# ABSTRACT

Title of dissertation: ON DEFINABILITY OF TYPES IN DEPENDENT THEORIES

Vincent Guingona, Doctor of Philosophy, 2011

Dissertation directed by: Professor Michael Chris Laskowski
Department of Mathematics

Using definability of types for stable formulas, one develops the powerful tools of stability theory, such as canonical bases, a nice forking calculus, and stable embeddability. When one passes to the class of dependent formulas, this notion of definability of types is lost. However, as this dissertation shows, we can recover suitable alternatives to definability of types for some dependent theories. Using these alternatives, we can recover some of the power of stability theory.

One alternative is uniform definability of types over finite sets (UDTFS). We show that all formulas in dp-minimal theories have UDTFS, as well as formulas with VC-density $< 2$. We also show that certain Henselian valued fields have UDTFS.

Another alternative is isolated extensions. We show that dependent formulas are characterized by the existence of isolated extensions, and show how this gives a weak stable embeddability result. We also explore the idea of UDTFS rank and show how it relates to VC-density.

Finally, we use the machinery developed in this dissertation to show that VC-minimal theories satisfy the Kueker Conjecture.

# ON DEFINABILITY OF TYPES IN DEPENDENT THEORIES

by

Vincent Guingona

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2011

Advisory Committee:
Professor Michael Chris Laskowski, Chair/Advisor
Professor David Kueker
Professor Edgar Lopez-Escobar
Professor Harry Tamvakis
Professor William Gasarch

# Dedication

To my wonderful spouse, Laura.

## Acknowledgments

First of all, I am very grateful for my advisor, Chris Laskowski, for helping me tremendously with this thesis. He is a great mentor, a wonderful teacher, and an excellent mathematician; I have learned a great deal from him. Secondly, I thank David Kueker for introducing me to model theory at Maryland. His first-year course in logic was a terrific introduction. I also thank the other senior logicians at Maryland during my time here, including John Goodrick, Alexei Kolesnikov, Justin Brody, Hunter Johnson, Greggo Johnson, and Chris Shaw, for listening to my research ideas and giving me helpful feedback. I thank the other members of my dissertation committee: William Gasarch, Ken Lopez-Escobar, and Harry Tamvakis. I appreciate the help of other logicians I have met outside of Maryland, including Joe Flenner, Maryanthe Malliaris, Tom Scanlon, and Sergei Starchenko.

I really appreciate the time and patience of my spouse, Laura. Without her love and support, I would not be where I am today. I thank my family: my mother, my father, my step-mother Laura, and my siblings Phillip and Sonya; and Laura's family: Roseann and Carl. I am in debt to them for their support. I thank my friends here at Maryland for playing games with me and providing me with a helpful distraction every once and a while. I thank Joe Galante for putting up with me as a roommate for two years. Finally, I also thank Bob Mastorakis, for introducing me to higher mathematics.

# Table of Contents

# List of Abbreviations

| | |
|---|---|
| EM | Ehrenfeucht-Mostowski |
| UDTFS | Uniform Definability of Types over Finite Sets |
| UDTFIS | Uniform Definability of Types over Finite Indiscernible Sets |
| UDTIS | Uniform Definability of Types over Indiscernible Sets |
| VC | Vapnik-Chervonekis |

# Chapter 1

# Introduction

## 1.1 Overview

As the purpose of this document is to generalize results from stability theory to the unstable setting, it is only natural to ask: What is stability theory? Stability theory was developed primarily by Shelah in the 1970's and 1980's. It is built out of a generalization of Morley's Categoricity Theorem, which states that if a complete theory in a countable language is categorical in some uncountable power, then it is categorical in all uncountable powers. Shelah generalized this result [22], categorizing when a theory has a certain number of isomorphism classes of models of a given cardinality (i.e., understanding what he calls $I(\lambda, T)$). This work brought out many model-theoretic tools for stable theories, including forking calculus, stable embeddability, and definability of types. This work also led to the development of other model-theoretic dividing lines, including superstability, $\aleph_0$-stability, dependence, strong dependence, and dp-minimality.

One of the main tools used in [22] and stability theory in general is the notion of definability of types. The existence of uniform definability of types allows one to show that, given any set $B$ and any externally definable subset $C \subseteq B$, there is a definition for $C$ over $B$ itself (i.e., $B$ is stably embedded). It also allows for a better understanding of types and type spaces. If a formula has uniform definability of types, then the number of types in that formula over any infinite set is bounded by the cardinality of that set. More generally, in a stable theory where all formulas have uniform definability of types, there exists a class of cardinals $\lambda$ such that, for all sets $B$ with $|B| = \lambda$, the number of all types over $B$ is bounded by $\lambda$.

Many important theories are stable, including the theory of algebraically closed fields, differentially closed fields, and modules over rings. However, there are many theories that are not stable, yet interesting to many mathematicians. These include, for example, the theory of the real field, the $p$-adic field, dense linear order without endpoints, and the random graph.

More recently, model theorists including Shelah have been studying a generalization of stability known as dependence. Shelah's study of dependence began with [22] and continued in [23, 24]. There are many interesting theories that are dependent yet unstable, including the theory of the real field, the $p$-adic field, and dense linear order without endpoints. However, a major problem that arises from working with dependent theories is that we lose much of the power of stability theory. For example, sets are not necessarily stably embedded and there is no control over the size of the type space. Most notably, we lose uniform definability of types for some formulas. The main question this thesis seeks to answer is: Can we find a suitable replacement for definability of types in dependent theories. We give partial answers to this question, suggesting several alternatives, and show how to recover some of the strength of stability theory in the process.

In Section 1.2, we introduce the background material necessary for this thesis, giving basic definitions and results, mostly from [22]. In Section 1.3, we discuss in more detail the issue of definability of types. In Section 1.4, we outline the body of this thesis and highlight key results.

## 1.2 Types, Stability, and Dependence

We use standard set-theoretic notation regarding ordinals and sets of functions. For example, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, and $\omega = \{0, 1, 2, ...\}$. So we see that $2 \in 5$ and we write "$<$" to mean "$\in$" for ordinals. For two fixed sets $A, B$, the set of functions from $A$ to $B$ is denoted ${}^{A}B$. Thus, ${}^{5}2$ is the set of functions from $5 = \{0, 1, 2, 3, 4\}$ to $2 = \{0, 1\}$, which has 32 elements. A function is a collection of ordered pairs, so, for example, $\{(0, 0)\}$ is the function from $\{0\}$ to itself. We use this fact to simplify notation in Section 2.7.

We begin with a language $L$ and a complete, first-order $L$-theory $T$ with infinite models. In our discussions, it benefits us to fix a large, sufficiently saturated model of $T$ called the "monster model." We denote this by $\mathfrak{C}$. The idea is that we only consider models $M \models T$ that have a small cardinality compared to the saturation of $\mathfrak{C}$. Therefore, we may assume that $M \preceq \mathfrak{C}$. Likewise, any "set" is a subset of $\mathfrak{C}$ of a small cardinality compared to the saturation of $\mathfrak{C}$ and arbitrary elements are contained in $\mathfrak{C}$. For any sentence $\theta$ over $\mathfrak{C}$, we abbreviate $\mathfrak{C} \models \theta$ by $\models \theta$. For most of our discussion, it suffices to consider models of size at most $2^{\aleph_0}$.

By a "formula" we mean an $\emptyset$-definable $L$-formula unless otherwise specified. For convenience, we sometimes write $\overline{x} = (x_0, ..., x_{n-1})$ for variables $x_0, ..., x_{n-1}$, so we would write the formula $\varphi(x_0, ..., x_{n-1})$ as $\varphi(\overline{x})$. The *length* of $\overline{x} = (x_0, ..., x_{n-1})$

is $n$ and is denoted $\lg(\overline{x})$. The same holds for tuples of elements from $\mathfrak{C}$. That is, if $a_0, ..., a_{n-1} \in \mathfrak{C}$, we may write $\overline{a} = (a_0, ..., a_{n-1})$ and say that the length of $\overline{a}$ is $\lg(\overline{a}) = n$. Of course, if $\varphi(\overline{x})$ is a formula and $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$, then $\varphi(\overline{a})$ denotes the $\overline{a}$-definable $L$-formula defined by substitution. Sometimes we partition the variables of a formula into two clusters. When we fix a partitioned formula

$$\varphi(\overline{x}; \overline{y}) = \varphi(x_0, ..., x_{n-1}; y_0, ..., y_{m-1}),$$

we mean to remember which elements are on the left and which are on the right. We often call the variables on the left the *free variables* and the variables on the right the *parameter variables*. When we have a list of tuples of variables, we sometimes denote this with a boldface variable to shorten notation. For example, we could write $\varphi(\overline{x}; \overline{y}_0, ..., \overline{y}_{k-1})$ as $\varphi(\overline{x}; \mathbf{y})$. If $\theta(\overline{x})$ is a formula, then denote $\theta(\overline{x})^0 = \neg\theta(\overline{x})$ and $\theta(\overline{x})^1 = \theta(\overline{x})$. For $\Theta(\overline{x})$ a set of formulas, let

$$\pm\Theta(\overline{x}) = \{\theta(\overline{x})^t : t < 2, \theta \in \Theta\}.$$

We define $|T|$ to be the cardinality of all $L$-formulas modulo $T$-equivalence.

Fix a partitioned formula $\varphi(\overline{x}; \overline{y})$. We say that a set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ is $\varphi$-*independent* if, for all maps $s \in {}^B2$, the set of formulas $\{\varphi(\overline{x}; \overline{b})^{s(\overline{b})} : \overline{b} \in B\}$ is consistent.[1] We say that $\varphi$ has *independence dimension* $N < \omega$, which we denote by $\mathrm{ID}(\varphi) = N$, if $N$ is maximal such that there exists a $\varphi$-independent set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| = N$. If no such maximal $N$ exists, we say that $\varphi$ is *independent* and write $\mathrm{ID}(\varphi) = \infty$.

[1] Saying that $B$ is $\varphi$-independent is the same as saying that $B$ is *shattered* by $\varphi^{\mathrm{opp}}$, the formula $\varphi$ with the opposite partitioning.

**Definition 1.2.1.** A partitioned formula $\varphi(\overline{x};\overline{y})$ is *dependent* if $\mathrm{ID}(\varphi) = N$ for some $N < \omega$. A theory $T$ is *dependent* if all partitioned formulas are dependent.

Notice that all of these notions are dependent on how the formula $\varphi$ is partitioned. For example, there exists a formula that is dependent when partitioned one way and independent when partitioned another (let $\psi(x;y)$ be an independent formula, let $\varphi(x,y,z) = (x = y) \wedge \psi(y,z)$, and consider $\varphi(x;y,z)$ and $\varphi(x,y;z)$).

Fix now a set of formulas

$$\Delta(\overline{x};\overline{y}) = \{\varphi_i(\overline{x};\overline{y}) : i \in I\}.$$

By a "$\Delta$-type over $B$" for some small subset $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ we mean a maximal consistent set of formulas of the form $\varphi_i(\overline{x};\overline{b})^t$ for $i \in I$, $t < 2$, and $\overline{b} \in B$. If $p$ is a $\Delta$-type over $B$, we say that $p$ has *domain* $\mathrm{dom}(p) = B$. For a set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, the space of all $\Delta$-types is denoted $S_\Delta(B)$. For any element $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$ and any set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, let $\mathrm{tp}_\Delta(\overline{a}/B)$ be the following $\Delta$-type (in $S_\Delta(B)$):

$$\mathrm{tp}_\Delta(\overline{a}/B) = \{\varphi(\overline{x};\overline{b})^t : \varphi \in \Delta, \overline{b} \in B, t < 2, \models \varphi(\overline{a};\overline{b})^t\}.$$

Any $\Delta$-type gives rise to a function $\delta \in {}^{(B\times\Delta)}2$ where, for all $\overline{b} \in B$ and $\varphi \in \Delta$,

$$\varphi(\overline{x};\overline{b})^{\delta(\overline{b},\varphi)} \in p(\overline{x}).$$

We call this $\delta$ the function *associated* to the $\Delta$-type $p$. For $B_0 \subseteq B$, $p \in S_\Delta(B)$, and $\delta$ associated to $p$, let

$$p_{B_0}(\overline{x}) = \{\varphi(\overline{x};\overline{b})^{\delta(\overline{b},\varphi)} : \overline{b} \in B_0, \varphi \in \Delta\}$$

denote the restriction of $p$ to $B_0$. If $\Delta = \{\varphi(\overline{x};\overline{y})\}$ (i.e., $\Delta$ is a single formula, $\varphi$), then we say "$\varphi$-type" for "$\{\varphi\}$-type," let $S_\varphi(B) = S_{\{\varphi\}}(B)$, etc. If $B_1 \subseteq B_0 \subseteq B$,

$p \in S_\varphi(B)$, and $\delta$ is the function associated to $p$ (note here that since $\{\varphi\}$ is a singleton, we may assume $\delta \in {}^B2$), let

$$p_{B_0,B_1}(\overline{x}) = \{\varphi(\overline{x};\overline{b})^{\delta(\overline{b})} : \overline{b} \in B_0 - B_1\} \cup \{\neg\varphi(\overline{x};\overline{b})^{\delta(\overline{b})} : \overline{b} \in B_1\}.$$

That is, $p_{B_0,B_1}$ is $p_{B_0}$ except we negate all instances of $\varphi$ on elements of $B_1$. Sometimes we call this *perturbing* $p_{B_0}$ *by* $B_1$. One should note that $p_{B_0,B_1}$ need not be a $\varphi$-type because it need not be consistent. This notation is used primarily in Section 2.4.

Notice that, since we are assuming that $\mathfrak{C}$ is sufficiently saturated, all $\Delta$-types have a realization in $\mathfrak{C}$ (so long as they are over a "small" set). Thus, for all $p \in S_\Delta(B)$, there exists $\overline{a}$ such that $p = \mathrm{tp}_\Delta(\overline{a}/B)$. For any such $\overline{a}$, we say $\overline{a} \models p$. For the moment, set $\Delta^*(\overline{x};\overline{y}_0,\overline{y}_1,...)$ equal to all formulas partitioned in this manner (though there are infinitely many variables, each formula in $\Delta^*$ uses only finitely many of them). Then, for any set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, we let $S_{\overline{x}}(B) = S_{\Delta^*}(B^\omega)$, the space of all types in the variables $\overline{x}$ over $B$. Since this really only depends on $\lg(\overline{x})$, we let $S_n(B)$ denote $S_{(x_0,\ldots,x_{n-1})}(B)$, the space of $n$-types over $B$. For any $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$, let $\mathrm{tp}(\overline{a}/B) = \mathrm{tp}_{\Delta^*}(\overline{a}/B^\omega)$, the (full) type of $\overline{a}$ over $B$. We say that $p$ is a *partial type* if $p \subseteq q$ for some type $q$.

The following two definitions are used in Section 3.3. We say that a partial type $p(\overline{x})$ is *finitely satisfied* over a set $A \subseteq \mathfrak{C}^{\lg(\overline{x})}$ if, for all finite partial subtypes $p_0 \subseteq p$, there exists $\overline{a} \in A$ such that $\overline{a} \models p_0$. Given sets $B \subseteq C \subseteq \mathfrak{C}^n$ and an ultrafilter $\mathcal{D}$ on $B$, the *average type* of $\mathcal{D}$ over $C$ is the type

$$\mathrm{Av}(\mathcal{D},C)(\overline{x}) = \{\delta(\overline{x}) \text{ parameter-definable type over } C : \delta(B) \in \mathcal{D}\}.$$

With most of the background out of the way, we begin with the definition of stable formulas.

**Definition 1.2.2.** We say that a partitioned formula $\varphi(\overline{x}; \overline{y})$ has the *order property* if there exists $\{\overline{a}_i : i < \omega\}$ with $\lg(\overline{a}_i) = \lg(\overline{x})$ for all $i < \omega$ and $\{\overline{b}_j : j < \omega\}$ with $\lg(\overline{b}_j) = \lg(\overline{y})$ for all $j < \omega$ such that, for all $i, j < \omega$, $\models \varphi(\overline{a}_i; \overline{b}_j)$ if and only if $i < j$. We say that $\varphi$ is *stable* if it does not have the order property. We say that a theory $T$ is *stable* if all partitioned formulas are stable.

Notice that if $\varphi$ is stable, then it is dependent (otherwise, independence provides a witness to the order property). Stable theories have many interesting properties. First, we define the notion of the Shelah 2-rank of a type (from [22], where Shelah denotes it by $R^{\lg(\overline{x})}(p, \Delta, 2)$). Let $\Delta(\overline{x}; \overline{y})$ be a finite partitioned collection of formulas and $p \in S_\Delta(B)$. Then the *Shelah 2-rank* of $p$, denoted $R_{2,\Delta}(p)$, is an ordinal-valued function on $\Delta$-types defined inductively as follows:

(i) $R_{2,\Delta}(p) \geq 0$ always.

(ii) $R_{2,\Delta}(p) \geq \delta$ for $\delta$ a limit ordinal if $R_{2,\Delta}(p) \geq \alpha$ for all $\alpha < \delta$.

(iii) $R_{2,\Delta}(p) \geq \alpha+1$ if, for all finite $\Delta$-types $q \subseteq p$, there are two $\Delta$-types $q_0, q_1 \supseteq q$ such that there exists a parameter-definable formula $\theta$ with $\theta \in q_0$ and $\neg\theta \in q_1$ and $R_{2,\Delta}(q_i) \geq \alpha$ for both $i < 2$.

Finally, we say that $R_{2,\Delta}(p) = \infty$ if $R_{2,\Delta}(p) \geq \alpha$ for all ordinals $\alpha$. *A priori* the Shelah 2-rank of a formula could be any ordinal or even $\infty$. However, if $\varphi$ is stable, then the 2-rank of all types are finite. Moreover, the size of type spaces are bounded.

**Theorem 1.2.3** (Theorem II.2.2 of [22]). *The following are equivalent for a partitioned formula $\varphi(\overline{x};\overline{y})$:*

*(i)* $\varphi(\overline{x};\overline{y})$ *is stable.*

*(ii) For all* $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ *with* $|B| \geq \aleph_0$, $|S_\varphi(B)| \leq |B|$.

*(iii)* $R_{2,\varphi}(p) < \omega$, *for all* $\varphi$*-types* $p$.

Chris Laskowski has a slightly different proof for Theorem 1.2.3 (iii) $\Rightarrow$ (i) that is useful in Section 3.2 when we introduce another rank (and show it is bounded by the Shelah 2-rank, see Theorem 3.2.8). This proof is presented in that section. The following theorem is known as "sufficiency of a single variable."

**Theorem 1.2.4** (Theorem II.2.13 and Theorem II.4.11 of [22]). *For a theory* $T$, *the following hold:*

*(i)* $T$ *is stable if and only if all partitioned formulas* $\varphi(x;\overline{y})$ *(with* $\lg(x) = 1$*) are stable.*

*(ii)* $T$ *is dependent if and only if all partitioned formulas* $\varphi(x;\overline{y})$ *are dependent.*

Instead of checking all formulas for stability (or dependence), it suffices to check only formulas of the form $\varphi(x;\overline{y})$. This is noteworthy because we show a similar result for UDTFS theories (see Lemma 2.3.6). This gives some evidence (albeit very little) that dependence is equivalent to UDTFS, as both have sufficiency of a single variable. Both dependence and stability of formulas are preserved under reduct, which is a question that is still open for formulas with UDTFS. For more on

UDTFS, see Section 2.3. More facts about stable formulas and theories are discussed in Section 1.3.

Another useful tool in studying theories is indiscernible sequences. For example, we use indiscernible sequences in Section 2.2 when studying definability of types. Fix a set of formulas $\Delta(\overline{y}_0, ..., \overline{y}_n)$ with $\lg(\overline{y}_i) = \lg(\overline{y}_j)$ for all $i, j \leq n$. Fix a linear order $(I, <)$ where $|I|$ is small in comparison to the saturation of $\mathfrak{C}$ and let $\langle \overline{b}_i : i \in I \rangle$ be a sequence of elements with each $\overline{b}_i \in \mathfrak{C}^{\lg(\overline{y}_0)}$.

**Definition 1.2.5.** We say that $\langle \overline{b}_i : i \in I \rangle$ is a $\Delta$-*indiscernible sequence* if, for all $i_0 < i_1 < ... < i_n$ and $j_0 < j_1 < ... < j_n$ from $I$ and for all $\delta \in \Delta$,

$$\models \delta(\overline{b}_{i_0}, ..., \overline{b}_{i_n}) \leftrightarrow \delta(\overline{b}_{j_0}, ..., \overline{b}_{j_n}). \tag{1.1}$$

We say that $\langle \overline{b}_i : i \in I \rangle$ is a $\Delta$-*indiscernible set* if, for all $i_0, i_1, ..., i_n \in I$ distinct, for all $j_0, ..., j_n \in I$ distinct (regardless of order), and for all $\delta \in \Delta$, (1.1) holds.

We say that $\langle \overline{b}_i : i \in I \rangle$ is a *indiscernible sequence* (respectively, *set*) if it is a $\Delta$-indiscernible sequence (respectively set) for all sets of formulas $\Delta$ of the appropriate free variables (i.e., arbitrarily many tuples of variables, each of length $\lg(\overline{b}_i)$).

We use the following fact, which follows from Erdös-Rado Theorem. Specifically, we use this in the proof of Theorem 2.2.1.

**Lemma 1.2.6.** *For all cardinals $\kappa$, there exists a $\lambda$ (depending on $\kappa$ and $|T|$) such that, for all sets $\{\overline{b}_i : i < \lambda\}$, there exists $I \subseteq \lambda$ with $|I| = \kappa$ such that $\langle \overline{b}_i : i \in I \rangle$ is an indiscernible sequence.*

Indiscernible sequences give us a means of analyzing the stability or dependence of a formula.

**Theorem 1.2.7** (Theorem II.2.20 and Theorem II.4.13 of [22]). *The following holds for a partitioned formula, $\varphi(\overline{x};\overline{y})$:*

*(i) If $\varphi$ is stable, then there exists $n < \omega$ such that, for all $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$ and all indiscernible sequences $\langle \overline{b}_i : i \in I \rangle$ of the appropriate length,*

$$|\{i \in I : \models \varphi(\overline{a};\overline{b}_i)\}| \leq n \text{ or } |\{i \in I : \models \neg\varphi(\overline{a};\overline{b}_i)\}| \leq n.$$

*(ii) If $\varphi$ is dependent, then there exists $n < \omega$ such that, for all $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$ and all indiscernible sequences $\langle \overline{b}_i : i \in I \rangle$, the truth value of $\varphi(\overline{a};\overline{b}_i)$ alternates at most $n$ times (i.e., there do not exist $i_0 < ... < i_n$ from $I$ such that $\models \varphi(\overline{a};\overline{b}_{i_\ell}) \leftrightarrow \neg\varphi(\overline{a};\overline{b}_{i_{\ell+1}})$ for all $\ell < n$).*

We use part (ii) of this theorem for Lemma 2.2.4 (and we provide a proof there). This gives rise to another rank to measure the complexity of a dependent formula, namely the alternation rank. We say that a formula has *alternation rank* $n < \omega$ if $n$ is minimal such that, for all $\overline{a}$ and $\langle \overline{b}_i : i \in I \rangle$, the truth value of $\varphi(\overline{a};\overline{b}_i)$ alternates at most $n$ times. We denote this by $\mathrm{alt}(\varphi) = n$. If no such $n$ exists, we say $\mathrm{alt}(\varphi) = \infty$.

Indeed there are other ranks by which to measure the complexity of a formula. For a partitioned formula $\varphi(\overline{x};\overline{y})$, we say that $\varphi$ has *VC-dimension* $n < \omega$ (denoted $\mathrm{VC}(\varphi) = n$) if $\mathrm{ID}(\varphi^{\mathrm{opp}}) = n$, where $\varphi^{\mathrm{opp}}(\overline{y};\overline{x}) = \varphi(\overline{x};\overline{y})$, i.e., $\varphi^{\mathrm{opp}}$ is $\varphi$ with the opposite partitioning. Here VC stands for Vapnik-Chervonenkis for two probabilists, Vladimir Vapnik and Alexey Chervonenkis.

Related is a notion called VC-density. We say that the formula $\varphi$ has *VC-density* $\ell$ for some $\ell \in \mathbb{R}$ if $\ell$ is the infimum over all $\ell' \in \mathbb{R}$ such that there exists $K < \omega$ such that, for all finite non-empty $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $\left|S_\varphi(B)\right| \leq K|B|^{\ell'}$. We denote this by $\mathrm{VCden}(\varphi) = \ell$. If no such $\ell$ exists, say $\mathrm{VCden}(\varphi) = \infty$. With this, we get the following theorem.

**Theorem 1.2.8** (Theorem 2.4 of [17], Sauer's Lemma [21], and others)**.** *The following are equivalent for a partitioned formula, $\varphi(\overline{x}; \overline{y})$:*

*(i)* $\mathrm{ID}(\varphi) < \infty$ *(i.e., $\varphi$ is dependent).*

*(ii)* $\mathrm{VC}(\varphi) < \infty$*.*

*(iii)* $\mathrm{alt}(\varphi) < \infty$*.*

*(iv)* $\mathrm{VCden}(\varphi) < \infty$*.*

Later, when we discuss UDTFS rank in Section 3.2, we analyze more closely how all of these ranks relate. Of course, there exist formulas with infinite Shelah 2-rank that are still dependent (take any dependent unstable formula). So Shelah 2-rank does not fit in nicely with these ranks measuring various levels of dependence.

We conclude this section with two definitions regarding subclasses of all dependent theories. The first deals with theories that define a linear order $<$ on $\mathfrak{C}$. The second one generalizes the first and does not require a linear order. One should note that any theory that defines an infinite linear order is necessarily unstable. However, it may still be dependent.

**Definition 1.2.9.** Suppose the language $L$ includes $<$ a binary relation and $T$ includes the axioms that $<$ is a linear order on $\mathfrak{C}$. We say that $T$ is *o-minimal* if, for all formulas $\varphi(x;\overline{y})$ and all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, the set $\{a \in \mathfrak{C} :\models \varphi(a;\overline{b})\}$ is a union of finitely many points and intervals of $\mathfrak{C}$. We say that $T$ is *weakly o-minimal* if, for all models $M \models T$, all formulas $\varphi(x;\overline{y})$, and all $\overline{b} \in M^{\lg(\overline{y})}$, $\{a \in M :\models \varphi(a;\overline{b})\}$ is a union of finitely many $<$-convex subsets of $M$.

**Definition 1.2.10.** A theory $T$ is *dp-minimal* if, for all formulas $\varphi(x;\overline{y})$ and $\psi(x;\overline{z})$ and all $\{\overline{b}_i : i < \omega\}$ and $\{\overline{c}_j : j < \omega\}$ (of the appropriate length), there exists $i_0, j_0 < \omega$ such that the following $\{\varphi,\psi\}$-type is inconsistent

$$\{\varphi(x;\overline{b}_{i_0}), \psi(x;\overline{c}_{j_0})\} \cup \{\neg\varphi(x;\overline{b}_i) : i < \omega, i \neq i_0\} \cup \{\neg\psi(x;\overline{c}_j) : j < \omega, j \neq j_0\}.$$

By compactness, if $T$ is dp-minimal and $\varphi(x;\overline{y})$ and $\psi(x;\overline{z})$ are partitioned formulas, then there exists $K < \omega$ such that the condition in Definition 1.2.10 holds for all sets $\{\overline{b}_i : i < K\}$ and $\{\overline{c}_j : j < K\}$. We use this fact in the proof of Theorem 2.4.1. We get the following relation of theories, which is easily checked.

**Proposition 1.2.11.** *If $T$ is o-minimal or weakly o-minimal, then $T$ is dp-minimal. If $T$ is dp-minimal, then $T$ is dependent.*

As we show in Section 3.2, there are theories which are not dp-minimal but are dependent (even with UDTFS). There are stable as well as unstable theories that are dp-minimal. Since stable theories are not o-minimal, there are dp-minimal theories that are not o-minimal.

## 1.3 Definability of Types

We now discuss the notion of definability of types and list some of the consequences of this property. The idea is the following: We want to better understand and control the space of types over a set. One way of understanding a type is finding a definition for it. Fix $\varphi(\overline{x};\overline{y})$ a partitioned formula and $p$ a $\varphi$-type.

**Definition 1.3.1.** A parameter-definable formula $\psi(\overline{y})$ *defines* $p$ if, for all $\overline{b} \in \mathrm{dom}(p)$, $\varphi(\overline{x};\overline{b}) \in p(\overline{x})$ if and only if $\models \psi(\overline{b})$.

On its face, this notion is quite trivial. In fact, for any $\varphi$-type $p$, let $\overline{a}$ be a realization of $p$. Then, the formula $\psi(\overline{y}) = \varphi(\overline{a};\overline{y})$ defines $p$. Our goal is three-fold:

(1) We want definitions for all $\varphi$-types $p$.

(2) We want to reduce the domain over which the definitions of $p$ are defined. Preferably, we want the definition to be defined over $\mathrm{dom}(p)$.

(3) We want our definition to be uniform. That is, for any given $\varphi$, we want a single $\emptyset$-definable formula $\psi(\overline{y};\overline{z})$ such that, when we plug in values for $\overline{z}$, we get definitions for any $\varphi$-type.

As we will see shortly, in stable formulas $\varphi$, (1), (2), and (3) hold in the strongest sense. However, when we pass to more general formulas, we lose some of this. In Chapter 2, we show that, for a large class of formulas $\varphi$ (possibly all dependent formulas), we achieve the strong form of (2) and (3), but fail on (1). We still have a uniform definition $\psi$, but it only works for finite $\varphi$-types. In Section 3.3, we take another approach to definability of types. There we get a version of definability of types for all dependent formulas, $\varphi$, that has (1), but we only achieve a weaker form of (2) and no (3). However, this is still strong enough to get a weak stable embeddability result (see Corollary 3.3.5).

As promised, we have the following result for stable formulas:

**Theorem 1.3.2** (Theorem II.2.12 of [22]). *The following are equivalent for a partitioned formula, $\varphi(\overline{x};\overline{y})$:*

*(i)* $\varphi$ *is stable.*

*(ii) There exists* $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ *with* $\lg(\overline{y}) = \lg(\overline{z}_i)$ *for all* $i < n$ *such that, for all* $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ *and all* $p \in S_\varphi(B)$*, there exists* $\overline{b}_0,...,\overline{b}_{n-1} \in B$ *such that* $\psi(\overline{y};\overline{b}_0,...,\overline{b}_{n-1})$ *defines* $p$*.*

This theorem is exactly what we try to capture with both UDTFS and Isolated Extensions (see Definition 2.1.1 and Theorem 3.3.3 below). Notice the similarities between the condition of Theorem 1.3.2 and Definition 2.1.1. Except for the word "finite," they are identical.

One immediate consequence of uniform definability of types is stable embeddability. We say that a subset $B \subseteq \mathfrak{C}^n$ is *stably embedded* if, for all parameter-definable formulas $\varphi(x_0,...,x_{n-1})$ (not necessarily defined over $B$), there exists a parameter-definable formula $\psi(x_0,...,x_{n-1})$ defined over $B$ such that, for all $\overline{b} \in B$, $\models \varphi(\overline{b}) \leftrightarrow \psi(\overline{b})$. That is, all externally definable subsets of $B$ are, in fact, internally definable. For stable theories, we get the following result:

**Corollary 1.3.3.** *If* $T$ *is stable, then all sets* $B$ *are stably embedded.*

Additionally, definability of types gives an explicit reason for the bound on type spaces. To illustrate this, let us sketch a proof of Theorem 1.2.3, (i) $\Rightarrow$ (ii) assuming Theorem 1.3.2. Fix $\varphi$ stable, so by Theorem 1.3.2, there exists $\psi(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1})$ that uniformly defines $\varphi$-types. Fix some $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| \geq \aleph_0$. For each $p \in S_\varphi(B)$, there exists $\overline{\mathbf{c}} \in B^n$ such that $\psi(\overline{y}; \overline{\mathbf{c}})$ defines $p$. It is easy to see, however, that one formula cannot define more than a single type over a given domain. Therefore, $\left|S_\varphi(B)\right| \leq |B^n| = |B|$.

When generalizing definability of types to dependent theories, one should note that a different version of a type space bound holds for dependent formulas. By Theorem 1.2.8 (i) $\Leftrightarrow$ (iv), we get the following:

**Corollary 1.3.4.** *A partitioned formula $\varphi(\overline{x}; \overline{y})$ is dependent if and only if there exists $n, K < \omega$ such that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $\left|S_\varphi(B)\right| \leq K \cdot |B|^n$.*

This finite version of Theorem 1.2.3, (i) $\Leftrightarrow$ (ii) for dependent formulas is part of the motivation for UDTFS. In fact, as we see in the proof of Proposition 2.3.3 below, as definability of types "explains" type space bounds for stable formulas, UDTFS "explains" type space bounds for some dependent formulas.

## 1.4 Outline of Thesis

The remainder of this thesis presents the work of this author. It should be noted that the material from Sections 2.2, 2.3, and 2.4 have been submitted for publication [11] and the material from Section 3.3 will appear in the *Proceedings of the American Mathematical Society* [9].

In Chapter 2 we introduce the notion of uniform definability of types over finite sets (UDTFS), motivate it, and show results regarding it. The following summarizes these results:

(i) (Theorem 2.2.1) A formula $\varphi$ is dependent if and only if it has uniform definability of types over finite indiscernible sequences.

(ii) (Lemma 2.3.6) A theory $T$ has UDTFS if and only if all formulas of the form $\varphi(x; \overline{y})$ has UDTFS.

(iii) (Theorem 2.4.1) If a theory $T$ is dp-minimal, then $T$ has UDTFS.

(iv) (Theorem 2.4.3) If a formula $\varphi$ has VC-density $< 2$, then $\varphi$ has UDTFS.

(v) (Theorem 2.5.3) If a Henselian valued field of equicharacteristic zero is such that the theory of the residue field and the theory of the value group have UDTFS, then the theory of the whole valued field has UDTFS.

(vi) (Theorem 2.6.6) A formula $\varphi$ is stable if and only if there exists $n < \omega$ such that, for all $\varphi$-types $p$, there exists $\overline{\mathbf{c}} \in (\mathrm{dom}(p))^n$ such that $p$ does not $\Delta_{n,\varphi}$-split over $\{\overline{\mathbf{c}}\}$ (see (2.5) below).

(vii) (Theorem 2.7.10) If a formula $\varphi$ is maximum, then $\varphi$ has UDTFS.

In Chapter 3 we discuss other definability of types notions, including UDTFS ranks and isolated extensions. The following summarizes the results of Chapter 3:

(viii) (Theorem 3.2.4) If there exists $k < \omega$ such that all formulas of the form $\varphi(x; \overline{y})$ have UDTFS rank $\leq k$, then all formulas of the form $\varphi(\overline{x}; \overline{y})$ have UDTFS rank $\leq k \cdot \lg(\overline{x})$. In particular, if $k = 1$, then $T$ has VC-density one.

(ix) (Theorem 3.3.3) A formula $\varphi$ is dependent if and only if, for all $\varphi$-types $p$, there exists an elementary $\varphi$-isolated $\varphi$-extension $p'$ (see Definition 3.3.2).

(x) (Corollary 3.3.5) If $T$ is dependent, then for any set $B$ in a model $M$, there exists $(N; B') \succeq (M; B)$ such that all externally parameter-definable subsets of $B$ are definable over $B'$.

(xi) (Theorem 3.4.3) For all formulas $\varphi(\overline{x}; \overline{y})$ from a dp-minimal theory with a linear order, there exists $N < \omega$ such that, for every finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and every $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$, there exists $B_0 \subseteq B$ with $|B_0| \leq N$ such that $\mathrm{tp}(\overline{a}/B_0) \vdash \mathrm{tp}_{\varphi}(\overline{a}/B)$.

In Chapter 4, we discuss VC-minimal theories in more detail. The following summarizes the results of Chapter 4:

(xii) (Theorem 4.2.2) If $T$ is VC-minimal, then $T$ is convexly orderable.

(xiii) (Theorem 4.4.4) If $T$ is such that there exists $(R, M)$ is a density $d$ rank (see Definition 4.4.3) on parameter-definable formulas (e.g., $d = 1$ for Morley rank and degree for strongly minimal theories), then $T$ has VC-density $d$.

(xiv) (Theorem 4.5.4) If $T$ is weakly VC-minimal, then either $T$ is stable or $T^{\mathrm{eq}}$ defines an infinite linear order.

(xv) (Corollary 4.5.6) If $T$ is weakly VC-minimal, then $T$ satisfies the Kueker Conjecture.

It is only fitting that we end this thesis with a result on the Kueker Conjecture, as this was the initial motivation for a large portion of this work.

# Chapter 2

# Uniform Definability of Types over Finite Sets

For this chapter, we work in a complete theory $T$ with monster model $\mathfrak{C}$.

