1}) < 0\}.$$

Thus $\mathcal{T}_+^{ord}(B)$ is a semialgebraic set. $\square$

Corollary 1.4.3. The following statements hold.

(a) $\mathcal{T}_+(B)$ has finitely many connected components.

(b) $\mathcal{S}_+(B)$ has finitely many connected components.

(c) There are finitely many SSE-$\mathbb{R}_+$ classes in the same similarity class.

Proof. (a) The map $(v_1, ..., v_n) \mapsto \{v_1, ..., v_n\}$ induces a fewer number of connected components in $\mathcal{T}_+(B)$ than the number of connected components in $\mathcal{T}_+^{ord}(B)$.

(b) This follows from Theorem 1.3.15 and Theorem 1.4.2.

(c) By Theorem 1.2.9, the number of SSE-$\mathbb{R}_+$ classes is at most the number of connected components of $\mathcal{S}_+(B)$. $\square$

Remark 1.4.4. We can show directly that $\mathcal{S}_+(B)$ has finitely many connected components by noticing that $\mathcal{S}_+(B)$ is a semialgebraic set. Let $p_B(t)$ be the characteristic polynomial of $B \oplus 1$. Let $p_B(t) = \prod_{k=1}^{m}(q_k(t))^{j_k}$ where the q_i are irreducible and distinct, and $j_k \in \mathbb{N}$. Then $P \in \mathcal{S}_+(B)$ if and only if

- $\sum_{j=1}^{n} p_{ij} = 1, 1 \leq i \leq n,$
- $p_{ij} > 0$ for $i, j = 1, ..., n,$
- $\text{rank}(q_k(P))^j = \text{rank}(q_k(B \oplus 1))^j, 1 \leq k \leq m, 1 \leq j \leq j_k.$

That a matrix M has a given rank r is equivalent to $r \times r$ being the size of the largest submatrix of M with nonzero determinant. This is a semialgebraic condition on M.

1.5 Same Connected Component Criteria

The purpose of this section is to prove the following theorems.

Theorem 1.5.1. Suppose $T_0, T_1 \in \mathcal{T}_+^{ord}(B)$. For $0 \le t \le 1$, set $T_t = (1-t)T_0 + tT_1$. If T_t is a positive tetrahedron for each $t \in [0,1]$ then T_0 and T_1 are in the same connected component.

Proof. Suppose $z \in T_t$. Then there exist $x \in T_0$ and $y \in T_1$ such that $z = (1-t)x + ty$, and $zB = (1-t)xB + tyB$. There exist $x' \in \text{int}(\text{Conv}(T_0))$ and $y' \in \text{int}(\text{Conv}(T_1))$ such that $xB = x'$ and $yB = y'$ and then for $0 < t < 1$, $zB = (1-t)x' + ty' \in \text{int}(\text{Conv}((1-t)T_0)) + \text{int}(\text{Conv}(tT_1)) \subset \text{Conv}(T_t)$. This shows that $T_t \in \mathcal{T}_+^{ord}(B)$ for all $t \in [0,1]$. So T_0 and T_1 are in the same connected component of $\mathcal{T}_+^{ord}(B)$. $\square$

Theorem 1.5.2. Each of the following pairs of positive invariant tetrahedra of B are in the same connected component of $\mathcal{T}_+^{ord}(B)$.

(a) $T_0 = (v_1, ..., v_n), T_1 = (c_1 v_1, ..., c_n v_n)$ where $c_1, ..., c_n$ are positive.

(b) $T_0 = (v_1, ..., v_{n-1}, v_n), T_1 = (v_1, ..., v_{n-1}, w_n)$

(c) $T_0 = (v_1, ..., v_{n-1}, v_n), T_1 = (w_1, ..., w_{n-1}, v_n)$ where v_i, w_i, v_n are colinear for all $i = 1, ..., n-1$.

Proof. (a) For $0 \le t \le 1$, define $T_t = ((1-t+c_1 t)v_1, ..., (1-t+c_n t)v_n)$. By Theorem 1.3.7 (b), T_t is a positive tetrahedron for all $t \in [0,1]$. Therefore, T_0 and T_1 are in the same connected component of $\mathcal{T}_+^{ord}(B)$ by Theorem 1.5.1.

(b) For $0 \le t \le 1$, define $T_t = (v_1, ..., v_{n-1}, (1-t)v_n + tw_n)$. By Theorem 1.3.7 (c), T_t is a positive tetrahedron for all $t \in [0,1]$. Therefore, T_0 and T_1 are in the same connected component of $\mathcal{T}_+^{ord}(B)$ by Theorem 1.5.1.

(c) Let $u_i \in \{v_i, w_i\}$ be such that

$$\|u_i - w_n\| = \max\{\|v_i - w_n\|, \|w_i - w_n\|\} \text{ for } i = 1, 2, ..., n-1.$$

Define $T_{\frac{1}{2}} = (u_1, u_2, ..., u_{n-1}, w_n)$. Note that v_i and w_i lie between u_i and w_n for all $i = 1, 2, ..., n-1$. Thus $\text{Conv}(T_{\frac{1}{2}}) = \text{Conv}(T_0) \cup \text{Conv}(T_1)$. This implies that $T_{\frac{1}{2}} \in \mathcal{T}_+^{ord}(B)$. For $0 \le t \le 1$, define $T_{\frac{t}{2}} = (1-t)T_0 + tT_{\frac{1}{2}}$. Then $\text{Conv}(T_0) \subseteq \text{Conv}(T_{\frac{t}{2}})$ for all $t \in [0,1]$. Thus $T_{\frac{t}{2}}$ is a positive tetrahedron for all $t \in [0,1]$. So T_0 and $T_{\frac{1}{2}}$ are in the same connected component of $\mathcal{T}_+^{ord}(B)$ by Theorem 1.5.1. Similarly, $T_{\frac{1+t}{2}} = (1-t)T_{\frac{1}{2}} + tT_1$ is a positive tetrahedron for any $t \in [0,1]$. Thus, $T_{\frac{1}{2}}$ and T_1 are in the same connected component of $\mathcal{T}_+^{ord}(B)$ and hence T_0 and T_1 are in the same connected component of $\mathcal{T}_+^{ord}(B)$. $\square$

1.6 Some Cases in Which $\mathcal{T}_+(B)$ Is Disconnected

In this section, we show that the space $\mathcal{T}_+(B)$ can be disconnected. In the next section we will show that $\mathcal{T}_+(B)$ has exactly 2 connected components when $B = \text{diag}(\alpha, -\beta)$ with $\alpha, \beta > 0$ and $\alpha + \beta \ge 1$.

Lemma 1.6.1. Let $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $-1 < \beta \le \alpha < 1$. Suppose that $T = \{(0,1), (-b, y), (c, z)\}$ where $b, c > 0$ is a positive tetrahedron. If $T \in \mathcal{T}_+(B)$ then $\alpha - \beta < 1$.

Proof. If $\alpha = \beta$ then we are done. Suppose that $\beta < \alpha$. The corresponding matrix for T is

$$P = \frac{1}{D}\begin{pmatrix} -(bz+cy)+\beta(b+c) & (1-\beta)c & (1-\beta)b \\ -(bz+cy)+\alpha b(z-y)+\beta y(b+c) & \alpha b(1-z)-\beta cy+c & \alpha b(y-1)-\beta by+b \\ -(bz+cy)+\alpha c(y-z)+\beta z(b+c) & \alpha c(z-1)-\beta cz+c & \alpha c(1-y)-\beta bz+b \end{pmatrix}$$

where $D = -(bz+cy)+b+c$. Note that $l = (l_1, l_2, l_3)$ where $l_1 = -\frac{bz+cy}{D}, l_2 = \frac{b}{D}, l_3 = \frac{c}{D}$ is the left Perron eigenvector of P. Thus $D > 0$. Since $p_{23} > 0$, we have $y > \frac{\alpha-1}{\alpha-\beta}$. Since $p_{32} > 0$, we have $z > \frac{\alpha-1}{\alpha-\beta}$. Since $p_{11} > 0$, we have $\beta > \frac{bz+cy}{b+c} > \frac{\alpha-1}{\alpha-\beta}$. Thus $\beta(\alpha-\beta) > \alpha-1, (1-\beta)(\alpha-\beta-1) < 0$, and hence $\alpha-\beta < 1$. $\square$

Proposition 1.6.2. If $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $\alpha-\beta \geq 1$ then $\mathcal{T}_+(B)$ is not path connected.

Proof. Let $T_0 = \{(-a,0),(1,1),(1,-1)\}$ and $T_1 = \{(a,0),(-1,-1),(-1,1)\}$ where $a > -\frac{\alpha+\beta}{1+\beta}$. Because $T_0 = -T_1$, both T_0 and T_1 correspond to the same positive stochastic matrix

$$P = \frac{1}{1+a}\begin{pmatrix} 1+\alpha a & \frac{(1-\alpha)a}{2} & \frac{(1-\alpha)a}{2} \\ 1-\alpha & \frac{\alpha+\beta+(1+\beta)a}{2} & \frac{\alpha-\beta+(1-\beta)a}{2} \\ 1-\alpha & \frac{\alpha-\beta+(1-\beta)a}{2} & \frac{\alpha+\beta+(1+\beta)a}{2} \end{pmatrix}.$$

Thus $T_0, T_1 \in \mathcal{T}_+(B)$. Suppose that there is a path T_t connecting T_0 and T_1. Then there is some $t_0 \in (0,1)$ such that $T_{t_0} = \{(0,x),(-b,y),(c,z)\}$ for some $x \neq 0$ and $b,c > 0$. Thus $\frac{1}{x}T_{t_0} \in \mathcal{T}_+(B)$. From Lemma 1.6.1, we get $\alpha-\beta < 1$ which is a contradiction. $\square$

Remark 1.6.3. Let T_0, T_1 be as in the last proof. Given P in $\mathcal{S}_+(B)$, we have $P = P_T$ for some $T \in \mathcal{T}_+(B)$. There is a path in $\mathcal{T}_+(B)$ to T_0 or T_1, and thus a path of similar positive stochastic matrices from P to $P_{T_0} = P_{T_1}$. Therefore, although $\mathcal{T}_+(B)$ is disconnected, the space $\mathcal{S}_+(B)$ is connected.

1.7 Connected Components of $\mathcal{T}_+(B)$ When B is 1×1 and 2×2

In this section, we determine the number of connected components of $\mathcal{T}_+(B)$ when B is a 1×1 and 2×2 matrix. It is easy to find the number of connected components of $\mathcal{T}_+(B)$ when B is 1×1. Unfortunately, we do not have a complete characterization of the number of connected components of $\mathcal{T}_+(B)$ when B is 2×2. We summarize the results as follows.

1. $\mathcal{T}_+(B)$ is path connected when B has the Jordan form

   (a) $(\lambda), \lambda \in (-1, 1)$,

   (b) $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \alpha, \beta \in [0, 1)$,

   (c) $\begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}, \alpha, \beta \in [0, 1)$ and $\alpha + \beta < 1$,

   (d) $\begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \alpha \in [0, \frac{1}{2})$,

   (e) $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \alpha \in [0, 1)$.

2. $\mathcal{T}_+(B)$ has 2 connected components when B has the Jordan form $\begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}, \alpha, \beta \in (0, 1)$ and $\alpha + \beta \geq 1$.

Whether $\mathcal{T}_+(B)$ is path connected is still unknown when B has one of the remaining Jordan forms which are compatible with $B \oplus 1$ being similar to a positive stochastic matrix:

(a) $\begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}$, $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$,

(b) $\begin{pmatrix} -\alpha & 1 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in (0, \frac{1}{2})$,

(c) $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $(\alpha, \beta) \in \mathrm{int}(\mathrm{Conv}(T))$ where $T = \{(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$.

We begin with the easy 1×1 case.

Proposition 1.7.1. [KR90] Let $B = (\lambda), -1 < \lambda < 1$. Then $\mathcal{T}_+(B)$ is path connected.

Proof. Let $T_0 = \{a, x\}, T_1 = \{b, y\}$ be positive invariant tetrahedra of B. We can assume that $a < 0 < x$ and $b < 0 < y$. Let $a_t = (1-t)a + tb$ and $x_t = (1-t)x + ty$. Define $T_t = \{a_t, x_t\}$. Note that $a_t < 0 < x_t$. Thus T_t is a positive tetrahedron for any $t \in [0,1]$. The corresponding stochastic matrix for T_t is

$$P_t = \frac{1}{a_t - x_t} \begin{pmatrix} \lambda x_t - a_t & (1-\lambda)x_t \\ (\lambda - 1)a_t & x_t - \lambda a_t \end{pmatrix}.$$

It is easy to check that P_t is positive for any $t \in [0,1]$. Thus T_t is positive invariant under B for any $t \in [0,1]$. Therefore $\mathcal{T}_+(B)$ is path connected. $\square$

The spectra of 3×3 stochastic matrices are completely characterized by Loewy and London(see e.g. [ELN04]). We give special cases for positive stochastic matrices.

Theorem 1.7.2. Let $\alpha, \beta \in (-1, 1)$. $\Lambda = \{1, \alpha, \beta\}$ is a spectrum of a 3×3 positive stochastic matrix if and only if $\alpha + \beta > -1$.

Theorem 1.7.3. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha^2+\beta^2 < 1$. Let $T = \{(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$. $\Lambda = \{1, \alpha + \beta i, \alpha - \beta i\}$ is a spectrum of a 3×3 positive stochastic matrix if and only if $(\alpha, \beta) \in \operatorname{int}(\operatorname{Conv}(T))$.

These results give all possible Jordan forms of a 2×2 matrix B for which $B \oplus 1$ is similar to a 3×3 positive stochastic matrix.

Theorem 1.7.4. Let $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $\alpha \geq |\beta|$.

(a) If $\alpha - \beta < 1$ then $\mathcal{T}_+(B)$ is path connected.

(b) If $\alpha - \beta \geq 1$ then $\mathcal{T}_+(B)$ contains exactly 2 path connected components.