## 2.1 Overview

The main goal of this chapter is to study a generalization of definability of types to a subclass of dependent theories known as uniform definability of types over finite sets (UDTFS). The study of UDTFS began with Johnson and Laskowski in [15]. They were analyzing compression schemes for concepts classes (see Section 2.7) and discovered that it was directly related to model theory. This motivated the definition of UDTFS (which was originally called *uniform type definition* in [15]). The following definition is due to Laskowski:

**Definition 2.1.1.** A partitioned formula $\varphi(\overline{x};\overline{y})$ has *uniform definability of types over finite sets* (*UDTFS*) if there exists a formula $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ (with $\lg(\overline{y}) = \lg(\overline{z}_i)$ for all $i < n$) such that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| \geq 2$ and all $p \in S_\varphi(B)$, there exists $\overline{c}_0,...,\overline{c}_{n-1} \in B$ such that, for all $\overline{b} \in B$, $\varphi(\overline{x};\overline{b}) \in p(\overline{x})$ if and only if

$$\models \psi(\overline{b};\overline{c}_0,...,\overline{c}_{n-1}).$$

We say that a theory $T$ has *uniform definability of types over finite sets* (*UDTFS*) if all partitioned formulas of $T$ have UDTFS.

Notice that this differs from Shelah's notion of uniform definability of types that characterizes stability (Theorem 1.3.2) only in the fact that we demand that $B$ be finite. This one simple change vastly expands the class of formulas and theories that have a notion of definability of types. It is clear that stable formulas (hence stable theories) have UDTFS and that formulas with UDTFS are dependent. The main open question, known as the *UDTFS Conjecture*, is: Does UDTFS characterizes dependence for formulas? This chapter gives various partial results to this conjecture.

In Section 2.2, we show that having uniform definability of types over finite *indiscernible sequences* (instead of merely sets) actually does characterize dependence for formulas. In Section 2.3, we develop basic properties of UDTFS, including sufficiency of a single variable and closure under boolean combinations. In Section 2.4, we prove that all dp-minimal theories have UDTFS. This implies that all o-minimal theories, all VC-minimal theories, and all VC-density one theories have UDTFS. In Section 2.5, we show that, given a Henselian valued field of equicharacteristic zero in the Denef-Pas language, if the residue field and value group have UDTFS, then the entire valued field has UDTFS. In Section 2.6, we give a characterization of UDTFS in terms of non-$\Delta$-splitting and discuss the Splitting Conjecture. Finally, in Section 2.7, we discuss the relationship between UDTFS and compression schemes for concept classes, as described in [15]. We show what the results of this chapter entail for compression schemes. We also use the results of Floyd and Warmuth in [7] to show that all maximum formulas have UDTFS.

## 2.2 Finite Indiscernible Sequences

Before launching into our discussion of UDTFS, let us first introduce another concept that actually proves to be equivalent to dependence. The following definition is only used in this section. We say that a partitioned formula $\varphi(\overline{x};\overline{y})$ has *uniform definability of types over (finite) indiscernible sequences* (*UDT(F)IS*) if there exists a formula $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ (with $\lg(\overline{y}) = \lg(\overline{z}_i)$ for all $i < n$) such that, for all (finite) indiscernible sequences $\langle \overline{b}_i : i \in I \rangle$ (with $\lg(\overline{b}_i) = \lg(\overline{y})$ for all $i \in I$) and all $p \in S_\varphi(\{\overline{b}_i : i \in I\})$, there exists $i_0, ..., i_{n-1} \in I$ such that $\psi(\overline{y}; \overline{b}_{i_0}, ..., \overline{b}_{i_{n-1}})$ defines $p$.

**Theorem 2.2.1.** *For a partitioned formula $\varphi(\overline{x};\overline{y})$, the following hold:*

*(i) $\varphi$ is stable if and only if $\varphi$ has UDTIS.*

*(ii) $\varphi$ is dependent if and only if $\varphi$ has UDTFIS.*

Notice that Theorem 2.2.1 (i) holds when we replace "indiscernible sequences" with arbitrary sets (see Theorem 1.3.2 above). However, it is still open whether or not (ii) holds with a similar modification. Studying this question is the basis of this chapter. We see by the analysis of this section that using indiscernible sequences instead of sets tends to smooth things out a bit.

Before we prove Theorem 2.2.1, let us deal with the case where $\varphi$ is dependent. First, fix a set of formulas $\Delta(\overline{z}_0, \overline{z}_1, ..., \overline{z}_n)$ with $\lg(\overline{z}_i) = \lg(\overline{z}_j)$ for all $i, j \leq n$.

**Definition 2.2.2.** We say that $\Delta$ is *closed under permutations* if, for all $\sigma \in S_{n+1}$

(where $S_i$ is the symmetric group on $i$) and for all $\delta \in \Delta$, the formula

$$\delta_\sigma(\overline{z}_0, \overline{z}_1, ..., \overline{z}_n) = \delta(\overline{z}_{\sigma(0)}, \overline{z}_{\sigma(1)}, ..., \overline{z}_{\sigma(n)}) \tag{2.1}$$

is also in $\Delta$.

Notice that if $\Delta$ is all formulas (of a fixed length and partitioning), then $\Delta$ is closed under permutations. The following lemma shows that if $\Delta$ is any set of formulas closed under permutations, for any $\Delta$-indiscernible sequence that is not a $\Delta$-indiscernible set, there exists an instance of $\pm\Delta$ that defines the linear order of the indiscernible sequence. The proof of this lemma is based on a modification of the proof of Theorem II.4.7 of [22].

**Lemma 2.2.3.** *If $\Delta(\overline{z}_0, \overline{z}_1, ..., \overline{z}_n)$ is a set of formulas that is closed under permutations, $(I, <)$ is a linear order with $|I| > n$, and $\langle \overline{b}_i : i \in I \rangle$ is a $\Delta$-indiscernible sequence that is not a $\Delta$-indiscernible set, then there exists $t < n-1$ and $\delta \in \pm\Delta$ such that*

$$\models \delta(\overline{b}_{i_0}, ..., \overline{b}_{i_t}, \overline{b}_{i_{t+1}}, ..., \overline{b}_{i_n}) \wedge \neg\delta(\overline{b}_{i_0}, ..., \overline{b}_{i_{t-1}}, \overline{b}_{i_{t+1}}, \overline{b}_{i_t}, \overline{b}_{i_{t+2}}, ..., \overline{b}_{i_n})$$

*for some (equivalently all) $i_0 < ... < i_n$ from $I$. In other words, $\delta$ is "order sensitive" at $t$.*

*Proof.* To simplify notation, assume that $0 < 1 < ... < n$ is in $I$ and we show this for $\overline{b}_0, ..., \overline{b}_n$. Since $\langle \overline{b}_i : i \in I \rangle$ is a $\Delta$-indiscernible sequence that is not a $\Delta$-indiscernible set, there exists some $\delta' \in \pm\Delta$ witnessing this fact. That is, $\models \delta'(\overline{b}_0, ..., \overline{b}_n)$ but $\models \neg\delta'(\overline{b}_{\sigma(0)}, ..., \overline{b}_{\sigma(n)})$ for some $\sigma \in S_{n+1}$. However, $S_{n+1}$ as a group is generated by elements of the form $(t\ t{+}1)$ for $t < n$ (i.e., the permutation that is the identity on all of $n+1$ except that it swaps $t$ and $t+1$). So, there exists another $\sigma \in S_{n+1}$ and $t < n$ such that

$$\models \delta'(\overline{b}_{\sigma(0)}, ..., \overline{b}_{\sigma(n)}) \wedge \neg\delta'(\overline{b}_{(\tau\circ\sigma)(0)}, ..., \overline{b}_{(\tau\circ\sigma)(n)})$$

where $\tau = (t\ t{+}1)$. Since $\Delta$ is closed under permutations, if we let $\delta = \delta'_\sigma$ as in (2.1), then we see that

$$\models \delta(\overline{b}_0, ..., \overline{b}_n) \wedge \neg\delta(\overline{b}_{\tau(0)}, ..., \overline{b}_{\tau(n)}),$$

which is exactly what we aimed to show. $\square$

Suppose now that $I = L < \omega$ (so $I$ is finite) and take $n$, $\Delta$, $\delta$, and $t$ as in Lemma 2.2.3. Then we can take the initial $t$ elements and final $n - t - 1$ elements of the $\Delta$-indiscernible sequence $\langle \overline{b}_i : i < L \rangle$ and get that the formula

$$\theta(\overline{y}_0; \overline{y}_1) = \delta(\overline{b}_0, ..., \overline{b}_{t-1}, \overline{y}_0, \overline{y}_1, \overline{b}_{L-n+t+1}, ..., \overline{b}_{L-1})$$

defines the linear order of the sequence $\langle \overline{b}_i : t \leq i \leq L - n + t \rangle$. That is, for all distinct $i, j$ with $t \leq i, j \leq L - n - t$, $\models \theta(\overline{b}_i, \overline{b}_j)$ if and only if $i < j$. A theme of this chapter is the following: creating a definable finite partial order is our main way of obtaining uniform definability of types. However, we should note here that we need a suitable choice of $\Delta$. In particular, if we can possibly hope for a uniform type definition, we need $\Delta$ to be finite.

Fix $\varphi(\overline{x}; \overline{y})$ a dependent partitioned formula with independence dimension $N$. Define $\Delta_{n,\varphi}$, a finite collection of formulas, as follows:

$$\Delta_{n,\varphi}(\overline{z}_0, ..., \overline{z}_n) = \left\{ \exists\overline{x} \left( \bigwedge_{i \leq n} \varphi(\overline{x}; \overline{z}_i)^{t(i)} \right) : t \in {}^{n+1}2 \right\}. \tag{2.2}$$

Notice that $\Delta_{n,\varphi}$ is closed under permutations. We now show that it suffices to consider $\Delta_{N,\varphi}$ to break up the finite index set $I$ into boundedly many pieces that are constant on $\varphi(\overline{a};\overline{y})$ for any choice of $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$.

**Lemma 2.2.4** (Theorem II.4.13 of [22]). *Let $\Delta = \Delta_{N,\varphi}$ (where $N = \mathrm{ID}(\varphi)$), $(I,<)$ be a linear order, $\langle \overline{b}_i : i \in I\rangle$ be a $\Delta$-indiscernible sequence, and $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$.*

*(i) There exists $K \leq N+1$ $<$-convex subsets of $I$, $I_0, ..., I_{K-1}$, such that, for all $i \in I$, $\models \varphi(\overline{a};\overline{b}_i)$ if and only if $i \in I_\ell$ for some $\ell < K$.*

*(ii) If $\langle \overline{b}_i : i \in I\rangle$ be a $\Delta$-indiscernible set and $|I| \geq 2N+1$, then there exists $I_0 \subseteq I$ with $|I_0| \leq N$ and $t < 2$ such that, for all $i \in I$, $\models \varphi(\overline{a};\overline{b}_i)^t$ if and only if $i \in I_0$.*

This is due to Shelah, but we sketch the proof here for completeness.

*Proof.* (i): If not, then there exists $i_0 < ... < i_{2N+1}$ from $I$ such that $\models \varphi(\overline{a};\overline{b}_{i_\ell})$ if and only if $\ell$ is odd. By $\Delta$-indiscernibility, for any $t \in {}^{N+1}2$,

$$\models \exists \overline{x} \bigwedge_{\ell \leq N} \varphi(\overline{x};\overline{b}_{i_\ell})^{t(\ell)}.$$

This contradicts the fact that $N = \mathrm{ID}(\varphi)$.

(ii): If not, then there exists distinct $i_0, ..., i_{2N+1} \in I$ such that $\models \varphi(\overline{a};\overline{b}_{i_\ell})$ if and only if $\ell$ is odd. The proof follows as in (i). □

We now introduce one more lemma before proving Theorem 2.2.1. This is proved exactly like Lemma 2.3.5, which in turn is proved like Theorem II.2.12 (1) of [22].

**Lemma 2.2.5.** *Fix $\varphi(\overline{x};\overline{y})$ a partitioned formula and $\{\psi_\ell(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) : \ell < L\}$ a finite collection of formulas such that, for all finite indiscernible sequences $\langle \overline{b}_i : i \in I\rangle$ and for all $p(\overline{x}) \in S_\varphi(\{\overline{b}_i : i \in I\})$, there exists $\ell < L$ and $i_0,...,i_{n-1} \in I$ such that $\psi_\ell(\overline{y};\overline{b}_{i_0},...,\overline{b}_{i_{n-1}})$ defines $p$. Then, $\varphi$ has UDTFIS.*

We are now ready to prove the main theorem. Fix any partitioned formula $\varphi(\overline{x};\overline{y})$ (not necessarily dependent).

*Proof of Theorem 2.2.1.* (i): Suppose first that $\varphi(\overline{x};\overline{y})$ is unstable, so $\varphi$ has the order property. By compactness and the Erdös-Rado Theorem, there exists an infinite indiscernible sequence in $\overline{y}$ and another infinite sequence in $\overline{x}$ witnessing the order property. By compactness, there exists an indiscernible sequence $\langle \overline{b}_q : q \in \mathbb{Q}\rangle$ and a set $\{\overline{a}_r : r \in \mathbb{R}\}$ such that $\models \varphi(\overline{a}_r;\overline{b}_q)$ if and only if $r < q$. Suppose, by means of contradiction, that there exists a formula $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ witnessing UDTIS. Then, for each $r \in \mathbb{R}$, there exists $q_0,...,q_{n-1} \in \mathbb{Q}$ such that $\psi(\overline{y};\overline{b}_{q_0},...,\overline{b}_{q_{n-1}})$ defines the type $\mathrm{tp}_\varphi(\overline{a}_r/\{\overline{b}_q : q \in \mathbb{Q}\})$. However, for different $r \in \mathbb{R}$, these types are different. Hence $|\mathbb{R}| = |\mathbb{Q}^n|$, a contradiction.

Conversely, suppose $\varphi$ is stable. By Theorem 1.3.2 (Theorem II.2.12 of [22]), there exists a formula $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ witnessing uniform definability of $\varphi$-types over all sets. Hence, this $\psi$ also witnesses that $\varphi$ has UDTIS.

(ii): Assume that $\varphi$ is independent. By compactness and the Erdös-Rado Theorem, there exists an indiscernible sequence $\langle \overline{b}_i : i < \omega\rangle$ such that the set $\{\overline{b}_i : i < \omega\}$ is $\varphi$-independent. Suppose, by means of contradiction, that there exists a formula $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ witnessing UDTFIS. Then, for each $L < \omega$, the number of $\varphi$-types over $\{\bar{b}_i : i < L\}$ is bounded by $L^n$. On the other hand, since $\{\bar{b}_i : i < L\}$ is $\varphi$-independent, the number of $\varphi$-types is exactly equal to $2^L$. Therefore, $2^L \leq L^n$. However, since $n$ is fixed and $L < \omega$ is arbitrary, this is a contradiction.

Conversely, assume that $\varphi$ is dependent. Let $N = \mathrm{ID}(\varphi)$ and let $\Delta = \Delta_{N,\varphi}$ as in (2.2). We now define finitely many formulas $\psi_\ell(\overline{y}; \overline{\mathbf{z}})$ that satisfy the hypotheses of Lemma 2.2.5, showing that $\varphi$ has UDTFIS. First, for each $t < 2$, let

$$\psi_t(\overline{y}; \overline{z}_0, ..., \overline{z}_N) = \left( \bigvee_{i \leq N} \overline{y} = \overline{z}_i \right)^t.$$

Now, for each $t < N$, $\delta \in \pm\Delta$, $K \leq N+1$ and $s \in {}^{N+1}2$, let

$$\theta_{t,\delta}(\overline{y}_0, \overline{y}_1; \overline{z}_0, ..., \overline{z}_N) = \delta(\overline{z}_0, ..., \overline{z}_{t-1}, \overline{y}_0, \overline{y}_1, \overline{z}_{t+2}, ..., \overline{z}_N)$$

and let

$$\begin{aligned}
&\psi_{t,\delta,K,s}(\overline{y}; \overline{z}_0, ..., \overline{z}_N, \overline{w}_0, ..., \overline{w}_{2K-1}) = \\
&\left( \bigvee_{i \leq N} \overline{y} = \overline{z}_i \to \bigvee_{i \leq N, s(i)=1} \overline{y} = \overline{z}_i \right) \wedge \left( \bigvee_{j < 2K} \overline{y} = \overline{w}_j \right) \wedge \\
&\left( \bigwedge_{i \leq N} \overline{y} \neq \overline{z}_i \wedge \bigwedge_{j < 2K} \overline{y} \neq \overline{w}_j \to \bigvee_{j < K} (\theta_{t,\delta}(\overline{w}_{2j}, \overline{y}; \overline{\mathbf{z}}) \wedge \theta_{t,\delta}(\overline{y}, \overline{w}_{2j+1}; \overline{\mathbf{z}})) \right).
\end{aligned}$$

Think of $\theta_{t,\delta}$ as defining the order of the sequence, as in Lemma 2.2.3. Also think of the $\overline{z}_i$ as the boundaries of the indiscernible sequence and think of $\overline{w}_j$ as the endpoints of the convex sets $I_0, ..., I_{K-1}$ given by Lemma 2.2.4 (i). So $\psi_{t,\delta,K,s}$ says that either $\overline{y}$ is equal to an appropriate $\overline{z}_i$ or $\overline{w}_j$, or it falls within one of the convex components $I_j$. We now show that these $\psi_t$ and $\psi_{t,\delta,K,s}$ satisfy the hypotheses of Lemma 2.2.5.

Fix a finite indiscernible sequence $\langle \bar{b}_i : i < L \rangle$, let $B = \{\bar{b}_i : i < L\}$ and fix $p \in S_\varphi(B)$. First, if $L \leq 2N$, we may use $\psi_t$ trivially to define $p$. Second, if

$\langle \overline{b}_i : i < L \rangle$ is a $\Delta$-indiscernible set, then, by Lemma 2.2.4 (ii), there exists $I_0 \subseteq L$ with $|I_0| \leq N$ and $t < 2$ such that, for all $i < L$, $\varphi(\overline{x}; \overline{b}_i)^t \in p(\overline{x})$ if and only if $i \in I_0$. Hence, $\psi_t(\overline{y}; \overline{b}_{i_0}, ..., \overline{b}_{i_N})$ defines $p$, where $I_0 = \{i_0, ..., i_N\}$.

Therefore, we may assume that $\langle \overline{b}_i : i < L \rangle$ is a $\Delta$-indiscernible sequence that is not a $\Delta$-indiscernible set. By Lemma 2.2.3, there exists $t < N$ and $\delta \in \pm\Delta$ such that $\theta_{t,\delta}(\overline{y}_0, \overline{y}_1; \overline{b}_0, ..., \overline{b}_t, \overline{b}_{L-N+t}, ..., \overline{b}_{L-1})$ defines the sequence order for $t \leq i \leq L - N + t$. By Lemma 2.2.4 (i) on the $\Delta$-indiscernible sequence $\langle \overline{b}_i : t \leq i \leq L - N + t \rangle$, there exists $K \leq N + 1$ and $<$-convex subsets of $\{t, ..., L - N + t\}$, $I_0, ..., I_{K-1}$ such that, for all $i$ with $t \leq i \leq L - N + t$, $\varphi(\overline{x}; \overline{b}_i) \in p(\overline{x})$ if and only if $i \in I_j$ for some $j < K$. Let $s \in {}^{N+1}2$ be such that $\varphi(\overline{x}; \overline{b}_i)^{s(i)} \in p(\overline{x})$ for all $i \leq t$ and $\varphi(\overline{x}; \overline{b}_{L-N+i-1})^{s(i)} \in p(\overline{x})$ for all $t < i \leq N$. Let $m_{2j}, m_{2j+1}$ be the endpoints of $I_j$ inclusive. Then, we can see that

$$\psi_{t,\delta,K,s}(\overline{y}; \overline{b}_0, ..., \overline{b}_t, \overline{b}_{L-N+t}, ..., \overline{b}_{L-1}, \overline{b}_{m_0}, ..., \overline{b}_{m_{2K-1}})$$

defines $p$, as desired. □

This proof also shows that if we assume that $\varphi$ is dependent with independence dimension $N$, then there exists a uniform definition of $\varphi$-types over finite $\Delta_{N,\varphi}$-indiscernible sequences. We use finiteness of the sequence in two places: our ability to define the linear order $\theta_{t,\delta}$ and our ability to choose endpoints of the convex sets, $m_{2j}$ and $m_{2j+1}$. Therefore, suitable generalizations can be made for indiscernible sequences with different index sets. For example, there exists a uniform definition of $\varphi$-types over $\Delta_{N,\varphi}$-indiscernible sequences indexed by $\omega + N$ (one needs to add a few definitions for the case where the truth value is cofinal in $\omega$). This clearly cannot generalize to all index sets (for example, if $\varphi$ is unstable, we see from the proof that the index set $\mathbb{Q}$ does not work).

This result relies heavily on our ability to define the linear order of the sequence. As the remainder of this chapter demonstrates, we can prove that a formula has UDTFS if we can come up with a similar notion of a definable order. For example, in Section 2.4, we use the ordering $\leq_p$ to produce a uniform definition. Is there a generalization of linear order that holds for dependent theories and can be used to prove that all dependent theories have UDTFS?

## 2.3 Basic Properties of UDTFS

In this section, we discuss basic properties of UDTFS. Some of these properties were worked out by Laskowski and Johnson in [15] and that is noted when we present them. We include proofs of these for completeness. Recall Definition 2.1.1, $\varphi(\overline{x};\overline{y})$ has *UDTFS* if there exists $\psi(\overline{y};\overline{\mathbf{z}})$ such that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $p \in S_\varphi(B)$, there exists $\overline{\mathbf{c}} \in B^n$ such that $\psi(\overline{y};\overline{\mathbf{c}})$ defines $p$. In this case, we say that $\psi$ is a *uniform definition of $\varphi$-types over finite sets.*

First, note that UDTFS is a property of the theory. This is because, for each formula $\varphi(\overline{x};\overline{y})$, the fact that $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ is a uniform definition for finite $\varphi$-types is expressible as the following collection of sentences: $\{\sigma_K : K < \omega\}$ where

$$\sigma_K = \forall \overline{y}_0 ... \overline{y}_K \forall \overline{x} \bigvee_{i \in {}^n K} \left( \bigwedge_{k<K} \varphi(\overline{x};\overline{y}_k) \leftrightarrow \psi(\overline{y}_k;\overline{y}_{i(0)},...,\overline{y}_{i(n-1)}) \right).$$

Second, note that UDTFS is dependent on the way we partition a formula. As before, if we let $\psi(x;y)$ be an independent formula, and let $\varphi(x,y,z) = (x = y) \wedge$

$\psi(y,z)$, then we see that $\varphi(x;y,z)$ has UDTFS but $\varphi(x,y;z)$ does not. It is still open whether or not the reduct of the a theory with UDTFS still has UDTFS; one could accidentally "throw out" the definition when taking reducts.

The following lemma follows easily by the arguments in [22]:

**Lemma 2.3.1.** *If $\varphi(\overline{x};\overline{y})$ and $\psi(\overline{x};\overline{z})$ have UDTFS, then the following formulas also have UDTFS:*

*(i)* $(\varphi \wedge \psi)(\overline{x};\overline{y},\overline{z}) = \varphi(\overline{x};\overline{y}) \wedge \psi(\overline{x};\overline{z})$.

*(ii)* $(\neg\varphi)(\overline{x};\overline{y}) = \neg\varphi(\overline{x};\overline{y})$.

*In other words, the class of formulas that have UDTFS and have the same free variables $\overline{x}$ is closed under boolean combinations.*

*Proof.* Fix $\varphi(\overline{x};\overline{y})$ and $\psi(\overline{x};\overline{z})$ with UDTFS, witnessed by $\gamma_\varphi(\overline{y};\overline{w}_0,...,\overline{w}_n)$ and $\gamma_\psi(\overline{z};\overline{v}_0,...,\overline{v}_n)$ respectively. Then, notice that

$$(\gamma_\varphi \wedge \gamma_\psi)(\overline{y},\overline{z};\overline{w}_0,\overline{v}_0,...,\overline{w}_n,\overline{v}_n) = \gamma_\varphi(\overline{y};\overline{\mathbf{w}}) \wedge \gamma_\psi(\overline{z};\overline{\mathbf{v}})$$

is a uniform definition of $(\varphi \wedge \psi)$-types over finite sets. Similarly, $\neg\gamma_\varphi(\overline{y};\overline{\mathbf{w}})$ is a uniform definition of $(\neg\varphi)$-types over finite sets. □

The next proposition follows by definition and Theorem 1.3.2 (also see Theorem II.2.12 of [22]):

**Proposition 2.3.2.** *If $\varphi(\overline{x};\overline{y})$ is a stable formula, then $\varphi$ has UDTFS. Thus, stable theories have UDTFS.*

The following proposition puts UDTFS between stability and dependence. It is due to Chris Laskowski (unpublished).

**Proposition 2.3.3.** *If a partitioned formula $\varphi(\overline{x};\overline{y})$ has UDTFS, then $\varphi$ is dependent.*

*Proof.* If $\varphi(\overline{x};\overline{y})$ has UDTFS, then it certainly has UDTFIS (as defined in Section 2.2). Therefore, by Theorem 2.2.1 (ii) ($\Leftarrow$), $\varphi$ is dependent. $\square$

It is still open whether or not all dependent formulas have UDTFS or even if all dependent theories have UDTFS. This is known as the *UDTFS Conjecture*, and was first proposed by Laskowski (unpublished). Laskowski classifies this as an "open question" and not a conjecture, but this author will go out on a limb:

**Conjecture 2.3.4** (UDTFS Conjecture)**.** *For a partitioned formula $\varphi(\overline{x};\overline{y})$, $\varphi$ is dependent if and only if $\varphi$ has UDTFS.*

We discuss the implications of this conjecture and the evidence for it at the end of this section.

The next lemma shows that we do not need a single uniform definition of $\varphi$-types over finite sets; it suffices to have a fixed finite number of them instead. This simplifies showing that formulas and theories have UDTFS. This is essentially due to Shelah in the proof of Theorem II.2.12 (1) in [22], where he shows it for standard definability of types.

**Lemma 2.3.5.** *Fix $\varphi(\overline{x};\overline{y})$ a partitioned formula and $\{\psi_\ell(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) : \ell < L\}$ a finite collection of formulas such that, for all finite non-empty $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and for* *all $p(\overline{x}) \in S_\varphi(B)$, there exists $\ell < L$ and $\overline{c}_0, ..., \overline{c}_{n-1} \in B$ such that $\psi_\ell(\overline{y}; \overline{c}_0, ..., \overline{c}_{n-1})$ defines $p$. Then, $\varphi$ has UDTFS.*

*Proof.* Let $\varphi$ and $\psi_\ell$ for $\ell < L$ be given as in the hypothesis. Consider the following formula:

$$\psi(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1}, \overline{w}, \overline{v}_0, ..., \overline{v}_{L-1}) = \bigwedge_{\ell < L} (\overline{w} = \overline{v}_\ell \to \psi_\ell(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1}).$$

We claim that this is a uniform definition of $\varphi$-types over finite sets, showing that $\varphi$ has UDTFS. Fix any finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| \geq 2$ and any $p \in S_\varphi(B)$. By the hypothesis, there exists $\ell < L$ and $\overline{\mathbf{c}} \in B^n$ such that $\psi_\ell(\overline{y}; \overline{\mathbf{c}})$ defines $p$. Fix any $\overline{b} \neq \overline{b}'$ from $B$ (this is where we use the hypothesis that $|B| \geq 2$) and let $\overline{b}_i = \overline{b}$ for all $i < L$, $i \neq \ell$ and let $\overline{b}_\ell = \overline{b}'$. Then, the following defines $p$:

$$\psi(\overline{y}; \overline{\mathbf{c}}, \overline{b}', \overline{b}_0, ..., \overline{b}_{L-1}).$$

□

Now we exhibit another lemma that reduces the difficulty in showing that a theory has UDTFS.

**Lemma 2.3.6** (Sufficiency of a single variable)**.** *A theory $T$ has UDTFS if and only if all formulas of the form $\varphi(x; \overline{y})$ have UDTFS (where* $\lg(x) = 1$*).*

*Proof.* One direction is trivial, so suppose that all formulas of the form $\varphi(x; \overline{y})$ have UDTFS. We show that a formula $\varphi(\overline{x}; \overline{y})$ has UDTFS by induction on $n = \lg(\overline{x})$. Of course, $n = 1$ is taken care of by assumption, so suppose $n > 1$.

Consider the repartitioned formula $\hat{\varphi}(x_0, ..., x_{n-2}; x_{n-1}, \overline{y}) = \varphi(\overline{x}; \overline{y})$. Since $\hat{\varphi}$ has only $n - 1$ free variables, by induction hypothesis, there exists a uniform definition of $\hat{\varphi}$-types over finite sets, say $\psi(x_{n-1}, \overline{y}; w_0, \overline{z}_0, ..., w_{k-1}, \overline{z}_{k-1})$ (where $\lg(\overline{z}_i) = \lg(\overline{y})$ for all $i < k$). Now let

$$\psi^*(x_n; \overline{y}, \overline{z}_0, ..., \overline{z}_{k-1}) = \psi(x_{n-1}, \overline{y}; x_{n-1}, \overline{z}_0, ..., x_{n-1}, \overline{z}_{k-1})$$

(where we substitute $x_{n-1}$ for each $w_i$ and repartition). Since this has only one free variable, by hypothesis, there exists a uniform definition of $\psi^*$-types over finite sets, say

$$\gamma(\overline{y}, \overline{z}_0, ..., \overline{z}_{k-1}; \overline{v}_0, \overline{u}_{0,0}, ..., \overline{u}_{0,k-1}, ..., \overline{v}_{\ell-1}, \overline{u}_{\ell-1,0}, ..., \overline{u}_{\ell-1,k-1})$$

(so each $(\overline{v}_j, \overline{u}_{j,0}, ..., \overline{u}_{j,k-1})$ corresponds to $(\overline{y}, \overline{z}_0, ..., \overline{z}_{k-1})$). Finally, let

$$\gamma^*(\overline{y}; \overline{z}_0, ..., \overline{z}_{k-1}, \overline{v}_0, ..., \overline{v}_{\ell-1}) =$$

$$\gamma(\overline{y}, \overline{z}_0, ..., \overline{z}_{k-1}; \overline{v}_0, \overline{z}_0, ..., \overline{z}_{k-1}, ..., \overline{v}_{\ell-1}, \overline{z}_0, ..., \overline{z}_{k-1})$$

(so we substitute $\overline{z}_i$ for each $\overline{u}_{j,i}$ and repartition). We claim that $\gamma^*$ is a uniform definition of $\varphi$-types over finite sets, completing the proof.

Fix $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite and fix $p \in S_\varphi(B)$. Let $a_i \in \mathfrak{C}$ be such that $(a_0, ..., a_{n-1}) \models p$. Consider the $\hat{\varphi}$-type

$$\hat{p}(x_0, ..., x_{n-2}) = \mathrm{tp}_{\hat{\varphi}}(a_0, ..., a_{n-2}/a_{n-1}{}^\frown B)$$

where $a_{n-1}{}^\frown B = \{(a_{n-1}, \overline{b}) : \overline{b} \in B\}$. As $\psi$ is a uniform definition of $\hat{\varphi}$-types over finite sets, there exists some $(a_{n-1}, \overline{c}_0), ..., (a_{n-1}, \overline{c}_{k-1}) \in a_{n-1}{}^\frown B$ (i.e., $\overline{c}_0, ..., \overline{c}_{k-1} \in$

$B$) such that

$$\text{for all } \overline{b} \in B,\ \varphi(\overline{x};\overline{b}) \in p \text{ if and only if}$$

$$\hat{\varphi}(x_0, ..., x_{n-2}; a_{n-1}, \overline{b}) \in \hat{p}(x_0, ..., x_{n-2}) \text{ if and only if}$$

$$\models \psi(a_{n-1}, \overline{b}; a_{n-1}, \overline{c}_0, ..., a_{n-1}, \overline{c}_{k-1}) \text{ if and only if}$$

$$\models \psi^*(a_{n-1}; \overline{b}, \overline{c}_0, ..., \overline{c}_{k-1}).$$

Now consider the $\psi^*$-type $q(x_{n-1}) = \mathrm{tp}_{\psi^*}(a_{n-1}/B^\frown(\overline{c}_0, ..., \overline{c}_{k-1}))$. As $\gamma$ is a uniform definition of $\psi^*$-types over finite sets, there exists some $\overline{d}_0, ..., \overline{d}_{\ell-1} \in B$ such that

$$\text{for all } \overline{b} \in B,\ \models \psi^*(a_{n-1}; \overline{b}, \overline{c}_0, ..., \overline{c}_{k-1}) \text{ if and only if}$$

$$\gamma(\overline{b}, \overline{c}_0, ..., \overline{c}_{k-1}; \overline{d}_0, \overline{c}_0, ..., \overline{c}_{k-1}, ..., \overline{d}_{\ell-1}, \overline{c}_0, ..., \overline{c}_{k-1}) \text{ if and only if}$$

$$\gamma^*(\overline{b}; \overline{c}_0, ..., \overline{c}_{k-1}, \overline{d}_0, ..., \overline{d}_{\ell-1}).$$

If we string all of these equivalent conditions together, we see that

$$\gamma^*(\overline{y}; \overline{c}_0, ..., \overline{c}_{k-1}, \overline{d}_0, ..., \overline{d}_{\ell-1})$$

defines $p$, as desired. □

We used nothing about finiteness, so this provides a new proof for the sufficiency of a single variable for stable formulas (i.e., Theorem 1.2.4 (i) above). If the UDTFS Conjecture holds, this provides another proof of the sufficiency of a single variable for dependent formulas as well (i.e., Theorem 1.2.4 (ii) above). Take note of the fact that, if $\hat{\varphi}$ has a finite type definition with $k$ parameter tuples and $\psi^*$ has a finite type definition with $\ell$ parameter tuples, then $\varphi$ has a finite type definition of $k + \ell$ tuples. We use this fact to prove Theorem 3.2.4 below.

Lemma 2.3.6 is used to prove UDTFS for a theory whose formulas with one free variable are well understood. For example, we use it to show that all dp-minimal theories have UDTFS (see Theorem 2.4.1 below).

The next two sections provide more examples of theories and formulas that have UDTFS. Using the methods of these proofs as a template, we suggest potential methods for proving the UDTFS Conjecture and provide more evidence for it. As we see in Section 2.7, the UDTFS Conjecture implies the Warmuth Conjecture (Conjecture 2.7.7 below). However, our main motivation is model-theoretic: The UDTFS Conjecture would provide a ideal generalization of uniform definability of types to dependent formulas.

## 2.4 dp-Minimal Theories have UDTFS

In this section, we prove the following theorem:

**Theorem 2.4.1.** *If $T$ is dp-minimal, then $T$ has UDTFS.*

This has the following corollary:

**Corollary 2.4.2.** *The following theories have UDTFS:*

*(i)* $T = \mathrm{Th}(\mathbb{Q}_p; +, \cdot, 0, 1)$ *(the theory of the p-adic field).*

*(ii)* $T = \mathrm{Th}(\mathbb{Z}; +, <)$ *(the theory of Presburger arithmetic).*

*(iii) Any VC-minimal $T$ (see Definition 4.1.1 below).*

*(iv) Any VC-density one theory $T$ (see Definition 3.2.5 below).*

(v) *Any o-minimal or weakly o-minimal theory $T$ (originally due to Johnson and Laskowski in [15]).*

Along the way, we also provide a proof of the following theorem:

**Theorem 2.4.3.** *If $\varphi(\overline{x}; \overline{y})$ is any formula and $N$ is a positive integer such that, for all $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| = N$, we have that $|S_\varphi(B)| \leq \frac{N(N+1)}{2}$, then $\varphi$ has UDTFS.*

Therefore, if $\varphi$ has VC-density $< 2$, then $\varphi$ has UDTFS. One should note that the bound given by the theorem, $N(N+1)/2$, is exactly one less than the maximum for independence dimension two given by Sauer's Lemma. We discuss this more in Subsection 2.7.2.

Recall the definition of dp-minimality (Definition 1.2.10). An *ICT-pattern* is a pair of formulas $\varphi(x; \overline{y})$ and $\psi(x; \overline{z})$ together with two sequences $\langle \overline{b}_i : i < \omega \rangle$ and $\langle \overline{c}_j : j < \omega \rangle$ such that, for all $i_0, j_0 < \omega$, the following type is consistent:

$$\{\varphi(x; \overline{b}_{i_0}), \psi(x; \overline{c}_{j_0})\} \cup \{\neg\varphi(x; \overline{b}_i) : i < \omega, i \neq i_0\} \cup \{\neg\psi(x; \overline{c}_j) : j < \omega, j \neq j_0\}.$$

Therefore, $T$ is dp-minimal if and only if there exists no ICT-pattern. For our purposes, ICT-patterns do not suffice. Instead, we look at a notion we call a TP-pattern.[1]

**Definition 2.4.4.** A *TP-pattern* is a formula $\varphi(x; \overline{y})$ with a single free variable $x$ together with a sequence of $\lg(\overline{y})$-tuples $\langle \overline{b}_i : i < \omega \rangle$ such that, for all $\ell < k < \omega$, the

[1]This author created this terminology for [11], intending TP to stand for *triangle pattern*, without realizing that the word *pattern* was used twice.

following formula holds:

$$\exists x \left( \varphi(x;\bar{b}_k) \wedge \varphi(x;\bar{b}_\ell) \wedge \bigwedge_{i<k, i\neq \ell} \neg\varphi(x;\bar{b}_i) \right).$$

We now show that having a TP-pattern is equivalent to having a ICT-pattern, providing a new characterization of dp-minimality and simplifying our proof of Theorem 2.4.1.

**Proposition 2.4.5.** *A theory $T$ is dp-minimal if and only if $T$ has no TP-pattern.*

*Proof.* Suppose first that $T$ has a TP-pattern, say $\varphi(x;\bar{y})$ and $\langle \bar{b}_i : i < \omega \rangle$. Define $\psi$ as follows:

$$\psi(x;\bar{y}_0,\bar{y}_1) = (\varphi(x;\bar{y}_0) \leftrightarrow \varphi(x;\bar{y}_1))$$

and let $K$ be any positive integer. By Ramsey's Theorem, we may assume that $\langle \bar{b}_i : i < \omega \rangle$ is $\Delta$-indiscernible for $\Delta = \Delta_{4K,\varphi}$ as in (2.2) (from Section 2.2). By the definition of a TP-pattern, the following is consistent:

$$\{\neg\varphi(x;\bar{b}_i) : i < 2K\} \cup \{\varphi(x;\bar{b}_{2K})\} \cup \{\neg\varphi(x;\bar{b}_i) : 2K < i \leq 6K\} \cup \{\varphi(x;\bar{b}_{6K+1})\}.$$

Let $a$ realize this type. By pigeon-hole principal, there exists some $t < 2$ such that, for infinitely many $i > 6K+1$, $\models \varphi(a;\bar{b}_i)^t$. By replacing $\langle \bar{b}_i : i < \omega \rangle$ with a subsequence, we may assume that $\models \varphi(a;\bar{b}_i)^t$ for *all* $i > 6K+1$ (notice that being $\Delta$-indiscernible is closed under subsequence). Therefore, we have that the following is consistent, witnessed by $a$:

$$\{\psi(x;\bar{b}_{2i},\bar{b}_{2i+1}) : i < K\} \cup \{\neg\psi(x;\bar{b}_{2K},\bar{b}_{2K+1})\} \cup \{\psi(x;\bar{b}_{2i},\bar{b}_{2i+1}) : K < i < 3K\} \cup$$
$$\{\neg\psi(x;\bar{b}_{6K},\bar{b}_{6K+1})\} \cup \{\psi(x;\bar{b}_{2i},\bar{b}_{2i+1}) : 3K < i < 4K\}.$$

By $\Delta$-indiscernibility of $\langle \overline{b}_i : i < \omega \rangle$, we have that, for all $\ell < K$ and $K \leq k < 2K$, the following is consistent:

$$\{\psi(x; \overline{b}_{2i}, \overline{b}_{2i+1}) : i < K, i \neq \ell\} \cup \{\neg\psi(x; \overline{b}_{2\ell}, \overline{b}_{2\ell+1})\} \cup$$

$$\{\psi(x; \overline{b}_{2i}, \overline{b}_{2i+1}) : K \leq i < 2K, i \neq k\} \cup \{\neg\psi(x; \overline{b}_{2k}, \overline{b}_{2k+1})\}.$$

Since $K < \omega$ was arbitrary, by compactness there exists $\overline{c}_i, \overline{d}_j \in \mathfrak{C}^{\lg(\overline{y})}$ for all $i, j < \omega$ such that, for all $\ell, k < \omega$, the following is consistent:

$$\{\psi(x; \overline{c}_{2i}, \overline{c}_{2i+1}) : i < \omega, i \neq \ell\} \cup \{\neg\psi(x; \overline{c}_{2\ell}, \overline{c}_{2\ell+1})\} \cup$$

$$\{\psi(x; \overline{d}_{2i}, \overline{d}_{2i+1}) : i < \omega, i \neq k\} \cup \{\neg\psi(x; \overline{d}_{2k}, \overline{d}_{2k+1})\}.$$

Then $\neg\psi$, $\neg\psi$, $\langle (\overline{c}_{2i}, \overline{c}_{2i+1}) : i < \omega \rangle$, and $\langle (\overline{d}_{2i}, \overline{d}_{2i+1}) : i < \omega \rangle$ form an ICT-pattern. Thus, $T$ is not dp-minimal.

Conversely, suppose that $T$ is not dp-minimal and let $\varphi(x; \overline{y})$, $\psi(x; \overline{z})$, $\langle \overline{b}_i : i < \omega \rangle$, and $\langle \overline{c}_j : j < \omega \rangle$ be a ICT-pattern witnessing this. Then define $\theta$ as follows:

$$\theta(x; \overline{y}, \overline{z}) = \neg(\varphi(x; \overline{y}) \leftrightarrow \psi(x; \overline{z})).$$

Then one easily checks that $\theta$ together with $\langle (\overline{b}_i, \overline{c}_i) : i < \omega \rangle$ form a TP-pattern. $\square$

Compactness together with Proposition 2.4.5 yields the following result:

**Lemma 2.4.6.** *Fix $T$ a dp-minimal theory. For all $\varphi(x; \overline{y})$, there exists $K < \omega$ such that, for all $\langle \overline{b}_i : i < K \rangle$ with $\overline{b}_i \in \mathfrak{C}^{\lg(\overline{y})}$ for all $i$, we have that the following two conditions hold:*

*(i) There exists $\ell < k < K$ such that*

$$\models \neg\exists x \left( \varphi(x; \overline{b}_k) \wedge \varphi(x; \overline{b}_\ell) \wedge \bigwedge_{i < k, i \neq \ell} \neg\varphi(x; \overline{b}_i) \right).$$

*(ii) There exists $\ell < k < K$ such that*

$$\models \neg \exists x \left( \neg\varphi(x; \bar{b}_k) \wedge \neg\varphi(x; \bar{b}_\ell) \wedge \bigwedge_{i<k, i\neq\ell} \varphi(x; \bar{b}_i) \right).$$

With TP-patterns defined, we move on to decision processes. Fix a partitioned formula $\varphi(\overline{x}; \overline{y})$. Fix $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite and $p \in S_\varphi(B)$. Let $\delta \in {}^B 2$ be the function associated to $p$ (i.e., $\varphi(\overline{x}; \overline{b})^{\delta(\overline{b})} \in p(\overline{x})$ for all $\overline{b} \in B$).

**Definition 2.4.7.** For any $\varphi$-type $q(\overline{x})$ and any $\overline{b} \in B$, we say that $q$ *decides* $\varphi(\overline{x}; \overline{b})$ if either $q(\overline{x}) \vdash \varphi(\overline{x}; \overline{b})$ or $q(\overline{x}) \vdash \neg\varphi(\overline{x}; \overline{b})$. We say that $q$ *decides* $\varphi(\overline{x}; \overline{b})$ *correctly* (*with respect to* $p$) if $q(\overline{x}) \vdash \varphi(\overline{x}; \overline{b})^{\delta(\overline{b})}$ (i.e., $q$ decides $\varphi(\overline{x}; \overline{b})$ and it implies the instance of $\pm\varphi(\overline{x}; \overline{b})$ contained in $p$).

Notice that $q$ need not be a $\varphi$-subtype of $p$. In fact, when it is, the following lemma is immediate since $p$ is consistent:

**Lemma 2.4.8.** *For any $\varphi$-subtype $q \subseteq p$ and any $\overline{b} \in B$, if $q$ decides $\varphi(\overline{x}; \overline{b})$ then it does so correctly.*

For any subsets $B_1 \subseteq B_0 \subseteq B$, recall the definitions of $p_{B_0}$ and $p_{B_0,B_1}$ from Section 1.2 ($p_{B_0}$ is the restriction of $p$ to $B_0$ and $p_{B_0,B_1}$ is $p_{B_0}$ perturbed by $B_1$).

**Definition 2.4.9.** Fix $B_0 \subseteq B$ and $\overline{b} \in B$. We say that $B_0$ $*$*-decides* $\varphi(\overline{x}; \overline{b})$ if $p_{B_0}$ decides $\varphi(\overline{x}; \overline{b})$ or there exists $\overline{b}_0 \in B_0$ such that $p_{B_0,\{\overline{b}_0\}}$ is consistent and decides $\varphi(\overline{x}; \overline{b})$. We say that $B_0$ $*$*-decides* $\varphi(\overline{x}; \overline{b})$ *correctly* if $B_0$ $*$-decides $\varphi(\overline{x}; \overline{b})$ and we have that one of the following holds:

(i) $p_{B_0}$ decides $\varphi(\overline{x}; \overline{b})$ (hence correctly by Lemma 2.4.8), or

(ii) For all $\bar{b}_0 \in B_0$ such that $p_{B_0,\{\bar{b}_0\}}$ is consistent and decides $\varphi(\overline{x};\overline{b})$, $p_{B_0,\{\bar{b}_0\}}$ decides $\varphi(\overline{x};\overline{b})$ correctly.

So $B_0$ $*$-decides $\varphi(\overline{x};\overline{b})$ if there exists a perturbation of $p_{B_0}$ of size at most one that decides $\varphi(\overline{x};\overline{b})$. By Lemma 2.4.8, if $p_{B_0}$ decides $\varphi(\overline{x};\overline{b})$, then it does so correctly. Therefore, the only way $B_0$ would $*$-decide $\varphi(\overline{x};\overline{b})$ incorrectly is if $p_{B_0}$ does not decide $\varphi(\overline{x};\overline{b})$ and, for some $\bar{b}_0 \in B_0$, $p_{B_0,\{\bar{b}_0\}}$ is consistent and decides $\varphi(\overline{x};\overline{b})$ incorrectly. That is, $p_{B_0,\{\bar{b}_0\}} \vdash \neg\varphi(\overline{x};\overline{b})^{\delta(\bar{b})}$.

One can generalize $*$-decides to perturbations of size at most $\ell$, called $*_\ell$-decides (so $*$-decides is $*_1$-decides). However, the Making Correct Decisions Lemma (Lemma 2.4.11 below) does not hold for any perturbation of size greater than 1. Still, somehow modifying the remaining argument to work for $*_\ell$-decisions may lead to a proof for the UDTFS Conjecture.

The concept of $*$-decides captures one possible way of constructing an algorithm to define the $\varphi$-type $p$. If we can construct, in a uniform manner, a small collection of small subsets of $B$ which, when chosen in a certain order, $*$-decides $\varphi(\overline{x};\overline{b})$ correctly for all $\overline{b} \in B$, we can get a uniform definition of $\varphi$-types over finite sets. We now show how to construct such a collection.

As orderings aided us in proving UDTFIS for dependent formulas in Section 2.2, we define a quasi-ordering on $\mathcal{P}(B)$, the powerset of $B$, as follows:

$$\text{For } B_0, B_1 \in \mathcal{P}(B), \text{ let } B_0 \leq_p B_1 \text{ if } p_{B_0}(\overline{x}) \vdash p_{B_1}(\overline{x}).$$

We say that $B_0$ is *p-equivalent* to $B_1$, denoted $B_0 \equiv_p B_1$, if $B_0 \leq_p B_1$ and $B_1 \leq_p B_0$. Clearly $\equiv_p$ is an equivalence relation on $\mathcal{P}(B)$ and $\leq_p$ is a partial ordering on

$\mathcal{P}(B)/\equiv_p$. In fact, $B_0 \leq_p B_1$ if and only if the set of realizations of $p_{B_0}$ is contained in $p_{B_1}$, so $B_0 \equiv_p B_1$ if and only if the set of realizations of $p_{B_0}$ equals the set of realizations of $p_{B_1}$. We say that $B_0 <_p B_1$ if $B_0 \leq_p B_1$ but $B_1 \not\leq_p B_0$. For completeness, $p_\emptyset$ is the empty $\varphi$-type, which is realized by all $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$. Therefore, $B_0 \leq_p \emptyset$ for all $B_0 \in \mathcal{P}(B)$ and $\emptyset \leq_p B_0$ if and only if $p_{B_0}$ is realized by all elements of $\mathfrak{C}^{\lg(\overline{x})}$. Now consider the following lemma, which is immediate from the definitions:

**Lemma 2.4.10.** *For the quasi-ordering $\leq_p$, the following hold:*

*(i) For all $B_0, B_1 \in \mathcal{P}(B)$, $B_0 \subseteq B_1$ implies that $B_1 \leq_p B_0$.*

*(ii) For all $B_0, B_1 \in \mathcal{P}(B)$ and all $B_0' \subseteq B_0$, $B_0 \leq_p B_1$ if and only if $B_0 \leq_p B_1 \cup B_0'$.*

*(iii) If $\mathcal{B} \subseteq \mathcal{P}(B)$ and $B_1 \in \mathcal{B}$, then there exists a $B_0 \leq_p B_1$ such that $B_0 \in \mathcal{B}$ and, for all other $B_2 \in \mathcal{B}$, $B_2 \leq_p B_0$ implies that $B_2 \equiv_p B_0$ (we call such $B_0$ $\leq_p$-minimal elements of $\mathcal{B}$).*

Notice that (iii) holds because $B$, hence $\mathcal{B}$, is finite. Surprisingly, this is our only use of finiteness in the proof of Theorem 2.4.1. That it, it is our only obstacle for showing general uniform definability of types for dp-minimal theories (since there are unstable dp-minimal theories, this is an unavoidable obstacle).

A great deal of mileage can be obtained by using $\leq_p$-minimal elements. Let $\mathcal{B}$ be any non-empty set of subsets of $B$. Consider the following lemma about correct decisions using $\leq_p$-minimality:

**Lemma 2.4.11** (Making Correct Decisions Lemma)**.** *Fix $\overline{b} \in B$, $B_0 \in \mathcal{B}$ $\leq_p$-minimal in $\mathcal{B}$, and $\overline{b}_0 \in B_0$. If* *(i)* $p_{B_0,\{\bar{b}_0\}}$ *is consistent and decides* $\varphi(\overline{x};\bar{b})$*,*

*(ii)* $p_{B_0}$ *does not decide* $\varphi(\overline{x};\bar{b})$*, and*

*(iii) there exists* $B_1 \leq_p B_0 - \{\bar{b}_0\}$ *such that* $B_1 \cup \{\bar{b}\} \in \mathcal{B}$*,*

*then* $p_{B_0,\{\bar{b}_0\}}$ *decides* $\varphi(\overline{x};\bar{b})$ *correctly.*

This lemma is the driving force behind creating an algorithm for correctly deciding $\varphi(\overline{x};\bar{b})$ for all $\bar{b} \in B$ given only a small amount of data from $p$. It says that, to get correct decisions, we need only find a $\mathcal{B}$ with good closure properties.

*Proof of Lemma 2.4.11.* Since $p_{B_0}$ does not decide $\varphi(\overline{x};\bar{b})$, by definition of $\leq_p$, we have that $B_0 \not\leq_p \{\bar{b}\}$. Therefore, by Lemma 2.4.10 (ii),

$$B_0 \not\equiv_p \big(B_0 - \{\bar{b}_0\}\big) \cup \{\bar{b}\}. \tag{2.3}$$

Now, by means of contradiction, suppose that $p_{B_0,\{\bar{b}_0\}}$ decides $\varphi(\overline{x};\bar{b})$ incorrectly. That is, suppose that

$$p_{(B_0-\{\bar{b}_0\})}(\overline{x}) \cup \left\{\neg\varphi(\overline{x};\bar{b}_0)^{\delta(\bar{b}_0)}\right\} \vdash \neg\varphi(\overline{x};\bar{b})^{\delta(\bar{b})}.$$

By contrapositive, we get that $p_{(B_0-\{\bar{b}_0\})\cup\{\bar{b}\}} \vdash p_{\{\bar{b}_0\}}$. Therefore, by definition of $\leq_p$,

$$\big(B_0 - \{\bar{b}_0\}\big) \cup \{\bar{b}\} \leq_p B_0.$$

Combined with (2.3), we get

$$\big(B_0 - \{\bar{b}_0\}\big) \cup \{\bar{b}\} <_p B_0.$$

Using hypothesis (iii), we note that

$$B_1 \cup \{\bar{b}\} \leq_p \big(B_0 - \{\bar{b}_0\}\big) \cup \{\bar{b}\} <_p B_0$$

and $B_1 \cup \{\bar{b}\} \in \mathcal{B}$. This contradicts the $\leq_p$-minimality of $B_0$ in $\mathcal{B}$. $\square$

We now construct a collection of subsets of $B$ for which the Making Correct Decisions Lemma shows that, when chosen in the correct order, these subsets correctly decide $\varphi(\overline{x};\bar{b})$ for all $\bar{b} \in B$. First, notice that everything worked out above for subsets of $B$ works for sequences in $B$ by considering the images of those sequences. With this in mind, we define $\mathcal{B}_n$, a set of sequences from $B$ of length $n$, inductively as follows:

For $n = 1$, let $\mathcal{B}_1 = \{\langle \bar{b} \rangle : \bar{b} \in B \text{ and } \varphi(\overline{x};\bar{b})^t \text{ is consistent for both } t < 2\}$ (i.e., for all $\bar{b} \in B$, $\langle \bar{b} \rangle \in \mathcal{B}_1$ if and only if $\emptyset \not\equiv_p \langle \bar{b} \rangle$ if and only if $p_{\langle \bar{b} \rangle, \langle \bar{b} \rangle}$ is consistent). For $n > 1$, let

$$\mathcal{B}_n = \{\beta^\frown \langle \bar{b} \rangle : \bar{b} \in B, \beta \in \mathcal{B}_{n-1}, \text{ and } \beta \text{ does not } *\text{-decide } \varphi(\overline{x};\bar{b})\}.$$

We state an equivalent definition for $\mathcal{B}_n$ in the following lemma:

**Lemma 2.4.12.** *The sequence $\beta = \langle \bar{b}_0, ..., \bar{b}_{n-1} \rangle \in \mathcal{B}_n$ if and only if*

*(i) for all $\ell < n$, $p_{\beta, \langle \bar{b}_\ell \rangle}$ is consistent, and*

*(ii) for all $\ell < k < n$, $p_{\langle \bar{b}_i : i \leq k \rangle, \langle \bar{b}_\ell, \bar{b}_k \rangle}$ is consistent.*

*Proof.* We prove this by induction on $n$. The case $n = 1$ is clear by definition. So suppose $n > 1$ and let $\beta' = \langle \bar{b}_i : i < n-1 \rangle$ (i.e., $\beta$ restricted to $n-1$).

First suppose that $\beta \in \mathcal{B}_n$. By definition of $\mathcal{B}_n$, we see that $\beta' \in \mathcal{B}_{n-1}$. Therefore, by induction, $p_{\beta', \langle \bar{b}_\ell \rangle}$ is consistent for each $\ell < n-1$ and $p_{\langle \bar{b}_i : i \leq k \rangle, \langle \bar{b}_\ell, \bar{b}_k \rangle}$ is consistent for each $\ell < k < n-1$. However, $\beta'$ does not $*$-decide $\varphi(\overline{x};\bar{b}_{n-1})$ by definition of $\mathcal{B}_n$. Thus, $p_{\beta'} \cup \{\varphi(\overline{x};\bar{b}_{n-1})^t\}$ is consistent for both $t < 2$ and

$p_{\beta',\langle\bar{b}_\ell\rangle} \cup \{\varphi(\bar{x};\bar{b}_{n-1})^t\}$ is consistent for each $\ell < n-1$ and $t < 2$. Therefore, $p_{\beta,\langle\bar{b}_\ell\rangle}$ is consistent for each $\ell < n$ and $p_{\langle\bar{b}_i:i\leq k\rangle,\langle\bar{b}_\ell,\bar{b}_k\rangle}$ is consistent for each $\ell < k < n$.

Conversely, suppose that $p_{\beta,\langle\bar{b}_\ell\rangle}$ is consistent for each $\ell < n$ and $p_{\langle\bar{b}_i:i\leq k\rangle,\langle\bar{b}_\ell,\bar{b}_k\rangle}$ is consistent for each $\ell < k < n$. Clearly this condition is closed downward, so it holds for $\beta'$. Therefore, by induction, $\beta' \in \mathcal{B}_{n-1}$. By means of contradiction, suppose that $\beta'^\frown\langle\bar{b}_{n-1}\rangle \notin \mathcal{B}_n$. This means exactly that $\beta'$ $*$-decides $\varphi(\bar{x};\bar{b}_{n-1})$ by definition of $\mathcal{B}_n$. As above, this implies that either $p_{\beta,\langle\bar{b}_\ell\rangle}$ is inconsistent for some $\ell < n$ or $p_{\beta,\langle\bar{b}_\ell,\bar{b}_{n-1}\rangle}$ is inconsistent for some $\ell < n-1$. This contradicts our assumption. $\square$

We get the following as a corollary:

**Corollary 2.4.13.** *If $\beta = \langle\bar{b}_0, ..., \bar{b}_{n-1}\rangle \in \mathcal{B}_n$, then the following hold:*

*(i) For all $k \leq n$ and all subsequences $\beta_0 \subseteq \beta$ (not necessarily initial sequences) of length $k$, $\beta_0 \in \mathcal{B}_k$.*

*(ii) $\left|S_\varphi(\{\bar{b}_i : i < n\})\right| > \frac{n(n+1)}{2}$.*

*(iii) For all $\bar{b} \in B$, if $\beta$ does not $*$-decide $\varphi(\bar{x};\bar{b})$, then $\beta^\frown\langle\bar{b}\rangle \in \mathcal{B}_{n+1}$.*

*Proof.* (i): This follows from the characterization of $\mathcal{B}_n$ in Lemma 2.4.12. Conditions (i) and (ii) of that lemma are clearly closed under subsequence.

(ii): Again this follows from Lemma 2.4.12. Condition (i) of that lemma yields $n$ new types and condition (ii) of that lemma yields $\binom{n}{2}$ new types. Together with the original $p$, there are at least $1 + n + \binom{n}{2} > n(n+1)/2$ types over $\{\bar{b}_i : i < n\}$.

(iii): This follows from the original definition of $\mathcal{B}_n$. $\square$

We now define, for each $n \geq 1$ and each $\leq_p$-minimal element $\beta$ of $\mathcal{B}_n$, a set $H(\beta)$ of non-empty sequences inductively as follows (note that $H$ stands for *history*, as one can think of it as showing the history of how $\beta$ is "built" up):

For $n = 1$ and any $\leq_p$-minimal element $\beta$ of $\mathcal{B}_1$, let $H(\beta) = \{\beta\}$. For $n > 1$, fix any $\leq_p$-minimal element $\beta$ of $\mathcal{B}_n$. Let $\beta = \langle \bar{b}_0, ..., \bar{b}_{n-1} \rangle$ and, for each $i < n$, let $\beta_i$ be the subsequence of $\beta$ given by

$$\beta_i = \langle \bar{b}_0, ..., \bar{b}_{i-1}, \bar{b}_{i+1}, ..., \bar{b}_{n-1} \rangle.$$

That is, $\beta_i$ is the $(n-1)$-element subsequence of $\beta$ obtained by removing the $i$th element. By Corollary 2.4.13 (i), $\beta_i \in \mathcal{B}_{n-1}$, so by Lemma 2.4.10 (iii), there exists $\beta_i' \in \mathcal{B}_{n-1}$ such that $\beta_i' \leq_p \beta_i$ and $\beta_i'$ is $\leq_p$-minimal. Fix any such choice of $\beta_i'$ for each $i < n$ and let

$$H(\beta) = \bigcup_{i<n} H(\beta_i') \cup \{\beta\}.$$

This defines $H$ on all $\leq_p$-minimal elements of $\mathcal{B}_n$ for each $n < \omega$, as desired. Note that $|H(\beta)|$ is a function only of $\lg(\beta)$. In fact, define $f_H : \omega \to \omega$ as follows: $f_H(0) = 0$ and $f_H(n) = n \cdot f_H(n-1) + 1$ for $n \geq 1$. Then $|H(\beta)| = f_H(\lg(\beta))$. We now show that elements of $H(\beta)$, when chosen in a particular manner, correctly $*$-decide $\varphi(\bar{x}; \bar{b})$.

**Lemma 2.4.14.** *Fix $\bar{b} \in B$, $n < \omega$, and $\beta \in \mathcal{B}_n$ $\leq_p$-minimal. Let $k \leq n$ be minimal such that there exists $\beta' \in H(\beta)$ with $\lg(\beta') = k$ and $\beta'$ $*$-decides $\varphi(\bar{x}; \bar{b})$. Then, any such $\beta'$ $*$-decides $\varphi(\bar{x}; \bar{b})$ correctly.*

*Proof.* First, it suffices to assume that $k = n$ and $\beta' = \beta$ by replacement. We now prove this statement by induction on $n$.

If $n = 1$, then take $\beta \in \mathcal{B}_1$ $\leq_p$-minimal. If $p_\beta$ decides $\varphi(\overline{x}; \overline{b})$, then it does so correctly by Lemma 2.4.8. So assume that $p_\beta$ does not decide $\varphi(\overline{x}; \overline{b})$. Therefore, $\overline{b} \not\equiv_p \emptyset$, so $\langle \overline{b} \rangle \in \mathcal{B}_1$. By Lemma 2.4.11 (Making Correct Decisions), if $p_{\beta,\beta}$ decides $\varphi(\overline{x}; \overline{b})$, then it does so correctly. Therefore, if $\beta$ $*$-decides $\varphi(\overline{x}; \overline{b})$, then it does so correctly.

Suppose now that $n > 1$. By Lemma 2.4.8, we may assume that $p_\beta$ does not decide $\varphi(\overline{x}; \overline{b})$. So assume that $p_{\beta, \langle \overline{b}_\ell \rangle}$ decides $\varphi(\overline{x}; \overline{b})$, where we let $\beta = \langle \overline{b}_0, ..., \overline{b}_{n-1} \rangle$. Consider $\beta'_\ell \leq_p \beta_\ell$ as defined above. We have that $\beta'_\ell \in H(\beta)$ and $\lg(\beta'_\ell) = n - 1 < n$. By minimality of $k = n$, $\beta'_\ell$ does not $*$-decide $\varphi(\overline{x}; \overline{b})$. By Lemma 2.4.13 (iii), $\beta'_\ell {}^\frown \langle \overline{b} \rangle \in \mathcal{B}_n$ (this follows by definition of $\mathcal{B}_n$). These are the exact hypotheses of the Making Correct Decisions Lemma (Lemma 2.4.11). Therefore, $p_{\beta, \langle \overline{b}_\ell \rangle}$ decides $\varphi(\overline{x}; \overline{b})$ correctly. Since $\ell < n$ was arbitrary such that $p_{\beta, \langle \overline{b}_\ell \rangle}$ decides $\varphi(\overline{x}; \overline{b})$, we see that $\beta$ $*$-decides $\varphi(\overline{x}; \overline{b})$ correctly. $\square$

All the pieces are now in place. In the following Theorem, we show how to put the pieces together to get UDTFS. Theorems 2.4.1 and 2.4.3 follow as a corollary.

**Theorem 2.4.15.** *Fix a partitioned formula $\varphi(\overline{x}; \overline{y})$ and a $K < \omega$. Suppose that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $p \in S_\varphi(B)$, the set $\mathcal{B}_K = \emptyset$ where $\mathcal{B}_n$ is defined as above for our choice of $B$ and $p$. Then, $\varphi$ has UDTFS.*

*Proof.* We need to consider $f_H(K)$ many sequences, each of length at most $K$. For each $i < f_H(K)$, we encode sequences of length $m_i \leq K$ as $\mathbf{\overline{z}}_i = (\overline{z}_{0,i}, ..., \overline{z}_{m_i - 1, i})$. Let $s \in {}^{K \times f_H(K)}2$ encode the fact that "$\varphi(x; \overline{z}_{j,i})^{s(j,i)}$ is in our type." There is a first order formula $\theta_{i,s}(\overline{y}; \mathbf{\overline{z}}_i)$ that says: "the sequence encoded by $\mathbf{\overline{z}}_i$ $*$-decides $\varphi(\overline{x}; \overline{y})$

with respect to the type denoted by $s$." For example,

$$\begin{aligned}\theta_{i,s}(\overline{y};\overline{\mathbf{z}}_i) = & \bigvee_{t<2} \forall \overline{x} \left( \bigwedge_{j<m_i} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \rightarrow \varphi(\overline{x};\overline{y})^t \right) \vee \\ & \bigvee_{j_0<m_i} \Bigg( \exists \overline{x} \left( \bigwedge_{j<m_i, j\neq j_0} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \wedge \neg\varphi(\overline{x};\overline{z}_{j_0,i})^{s(j_0,i)} \right) \wedge \bigvee_{t<2} \\ & \forall \overline{x} \left( \bigwedge_{j<m_i, j\neq j_0} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \wedge \neg\varphi(\overline{x};\overline{z}_{j_0,i})^{s(j_0,i)} \rightarrow \varphi(\overline{x};\overline{y})^t \right) \Bigg).\end{aligned}$$

In a similar manner, there exists a formula $\theta^*_{i,s}(\overline{y};\overline{\mathbf{z}}_i)$ that says that "the sequence encoded by $\overline{\mathbf{z}}_i$ $*$-decides $\varphi(\overline{x};\overline{y})$ positively with respect to the type denoted by $s$" (i.e., $\theta^*_{i,s}$ holds when the sequence $*$-decides $\varphi(\overline{x};\overline{y})$ correctly and $\varphi(\overline{x};\overline{y})$ is in our type and $\theta^*_{i,s}$ fails when the sequence $*$-decides $\varphi(\overline{x};\overline{y})$ correctly and $\neg\varphi(\overline{x};\overline{y})$ is in our type). For example,

$$\begin{aligned}\theta^*_{i,s}(\overline{y};\overline{\mathbf{z}}_i) = & \forall \overline{x} \left( \bigwedge_{j<m_i} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \rightarrow \varphi(\overline{x};\overline{y}) \right) \vee \\ & \Bigg( \bigwedge_{t<2} \neg\forall \overline{x} \left( \bigwedge_{j<m_i} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \rightarrow \varphi(\overline{x};\overline{y})^t \right) \wedge \\ & \bigvee_{j_0<m_i} \Bigg( \exists \overline{x} \left( \bigwedge_{j<m_i, j\neq j_0} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \wedge \neg\varphi(\overline{x};\overline{z}_{j_0,i})^{s(j_0,i)} \right) \wedge \\ & \forall \overline{x} \left( \bigwedge_{j<m_i, j\neq j_0} \varphi(\overline{x};\overline{z}_{j,i})^{s(j,i)} \wedge \neg\varphi(\overline{x};\overline{z}_{j_0,i})^{s(j_0,i)} \rightarrow \varphi(\overline{x};\overline{y}) \right) \Bigg) \Bigg).\end{aligned}$$

Now we can encode an algorithm that finds the first $*$-decision, and checks whether or not it holds positively. Namely,

$$\psi_s(\overline{y};\overline{\mathbf{z}}_0, ..., \overline{\mathbf{z}}_{f_H(K)-1}) = \bigwedge_{i<f_H(K)} \left( \theta_{i,s}(\overline{y};\overline{\mathbf{z}}_i) \wedge \bigwedge_{i'<i} \neg\theta_{i',s}(\overline{y};\overline{\mathbf{z}}_{i'}) \rightarrow \theta^*_{i,s}(\overline{y};\overline{\mathbf{z}}_i) \right).$$

We claim that the set $\{\psi_s : s \in {}^{K\times f_H(K)}2\}$ is a uniform definition of $\varphi$-types over finite sets, completing the proof.

Fix a finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $p \in S_\varphi(B)$. Choose $n$ maximum such that $\mathcal{B}_n \neq \emptyset$ for our choice of $B$ and $p$. This exists and $n < K$ by hypothesis. Choose any $\leq_p$-minimal $\beta \in \mathcal{B}_n$ and let $H = H(\beta)$. Let $H = \{\gamma_i : i < f_H(n)\}$ ordered by length of $\gamma_i$ (that is, if $i < j$, then $\lg(\gamma_i) \leq \lg(\gamma_j)$). So $\gamma_i = \langle \overline{b}_{j,i} : j < m_i \rangle$ for some $m_i \leq n$. Choose $\overline{b}_{j,i}$ arbitrarily for $i < f_H(n), m_i \leq j < K$ and $f_H(n) \leq i < f_H(K), j < K$. Choose $s \in {}^{K \times f_H(K)}2$ such that

$$p_{\{\overline{b}_{j,i}\}} = \left\{ \varphi(\overline{x}; \overline{b}_{j,i})^{s(j,i)} \right\}$$

for all $(j,i) \in K \times f_H(K)$. So, if $\delta$ is associated to $p$, then $s(j,i) = \delta(\overline{b}_{j,i})$. Finally, we claim that $\psi_s(\overline{y}; \overline{b}_{0,0}, ..., \overline{b}_{K-1, f_H(K)-1})$ defines $p$, as desired.

Fix $\overline{b} \in B$. By Lemma 2.4.14, if $\beta' \in H(\beta)$ is ever minimal such that $\beta'$ $*$-decides $\varphi(\overline{x}; \overline{b})$, then it does so correctly. Therefore, $\models \psi_s(\overline{b}; \overline{b}_{0,0}, ..., \overline{b}_{K-1, f_H(K)-1})$ if and only if $\varphi(\overline{x}; \overline{y}) \in p(\overline{x})$. On the other hand, if no $\beta' \in H(\beta)$ $*$-decides $\varphi(\overline{x}; \overline{b})$, then by Lemma 2.4.13 (iii), $\beta^\frown\langle \overline{b} \rangle \in \mathcal{B}_{n+1}$. However, this contradicts our choice of $n$ (since $\mathcal{B}_{n+1} = \emptyset$). Therefore, $\psi_s$ works. □

We can now prove Theorems 2.4.1 and 2.4.3.

*Proof of Theorem 2.4.3.* Fix a partitioned formula $\varphi(\overline{x}; \overline{y})$ and $N < \omega$ such that, for all $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| = N$, $\left|S_\varphi(B)\right| \leq N(N+1)/2$. Fix any $p \in S_\varphi(B)$. By Lemma 2.4.13 (ii), if $\mathcal{B}_N \neq \emptyset$, then for any $\beta \in \mathcal{B}_n$, $|S_\varphi(\beta)| > N(N+1)/2$, a contradiction to our assumption. Therefore, $\mathcal{B}_N = \emptyset$. Thus, by Theorem 2.4.15, $\varphi$ has UDTFS. □

*Proof of Theorem 2.4.1.* By sufficiency of a single variable (Lemma 2.3.6), it suffices to check that formulas of the form $\varphi(x; \overline{y})$ have UDTFS, so fix such a $\varphi$. Let $K < \omega$

be given for $\varphi$ as in Lemma 2.4.6. We claim that $\mathcal{B}_{2K-1} = \emptyset$ for any choice of finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $p \in S_\varphi(B)$. Thus, by Theorem 2.4.15, $\varphi$ has UDTFS.

Suppose, by means of contradiction, that $\mathcal{B}_{2K-1} \neq \emptyset$ for some choice of $B$ and $p$. Fix $\beta \in \mathcal{B}_{2K-1}$. By pigeon-hole principal, there exists $t < 2$ and a subsequence $\langle \overline{b}_0, ..., \overline{b}_{K-1} \rangle = \beta_0 \subseteq \beta$ of length $K$ such that, for all $i < K$, $\varphi(x; \overline{b}_i)^t \in p(x)$ (i.e., $p$ is constant on $\beta_0$). By Lemma 2.4.13 (i), $\beta_0 \in \mathcal{B}_K$. Therefore, by Lemma 2.4.12, for all $\ell < k < K$,

$$p_{\{\overline{b}_i : i \leq k\}, \{\overline{b}_\ell, \overline{b}_k\}}(x)$$

is consistent. However, since $p$ is constant on $\beta_0$, we have that, for all $\ell < k < K$,

$$\{\varphi(x; \overline{b}_i)^t : i < k, i \neq \ell\} \cup \{\neg\varphi(x; \overline{b}_\ell)^t, \neg\varphi(x; \overline{b}_k)^t\}$$

is consistent. This contradicts our choice of $K$ as in Lemma 2.4.6 (i.e., it contradicts the non-existence of a TP-pattern in $\pm\varphi$ of length $K$). □

In Chapter 4, we define VC-minimal theories and note that all VC-minimal theories are dp-minimal. This implies Corollary 2.4.2 (iii). By Theorem 4.1.2 (i), all o-minimal and weakly o-minimal theories are VC-minimal, so this implies Corollary 2.4.2 (v). In Section 4.4, we define VC-density one theories and note that all such theories are dp-minimal. This implies Corollary 2.4.2 (iv). Theorem 6.6 of [6] says that the $p$-adic field is dp-minimal, thus Corollary 2.4.2 (i) holds. Finally, one can show that Presburger arithmetic is dp-minimal (see Example 2.5 in [8]), thus Corollary 2.4.2 (ii) holds.