Proof. Let $T_0 = \{v_1, v_2, v_3\} \in \mathcal{T}_+(B)$, L_{ij} be the line segment connecting v_i and v_j for all $1 \leq i < j \leq 3$, and W_{ij} be the convex hull of $0, v_i, v_j$ for all $1 \leq i < j \leq 3$. Then one of W_{ij} intersects the x-axis at the origin only, say W_{12}. If L_{12} is parallel to the x-axis then we can perturb L_{12} so that L_{12} is not parallel to the x-axis. Suppose without loss of generality that L_{12} has positive slope or a vertical line. Let u_1 be the intersection between the line connecting v_1 and v_2 and the x-axis. We can also assume that v_1 is on a line segment connecting u_1 and v_2. Let $T_1 = \{u_1, v_2, v_3\}$. Then $\operatorname{Conv}(T_0) \subseteq \operatorname{Conv}(T_1)$. Since u_1 is on the x-axis, $u_1 B = \alpha u_1$ which is in the interior of T_1 and hence $T_1 \in \mathcal{T}_+(B)$. Thus every positive invariant tetrahedron of B is in the same connected component as an invariant tetrahedron whose one vertex is on the x-axis. Next, suppose that $T_2 = \{(a, 0), (b, y), (c, z)\}$ where $a \neq 0$. We consider 2 cases.

Case 1: $a < 0$. We can assume without loss of generality that $b \leq c$. Then $c > 0$. Let $T_3 = \{(-1,0), (-\frac{b}{a}, -\frac{y}{a}), (-\frac{c}{a}, -\frac{z}{a})\}$. The positive tetrahedron of the form $\{(-1,0), (d,w), (d,-w)\}, d > 0$ is in $\mathcal{T}_+(B)$ because it corresponds to the matrix

$$\frac{1}{2(1+d)} \begin{pmatrix} 2(\alpha+d) & 1-\alpha & 1-\alpha \\ 2(1-\alpha)d & (1-\beta)+(\alpha-\beta)d & (1+\beta)+(\alpha+\beta)d \\ 2(1-\alpha)d & (1+\beta)+(\alpha+\beta)d & (1-\beta)+(\alpha-\beta)d \end{pmatrix}.$$

Then T_3 is in the same connected component as $T_4 = \{(-1,0), (-\frac{c}{a}, -\frac{z}{a}), (-\frac{c}{a}, \frac{z}{a})\}$. Thus T_4 is in the same connected component as $T_5 = \{(-1,0), (1,1), (1,-1)\}$ via the path

$$T_{4+t} = \{(-1,0), (\frac{at-(1-t)c}{a}, \frac{at-(1-t)|z|}{a}), (\frac{at-(1-t)c}{a}, \frac{at+(1-t)|z|}{a})\}.$$

Case 2: $a > 0$. By using similar arguments as in case 1, T_2 is in the same connected component as $T_6 = \{(1,0), (-1,1), (-1,-1)\}$. Therefore every positive invariant tetrahedron of B is in the same connected component as either T_5 or T_6.

(a) If $\alpha - \beta < 1$ then a positive tetrahedron $T_7 = \{(0,1), (-1, \alpha-1), (1, \alpha-1)\}$ is in $\mathcal{T}_+(B)$. Let L be the line segment connecting $(-1, \alpha-1)$ and $(1, \alpha-1)$. We can perturb L so that it has positive or negative slope after perturbation. We denote L' and T_7' as the line segment L and the positive tetrahedron T_7 after perturbation respectively. By continuity, T_7' is still in $\mathcal{T}_+(B)$ and in the same connected component as T_7. If L' has positive slope then T_7' is in the same connected component as T_6. If L' has negative slope then T_7' is in the same connected component as T_5. Thus T_5

and T_6 are in the same connected component. Therefore $\mathcal{T}_+(B)$ is path connected.

(b) If $\alpha - \beta \geq 1$ then Proposition 1.6.2 implies that T_5 and T_6 are not in the same connected component. Thus $\mathcal{T}_+(B)$ has exactly 2 path connected components.

$\square$

Lemma 1.7.5. Let $B = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$ where $0 < \alpha < \beta < 1$ and $a, b, c, x, y > 0$.

(a) If $T_0 = \{(1, 0), (a, x), (-b, -y)\} \in \mathcal{T}_+(B)$ then

$$T_1 = \{(1, 0), (0, x), (-b, -y)\} \in \mathcal{T}_+(B).$$

(b) If $T = \{(1, 0), (-a, x), (-b, -y)\} \in \mathcal{T}_+(B)$ then $a < \frac{1-\beta^2}{\beta^2-\alpha^2}$.

(c) If $0 < a < \frac{1-\beta^2}{\beta^2-\alpha^2}$ and $1 < c < \frac{1}{\beta^2+(\beta^2-\alpha^2)a}$ then

$$T = \{(1, 0), (-a, x), c(-\alpha a, -\beta x)\} \in \mathcal{T}_+(B).$$

(d) If $\alpha + \beta < 1$ and $\beta x < y < \left(\frac{1-\alpha}{\alpha+\beta}\right)x$ then

$$T = \{(0, x), (-a, -y), (b, -y)\} \in \mathcal{T}_+(B).$$

Proof. (a) Let $v_1 = (1, 0), v_2 = (a, x), v_3 = (-b, -y)$ and H_1, H_2, H_3 be lines passing through $T_0 - \{v_1\}, T_0 - \{v_1\}, T_0 - \{v_2\}$ respectively. Then H_1, H_2, H_3 can be described by the equations

$$H_1 : \Big(\frac{x + y}{ay - bx}\Big)X - \Big(\frac{a + b}{ay - bx}\Big)Y = 1$$
$$H_2 : X - \Big(\frac{b + 1}{y}\Big)Y = 1$$
$$H_3 : X + \Big(\frac{1 - a}{x}\Big)Y = 1.$$

Since v_1 is an eigenvector of B corresponding to a positive eigenvalue $\alpha < 1, v_1B \in \mathrm{int}(\mathrm{Conv}(T_1))$. The line H_1 intersects the y-axis at $v_4 = (0, \frac{bx-ay}{a+b})$. Since $v_3B =$

$(-\alpha b, \beta y)$ which is in the second quadrant, $v_3 B \in \mathrm{int}(\mathrm{Conv}(v_1, v_3, v_4))$. Let $v_5 = (0, x)$. Then

$$\begin{aligned}\mathrm{Conv}(v_1, v_3, v_4) &\subseteq \mathrm{Conv}(v_1, v_3, v_5)\\ &= \mathrm{Conv}(T_1).\end{aligned}$$

because $\frac{bx-ay}{a+b} < x$. Thus $v_3 B \in \mathrm{int}(\mathrm{Conv}(T_1))$. The line H_2 intersects the y-axis at $v_6 = (0, -\frac{y}{b+1})$. Since $v_2 B \in \mathrm{int}(\mathrm{Conv}(T_1)) \subseteq H_2^+$, we have $\alpha a + \beta x(\frac{b+1}{y}) < 1$ which is equivalent to $-\frac{y}{b+1} < -\beta x$. Thus $v_5 B = (0, -\beta x) \in \mathrm{int}(\mathrm{Conv}(T_1))$. Therefore, $T_1 \in \mathcal{T}_+(B)$.

(b) Let $v_1 = (1, 0), v_2 = (-a, x), v_3 = (-b, -y)$ and H_1, H_2, H_3 be lines passing through $T_0 - \{v_1\}, T_0 - \{v_1\}, T_0 - \{v_2\}$ respectively. Then H_1, H_2, H_3 can be described by the equations

$$\begin{aligned}&H_1 : -\Big(\frac{x+y}{ay+bx}\Big)X - \Big(\frac{a-b}{ay+bx}\Big)Y = 1\\ &H_2 : X - \Big(\frac{b+1}{y}\Big)Y = 1\\ &H_3 : X + \Big(\frac{a+1}{x}\Big)Y = 1.\end{aligned}$$

Since $v_2 B^2 \in \mathrm{int}(\mathrm{Conv}(T)) \subseteq H_3^+$, we must have $-\alpha^2 a + (\frac{a+1}{x})q^2 x < 1$ which is equivalent to

$$a < \frac{1-\beta^2}{\beta^2 - \alpha^2}.$$

(c) Let $v_1 = (1, 0), v_2 = (-a, x), v_3 = c(-\alpha a, -\beta x)$ and H_1, H_2, H_3 be lines passing through $T_0 - \{v_1\}, T_0 - \{v_1\}, T_0 - \{v_2\}$ respectively. Then H_1, H_2, H_3 can be described by the equations

$$\begin{aligned}
&H_1 : -\Big(\frac{\beta c+1}{(\alpha+\beta)ac}\Big)X - \Big(\frac{\alpha c-1}{(\alpha+\beta)cx}\Big)Y = 1\\
&H_2 : X - \Big(\frac{\alpha ac+1}{\beta cx}\Big)Y = 1\\
&H_3 : X + \Big(\frac{1+a}{x}\Big)Y = 1.
\end{aligned}$$

Since v_1 is an eigenvector of B corresponding to a positive eigenvalue $\alpha < 1, v_1B \in \operatorname{int}(\operatorname{Conv}(T))$. Since $c > 1, v_2B$ lies on the line segment between v_3 and the origin. Thus $v_2B \in \operatorname{int}(\operatorname{Conv}(T))$. Note that $v_3B = c(-\alpha^2 a, \beta^2 x)$. To show that $v_3B \in \operatorname{int}(\operatorname{Conv}(T))$, it suffice to check that $v_3B \in H_1^+ \cap H_2^+ \cap H_3^+$. Note that

$$\begin{aligned}
\Big(\frac{\beta c+1}{(\alpha+\beta)ac}\Big)(\alpha^2 ac) + \Big(\frac{\alpha c-1}{(\alpha+\beta)cx}\Big)(\beta^2 cx) &= \frac{\alpha^2(\beta c+1)+\beta^2(\alpha c-1)}{\alpha+\beta}\\
&= \frac{\alpha\beta c(\alpha+\beta)+(\alpha-\beta)(\alpha+\beta)}{\alpha+\beta}\\
&= \alpha\beta c+\alpha-\beta\\
&< 1, \text{ since } c < \frac{1}{\beta^2+(\beta^2-\alpha^2)a}.
\end{aligned}$$

Thus $v_3B \in H_1^+$. We also have that

$$\begin{aligned}
-\alpha^2 ac - \Big(\frac{1+\alpha ac}{\beta cx}\Big)(\beta^2 cx) &< 0\\
&< 1,\\
-\alpha^2 ac + \Big(\frac{1+a}{x}\Big)(\beta^2 cx) &= -\alpha^2 ac + (1+a)\beta^2 c\\
&= (\beta^2+(\beta^2-\alpha^2)a)c\\
&< 1 \text{ by assumption.}
\end{aligned}$$

Hence $v_3B \in H_2^+ \cap H_3^+$. This completes the proof for 3).

(d) Let $v_1 = (0, x), v_2 = (-a, -y), v_3 = (b, -y)$ and H_1, H_2, H_3 be lines passing through $T - \{v_1\}, T - \{v_1\}, T - \{v_2\}$ respectively. Then H_1, H_2, H_3 can be described by the equations

$$
\begin{aligned}
&H_1 : -\frac{1}{y} Y = 1 \\
&H_2 : \Big(\frac{x+y}{bx}\Big) X + \frac{1}{x} Y = 1 \\
&H_3 : -\Big(\frac{x+y}{ax}\Big) X + \frac{1}{x} Y = 1.
\end{aligned}
$$

The line H_1 intersects the y-axis at $(0, -y)$. Note that $v_1 B = (0, -\beta x)$. Since $-\beta x > -y, (0, -\beta x)$ is on the line segment between $(0, -y)$ and the origin. Hence $v_1 B \in \operatorname{int}(\operatorname{Conv}(T))$. Note that $v_2 B = (-\alpha a, \beta y)$ is in the second quadrant. Thus it suffices to check that $v_2 B \in H_3^+$. We have

$$
\begin{aligned}
-\Big(\frac{x+y}{ax}\Big)(-\alpha a) + \frac{1}{x}(\beta y) &= \alpha\Big(1 + \frac{y}{x}\Big) + \beta\Big(\frac{y}{x}\Big) \\
&= \alpha + (\alpha + \beta)\frac{y}{x} \\
&< \alpha + 1 - \alpha \\
&= 1.
\end{aligned}
$$

Thus $v_2 B \in H_3^+$. For v_3, we have $v_3 B = (\alpha b, \beta y)$ which is in the first quadrant. Thus it suffices to check that $v_3 B \in H_2^+$. We have

$$
\begin{aligned}
\Big(\frac{x+y}{bx}\Big)(\alpha b) + \frac{1}{x}(\beta y) &= \alpha\Big(1 + \frac{y}{x}\Big) + \beta\Big(\frac{y}{x}\Big) \\
&= \alpha + (\alpha + \beta)\frac{y}{x} \\
&< \alpha + 1 - \alpha \\
&= 1.
\end{aligned}
$$

Hence $v_3 B \in H_2^+$. Therefore, $T \in \mathcal{T}_+(B)$. $\square$

Theorem 1.7.6. Let $B = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$ where $0 < \alpha < \beta < 1$.

(a) If $\alpha + \beta < 1$ then $\mathcal{T}_+(B)$ is path connected.

(b) If $\alpha + \beta \geq 1$ then $\mathcal{T}_+(B)$ has exactly 2 connected components.