One should note that UDTFS does not characterize dp-minimality. There are examples of stable theories that are not dp-minimal (see Theorem 3.5 (iii) of [18], for example). However, since stable theories have UDTFS, this shows that UDTFS does not imply dp-minimal. Even unstable UDTFS does not imply dp-minimal, as we show in Section 4.3. For more examples of dp-minimal theories, see [6], [8], [18], and [27].

## 2.5 UDTFS and Valued Fields

As in Shelah's work on strongly dependent theories of valued fields in [26], we consider a language called the Denef-Pas language. The Denef-Pas language has three sorts: the valued field sort, which we denote $K$, the value group sort, which we denote $\Gamma$, and the residue field sort, which we denote $k$. We put the ring language and a partial function for the inverse on the sorts $K$ and $k$ and the language of ordered abelian groups (written additively, with a function for subtraction) on $\Gamma$. Finally, we have two maps between sorts: the valuation map, $v : K \to \Gamma$ and the angular component map, $\mathrm{ac} : K \to k$. We also consider a variation on this language which we call the "Denef-Pas $\omega$-language" (which is still a first-order language). In this language, we replace ac with a collection of formulas $\mathrm{ac}_n$ indexed by $n \in \omega$. These are again maps of the form $\mathrm{ac}_n : K \to k$. The theory of valued fields in the Denef-Pas language (respectively the Denef-Pas $\omega$-language) says that $\Gamma$ is an ordered abelian group, $K$ and $k$ are fields, and $v$ and ac (respectively $\mathrm{ac}_n$ for all $n \in \omega$) satisfy the natural demands on them. The theory of valued fields in the Denef-Pas $\omega$-language also requires that $\Gamma$ have a least positive element, called $1_\Gamma$ (or simply 1).

**Definition 2.5.1.** We say that a theory $T$ in the Denef-Pas language (or the Denef-Pas $\omega$-language) has *elimination of field quantifiers* if all formulas are $T$-equivalent to a boolean combination of atomic formulas, formulas only on the sort $k$ (i.e., all free and bounded variables in the formula are from the $k$ sort), and formulas only on the sort $\Gamma$.

**Theorem 2.5.2** (Theorem 4.2 of [5]). *(i) If $p$ is any prime, and $T$ is the theory, in the Denef-Pas $\omega$-language, of the $p$-adic field, then $T$ has elimination of field quantifiers.*

*(ii) If $\mathrm{char}(k) = 0$, $\Gamma$ has a least positive element, $1_\Gamma$, and $K$ is a Henselian valued field, then the theory of $(K, k, \Gamma)$ in the Denef-Pas $\omega$-language has elimination of field quantifiers.*

*(iii) If $\mathrm{char}(k) = 0$, $\Gamma$ is divisible, and $K$ is a Henselian valued field, then the theory of $(K, k, \Gamma)$ in the Denef-Pas language has elimination of field quantifiers.*

As in Shelah's paper on strongly dependent theories, we now prove an Ax-Kochen-style transfer result about UDTFS for valued fields in the Denef-Pas language (and $\omega$-language).

**Theorem 2.5.3.** *Let $T$ be any complete theory of valued fields, in either the Denef-Pas language or the Denef-Pas $\omega$-language, that has elimination of field quantifiers. If $T|_k$ (i.e., the theory $T$ only on the sort $k$ with the induced language) and $T|_\Gamma$ have UDTFS, then $T$ has UDTFS.*

By abuse of notation, let $K = K^{\mathfrak{C}}$, $\Gamma = \Gamma^{\mathfrak{C}}$, and $k = k^{\mathfrak{C}}$ (the three sorts of $\mathfrak{C}$).

Let $F$ be the prime field of $K$ (so $F = \mathbb{Q}$ if $\text{char}(K) = 0$ and $\mathbb{F}_p$ if $\text{char}(K) = p > 0$). Similarly, let $F'$ be the prime field of $k$. To prove the theorem, we must first understand the formulas of $T$. By elimination of field quantifiers, all formulas are $T$-equivalent to a boolean combination of formulas of the form:

(i) $\varphi_k$ a formula only in the sort $k$,

(ii) $\varphi_\Gamma$ a formula only in the sort $\Gamma$,

(iii) $g(\overline{x}) = 0$ for some $g \in F(\overline{x})$ with $\overline{x}$ variables in the $K$ sort,

(iv) $\tau(v(g_0(\overline{x})), ..., v(g_{n-1}(\overline{x})), \overline{y}) = 0$ for some $g_0, ..., g_{n-1} \in F(\overline{x})$ and some term $\tau(z_0, ..., z_{n-1}, \overline{y})$ from the $\Gamma$ sort,

(v) $\tau(v(g_0(\overline{x})), ..., v(g_{n-1}(\overline{x})), \overline{y}) < 0$ for some $g_0, ..., g_{n-1} \in F(\overline{x})$ and some term $\tau(z_0, ..., z_{n-1}, \overline{y})$ from the $\Gamma$ sort, and

(vi) $f(\text{ac}(g_0(\overline{x})), ..., \text{ac}(g_{n-1}(\overline{x})), \overline{y}) = 0$ for $g_i \in F(\overline{x})$ with $\overline{x}$ from the $K$ sort and for $f \in F'(z_0, ..., z_{n-1}, \overline{y})$ with $z_i$ and $\overline{y}$ from the $k$ sort.

(vi)* If we are working in the $\omega$-language, replace $\text{ac}(g_i)$ with $\text{ac}_{m_i}(g_i)$ for some $m_i \in \omega$ for $i < n$.

This is true because (iii) through (vi) enumerate all possible atomic formulas, up to $T$-equivalence. For any atomic formula, the relation symbol is either $=$ in the sort $K$ (i.e., formula (iii)), $=$ or $<$ in the sort $\Gamma$ (i.e., formulas (iv) and (v) respectively), or $=$ in the sort $k$ (i.e., formula (vi)).

By Lemma 2.3.6, it suffices to check that each of these formulas individually has UDTFS. By assumption, formulas as in (i) and (ii) already have UDTFS. For (iii), we can reduce to the case where $g \in F[\overline{x}]$. For (iv), since $v$ is a homomorphism from $K$ to $\Gamma$, we may combine all $g_i$'s into a single $g \in F(\overline{x})$ and move the remaining variables to the other side of the equation. We can do this similarly for (v), so this yields:

(iv)' $v(g(\overline{x})) = \tau(\overline{y})$ for some $g \in F(\overline{x})$ and some term $\tau(\overline{y})$ from the $\Gamma$ sort,

(v)' $v(g(\overline{x})) < \tau(\overline{y})$ for some $g \in F(\overline{x})$ and some term $\tau(\overline{y})$ from the $\Gamma$ sort.

Furthermore, (iv)' can actually be broken down into boolean combinations of instances of (v)'. That is, $v(g(\overline{x})) = \tau(\overline{y})$ is $T$-equivalent to $\neg(v(g(\overline{x})) < \tau(\overline{y})) \wedge \neg(v(g(\overline{x})^{-1}) < -\tau(\overline{y}))$. So it suffices to show that formulas of the form (iii) (for $g \in F[\overline{x}]$), (v)', and (vi) have UDTFS (or (vi)* if we are working in the $\omega$-language).

**Lemma 2.5.4.** *If $\varphi(x;\overline{y}) = [g(x,\overline{y}) = 0]$ for some $g \in F[x,\overline{y}]$, then $\varphi$ has UDTFS.*

*Proof.* Let $n$ be the $x$-degree of $g$. For any $\overline{b} \in K^{\lg(\overline{y})}$, $g(x,\overline{b})$ has at most $n$ solutions in $K$. Therefore, the following formula can define any non-trivial $\varphi$-type:

$$\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) = \forall x \left( \bigwedge_{i<n} g(x,\overline{z}_i) = 0 \rightarrow g(x,\overline{y}) = 0 \right)$$

(because a non-empty intersection of sets of size at most $n$ is equal to an intersection of at most $n$ of them). This shows $\varphi$ has UDTFS. □

This takes care of the formulas in (iii) with one free variable. For (v)', consider the following lemma:

**Lemma 2.5.5.** *If $\varphi(x;\overline{y},\overline{z}) = [v(h(x,\overline{y})) < \tau(\overline{z})]$ for some $h \in F(x,\overline{y})$ and $\tau(\overline{z})$ is some term in the $\Gamma$ sort, then $\varphi$ has UDTFS.*

*Proof.* First, let $h(x;\overline{y}) = f(x;\overline{y})/g(x;\overline{y})$ for $f, g \in F[x;\overline{y}]$, let $n$ be the $x$-degree of $f$ and $m$ the $x$-degree of $g$. Let $f(x;\overline{y}) = f_n^*(\overline{y})x^n + ... + f_0^*(\overline{y})$ and let $g(x;\overline{y}) = g_m^*(\overline{y})x^m + ... + g_0^*(\overline{y})$ for $f_i^*, g_j^* \in F(\overline{y})$. Therefore, the formula $\varphi$ reduces to

$$v(f_n^*(\overline{y})x^n + ... + f_0^*(\overline{y})) < \tau(\overline{z}) + v(g_m^*(\overline{y})x^m + ... + g_0^*(\overline{y})).$$

We now show, by induction on $n+m$, that formulas of this form have UDTFS. If $n + m = 0$, then we have a formula where $x$ is a dummy variable. In this case, $\varphi(x;\overline{y},\overline{z})$ clearly has UDTFS. Now suppose $n + m > 0$.

This breaks down further into three cases, $n = m$, $n < m$, and $n > m$.

**Case 1.** $m = n$.

Notice that in this case $\varphi(x;\overline{y},\overline{z})$ is $T$-equivalent to

$$v\left(x^n + \sum_{i<n}(f_i^*(\overline{y})/f_n^*(\overline{y}))x^i\right) - v\left(x^n + \sum_{i<n}(g_i^*(\overline{y})/g_n^*(\overline{y}))x^i\right) <$$
$$\tau(\overline{z}) - v(f_n^*(\overline{y})) + v(g_n^*(\overline{y})).$$

For notational simplicity, let $f_i^{**} = f_i^*/f_n^*$ and $g_i^{**} = g_i^*/g_n^*$ for $i \leq n$ and let $f^{**} = \sum_{i\leq n} f_i^{**}x^i$ and $g^{**} = \sum_{i\leq n} g_i^{**}x^i$. Hence, $\varphi$ is $T$-equivalent to

$$v(f^{**}(x;\overline{y})) - v(g^{**}(x;\overline{y})) < \tau(\overline{z}) - v(f_n^*(\overline{y})) + v(g_n^*(\overline{y})).$$

Notice that $f^{**}(x;\overline{y}) - g^{**}(x;\overline{y})$ has $x$-degree less than $n$ as

$$f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}) = \sum_{i<n}[f_i^{**}(\overline{y}) - g_i^{**}(\overline{y})]x^i.$$

Therefore, the formula $v(g^{**}(x;\overline{y})) < v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}))$ has UDTFS, by induction, say witnessed by $\psi_+(\overline{y};\overline{\mathbf{w}}_+)$. Similarly, $v(g^{**}(x;\overline{y})) > v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}))$ has UDTFS, say witnessed by $\psi_-(\overline{y};\overline{\mathbf{w}}_-)$. If $v(g^{**}(x;\overline{y})) < v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}))$ holds, then, by the strong triangle inequality, $v(f^{**}(x;\overline{y})) = v(g^{**}(x;\overline{y}))$ holds. Therefore, in this case, $\varphi(x;\overline{y},\overline{z})$ is $T$-equivalent to $\tau(\overline{z}) - v(f^*_n(\overline{y})) + v(g^*_n(\overline{y})) > 0$, which is a formula without $x$. Similarly, if $v(g^{**}(x;\overline{y})) > v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}))$ holds, then

$$v(f^{**}(x;\overline{y})) = v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y})).$$

Therefore, in this case, $\varphi(x;\overline{y},\overline{z})$ is $T$-equivalent to

$$v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y})) - v(g^{**}(x;\overline{y})) < \tau(\overline{z}) - v(f^*_n(\overline{y})) + v(g^*_n(\overline{y})).$$

This is $T$-equivalent to

$$v((f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}))f^*_n(\overline{y})) < \tau(\overline{z}) + v(g^{**}(x;\overline{y})g^*_n(\overline{y})),$$

which has UDTFS by induction, say witnessed by $\gamma_1(\overline{y},\overline{z};\overline{\mathbf{w}}_1)$. Finally, if $v(g^{**}(x;\overline{y}))$ $= v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y}))$ holds, then $\varphi(x;\overline{y},\overline{z})$ is $T$-equivalent to

$$v(f^{**}(x;\overline{y})) - v(f^{**}(x;\overline{y}) - g^{**}(x;\overline{y})) < \tau(\overline{z}) - v(f^*_n(\overline{y})) + v(g^*_n(\overline{y})),$$

which again has UDTFS by induction, say witnessed by $\gamma_2(\overline{y},\overline{z};\overline{\mathbf{w}}_2)$. Collecting these three possibilities together into a single formula, we see that the following formula witnesses the fact that $\varphi$ has UDTFS:

$$(\psi_+ \to [\tau(\overline{z}) + v(f^*_n(\overline{y})) - v(g^*_n(\overline{y})) > 0]) \wedge (\psi_- \to \gamma_1) \wedge (\neg\psi_+ \wedge \neg\psi_- \to \gamma_2). \quad (2.4)$$

**Case 2.** $m < n$.

First, by dividing both sides by $f_n^*(\overline{y})$ (i.e., subtracting $v(f_n^*(\overline{y}))$), we may assume that $f_n^* = 1$. Fix $B \subseteq K^{\lg(\overline{y})} \times \Gamma^{\lg(\overline{z})}$ finite and $a \in K$. Choose $(\overline{b}_0, \overline{c}_0) \in B$ such that $v(f(a; \overline{b}_0)) \geq v(g(a; \overline{b}_0)) + \tau(\overline{c}_0)$ (i.e., $\neg\varphi(a; \overline{b}_0, \overline{c}_0)$ holds) and $v(g(a; \overline{b}_0)) + \tau(\overline{c}_0)$ is $\leq$-maximal such (if no such $(\overline{b}_0, \overline{c}_0)$ exists, then the type is trivial). Now fix any $(\overline{b}, \overline{c}) \in B$ and we describe a uniform algorithm to determine $\varphi(a; \overline{b}, \overline{c})$, showing that $\varphi$ has UDTFS.

**Subcase 2(a).** $v(g(a; \overline{b})) + \tau(\overline{c}) > v(g(a; \overline{b}_0)) + \tau(\overline{c}_0)$.

First notice that this formula has UDTFS, since $m < n$ and using induction. If this holds, then the maximality of the choice of $(\overline{b}_0, \overline{c}_0) \in B$ implies that $\varphi(a; \overline{b}, \overline{c})$ holds.

**Subcase 2(b).** $v(g(a; \overline{b})) + \tau(\overline{c}) \leq v(g(a; \overline{b}_0)) + \tau(\overline{c}_0)$ and $v(f(a; \overline{b}) - f(a; \overline{b}_0)) < v(g(a; \overline{b})) + \tau(\overline{c})$.

Notice that, since $f_n^* = 1$,

$$f(x; \overline{y}) - f(x; \overline{y}') = \sum_{i<n} [f_i^*(\overline{y}) - f_i^*(\overline{y}')] x^i.$$

Therefore, the formula $v(f(x; \overline{y}) - f(x; \overline{y}')) < v(g(x; \overline{y})) + \tau(\overline{z})$ has UDTFS by induction. Now, if Subcase 2(b) holds, suppose by way of contradiction that $\neg\varphi(a; \overline{b}, \overline{c})$ also holds. Then we would have $v(f(a; \overline{b})) \geq v(g(a; \overline{b})) + \tau(\overline{c})$, hence

$$v(f(a; \overline{b})) > v(f(a; \overline{b}) - f(a; \overline{b}_0)).$$

By strong triangle inequality, we get that $v(f(a; \overline{b}_0)) = v(f(a; \overline{b}) - f(a; \overline{b}_0))$. By assumption,

$$v(f(a; \overline{b}_0)) < v(g(a; \overline{b})) + \tau(\overline{c}) \leq v(g(a; \overline{b}_0)) + \tau(\overline{c}_0),$$

contrary to the assumption that $\neg\varphi(a;\overline{b}_0,\overline{c}_0)$ holds. Therefore, Subcase 2(b) implies that $\varphi(a;\overline{b},\overline{c})$ holds.

**Subcase 2(c).** $v(g(a;\overline{b}))+\tau(\overline{c}) \leq v(g(a;\overline{b}_0))+\tau(\overline{c}_0)$ and $v(f(a;\overline{b})-f(a;\overline{b}_0)) \geq v(g(a;\overline{b}))+\tau(\overline{c})$.

Suppose, by way of contradiction, that Subcase 2(c) holds and $\varphi(a;\overline{b},\overline{c})$ holds. Then $v(f(a;\overline{b})) < v(g(a;\overline{b}))+\tau(\overline{c})$, hence $v(f(a;\overline{b})) < v(f(a;\overline{b})-f(a;\overline{b}_0))$. By strong triangle inequality, $v(f(a;\overline{b}_0)) = v(f(a;\overline{b}))$, hence

$$v(f(a;\overline{b}_0)) < v(g(a;\overline{b}))+\tau(\overline{c}) \leq v(g(a;\overline{b}_0))+\tau(\overline{c}_0),$$

contrary to the assumption that $\neg\varphi(a;\overline{b}_0,\overline{c}_0)$ holds. Therefore, Subcase 2(c) implies that $\neg\varphi(a;\overline{b},\overline{c})$ holds.

As in Case 1, since all formulas distinguishing the three subcases have UDTFS, we can replace them with the formulas witnessing their UDTFS, thus showing that $\varphi$ itself has UDTFS.

**Case 3.** $m > n$.

This is similar to Case 2. As before, we may assume that $g^*_m = 1$. Fix $B \subseteq K^{\lg(\overline{y})} \times \Gamma^{\lg(\overline{z})}$ finite and $a \in K$. Choose $(\overline{b}_0,\overline{c}_0) \in B$ such that $v(f(a;\overline{b}_0)) < v(g(a;\overline{b}_0))+\tau(\overline{c}_0)$ (i.e., $\varphi(a;\overline{b}_0,\overline{c}_0)$ holds) and $v(f(a;\overline{b}_0))-\tau(\overline{c}_0)$ is $\leq$-maximal such (if no such $(\overline{b}_0,\overline{c}_0)$ exists, then the type is trivial). Now fix any $(\overline{b},\overline{c}) \in B$ and we describe a uniform algorithm to determine $\varphi(a;\overline{b},\overline{c})$, showing that $\varphi$ has UDTFS.

**Subcase 3(a).** $v(f(a;\overline{b}))-\tau(\overline{c}) > v(f(a;\overline{b}_0))-\tau(\overline{c}_0)$.

In this case $\neg\varphi(a;\overline{b},\overline{c})$ holds.

**Subcase 3(b).** $v(f(a;\overline{b}))-\tau(\overline{c}) \leq v(f(a;\overline{b}_0))-\tau(\overline{c}_0)$ and $v(f(a;\overline{b}))-\tau(\overline{c}) <$

$v(g(a;\bar{b}) - g(a;\bar{b}_0))$.

As before, we can show that, in this case, $\varphi(a;\bar{b},\bar{c})$ holds.

**Subcase 3(c).** $v(g(a;\bar{b})) + \tau(\bar{c}) \leq v(g(a;\bar{b}_0)) + \tau(\bar{c}_0)$ and $v(f(a;\bar{b})) - \tau(\bar{c}) \geq v(g(a;\bar{b}) - g(a;\bar{b}_0))$.

In this case, $\neg\varphi(a;\bar{b},\bar{c})$ holds.

As in Case 1 and Case 2, this algorithm shows that $\varphi$ has UDTFS. $\square$

In addition to the form taken in Lemma 2.5.5, formulas in (v)' with a single non-parameter variable can also have the form $\varphi(x;\bar{y},\bar{z}) = [v(h(\bar{y})) < \tau(x,\bar{z})]$. The terms of an abelian group are very simple; we can show that this formula takes the form

$$mx > v(h(\bar{y})) + \tau_0(\bar{z})$$

for some $m \in \mathbb{Z}$ and some term $\tau_0$. Now given $B \subseteq K^{\lg(\bar{y})} \times \Gamma^{\lg(\bar{z})}$ finite and $a \in \Gamma$, we simply choose $(\bar{b}_0,\bar{c}_0) \in B$ such that $ma > v(h(\bar{b}_0)) + \tau_0(\bar{c}_0)$ and $v(h(\bar{b}_0)) + \tau_0(\bar{c}_0)$ is $\leq$-maximal such (if none exists, the type is trivial). Then, for any $(\bar{b},\bar{c}) \in B$,

$$\varphi(a;\bar{b},\bar{c}) \Leftrightarrow [v(h(\bar{b})) + \tau_0(\bar{c}) \leq v(h(\bar{b}_0)) + \tau_0(\bar{c}_0)].$$

This shows that formulas of the form $\varphi(x;\bar{y},\bar{z}) = [v(h(\bar{y})) < \tau(x,\bar{z})]$ have UDTFS. In combination with Lemma 2.5.5, we see that all formulas of the form (v)' with one free variable have UDTFS. It remains to show that formulas of the form (vi) (or, in the case of the Denef-Pas $\omega$-language, (vi)$^*$) with one free variable have UDTFS.

**Lemma 2.5.6.** *If $\varphi(x;\bar{y},\bar{z}) = [f(\mathrm{ac}(g_0(x,\bar{y})),...,\mathrm{ac}(g_{n-1}(x,\bar{y})),\bar{z}) = 0]$ as in (vi), then $\varphi$ has UDTFS.*

Before giving the full proof of Lemma 2.5.6, we give a simple example. Suppose that $\varphi(x; y, z) = [\mathrm{ac}(x - y) = z]$. If we are given $B \subseteq K \times k$ finite and $a \in K$, the first step toward developing a uniform algorithm for determining the truth value of $\varphi(a, b, c)$ for $(b, c) \in B$ is to choose a closest approximation $b_0$ to $a$ (then we can replace $a$ with $b_0$ in most cases). Consider the following facts about ac in valued fields:

*Remark* 2.5.7. For $K$ a valued field, the following hold for all $a, b \in K$:

(i) $\mathrm{ac}(ab) = \mathrm{ac}(a) \cdot \mathrm{ac}(b)$,

(ii) $v(a) = v(b)$ and $v(a - b) > v(a)$ implies that $\mathrm{ac}(a) = \mathrm{ac}(b)$,

(iii) $v(a) = v(b) = v(a - b)$ implies that $\mathrm{ac}(a - b) = \mathrm{ac}(a) - \mathrm{ac}(b)$, and

(iv) $v(a) < v(b)$ implies that $\mathrm{ac}(a - b) = \mathrm{ac}(a)$.

So to approximate $a$ with an element $b_0$ requires $v(a - b_0)$ to be large. With this is mind, choose $(b_0, c_0) \in B$ such that $v(a - b_0)$ is $\leq$-maximal. For any other $(b, c) \in B$, we have $v(a - b) \leq v(a - b_0)$. If $v(a - b) < v(a - b_0)$, then, by Remark 2.5.7 (iv), $\mathrm{ac}(a - b) = \mathrm{ac}(b_0 - b)$. Therefore, in this case, $\varphi(a; b, c)$ holds if and only if $\mathrm{ac}(b_0 - b) = c$. If $v(a - b) = v(a - b_0)$, then, by the strong triangle inequality, $v(b - b_0) \geq v(a - b)$, yielding another two cases. If $v(b_0 - b) > v(a - b)$, then, by Remark 2.5.7 (ii), $\mathrm{ac}(a - b) = \mathrm{ac}(a - b_0)$. Fix $(b_1, c_1) \in B$ such that $\mathrm{ac}(a - b_0) = c_1$, if such an element exists (otherwise, the type in this case is trivial). Therefore, in this case, $\varphi(a; b, c)$ holds if and only if $c_1 = c$. The only remaining case is when $v(a - b) = v(a - b_0) = v(b_0 - b)$. In this case, by Remark 2.5.7 (iii), $\mathrm{ac}(a - b) =$

$\mathrm{ac}(a-b_0)+\mathrm{ac}(b_0-b)$. Now fix $(b_2, c_2) \in B$ such that $\mathrm{ac}(a-b_0)+\mathrm{ac}(b_2-b_0) = c_2$, if such an element in $B$ exists (if not, then the type in this case is trivially false). Hence $\mathrm{ac}(a-b_0) = c_2 - \mathrm{ac}(b_2-b_0)$. Then, $\varphi(a; b, c)$ holds if and only if $\mathrm{ac}(a-b) = c$ if and only if $\mathrm{ac}(a-b_0)+\mathrm{ac}(b_0-b) = c$ if and only if

$$c_2 - \mathrm{ac}(b_2 - b_0) + \mathrm{ac}(b_0 - b) = c.$$

Since these three cases are determined by formulas with UDTFS (namely of the form (iv) and (v) above), this gives us a uniform algorithm for determining the truth value of $\varphi(a, b, c)$. Therefore, $\varphi$ has UDTFS. Now consider the general case.

*Proof of Lemma 2.5.6.* Let $\varphi(x; \overline{y}, \overline{z}) = [f(\mathrm{ac}(g_0(x, \overline{y})), ..., \mathrm{ac}(g_{n-1}(x, \overline{y})), \overline{z}) = 0]$ as in (vi). We first construct polynomials $g_i^j \in F[x, \overline{y}_0, ..., \overline{y}_j]$ and $h_i^j \in F[\overline{y}_0, ..., \overline{y}_j]$ by induction on $j$. First, let $h_i^0$ be the leading polynomial coefficient of $g_i(x, \overline{y}_0)$ in the $x$ variable and let $g_i^0(x, \overline{y}_0) = g_i(x, \overline{y}_0)/h_i^0(\overline{y}_0)$ (so $g_i^0$ has leading polynomial coefficient 1 in the $x$ variable). Now suppose $g_i^{j-1}$ and $h_i^{j-1}$ are constructed. Temporarily, define

$$g^*(x; \overline{y}_0, ..., \overline{y}_j) = [g_i^{j-1}(x; \overline{y}_0, ..., \overline{y}_{j-2}, \overline{y}_j) - g_i^{j-1}(x; \overline{y}_0, ..., \overline{y}_{j-2}, \overline{y}_{j-1})].$$

Note that $\deg_x(g^*) < \deg_x(g_i^{j-1})$ because $g_i^{j-1}$ has leading coefficient 1 in the $x$ variable. If $\deg_x(g^*) = 0$, we terminate the construction, let $g_i^j = g^*$, $h_i^j = 1$, and set $N_i = j$ (in this case, notice that $g_i^j \in F[\overline{y}_0, ..., \overline{y}_j]$, as there is no $x$ variable). Otherwise, let $h_i^j$ be the leading coefficient of $g^*$ in the $x$ variable and set $g_i^j = g^*/h_i^j$. Since $\deg_x(g_i) = \deg_x(g_i^0)$ is finite, this process must terminate for all $i < n$, producing a sequence $(N_0, ..., N_{n-1})$.

Using these $g_i^j$ and $h_i^j$, along with the approximation trick used in the example above, we produce a uniform algorithm for determining $\varphi$-types over finite sets, showing $\varphi$ has UDTFS. Fix $B \subseteq K^{\lg(\overline{y})} \times k^{\lg(\overline{z})}$ finite and $a \in K$. We construct $(\overline{b}_i^j, \overline{c}_i^j) \in B$ inductively as follows: Choose $(\overline{b}_i^0, \overline{c}_i^0) \in B$ such that $v(g_i^0(a; \overline{b}_i^0))$ is $\leq$-maximal. If $(\overline{b}_i^{j-1}, \overline{c}_i^{j-1}) \in B$ and $j < N_i$, let $(\overline{b}_i^j, \overline{c}_i^j) \in B$ such that $v(g_i^j(a; \overline{b}_i^0, ..., \overline{b}_i^j))$ is $\leq$-maximal. Let

$$\overline{\mathbf{b}} = \overline{b}_0^0 {}^\frown ... {}^\frown \overline{b}_0^{N_0-1} {}^\frown ... {}^\frown \overline{b}_{n-1}^0 {}^\frown ... {}^\frown \overline{b}_{n-1}^{N_{n-1}-1}$$

and, by abuse of notation, let

$$g_i^j(a; \overline{\mathbf{b}}, \overline{y}) = g_i^j(a; \overline{b}_i^0, ..., \overline{b}_i^{j-1}, \overline{y}),$$

$$g_i^j(a; \overline{\mathbf{b}}) = g_i^j(a; \overline{b}_i^0, ..., \overline{b}_i^{j-1}, \overline{b}_i^j), \text{ and}$$

$$h_i^j(a; \overline{\mathbf{b}}, \overline{y}) = h_i^j(a; \overline{b}_i^0, ..., \overline{b}_i^{j-1}, \overline{y})$$

for all $i < n$ and appropriate $j$. This simplifies further notation. When $j = N_i$, we may drop the $a$ as $x$ is a dummy variable in $g_i^{N_i}$. Fix $(\overline{b}, \overline{c}) \in B$. Now, for any $i < n$ and $j < N_i$, we have three cases (just as in the example above):

**Case 1**$_{(i,j)}$. $v(g_i^j(a; \overline{\mathbf{b}}, \overline{b})) < v(g_i^j(a; \overline{\mathbf{b}}))$.

In this case, using Remark 2.5.7 (iv), we see that

$$\mathrm{ac}(g_i^j(a; \overline{\mathbf{b}}, \overline{b})) = \mathrm{ac}(g_i^j(a; \overline{\mathbf{b}}, \overline{b}) - g_i^j(a; \overline{\mathbf{b}})) = \mathrm{ac}(h_i^{j+1}(a; \overline{\mathbf{b}}, \overline{b})) \cdot \mathrm{ac}(g_i^{j+1}(a; \overline{\mathbf{b}}, \overline{b})).$$

**Case 2**$_{(i,j)}$. $v(g_i^j(a; \overline{\mathbf{b}}, \overline{b}) - g_i^j(a; \overline{\mathbf{b}})) > v(g_i^j(a; \overline{\mathbf{b}}, \overline{b})) = v(g_i^j(a; \overline{\mathbf{b}}))$.

By Remark 2.5.7 (ii), we see that $\mathrm{ac}(g_i^j(a; \overline{\mathbf{b}}, \overline{b})) = \mathrm{ac}(g_i^j(a; \overline{\mathbf{b}}))$.

**Case 3**$_{(i,j)}$. $v(g_i^j(a; \overline{\mathbf{b}}, \overline{b}) - g_i^j(a; \overline{\mathbf{b}})) = v(g_i^j(a; \overline{\mathbf{b}}, \overline{b})) = v(g_i^j(a; \overline{\mathbf{b}}))$.

By Remark 2.5.7 (iii), we see that

$$\mathrm{ac}(g_i^j(a;\overline{\mathbf{b}},\overline{b})) = \mathrm{ac}(g_i^j(a;\overline{\mathbf{b}})) + \mathrm{ac}(g_i^j(a;\overline{\mathbf{b}},\overline{b}) - g_i^j(a;\overline{\mathbf{b}}))$$

hence

$$\mathrm{ac}(g_i^j(a;\overline{\mathbf{b}},\overline{b})) = \mathrm{ac}(g_i^j(a;\overline{\mathbf{b}})) + \mathrm{ac}(h_{i+1}^j(\overline{\mathbf{b}},\overline{b})) \cdot \mathrm{ac}(g_i^{j+1}(a;\overline{\mathbf{b}},\overline{b})).$$

In this fashion, we can reduce the entire polynomial equation of $\varphi$ depending on only $3^{\left(\sum_{i<n} N_i\right)}$ many cases, each of which is distinguishable by formulas with UDTFS. Encode these cases by $\rho : \{(i,j) : i < n, j < N_i\} \to \{1,2,3\}$ where $\rho(i,j) = 1$ if Case $1_{(i,j)}$ holds, $\rho(i,j) = 2$ if Case $2_{(i,j)}$ holds, and $\rho(i,j) = 3$ if Case $3_{(i,j)}$ holds. We then encode $\mathrm{ac}(g_i^j(a;\overline{\mathbf{b}},\overline{b}))$ by the variable $w_{i,j}$, $\mathrm{ac}(g_i^j(a;\overline{\mathbf{b}}))$ by the variable $v_{i,j}$, and $\mathrm{ac}(h_i^j(a;\overline{\mathbf{b}},\overline{b}))$ by the variable $u_{i,j}$ in the following formula:

$$\psi(v_{i,j}; u_{i,j}, w_{i,N_i}, \overline{z}) = (\exists w_{i,j})_{i<n,j<N_i}\Big[\bigwedge_{\rho(i,j)=1} w_{i,j} = u_{i,j+1}w_{i,j+1} \wedge \bigwedge_{\rho(i,j)=2} w_{i,j} = v_{i,j} \wedge$$
$$\bigwedge_{\rho(i,j)=3} v_{i,j} + u_{i,j+1}w_{i,j+1} \wedge f(w_{0,0}, \dots, w_{n-1,0}, \overline{z}) = 0\Big]$$

(where $u_{i,j}$ ranges over $j \leq N_i$ but $v_{i,j}$ only ranges over $j < N_i$). Notice that $\psi$ is a formula only in the $k$ sort. By the assumption that $T|_k$ has UDTFS, we know $\psi$ has UDTFS, say by $\gamma$. Let

$$B^* = \{\mu(\overline{b},\overline{c}) : (\overline{b},\overline{c}) \in B\}$$

where

$$\mu(\overline{b},\overline{c}) = \mu(\overline{b},\overline{c};\overline{\mathbf{b}}) = \langle \mathrm{ac}(h_i^j(\overline{\mathbf{b}},\overline{b}))\rangle_{i<n,j\leq N_i}{}^\frown\langle \mathrm{ac}(g_i^{N_i}(\overline{\mathbf{b}},\overline{b}))\rangle_{i<n}{}^\frown\overline{c}$$

(notice that $\mu$ is a definable function). Let

$$\overline{a}^* = \left\langle \mathrm{ac}\left(g_i^j(a;\overline{\mathbf{b}})\right)\right\rangle_{i<n,j<N_i}.$$

By UDTFS on $\mathrm{tp}_\psi(\overline{a}^*/B^*)$, there exists $(\overline{d}_\ell, \overline{e}_\ell) \in B$ for $\ell < L$ such that, for all $(\overline{b}, \overline{c}) \in B$, $\psi(\overline{a}^*; \mu(\overline{b}, \overline{c}))$ holds if and only if $\gamma(\mu(\overline{b}, \overline{c}); \mu(\overline{d}_0, \overline{e}_0), ..., \mu(\overline{d}_{L-1}, \overline{e}_{L-1}))$ holds.

Therefore, $\varphi(a; \overline{b}, \overline{c})$ holds if and only if $f(\mathrm{ac}(g_0(a; \overline{b})), ..., \mathrm{ac}(g_{n-1}(a; \overline{b})), \overline{c}) = 0$ if and only if $\psi(\overline{a}^*; \mu(\overline{b}, \overline{c}))$ holds if and only if

$$\gamma(\mu(\overline{b}, \overline{c}); \mu(\overline{d}_0, \overline{e}_0), ..., \mu(\overline{d}_{L-1}, \overline{e}_{L-1}))$$

holds. This shows that $\varphi$ has UDTFS. $\square$

In a manner very similar to Lemma 2.5.4, we can prove that formulas of the form

$$\varphi(x; \overline{y}, \overline{z}) = [f(\mathrm{ac}(g_0(\overline{y})), ..., \mathrm{ac}(g_{n-1}(\overline{y})), x, \overline{z}) = 0]$$

have UDTFS. Thus, all formulas of the form (vi) with one free variable have UDTFS.