Proof. By using the same idea as in Theorem 1.7.4, any positive tetrahedron of B is in the same connected component as an invariant tetrahedron whose one vertex is on the x-axis and by scaling we can assume that the vertex is either $(1,0)$ or $(-1,0)$. Without loss of generality, we assume that the vertex is $(1,0)$. Let

$$T_0 = \{(1,0),(a,x),(b,-y)\} \text{ where } x,y > 0.$$

If $a > 0$ then $b < 0$ and $T_1 = \{(1,0),(0,x),(b,-y)\} \in \mathcal{T}_+(B)$ and is in the same connected component as T_0. If $b > 0$ then $a < 0$ and $T_1' = \{(1,0),(a,x),(0,-y)\} \in \mathcal{T}_+(B)$ and is in the same connected component as T_0. By using perturbation, T_1 (or T_1') can be moved to T_2 of the form

$$T_2 = \{(1,0),(-a,x),(-b,-y)\} \text{ where } a,b,x,y > 0.$$

T_2 is still in the same connected component as T_0. Next, we show that if

$$T_3 = \{(1,0),(-c,z),(-d,-w)\} \in \mathcal{T}_+(B)$$

then it is in the same connected component as T_2. From Lemma 1.7.5, T_2 and T_3 are in the same connected component as

$$T_4 = \{(1,0),(-a,x),c_1(-\alpha a,-\beta x)\} \text{ where } c_1 = \frac{1}{2}\Big(1 + \frac{1}{\beta^2 + (\beta^2 - \alpha^2)a}\Big)$$

$$T_5 = \{(1,0),(-c,z),c_2(-\alpha c,-\beta z)\} \text{ where } c_2 = \frac{1}{2}\Big(1 + \frac{1}{\beta^2 + (\beta^2 - \alpha^2)c}\Big),$$

respectively. Let $x_t = (1-t)a + tc, y_t = (1-t)x + tz$, and $c_t = \frac{1}{2}\left(1 + \frac{1}{\beta^2 + (\beta^2 - \alpha^2)x_t}\right)$ for $t \in [0,1]$. Then $T_{4+t} = \{(1,0), (-x_t, y_t), c_t(-\alpha x_t, -\beta y_t)\}$ is a path in $\mathcal{T}_+(B)$ connecting T_4 and T_5. This shows that all positive invariant tetrahedra of B which have $(1,0)$ in their vertex are in the same connected component. Similarly, all positive invariant tetrahedra of B having $(-1,0)$ in their vertex are in the same connected component.

(a) If $\alpha + \beta < 1$ then $T_6 = \{(0,1), (-1, \alpha - 1), (1, \alpha - 1)\} \in \mathcal{T}_+(B)$. The line segment connecting $(-1, \alpha - 1)$ and $(1, \alpha - 1)$ can be perturbed to have both positive and negative slopes. Thus T_8 connects all invariant tetrahedra of B. Therefore, $\mathcal{T}_+(B)$ is path connected.

(b) If $\alpha + \beta \geq 1$ then an invariant tetrahedron whose one vertex is $(1,0)$ cannot be in the same connected component as its reflection about the y axis. Thus $\mathcal{T}_+(B)$ has 2 connected components. □

Theorem 1.7.7. Let $B = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}, 0 \leq \lambda < \frac{1}{2}$. Then $\mathcal{T}_+(B)$ is path connected.

Proof. This is a special case of Theorem 1.10.5. □

Theorem 1.7.8. Let $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \lambda \geq 0$. Then $\mathcal{T}_+(B)$ is path connected.

Proof. This is a special case of Theorem 1.10.7. □

1.8 Positive Stochastic Matrices Strong Shift Equivalent to Positive Doubly Stochastic Matrices

Throughout this section, we assume that $\mathbb{S}$ is a subring of $\mathbb{R}$ containing $\mathbb{Q}$. We will prove that every positive stochastic matrix over $\mathbb{S}$ is strong shift equivalent over $\mathbb{S}_+$ to a positive doubly stochastic matrix. As a corollary, the nonzero spectra of primitive doubly stochastic matrices of positive trace are the same as the nonzero spectra of primitive stochastic matrices of positive trace.

Lemma 1.8.1. Every positive stochastic matrix over $\mathbb{S}$ is similar and strong shift equivalent over to a positive stochastic matrix over $\mathbb{S}_+$ whose left Perron eigenvector is rational.

Proof. Let P be an $n \times n$ positive stochastic matrix with the left Perron eigenvector $l = (l_1, ..., l_n)$. For each k, let $r_k \in \mathbb{Q}_+^{n-1}$ be such that $r_{kj} \leq l_j$ for all $j = 1, ..., n-1$ and $\lim_{n \to \infty} r_k = (l_1, ..., l_{n-1})$. Define

$$M_k = \begin{pmatrix} \frac{l_1}{r_{k1}} & 0 & \cdots & 0 & 1 - \frac{l_1}{r_{k1}} \\ 0 & \frac{l_2}{r_{k2}} & \cdots & 0 & 1 - \frac{l_2}{r_{k2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{l_{n-1}}{r_{k,n-1}} & 1 - \frac{l_{n-1}}{r_{k,n-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and $P_k = M_k P M_k^{-1}$. Note that $M_k \to I_n$ and $P_k \to P$ as $k \to \infty$ and

$$M_k^{-1} = \begin{pmatrix} \frac{r_{k1}}{l_1} & 0 & \cdots & 0 & 1 - \frac{r_{k1}}{l_1} \\ 0 & \frac{r_{k2}}{l_2} & \cdots & 0 & 1 - \frac{r_{k2}}{l_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{r_{k,n-1}}{l_{n-1}} & 1 - \frac{r_{k,n-1}}{l_{n-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \geq 0$$

for all $k \in \mathbb{N}$. Choose N such that $M_N P > 0$. Let $\hat{l} = (\hat{l_1}, ..., \hat{l_n})$ where $\hat{l_j} = r_{Nj}$ for $j = 1, ..., n-1$ and $\hat{l_n} = 1 - r_{N1} - \cdots - r_{N,n-1}$. Then $\hat{l}$ is rational, $\hat{l} M_N = l$, and

$$\begin{aligned} \hat{l} P_N &= \hat{l} M_N P M_N^{-1} \\ &= l P M_N^{-1} \\ &= l M_N^{-1} \\ &= \hat{l}. \end{aligned}$$

Thus $\hat{l}$ is the left Perron eigenvector of P_N. Since $M_N P > 0$, P is similar and strong shift equivalent over to P_N. $\square$

Theorem 1.8.2. Every positive stochastic matrix over $\mathbb{S}$ is strong shift equivalent over $\mathbb{S}_+$ to a positive doubly stochastic matrix over $\mathbb{S}$.

Proof. Let P be an $n \times n$ stochastic matrix over $\mathbb{S}$. By Lemma 1.8.1, we can assume that P has rational left Perron eigenvector $l = (\frac{r_1}{s_1}, .., \frac{r_n}{s_n})$ where $r_i, s_i \in \mathbb{N}$ for all $i = 1, ..., n$. Let $M = \text{lcm}(s_1, ..., s_n)$. Then l can be written as $l = (\frac{m_1}{M}, ..., \frac{m_n}{M})$ where $m_i = \frac{r_i M}{s_i} \in \mathbb{N}$ for all $i = 1, ..., n$. If $m_1 \neq 1$, we perform a column splitting on the first column of P as follows:

$$P^{(1)} = \begin{pmatrix} \frac{1}{m_1}p_{11} & (1-\frac{1}{m_1})p_{11} & p_{12} & \cdots & p_{1n} \\ \frac{1}{m_1}p_{11} & (1-\frac{1}{m_1})p_{11} & p_{12} & \cdots & p_{1n} \\ \frac{1}{m_1}p_{21} & (1-\frac{1}{m_1})p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m_1}p_{n1} & (1-\frac{1}{m_1})p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}.$$

The left Perron eigenvector of $P^{(1)}$ is $l^{(1)} = (\frac{1}{M}, \frac{m_1-1}{M}, \frac{m_2}{M}, ..., \frac{m_n}{M})$. If $m_1 - 1 \neq 1$ we perform a column splitting on the second column of $P^{(1)}$ by splitting the second column of $P^{(1)}$ as $\frac{1}{m_1-1}C_2^{(1)}$ and $(1-\frac{1}{m_1-1})C_2^{(1)}$ where $C_2^{(1)}$ is the second column of $P^{(1)}$. Suppose $P^{(2)}$ is the matrix after splitting $P^{(1)}$. Then the left Perron eigenvector of $P^{(2)}$ is $l^{(2)} = (\frac{1}{M}, \frac{1}{M}, \frac{m_1-2}{M}, ..., \frac{m_n}{M})$. Continuing in this manner, we finally get an $M \times M$ matrix $P^{(k)}$ whose the left Perron eigenvector $l^{(k)}$ is $\frac{1}{M}(1, 1, ..., 1)$ for some $k \in \mathbb{N}$. Note that $P^{(i)}$ is stochastic for all $i = 1, .., k$. Therefore, $P^{(k)}$ is doubly stochastic. This completes the proof. $\square$

Corollary 1.8.3. The set of nonzero spectra of positive doubly stochastic matrices over $\mathbb{S}$ and the set of nonzero spectra of primitive stochastic matrices over $\mathbb{S}$ with positive trace coincide.

It is not true in general that every positive stochastic matrix is strong shift equivalent over $\mathbb{S}_+$ to a positive doubly stochastic matrix of the same size, because there are positive stochastic matrices whose nonzero spectra cannot be the nonzero spectra of doubly stochastic matrices of the same size. An example can be found in [J81]. We will reprove it. We first reprove the following result of Johnson [J81].

Proposition 1.8.4. [J81] There is no 3×3 doubly stochastic matrix with the characteristic polynomial $t(t-1)(t+1)$.

Proof. Suppose there is such a matrix

$$A = \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{pmatrix}.$$

Observe that $\det(A) = 0$ and $\mathrm{Tr}(A) = 0$. Since A is nonnegative and $\mathrm{Tr}(A) = 0$, we have $a = d = a+b+c+d-1 = 0$. Thus $b + c = 1$. Then A can be rewritten as

$$A = \begin{pmatrix} 0 & b & c \\ c & 0 & b \\ b & c & 0 \end{pmatrix}.$$

Hence $b^3 + c^3 = \det(A) = 0$. This implies $b = c = 0$ which is a contradiction. $\square$

Next, we define for any $n \in \mathbb{N}$ the matrix

$$A_n = \begin{pmatrix} \frac{1}{n+2} & \frac{n}{n+2} & \frac{1}{n+2} \\ \frac{n}{n+2} & \frac{1}{n+2} & \frac{1}{n+2} \\ \frac{n}{n+2} & \frac{1}{n+2} & \frac{1}{n+2} \end{pmatrix}.$$

Suppose that there is a sequence of 3×3 doubly stochastic matrices $\{B_n\}$ such that B_n and A_n are similar for all $n \in \mathbb{N}$. By compactness, $\{B_n\}$ has a convergent subsequence $\{B_{n_k}\}$. Suppose that $\{B_{n_k}\}$ converges to a matrix B. Then B is doubly stochastic since the set of doubly stochastic matrices is closed. For any $n \in \mathbb{N}$, the characteristic polynomial of A_n is $p_n(t) = t(t-1)(t + \frac{n-1}{n+2})$ which converges to $t(t-1)(t+1)$ as $n \to \infty$. Thus B must have the characteristic polynomial

$t(t-1)(t+1)$ which is a contradiction. So there must be some matrix A_{n_0} which is not similar to a doubly stochastic matrix. Since strong shift equivalence preserves the Jordan form away from zero(and in this case it is the Jordan form), A_{n_0} is not strong shift equivalent over $\mathbb{S}_+$ to a 3×3 doubly stochastic matrix.

1.9 Unbounded Lag of SSE

The purpose of this section is to provide the following example.

Theorem 1.9.1. For $t \in [0,1]$, define

$$P_t = \frac{1}{4} \begin{pmatrix} 3+t & 1-t \\ 1+t & 3-t \end{pmatrix}.$$

For $0 \leq t < 1$, the matrices P_t are positive, similar, and SSE-$\mathbb{R}_+$. However, for any $L > 0$, there exists $t \in (0,1)$ such that there is no SSE-$\mathbb{R}_+$ of lag less than L using matrices with size fewer than L.

Definition 1.9.2. A (not necessarily square) nonnegative matrix P is called *generalized row stochastic* if every row sum of P is 1.

We recall the stochasticization of an irreducible matrix A, $P(A) = \frac{1}{\lambda} D^{-1} A D$ where λ is the Perron eigenvalue of A and D is the diagonal matrix whose vector of diagonal entries is the stochastic right eigenvector of A. We need the following theorem for the proof of Theorem 1.9.1.

Theorem 1.9.3. Let A and B be respectively $m \times m$ and $n \times n$ irreducible matrices over $\mathbb{R}$. If A and B are ESSE-$\mathbb{R}_+$ then $P(A)$ and $P(B)$ are also ESSE-$\mathbb{R}_+$. Moreover, there exist generalized row stochastic matrices R, S such that $P(A) = RS$ and $P(B) = SR$.