Now consider the Denef-Pas $\omega$-language and formulas of the form (vi)$^*$. We have a remark for $\mathrm{ac}_n$ corresponding to Remark 2.5.7 on ac:

*Remark* 2.5.8. For $K$ a valued field, the following hold for all $a, b \in K$ and all $n \in \omega$:

(i) $\mathrm{ac}_n(ab) = \sum_{i \leq 2n} \mathrm{ac}_i(a) \cdot \mathrm{ac}_{2n-i}(b)$,

(ii) $v(a) = v(b)$ and $v(a - b) > v(a) + n$ implies that $\mathrm{ac}_n(a) = \mathrm{ac}_n(b)$,

(iii) $v(a) = v(b)$ and $v(a-b) = v(a)+i$ for some $0 \leq i \leq n$ implies that $\mathrm{ac}_n(a-b) = \mathrm{ac}_{n+i}(a) - \mathrm{ac}_{n+i}(b)$,

(iv) $v(a) < v(b) - n$ implies that $\mathrm{ac}_n(a - b) = \mathrm{ac}_n(a)$, and

(v) $v(a) = v(b) - i$ for some $0 < i < n$ implies that $\mathrm{ac}_n(a - b) = \mathrm{ac}_n(a) - \mathrm{ac}_{n-i}(b)$.

A proof similar to the one for Lemma 2.5.6 show that formulas of the form

$$\varphi(x;\overline{y},\overline{z}) = [f(\mathrm{ac}_{\ell_0}(g_0(x,\overline{y})),...,\mathrm{ac}_{\ell_{n-1}}(g_{n-1}(x,\overline{y})),\overline{z}) = 0]$$

have UDTFS. The argument is more complicated, mainly due to the fact that $\mathrm{ac}_n(a\cdot b)$ is not necessarily equal to $\mathrm{ac}_n(a)\cdot\mathrm{ac}_n(b)$ for $n>0$. Instead, when we decompose products inside $\mathrm{ac}_n$ as we did with ac in the proof of Lemma 2.5.6, we get instead a linear combination of $\mathrm{ac}_i$ (as in Remark 2.5.8 (i)). This complicates the argument, but it remains essentially the same. As in Lemma 2.5.4, we can also show that formulas of the form

$$\varphi(x;\overline{y},\overline{z}) = [f(\mathrm{ac}_{\ell_0}(g_0(\overline{y})),...,\mathrm{ac}_{\ell_{n-1}}(g_{n-1}(\overline{y})),x,\overline{z}) = 0]$$

have UDTFS. Thus, all formulas of the form (vi)$^*$ with one free variable have UDTFS. We therefore conclude that Theorem 2.5.3 holds.

We get the following corollary of the theorem:

**Corollary 2.5.9.** *The following theories have UDTFS:*

*(i) The theory of the p-adic field in the Denef-Pas $\omega$-language.*

*(ii) The theory of the field $\mathbb{R}((t))$ in the Denef-Pas $\omega$-language.*

*(iii) The theory of the field $\mathbb{C}((t^{\mathbb{Q}}))$ in the Denef-Pas language.*

*(iv) Algebraically closed valued fields of equicharacteristic zero (with a non-trivial valuation) in the Denef-Pas language.*

*Proof.* (i): Note that $(\mathbb{Z};+,<)$ has UDTFS by 2.4.2 (ii). Also, $k=\mathbb{F}_p$ which is finite, hence has UDTFS. By Theorem 2.5.2 (i) and Theorem 2.5.3, the theory of the $p$-adic field has UDTFS in the Denef-Pas $\omega$-language.

(ii): Again $(\mathbb{Z};+,<)$ has UDTFS. The real field $\mathbb{R}$ is o-minimal, hence has UDTFS by Corollary 2.4.2 (v). Therefore, by Theorem 2.5.2 (ii) and Theorem 2.5.3 the theory of $\mathbb{R}((t))$ has UDTFS in the Denef-Pas $\omega$-language.

(iii): Note that $(\mathbb{Q};+,<)$ is o-minimal, hence has UDTFS by Corollary 2.4.2 (v). The theory of the complex field is stable (in fact $\aleph_0$-stable), hence has UDTFS. Therefore, the theory of $\mathbb{C}((t^{\mathbb{Q}}))$ has UDTFS by Theorem 2.5.2 (iii) and Theorem 2.5.3.

(iv): One can show that such fields have a divisible value group and a algebraically closed residue field. Hence, this follows similarly to (iii). $\square$

Instead of the Denef-Pas ($\omega$-)language, we can use a simpler, one-sorted language, $L_{\text{val}} = \{0, 1, +, -, \cdot, |\}$. Here we interpret $0, 1, +, -, \cdot$ in the standard way and $|$ is the binary relation where $x|y$ if and only if $v(x) \leq v(y)$. In this language, the theory of algebraically closed valued fields of characteristics $(p, q)$, denoted $\mathrm{ACVF}_{(p,q)}$, has elimination of quantifiers. As in Lemma 2.5.5, we get the following:

**Lemma 2.5.10.** *If $f(x;\overline{y}), g(x;\overline{y}) \in \mathbb{Z}[x;\overline{y}]$ and $\varphi(x;\overline{y}) = f(x;\overline{y})|g(x;\overline{y})$, then $\varphi$ has UDTFS.*

And thus, we get the following result:

**Theorem 2.5.11.** *If $T$ is an $L_{\text{val}}$-theory of valued fields that eliminates quantifiers, then $T$ has UDTFS.*

Therefore, $\mathrm{ACVF}_{(p,q)}$ in the language $L_{\text{val}}$ has UDTFS. Since UDTFS is not closed under reducts, this does not follow immediately from Corollary 2.5.9 (iv).

However, it can be shown that $\mathrm{ACVF}_{(p,q)}$ is VC-minimal (see Theorem 3.7 of [12]), hence dp-minimal, so this also follows as a corollary of Theorem 2.4.1.

## 2.6 $\Delta$-Splitting and Coherence

UDTFS is closely related to a notion developed by Shelah called non-splitting. In our context, fix a formula $\varphi(\overline{x};\overline{y})$, a set of formulas $\Delta(\overline{y};\overline{z})$, two sets $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $C \subseteq \mathfrak{C}^{\lg(\overline{z})}$, and a $\varphi$-type $p \in S_\varphi(B)$.

**Definition 2.6.1.** We say that $p$ *does not $\Delta$-split over $C$* if, for all $\overline{b}_0, \overline{b}_1 \in B$ with $\mathrm{tp}_\Delta(\overline{b}_0/C) = \mathrm{tp}_\Delta(\overline{b}_1/C)$, we have that $\varphi(\overline{x};\overline{b}_0) \in p(\overline{x})$ if and only if $\varphi(\overline{x};\overline{b}_1) \in p(\overline{x})$.

Note that, in [22], Shelah calls this $(\{\varphi\},\Delta)$*-non-splitting*.

In our context, we usually consider finite sets $\Delta$ of the form $\Delta(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ where $\lg(\overline{z}_i) = \lg(\overline{y})$ and $C = \{\overline{\mathbf{c}}\}$. For the moment, let us make a definition measuring the complexity of a formula in terms of non-splitting. We say that a partitioned formula $\varphi(\overline{x};\overline{y})$ has *finite non-splitting* if there exists a finite set of formulas $\Delta(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$ such that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $p \in S_\varphi(B)$, there exists $\overline{\mathbf{c}} \in B^n$ such that $p$ does not $\Delta$-split over $\{\overline{\mathbf{c}}\}$.

**Theorem 2.6.2.** *Fix a partitioned formula $\varphi(\overline{x};\overline{y})$. Then $\varphi$ has finite non-splitting if and only if $\varphi$ has UDTFS.*

*Proof.* Suppose first that $\varphi(\overline{x};\overline{y})$ has UDTFS, witnessed by $\psi(\overline{y};\overline{z}_0,...,\overline{z}_{n-1})$. Then we claim that $\Delta = \{\psi\}$ suffices to show that $\varphi$ has finite non-splitting. Fix any $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $p \in S_\varphi(B)$. By UDTFS, there exists $\overline{\mathbf{c}} \in B^n$ such that $\psi(\overline{y};\overline{\mathbf{c}})$ defines

$p$. Now fix $\overline{b}_0, \overline{b}_1 \in B$ such that $\mathrm{tp}_\psi(\overline{b}_0/\{\overline{\mathbf{c}}\}) = \mathrm{tp}_\psi(\overline{b}_1/\{\overline{\mathbf{c}}\})$. This means exactly that

$$\models \psi(\overline{b}_0; \overline{\mathbf{c}}) \leftrightarrow \psi(\overline{b}_1; \overline{\mathbf{c}}).$$

Therefore, by the definition of defines, $\varphi(\overline{x}; \overline{b}_0) \in p(\overline{x})$ if and only if $\varphi(\overline{x}; \overline{b}_1) \in p(\overline{x})$, showing that $p$ does not $\psi$-split over $\{\overline{\mathbf{c}}\}$.

Conversely, suppose that $\varphi(\overline{x}; \overline{y})$ has finite non-splitting. Let $\Delta(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1})$ witness this. For each $I \subseteq {}^{\Delta}2$, consider the following formula:

$$\psi_I(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1}) = \bigvee_{\mu \in I} \bigwedge_{\delta \in \Delta} \delta(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1})^{\mu(\delta)}.$$

We claim that these $\psi_I$ witness the fact that $\varphi$ has UDTFS. Fix any finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $p \in S_\varphi(B)$. By finite non-splitting, there exists $(\overline{c}_0, ..., \overline{c}_{n-1}) = \overline{\mathbf{c}} \in B^n$ such that $p$ does not $\Delta$-split over $\{\overline{\mathbf{c}}\}$. Let

$$I^* = \left\{ \mu \in {}^{\Delta}2 : \text{ for all } \overline{b} \in B \text{ with } \bigwedge_{\delta \in \Delta} \delta(\overline{b}; \overline{c}_0, ..., \overline{c}_{n-1})^{\mu(\delta)}, \varphi(\overline{x}; \overline{b}) \in p(\overline{x}) \right\}.$$

Since $p$ does not $\Delta$-split over $\{\overline{\mathbf{c}}\}$, if any $\overline{b} \in B$ is such that $\varphi(\overline{x}; \overline{b}) \in p(\overline{x})$, then all $\overline{b}' \in B$ with the same $\Delta$-type over $\{\overline{\mathbf{c}}\}$ as $\overline{b}$ are also such that $\varphi(\overline{x}; \overline{b}') \in p(\overline{x})$. So we see that this actually partitions $B$ and $I^*$ describes this partition. Therefore, it is easy to check that $\psi_{I^*}(\overline{y}; \overline{\mathbf{c}})$ defines $p$, as desired. $\square$

A question one could ask is which $\Delta$ are needed to get finite non-splitting. A particularly nice set $\Delta$ is the following: Given $\varphi(\overline{x}; \overline{y})$ a partitioned formula and $n < \omega$, let

$$\Delta_{n,\varphi}(\overline{y}; \overline{z}_0, ..., \overline{z}_{n-1}) = \left\{ \exists \overline{x} \left( \varphi(\overline{x}; \overline{y})^s \wedge \bigwedge_{i<n} \varphi(\overline{x}; \overline{z}_i)^{t(i)} \right) : s < 2, t \in {}^n 2 \right\}. \tag{2.5}$$

This is the same as the $\Delta_{n,\varphi}$ defined in (2.2) except that we have partitioned the formulas. This particular set $\Delta_{n,\varphi}$ is used in Section 3.3 when we discuss the Isolated Extension Theorem. For this discussion, we show how $\Delta_{n,\varphi}$ is related to the notion of coherence from [14]. In this paper, Johnson defines coherence as follows: Fix a partitioned formula $\varphi(\overline{x};\overline{y})$, $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $B_0 \subseteq B$, and $p \in S_\varphi(B)$ (we use the language of Section 2.4, see Definition 2.4.7).

**Definition 2.6.3.** We say that $B_0$ is *coherent at $p$* if

(i) For all $\overline{b} \in B$, there exists $q \in S_\varphi(B_0)$ such that $q$ decides $\varphi(\overline{x};\overline{b})$; and

(ii) For any $\overline{b}_0, \overline{b}_1 \in B$ such that, for all $q \in S_\varphi(B_0)$, $q$ decides $\varphi(\overline{x};\overline{b}_0)$ if and only if $q$ decides $\varphi(\overline{x};\overline{b}_1)$, we have that there exists $q_0 \in S_\varphi(B_0)$, $t(0), t(1) < 2$ such that $q_0(\overline{x}) \vdash \varphi(\overline{x};\overline{b}_0)^{t(0)} \wedge \varphi(\overline{x};\overline{b}_1)^{t(1)}$ and either

   (a) $\varphi(\overline{x};\overline{b}_0)^{t(0)}, \varphi(\overline{x};\overline{b}_1)^{t(1)} \in p(\overline{x})$ or

   (b) $\neg\varphi(\overline{x};\overline{b}_0)^{t(0)}, \neg\varphi(\overline{x};\overline{b}_1)^{t(1)} \in p(\overline{x})$.

Finally, say that $\varphi$ is *coherent* if there exists $k < \omega$ such that, for all finite $B$ and $p \in S_\varphi(B)$, there exists $B_0 \subseteq B$ with $|B_0| \leq k$ such that $B_0$ is coherent at $p$.

We get the following result relating non-$\Delta_{n,\varphi}$-splitting to coherence:

**Proposition 2.6.4.** *Fix $B_0 \subseteq B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $B_0$ finite and $p \in S_\varphi(B)$ and let $n = |B_0|$. If $B_0$ is coherent at $p$, then $p$ does not $\Delta_{n,\varphi}$-split over $\{\overline{\mathbf{c}}\}$, where $\overline{\mathbf{c}}$ is an enumeration (in any order) of the elements of $B_0$.*

*Proof.* Fix $B_0 \subseteq B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $B_0$ finite, $p \in S_\varphi(B)$, and suppose that $B_0$ is coherent at $p$. Let $n = |B_0|$, $\overline{\mathbf{c}}$ be an enumeration of all of the elements in $B_0$, and let $\Delta = \Delta_{n,\varphi}$. Fix $\bar{b}_0, \bar{b}_1 \in B$ such that $\mathrm{tp}_\Delta(\bar{b}_0/\{\bar{\mathbf{c}}\}) = \mathrm{tp}_\Delta(\bar{b}_1/\{\bar{\mathbf{c}}\})$. Therefore, for all $q \in S_\varphi(B_0)$, $q$ decides $\varphi(\bar{x}; \bar{b}_0)$ if and only if $q$ decides $\varphi(\bar{x}; \bar{b}_1)$ (because, in the type $\mathrm{tp}_\Delta(\bar{b}_0/\{\bar{\mathbf{c}}\})$ are formulas of the form

$$\pm\forall\bar{x}\left(\bigwedge q(\bar{x}) \to \varphi(\bar{x}; \bar{y})^t\right)$$

for all $q \in S_\varphi(B_0)$ and $t < 2$). By clause (ii) of Definition 2.6.3, this implies that there exists $q_0 \in S_\varphi(B_0)$, $t(0), t(1) < 2$ such that $q_0 \vdash \varphi(\bar{x}; \bar{b}_\ell)^{t(\ell)}$ for both $\ell < 2$ and either $\varphi(\bar{x}; \bar{b}_\ell)^{t(\ell)} \in p(\bar{x})$ for both $\ell < 2$ or $\neg\varphi(\bar{x}; \bar{b}_\ell)^{t(\ell)} \in p(\bar{x})$ for both $\ell < 2$. However, since $\mathrm{tp}_\Delta(\bar{b}_0/\{\bar{\mathbf{c}}\}) = \mathrm{tp}_\Delta(\bar{b}_1/\{\bar{\mathbf{c}}\})$, we see that $t(0) = t(1)$. Therefore, $\varphi(\bar{x}; \bar{b}_0) \in p(\bar{x})$ if and only if $\varphi(\bar{x}; \bar{b}_1) \in p(\bar{x})$. This means exactly that $p$ does not $\Delta$-split over $\{\bar{\mathbf{c}}\}$, as desired. $\square$

This generalizes Theorem 4.2 of [14], which states that if a formula $\varphi$ is coherent, then it has UDTFS. If a formula $\varphi$ is coherent, then $\Delta = \Delta_{n,\varphi}$ suffices to show that $\varphi$ has finite non-splitting by Proposition 2.6.4. Therefore, by Theorem 2.6.2, $\varphi$ has UDTFS. By Theorem 5.4 of [14], formulas of the form $\varphi(x; \bar{y})$ in dp-minimal theories are coherent. Therefore, $\Delta = \Delta_{n,\varphi}$ suffices for such formulas to show they have finite non-splitting. This leads to the following open question, which is far stronger than the UDTFS Conjecture:

**Open Question 2.6.5** (Splitting Conjecture). *If $\varphi$ is dependent, then does there exist $n < \omega$ such that $\Delta = \Delta_{n,\varphi}$ witnesses that $\varphi$ has finite non-splitting?*

Surprisingly enough, this holds for stable formulas. In fact, we have the following characterization of stability:

**Theorem 2.6.6.** *A partitioned formula $\varphi(\overline{x};\overline{y})$ is stable if and only if there exists $n<\omega$ such that, for all $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $p \in S_{\varphi}(B)$, there exists $\overline{\mathbf{c}} \in B^n$ such that $p$ does not $\Delta_{n,\varphi}$-split over $\{\overline{\mathbf{c}}\}$.*

Our proof of this theorem is based on the proof of Lemma 2.2 in [19]. In that lemma, Pillay shows that, over models, one can take the type definition of a stable formula to be a boolean combination of instances of the that formula.

One direction of Theorem 2.6.6 is clear by definability of types (see Theorem 1.3.2 and the proof of Theorem 2.6.2 above). So assume that $\varphi$ is stable and let $N<\omega$ be such that there exists no $\overline{a}_i \in \mathfrak{C}^{\lg(\overline{x})}$ and $\overline{b}_j \in \mathfrak{C}^{\lg(\overline{y})}$ for $i,j<N$ such that, for all $i,j<N$, $\models \varphi(\overline{a}_i;\overline{b}_j)$ if and only if $i<j$ and there exists no $\overline{a}_i' \in \mathfrak{C}^{\lg(\overline{x})}$ and $\overline{b}_j' \in \mathfrak{C}^{\lg(\overline{y})}$ for $i,j<N$ such that, for all $i,j<N$, $\models \neg\varphi(\overline{a}_i';\overline{b}_j')$ if and only if $i<j$.

For each $n<2N$, let

$$X_n = \{0,1\} \times \mathcal{P}(\{0,1,...,n-1\})$$

and let $M_n = |X_n| = 2^{n+1}$. Let $f^n : M_n \to X_n$ be any bijection (to put an ordering on $X_n$). For any $m<M_n$, let $f^n(m) = (f_0^n(m), f_1^n(m))$ for $f_0^n(m)<2$ and $f_1^n(m) \in \mathcal{P}(n)$. Finally, let $M = \sum_{n<2N} M_n$.

Now start with any $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and any $p \in S_{\varphi}(B)$. For simplicity, let $\overline{a} \models p$. Assume that there exists $\overline{b}, \overline{b}' \in B$ such that $\models \varphi(\overline{a};\overline{b})$ and $\models \neg\varphi(\overline{a};\overline{b}')$, otherwise $p$ does not $\Delta_{M,\varphi}$-split over any subset of $B^M$. We now construct inductively, for each $n<2N$ and $m<M_n$, $I_m^n \subseteq M_n$, $q^n(\overline{x}_0,...,\overline{x}_n)$ and $q_m^n(\overline{x}_0,...,\overline{x}_n)$ (consistent) partial types over $B$, and $\overline{b}_m^n \in B$ as follows:

For $n=0$ and $m<M_0=2$, let $\overline{b}_m^0 \in B$ be such that $\models \neg\varphi(\overline{a};\overline{b}_m^0)^{f_0^0(m)}$ (these

exists by hypothesis). Let $I_0 = \{0, 1\}$ and let $q^0_0 = q^0_1 = \emptyset$.

Suppose now that $n \geq 0$ and, for all $k < n$, $I^k_m$, $q^k_m$, and $\overline{b}^k_m$ are defined for all $m < M_k$. Then, let

$$q^n(\overline{x}_0, ..., \overline{x}_n) = q^n_{M_n-1}(\overline{x}_0, ..., \overline{x}_{n-1}) \cup \{\neg\varphi(\overline{x}_n; \overline{b}^k_m)^{f^k_0(m)} : k \leq n, m < M_k\}.$$

In other words, $q^n$ is $q^n_{M_n-1}$ together with $\mathrm{tp}_\varphi(\overline{a}/\{\overline{b}^k_m : k \leq n, m < M_k\})(\overline{x}_n)$. Since $q^n_{M_n-1}$ is consistent and $\overline{a}$ realizes the $\overline{x}_n$ portion of the type, then we see that $q^n$ is consistent.

Suppose now that $n \geq 1$, $m < M_n$, $q^{n-1}$ is defined, and $I^n_{m'}$, $q^n_{m'}$, and $\overline{b}^n_{m'}$ are defined for all $m' < m$ (when $m = 0$, set $I^n_{-1} = I^{n-1}_{M_{n-1}-1}$ and $q^n_{-1} = q^{n-1}$, so we simply include the items from the previous level of the construction). Let $(t, W) = f^n(m)$, so $t = f^n_0(m) < 2$ and $W = f^n_1(m) \in \mathcal{P}(n)$. If there exists $\overline{b}^n_m \in B$ such that

(i) $\models \neg\varphi(\overline{a}; \overline{b}^n_m)^t$, and

(ii) $q^n_{m-1}(\overline{x}_0, ..., \overline{x}_{n-1}) \cup \{\varphi(\overline{x}_i; \overline{b}^n_m)^t : i \in W\}$ is consistent,

then let $\overline{b}^n_m \in B$ be such a witness, let $I^n_m = I^n_{m-1} \cup \{m\}$, and let

$$q^n_m(\overline{x}_0, ..., \overline{x}_{n-1}) = q^n_{m-1}(\overline{x}_0, ..., \overline{x}_{n-1}) \cup \{\varphi(\overline{x}_i; \overline{b}^n_m)^t : i \in W\}.$$

By assumption, $q^n_m$ is consistent. If there exists no such $\overline{b}^n_m \in B$ satisfying (i) and (ii), then let $I^n_m = I^n_{m-1}$, let $q^n_m = q^n_{m-1}$, and let $\overline{b}^n_m \in B$ be arbitrary so that only (i) holds (again, this exists by hypothesis).

Finally, let $I^n = I^n_{M_n-1}$ and let $q = q^{2N-1}$. We have now constructed $q$ a partial type over $B$ with at most $M$ elements from $B$. Now consider the following lemma.

**Lemma 2.6.7.** *For any $W \subseteq 2N$ and any $t < 2$, if there exists $\overline{b} \in B$ such that*

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b})^t : i \in W\}$$

*is consistent and $\models \neg\varphi(\overline{a}; \overline{b})^t$, then, for any realization $(\overline{a}_0, ..., \overline{a}_{2N-1}) \models q$, there exists $m_k \in M_k$ for $k \in W$ such that, for all $i, k \in W$,*

$$\models \varphi(\overline{a}_i; \overline{b}^k_{m_k})^t \text{ if and only if } i < k.$$

*That is, $\langle \overline{a}_i : i \in W\rangle$ and $\langle \overline{b}^k_{m_k} : k \in W\rangle$ witness that $\varphi^t$ has the order property of length $|W|$.*

*Proof.* Fix $(\overline{a}_0, ..., \overline{a}_{2N-1}) \models q$. Fix any $k \in W$ and let $W' = W \cap k$. Choose $m = m_k < M_k$ such that $f^k_0(m) = t$ and $f^k_1(m) = W'$. We first claim that $m \in I^k$.

Since

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b})^t : i \in W\}$$

is consistent, so is the subtype

$$q^k_{m-1}(\overline{x}_0, ..., \overline{x}_{k-1}) \cup \{\varphi(\overline{x}_i; \overline{b})^t : i \in W'\}$$

(if $m = 0$, let $q^k_{-1} = q^{k-1}$ as in the construction above). Thus cases (i) and (ii) of the construction are met (for condition (i), note that $\models \neg\varphi(\overline{a}; \overline{b})^t$). Therefore, $m \in I^k$ and, furthermore, $\overline{b}^k_m$ is a witness to this fact. That is, $\models \neg\varphi(\overline{a}; \overline{b}^k_m)^t$ and

$$\{\varphi(\overline{x}_i; \overline{b}^k_m)^t : i \in W'\} \subseteq q.$$

Thus, by construction, for all $i < k$ with $i \in W$ (i.e. all $i \in W'$), $\varphi(\overline{x}_i; \overline{b}^k_m)^t \in q$. Hence, $\models \varphi(\overline{a}_i; \overline{b}^k_m)^t$. On the other hand, for all $i \geq k$, $\neg\varphi(\overline{x}_i; \overline{b}^k_m)^t \in q$ (since, by construction, in $q$ the variables $\overline{x}_i$ for $i \geq k$ match $\overline{a}$ on $\overline{b}^k_m$). Thus we see that, for all $i \in W$,

$$\models \varphi(\overline{a}_i; \overline{b}^k_{m_k})^t \text{ if and only if } i < k.$$

This yields the desired conclusion. □

Therefore, since we chose $N$ to witness the fact that the order property fails for $\varphi$ and $\neg\varphi$, if $W \subseteq 2N$ with $|W| = N$, then, for any $t < 2$, if there exists $\overline{b} \in B$ such that

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b})^t : i \in W\}$$

is consistent, then $\models \varphi(\overline{a}; \overline{b})^t$. However, by pigeon-hole principal, for any $\overline{b} \in B$, there exists $t < 2$ and $W \subseteq 2N$ with $|W| = N$ such that

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b})^t : i \in W\}$$

is consistent. Therefore, for all $\overline{b} \in B$, $\models \varphi(\overline{a}; \overline{b})$ if and only if, for some $W \subseteq 2N$ with $|W| = N$,

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b}) : i \in W\}$$

is consistent. With this, we are ready to prove Theorem 2.6.6.

*Proof of Theorem 2.6.6.* Suppose that $\varphi(\overline{x}; \overline{y})$ is stable and $N$ and $M$ are as above. We claim that $n = M$ suffices. Fix any $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $p \in S_\varphi(B)$. Let $q$ and $\overline{b}^k_m \in B$ for $k < 2N$ and $m < M_k$ be as above. Finally, let $\overline{\mathbf{c}} = (\overline{b}^k_m : k < 2N, m < M_k)$. We claim that $p$ does not $\Delta_{M,\varphi}$-split over $\{\overline{\mathbf{c}}\}$.

Fix $\overline{b}, \overline{b}' \in B$ such that $\varphi(\overline{x}; \overline{b}) \in p(\overline{x})$ and $\neg\varphi(\overline{x}; \overline{b}') \in p(\overline{x})$. By the characterization above, there exists $W \subseteq 2N$ with $|W| = N$ such that

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b}) : i \in W\}$$

is consistent and

$$q(\overline{x}_0, ..., \overline{x}_{2N-1}) \cup \{\varphi(\overline{x}_i; \overline{b}') : i \in W\}$$

is inconsistent (since it is for all such $W$). This implies that

$$\theta(\overline{y}) = \exists \overline{x}_0 ... \exists \overline{x}_{2N-1} \left( \bigwedge q(\overline{x}_0, ..., \overline{x}_{2N-1}) \wedge \bigwedge_{i \in W} \varphi(\overline{x}_i; \overline{y}) \right)$$

holds for $\overline{b}$ and fails for $\overline{b}'$. For each $k < 2N$, let

$$q_k(\overline{x}) = \{\varphi(\overline{x}; \overline{b}^k_m)^t : m < M_k, t < 2, \varphi(\overline{x}_k; \overline{b}^k_m)^t \in q\}$$

be the restriction of $q$ to the variable $\overline{x}_k$. Then, we see that $\theta(\overline{y})$ is equivalent to

$$\theta'(\overline{y}) = \bigwedge_{k < 2N, k \in W} \exists \overline{x} \left( \bigwedge q_k(\overline{x}) \wedge \varphi(\overline{x}; \overline{y}) \right) \wedge \bigwedge_{k < 2N, k \notin W} \exists \overline{x} \left( \bigwedge q_k(\overline{x}) \right).$$

However, $\theta'$ is equivalent to a boolean combination of formulas from $\Delta_{M,\varphi}(\overline{y}; \overline{\mathbf{c}})$. Therefore, since $\models \theta'(\overline{b}) \wedge \neg\theta'(\overline{b})$, we see that

$$\mathrm{tp}_{\Delta_{M,\varphi}}(\overline{b}/\{\overline{\mathbf{c}}\}) \neq \mathrm{tp}_{\Delta_{M,\varphi}}(\overline{b}'/\{\overline{\mathbf{c}}\}).$$

This means exactly that $p$ does not $\Delta_{M,\varphi}$-split over $\{\overline{\mathbf{c}}\}$. $\square$

## 2.7 Applications to Compression Schemes

### 2.7.1 Concept Classes and Compression Schemes

In [15], Johnson and Laskowski discover a relationship between UDTFS and (extended) compression schemes from computer science. In fact, the idea of compression schemes was the inspiration for Laskowski's definition of UDTFS. This chapter summarizes the work of Johnson and Laskowski from [15] and discusses how the results of this thesis contribute to the field of computer science.

The basic idea is the following: We have a set $X$ and a set of subsets of $X$, $\mathcal{C}$, and we wish to measure the complexity of this set on finite subsets of $X$ by sampling points in $X$. For simplicity of notation, we actually take $\mathcal{C} \subseteq {}^X 2$ (where $2 = \{0, 1\}$) and think of these as the characteristic functions of the subsets of $X$ in question. We call such $\mathcal{C}$ *concept classes* on $X$. For this discussion, we are only interested in finite subsets of $X$, so define $\mathcal{C}|_{\text{fin}}$ as follows:

$$\mathcal{C}|_{\text{fin}} = \{f|_Y : f \in \mathcal{C}, Y \subseteq X \text{ finite}\}.$$

Consider the following definition from [15] which is an adaptation of the definition of an extended compression scheme introduced by Floyd and Warmuth in [7]:

**Definition 2.7.1.** Fix $\mathcal{C} \subseteq {}^X 2$ and $d < \omega$. We say that $\mathcal{C}$ has an (*extended $d$-sequence*) *compression scheme* if there exists a compression function $\kappa : \mathcal{C}|_{\text{fin}} \to X^d$ and a set of finitely many recovery functions $\mathcal{R}$ (with $\rho : X^d \to {}^X 2$ for each $\rho \in \mathcal{R}$) such that, for every $f \in \mathcal{C}$ and $Y \subseteq X$ finite, there exists $\rho \in \mathcal{R}$ such that

(i) $\kappa(f|_Y) \in Y^d$; and

(ii) $\rho(\kappa(f|_Y))$ extends $f|_Y$ (as functions).

Finally, we simply say that $\mathcal{C}$ has a *compression scheme* if it has a extended $d$-sequence compression scheme for some $d < \omega$. The *dimension* of a compression scheme is the minimal such $d$.

This captures one method of measuring the complexity of $\mathcal{C}$. Another way of capturing the complexity of a concept class is the VC-dimension. We say that a concept class $\mathcal{C} \subseteq {}^X2$ has *VC-dimension* $d$ for some $d < \omega$ if $d$ is maximal such that there exists $Y \subseteq X$ with $|Y| = d$ and $\left|\mathcal{C}|_Y\right| = 2^d$ (where $\mathcal{C}|_Y = \{f|_Y : f \in \mathcal{C}\}$). We say that $\mathcal{C}$ is a *VC-class* if it has a finite VC-dimension.

Another measure of complexity is VC-density. A concept class $\mathcal{C} \subseteq {}^X2$ has *VC-density* $r$ for some $r \in \mathbb{R}$ if $r$ is the infimum of all $r' \in \mathbb{R}$ such that there exists $K < \omega$ such that, for all finite $Y \subseteq X$, $\left|\mathcal{C}|_Y\right| < K \cdot |Y|^{r'}$. By Sauer's Lemma [21], we get that if $\mathcal{C}$ has VC-dimension $d$, then $\mathcal{C}$ has VC-density $\leq d$. Conversely, it is clear that if $\mathcal{C}$ has finite VC-density, then $\mathcal{C}$ is a VC-class. One should also note that if $\mathcal{C}$ has a $d$-dimensional compression scheme, then it has VC-density $\leq d$. Therefore, if $\mathcal{C}$ has a compression scheme, then $\mathcal{C}$ is a VC-class.

**Example 2.7.2.** Let $X = \mathbb{R}$ and let $\mathcal{C} = \{\chi_{[a,b]} : a, b \in \mathbb{R}, a < b\}$, where $\chi_A$ denotes the characteristic function of $A \subseteq X$. So $\mathcal{C}$ is the concept class of closed intervals in $\mathbb{R}$. One checks that $\mathcal{C}$ has VC-dimension 2, VC-density 1, and a compression scheme of dimension 2 (the last fact actually follows from the discussion of UDTFS rank in Section 3.2).

Compression schemes are intimately related to UDTFS. Starting with $X$ any set and $\mathcal{C} \subseteq {}^X2$, construct an $L$-structure $M_{\mathcal{C}}$ in the language $L = \{U, R\}$ (for $U$ a unary relation symbol and $R$ a binary relation symbol) as follows: Let $M_{\mathcal{C}}$ have universe $\mathcal{C} \sqcup X$ (where $\sqcup$ denotes disjoint union), let $U^{M_{\mathcal{C}}} = \mathcal{C}$, and let $R(f, x)$ hold in $M_{\mathcal{C}}$ if and only if $f \in \mathcal{C}$, $x \in X$, and $f(x) = 1$. This provides a way to encode $\mathcal{C}$ into a model-theoretic structure. We now get the following result relating compression schemes to UDTFS (for a proof, see the proof of Proposition 3.2.9 below):

**Proposition 2.7.3** (Johnson and Laskowski, [15])**.** *If $\mathcal{C} \subseteq {}^X 2$ is such that $R$ has UDTFS in $M_{\mathcal{C}}$, then $\mathcal{C}$ has a compression scheme.*

We can also go the other direction, starting with a formula with UDTFS and producing a concept class with a compression scheme. Suppose $\varphi(\overline{x}; \overline{y})$ is a partitioned formula in $T$ with UDTFS (and recall that $\mathfrak{C}$ is a monster model for $T$). Let $X = \mathfrak{C}^{\lg(\overline{y})}$ and let

$$\mathcal{C}_\varphi = \left\{ \chi_{\varphi(\overline{a}; \mathfrak{C})} : \overline{a} \in \mathfrak{C}^{\lg(\overline{x})} \right\}.$$

We get the following result:

**Proposition 2.7.4** (Johnson and Laskowski, [15])**.** *If $\varphi$ has UDTFS, then $\mathcal{C}_\varphi$ has a compression scheme.*

In [15], they mention that $\mathcal{C}_\varphi$ has a compression scheme for any stable $\varphi$ and prove that $\mathcal{C}_\varphi$ has a compression scheme for any $\varphi$ from an o-minimal theory, $T$ (by showing that such theories have UDTFS).

The results of this thesis produce more concept classes that have compression schemes. By Theorem 2.4.1, all concept classes $\mathcal{C}_\varphi$ with $\varphi$ from a dp-minimal theory have a compression scheme. For example, by the results of [6], this implies that concept classes $\mathcal{C}_\varphi$ with $\varphi$ from $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |, 0, 1)$ have compression schemes.

**Example 2.7.5.** Working in $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |, 0, 1)$, let $\varphi(x_0, x_1; y) = x_1 | (y - x_0)$. Then $\mathcal{C}_\varphi$ is the following concept class on $\mathbb{Q}_p$:

$$\mathcal{C}_\varphi = \{\chi_D : D \subseteq \mathbb{Q}_p, D \text{ is a closed ball in } \mathbb{Q}_p\}.$$

Therefore, we see that the concept class of closed balls in $\mathbb{Q}_p$ has a compression scheme. Similarly, the concept class of open balls in $\mathbb{Q}_p$ has a compression scheme.

By the results of Section 2.5, fields such as $\mathbb{R}((t))$ have UDTFS. Therefore, concept classes of uniformly definable subsets of $\mathbb{R}((t))$ also have compression schemes. The same is true of other fields such as $\mathbb{C}((t))$ and $\mathbb{C}((t^{\mathbb{Q}}))$. Considering UDTFS rank in Section 3.2 allows us to bound the dimension of the compression scheme. If we suppose that $\varphi$ has UDTFS rank $r$, then $\mathcal{C}_\varphi$ has a compression scheme of dimension $\leq r$.