Proof. Since A and B are elementary strong shift equivalent over $\mathbb{R}_+$, they have the same Perron eigenvalue λ. Let $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ be such that $Av = \lambda v$ and $Bw = \lambda w$. Let $D = \operatorname{diag}(v_1, ..., v_m)$ and $E = \operatorname{diag}(w_1, .., w_n)$. Then $P(A) = \frac{1}{\lambda} D^{-1} AD$ and $P(B) = \frac{1}{\lambda} E^{-1} BE$. Suppose that $A = XY$ and $B = YX$. Then

$$P(A) = \left(\tfrac{1}{\lambda} D^{-1} XE\right)\left(E^{-1} Y D\right) \text{ and } P(B) = \left(E^{-1} Y D\right)\left(\tfrac{1}{\lambda} D^{-1} XE\right).$$

Thus $P(A)$ and $P(B)$ are elementary strong shift equivalent over $\mathbb{R}_+$. Next, suppose that $P(A) = UV$ and $P(B) = VU$. Let $e_m = (1, ..., 1)^t \in \mathbb{R}^m$ and $e_n = (1, ..., 1)^t \in \mathbb{R}^n$. Since $P(A)U = UP(B)$, we have $P(A)Ue_n = UP(B)e_n = Ue_n$. Thus Ue_n is a right eigenvector of $P(A)$ corresponding to an eigenvalue 1 and hence $Ue_n = \alpha e_m$ for some $\alpha > 0$. Similarly, $Ve_m = VP(A)e_m = P(B)Ve_m$, so $Ve_m = \beta e_n$ for some $\beta > 0$. Let $R = \frac{1}{\alpha} U$ and $S = \frac{1}{\beta} V$. Then $Re_n = \frac{1}{\alpha} Ue_n = e_m$ and $Se_m = \frac{1}{\beta} Ve_m = e_n$. Thus R, S are generalized row stochastic matrices. Furthermore, we have $P(A) = UV = (\alpha\beta)RS$ and $P(B) = VU = (\alpha\beta)SR$. Note that

$$\begin{aligned}
m &= e_m^t P(A) e_m \\
&= \alpha\beta e_m^t RS e_m \\
&= \alpha\beta e_m^t R e_n \\
&= \alpha\beta e_m^t e_m \\
&= m\alpha\beta.
\end{aligned}$$

Thus $\alpha\beta = 1$ and hence $P(A) = RS$ and $P(B) = SR$. $\square$

Proof of Theorem 1.9.1. The similarity holds because $\mathrm{Tr}(P_t) = 6, \det(P_t) = 8$ for all $0 \le t < 1$. By Theorem 1.2.9, P_t and P_0 are SSE-$\mathbb{R}_+$ for all $0 \le t < 1$. It is well-known that strong shift equivalence preserves irreducibility [LM95, Proposition 7.4.1]. So P_0 and P_1 are not strong shift equivalent over $\mathbb{R}_+$ because P_0 is irreducible whereas P_1 is reducible. Next, suppose that P_0 and P_t are SSE over $\mathbb{R}_+$ via 2×2 matrices with lag $l \le k$ and size $n \le k$ for all $t \in (0, 1)$. Without loss of generality, we assume that the lag $l = k$ for all $t \in (0, 1)$. For each $t \in (0, 1)$ we have a chain of ESSEs over $\mathbb{R}_+$ $P_0, A_1(t), ..., A_{k-1}(t), P_t$ together with a chain of intermediate matrices $(R_1(t), S_1(t)), ..., (R_k(t), S_k(t))$. Since P_t is positive for $0 \le t < 1$, each A_i has a unique maximal irreducible submatrix, say A_i^0. The given SSE restricts to an SSE of the A_i^0. So, without loss of generality, we assume $A_i^0 = A_i$. By passing through Theorem 1.9.3, we can assume that $A_j(t), R_j(t), S_j(t)$ are generalized row stochastic for all $j = 1, ..., k$ and all $t \in (0, 1)$. Then all matrices are bounded (by 1), so there is a subsequence $t_n \to 1$ such that $A_j(t_n) \to A_j, R_j(t_n) \to R_j, S_j(t_n) \to S_j$ for some A_j, R_j, S_j. But then we get a strong shift equivalence over $\mathbb{R}_+$ between P_0 and P_1 which is a contradiction. $\square$

1.10 Some Cases in Which $\mathcal{T}_+(B)$ Is Connected

In this section, we collect some miscellaneous results which show that the space $\mathcal{T}_+(B)$ is path connected.

Theorem 1.10.1. Let $n \in \mathbb{N}$. $\mathcal{T}_+(B)$ is path connected if B has the following forms

(a) $B = 0_n$.

(b) $B = \lambda I_n$ where $-\frac{1}{n} < \lambda < 1$.

(c) B is nilpotent [KR90].

(d) $B = \lambda I_n + N$ where $0 \leq \lambda < 1$ and N is nilpotent.

(e) $B = (\lambda) \oplus 0_n$ where $-1 < \lambda < 1$.

Theorem 1.10.1 is a combination of the following theorems.

Theorem 1.10.2. For $n > 1, \mathcal{T}_+(0_{n-1})$ is path connected.

Proof. Let $T_0 = \{v_1, ..., v_n\}$ be a positive tetrahedron. Suppose that

$l_1 v_1 + \cdots + l_n v_n = 0$. Let $T_1 = \{v_1, ..., v_{n-1}, u_n\}$ where $u_n = -(v_1 + \cdots + v_{n-1})$. Then T_1 is a positive tetrahedron.

Define $T_t = \{v_1, ..., v_{n-1}, v_n(t)\}$ where $v_n(t) = (1-t)v_n + tu_n$. Then T_t is a positive tetrahedron for all $t \in [0,1]$ because $l_1(t)v_1 + \cdots + l_{n-1}(t)v_n(t) = 0$ where $l_j(t) = \frac{(1-t)l_j + tl_n}{1-t+nl_nt}$ for $j = 1, ..., n-1$ and $l_n(t) = \frac{l_n}{1-t+nl_nt}$. This proves that every positive tetrahedron is in the same connected component as a positive tetrahedron which has zero vertex sum.

Next, suppose that $T_0 = \{v_1, ..., v_n\}$ and $T_1 = \{w_1, ..., w_n\}$ where $v_1 + \cdots + v_n = 0$ and $w_1 + \cdots + w_n = 0$. Define a linear transformation $L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$L(v_i) = w_i \text{ for all } i = 1, ..., n-1.$$

Let A be the matrix of L with respect to the basis $\{v_1, ..., v_{n-1}\}$. If $\det(A) < 0$, we can define $L(v_1) = w_2$ and $L(v_2) = w_1$ so that $\det(A) > 0$. Thus there is a path A_t in $GL_{n-1}(\mathbb{R})$ such that $A_0 = I_{n-1}$ and $A_1 = A$. Then the path $T_t = \{v_1 A_t, ..., v_n A_t\}$ is a path of positive tetrahedra connecting T_0 and T_1. This completes the proof. $\square$

There are other matrices whose the space of invariant tetrahedra coincide with $\mathcal{T}_+(0_{n-1})$. The following results give all such possible matrices.

Lemma 1.10.3. Let $T = \{v_1, ..., v_n\} \in \mathcal{T}_+(B)$ with $l_1 v_1 + \cdots + l_n v_n = 0$. Then the following statements are equivalent:

(a) $T_0(d) = \{v_1, ..., v_{n-1}, dv_n\} \in \mathcal{T}_+(B)$ for all $d \geq 1$

(b) v_n is an eigenvector of B corresponding to an eigenvalue $\lambda \geq 0$.

Proof. (b) $\Rightarrow$ (a) This direction is obvious.

(a) $\Rightarrow$ (b) Suppose that $T_0(d) = \{v_1, ..., v_{n-1}, dv_n\} \in \mathcal{T}_+(B)$ for all $d \geq 1$. Let $Q(d) = (q_{ij}(d)$ be the corresponding positive stochastic matrix of $T_0(d)$. Applying Theorem 1.3.7 (b) with $c_1 = c_2 = \cdots = c_{n-1} = 1$ and $c_n = \frac{1}{d}$, we get

$$\begin{aligned} q_{nj}(d) &= \frac{c_j}{c_n}\Big[p_{nj} + \frac{l_j\{c_n - (c_1 p_{n1} + \cdots + c_n p_{nn})\}}{c_1 l_1 + \cdots + c_n l_n}\Big] \\ &= dc_j\Big[p_{nj} + \frac{l_j\{\frac{1}{d} - (1 - p_{nn} + \frac{p_{nn}}{d})\}}{1 - l_n + \frac{l_n}{d}}\Big] \\ &= dc_j\Big[p_{nj} + \frac{(1-d)l_j(1-p_{nn})}{l_n + d(1-l_n)}\Big] \end{aligned}$$

for all $j = 1, 2, ..., n$. For $j = n$ we have

$$\begin{aligned} q_{nn}(d) &= p_{nn} + \frac{(1-d)l_n(1-p_{nn})}{l_n + d(1-l_n)} \\ &= \frac{l_n + d(p_{nn} - l_n)}{l_n + d(1-l_n)}. \end{aligned}$$

For $j \in \{1, 2, ..., n-1\}$ we have

$$\begin{aligned} q_{nj}(d) &= d\Big[p_{nj} + \frac{(1-d)l_j(1-p_{nn})}{l_n + d(1-l_n)}\Big] \\ &= d\Big[\frac{l_n p_{nj} + l_j(1-p_{nn}) + d\{p_{nj}(1-l_n) - l_j(1-p_{nn})\}}{l_n + d(1-l_n)}\Big]. \end{aligned}$$

Letting $d \to \infty$ we have $q_{nn}(d) \to \frac{p_{nn}-l_n}{1-l_n}$. Thus $p_{nn} \geq l_n$. If $p_{nj}(1-l_n) \neq l_j(1-p_{nn})$ for some $j \in \{1, 2, ..., n-1\}$ then $q_{nj}(d) \to \pm\infty$ as $d \to \infty$ which is a contradiction. Thus $p_{nj} = \frac{l_j(1-p_{nn})}{1-l_n}$. Moreover, we have

$$\begin{aligned} q_{nj}(d) &= \frac{d[p_{nj}(1-l_n) + l_n p_{nj}]}{l_n + d(1-l_n)} \\ &= \frac{dp_{nj}}{l_n + d(1-l_n)}. \end{aligned}$$

Let $\lambda = \frac{p_{nn}-l_n}{1-l_n} \geq 0$. Then

$$\begin{aligned} v_n B &= p_{n1}v_1 + \cdots + p_{n,n-1}v_{n-1} + p_{nn}v_n \\ &= \left(\frac{1-p_{nn}}{1-l_n}\right)(l_1 v_1 + \cdots l_{n-1}v_{n-1}) + p_{nn}v_n \\ &= \left(\frac{1-p_{nn}}{1-l_n}\right)(-l_n v_n) + p_{nn}v_n \\ &= \left(\frac{p_{nn}-l_n}{1-l_n}\right)v_n \\ &= \lambda v_n. \end{aligned}$$

Thus v_n is an eigenvalue of B corresponding to the eigenvalue $\lambda \geq 0$. $\square$

Theorem 1.10.4. $\mathcal{T}_+(B) = \mathcal{T}_+(0_{n-1})$ if and only if $B = \lambda I_{n-1}$ for some $\lambda \geq 0$.

Proof. Suppose that $B = \lambda I_{n-1}$ for some $0 \leq \lambda < 1$. Let $T = \{v_1, ..., v_n\}$ be a positive tetrahedron. Then $v_i B = \lambda v_i$ is in the interior of T because it is on the line between the origin and v_i. Thus $T \in \mathcal{T}_+(B)$ and hence $\mathcal{T}_+(B) = \mathcal{T}_+(0)$.

Suppose that $\mathcal{T}_+(B) = \mathcal{T}_+(0)$. Let v_1 be a nonzero vector in $\mathbb{R}^{n-1}$. Choose a basis of $\mathbb{R}^{n-1}$ which has v_1 as a basis element, say $\{v_1, ..., v_{n-1}\}$. Let

$$v_n = -(v_1 + \cdots + v_{n-1}).$$

Then $T = \{v_1, ..., v_n\}$ is a positive tetrahedron and hence $T(d) = \{dv_1, v_2, ..., v_n\}$ are also positive tetrahedra for all $d \geq 1$ by Theorem 1.3.7 (b). From assumption $T(d) \in \mathcal{T}_+(B)$ for all $d \geq 1$. Lemma 1.10.3 implies that v_1 is an eigenvector of B corresponding to a nonnegative eigenvalue. This shows that every nonzero vector of $\mathbb{R}^{n-1}$ is an eigenvector of B corresponding to some nonnegative eigenvalue. Therefore $B = \lambda I_{n-1}$ for some $0 \leq \lambda < 1$. □

Theorem 1.10.5. Let $B = -\lambda I_{n-1}, 0 \leq \lambda < \frac{1}{n-1}$. Then $\mathcal{T}_+(B)$ is path connected.

Proof. Let $T_0 = \{v_1, ..., v_n\} \in \mathcal{T}_+(B)$ with $l_1 v_1 + \cdots + l_n v_n = 0$. Then T_0 corresponds to the matrix $P = (1+\lambda)L - \lambda I_n$ where L is a positive stochastic matrix having every row equals $l = (l_1, ..., l_n)$. Thus $l_i \in \left(\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda}\right)$ for all $i = 1, ..., n$. Then the set of vectors $l \in \operatorname{int}(\Delta^{n-1})$ such that $T = \{v_1, ..., v_n\} \in \mathcal{T}_+(B)$ and $l_1 v_1 + \cdots + l_n v_n = 0$ is convex. In particular, we have $\frac{1}{n} \in \left(\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda}\right)$. Thus T_0 and $T_1 = \{v_1, ..., v_{n-1}, -(v_1 + \cdots + v_{n-1})\}$ are in the same connected component. Next, suppose that $T_2 = \{u_1, ..., u_n\}, T_3 = \{w_1, ..., w_n\} \in \mathcal{T}_+(B)$ where $u_1 + \cdots + u_n = 0$ and $w_1 + \cdots + w_n = 0$. Let A_1 be an invertible matrix with positive determinant such that $u_i A_1 = w_i$ for all $i = 1, ..., n$. Then T_2 and T_3 are in the same connected component via the path $T_{2+t} = \{u_1 A_t, ..., u_n A_t\}$ where A_t is a path of invertible matrices connecting $A_0 = I_{n-1}$ and A_1. Thus $\mathcal{T}_+(B)$ is path connected. □

Theorem 1.10.6. [KR90]

If B is an $(n-1) \times (n-1)$ nilpotent matrix then $\mathcal{T}_+(B)$ is path connected.

Proof. The original proof of this theorem can be found in [KR90]. In this thesis, we give another proof. We can assume without loss of generality that B is of the form

$$\begin{pmatrix} 0 & \epsilon_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \epsilon_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \epsilon_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\epsilon_i \in \{0,1\}$ for all $i = 1,2,...,n-2$. Let $T_0, T_1 \in \mathcal{T}_+(B)$. Let T_t be a path of positive tetrahedra connecting T_0 and T_1. Then there is a path P_t of quasi-stochastic matrices corresponding to the path T_t. For $\theta \in (0,1]$, define $D(\theta) = \mathrm{diag}(\theta^{-1}, \theta^{-2}, ..., \theta^{1-n})$. Then $D(\theta)B = \theta B D(\theta)$ for all $\theta \in (0,1]$. Let $l_t = (l_1(t), ..., l_n(t)) \in \mathrm{int}(\Delta^{n-1})$ be such that $l_1(t)v_1(t) + \cdots + l_n(t)v_n(t) = 0$ and L_t be the matrix whose every row equals l_t. Then observe that $L_t T_t = 0$. Thus

$$\begin{aligned} T_t D(\theta) B &= \theta T_t B D(\theta) \\ &= \theta P_t T_t D(\theta) \\ &= [(1-\theta)L_t + \theta P_t] T_t D(\theta) \\ &= [L_t + \theta(P_t - L_t)] T_t D(\theta). \end{aligned}$$

By compactness, we choose $\theta_0 > 0$ such that $L_t + \theta_0(P_t - L_t) > 0$ for all $t \in [0,1]$. Then $T_t D(\theta_0) \in \mathcal{T}_+(B)$ for all $t \in [0,1]$ and, consequently, $T_0 D(\theta_0)$ and $T_1 D(\theta_0)$ are in the same connected component. To finish the proof, we show that T_i and $T_i D(\theta_0)$ are in the same connected component for $i = 0,1$. Fix $i \in \{0,1\}$. For $t \in [0,1]$, define $T_i(t) = T_i D(\theta_0^t)$. Then

$$
\begin{aligned}
T_i(t)B &= T_i D(\theta_0^t)B \\
&= \theta_0^t T_i B D(\theta_0^t) \\
&= \theta_0^t P_i T_i D(\theta_0^t) \\
&= \theta_0^t P_i T_i(t) \\
&= [(1-\theta_0^t)L_i + \theta_0^t P_i]T_i(t).
\end{aligned}
$$

One can easily check that $(1-\theta_0^t)L_i + \theta_0^t P_i > 0$ for all $t \in [0,1]$. Thus T_i and $T_i D(\theta_0)$ are in the same connected component. The proof is completed. $\square$

A slight generalization of Theorem 1.10.6 is the following result.

Theorem 1.10.7. Let $B = \lambda I_{n-1} + N$ where $0 \leq \lambda < 1$ and N is the Jordan form of a nilpotent matrix. Then $\mathcal{T}_+(B)$ is path connected.

Proof. Let $\lambda < \alpha < 1$. Then $B = \alpha\left(\frac{\lambda}{\alpha} I_{n-1}\right) + (1-\alpha)\left(\frac{1}{1-\alpha}N\right)$. First, we show that $\mathcal{T}_+((1-\alpha)^{-1}N) \subseteq \mathcal{T}_+(B)$. Suppose that $T \in \mathcal{T}_+((1-\alpha)^{-1}N)$. Then $T(1-\alpha)^{-1}N = QT$ for some positive stochastic matrix Q and $TB = (\alpha P + (1-\alpha)Q)T$ where P is similar to $\frac{\lambda}{\alpha} I_{n-1} \oplus 1$. Observe that P is positive and stochastic since $T \in \mathcal{T}_+(\lambda\alpha^{-1} I_{n-1})$. Thus $T \in \mathcal{T}_+(B)$. This proves the claim. Next, we show that any $T_0 \in \mathcal{T}_+(B)$ is in the same connected component as some $T_1 \in \mathcal{T}_+((1-\alpha)^{-1}N)$. Let $T_0 = \{v_1, ..., v_n\} \in \mathcal{T}_+(B)$. Then

$$T_0 B = (\alpha P + (1-\alpha)Q)T_0$$

where P is positive, stochastic, and similar to $(\lambda\alpha^{-1})I_{n-1} \oplus 1$ and Q is quasi-stochastic and similar to $(1-\alpha)^{-1}N \oplus 1$. Note that Q is not necessarily positive.

For $\theta \in (0,1]$, define $D(\theta) = \operatorname{diag}(\theta^{-1}, \theta^{-2}, ..., \theta^{1-n})$. Then $D(\theta)N = \theta N D(\theta)$ for all $\theta \in (0,1]$. Let $l = (l_1, ..., l_n) \in \operatorname{int}(\Delta^{n-1})$ be such that $l_1 v_1 + \cdots + l_n v_n = 0$ and L be the matrix whose every row equals l. Then $LT_0 = 0$ and

$$\begin{aligned} T_0 D(\theta) B &= \Big[\alpha T_0 \Big(\lambda \alpha^{-1} I_{n-1}\Big) + (1-\alpha)\Big(T_0(\theta(1-\alpha)^{-1}N)\Big)\Big] D(\theta) \\ &= \Big[\alpha P + (1-\alpha)\Big((1-\theta)L + \theta Q\Big)\Big] T_0 D(\theta). \end{aligned}$$

One can see that each entry of $\alpha P + (1-\alpha)[(1-\theta)L + \theta Q]$ is a linear function of θ.

If $\theta = 0$ then $(\alpha P + (1-\alpha)((1-\theta)L + \theta Q)) = \alpha P + (1-\alpha)L > 0$.

If $\theta = 1$ then $(\alpha P + (1-\alpha)((1-\theta)L + \theta Q)) = \alpha P + (1-\alpha)Q > 0$.

Consequently, $\alpha P + (1-\alpha)\Big((1-\theta)L + \theta Q\Big)$ is positive for all $\theta \in [0,1]$. Thus $T_0 D(\theta) \in \mathcal{T}_+(B)$ for all $\theta \in (0,1]$. Choose θ_0 sufficiently small so that $(1-\theta_0)L + \theta_0 Q > 0$ and let $T_1 = T_0 D(\theta_0)$. Then $T_1 \in \mathcal{T}_+((1-\alpha)^{-1}N)$ since $T_1((1-\alpha)^{-1}N) = [(1-\theta_0)L + \theta_0 Q]T_1$.

Furthermore, T_0 and T_1 are in the same connected component of $\mathcal{T}_+(B)$ via the path

$$T_t = T_0 D(\theta_0^t), t \in [0,1].$$

Since $\mathcal{T}_+((1-\alpha)^{-1}N)$ is path connected, we complete the proof. $\square$

Let $\lambda \in (-1,1)$ and define $B_n = (\lambda) \oplus 0_{n-1}$.

Theorem 1.10.8. $\mathcal{T}_+(B_{n-1})$ is path connected for all $n \in \mathbb{N}$.

Proof. We can assume without loss of generality that $0 \leq \lambda < 1$. Let $T_0 = \{v_1, ..., v_n\}$ be a positive invariant tetrahedron of B_{n-1}. Suppose without loss of generality that $\min_{1\le i\le n} v_{i1} = v_{11}$ and $\max_{1\le i\le n} v_{i1} = v_{n1}$. We divide the proof into 5 steps.

Step 1: Since $T_0 \in \mathcal{T}_+(B_{n-1})$, we can extend v_1 along the line joining v_1 and v_n to $\hat{v}_1$ so that $\hat{v}_{11} < v_{11}$ and $\hat{v}_1 B \subseteq \mathrm{int}(\mathrm{Conv}(T_0))$. Then $T_1 = \{\hat{v}_1, v_2, ..., v_n\} \in \mathcal{T}_+(B_{n-1})$. By Theorem 1.5.2 (b) T_1 is in the same connected component as T_0.

Step 2: We extend v_i along the line joining $\hat{v}_1$ and v_i to the point $\hat{v}_i$ which has $\hat{v}_{i1} = v_{n1}$ for any $i = 2, ..., n-1$. For convenience, we also define $\hat{v}_n = v_n$. Let $T_2 = \{\hat{v}_1, \hat{v}_2, ..., \hat{v}_n\}$. Then $T_2 \in \mathcal{T}_+(B_{n-1})$ since $T_2 B_{n-1} = T_1 B_{n-1}$ and $\mathrm{Conv}(T_1) \subseteq \mathrm{Conv}(T_2)$. T_2 is in the same connected component as T_1 by Theorem 1.5.2 (c).

Step 3: Define $v_b = \frac{1}{n}\sum_{i=2}^{n} \hat{v}_i$ and set $w_i = v_b + a(\hat{v}_i - v_b)$ for $i = 2, ..., n$ where $a \ge 1$ is large enough so that $(v_{n1}, 0, 0, ..., 0)$ is in the interior of the convex hull of $w_2, ..., w_n$. Let $T_3 = \{\hat{v}_1, w_2, ..., w_n\}$ and define $T_{2+t} = \{v_1(t), ..., v_n(t)\}$ for $t \in [0,1]$ where

$$v_1(t) = \hat{v}_1 \text{ and } v_i(t) = \hat{v}_i + t(w_i - \hat{v}_i) \text{ for } i = 2, ..., n.$$

Observe that

$$\begin{aligned}
\frac{v_1(t) + \cdots + v_n(t)}{n} &= \frac{1}{n}\sum_{i=1}^{n}(\hat{v}_i + t(w_i - \hat{v}_i)) \\
&= \frac{t}{n}\sum_{i=1}^{n} w_i + \frac{(1-t)}{n}\sum_{i=1}^{n}\hat{v}_i \\
&= \frac{t}{n}\sum_{i=1}^{n}(a\hat{v}_i + (1-a)v_b) + \frac{(1-t)}{n}\sum_{i=1}^{n}\hat{v}_i \\
&= [atv_b + (1-a)tv_b] + (1-t)v_b \\
&= v_b.
\end{aligned}$$

Then we have

$$\begin{aligned}
\hat{v}_i &= \frac{1}{1+t(a-1)}v_i(t) + \frac{t(a-1)}{1+t(a-1)}v_b \\
&= \frac{1}{1+t(a-1)}v_i(t) + \frac{t(a-1)}{1+t(a-1)}\left[\frac{v_1(t)+\cdots+v_n(t)}{n}\right] \quad \text{for all } i = 1, ..., n.
\end{aligned}$$

Thus $\mathrm{Conv}(T_2) \subseteq \mathrm{Conv}(T_2(t))$ for all $t \in [0,1]$. Hence $T_2(t)$ is a positive tetrahedron for any $t \in [0,1]$. It is easy to see that

$$\begin{aligned}
T_2(t)B_{n-1} &= [\lambda\hat{v}_{11}, \lambda v_{n1}] \times \{0\} \times \cdots \times \{0\} \\
&\subseteq \mathrm{int}(\mathrm{Conv}(T_2)) \\
&\subseteq \mathrm{int}(\mathrm{Conv}(T_2(t))) \text{ for all } t \in [0,1].
\end{aligned}$$

Then $T_{2+t} \in \mathcal{T}_+(B_{n-1})$ for all $t \in [0,1]$. This shows that T_2 and T_3 are in the same connected component.

Step 4: Let w_1 be the point in $\mathrm{Conv}(T_3) \cap \{(x, 0, ..., 0) : x < 0\}$ which has maximum norm. Then w_1 is on the boundary of $\mathrm{Conv}(T_3)$ and $\hat{v}_{11} \leq w_{11} < \lambda\hat{v}_{11} \leq 0$. Let $T_4 = \{w_1, ..., w_n\}$. Note that $\{w_2, ..., w_n\}$ is a basis of $\mathbb{R}^{n-1}$. The origin is in the interior of T_4 because it is in $T_3B = T_4B$. Thus T_4 is a positive tetrahedron. We also have

$$\begin{aligned}
T_4B_{n-1} &= [\lambda w_{11}, \lambda v_{n1}] \times \{0\} \times \cdots \times \{0\} \\
&\subseteq (w_{11}, v_{n1}) \times \{0\} \times \cdots \times \{0\} \\
&\subseteq \mathrm{int}(\mathrm{Conv}(T_4)).
\end{aligned}$$

Thus $T_4 \in \mathcal{T}_+(B_{n-1})$. Theorem 1.5.2 (b) guarantees that T_4 is still in the same connected component as T_3.

Step 5: Let $u_i = \frac{1}{v_{n1}} w_i$ for any $i = 1, ..., n$. Define $T_5 = \{u_1, ..., u_n\}$. T_5 is in the same connected component as T_4 by Theorem 1.5.2 (a). Let $x_i = (u_{i2}, ..., u_{in})$ for $i = 2, ..., n$ and define $T_5' = \{x_2, ..., x_n\}$. Let y_i be the $(i-1)$th standard basis element of $\mathbb{R}^{n-2}$ for $i = 2, ..., n-1$ and $y_n = -(y_2 + \cdots + y_{n-1})$. Define $T_6' = \{y_2, ..., y_n\}$. Then T_5' and T_6' are positive tetrahedra. By Theorem 1.10.2 there is a path $T_{5+t}' = \{x_2(t), ..., x_n(t)\}$ connecting T_5' and T_6'. The path

$$T_{5+t} = \{u_1, u_2(t), ..., u_n(t)\}$$

where $u_{i1}(t) = 1$ and $u_{ij}(t) = x_{ij}(t)$ for all $i, j = 2, ..., n$ is the path in $\mathcal{T}_+(B_{n-1})$ connecting T_5 and $T_6 = \{u_1, z_2, ..., z_n\}$ where $z_{i1} = 1$ and $z_{ij} = y_{ij}$ for all $i, j = 2, ..., n$. Therefore, $\mathcal{T}_+(B_{n-1})$ is path connected, as required. $\square$

Chapter 2

The Mapping Class Group of a Shift of Finite Type

2.1 Introduction

One of the interesting problems in symbolic dynamics is the classification of SFTs up to flow equivalence. Given any discrete dynamical system we can also construct a corresponding continuous-time dynamical system by using suspension flows defined on the mapping torus of the original discrete system. Given a dynamical system (X, T) where X is a compact metric space and $T : X \to X$ is a homeomorphism, we define the *mapping torus* Y_T of (X, T) as

$$Y_T = \{(x, t) : x \in X, t \in \mathbb{R}\}/\sim$$

where $(x, 1) \sim (T(x), 0)$. Distinct equivalence classes may be uniquely represented by $\{[x, t] : x \in X, 0 \leq t < 1\}$. For any $s \in \mathbb{R}$, the *suspension flow* α on Y_T is defined by $\alpha_s([x, t]) = [x, s + t]$ for any $[x, t] \in Y_T$. Two discrete dynamical systems are *flow equivalent* if the corresponding suspensions are conjugate as flows. Any conjugacy of discrete dynamical systems induces a flow equivalence of the corresponding suspension flows, but flow equivalence is a much weaker equivalence relation in general. For shifts of finite type, Parry and Sullivan [PS75] showed that flow equivalence of SFTs is generated by conjugacy, state stretching, and state contracting. For an SFT (X_A, σ_A), we define a *state stretching* as follows: Pick any symbol a in the alphabet of X_A and then replace a by a word a_1a_2 where a_1, a_2 are new symbols. The inverse of a state stretching is called a *state contracting*. Using a matrix interpretation of state stretching, they also showed that $\det(I-A)$ is an invariant of flow equivalence. Bowen and Franks [BowF77] then showed that the Bowen-Franks group $\operatorname{cok}(I-A) = \mathbb{Z}^n/(I-A)\mathbb{Z}^n$, if A is $n \times n$, is also an invariant of a flow equivalence. Then Franks [F84] completely solved the flow equivalence problem for nontrivial ISFTs by showing these two invariants are complete. Huang has completely characterized reducible SFTs up to flow equivalence [Huang94, Huang95, Bo02b, BoHuang03]. Boyle [Bo02b] also gave an alternative approach via positive equivalence.