Theorem 2.4.3 provides an interesting result on concept classes. Fix $X$ any set and $\mathcal{C} \subseteq {}^X 2$ any concept class. Then, we get the following result:

**Corollary 2.7.6.** *If there exists $N < \omega$ such that, for all $Y \subseteq X$ with $|Y| = N$, $\left|\mathcal{C}|_Y\right| \leq \frac{N(N+1)}{2}$, then $\mathcal{C}$ has a compression scheme. In particular, if $\mathcal{C}$ has VC-density $< 2$, then $\mathcal{C}$ has a compression scheme.*

This analysis of compression schemes leads to several open questions. The biggest of which is known as the Warmuth Conjecture (from [7]):

**Conjecture 2.7.7** (Warmuth Conjecture)**.** *If $\mathcal{C}$ is a VC-class, then $\mathcal{C}$ has a compression scheme.*

Of course, the UDTFS Conjecture implies the Warmuth Conjecture, but not conversely. In fact, this leads to another interesting open question:

**Open Question 2.7.8.** *If $\varphi$ is such that $\mathcal{C}_\varphi$ has a compression scheme, then does $\varphi$ have UDTFS?*

Certainly UDTFS implies a compression scheme, but the converse need not hold. In the definition of compression schemes, there is no mention of "definability" of the compression scheme. In fact, we could even alter the language of $M_{\mathcal{C}}$ and, if $R$ has UDTFS in that expanded language, then $\mathcal{C}$ would still have a compression scheme.

### 2.7.2 Maximum Formulas

Going the other direction, we use a result about concept classes to prove that a certain class of formulas has UDTFS. First, by Sauer's Lemma, we know that if a partitioned formula $\varphi(\overline{x};\overline{y})$ has independence dimension $d < \omega$, then, for any finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| \geq d$,

$$\left|S_{\varphi}(B)\right| \leq \sum_{i \leq d} \binom{|B|}{i}.$$

When a formula attains this maximum bound for all such $B$, then we say that this formula is *maximum of dimension* $d$. For example, consider any infinite set $X$ and let $\mathcal{C}_d = \{f \in {}^X2 : |\mathrm{supp}(f)| \leq d\}$ for some fixed $d < \omega$ (where $\mathrm{supp}(f) = \{x \in X : f(x) = 1\}$). Finally, let $M = M_{\mathcal{C}_d}$ in the language $L = \{U, R\}$ as defined in the previous subsection. Then one can see that $R$ is a maximum formula of dimension $d$ in $M$. There are other examples of maximum formulas, and these were studied in the context of concept classes by Floyd and Warmuth in [7].

Given a concept class $\mathcal{C}$ on a set $X$, we say that $\mathcal{C}$ is *maximum of dimension* $d < \omega$ if $R$ is maximum with dimension $d$ in $M_{\mathcal{C}}$. That is, for all finite $Y \subseteq X$ with

$|Y| \geq d$,

$$\left|\mathcal{C}|_Y\right| = \sum_{i \leq d} \binom{|Y|}{i}.$$

We get the following result about maximum concept classes.

**Theorem 2.7.9** (Theorem 10 of [7])**.** *Fix $d < \omega$, $X$ a finite set with $|X| \geq d$, and let $\mathcal{C} \subseteq {}^X2$ be a maximum concept class of dimension $d$. Then, for each $f \in \mathcal{C}$, there exists $Y \subseteq X$ with $|Y| = d$ such that, for all $x \in (X - Y)$ and all $g \in {}^Y2$, $g \cup f|_{\{x\}} \in \mathcal{C}|_{Y \cup \{x\}}$.*

That is, the value of $f$ on any $x \in (X - Y)$ is determined, since there is at most one $t < 2$ such that $g \cup \{(x, t)\}$ is extendable to a function in $\mathcal{C}$ for all $g \in {}^Y2$ (otherwise, $Y \cup \{x\}$ would witness that $\mathcal{C}$ has VC-dimension $d+1$, a contradiction). Using the dictionary between concept classes and formulas, we can use Theorem 2.7.9 to get a result for maximum formulas.

**Theorem 2.7.10.** *If $\varphi(\overline{x}; \overline{y})$ is maximum of dimension $d < \omega$, then $\varphi$ has UDTFS. Furthermore, $\varphi$ has UDTFS rank $\leq d$ (see Section 3.2).*

*Proof.* Fix $\varphi(\overline{x}; \overline{y})$ a maximum formula of dimension $d < \omega$. Consider, for each $s \in {}^d2$, the formula

$$\psi_s(\overline{y}; \overline{z}_0, ..., \overline{z}_{d-1}) = \left( \bigwedge_{i<d} \overline{y} \neq \overline{z}_i \wedge \bigwedge_{t \in {}^d2} \exists \overline{x} \left( \varphi(\overline{x}; \overline{y}) \wedge \bigwedge_{i<d} \varphi(\overline{x}; \overline{z}_i)^{t(i)} \right) \right) \vee$$
$$\bigvee_{i<d, s(i)=1} (\overline{y} = \overline{z}_i).$$

We claim that these define $\varphi$-types over finite sets. Fix any $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite and $p \in S_\varphi(B)$. If $|B| < d$, we can use $\psi_s$ trivially to define $p$. For each $q \in S_\varphi(B)$, let

$\delta_q \in {}^B 2$ be the function associated to $q$ (so $\varphi(\overline{x};\overline{b})^{\delta_q(\overline{b})} \in q(\overline{x})$ for all $\overline{b} \in B$). Let $X = B$ and let

$$\mathcal{C} = \{\delta_q : q \in S_\varphi(B)\}.$$

It is easy to show that $\mathcal{C}$ is of maximum dimension $d$, so the hypotheses of Theorem 2.7.9 are met. Therefore, if we let $f = \delta_p$, then there exists $B_0 \subseteq B$ with $|B_0| = d$ such that, for all $\overline{b} \in (B - B_0)$, $\varphi(\overline{x};\overline{b}) \in p(\overline{x})$ if and only if $\delta_p(\overline{b}) = 1$ if and only if $f(\overline{b}) = 1$ if and only if $g \cup \{(\overline{b}, 1)\}$ extends to a function in $\mathcal{C}$ for all $g \in {}^{B_0}2$. This is true if and only if

$$\varphi(\overline{x};\overline{b}) \wedge \bigwedge_{i<d} \varphi(\overline{x};\overline{c}_i)^{t(i)}$$

is consistent for every $t \in {}^d 2$ where $B_0 = \{\overline{c}_i : i < d\}$. Therefore, letting $s \in {}^d 2$ be such that $s(i) = \delta_p(\overline{c}_i)$, we see that

$$\psi_s(\overline{y}; \overline{c}_0, ..., \overline{c}_{d-1})$$

defines $p$. Thus, $\varphi$ has UDTFS, as desired.

Furthermore, since $\psi_s$ has only $d$ parameter tuples, we see that the UDTFS rank of $\varphi$ is $\leq d$. $\square$

Moreover, we see from the proof above that if $\varphi$ is maximum of dimension $d < \omega$, then the Splitting Conjecture (Open Question 2.6.5) holds for $\varphi$ (with $n = d$).

Combined with Theorem 2.4.3, one may be tempted to conclude that this implies all formulas $\varphi$ with independence dimension two have UDTFS. However, this does not necessarily follow.

**Example 2.7.11.** Let $X_n$ be an $n$-element set for all $n \geq 2$ and let $X = \coprod_{n\geq 2} X_n$. Let $\mathcal{C}_n$ be maximum of dimension 2 on $X_n$. Finally, let

$$\mathcal{C} = \{c \in {}^{X}2 : (\exists n \geq 2)(c|_{X_n} \in \mathcal{C}_n \wedge (\forall m \neq n)(c|_{X_m} = 0))\}.$$

Let $M = M_{\mathcal{C}}$ as above. Then, for any $n \geq 2$,

$$\left|S_R(X_n)\right| > n(n+1)/2.$$

So the condition of Theorem 2.4.3 is not met. However, $R$ is clearly not maximal of dimension 2, so the condition of Theorem 2.7.10 above is not met. So we cannot conclude from these that $R$ has UDTFS. However, one can see that this particular $R$ actually does have UDTFS.

It is the opinion of this author that we should somehow be able to amalgamate the procedure of Theorem 2.7.10 and the procedure of Theorem 2.4.3 to produce a means of showing that formulas of independence dimension two have UDTFS. With any luck, this may also give a means of showing, in general, that dependent formulas have UDTFS.

# Chapter 3

# Other Definability of Types Notions in Dependent Theories

## 3.1 Overview

The main goal of this chapter is to study other generalizations of definability of types to subclass of dependent theories. The study of these notions branched off of this author's study into UDTFS.

In Section 3.2, we study UDTFS rank and show how it relates to VC-density and other ranks. This ties up some loose ends from the last chapter, including how to compute compression scheme dimensions from model-theoretic notions. Our main result is Theorem 3.2.4, where we show how to calculate the UDTFS rank of all formulas from only those with one free variable. This has applications to showing certain theories have VC-density one.

In Section 3.3, we examine isolated extensions as another alternative to definability of types, this time for all dependent formulas. The main result of this section is the Isolated Extension Theorem, Theorem 3.3.3. This has several corollaries, including a new notion of definability of types and a weak notion of stable embeddability for dependent theories (see Corollary 3.3.5).

In Section 3.4, we talk briefly about the work of Pierre Simon in [28] and give some results relating to dp-minimal theories. In particular, we show that Conjecture 4.7 of [28] holds for dp-minimal theories with a linear order (see Theorem 3.4.3).

## 3.2 UDTFS ranks and VC-Density

### 3.2.1 UDTFS rank

In this section, the main object of our study is UDTFS rank. This rank is closely related to the VC$m$ property from [3].

**Definition 3.2.1.** A formula $\varphi(\overline{x};\overline{y})$ has *UDTFS rank* $n < \omega$ (denoted $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) = n$) if $n$ is minimal such that there exists finitely many formulas $\{\psi_\ell(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) : \ell < L\}$ such that, for all finite non-empty $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and for all $p(\overline{x}) \in S_\varphi(B)$, there exists $\ell < L$ and $\overline{c}_0,...,\overline{c}_{n-1} \in B$ such that $\psi_\ell(\overline{y};\overline{c}_0,...,\overline{c}_{n-1})$ defines $p$. If no such $n$ exists, we let $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) = \infty$.

Clearly $\varphi$ has UDTFS if and only if $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) < \infty$. We list several other simple observations:

**Proposition 3.2.2.** *For a fixed partitioned formula $\varphi(\overline{x};\overline{y})$ with UDTFS, the following hold:*

*(i)* $\mathrm{VCden}(\varphi) \leq \mathrm{R}_{\mathrm{UDTFS}}(\varphi)$.

*(ii)* $\mathrm{alt}(\varphi) \leq 2 \cdot \mathrm{R}_{\mathrm{UDTFS}}(\varphi)$.

*Proof.* (i): Let $\{\psi_\ell(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) : \ell < L\}$ witness that $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) = n$ and we show that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $\left|S_\varphi(B)\right| \leq L \cdot |B|^n$, as desired. Each type $p \in S_\varphi(B)$ is associated to $\ell < L$ and $\overline{\mathbf{c}} \in B^n$ via a choice such that $\psi_\ell(\overline{y};\overline{\mathbf{c}})$ defines $p$. This is clearly an injective map, so $\left|S_\varphi(B)\right| \leq |L \times B^n| = L \cdot |B|^n$.

(ii): Again, let $\{\psi_\ell(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) : \ell < L\}$ witness that $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) = n$. Suppose, by means of contradiction, that $\mathrm{alt}(\varphi) > 2n$. Let $\langle \overline{b}_i : i \in I\rangle$ be an indiscernible sequence and $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$ witnessing $\mathrm{alt}(\varphi) > 2n$. Therefore, there exists $i_0 < i_1 < ... < i_{2n}$ from $I$ such that

$$\models \varphi(\overline{a}; \overline{b}_{i_j}) \leftrightarrow \neg\varphi(\overline{a}; \overline{b}_{i_{j+1}})$$

for all $j < 2n$ (i.e., the subsequence alternates with respect to the truth value of $\varphi(\overline{a}; \overline{y})$). Let $B = \{\overline{b}_{i_j} : j \leq 2n\}$ and fix $p = \mathrm{tp}_{\varphi}(\overline{a}/B)$. By definition of UDTFS rank, there exists $\ell < L$ and $\overline{c}_0, ..., \overline{c}_{n-1} \in B$ such that $\psi_\ell(\overline{y}; \overline{c}_0, ..., \overline{c}_{n-1})$ defines $p$. By pigeon-hole principal, there exists $j_0 < 2n$ such that $\overline{b}_{j_0}$ and $\overline{b}_{j_0+1}$ are not among the $\overline{c}_t$'s. Therefore, by indiscernibility,

$$\models \psi_\ell(\overline{b}_{j_0}; \overline{c}_0, ..., \overline{c}_{n-1}) \leftrightarrow \psi_\ell(\overline{b}_{j_0+1}; \overline{c}_0, ..., \overline{c}_{n-1}).$$

That is,

$$\varphi(\overline{x}; \overline{b}_{j_0}) \in p(\overline{x}) \text{ if and only if } \varphi(\overline{x}; \overline{b}_{j_0+1}) \in p(\overline{x}).$$

This contradicts the fact that the sequence alternates with respect to the truth value of $\varphi(\overline{a}; \overline{y})$. Therefore, $\mathrm{alt}(\varphi) \leq 2n$, as desired. $\square$

The result of Proposition 3.2.2 (ii) is surprising, since the exact same result holds if we replace $\mathrm{R}_{\mathrm{UDTFS}}$ with ID. That is, for any dependent formula $\varphi$, $\mathrm{alt}(\varphi) \leq 2 \cdot \mathrm{ID}(\varphi)$. This follows from the proof of Lemma 2.2.4 (i) and is originally due to Shelah in [22]. The relationship between the independence dimension and the UDTFS rank is unknown (they could, in fact, be equal). However, it is known that the UDTFS rank can be much higher than the VC-density. Consider the following example, due to Alfred Dolich (unpublished):

**Example 3.2.3.** Fix any $n \geq 2$. We exhibit an example where $\mathrm{VCden}(\varphi) = 1$ but $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) = n$.

Let $L = \{E, U, R\}$ for $E, R$ binary relation symbols and $X$ a unary relation symbol. Let $M$ be the $L$-structure given by

$$M = \omega \times (2n \sqcup \{I \subset 2n : |I| = n\}),$$

where we let $E^M((i,a),(j,b))$ hold if and only if $i = j$, $X^M(i,b)$ holds if and only if $b \in 2n$ (the first part of the disjoint union), and $R^M((i,a),(j,b))$ holds if and only if $i = j$, $a \in \{I \subset 2n : |I| = n\}$, $b \in 2n$, and $b \in a$. We claim that $R(x;y)$ has VC-density 1 and UDTFS rank $n$.

For any $(i,b) \in X^M$, there are at most $\binom{2n}{n}$ elements $(i,a)$ such that $M \models R((i,a),(i,b))$. From this, one easily checks that, for any finite $B \subseteq M$, $\left|S_R(B)\right| \leq \binom{2n}{n} \cdot |B|$, hence showing that $\mathrm{VCden}(R(x;y)) \leq 1$. One easily checks that it is, in fact, equal to one. Next, for any permutation $\sigma \in S_{2n}$ and any $i_0 < \omega$, there exists an $L$-automorphism of $M$, which we call $\sigma^{i_0}$, where $\sigma^{i_0}(i,x) = (i,x)$ for all $i \neq i_0$, $\sigma^{i_0}(i_0,x) = (i_0,\sigma(x))$ for $x \in 2n$, and $\sigma^{i_0}(i_0,\{p_0,...,p_{n-1}\}) = (i_0,\{\sigma(p_0),...,\sigma(p_{n-1})\})$. Fix $B = \{(0,b) : b \in 2n\}$ and let $a = (0,I)$ for any $I \subseteq 2n$, $|I| = n$, and consider $p = \mathrm{tp}_R(a/B)$. If we assume, by means of contradiction, that $\mathrm{R}_{\mathrm{UDTFS}}(R(x;y)) < n$, then there exists a defining formula $\psi(y, z_0, ..., z_{n-2})$ for $p$. That is, there exists $b_0, ..., b_{n-2} \in 2n$ such that $\psi(y,(0,b_0),...,(0,b_{n-2}))$ defines $p$. Choose any $c_0 \in I - \{b_0,...,b_{n-2}\}$ (which exists since $|I| = n$ and $|\{b_0,...,b_{n-2}\}| \leq n-1$) and $c_1 \in 2n - (I \cup \{b_0,...,b_{n-2}\})$ (which also exists by counting). Let $\sigma = (c_0\ c_1)$, the permutation of $S_{2n}$ swapping $c_0$ and $c_1$ but fixing everything else.

Then, since $c_0 \in I$, $R((0,I),(0,c_0))$ holds, hence $\psi((0,c_0),(0,b_0),...,(0,b_{n-2}))$ holds as this defines $p$. So we get

$$c_0 \in I \text{ hence } \models \sigma^0[\psi((0,c_0),(0,b_0),...,(0,b_{n-2}))] \text{ hence}$$
$$\models \psi((0,c_1),(0,b_0),...,(0,b_{n-2})) \text{ hence } c_1 \in I.$$

contrary to our choice of $c_1 \notin I$. Therefore, no such definition exists, so we see that $\mathrm{R}_{\mathrm{UDTFS}}(R(x;y)) \geq n$. One can check, however, that a definition with $n$ parameters exists. Also note that $\mathrm{ID}(R(x;y)) = n$.

We see from this example that both UDTFS rank and independence dimension are sensitive to small portions of the theory, while VC-density ignores smaller subpieces of the structure for a more big-picture look. Depending on the context, one of these viewpoints may be preferable.

Now that we have an understanding of how UDTFS rank relates to other ranks, we present the main result of this section.[1]

**Theorem 3.2.4.** *Fix $T$ a theory and $k < \omega$. If all formulas of the form $\varphi(x;\overline{y})$ have UDTFS rank $\leq k$, then all formulas of the form $\varphi(\overline{x};\overline{y})$ have UDTFS rank $\leq k \cdot \lg(\overline{x})$.*

Before proving this, we examine some of the consequences. Most notably, it allows us to measure the density of a theory.

**Definition 3.2.5.** For a fixed $k < \omega$, a theory $T$ has *VC-density* $k$ if, for all partitioned formulas $\varphi(\overline{x};\overline{y})$, $\mathrm{VCden}(\varphi) \leq k \cdot \lg(\overline{x})$.

[1] This result is similar to the sufficiency of a single variable result for the VC$m$ property in [3].

The following is an immediate consequence of Theorem 3.2.4 and Proposition 3.2.2 (i):

**Corollary 3.2.6.** *Fix $T$ a theory and $k < \omega$. If, all formulas of the form $\varphi(x;\overline{y})$ have UDTFS rank $\leq k$, then $T$ has VC-density $k$.*

A simple calculation shows the following, originally due to Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko in [3]:

**Corollary 3.2.7.** *If $T$ is weakly o-minimal, then $T$ has VC-density one.*

*Proof.* We actually show that any formula of the form $\varphi(x;\overline{y})$ have UDTFS rank $\leq 1$. By Corollary 3.2.6, this suffices to show that $T$ has VC-density one.

Fix any such formula, $\varphi(x;\overline{y})$. Then, by weak o-minimality of $T$, there exists some $K < \omega$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, $\varphi(\mathfrak{C};\overline{b})$ is a union of at most $K$ convex subsets of $\mathfrak{C}$. Let $\psi_i(x;\overline{y})$ be so that, for each $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, $\psi_i(\mathfrak{C};\overline{b})$ is the $i$th convex subset of $\pm\varphi(\mathfrak{C};\overline{b})$ (this is clearly definable, and there are at most $2K+1$ of them; see, for example, the proof of Proposition 4.3.3). For each $i < 2K+1$, let

$$\gamma_i(\overline{y};\overline{z}) = \forall x(\psi_i(x;\overline{z}) \to \exists x'(x' \geq x \wedge \psi_i(x';\overline{z}) \wedge \varphi(x';\overline{y}))).$$

We claim that $\{\gamma_i(\overline{y};\overline{z}) : i < 2K+1\}$ can be used to define $\varphi$-types over finite sets, thus showing that $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) \leq 1$. Fix any finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and $a \in \mathfrak{C}$. Then, for each $\overline{b} \in B$, let $V(\overline{b}) = \psi_i(\mathfrak{C};\overline{b})$ for the $i < 2K+1$ such that $\models \psi_i(a;\overline{b})$ (there exists exactly one such $i$). Let $\mathcal{V} = \{V(\overline{b}) \cap [a,\infty) : \overline{b} \in B\}$ and choose $\overline{b}_0 \in B$ such that $V(\overline{b}_0) \cap [a,\infty)$ is $\subseteq$-minimal in $\mathcal{V}$. Let $i_0$ be such that $V(\overline{b}_0) = \psi_{i_0}(\mathfrak{C};\overline{b}_0)$. Then we claim $\gamma_{i_0}(\overline{y};\overline{b}_0)$ defines $\mathrm{tp}_\varphi(a/B)$, as desired. For any $\overline{b} \in B$, $\gamma_{i_0}(\overline{b};\overline{b}_0)$ holds if and only if $V(\bar{b}_0)$ is cofinal in $\varphi(\mathfrak{C};\bar{b})$ if and only if $\varphi(a;\bar{b})$ holds (by minimality of $V(\bar{b}_0) \cap [a,\infty)$ in $\mathcal{V}$). $\square$

We now prove the main theorem, which essentially follows from the proof of Lemma 2.3.6.

*Proof of Theorem 3.2.4.* Fix a partitioned formula $\varphi(\overline{x};\overline{y})$ with $n = \lg(\overline{x})$. Suppose that $\hat{\varphi}(x_0,...,x_{n-2};x_{n-1},\overline{y}) = \varphi(\overline{x};\overline{y})$ has UDTFS rank $\leq K_0$ and we take a set of formulas

$$\{\psi_\ell(x_{n-1},\overline{y};w_0,\overline{z}_0,...,w_{K_0-1},\overline{z}_{K_0-1}) : \ell < L_0\}$$

witnessing this. As in the proof of Lemma 2.3.6, for each $\ell < L_0$, let

$$\psi^*_\ell(x_n;\overline{y},\overline{z}_0,...,\overline{z}_{k-1}) = \psi_\ell(x_{n-1},\overline{y};x_{n-1},\overline{z}_0,...,x_{n-1},\overline{z}_{k-1})$$

and suppose that $\psi^*_\ell(x_n;\overline{y},\mathbf{z})$ has UDTFS rank $\leq K_1$ for each $\ell < L_0$. For each $\ell < L_0$, suppose this is witnessed by

$$\left\{\gamma_{\ell,\ell'}(\overline{y},\overline{z}_0,...,\overline{z}_{K_0-1};\overline{v}_0,\overline{u}_{0,0},...,\overline{u}_{0,K_0-1},...,\overline{v}_{K_1-1},\overline{u}_{K_1-1,0},...,\overline{u}_{K_1-1,K_0-1}) : \ell' < L_1\right\}.$$

Again, as in the proof of Lemma 2.3.6, for each $\ell < L_0$ and $\ell' < L_1$, let

$$\begin{aligned}&\gamma^*_{\ell,\ell'}(\overline{y};\overline{z}_0,...,\overline{z}_{K_0-1},\overline{v}_0,...,\overline{v}_{K_1-1}) = \\ &\gamma_{\ell,\ell'}(\overline{y},\overline{z}_0,...,\overline{z}_{K_0-1};\overline{v}_0,\overline{z}_0,...,\overline{z}_{K_0-1},...,\overline{v}_{K_1-1},\overline{z}_0,...,\overline{z}_{K_0-1}).\end{aligned}$$

Then, just as in the proof of Lemma 2.3.6, we see that $\{\gamma^*_{\ell,\ell'} : \ell < L_0, \ell' < L_1\}$ is a witness to the fact that $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) \leq K_0 + K_1$.

Now, given the above information, we prove the theorem by induction on $n = \lg(\overline{x})$. For $n = 1$, this is given by hypothesis. Fix $n > 1$, fix $\varphi(\overline{x};\overline{y})$ with

$n = \lg(\overline{x})$. Note that $\hat{\varphi}$ has UDTFS rank $\leq k \cdot (n-1)$ by induction and each $\psi^*_\ell$ has UDTFS rank $\leq k$ by assumption. Therefore, $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) \leq k(n-1) + k = kn$, as desired. $\square$

This gives us one way of showing that theories have VC-density $k$ for various $k < \omega$. Are there other ways of proving theories have bounded VC-density? We come back to this question in Section 4.4 when we show that, in particular, all strongly minimal theories have VC-density one. This does not follow from Theorem 3.2.4; Example 4.3.5 below shows that strongly minimal theories can have formulas with UDTFS rank 2.

For a moment, we move back to the stable setting. The following theorem is due to Laskowksi (unpublished).

**Theorem 3.2.8** (Laskowski)**.** *Fix a stable formula $\varphi(\overline{x}; \overline{y})$. Then,*

$$\mathrm{R}_{\mathrm{UDTFS}}(\varphi) \leq R_{2,\varphi}(\emptyset),$$

*where $\emptyset$ is the empty $\varphi$-type.*

*Proof.* Since $\varphi$ is stable, $R_{2,\varphi}(\emptyset)$ is finite, say $n = R_{2,\varphi}(\emptyset)$. For each $s \in {}^n 2$, define

$$\theta_s(\overline{x}; \overline{z}_0, ..., \overline{z}_{n-1}) = \bigwedge_{i<n} \varphi(\overline{x}; \overline{z}_i)^{s(i)}$$

(so $\theta_s$ is used to encode a finite $\varphi$-type of size $\leq n$). For each $K < \omega$, let

$$\psi_{K,s}(\overline{y}; \mathbf{\overline{z}}) = \exists(\overline{w}_\nu : \nu \in {}^{\leq K}2)$$
$$\bigwedge_{\eta \in {}^K 2} \exists \overline{x} \left( \theta_s(\overline{x}; \mathbf{\overline{z}}) \wedge \varphi(\overline{x}; \overline{y}) \wedge \bigwedge_{i<K} \varphi(\overline{x}; \overline{w}_{\eta|_i})^{\eta(i)} \right).$$

So $\psi_{K,s}$ encodes that the type encoded by $\theta_s$, together with the positive instance of $\varphi(\overline{x};\overline{y})$, has Shelah 2-rank $\geq K$. We claim that $\{\psi_{K,s} : K \leq n, s \in {}^n2\}$ is a witness to the fact that $\varphi$ has UDTFS rank $\leq n$.

Fix any $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite and $p \in S_\varphi(B)$. Say that $R_{2,\varphi}(p) = \ell \leq n$. Let $q_0 = \emptyset$ and inductively define $q_i \subseteq p$ as follows: Fix $i > 0$. If there exists $\overline{b} \in B$ so that $q_{i-1} \cup p_{\{\overline{b}\}}$ has Shelah 2-rank $< R_{2,\varphi}(q_{i-1})$, then let $q_i = q_{i-1} \cup p_{\{\overline{b}\}}$ for any such $\overline{b}$. Otherwise, the construction halts and let $q = q_{i-1}$ and $K = R_{2,\varphi}(q)$. Since $R_{2,\varphi}(q_0) = n$ and $R_{2,\varphi}(p) = \ell$, this construction halts in at most $n - \ell$ steps and $K \leq n$. Now, by construction, for each $\overline{b} \in B$,

$$R_{2,\varphi}(q \cup p_{\{\overline{b}\}}) = R_{2,\varphi}(q) = K.$$

Therefore, $\varphi(\overline{x};\overline{b}) \in p(\overline{x})$ if and only if $q(\overline{x}) \cup \{\varphi(\overline{x};\overline{b})\}$ has Shelah 2-rank $\geq K$. Therefore, if we choose $\overline{c}_i \in B$ for $i < n$ and $s \in {}^n2$ so that

$$q(\overline{x}) = \{\varphi(\overline{x};\overline{c}_i)^{s(i)} : i < n\}$$

(allowing for some overlap if $|q| < n$), then we see that $\psi_{K,s}(\overline{y};\mathbf{\overline{c}})$ defines $p$, as desired. $\square$

### 3.2.2 Applications to Compression Schemes

We now discuss applications of UDTFS rank to compression schemes. Recall the definitions of concept classes and compression schemes from Subsection 2.7.1. Fix $X$ any set and let $\mathcal{C} \subseteq {}^X2$ be a concept class. Use the same definition of $M_\mathcal{C}$ as in Subsection 2.7.1. We get the following refinement of Proposition 2.7.3, which essentially follows from the proof in [15], but we include it here for completeness:

**Proposition 3.2.9** (Johnson and Laskowski, [15]). *If* $\mathrm{R}_{\mathrm{UDTFS}}(R(x;y)) = d$ *in* $M_{\mathcal{C}}$, *then* $\mathcal{C}$ *has an extended* $d$*-sequence compression scheme.*

*Proof.* Let $\{\psi_\ell(y; z_0, ..., z_{d-1}) : \ell < L\}$ witness the fact that $R(x;y)$ has UDTFS rank $d$. Each $\psi_\ell$ gives rise to a function $\rho_\ell : X^d \to {}^X 2$ in the following manner:

$$[\rho_\ell(c_0, ..., c_{d-1})](b) = 1 \text{ if and only if } M_{\mathcal{C}} \models \psi_\ell(b; c_0, ..., c_{d-1})$$

for all $b, c_0, ..., c_{d-1} \in X$. Similarly, we have $\kappa : \mathcal{C}|_{\mathrm{fin}} \to X^d$ defined as follows: Start with $f \in \mathcal{C}$ and $Y \subseteq X$ finite. Then consider the type $p_{f,Y} = \mathrm{tp}_R(f/Y)$. By definition, there exists $c_0, ..., c_{d-1} \in Y$ such that $\psi_\ell(y; c_0, ..., c_{d-1})$ defines $p$. Set $\kappa(f|Y) = (c_0, ..., c_{d-1})$. Now, it is easy to check that $\kappa$ is a compression function and $\{\rho_\ell : \ell < L\}$ are recovery functions witnessing the fact that $\mathcal{C}$ has an extended $d$-sequence compression scheme. $\square$

Coupled with Theorem 3.2.4 above, we can now compute the dimension of many compression schemes. For example, consider $\mathrm{Th}(\mathbb{R}; <, +, \cdot, 0, 1)$. This is o-minimal, so by the proof of Corollary 3.2.7, all formulas of the form $\varphi(\overline{x}; \overline{y})$ have UDTFS rank $\leq \lg(\overline{x})$. Therefore,

(i) The concept class $\mathcal{C}_{\text{disks}} = \{\chi_D : D \subseteq \mathbb{R}^2, D \text{ is an open disk }\}$ has a compression scheme of dimension $\leq 3$. This is because $\mathcal{C}_{\text{disks}} = \mathcal{C}_\varphi$ for

$$\varphi(x_0, x_1, x_2; y_0, y_1) = [(y_0 - x_0)^2 + (y_1 - x_1)^2 < x_2].$$

(ii) The concept class $\mathcal{C}_{\text{rect}}$ of axis-parallel rectangles has a compression scheme of dimension $\leq 4$, since $\mathcal{C}_{\text{rect}} = \mathcal{C}_\varphi$ for

$$\varphi(x_0, x_1, x_2, x_3; y_0, y_1) = [(x_0 < y_0 < x_1) \wedge (x_2 < y_1 < x_3)].$$

Both of these particular examples are already known, but we illustrate here a general method for computing the dimension of any compression scheme uniformly definable in the real ordered field. As another example, consider the following:

(iii) Consider the concept class $\mathcal{C}_{p-\text{disks}} = \{\chi_D : D \subseteq \mathbb{Q}_p, D \text{ is an closed disk}\}$ of closed disks in the $p$-adic field. Then $\mathcal{C}_{p-\text{disks}} = \mathcal{C}_\varphi$ where $\varphi(x_0, x_1; y) = x_1 | (y - x_0)$ (in the language $L_{\text{val}} = \{0, 1, +, -, \cdot, |\}$ as defined in Section 2.5). One easily computes that $\mathrm{R}_{\mathrm{UDTFS}}(\varphi) = 2$. Therefore, $\mathcal{C}_{p-\text{disks}}$ has a compression scheme of dimension $\leq 2$.

## 3.3 Isolated Extension Theorem

The main theorem of this section, Theorem 3.3.3, characterizes dependent formulas using the notions of $\varphi$-isolation and $\varphi$-definability. These differ from the concept of UDTFS dramatically. First of all, we are no longer only working over finite domains, but types of any size. Secondly, the definition one obtains is not uniform and it is not over the original domain of the type. However, it does have a very special form (namely, it is a $\varphi$-definition), and it holds for *all* dependent formulas.

One interesting thing about Theorem 3.3.3 is that it is a local result; it describes the behavior of a specific $\varphi$-type for a dependent formula $\varphi$ regardless of the complexities of the ambient theory (e.g., even when the whole theory is independent). Another noteworthy fact is that this gives a new result even for stable formulas. This is discussed in Subsection 3.3.3.

### 3.3.1 The Isolated Extension Theorem

We begin with some definitions that are relevant to this section. Fix a partitioned formula $\varphi(\overline{x};\overline{y})$. First, we define $\varphi$-isolation and $\varphi$-definitions:

**Definition 3.3.1.** We say that a $\varphi$-type $p$ is *$\varphi$-isolated* if there exists a finite $\varphi$-subtype $p_0(\overline{x}) \subseteq p(\overline{x})$ such that $p_0(\overline{x}) \vdash p(\overline{x})$. We say that a parameter-definable formula $\psi(\overline{x})$ is a *$\varphi$-formula* if it is of the form $\psi(\overline{x}) = \bigwedge_{i<n} \varphi(\overline{x};\overline{b}_i)^{s(i)}$ for some $n < \omega$, elements $\overline{b}_i$, and some $s \in {}^n2$. We say that a parameter-definable formula $\gamma(\overline{y})$ *$\varphi$-defines* a $\varphi$-type $p$ if $\gamma$ defines $p$ and it is of the form $\gamma(\overline{y}) = \forall\overline{x}(\psi(\overline{x}) \rightarrow \varphi(\overline{x};\overline{y}))$ for some $\varphi$-formula $\psi$.

Notice that a $\varphi$-type is $\varphi$-isolated if and only if there exists a $\varphi$-formula $\psi(\overline{x})$ over $\mathrm{dom}(p)$ such that $p(\overline{x})$ is equivalent to $\psi(\overline{x})$ (namely, take $\psi$ to be the conjunction of the witnessing finite $\varphi$-subtype of $p$). This holds if and only if there exists a $\varphi$-definition of $p$ over $\mathrm{dom}(p)$, namely $\forall\overline{x}(\psi(\overline{x}) \rightarrow \varphi(\overline{x};\overline{y}))$.

For $M \models T$ and $B \subseteq M^{\lg(\overline{y})}$, consider the language $L_B = L \cup \{P_B\}$, an expansion of $L$ by adding a single $\lg(\overline{y})$-ary predicate, $P_B$. Let $(M;B)$ be the obvious $L_B$-structure. By "$(N;B') \succeq (M;B)$" we mean that $(N;B')$ is an elementary extension of $(M;B)$ in the language $L_B$.

**Definition 3.3.2.** Fix $M \models T$ and $B \subseteq M^{\lg(\overline{y})}$. We say that a $\varphi$-type $p'$ is an *elementary $\varphi$-extension* of the $\varphi$-type $p \in S_\varphi(B)$ if $p'$ extends $p$ and $\mathrm{dom}(p') \subseteq B'$ for some $(N;B') \succeq (M;B)$.

We are now ready to state the main theorem of this section. The proof is presented in Subsection 3.3.2.