Two discrete dynamical systems (X, T) and (X', T') are *flow equivalent* if there is a homeomorphism $F : Y_T \to Y_{T'}$ whose restriction to any orbit is an orientation preserving homeomorphism onto some orbit of the range flow. F is called a *flow equivalence*. Two flow equivalences $F_0, F_1 : Y_T \to Y_{T'}$ are *isotopic* if there is a path ϕ_t in the space of flow equivalences $Y_T \to Y_{T'}$ such that $\phi_0 = F_0$ and $\phi_1 = F_1$. The mapping class group $\mathcal{M}_A$ of an ISFT (X_A, σ_A) is the group of flow equivalences on the mapping torus Y_A of (X_A, σ_A) modulo the subgroup of flow equivalences which are isotopic to the identity. It is the analogue of the automorphism group $\operatorname{Aut}(\sigma_A)$, the group of homeomorphisms of (X_A, σ_A) which commute with σ_A. $\mathcal{M}_A$ is even more complicated than $\operatorname{Aut}(\sigma_A)$, although it is still countable. In Section 2.3, we show that $\mathcal{M}_A$ acts n-transitively and faithfully on the set of circles in Y_A for every $n \in \mathbb{N}$, and the center of $\mathcal{M}_A$ is trivial. In Section 2.4, we show that $\mathcal{M}_A$ contains an embedded copy of $\operatorname{Aut}(\sigma_B)/ < \sigma_B >$ for any SFT (X_B, σ_B) flow equivalent to (X_A, σ_A). Also, a flow equivalence $F : Y_A \to Y_A$ has an invariant cross section if and only if F is induced by an automorphism of the first return map to some cross section of Y_A (which is an irreducible SFT flow equivalent to (X_A, σ_A)). However, we will show that not every flow equivalence has an invariant cross section. In Section 2.5, we show that every flow equivalence on Y_A is compatible with every right projection of a positive equivalence to $\mathrm{SL}(\mathbb{Z})$.

Altogether, these results provide supporting evidence for the possibility that the kernel of the Bowen-Franks representation (described below in Section 2.2.3) is simple.

2.2 Definitions and Background

2.2.1 Suspensions, Cross Sections, and Flow Equivalences

Let X be a compact metric space. Let $T : X \to X$ be a homeomorphism and $f : X \to \mathbb{R}$ be continuous and positive. Define the *suspension* $Y_{f,T}$ by

$$Y_{f,T} = \{(x,t) : x \in X, 0 \leq t \leq f(x)\} \diagup \sim$$

where $(x, f(x)) \sim (T(x), 0)$. Distinct equivalence classes may be represented uniquely by $\{[x,t] : x \in X, 0 \leq t < f(x)\}$. For $n \geq 0$, define $f_0 \equiv 0, f_n(x) = \sum_{j=0}^{n-1} f(T^j(x))$, and $f_{-n}(x) = -\sum_{j=1}^{n} f(T^{-j}(x))$ for all $x \in X$. For any $s \in \mathbb{R}$, the *suspension flow* α on $Y_{f,T}$ is defined by $\alpha_s([x,t]) = [T^n(x), s+t-f_n(x)]$ where $n \in \mathbb{Z}$ is such that $f_n(x) \leq s+t < f_{n+1}(x)$. If $f \equiv 1$ on X then $Y_{f,T}$ is called *the mapping torus of* (X,T) and is denoted by Y_T. The suspension flow α on Y_T can be simply defined by $\alpha_t([x,s]) = [x, s+t]$ for any $t \in \mathbb{R}$. $X \times \mathbb{R}$ carries the "vertical"flow, $\widetilde{\alpha}$, for which

$\widetilde{\alpha}_s : (x,t) \mapsto (x, t+s)$. The rule $(x,t) \mapsto [x,t]$ defines a surjective local homeomorphism $\pi_T : X \times \mathbb{R} \to Y_T$ which intertwines the vertical and suspension flows. Two discrete dynamical systems (X,T) and (X',T') are *flow equivalent* if there is a homeomorphism $F : Y_T \to Y_{T'}$ whose restriction to any orbit is an orientation preserving homeomorphism onto some orbit of the range flow. F is called a *flow equivalence*. If $F : Y_T \to Y_{T'}$ is a flow equivalence, then there is a homeomorphism $\widetilde{F}$ such that

$$\begin{array}{ccc} X \times \mathbb{R} & \xrightarrow{\widetilde{F}} & X' \times \mathbb{R} \\ \Big\downarrow{\scriptstyle \pi_T} & & \Big\downarrow{\scriptstyle \pi_{T'}} \\ Y_T & \xrightarrow{F} & Y_{T'} \end{array}$$

commutes. The lift $\widetilde{F}$ is not unique.

A *cross section* C of the suspension flow α on Y_T is a closed set of Y_T such that $\alpha : C \times \mathbb{R} \to Y_T$ is a local homeomorphism onto Y_T. It follows that every orbit hits C in forward time and in backward time, *the first return time* defined by $f_c(x) = \inf\{s > 0 : \alpha_s(x) \in C\}$ is continuous and strictly positive on C, and *the first return map* $T_c : C \to C$ defined by $T_c(x) = \alpha_{f_c(x)}(x)$ is a homeomorphism. Discrete systems (X,T) and (X',T') are flow equivalent if and only if there is a flow Y with two cross sections whose return maps are conjugate respectively to T and T'. Two flow equivalences $F_0, F_1 : Y_T \to Y_{T'}$ are *isotopic* if there is a path ϕ_t in the space of flow equivalences $Y_T \to Y_{T'}$ such that $\phi_0 = F_0$ and $\phi_1 = F_1$.

Let (X_A, σ_A) be a shift of finite type. The mapping torus of (X_A, σ_A) is denoted by Y_A. The *mapping class group* of Y_A, denoted by $\mathcal{M}_A$, is the group of flow equivalences $Y_A \to Y_A$ modulo the subgroup of flow equivalences which are isotopic to the identity. Abusing notation, given a flow equivalence $F : Y_A \to Y_A$, we may still refer to F (rather than its equivalence class $[F]$) as an element of $\mathcal{M}_A$.

2.2.2 The Parry-Sullivan Theorem and Invariants for Flow Equivalence

Definition 2.2.1. Let (X_A, σ_A) be a shift of finite type. We define a *state stretching* as follows: Pick any symbol a in the alphabet of X_A and then replace a by a word a_1a_2 where a_1, a_2 are new symbols. The inverse of a state stretching is called a *state contracting.*

Example 2.2.2. Suppose X is the 2-shift $\{0,1\}^{\mathbb{Z}}$. We replace every 0 with $0_1, 0_2$, e.g.,

$$\cdots 101011001 \cdots \;\Rightarrow\; \cdots 10_10_210_10_2110_10_20_10_21 \cdots$$

We can describe the return map as the subshift obtained from the 2-shift by stretching the symbol 0 to 0_1 and 0_2. This subshift is the golden mean shift.

Theorem 2.2.3. [PS75] Let (X_A, σ_A) be an SFT. Then $F : Y_A \to Y_A$ is a flow equivalence if and only if there exist SFTs $(X_1, T_1), (X_2, T_2)$ which are conjugate to (X_A, σ_A) and T_1 becomes T_2 by a finite sequence of state stretchings and state contractings.

As a consequence of the Parry-Sullivan Theorem, we state the following fact.

Proposition 2.2.4. For any shift of finite type (X_A, σ_A), $\mathcal{M}_A$ is countable.

Proof. By Theorem 2.2.3, a flow equivalence (up to isotopy) can be obtained by a conjugacy followed by a series of state stretchings or state contractings and followed by a conjugacy. There are only countably many ways to obtain each step. Thus $\mathcal{M}_A$ is a subset of the product of 3 countable sets. Therefore, $\mathcal{M}_A$ is countable. $\square$

Given an $n \times n$ integral matrix A, we define the *Bowen-Franks group of A* as $\operatorname{cok}(I - A) = \mathbb{Z}^n/(I - A)\mathbb{Z}^n$. For a shift of finite type (X_A, σ_A), it is known that $\det(I - A)$ [PS75] and $\operatorname{cok}(I - A)$ [BowF77] are invariants of flow equivalence. If (X_A, σ_A) is irreducible and nontrivial, then they are complete invariants.

Example 2.2.5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\operatorname{cok}(I - A) \cong \operatorname{cok}(I - B) \cong 0$ and $\det(I - A) = \det(I - B) = -1$. Thus the full 2-shift and the golden mean shift are flow equivalent. However, they are not shift equivalent.

2.2.3 Positive Equivalences and the Bowen-Franks Representation

Let A and B be irreducible matrices. We embed A and B to the set of *essentially irreducible* infinite matrices over $\mathbb{Z}_+$, those which have only one irreducible component. Mike Boyle [Bo02b], building on [F84] within the "positive K-Theory" approach to symbolic dynamics [Wa00, BoW04, Bo02a], developed a general method to construct flow equivalences $F : Y_A \to Y_B$ given that A and B are flow equivalent. A *basic elementary matrix* E is a matrix in $\mathrm{SL}(\mathbb{Z})$ which has off-diagonal entry $E_{ij} = 1$ where $i \neq j$ and 1 on the main diagonal and 0 elsewhere, e.g.

$$E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define 4 *basic positive equivalences* as follows: suppose $A_{ij} > 0$,

$$(E, I) : I - A \to E(I - A), \qquad (E^{-1}, I) : E(I - A) \to I - A$$

$$(I, E) : I - A \to (I - A)E, \qquad (I, E^{-1}) : (I - A)E \to I - A.$$

A *positive equivalence* is the composition of basic positive equivalences. We will only discuss the flow equivalence induced by the basic positive equivalence $(E, I) : I - A \to E(I - A)$. We can apply the same idea with the others. Define A' from the equation $E(I - A) = I - A'$. Then A and A' agree except in row i, where we have

$$A'_{ik} = A_{ik} + A_{jk} \text{ if } j \neq k, \text{ and}$$

$$A'_{ij} = A_{ij} + A_{jj} - 1.$$

Let $\mathcal{G}_A$ be a directed graph having A as the adjacency matrix with edge set $\mathcal{E}_A$. We can describe a directed graph $\mathcal{G}_{A'}$ which has A' as its adjacency matrix as follows. Pick an edge e which runs from a vertex i to a vertex j in $\mathcal{G}_A$(e exists because $A_{ij} > 0$ by assumption). The edge set $\mathcal{E}_{A'}$ will be obtained from $\mathcal{E}_A$ as follows:

a) remove e from $\mathcal{E}_A$.

b) For every vertex k, for every edge f in $\mathcal{E}_A$ from j to k add a new edge named $[ef]$ from i to k.

Let $\mathcal{E}^*_A$ be the set of new edges obtained from the above construction. Define a map $\gamma : \mathcal{E}_{A'} \to \mathcal{E}^*_A$ by $\gamma(f) = f$ and $\gamma([ef]) = ef$. Then γ induces a map $\widehat{\gamma} : X_A \to X_{A'}$ defined naturally by the rule

$$\widehat{\gamma} : \cdots x'_{-2}x'_{-1}.x'_0x'_1 \cdots \mapsto \cdots \gamma(x'_{-2})\gamma(x'_{-1}).\gamma(x'_0)\gamma(x'_1) \cdots$$

Define a flow equivalence $F_\gamma : Y_{A'} \to Y_A$ by

$$F([x,t]) = \begin{cases} [\widehat{\gamma}(x), t], & \text{if } x \in X_e \text{ for every single edge } e \\ [\widehat{\gamma}(x), 2t], & \text{if } x \in X_{[ef]} \text{ for every edge of the form } [ef]. \end{cases}$$

One can check that F_γ is a flow equivalence.

Example 2.2.6. Suppose

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, A' = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \text{and } E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $E(I - A) = I - A'$. Label edges on $\mathcal{G}_A$ and write

$$A = \begin{pmatrix} a+b & c+d & 0 \\ 0 & e & f \\ g & h & 0 \end{pmatrix}.$$

To get the graph $\mathcal{G}_{A'}$, we pick an edge c (or d) and write

$$E = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$I - A = \begin{pmatrix} 1-a-b & -c-d & 0 \\ 0 & 1-e & -f \\ -g & -h & 1 \end{pmatrix}, \text{so } E(I-A) = \begin{pmatrix} 1-a-b & -d-ce & -cf \\ 0 & 1-e & -f \\ -g & -h & 1 \end{pmatrix} = I - A'.$$

Thus

$$A' = \begin{pmatrix} a+b & d+ce & cf \\ 0 & e & f \\ g & h & 0 \end{pmatrix}.$$

Note that this idea is compatible with the construction of $\mathcal{G}_{A'}$ described before. The flow equivalence $F : Y_{A'} \to Y_A$ is defined by

$$F([x,t]) = \begin{cases} [\widehat{\gamma}(x), t], & \text{if } x \in X_a \cup X_b \cup X_d \cup X_f \cup X_g \cup X_h \\ [\widehat{\gamma}(x), 2t], & \text{if } x \in X_{[ce]} \cup X_{[cf]}. \end{cases}$$

We will be considering the following result from [Bo02b].

Theorem 2.2.7. Suppose A, B are nontrivial essentially irreducible matrices defining SFTs with more than a single orbit; U, V are in $\mathrm{SL}(\mathbb{Z})$; and $U(I-A)V = I-B$.

Then for some positive integer k, there are positive equivalences (E_i, F_i) from $I - A_i$ to $I - A_{i+1}$, $0 \leq i \leq k$, such that $A_0 = A, A_k = B$ and $(U, V) = (E_k \cdots E_1, F_1 \cdots F_k)$.

In other words, every $\mathrm{SL}(\mathbb{Z})$ equivalence from $I - A$ to $I - B$ is a composition of basic positive equivalences.

Let (X_A, σ_A) be a nontrivial irreducible SFT. Let $(U, V) : (I - A) \to (I - A)$ be a positive equivalence and $F_{(U,V)}$ be an associated flow equivalence. We define $F^*_{(U,V)} : \mathrm{cok}(I - A) \to \mathrm{cok}(I - A)$ by the rule $[u] \mapsto [uV]$ (we use the action on row vectors to define $\mathrm{cok}(I - A)$). Then $F^*_{(U,V)}$ is an isomorphism. Given any flow equivalence $F : Y_A \to Y_A$, there is a positive equivalence $(U, V) : (I - A) \to (I - A)$ such that $F = F_{(U,V)}$. Let $F^* = F^*_{(U,V)}$. Let $\mathrm{Aut}(\mathrm{cok}(I - A))$ denote the group of group automorphisms of $\text{cok}(I - A)$. We define the map $\rho : \mathcal{M}_A \to \text{Aut}(\text{cok}(I-A))$ by the rule $\rho : F \mapsto F^*$. We call ρ the *Bowen-Franks representation* of (X_A, σ_A). It was proved in [Bo02b] that this rule indeed gives a well defined group homomorphism. If Y_A is not a circle then ρ is surjective [Bo02b].

2.3 The Mapping Class Group, Circles, and The Center

Theorem 2.3.1. Let (X_A, σ_A) be an irreducible shift of finite type. For $F \in \mathcal{M}_A$, the following are equivalent

a) F is isotopic to the identity.

b) $F(\mathcal{O}) = \mathcal{O}$ for all suspension flow orbits $\mathcal{O}$ in Y_A.

c) $F(\mathcal{C}) = \mathcal{C}$ for all but finitely many circles $\mathcal{C}$ in Y_A.

Proof. The equivalence is obvious if X_A is a single orbit (i.e., X_A has a single circle). So, we may suppose X_A is nontrivial (i.e., contains more than one orbit).

a) $\Rightarrow$ b) Suppose there is an isotopy F_t such that $F_0 = F$ and $F_1 =$ the identity on Y_A. For any $x \in X_A$, $F_t([x, 0])$ is a path of points in Y_A. Thus $F([x, 0]) = F_0([x, 0])$ is in the same connected component of Y_A as $F_1([x, 0]) = [x, 0]$. These components are precisely the flow orbits.

b) $\Rightarrow$ a) Let $\widetilde{F} : X_A \times \mathbb{R} \to X_A \times \mathbb{R}$ be a lift of F (i.e., $\pi_{\sigma_A}\widetilde{F} = F\pi_{\sigma_A}$, and $\widetilde{F}$ is a homeomorphism). This gives a continuous function $\delta : (x, t) \to \mathbb{R}$ such that $\widetilde{F} : (x, t) \mapsto \widetilde{\alpha}_{\delta(x,t)}(x, t)$. By the equivariance of π_{σ_A}, $\delta(x, t)$ depends only on $y = [x, t]$. So, for y in Y_A, $F(y) = \alpha_{\delta(y)}(y)$, where $\delta : Y_A \to \mathbb{R}$ is continuous. Now for $0 \le t \le 1$ define $F_t(y) = \alpha_{t\delta(y)}(y)$. This gives the isotopy from F to the identity.

b) $\Rightarrow$ c) This is trivial.

c) $\Rightarrow$ b) Let $x \in X_A$. Since σ_A is irreducible, $\mathrm{Per}(\sigma_A)$ is dense in X_A. There is a sequence of distinct points x_n in $\mathrm{Per}(\sigma_A)$ such that $x_n \to x$ as $n \to \infty$. Then $F([x_n, 0]) \to F([x, 0])$ as $n \to \infty$. Since $F(\mathcal{C}) = \mathcal{C}$ for all but finitely many circles, we have, for all but finitely many n, that there exists t_n such that $F([x_n, 0]) = [x_n, t_n]$. Appealing to the lift $\widetilde{F}$ of F, we have that there exists $T > 0$ such that for all n we may require $|t_n| \leq T$. Taking a convergent subsequence of (t_n) with limit t, we conclude $F([x, 0]) = [x, t]$. This shows that $F(\mathcal{O}_x) = \mathcal{O}_x$ where $\mathcal{O}_x$ represents the flow orbit containing $[x, 0]$. □

Corollary 2.3.2. The mapping class group $\mathcal{M}_A$ of an irreducible SFT σ_A acts by permutations on the set of circles of Y_A. This action is faithful.

Proof. This follows immediately from Theorem 2.3.1. □

Remark 2.3.3. Theorem 2.3.1 also implies that the action of $\mathcal{M}_A$ on the (ordered) cohomology group $C(X_A, \mathbb{Z})/(I - \sigma_A)C(X_A, \mathbb{Z})$ (considered in [BoH96] and [KRW01]) is faithful.

Theorem 2.3.4. Let (X_A, σ_A) be an irreducible shift of finite type. Then $\mathcal{M}_A$ acts n-transitively on the set of circles in Y_A for all $n \in \mathbb{N}$.

Proof. Let $\{\mathcal{C}_1, ..., \mathcal{C}_n\}$ and $\{\mathcal{C}'_1, ..., \mathcal{C}'_n\}$ be sets of n distinct circles. For each $i \in \{1, 2, ..., n\}$, let x_i, x'_i be representatives of the circles $\mathcal{C}_i, \mathcal{C}'_i$ respectively. We take a k-block presentation of (X_A, σ_A) where k is large enough that any point of period p comes from a path of length p without repeated vertices except initial and terminal vertices and no two of these loops share a vertex. If one of these loops, say L, has length greater than 1, then we apply a basic positive equivalence which corresponds to cutting out an edge e on the loop L and replacing it with edges labeled $[ef]$, for the edge f following e. The new loop will have length $p-1$ in the new graph. Continuing in the same fashion, we get a loop of length 1. Since no two of these loops share a vertex, we can apply the same idea to another loop without changing the former loop. Continuing in this way, we get a graph with loops $y_1, ..., y_n, y'_1, ..., y'_n$ of length 1, each of which comes from the loop containing $x_1, ..., x_n, x'_1, ..., x'_n$. If necessary we continue to apply basic positive equivalences until we get a graph with at least one point of least period n, for every positive integer n. Let (X_B, σ_B) be the SFT induced by the graph $\mathcal{G}_B$. (X_B, σ_B) is flow equivalent to (X_A, σ_A). Since $y_1, ..., y_n, y'_1, ..., y'_n$ are fixed points in (X_B, σ_B) and σ_B is mixing with points of all least periods, there is an inert automorphism $u \in \mathrm{Aut}(\sigma_B)$ such that $u(y_i) = y'_i$ for all $i = 1, ..., n$ [BoF91]. Extend u to a flow equivalence $\widehat{u} : Y_B \to Y_B$ by $\widehat{u}([x,t]) = [u(x), t]$. Let $G : Y_A \to Y_B$ be a flow equivalence arising from the construction. Then $F = G^{-1}\widehat{u}G$ is the required flow equivalence, i.e., $F(\mathcal{C}_i) = \mathcal{C}'_i$ for all $i = 1, ..., n$. □

Theorem 2.3.5. The center of $\mathcal{M}_A$ is trivial.

Proof. Let $\mathcal{C}$ be a circle in Y_A and F be an element in the center of $\mathcal{M}_A$. Suppose that $F(\mathcal{C}) \neq \mathcal{C}$. Note that $F(\mathcal{C})$ is also a circle. Then there is a flow equivalence G such that $G(\mathcal{C}) = \mathcal{C}$ and $G(F(\mathcal{C})) \neq F(\mathcal{C})$ by Theorem 2.3.4. Thus $FG(\mathcal{C}) = F(\mathcal{C}) \neq GF(\mathcal{C})$ which is a contradiction. Hence $F(\mathcal{C}) = \mathcal{C}$ for all circles $\mathcal{C}$ in Y_A. Therefore, F is isotopic to the identity by Theorem 2.3.1. □

2.4 The Mapping Class Group, Cross Sections, and Automorphisms of The Shifts

Let (X_A, σ_A) be an ISFT. For $u \in \mathrm{Aut}(\sigma_A)$, define $\widehat{u} : Y_A \to Y_A$ by $\widehat{u}([x,t]) = [ux, t]$. Clearly, $\widehat{u} \in \mathcal{M}_A$. Define $\phi : \mathrm{Aut}(\sigma_A) \to \mathcal{M}_A$ by $\phi(u) = \widehat{u}$.

Theorem 2.4.1. Let ϕ be defined as above. Then

a) ϕ is a group homomorphism.

b) $\mathrm{Ker}(\phi) =< \sigma_A >$, the cyclic group generated by σ_A.

Proof. a) Let $u, v \in \mathrm{Aut}(\sigma_A)$ and $[x,t] \in Y_A$. Then

$$\begin{aligned} \widehat{uv}([x,t]) &= [uv(x), t] \\ &= \widehat{u}([v(x), t]) \\ &= \widehat{u}\widehat{v}([x,t]). \end{aligned}$$

Thus $\widehat{uv} = \widehat{u}\widehat{v}$. This means that ϕ is a homomorphism.

b) Let $u \in \mathrm{Ker}(\phi)$. Then $\widehat{u}(\mathcal{O}) = \mathcal{O}$ for all flow orbits $\mathcal{O}$ in Y_A by Theorem 2.3.1. Thus $[u(x), 0] = \widehat{u}([x,0]) = [x, h(x)]$ for some continuous function $h : X_A \to \mathbb{R}$. This shows that $h(x) \in \mathbb{Z}$ for all $x \in X_A$, and $[u(x), 0] = [\sigma_A^{h(x)}(x), 0]$. Let x be a point with a dense orbit in X_A under the shift, and set $M = h(x)$. Since $u(x) = \sigma_A^M(x)$, for $n \in \mathbb{Z}$ we have $u(\sigma_A^n(x)) = \sigma_A^n u(x) = \sigma_A^n \sigma_A^M(x) = \sigma_A^M(\sigma_A^n(x))$. Thus $u = \sigma_A^M$ on the shift orbit of x, and by continuity $u = \sigma_A^M$ everywhere. If $u = \sigma_A^n$ for some $n \in \mathbb{Z}$ then $\widehat{u}$ is isotopic to the identity on Y_A. Define isotopy by going to lift and $\widetilde{F}_t(x,s) = (x, s+nt), 0 \le t \le 1$. Thus $\mathrm{Ker}(\phi) =< \sigma_A >$. □

Theorem 2.4.2. If A and B are flow equivalent then $\mathrm{Aut}(\sigma_B)/ < \sigma_B >$ is embedded to $\mathcal{M}_A$.

Proof. Let $F : Y_B \to Y_A$ be a flow equivalence. Then F induces an isomorphism $\widehat{F} : \mathcal{M}_B \to \mathcal{M}_A$ defined by $\widehat{F}(G) = FGF^{-1}$ for all $G \in \mathcal{M}_B$. By Theorem 2.4.1 b), there is an embedding of $\mathrm{Aut}(\sigma_B)/ < \sigma_B >$ into $\mathcal{M}_B$. Then we have an embedding $\mathrm{Aut}(\sigma_B)/ < \sigma_B > \to \mathcal{M}_B \xrightarrow{\widehat{F}} \mathcal{M}_A$. $\square$

Example 2.4.3. If (X_A, σ_A) and (X_B, σ_B) are flow equivalent then it is not necessarily true that their groups $\mathrm{Aut}(\sigma_A)/ < \sigma_A >$ and $\mathrm{Aut}(\sigma_B)/ < \sigma_B >$ are isomorphic as groups. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = A^2, \text{ and}, C = [2].$$

By the invariants for the classification of irreducible SFTs up to flow equivalence, they are flow equivalent (if $D = A, B$, or C, then $\mathrm{cok}(I-D)$ is trivial and det $(I - D)$=-1). But in $\mathrm{Aut}(\sigma_B)$, the center has a square root and in the others it does not.

Definition 2.4.4. Let $F : Y_A \to Y_A$ be a flow equivalence. A cross section C of Y_A is called an *invariant cross section* for F if $F(C) = C$. When F is used to denote the element $[F]$ of $\mathcal{M}_A$, we say F has an invariant cross section if any element of $[F]$ (any equivalence isotopic to F) has an invariant cross section.

For example, $\{[x, 0] : x \in X_A\}$ is an invariant cross section for any F induced by an element of $\mathrm{Aut}(\sigma_A)$. If equivalences F, F' have the same invariant cross section C, and $F(y) = F'(y)$ for all y in C, then F and F' are isotopic.

Theorem 2.4.5. Let $F : Y_A \to Y_A$ be a flow equivalence. If F has an invariant cross section C then F is isotopic to an equivalence induced by an automorphism of the first return map T_c.