**Theorem 3.3.3** (Isolated Extension Theorem). *For any partitioned formula $\varphi(\overline{x};\overline{y})$, the following are equivalent:*

*(i) $\varphi$ is dependent.*

*(ii) For all $\varphi$-types $p$, there exists a $\varphi$-isolated elementary $\varphi$-extension of $p$.*

*Moreover, if the above conditions hold, we can choose $p'$ a $\varphi$-isolated elementary $\varphi$-extension of $p \in S_\varphi(B)$ such that $\left|\mathrm{dom}(p') - B\right| \leq 2 \cdot \mathrm{ID}(\varphi)$. Finally, fixing $m < \omega$ and $B \subseteq M^m$, if we let $(N; B') \succeq (M; B)$ be $|B|^+$-saturated, then for all dependent formulas $\varphi(\overline{x};\overline{y})$ with $\lg(\overline{y}) = m$ and all $p \in S_\varphi(B)$, there exists $p'$ a $\varphi$-isolated extension of $p$ with $\mathrm{dom}(p') \subseteq B'$.*

We get the following corollary:

**Corollary 3.3.4** (Elementary $\varphi$-definability of types). *If $M \models T$, $m < \omega$, and $B \subseteq M^m$, then there exists $(N; B') \succeq (M; B)$ such that, for all dependent formulas $\varphi(\overline{x};\overline{y})$ with $\lg(\overline{y}) = m$, for all $p(\overline{x}) \in S_\varphi(B)$, there exists $\gamma(\overline{y})$ defined over $B'$ that $\varphi$-defines $p$.*

*Proof.* Fix $M \models T$, $m < \omega$, and $B \subseteq M^m$ and let $(N; B') \succeq (M; B)$ be $|B|^+$-saturated. Then, by Theorem 3.3.3, there exists $p'$ a $\varphi$-isolated extension of $p$ with $\mathrm{dom}(p') \subseteq B'$. Since $p'$ is $\varphi$-isolated, there exists $\gamma$ over $\mathrm{dom}(p') \subseteq B'$ that $\varphi$-defines $p'$. Clearly, this $\gamma$ also $\varphi$-defines $p$. $\square$

As in the stable case, we get a weak form of stable embeddability:

**Corollary 3.3.5** (Weak Stable Embedability). *If $M \models T$ for a dependent theory $T$, $m < \omega$, and $B \subseteq M^m$, then there exists $(N; B') \succeq (M; B)$ such that, for all* *parameter-definable formulas $\varphi(\overline{y})$ with $\lg(\overline{y}) = m$ defined over any elementary supermodel of $M$, there exists a formula $\gamma(\overline{y})$ defined over $B'$ such that $\varphi(B) = \gamma(B)$. Moreover, we may take $\gamma$ to be of the form $\forall \overline{x}(\psi(\overline{x}) \to \varphi(\overline{x};\overline{y}))$ for some $\varphi$-formula $\psi$.*

*Proof.* Fix $(N; B') \succeq (M; B)$ $|B|^+$-saturated. For any parameter-definable formula $\varphi(\overline{y})$, say $\varphi(\overline{y}) = \varphi_0(\overline{a};\overline{y})$ for some $\emptyset$-definable formula $\varphi_0$ and some $\overline{a}$. Let $p = \mathrm{tp}_{\varphi_0}(\overline{a}/B)$. As $\varphi_0$ is dependent, by Corollary 3.3.4, there exists $\gamma(\overline{y})$ that $\varphi_0$-defines $p$. Then, by definition, $\varphi(B) = \gamma(B)$. □

### 3.3.2 Proof of the Isolated Extension Theorem

First, to show (ii) implies (i), we exhibit the contrapositive. Assume that $\varphi(\overline{x};\overline{y})$ is independent. By compactness, there exists a model $M$ with an infinite $\varphi$-independent set $B$. Let $(N; B') \succeq (M; B)$ by any extension. By elementarily, it follows that all finite subsets of $B'$ are $\varphi$-independent (for each $n < \omega$, consider the $L_B$-sentence

$$\sigma_n = \forall \overline{y}_0, ..., \overline{y}_{n-1} \left( \bigwedge_{i<j<n} \overline{y}_i \neq \overline{y}_j \wedge \bigwedge_{i<n} P_B(\overline{y}_i) \to \bigwedge_{s \in {}^n 2} \exists \overline{x} \left( \bigwedge_{i<n} \varphi(\overline{x};\overline{y}_i)^{s(i)} \right)\right)$$

and note that $\{\sigma_n : n < \omega\}$ holds in $(M; B)$). Let $p'$ be any extension of $p$ to a $\varphi$-type with $\mathrm{dom}(p') \subseteq B'$. Fix any finite subtype $p_0(\overline{x}) \subseteq p'(\overline{x})$. For any $p_1$ a finite $\varphi$-subtype of $p'$ with $p_0(\overline{x}) \subset p_1(\overline{x})$, since $\mathrm{dom}(p_1)$ is $\varphi$-independent, we cannot have $p_0 \vdash p_1$. Therefore, $p_0 \not\vdash p'$. This shows that no elementary $\varphi$-extension of $p$ is $\varphi$-isolated. Therefore, (ii) implies (i).

To show (i) implies (ii), we first establish the following proposition:

**Proposition 3.3.6.** *Fix a theory $T$, $\Theta(\overline{y})$ a partial type over $\emptyset$, $\varphi(\overline{x};\overline{y})$ a dependent formula, $M \models T$, $B \subseteq \Theta(M)$, and $N \succeq M$ that is $|B|^+$-saturated. Then, there exists $C \subseteq \Theta(N)$ with $|C| \leq 2 \cdot \mathrm{ID}(\varphi)$ and an extension $p'(\overline{x}) \in S_\varphi(B \cup C)$ of $p(\overline{x})$ that is $\varphi$-isolated.*

Assuming Proposition 3.3.6, we finish the proof of Theorem 3.3.3. To see (i) implies (ii), fix a dependent formula $\varphi(\overline{x};\overline{y})$, $M \models T$, $B \subseteq M^{\lg(\overline{y})}$, and $p \in S_\varphi(B)$. We work in the language $L_B$ and let $\Theta(\overline{y}) = \{P_B(\overline{y})\}$. By upward Löwenheim-Skolem, there exists $(N;B') \succeq (M;B)$ that is $|B|^+$-saturated. By Proposition 3.3.6, there exists $C \subseteq \Theta((N;B')) = B'$ with $|C| \leq 2 \cdot \mathrm{ID}(\varphi)$ and a type $p' \in S_\varphi(B \cup C)$ extending $p$ that is $\varphi$-isolated. Hence $p'$ is an elementary $\varphi$-extension of $p$ that is $\varphi$-isolated, so condition (ii) holds. Moreover, $\left|\mathrm{dom}(p') - B\right| \leq 2 \cdot \mathrm{ID}(\varphi)$, as desired. Finally, this holds for any choice of $(N;B') \succeq (M;B)$ that is $|B|^+$-saturated. This completes the proof of Theorem 3.3.3.

So we now aim to prove Proposition 3.3.6. Fix a dependent formula $\varphi(\overline{x};\overline{y})$ in a theory $T$ and $\Theta(\overline{y})$ any partial type over $\emptyset$. Let $n = \mathrm{ID}(\varphi)$, the independence dimension of $\varphi$, and let $\Delta(\overline{y};\overline{z}_0,...,\overline{z}_{n-1}) = \Delta_{n,\varphi}$ as in (2.5). Fix $M \models T$, $B \subseteq \Theta(M)$, $N \succeq M$ that is $|B|^+$-saturated, and $p \in S_\varphi(B)$. If $B$ is finite, then $p$ is already $\varphi$-isolated, so assume that $B$ is infinite. We now define the notion of a good configuration. This allows us to build up the external $C$ in, at most, $\mathrm{ID}(\varphi)$ steps (adding two elements at a time):

**Definition 3.3.7.** A *good configuration* of $p$ of size $K < \omega$ is a sequence $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2\rangle$ such that the following conditions hold:

(i) $\overline{c}_{i,t} \models \Theta(\overline{y})$ for all $i < K, t < 2$;

(ii) $p(\overline{x}) \cup \{\varphi(\overline{x}; \overline{c}_{i,t})^t : i < K, t < 2\}$ is consistent; and

(iii) For all $s \in {}^K 2$ and all $j < K$, $\overline{c}_{j,0}$ and $\overline{c}_{j,1}$ have the same $\Delta$-type over $(B \cup \{\overline{c}_{i,s(i)} : i \neq j\})^n$.

If $\overline{\mathbf{c}}$ is a good configuration of $p$ of size $K$, let $p_{\overline{\mathbf{c}}} = p(\overline{x}) \cup \{\varphi(\overline{x}; \overline{c}_{i,t})^t : i < K, t < 2\}$ as in (ii).

The first thing to note is that these good configurations are used to extend the $\varphi$-type $p$ in a very specific way. These could, *a priori*, be arbitrarily large. However, the fact that $\varphi$ is dependent forces good configurations to be of bounded size.

**Lemma 3.3.8.** *If $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2\rangle$ is a good configuration of $p$ of size $K$, then $K \leq n = \mathrm{ID}(\varphi)$.*

*Proof.* Suppose not, i.e., $K > n$. A good configuration of size $n+1$ explicitly gives us a special $\varphi$-type over $2(n+1)$ parameters. Using condition (iii), we can translate this to $2^{n+1}$ distinct $\varphi$-types over $n+1$ parameters, producing a $\varphi$-independent set of size $n+1$, contrary to the fact that $n = \mathrm{ID}(\varphi)$. Specifically, for each $s \in {}^{n+1}2$, notice that

$$\models \exists \overline{x} \bigwedge_{i<n+1} \varphi(\overline{x}; \overline{c}_{i,s(i)})^{s(i)} \tag{3.1}$$

because $\{\varphi(\overline{x}; \overline{c}_{i,s(i)})^{s(i)} : i < n+1\}$ is a consistent type. Now notice that, for any

$j \leq n$,

$$\models \exists \overline{x} \left( \bigwedge_{i<j} \varphi(\overline{x}; \overline{c}_{i,0})^{s(i)} \wedge \bigwedge_{j \leq i < n+1} \varphi(\overline{x}; \overline{c}_{i,s(i)})^{s(i)} \right) \Rightarrow$$

$$\models \exists \overline{x} \left( \bigwedge_{i \leq j} \varphi(\overline{x}; \overline{c}_{i,0})^{s(i)} \wedge \bigwedge_{j < i < n+1} \varphi(\overline{x}; \overline{c}_{i,s(i)})^{s(i)} \right) \tag{3.2}$$

because $\overline{c}_{j,0}$ and $\overline{c}_{j,1}$ have the same $\Delta$-type over $(\overline{c}_{i,0} : i < j)^\frown(\overline{c}_{i,s(i)} : j < i \leq n)$ (to see this, use condition (iii) of Definition 3.3.7 with the function $s' = \{(i, 0) : i < j\} \cup \{(i, s(i)), j \leq i \leq n\}$). Starting with (3.1), and using (3.2) and induction, we get that

$$\models \exists \overline{x} \bigwedge_{i<n+1} \varphi(\overline{x}; \overline{c}_{i,0})^{s(i)}.$$

As $s \in {}^{n+1}2$ was arbitrary, we see that $\{\overline{c}_{i,0} : i < n+1\}$ is a $\varphi$-independent set, contradicting the fact that $n = \mathrm{ID}(\varphi)$. $\square$

If we fix $\overline{\mathbf{c}}$ a maximal good configuration for the $\varphi$-type $p$, then we argue in the remaining lemmas that $p_{\overline{\mathbf{c}}}$ is $\varphi$-isolated. If no such maximal good configuration exists, then use Lemma 3.3.8 to show that $\varphi$ is not actually dependent.

Now that we have defined good configurations, we need a sufficient condition for taking a good configuration and building a larger one out of it. Clearly any new $\overline{d}_0$ and $\overline{d}_1$ we would like to add must realize $\Theta$ and must be so that $\neg\varphi(\overline{x}; \overline{d}_0) \wedge \varphi(\overline{x}; \overline{d}_1)$ is consistent with $p_{\overline{\mathbf{c}}}(\overline{x})$. However, condition (iii) of Definition 3.3.7 is trickier to satisfy. Not only do $\overline{d}_0$ and $\overline{d}_1$ have to have the same $\Delta$-type over $(B \cup \{\overline{c}_{i,s(i)} : i < K\})^n$ for all $s \in {}^K2$, but, for each $j < K$, $\overline{c}_{j,0}$ and $\overline{c}_{j,1}$ have to have the same $\Delta$-type over $(B \cup \{\overline{c}_{i,s(i)} : i \neq j\} \cup \{\overline{d}_t\})^n$ for all $s \in {}^K2$ and $t < 2$. With this in mind, we now give a sufficient condition for adding to a good configuration.

**Lemma 3.3.9.** *If $\mathbf{\overline{c}} = \langle \overline{c}_{i,t} : i < K, t < 2\rangle$ is a good configuration of $p$ and there exists $\overline{d}_0$ and $\overline{d}_1$ such that:*

*(i)* $\overline{d}_0, \overline{d}_1 \models \Theta(\overline{y})$*;*

*(ii)* $p_{\mathbf{\overline{c}}}(\overline{x}) \cup \{\varphi(\overline{x}; \overline{d}_t)^t : t < 2\}$ *is consistent;*

*(iii)* $\mathrm{tp}_\Delta(\overline{d}_0/(B \cup C)^n) = \mathrm{tp}_\Delta(\overline{d}_1/(B \cup C)^n)$*; and*

*(iv)* $\mathrm{tp}_\Delta(\overline{d}_0/(B \cup C)^n)$ *is finitely satisfiable in $B$.*

*Then, $\mathbf{\overline{c}}^\frown\langle \overline{d}_0, \overline{d}_1\rangle$ is a good configuration of $p$ of size $K+1$ (where $C = \{\overline{c}_{i,t} : i < K, t < 2\}$).*

*Proof.* Clearly all conditions for $\mathbf{\overline{c}}^\frown\langle \overline{d}_0, \overline{d}_1\rangle$ to be a good configuration of $p$ are met except perhaps the condition that $\overline{c}_{j,0}$ and $\overline{c}_{j,1}$ have the same $\Delta$-type over $(B \cup \{\overline{c}_{i,s(i)} : i \neq j\} \cup \{\overline{d}_t\})^n$ for all $s \in {}^K2$, $t < 2$. So suppose this fails, and fix the $s \in {}^K2$ and $t < 2$ where this fails.

Then there exists $\delta \in \pm\Delta$ such that $N \models \delta(\overline{c}_{j,0}, \mathbf{\overline{e}}) \wedge \neg\delta(\overline{c}_{j,1}, \mathbf{\overline{e}})$ for some $\mathbf{\overline{e}} \in (B \cup \{\overline{c}_{i,s(i)} : i \neq j\} \cup \{\overline{d}_t\})^n$. Since $\overline{c}_{j,0}$ and $\overline{c}_{j,1}$ have the same $\Delta$-type over $(B \cup \{\overline{c}_{i,s(i)} : i \neq j\})^n$, we may assume that $\mathbf{\overline{e}} = (\overline{d}_t, \mathbf{\overline{e}}')$ for some $\mathbf{\overline{e}}'$ from $(B \cup \{\overline{c}_{i,s(i)} : i \neq j\})^{n-1}$ (we could have repeated instances of $\overline{d}_t$, but this would only trivially change the argument, so we may assume not). Therefore, we get that:

$$N \models \delta(\overline{c}_{j,0}, \overline{d}_t, \mathbf{\overline{e}}') \wedge \neg\delta(\overline{c}_{j,1}, \overline{d}_t, \mathbf{\overline{e}}'). \tag{3.3}$$

By condition (iii), we may assume that $t = 0$. Then by condition (iv), there exists $\overline{b} \in B$ such that:

$$N \models \delta(\overline{c}_{j,0}, \bar{b}, \overline{\mathbf{e}}') \wedge \neg\delta(\overline{c}_{j,1}, \bar{b}, \overline{\mathbf{e}}'). \tag{3.4}$$

But, as $(\bar{b}, \overline{\mathbf{e}}') \in (B \cup \{\overline{c}_{i,s(i)} : i \neq j\})^n$, this contradicts the fact that $\overline{c}_{j,0}$ and $\overline{c}_{j,1}$ have the same $\Delta$-type over $(B \cup \{\overline{c}_{i,s(i)} : i \neq j\})^n$. □

So fix $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2 \rangle$ a maximal good configuration of $p$ and let $C = \{\overline{c}_{i,t} : i < K, t < 2\}$. Therefore, $p_{\overline{\mathbf{c}}}$ is a $\varphi$-type over $B \cup C$ Let $s(\overline{x})$ be any extension of $p_{\overline{\mathbf{c}}}(\overline{x})$ to a complete type in $S_{\overline{x}}(B \cup C)$. Define the partial type $r_s(\overline{y})$ as follows:

$$r_s(\overline{y}) = \{\exists \overline{x}(\varphi(\overline{x}; \overline{y})^t \wedge \psi(\overline{x})) : \psi \in s, t < 2\} \cup \Theta(\overline{y}).$$

**Lemma 3.3.10.** *Given $r_s$ as above, $r_s(\overline{y})$ is not finitely satisfied in $B$.*

*Proof.* Suppose, by means of contradiction, that $r_s$ is finitely satisfied in $B$. Let $\mathcal{D}$ be an ultrafilter on $B$ such that for all parameter-definable $\delta(\overline{y}) \in r_s(\overline{y})$, $\delta(B) \in \mathcal{D}$ (by finite satisfiability of $r_s$ in $B$, the set $\{\delta(B) : \delta \in r_s\}$ has the finite intersection property, hence can be completed to an ultrafilter by Zorn's Lemma). Let $q(\overline{y}) = \mathrm{Av}(\mathcal{D}, B \cup C)$, the average type of $\mathcal{D}$ over $B \cup C$. That is, for any parameter-definable formula $\delta(\overline{y})$ over $B \cup C$, $\delta(\overline{y}) \in q(\overline{y})$ if and only if $\delta(B) \in \mathcal{D}$. Then $q \in S_{\overline{y}}(B \cup C)$, $q$ extends $r_s$, and $q$ is finitely satisfied in $B$. Let $q' = q|_\Delta$ (the restriction to the $\Delta$-type over $(B \cup C)^n$).

Now notice that $\{\exists \overline{x}(\varphi(\overline{x}; \overline{y})^t \wedge \psi(\overline{x}))\} \cup q(\overline{y})$ is consistent for each $\psi \in s$ and each $t < 2$ (as $q$ extends $r_s$). Since $s$ is closed under conjunction, by compactness we get that $s(\overline{x}) \cup \{\varphi(\overline{x}; \overline{y})^t\} \cup q(\overline{y})$ is consistent for each $t < 2$. Therefore,

$s(\overline{x}) \cup \{\varphi(\overline{x};\overline{y})^t\} \cup q'(\overline{y}) \cup \{\theta(\overline{y})\}$ is consistent for each $t < 2$ and each $\theta(\overline{y})$ a finite conjunction of formulas from $\Theta(\overline{y})$ (as $q'(\overline{y}) \cup \Theta(\overline{y}) \subseteq q(\overline{y})$). This means that $s(\overline{x}) \cup \{\exists\overline{y}(\varphi(\overline{x};\overline{y})^t \wedge \theta(\overline{y}) \wedge \psi(\overline{y}))\}$ is consistent for each $\psi(\overline{y})$ a finite conjunction of formulas from $q'(\overline{y})$ and each $\theta(\overline{y})$ a finite conjunction of formulas from $\Theta(\overline{y})$. But, since $s$ is a complete type in the variables $\overline{x}$, $s$ decides all formulas of the form $\exists\overline{y}(\varphi(\overline{x};\overline{y})^t \wedge \theta(\overline{y}) \wedge \psi(\overline{y}))$. Therefore, we get that:

$$\exists\overline{y}(\varphi(\overline{x};\overline{y})^t \wedge \theta(\overline{y}) \wedge \psi(\overline{y})) \in s(\overline{x}).$$

Choose $\psi_t(\overline{x})$ a finite conjunction of formulas from $q'(\overline{y})$ and $\theta_t(\overline{y})$ a finite conjunction of formulas from $\Theta(\overline{y})$ for both $t < 2$. Then $\exists\overline{y}_t(\varphi(\overline{x};\overline{y}_t)^t \wedge \theta_t(\overline{y}_t) \wedge \psi_t(\overline{y}_t)) \in s(\overline{x})$ for both $t < 2$. Therefore, we get that:

$$s(\overline{x}) \cup \{\exists\overline{y}_0(\neg\varphi(\overline{x};\overline{y}_0) \wedge \theta_0(\overline{y}_0) \wedge \psi_0(\overline{y}_0))\} \cup \{\exists\overline{y}_1(\varphi(\overline{x};\overline{y}_1) \wedge \theta_1(\overline{y}_1) \wedge \psi_1(\overline{y}_1))\}$$

is consistent. Now, by compactness,

$$u(\overline{x},\overline{y}_0,\overline{y}_1) = s(\overline{x}) \cup \{\neg\varphi(\overline{x};\overline{y}_0) \wedge \varphi(\overline{x};\overline{y}_1)\} \cup q'(\overline{y}_0) \cup q'(\overline{y}_1) \cup \Theta(\overline{y}_0) \cup \Theta(\overline{y}_1)$$

is consistent. So, taking any realization $(\overline{a},\overline{d}_0,\overline{d}_1)$ of $u(\overline{x},\overline{y}_0,\overline{y}_1)$ from $N$, we see that $\overline{d}_0,\overline{d}_1 \models \Theta(\overline{y})$, $\overline{d}_0,\overline{d}_1 \models q'(\overline{y})$, and $p_{\overline{\mathbf{c}}}(\overline{x}) \cup \{\varphi(\overline{x};\overline{d}_t)^t : t < 2\}$ is consistent. So conditions (i), (ii), and (iii) of Lemma 3.3.9 are met. However, since $q$ is finitely satisfied in $B$, $q'$ is finitely satisfied in $B$. Therefore, condition (iv) of Lemma 3.3.9 is met, so $\overline{\mathbf{c}}^\frown\langle\overline{d}_0,\overline{d}_1\rangle$ is a good configuration of $p$. This contradicts the maximality of $\overline{\mathbf{c}}$. $\square$

We now show how the non-finite-satisfiability of $r_s$ in $B$ leads to a formula definition of $p_{\overline{\mathbf{c}}}(\overline{x})$. Again, fix $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2\rangle$ a maximal good configuration of $p$ and let $C = \{\overline{c}_{i,t} : i < K, t < 2\}$.

**Lemma 3.3.11.** *For any $s(\overline{x}) \in S_{\overline{x}}(B \cup C)$ an extension of $p_{\overline{\mathbf{c}}}(\overline{x})$, there exists a formula $\gamma(\overline{x}) \in s(\overline{x})$ such that $\gamma(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$.*

*Proof.* Consider $r_s$ as given above. By Lemma 3.3.10, $r_s$ is not finitely satisfiable in $B$. Thus, there exists $\psi_0(\overline{x}), ..., \psi_{L-1}(\overline{x}) \in s(\overline{x})$ and $t(0), ..., t(L-1) < 2$ such that, for all $\overline{b} \in B$,

$$N \models \neg\left(\bigwedge_{\ell<L} \exists\overline{x}(\varphi(\overline{x};\overline{b})^{t(\ell)} \wedge \psi_\ell(\overline{x}))\right)$$

(notice here that $\overline{b} \models \Theta(\overline{y})$ for all $\overline{b} \in B$, so all the formulas in $\Theta(\overline{y}) \subseteq r_s(\overline{y})$ are always realized in $B$). Taking $\psi(\overline{x}) = \bigwedge_{\ell<L} \psi_\ell(\overline{x})$, we see that, for all $\overline{b} \in B$, there exists $t < 2$ such that

$$N \models \neg\exists\overline{x}(\varphi(\overline{x};\overline{b})^t \wedge \psi(\overline{x})).$$

Therefore, for all $\overline{b} \in B$, $N \models \forall\overline{x}(\psi(\overline{x}) \to \varphi(\overline{x};\overline{b}))$ or $N \models \forall\overline{x}(\psi(\overline{x}) \to \neg\varphi(\overline{x};\overline{b}))$. Let $\gamma(\overline{x})$ be defined as follows:

$$\gamma(\overline{x}) = \psi(\overline{x}) \wedge \bigwedge_{i<K,u<2} \varphi(\overline{x};\overline{c}_{i,u})^u.$$

Since $s$ is closed under conjunction and $s$ extends $p_{\overline{\mathbf{c}}}$, we get that $\gamma(\overline{x}) \in s(\overline{x})$. To prove that $\gamma(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$, notice that, for all $\overline{b} \in B$, there exists $t < 2$ such that $\psi(\overline{x}) \vdash \varphi(\overline{x};\overline{b})^t$, hence $\varphi(\overline{x};\overline{b})^t \in s(\overline{x})$. But $s$ extends $p_{\overline{\mathbf{c}}}$, so we get that $\varphi(\overline{x};\overline{b})^t \in p_{\overline{\mathbf{c}}}(\overline{x})$. Similarly, $\gamma(\overline{x}) \vdash \varphi(\overline{x};\overline{c}_{i,u})^u$ for all $i < K$ and $u < 2$. Therefore, $\gamma(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$. $\square$

Now that we have a formula definition for $p_{\overline{\mathbf{c}}}(\overline{x})$ for each $s \in S_{\overline{x}}(B \cup C)$ extending $p_{\overline{\mathbf{c}}}$, we see that a single formula is equivalent to $p_{\overline{\mathbf{c}}}(\overline{x})$ using compactness. After that, we show that this implies that a finite $\varphi$-subtype of $p_{\overline{\mathbf{c}}}(\overline{x})$ is equivalent to the whole of $p_{\overline{\mathbf{c}}}(\overline{x})$.

**Lemma 3.3.12.** *If $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2 \rangle$ is a maximal good configuration of $p$ (and $C = \{\overline{c}_{i,t} : i < K, t < 2\}$), then there exists a formula $\psi(\overline{x})$ over $B \cup C$ such that $\psi(\overline{x})$ is equivalent to $p_{\overline{\mathbf{c}}}(\overline{x})$.*

*Proof.* For each such $s \in S_{\overline{x}}(B \cup C)$ extending $p_{\overline{\mathbf{c}}}(\overline{x})$, define $\gamma_s(\overline{x})$ to be a formula such that $\gamma_s(\overline{x}) \in s(\overline{x})$ and $\gamma_s(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$ as given in Lemma 3.3.11.

Consider the following partial type over $B \cup C$:

$$\Sigma(\overline{x}) = \{\neg\gamma_s(\overline{x}) : s \in S_{\overline{x}}(B \cup C) \text{ and } s(\overline{x}) \supseteq p_{\overline{\mathbf{c}}}(\overline{x})\} \cup p_{\overline{\mathbf{c}}}(\overline{x}).$$

Note that $\Sigma(\overline{x})$ is inconsistent, since otherwise we would have $\overline{a} \models p_{\overline{\mathbf{c}}}(\overline{x})$ yet $\overline{a} \models \neg\gamma_s(\overline{x})$ for any $s(\overline{x})$ extending $p_{\overline{\mathbf{c}}}(\overline{x})$. In particular, $\overline{a} \models \neg\gamma_{s_0}(\overline{x})$ for $s_0 = \mathrm{tp}(\overline{a}/B \cup C)$. This contradicts the fact that $s_0(\overline{x}) \vdash \gamma_{s_0}(\overline{x})$. Therefore, by compactness, there exists some finite set $S_0 \subseteq S_{\overline{x}}(B \cup C)$ of types extending $p_{\overline{\mathbf{c}}}$ so that $\Sigma_0(\overline{x}) = \{\neg\gamma_s(\overline{x}) : s \in S_0\} \cup p_{\overline{\mathbf{c}}}(\overline{x})$ is inconsistent. Let $\psi(\overline{x}) = \bigvee_{s \in S_0} \gamma_s(\overline{x})$.

Certainly $\psi(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$ as $\gamma_s(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$ for all $s \in S_0$. Conversely, if $\overline{a} \models p_{\overline{\mathbf{c}}}(\overline{x})$, then $\overline{a} \not\models \{\neg\gamma_s(\overline{x}) : s \in S_0\}$ (by the inconsistency of $\Sigma_0(\overline{x})$). Therefore, $\overline{a} \models \psi(\overline{x})$. Hence, $p_{\overline{\mathbf{c}}}(\overline{x}) \vdash \psi(\overline{x})$, as desired. $\square$

**Lemma 3.3.13.** *If $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2 \rangle$ is a maximal good configuration of $p$ (and $C = \{\overline{c}_{i,t} : i < K, t < 2\}$), then there exists a finite $\varphi$-subtype $p_0(\overline{x}) \subseteq p_{\overline{\mathbf{c}}}(\overline{x})$ so that $p_0(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$.*

*Proof.* First let $\psi(\overline{x})$ be a formula over $B \cup C$ that is equivalent to $p_{\overline{\mathbf{c}}}(\overline{x})$, given by Lemma 3.3.12. Then consider $\{\neg\psi(\overline{x})\} \cup p_{\overline{\mathbf{c}}}(\overline{x})$, a partial type over $B \cup C$. This is clearly inconsistent. Therefore, there exists a finite subset $p_0(\overline{x}) \subseteq p_{\overline{\mathbf{c}}}(\overline{x})$ such that $\{\neg\psi(\overline{x})\} \cup p_0(\overline{x})$ is inconsistent. That is, $p_0(\overline{x}) \vdash \psi(\overline{x})$ and, therefore, we get that $p_0(\overline{x}) \vdash p_{\overline{\mathbf{c}}}(\overline{x})$. $\square$

We are now ready to prove Proposition 3.3.6, hence finish our proof of Theorem 3.3.3.

*Proof of Proposition 3.3.6.* Take $\overline{\mathbf{c}} = \langle \overline{c}_{i,t} : i < K, t < 2 \rangle$ any maximal good configuration of $p$ and let $C = \{\overline{c}_{i,t} : i < K, t < 2\}$. By definition, $C \subseteq \Theta(N)$. By Lemma 3.3.8, $K \leq n$, hence $|C| \leq 2 \cdot n$. Let $p'(\overline{x}) = p_{\overline{\mathbf{c}}}(\overline{x}) = p(\overline{x}) \cup \{\varphi(\overline{x}; \overline{c}_{i,t})^t : i < K, t < 2\}$. By Lemma 3.3.13, there exists a finite $p_0(\overline{x}) \subseteq p'(\overline{x})$ so that $p_0(\overline{x}) \vdash p'(\overline{x})$. Therefore, $p'(\overline{x})$ is $\varphi$-isolated. $\square$

This concludes our proof of the Isolated Extension Theorem. In the next subsection, we discuss the applications of this theorem to the stable setting.

### 3.3.3 Stable Case

Since stable formulas are, in particular, dependent, all stable formulas have the property of Theorem 3.3.3 (ii). But what is the $\varphi$-isolated elementary $\varphi$-extension $p'(\overline{x})$ of a given $\varphi$-type $p(\overline{x})$? In the interesting case when $p(\overline{x})$ is not already $\varphi$-isolated, $p'(\overline{x})$ is a *forking* extension of $p(\overline{x})$. This follows from the Open Mapping Theorem (i.e., the fact that the restriction map from non-forking $\varphi$-extensions of

$S_\varphi(B)$ to $S_\varphi(B)$ is open) as, if $p$ has a non-forking $\varphi$-isolated extension, then it is already $\varphi$-isolated.

On the issue of uniformity, the results of Theorem 3.3.3 differ strongly from the standard definability of $\varphi$-types in the stable case, as in Theorem 1.3.2. In the case where $\varphi$ is stable, we can use a compactness argument to get a uniform definition of $\varphi$-types. Note, however, that this uniform definition is not necessarily a $\varphi$-definition. One cannot, in general, get a uniform $\varphi$-definition of all $\varphi$-types, even in the case where $\varphi$ is stable.

**Example 3.3.14.** We use a classic example to illustrate this point. Let $T$ be the theory, in the language $L = \{E\}$ with a single binary relation $E$, stating that $E$ is an equivalence relation with infinitely many $E$-equivalence classes all of infinite size. This theory is certainly stable, and even $\aleph_0$-stable. Fix $M \models T$ and let $B \subset M$ be a set containing one element from one class, two from another, three from a third class, and so on. Finally, let $\varphi(x; y, z, w)$ be the formula given by:

$$\varphi(x; y, z, w) = [(z = w \rightarrow x = y) \wedge (z \neq w \rightarrow E(x, y))]$$

(so $\varphi$ encodes the two formulas $x = y$ and $E(x, y)$ into a single formula). Now let $n \in \omega$ be arbitrary and let $a \in M - B$ be in the $E$-equivalence class with exactly $n$ elements of $B$ in it; call this class $[a]_E$. Finally, let $p_n(x) = \mathrm{tp}_\varphi(a/B)$. Now, for any $(N; B') \succeq (M; B)$, notice that the $E$-equivalence class with exactly $n$ elements from $B$ still has exactly $n$ elements from $B'$, so $[a]_E \cap B' = [a]_E \cap B$. However, this shows that any $\varphi$-extension of $p_n$ to some $p'$ with $\mathrm{dom}(p') \subseteq B'$ is $\varphi$-isolated only by a finite subtype whose domain contains $[a]_E \cap B$ (this is because we need the full set $[a]_E \cap B$ to say that $x \neq b$ for each $b \in ([a]_E \cap B)$ yet $E(x, b)$ for some (all) $b \in ([a]_E \cap B)$). As $|[a]_E \cap B| = n$ and $n < \omega$ was arbitrary, we see from this example that there is no *uniform* bound on the size of the $\varphi$-isolating $\varphi$-subtype of the elementary $\varphi$-extension given by Theorem 3.3.3, even in the stable case.

## 3.4 Distal Theories and Finite Type Implications

This chapter gives an answer to the conjecture of Simon in [28] in the case of dp-minimal theories with a linear order and discusses the notion of finite type definitions.

We say that two indiscernible sequences $\langle \bar{b}_i : i \in I \rangle$ and $\langle \bar{c}_j : j \in J \rangle$ have the same *EM-type* if, for all $n < \omega$, all $i_0 < ... < i_n$ from $I$, and all $j_0 < ... < j_n$ from $J$, $\mathrm{tp}(\bar{b}_{i_0}, ..., \bar{b}_{i_n}/\emptyset) = \mathrm{tp}(\bar{c}_{j_0}, ..., \bar{c}_{j_n}/\emptyset)$.[2] For any linear order $(I, <)$, a *non-principal cut* $C$ of $I$ is a subset of $I$ such that $C$ is downward closed, $C$ has no largest element, and $I - C$ has no smallest element. Using these definitions, we define distal sequences and distal theories:

**Definition 3.4.1** (Definition 2.1 of [28])**.** An infinite indiscernible sequence $\langle \bar{b}_i : i \in I \rangle$ is *distal* if, for every indiscernible sequence $\langle \bar{c}_j : j \in J \rangle$ with the same EM-type as $\langle \bar{b}_i : i \in I \rangle$ and $(J, <)$ dense, for every distinct non-principal cuts $C_1$ and $C_2$ of $J$ with $C_1 \subseteq C_2$, and for every $\bar{d}_1, \bar{d}_2$, if the sequences

$$\langle \bar{c}_j : j \in C_1 \rangle^\frown \langle \bar{d}_1 \rangle^\frown \langle \bar{c}_j : j \in I - C_1 \rangle \text{ and } \langle \bar{c}_j : j \in C_2 \rangle^\frown \langle \bar{d}_2 \rangle^\frown \langle \bar{c}_j : j \in I - C_2 \rangle$$

[2]Here EM stands for Ehrenfeucht-Mostowski.

are each indiscernible, then the sequence

$$\langle \overline{c}_j : j \in C_1 \rangle ^\frown \langle \overline{d}_1 \rangle ^\frown \langle \overline{c}_j : j \in C_2 - C_1 \rangle ^\frown \langle \overline{d}_2 \rangle ^\frown \langle \overline{c}_j : j \in I - C_2 \rangle$$

is indiscernible. We say that a theory $T$ is *distal* if all infinite indiscernible sequences are distal.

Simon says that distal theories are intended to be, in some sense, purely instable. He goes on to show a remarkable result about distal theories, namely that $T$ is distal if and only if all generically stable measures are smooth (see Theorem 1.1 of [28]). For this discussion, we are primarily interested in the conjecture at the end of the paper. Namely,

**Conjecture 3.4.2** (Conjecture 2.29 of [28])**.** *Let $T$ be distal and $\varphi(\overline{x};\overline{y})$ be any partitioned formula. Then, there exists $N < \omega$ such that, for every finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and every $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$, there exists a subset $B_0 \subseteq B$ with $|B_0| \leq N$ such that $\mathrm{tp}(\overline{a}/B_0) \vdash \mathrm{tp}_\varphi(\overline{a}/B)$.*

The converse of this conjecture is true, as noted by Simon in [28]. By this conjecture, Simon intends to show how UDTFS can be separated into isolation in the distal case and "pure definability" in the non-distal dependent case. By Corollary 2.28 of [28], dp-minimal theories with a linear order are distal. We now exhibit a partial solution to Conjecture 3.4.2:

**Theorem 3.4.3.** *Fix $T$ a dp-minimal theory with a definable linear order on $\mathfrak{C}$. For every partitioned formula $\varphi(\overline{x};\overline{y})$, there exists $N < \omega$ such that, for every finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and every $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$, there exists a subset $B_0 \subseteq B$ with $|B_0| \leq N$ such that $\mathrm{tp}(\overline{a}/B_0) \vdash \mathrm{tp}_\varphi(\overline{a}/B)$.*

Before proving the theorem, we discuss a notion we call finite type implication.

**Definition 3.4.4.** We say that a partitioned formula $\varphi(\overline{x};\overline{y})$ has *finite type implications* if there exists a finite set of partitioned formulas $\{\psi_\ell(\overline{x};\overline{z}_0,...,\overline{z}_{N-1}) : \ell < L\}$ such that, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $\overline{a} \in \mathfrak{C}^{\lg(\overline{x})}$, there exists $\overline{b}_0,...,\overline{b}_{N-1} \in B$ and $\ell < L$ such that

(i) $\models \psi_\ell(\overline{a};\overline{b}_0,...,\overline{b}_{N-1})$; and

(ii) $\psi_\ell(\overline{x};\overline{b}_0,...,\overline{b}_{N-1}) \vdash \mathrm{tp}_\varphi(\overline{a}/B)(\overline{x})$.