Proof. Let $u = F|_C$. Then $u : C \to C$ is a homeomorphism. We will show that $uT_c = T_c u$. Let $y \in C$. Then $uT_c(y) = u\alpha_{f_c(y)}(y)$ and $T_c u(y) = \alpha_{f_c(u(y))}(u(y))$. Observe that y and $\alpha_{f_c(y)}(y)$ are closest points of C in the same flow orbit. Also, $u(y)$ and $\alpha_{f_c(u(y))}(u(y))$ are closest points of C in the same flow orbit. Since F is orientation preserving and $u : y \mapsto u(y), u : \alpha_{f_c(y)}(y) \mapsto \alpha_{f_c(u(y))}(u(y))$. This shows that $uT_c = T_c u$ as required. Therefore, $u \in \mathrm{Aut}(T_c)$. $\square$

Proposition 2.4.6. Let (X_A, σ_A) be an irreducible SFT. Suppose (X', σ') is an irreducible subshift of finite type of (X_A, σ_A), $F \in \mathcal{M}_A$, and F maps $Y_{\sigma'}$ into itself but not onto itself. Then F has no invariant cross section.

Proof. Suppose $F : Y_A \to Y_A$ is induced by an automorphism u of the return map T_c to some cross section C. The restriction of T_c to $C \cap Y_{\sigma'}$ defines an irreducible SFT, because it is flow equivalent to (X', σ'), since $C \cap Y_{\sigma'}$ is a cross section of $Y_{\sigma'}$. Therefore the restriction of u to $C \cap Y_{\sigma'}$, being an injection into $C \cap Y_{\sigma'}$, must be a surjection. But this implies $\widehat{u}$ maps $Y_{\sigma'}$ onto itself, which is a contradiction. $\square$

We can construct a flow equivalence which satisfies the conditions of Proposition 2.4.6 and therefore has no invariant cross section.

Example 2.4.7. Given a finite set F of words, let F^* denote the space of doubly infinite sequences formed by all possible concatenations of those words. Let $S =$

$\{a, b, c, d_1 d_2\}^*$. Define a flow equivalence $F : Y_S \to Y_S$ to be the induced flow equivalence of the automorphism defined by permuting a and c and permuting b and $d_1 d_2$. Let $\overline{S} = \{c, d_1 d_2\}^*$ and $T = \{a, b\}^*$. Define $\phi : \overline{S} \to T$ by the rule $c \mapsto a$ and $d_1 d_2 \mapsto ab$. Theorem 1.5 in [BoK93] states that if there exists some N such that for all $n > N$, $\pi_n(S) - \pi_n(\overline{S}) \geq 2n$ then ϕ extends to an automorphism of S where $\pi_n(S), \pi_n(\overline{S})$ represent the number of periodic points of $S, \overline{S}$ with least period n, respectively. For any $n \geq 3$, the number of periodic points of $\overline{S}$ is at least $n + 1$. Given any periodic point of $\overline{S}$ which comes from a word of length $n - 1$, we can construct 2 periodic points of S with least period n by adding a or b at the end of the word. Thus $\pi_n(S) - \pi_n(\overline{S}) \geq 2\pi_{n-1}(\overline{S}) \geq 2n$ for all $n \geq 4$. This implies that ϕ extends to an automorphism u of S by [BoK93, Theorem 1.5]. Then $\widehat{F} = \widehat{u}F$ maps Y_T properly into itself since there is no flow orbit of Y_T that maps to the flow orbit containing the point $[b^\infty, 0]$. By Proposition 2.4.6, $\widehat{F}$ has no invariant cross section.

Proposition 2.4.8. Let (X_A, σ_A) be an irreducible SFT. Let $F \in \mathcal{M}_A$. If there is a circle $\mathcal{C}$ such that $\{F^n(\mathcal{C}) : n \in \mathbb{N}\}$ is an infinite collection of circles then F has no invariant cross section.

Proof. If $u \in \mathrm{Aut}(\sigma_A)$ then any periodic orbit of σ_A is mapped into the finite set of periodic orbits of equal period. Therefore the orbit of a circle under $\widehat{u}$ must equal finitely many circles. □

Example 2.4.9. Let $\widehat{F}$ be defined as in Example 2.4.7. Let $\mathcal{C}$ be the circle containing the point $[b^\infty, 0]$. For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the circle containing the point $(a^n b)^\infty$ in X_A. Then $\{\widehat{F}^n(\mathcal{C}) : n \in \mathbb{N}\} = \{\mathcal{C}_n : n \in \mathbb{N}\}$ which is infinite.

We close this section with the following problem.

Problem 2.4.10. Characterize flow equivalences which have an invariant cross section.

2.5 The Mapping Class Group and The Positive Equivalence Groupoid

Consider triples $[(I-A),(U,V),(I-B)]$ such that A,B,U,V have entries in $\mathbb{Z}_+$, A and B define infinite irreducible SFTs, one of U,V is Id and the other is a basic elementary matrix (at most one entry differs from I, and it can only be off-diagonal), the matrix I is the infinite identity matrix, the matrices A,B have only finitely many nonzero entries. We picture the triple as a directed edge labeled (U,V) from a vertex $I-A$ to a vertex $I-B$, in a countably infinite directed graph.

Now we define a groupoid $G(\mathbb{Z}_+)$, as a groupoid of morphisms in a category. The objects of the category are the matrices $I-A$. A triple $[(I-A),(U,V),(I-B)]$ as described in the previous paragraph is a morphism from $I-A$ to $I-B$. Its formal inverse $[(I-A),(U,V),(I-B)]^{-1}$ is a morphism from $I-B$ to $I-A$. A general element of the groupoid is a concatenation $g_1\cdots g_n$ of such elementary morphisms, with g_i a morphism from $I-A_i$ to $I-B_i$, such that $I-B_i=I-A_{i+1}, 1\le i\le n$. The identity morphism 1_{I-A} from $I-A$ to $I-A$ is $[(I-A),(I,I),(I-A)]$.

The description of the groupoid in the infinite directed graph is that the elements are finite concatenations of edges, where the concatenation must be legal; and adjacent inverses may be cancelled; legal addition of edges 1_{I-A} does not change a group element. A path P from $I-A$ to $I-B$ determines a flow equivalence $\rho(P)$

from Y_A to Y_B (see the construction in Section 2.2.3). Now let $PE_{\mathbb{Z}}(A)$ be the subgroupoid of $G(\mathbb{Z}_+)$ corresponding to elements which begin and end at $I - A$ (we use $\mathbb{Z}$ to indicate that the entries of A are not restricted to $\{0, 1\}$). This $PE_{\mathbb{Z}}(A)$ is a group. Let $\mathcal{S}_A$ be the subgroup of $\mathcal{M}_A$ generated by "simple"flow equivalences, of the form GFG^{-1} where G is in $\mathcal{M}_A$ and F is induced by a basic simple automorphism of a return map to a cross section. A *simple automorphism* of an SFT is an automorphism which is conjugate to a code generated by a graph automorphism which fixes all vertices. A *basic simple automorphism* is an automorphism conjugate to a 1-block code defined by a permutation of edges which exchanges two edges (with the same initial and terminal vertices) and leaves the other fixed. Every simple automorphism is a composition of basic simple automorphisms. The rule ρ above defines a homomorphism ρ_A from $PE_{\mathbb{Z}}(A)$ to $\mathcal{M}_A/\mathcal{S}_A$. We know that ρ_A is well defined and surjective [Bo02b]. There is another homomorphism, π_A, from $PE_{\mathbb{Z}}(A)$ to $\mathrm{SL}(\mathbb{Z})$. This is the homomorphism determined by sending each generator $[(I - A), (U, V), (I - B)]$ to V. We will need the following background.

Definition 2.5.1. Let $M \in \mathrm{GL}_n(\mathbb{Z})$ and I be the $\mathbb{N} \times \mathbb{N}$ identity matrix . By identifying M as $\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}$, we have an embedding of $\mathrm{GL}_n(\mathbb{Z})$ into the group of $\mathbb{N} \times \mathbb{N}$ invertible matrices. Then we have an ascending chain of subgroups $\mathrm{GL}_1(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z}) \subset \cdots$. The *stable linear group over* $\mathbb{Z}$ is defined by $\mathrm{GL}(\mathbb{Z}) = \bigcup_{n=1}^{\infty} \mathrm{GL}_n(\mathbb{Z})$. If $A \in \mathrm{GL}_n(\mathbb{Z})$ and $B \in \mathrm{GL}_m(\mathbb{Z})$ we define AB as the matrix multiplication of the embedding of A and B in $\mathrm{GL}_k(\mathbb{Z})$ for any $k \geq m, n$. The subgroup $\mathrm{SL}(\mathbb{Z})$ generated similarly by the determinant 1 matrices over $\mathbb{Z}$ is called the *stable special linear group over* $\mathbb{Z}$, and is generated by the elementary matrices in $\mathbb{Z}$ (those equal to Id except possibly in one off-diagonal entry). For each $n \geq 0$ and $k \geq 1$, we define the group homomorphism $g_{nk} : \mathrm{GL}_k(\mathbb{Z}) \to \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z})$ by $g_{nk}(M) = \overline{M}$. The *n-congruence subgroup* of $\mathrm{GL}_k(\mathbb{Z})$ is defined as $\mathrm{CL}_k(n, \mathbb{Z}) = \mathrm{Ker}(g_{nk})$. Let $\mathrm{E}_k(n, \mathbb{Z})$ denote the group generated by all elementary matrices in $\mathrm{CL}_k(n, \mathbb{Z})$. Set $\mathrm{CL}(n, \mathbb{Z}) = \bigcup_{k=1}^{\infty} \mathrm{CL}_k(n, \mathbb{Z})$ and $\mathrm{E}(n, \mathbb{Z}) = \bigcup_{k=1}^{\infty} \mathrm{E}_k(n, \mathbb{Z})$.

Theorem 2.5.2. [S76] Let H be a normal subgroup of $\mathrm{SL}(\mathbb{Z})$. Then there exists a unique integer $n \geq 0$ such that

$$\mathrm{E}(n, \mathbb{Z}) \subseteq H \subseteq \mathrm{CL}(n, \mathbb{Z}).$$

Theorem 2.5.3. Suppose that (X_A, σ_A) is a mixing SFT with positive entropy and $\mathrm{cok}(I - A) = 0$. For every element $F \in \mathcal{M}_A/\mathcal{S}_A$, and every V in $\mathrm{SL}(\mathbb{Z})$, there is an element $g \in PE_{\mathbb{Z}}(A)$ such that $\rho_A(g) = F$ and $\pi_A(g) = V$. Equivalently, the restriction of π_A to $(\rho_A)^{-1}(Id)$ is surjective.

Proof. First, we show that π_A is surjective. Since $\mathrm{cok}(I - A) = 0$, $I - A$ is invertible. Given $V \in \mathrm{SL}(\mathbb{Z})$, we choose $U = (I - A)V^{-1}(I - A)^{-1} \in \mathrm{SL}(\mathbb{Z})$. Then $U(I - A)V = I - A$ and is an $\mathrm{SL}(\mathbb{Z})$ equivalence of $I - A$ to itself, and by Theorem 2.2.7 it is induced by a positive equivalence. Thus π_A is surjective. Next, we assume without loss of generality that A has an off-diagonal entry 1. Let $K = \rho_A^{-1}(Id)$. Then K is a normal subgroup of $PE_{\mathbb{Z}}(A)$. Since π_A is surjective, $\pi_A(K)$ is a normal subgroup of $\mathrm{SL}(\mathbb{Z})$. The next three equations describe positive equivalences whose composition is the identity map.

$$\begin{pmatrix} I-A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -A & I \end{pmatrix} = \begin{pmatrix} I-A & 0 \\ -A & I \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} I-A & 0 \\ -A & I \end{pmatrix} = \begin{pmatrix} I-A & 0 \\ -I & I \end{pmatrix}$$

$$\begin{pmatrix} I-A & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} = \begin{pmatrix} I-A & 0 \\ 0 & I \end{pmatrix}.$$

Thus $\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} I-A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I-A & I \end{pmatrix} = \begin{pmatrix} I-A & 0 \\ 0 & I \end{pmatrix}$. Let

$$U = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}, V = \begin{pmatrix} I & 0 \\ I-A & I \end{pmatrix}.$$

Note that $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ represents the same infinite matrix as A. By Theorem 2.2.7, there is a composition of basic elementary positive equivalences $(U_i, V_i), 1 \leq i \leq n$, such that $U = U_n U_{n-1} \cdots U_1$ and $V = V_1 V_2 \cdots V_n$. Then the concatenation of edges $[(I - A_i), (U_i, V_i), (I - B_i)], 1 \leq i \leq n$, induces the identity on $\mathcal{M}_A/\mathcal{S}_A$. Since A has off-diagonal entry 1, V is not in the n-congruence subgroup of $\mathrm{SL}(\mathbb{Z})$ for any $n \neq 1$. By Theorem 2.5.2, we must have $\mathrm{E}(1, \mathbb{Z}) \subseteq \pi_A(K) \subseteq \mathrm{CL}(1, \mathbb{Z})$. But $\mathrm{CL}(1, \mathbb{Z}) = \mathrm{GL}(\mathbb{Z})$ and hence $\mathrm{E}(1, \mathbb{Z}) = \mathrm{SL}(\mathbb{Z})$. So, $\pi_A(K) = \mathrm{SL}(\mathbb{Z})$. This completes the proof. $\square$

We finish this section with the following question.

Question 2.5.4. Is the kernel of the Bowen-Franks homomorphism from the mapping class group of a positive entropy irreducible shift of finite type simple?

The result in Theorem 2.5.3 shows that in the case the Bowen-Franks group is trivial, we can get no more information about the normal subgroup structure of $\mathcal{M}_A/\mathcal{S}_A$ from the natural projection onto $\mathrm{SL}(\mathbb{Z})$ of $PE_{\mathbb{Z}}(A)$. The extension of this result to $\mathcal{M}_A$ has not yet been established.

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