This is clearly a stronger condition than the conclusion of Conjecture 3.4.2. Therefore, to prove Theorem 3.4.3, it suffices to show that all partitioned formulas in a dp-minimal theory with a linear order have finite type implications. Finite type implications is clearly strictly stronger than UDTFS.

**Proposition 3.4.5.** *If a partitioned formula $\varphi(\overline{x};\overline{y})$ has finite type implications, then $\varphi$ has UDTFS.*

*Proof.* Fix $\psi_\ell(\overline{x};\overline{z}_0,...,\overline{z}_{N-1})$ as in Definition 3.4.4 and let $\gamma_\ell(\overline{y};\overline{\mathbf{z}})$ be defined as follows:

$$\gamma_\ell(\overline{y};\overline{\mathbf{z}}) = \forall\overline{x}\big(\psi_\ell(\overline{x};\overline{\mathbf{z}}) \to \varphi(\overline{x};\overline{y})\big).$$

It is clear that the $\gamma_\ell$ are uniform definitions for finite $\varphi$-types. By Lemma 2.3.5, $\varphi$ has UDTFS. $\square$

Just as the sufficiency of a single variable holds for UDTFS (Lemma 2.3.6), it also holds for finite type implications.

**Proposition 3.4.6.** *If $T$ is a theory such that all partitioned formulas of the form $\varphi(x;\overline{y})$ have finite type implications, then all partitioned formulas have finite type implications.*

*Proof.* By induction on $n = \lg(\overline{x})$. The case $n = 1$ is given by hypothesis, so assume $n > 1$. Fix $\varphi(x_0, ..., x_{n-1}; \overline{y})$ any partitioned formula. Let $\hat{\varphi}(x_0, ..., x_{n-2}; x_{n-1}, \overline{y}) = \varphi(\overline{x}; \overline{y})$. By induction, there exists $\psi_\ell(x_0, ..., x_{n-2}; w_0, \overline{z}_0, ..., w_{N_0-1}, \overline{z}_{N_0-1})$ for $\ell < L_0$ exhibiting the fact that $\hat{\varphi}$ has finite type implications. For each $\ell < L_0$ and $t < 2$, let

$$\gamma_{t,\ell}(x_{n-1}; \overline{y}, \overline{z}_0, ..., \overline{z}_{N_0-1}) = \forall x_0 ... x_{n-2}$$

$$\big(\psi_\ell(x_0, ..., x_{n-2}; x_{n-1}, \overline{z}_0, ..., x_{n-1}, \overline{z}_{N_0-1}) \to \varphi(\overline{x}; \overline{y})^t\big).$$

By hypothesis, $\gamma_{t,\ell}$ has finite type implications, so there exists

$$\rho_{t,\ell,\ell'}(x_{n-1}; \overline{w}_0, \overline{\mathbf{v}}_0, ..., \overline{w}_{N_1-1}, \overline{\mathbf{v}}_{N_1-1})$$

witnessing this for $\ell' < L_1$. Finally, let $\delta_{t,\ell,\ell'}$ be given as follows:

$$\delta_{t,\ell,\ell'}(\overline{x}; \overline{\mathbf{w}}, \overline{\mathbf{z}}) = \rho_{t,\ell,\ell'}(x_{n-1}; \overline{w}_0, \overline{\mathbf{z}}, ..., \overline{w}_{N_1-1}, \overline{\mathbf{z}}) \wedge$$

$$\psi_\ell(x_0, ..., x_{n-2}; x_{n-1}, \overline{z}_0, ..., x_{n-1}, \overline{z}_{N_0-1}).$$

We claim that $\{\delta_{0,\ell,\ell'} \wedge \delta_{1,\ell,\ell'} : \ell < L_0, \ell' < L_1\}$ witnesses that $\varphi$ has finite type implications.

Fix $\overline{a} \in \mathfrak{C}^n$ and $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite. By definition of finite type implications on $(a_0, ..., a_{n-2})$ and $a_{n-1}{}^\frown B$, there exists $\overline{c}_i \in B$ for $i < N_0$ and $\ell < L_0$ such that

$$(a_0, ..., a_{n-2}) \models \psi_\ell(x_0, ..., x_{n-2}; a_{n-1}, \overline{c}_0, ..., a_{n-1}, \overline{c}_{N_0-1}) \vdash$$

$$\mathrm{tp}_{\hat{\varphi}}((a_0, ..., a_{n-2})/a_{n-1}{}^\frown B).$$

That is, for any $\overline{b} \in B$ and $t < 2$ such that $\models \varphi(\overline{a}; \overline{b})^t$,

$$\models \forall x_0 ... \forall x_{n-1} \big( \psi_\ell(x_0, ..., x_{n-2}; a_{n-1}, \overline{c}_0, ..., a_{n-1}, \overline{c}_{N_0-1}) \to \varphi(x_0, ..., x_{n-2}, a_{n-1}; \overline{b})^t \big).$$

That is, $\models \gamma_{t,\ell}(a_{n-1}; \overline{b}, \mathbf{\overline{c}})$. By the definition of finite type implications on $a_{n-1}$ and $B^\frown \mathbf{\overline{c}}$, for each $t < 2$ there exists $\overline{d}_{t,j}$ for $j < N_1$ and $\ell' < L_1$ such that

$$\begin{aligned} a_{n-1} \models & \rho_{t,\ell,\ell'}(x_{n-1}; \overline{d}_{t,0}, \mathbf{\overline{c}}, ..., \overline{d}_{t,N_1-1}, \mathbf{\overline{c}}) \vdash \\ & \mathrm{tp}_{\gamma_{t,\ell}}(a_{n-1}/B^\frown \mathbf{\overline{c}}) \end{aligned}$$

Clearly $\overline{a} \models \delta_{0,\ell,\ell'}(\overline{x}; \overline{\mathbf{d}}_0, \mathbf{\overline{c}}) \wedge \delta_{1,\ell,\ell'}(\overline{x}; \overline{\mathbf{d}}_1, \mathbf{\overline{c}})$, by construction. Now suppose

$$(e_0, ..., e_{n-1}) \models \delta_{0,\ell,\ell'}(\overline{x}; \overline{\mathbf{d}}_0, \mathbf{\overline{c}}) \wedge \delta_{1,\ell,\ell'}(\overline{x}; \overline{\mathbf{d}}_1, \mathbf{\overline{c}})$$

and fix $\overline{b} \in B$ and $t < 2$ such that $\models \varphi(\overline{a}; \overline{b})^t$. Then, in particular, $(e_0, ..., e_{n-1}) \models \delta_{t,\ell,\ell'}(\overline{x}; \overline{\mathbf{d}}_t, \mathbf{\overline{c}})$, so by definition,

$$e_{n-1} \models \mathrm{tp}_{\gamma_{t,\ell}}(a_{n-1}/B^\frown \mathbf{\overline{c}}).$$

Therefore, $\models \gamma_{t,\ell}(e_{n-1}; \overline{b}, \mathbf{\overline{c}})$. Hence, by definition of $\gamma_{t,\ell}$,

$$\models \forall x_0 ... \forall x_{n-1} \big( \psi_\ell(x_0, ..., x_{n-2}; e_{n-1}, \overline{c}_0, ..., e_{n-1}, \overline{c}_{N_0-1}) \to \varphi(x_0, ..., x_{n-2}, e_{n-1}; \overline{b})^t \big).$$

In particular, since $(e_0, ..., e_{n-2}) \models \psi_\ell(x_0, ..., x_{n-2}; e_{n-1}, \overline{c}_0, ..., e_{n-1}, \overline{c}_{N_0-1})$, we see that $\models \varphi(\overline{e}; \overline{b})^t$. Since $\overline{b} \in B$ was arbitrary, we see that $\overline{e}$ realizes the type $\mathrm{tp}_\varphi(\overline{a}/B)$. This is the desired conclusion. $\square$

In light of Proposition 3.4.6, in order to prove Theorem 3.4.3, it suffices to check that, in a dp-minimal theory with a linear order, all formulas of the form $\varphi(x; \overline{y})$ have finite type implications. For the remainder of this section, let $T$ be a dp-minimal theory and let $<$ be a definable linear order on $\mathfrak{C}$. Fix $\varphi(x;\overline{y})$ any formula.

Since $T$ is dp-minimal, the formulas $\{\varphi(x;\overline{y}), \psi(x;z_0,z_1)\}$ do not form an ICT-pattern, where $\psi(x;z_0,z_1) = (z_0 < x \leq z_1)$ (see Section 2.4 for a definition). By compactness, there exists $K < \omega$ such that, for all $\overline{b}_0, ..., \overline{b}_{K-1} \in \mathfrak{C}^{\lg(\overline{y})}$, all $c_0 < ... < c_K$ from $\mathfrak{C}$, and both $t < 2$, there exists $i_0, j_0 < K$ such that

$$\models \neg\exists x \left( c_{j_0} < x \leq c_{j_0+1} \wedge \neg\varphi(x;\overline{b}_{i_0})^t \wedge \bigwedge_{i \neq i_0} \varphi(x;\overline{b}_i)^t \right). \tag{3.5}$$

For simplicity, define $\gamma_{i_0,t}(x;\overline{\mathbf{b}})$ for each $i_0 < K$ and $t < 2$ as follows:

$$\gamma_{i_0,t}(x;\overline{b}_0, ..., \overline{b}_{K-1}) = \neg\varphi(x;\overline{b}_{i_0})^t \wedge \bigwedge_{i \neq i_0} \varphi(x;\overline{b}_i)^t.$$

For each $\overline{b}_0, ..., \overline{b}_{K-1} \in \mathfrak{C}^{\lg(\overline{y})}$, $t < 2$, and $i < K$, consider the following definition:

**Definition 3.4.7.** We say that a sequence $d_0 < ... < d_L$ from $\mathfrak{C}$ is $(i,t)$*-full for* $\overline{\mathbf{b}} = (\overline{b}_0, ..., \overline{b}_{K-1})$ if the following conditions hold:

(i) $\models \gamma_{i,t}(d_j;\overline{\mathbf{b}})$ for all $j \leq L$; and

(ii) $\models \bigwedge_{i'<K, i' \neq i} \exists x (d_j < x < d_{j+1} \wedge \gamma_{i',t}(x;\overline{\mathbf{b}}))$ for all $j < L$.

So not only do we demand that each $d_j$ satisfy $\gamma_{i,t}(x;\overline{\mathbf{b}})$, but between each of them there is a witness to $\gamma_{i',t}(x;\overline{\mathbf{b}})$ for all other $i' < K$. Because of dp-minimality, there is a restriction on the length of $(i,t)$-full sequences.

**Lemma 3.4.8.** *If* $d_0 < ... < d_L$ *is a sequence that is* $(i,t)$*-full for* $\overline{\mathbf{b}}$*, then* $L < K$*.*

*Proof.* Suppose that $L \geq K$. Then, notice that the following is true for all $i, j < K$:

$$\models \exists x \left( d_j < x \leq d_{j+1} \wedge \gamma_{i,t}(x; \overline{\mathbf{b}}) \right).$$

By definition of $\gamma_{i,t}$, this contradicts (3.5). $\square$

For each $\overline{\mathbf{b}} \in (\mathfrak{C}^{\lg(\overline{y})})^K$, $a \in \mathfrak{C}$, $i < K$, and $t < 2$, let $W(a, \overline{\mathbf{b}}, i, t)$ be the largest (ordered by inclusion) convex subset of $\mathfrak{C}$ containing $a$ such that, for all $e \in W(a, \overline{\mathbf{b}}, i, t)$, $\models \neg\gamma_{i,t}(e; \overline{\mathbf{b}})$ (if $\models \gamma_{i,t}(a; \overline{\mathbf{b}})$, then $W(a, \overline{\mathbf{b}}, i, t) = \emptyset$, but this can only happen for at most one $i < K$). These convex sets are themselves partially ordered by inclusion. So let $W(a, \overline{\mathbf{b}}, t)$ be a maximal such convex set and let $i(a, \overline{\mathbf{b}}, t) < K$ be so that $W(a, \overline{\mathbf{b}}, t) = W(a, \overline{\mathbf{b}}, i(a, \overline{\mathbf{b}}, t), t)$. We now show that these sets are uniformly definable over $\overline{\mathbf{b}}$.

Fix $t < 2$. For each $i < K$ and $\ell < K$, let $\theta_{i,\ell,t}(w_0, ..., w_\ell; \overline{\mathbf{z}})$ encode the fact that $w_0 < ... < w_\ell$ is $(i,t)$-full for $\overline{\mathbf{z}}$, which is expressible as a $\emptyset$-definable first-order formula. For example,

$$\theta_{i,\ell,t}(w_0, ..., w_\ell; \overline{\mathbf{z}}) = \bigwedge_{j \leq \ell} \gamma_{i,t}(w_j; \overline{\mathbf{z}}) \wedge \bigwedge_{j < \ell} \left( \bigwedge_{i' < K, i' \neq i} \exists x (w_j < x < w_{j+1} \wedge \gamma_{i',t}(x; \overline{\mathbf{z}})) \right).$$

For each $I \subseteq K$, $L \in {}^K K$, and $j \in {}^K(K+1)$, let

$$\psi_{I,L,j,t}(x; \overline{\mathbf{z}}) = \bigwedge_{i \in I} \forall w_0 ... w_{L(i)} \left( \theta_{i,L(i),t}(w_0, ..., w_L(i); \overline{\mathbf{z}}) \rightarrow w_{j(i)-1} < x < w_{j(i)} \right),$$

where we let $w_{-1} = -\infty$ and $w_{j(i)} = \infty$ for $j(i) > L(i)$. Finally, let $\psi_0(x; \overline{\mathbf{z}}) = (x = x)$.

**Lemma 3.4.9.** *For each $\overline{\mathbf{b}} \in (\mathfrak{C}^{\lg(\overline{y})})^K$, $a \in \mathfrak{C}$, and $t < 2$, there exists $I \subseteq K$, $L \in {}^K K$, and $j \in {}^K(K+1)$ such that $\psi_{I,L,j,t}(\mathfrak{C}; \overline{\mathbf{b}}) = W(a, \overline{\mathbf{b}}, t)$ (or $\psi_0(\mathfrak{C}; \overline{\mathbf{b}}) = \mathfrak{C} = W(a, \overline{\mathbf{b}}, t)$).*

*Proof.* Let $W = W(a, \overline{\mathbf{b}}, t)$ and let $I = \{i < K : (\forall e \in W)(\models \neg\gamma_{i,t}(e, \overline{\mathbf{b}}))\}$. Since $i(a, \overline{\mathbf{c}}, t) \in I$, we know that $|I| \geq 1$. If $W = \mathfrak{C}$, then $\psi_0(x; \overline{\mathbf{b}})$ defines $W$, so suppose not. Therefore, for all $i \in I$, there exists at least one $d_i \in \mathfrak{C}$ with $\models \gamma_{i,t}(d_i; \overline{\mathbf{b}})$.

**Claim 1.** If $i \in I$, $d_0, d_1 \models \gamma_{i,t}(x; \overline{\mathbf{b}})$, and $d_0 < W < d_1$, then $d_0 < d_1$ is $(i,t)$-full for $\overline{\mathbf{b}}$.

To see this, it suffices to show that condition (ii) of Definition 3.4.7 is satisfied, as condition (i) is given by hypothesis. If, by means of contradiction, there exists $i' < K$ with $i' \neq i$ so that, for all $e$ with $d_0 < e < d_1$, $\models \neg\gamma_{i',t}(e; \overline{\mathbf{b}})$, then the interval $[d_0, d_1] \subseteq W(a, \overline{\mathbf{b}}, i', t)$. However, since $W \subsetneq (d_0, d_1)$, we get a contradiction to the maximality of $W = W(a, \overline{\mathbf{c}}, t)$.

Now, for each $i \in I$, let $L(i)$ be maximal so that there exists $d_{i,0} < ... < d_{i,L(i)}$ a sequence that is $(i,t)$-full for $\overline{\mathbf{b}}$. Then, for each $i \in I$, there exists $j(i)$ with $0 \leq j(i) \leq L(i) + 1$ such that $d_{i,j(i)-1} < W < d_{i,j(i)}$ (where we set $d_{i,-1} = -\infty$ and $d_{i,L(i)+1} = \infty$).

**Claim 2.** For all $e \in \mathfrak{C}$, $e \in W$ if and only if, for all $i \in I$, for all sequences $d'_0 < ... < d'_{L(i)}$ that are $(i,t)$-full for $\overline{\mathbf{b}}$, $d'_{j(i)-1} < e < d'_{j(i)}$ (where again we use the convention that $d'_{-1} = -\infty$ and $d'_{L(i)+1} = \infty$).

To see this, first suppose that $e \in W$ and fix $i \in I$ and a sequence $d'_0 < ... < d'_{L(i)}$ that is $(i,t)$-full for $\overline{\mathbf{b}}$. Suppose, by means of contradiction, that $d'_{j'-1} < e < d'_{j'}$ for some $j' \neq j(i)$. Then, $d'_{j'-1} < W < d'_{j'}$. Without loss of generality, suppose $j' > j(i)$ (the other direction follows by symmetry). By our first claim, $d'_0 < ... < d'_{j'-1} < d_{i,j(i)} < ... < d_{i,L(i)}$ is $(i,t)$-full for $\overline{\mathbf{b}}$. As $j' > j(i)$, this sequence has length $> L(i)$, contradicting the maximality of $L(i)$.

Conversely, suppose that, for all $i \in I$, for all sequences $d'_0 < ... < d'_{L(i)}$ that are $(i,t)$-full for $\overline{\mathbf{b}}$, $d'_{j(i)-1} < e < d'_{j(i)}$. By means of contradiction, suppose that $e \notin W$. Without loss of generality, we may suppose $e < W$ (the other direction follows by symmetry). Then, there exists $i' \in I$ and $e'$ with $e \leq e' < W$ so that $\models \gamma_{i',t}(e';\overline{\mathbf{b}})$ (if not, then for any $f \in W$, $[e,f] \cup W \subseteq W(a,\overline{\mathbf{b}},i',t)$, which would contradict the maximality of $W = W(a,\overline{\mathbf{b}},t)$). Fix any $d'_0 < ... < d'_{L(i')}$ that are $(i',t)$-full for $\overline{\mathbf{b}}$. Thus $d'_{j(i')-1} < e \leq e'$ by hypothesis on $i'$. However, by the other direction, we have that $W < d'_{j(i')}$, hence $d'_{j(i')-1} < e' < W < d'_{j(i')}$. By our first claim, $d'_0 < ... < d'_{j(i')-2} < e' < d'_{j(i')} < ... < d'_{L(i)}$ is $(i,t)$-full for $\overline{\mathbf{b}}$. By hypothesis, this means that $e' < e < d'_{j(i')}$. However, this contradicts the fact that $e \leq e'$.

By our second claim, it is clear that $\psi_{I,L,j,t}(\mathfrak{C};\overline{\mathbf{b}}) = W$. $\square$

Let $\mathbf{I} : (2 \times 2) \to \mathcal{P}(K)$, $\mathbf{L} : (2 \times 2) \to {}^K K$, and $\mathbf{j} : (2 \times 2) \to {}^K(K+1)$. Define

$$\theta_{\mathbf{I},\mathbf{L},\mathbf{j}}(x;\overline{\mathbf{z}}_{0,0},\overline{\mathbf{z}}_{0,1},\overline{\mathbf{z}}_{1,0},\overline{\mathbf{z}}_{1,1}) = \bigwedge_{s,t<2} \psi_{\mathbf{I}(s,t),\mathbf{L}(s,t),\mathbf{j}(s,t),t}(x;\overline{\mathbf{z}}_{s,t}).$$

Finally, let

$$\delta_{\mathbf{I},\mathbf{L},\mathbf{j}}(x;\overline{\mathbf{z}}_{0,0},\overline{\mathbf{z}}_{0,1},\overline{\mathbf{z}}_{1,0},\overline{\mathbf{z}}_{1,1},\overline{\mathbf{w}}_0,\overline{\mathbf{w}}_1) = \theta_{\mathbf{I},\mathbf{L},\mathbf{j}} \wedge \bigwedge_{i<K-1,t<2} \varphi(x;\overline{w}_{i,t})^t.$$

We claim that the $\delta_{\mathbf{I},\mathbf{L},\mathbf{j}}$ witness the fact that $\varphi(x;\overline{y})$ has finite type implications.

Fix $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite, $a \in \mathfrak{C}$, and $t < 2$ and let

$$W_t^* = \bigcap_{\overline{\mathbf{b}} \in B^K} W(a,\overline{\mathbf{b}},t).$$

Since this is the intersection of finitely many convex subsets of $\mathfrak{C}$, there exists $\overline{\mathbf{b}}_{0,t}, \overline{\mathbf{b}}_{1,t} \in B^K$ such that

$$W_t^* = W(a,\overline{\mathbf{b}}_{0,t},t) \cap W(a,\overline{\mathbf{b}}_{1,t},t).$$

So we see that $W_0^* \cap W_1^* = \bigcap_{s,t<2} W(a, \overline{\mathbf{b}}_{s,t}, t)$, which is defined by the formula

$$\theta(x) = \theta_{\mathbf{I},\mathbf{L},\mathbf{j}}(x; \overline{\mathbf{b}}_{0,0}, \overline{\mathbf{b}}_{0,1}, \overline{\mathbf{b}}_{1,0}, \overline{\mathbf{b}}_{1,1})$$

for an appropriate choice of $\mathbf{I}(s,t), \mathbf{L}(s,t), \mathbf{j}(s,t)$, as given by Lemma 3.4.9. Clearly $\models \theta(a)$ and, for all $\overline{\mathbf{c}} \in B^K$ and $t < 2$, $\theta(\mathfrak{C}) \subseteq W(a, \overline{\mathbf{c}}, t)$. For each $t < 2$, let $B_t = \{\overline{b} \in B : \models \varphi(a; \overline{b})^t\}$. Now, as we did in Section 2.4, we put a quasi-order on a subset of $\mathcal{P}(B_t)$. Notice that, for all $\overline{c}_0, ..., \overline{c}_{K-1} \in B$, there exists $i_0 < K$ such that

$$\models \forall x \left( \theta(x) \wedge \bigwedge_{i \neq i_0} \varphi(x; \overline{c}_i)^t \rightarrow \varphi(x; \overline{c}_{i_0})^t \right) \tag{3.6}$$

(this is because $\theta(\mathfrak{C}) \subseteq W(a, \overline{\mathbf{c}}, t)$ by construction, hence there exists $i_0 < K$ such that $\models \forall x(\theta(x) \rightarrow \neg\gamma_{i_0,t}(x; \overline{\mathbf{c}}))$, but this is exactly (3.6)).

**Lemma 3.4.10.** *Fix $t < 2$. Then, either for all $\overline{b} \in B_t$, $\models \neg\varphi(a; \overline{b})^t$ or there exists $\overline{c}_0, ..., \overline{c}_{K-2} \in B_t$ such that*

$$\theta(x) \wedge \bigwedge_{i<K-1} \varphi(x; \overline{c}_i)^t \vdash \mathrm{tp}_\varphi(a/B_t)(x).$$

*Proof.* If $B_t = \emptyset$, then clearly $\models \neg\varphi(a; \overline{b})^t$ for all $\overline{b} \in B$, so we may assume that $B_t \neq \emptyset$. If $|B_t| < K$, then we can set $B_t = \{\overline{c}_0, ..., \overline{c}_{K-2}\}$ and this choice of $\overline{c}_0, ..., \overline{c}_{K-2} \in B_t$ clearly works. So we may also assume that $|B_t| \geq K$. Let $[B_t]^{K-1} = \{C \subseteq B : |C| = K-1\}$ and define a partial order, $\leq_a$, on the set $[B_t]^{K-1}$, as follows:

$$C_0 \leq_a C_1 \text{ if and only if } \mathrm{tp}_\varphi(a/C_0)(x) \cup \{\theta(x)\} \vdash \mathrm{tp}_\varphi(a/C_1)(x).$$

Notice that this is the same as $\leq_p$ defined in Section 2.4.1 where we let $p(x) = \mathrm{tp}_\varphi(a/B)(x) \cup \{\theta(x)\}$. Therefore, it is easy to see that Lemma 2.4.10 holds for $\leq_a$, and $\leq_a$-minimal elements exist. Choose $C_0 \in [B_t]^{K-1}$ $\leq_a$-minimal in $[B_t]^{K-1}$ and let $C_0 = \{\overline{c}_0, ..., \overline{c}_{K-2}\}$. We claim that these $\overline{c}_i$'s work.

By way of contradiction, suppose $\theta(x) \wedge \bigwedge_{i<K-1} \varphi(x; \overline{c}_i)^t \not\vdash \mathrm{tp}_\varphi(a/B_t)(x)$. Fix $\overline{b} \in B_t$ such that

$$\models \neg \forall x \left( \theta(x) \wedge \bigwedge_{i<K-1} \varphi(x; \overline{c}_i)^t \to \varphi(x; \overline{b})^t \right).$$

However, as established in (3.6), there exists $i_0 < K-1$ such that

$$\models \forall x \left( \theta(x) \wedge \bigwedge_{i<K-1, i \neq i_0} \varphi(x; \overline{c}_i)^t \wedge \varphi(x; \overline{b})^t \to \varphi(x; \overline{c}_{i_0})^t \right).$$

Therefore, we see that $C_0 - \{\overline{c}_{i_0}\} \cup \{\overline{b}\} <_a C_0$, contrary to the minimality of $C_0$. Hence, $\overline{c}_0, ..., \overline{c}_{K-2}$ works to prove the lemma. $\square$

Finally, for each $t < 2$, let $\overline{\mathbf{c}}_t = (\overline{c}_{0,t}, ..., \overline{c}_{K-2,t})$ be given as in Lemma 3.4.10 (in the case where $\models \neg\varphi(a; \overline{b})^t$ for all $\overline{b} \in B$, we ignore this value of $t < 2$). By Lemma 3.4.10 on both $t < 2$ simultaneously, we see that

$$\theta(x) \wedge \bigwedge_{i<K-1, t<2} \varphi(x; \overline{c}_{i,t})^t \vdash \mathrm{tp}_\varphi(a/B)(x).$$

That is,

$$\delta_{\mathbf{I},\mathbf{L},\mathbf{j}}(x; \overline{\mathbf{b}}_{0,0}, \overline{\mathbf{b}}_{0,1}, \overline{\mathbf{b}}_{1,0}, \overline{\mathbf{b}}_{1,1}, \overline{\mathbf{c}}_0, \overline{\mathbf{c}}_1) \vdash \mathrm{tp}_\varphi(a/B)(x).$$

Hence, we see that $\varphi(x; \overline{y})$ has finite type implications. Since $\varphi(x; \overline{y})$ was arbitrary, by Proposition 3.4.6, all formulas $\varphi(\overline{x}; \overline{y})$ in a dp-minimal theory with a definable linear order on $\mathfrak{C}$ have finite type implications. This completes the proof of Theorem 3.4.3 and answers Conjecture 3.4.2 for dp-minimal theories with a definable linear order on $\mathfrak{C}$.

# Chapter 4

# VC-Minimal Theories and Variants

## 4.1 Overview

In this chapter, we take a slight detour from our main goal of investigating definability of types to discuss VC-minimal theories. VC-minimal theories were invented by Hans Adler in [2]. In that paper, he shows that VC-minimal theories generalize o-minimal and weakly o-minimal theories. Additionally, VC-minimal theories are dp-minimal, hence they have UDTFS by Theorem 2.4.1.

For the remainder of this chapter, fix a theory $T$ with monster model $\mathfrak{C}$.

**Definition 4.1.1.** A set of formulas $\Phi = \{\varphi_i(x; \overline{y}_i) : i \in I\}$ is a *VC-instantiable family* if, for all $i, j \in I$, $\overline{b} \in \mathfrak{C}^{\lg(\overline{y}_i)}$, and $\overline{c} \in \mathfrak{C}^{\lg(\overline{y}_j)}$, one of the following four conditions holds:

(i) $\models \neg\exists x(\varphi_i(x; \overline{b}) \wedge \varphi_j(x; \overline{c}))$,

(ii) $\models \neg\exists x(\neg\varphi_i(x; \overline{b}) \wedge \varphi_j(x; \overline{c}))$,

(iii) $\models \neg\exists x(\varphi_i(x; \overline{b}) \wedge \neg\varphi_j(x; \overline{c}))$, or

(iv) $\models \neg\exists x(\neg\varphi_i(x; \overline{b}) \wedge \neg\varphi_j(x; \overline{c}))$.

An *instance* of $\Phi$ means $\varphi_i(x; \overline{b})$ for some $i \in I$ and $\overline{b} \in \mathfrak{C}^{\lg(\overline{y}_i)}$. We say that a theory $T$ is *VC-minimal* if there exists a VC-minimal instantiable family $\Phi$ such that all parameter-definable formulas $\psi(x)$ are $T$-equivalent to a boolean combination of instances of $\Phi$.

VC-minimal theories have a "backbone" of formulas that have independence dimension $\leq 1$. VC-minimality was developed primarily to generalize the appearance of swiss cheeses (see Proposition 7 of [2]). We get the following theorem relating VC-minimality to other model-theoretic properties:

**Theorem 4.1.2** ([2])**.** *The following hold:*

*(i) If $T$ is strongly minimal, o-minimal, or weakly o-minimal, then $T$ is VC-minimal.*

*(ii) The theory* ACVF *in the language $L_{\mathrm{val}}$ (as in Theorem 2.5.11) is VC-minimal.*

*(iii) If $T$ is VC-minimal, then $T$ is dp-minimal.*

In Section 4.2, we explore a notion called convexly orderable. Our main result is to show that any model of a VC-minimal theory is convexly orderable. In Section 4.3, we develop several alternatives to VC-minimality, including full VC-minimality and weak VC-minimality. In Section 4.4, we discuss the conjecture from [3]: All VC-minimal theories have VC-density one (Conjecture 4.4.1 below). Although this question is still open, we present evidence for the truth of the conjecture in this section. Finally, we conclude this thesis with a proof of Kueker's Conjecture for weakly VC-minimal theories in Section 4.5.

## 4.2 Convexly Orderable Structures

The main objective of this section is to show that VC-minimal theories are "like" weakly o-minimal theories. In a weakly o-minimal theory $T$, for any formula $\varphi(x;\overline{y})$, there exists $K < \omega$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{b})}$, $\varphi(\mathfrak{C};\overline{b})$ is a union of at most $K$ convex subsets of $\mathfrak{C}$. Thus, models of a weakly o-minimal theories are convexly orderable, defined as follows:

**Definition 4.2.1.** Fix an $L$-structure $M$. We say that $M$ is *convexly orderable* if there exists a linear order $<$ on $M$ (not necessarily definable) such that, for all formulas $\varphi(x;\overline{y})$, there exists $K < \omega$ such that, for all $\overline{b} \in M^{\lg(\overline{b})}$, $\varphi(M;\overline{b})$ is a union of at most $K$ $<$-convex subsets of $M$.

In fact, it is clear that if $M$ is a reduct of a weakly o-minimal theory, then $M$ is convexly orderable. More is true.

**Theorem 4.2.2.** *If $T$ is a VC-minimal theory and $M \models T$, then $M$ is convexly orderable.*

In order to prove this, we actually show something more general about sets with independence dimension $\leq 1$. Fix $X$ any set and $\mathcal{A} \subseteq \mathcal{P}(X)$ (so $\mathcal{A}$ is a set of subsets of $X$). Since $X$ is fixed beforehand, for any $A \subseteq X$, let $A^0 = X - A$ and $A^1 = A$. We say that $\mathcal{A}$ has *independence dimension* $n < \omega$ if $n$ is maximal such that there exists $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = n$ and, for all $s \in {}^{\mathcal{B}}2$, we have $\bigcap_{B \in \mathcal{B}} B^{s(i)} \neq \emptyset$. We denote this by $\mathrm{ID}_X(\mathcal{A}) = n$. Let $L = \{U, R\}$ as in Section 2.7 and let $M_{\mathcal{A}}$ be the $L$-structure where $M_{\mathcal{A}} = M = X \sqcup \mathcal{A}$, $U^M = X$, and $R^M(x, A)$ holds if and only if

$x \in X$, $A \in \mathcal{A}$, and $x \in A$. Then, $\mathcal{A}$ has independence dimension $n$ if and only if $R(x; y)$ has independence dimension $n$ in $M_{\mathcal{A}}$ in the usual sense. This construction is similar to the one in Section 2.7, except we demand here that $\mathcal{A}$ is a set of subsets of $X$ and not a set of functions from ${}^{X}2$ and, when constructing $M_{\mathcal{A}}$, we reverse the roles of $X$ and $\mathcal{A}$ with respect to $R$. Both of these changes are superficial, but aid in the presentation of this section.

**Proposition 4.2.3.** *If $X$ is any set and $\mathcal{A} \subseteq \mathcal{P}(X)$ has independence dimension $\leq 1$, then there exists $<$ a linear order on $X$ such that, for all $A \in \mathcal{A}$, either $A^0$ is a $<$-convex subset of $X$ or $A^1$ is a $<$-convex subset of $X$.*

First we show this for finite $\mathcal{A}$, then use compactness to yield the general result.

**Lemma 4.2.4.** *Fix $X$ any set and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be any finite set of subsets of $X$ that has independence dimension at most one. Then, there exists a linear order $<$ on $X$ such that, for all $A \in \mathcal{A}$, either $A^0$ or $A^1$ is a $<$-convex subset of $X$.*

*Proof of Lemma 4.2.4.* Notice that it suffices to show this for $\mathcal{A}$ such that $\emptyset \notin \mathcal{A}$ and $X \notin \mathcal{A}$ (regardless of the order we put on $X$, these two sets are convex). We first prove the following claim via induction on the size of $\mathcal{A}$:

**Claim 1.** If $\mathcal{A}$ is such that there exists no $A, B \in \mathcal{A}$ where $A \cap B \neq \emptyset$, $(B - A) \neq \emptyset$, and $(A - B) \neq \emptyset$, then there exists a linear order $<$ on $X$ such that, for all $A \in \mathcal{A}$, $A$ is a $<$-convex subset of $X$.

*Proof of Claim 1.* This proceeds by induction on $|\mathcal{A}|$. If $|\mathcal{A}| = 1$, this is trivial, so suppose not. Fix $A \in \mathcal{A}$ that is $\subseteq$-maximal. As stated above, we may assume

$A \neq X$. Then, for all $B \in \mathcal{A}$, notice that either $B \subseteq A^0$ or $B \subseteq A^1$, but not both. To see this, suppose that $A^0 \cap B^1 \neq \emptyset$ and $A^1 \cap B^1 \neq \emptyset$. Further, suppose that $A^1 \cap B^0 = \emptyset$ (i.e., $A \subset B$). This contradicts the maximality of $A$, so $A^1 \cap B^0 \neq \emptyset$. Hence $A \cap B \neq \emptyset$, $(B - A) \neq \emptyset$, and $(A - B) \neq \emptyset$, contrary to assumption.

Therefore, we can break up $\mathcal{A} - \{A\}$ into two disjoint subsets:

$$\mathcal{A}_t = \{B \in \mathcal{A} : B \neq A \wedge B \subseteq A^t\}$$

for each $t < 2$. Now, as $\mathcal{A}_t$ has independence dimension $\leq 1$ with respect to $A^t$, by induction hypothesis there exists an ordering $<_t$ on $A^t$ such that, for all $B \in \mathcal{A}_t$, $B$ is a $<_t$-convex subset of $A^t$. Define $<$ globally by letting $<$ extend $<_0$ and $<_1$ and setting $A^0 < A^1$ (i.e., for all $a_0 \in A^0$ and $a_1 \in A^1$, $a_0 < a_1$). Then, for all $B \in \mathcal{A}$, either:

(i) $B = A^t$ for some $t < 2$ and $B$ is clearly $<$-convex;

(ii) $B \in \mathcal{A}_0$ and $B$ is $<_0$-convex, hence since $B \subseteq A^0$ and $<$ extends $<_0$, $B$ is $<$-convex; or

(iii) $B \in \mathcal{A}_1$ and $B$ is $<_1$-convex, hence since $B \subseteq A^1$ and $<$ extends $<_1$, $B$ is $<$-convex.

Claim 1 $\square$

Now, more generally, fix $\mathcal{A} \subseteq \mathcal{P}(X)$ finite with $\mathrm{ID}_X(\mathcal{A}) \leq 1$. Fix any $A \in \mathcal{A}$ (recall that we may assume $A \neq X$ and $A \neq \emptyset$). Now, by independence dimension $\leq 1$, for any $B \in \mathcal{A}$, one of the following four conditions hold:

(i) $B^0 \subseteq A^0$,

(ii) $B^0 \subseteq A^1$,

(iii) $B^1 \subseteq A^0$, or

(iv) $B^1 \subseteq A^1$,

Let $\mathcal{A}_t = \{B^s : B \in \mathcal{A} - \{A\}, s < 2, B^s \subseteq A^t\}$ for each $t < 2$. By the statement above, for any $B \in \mathcal{A} - \{A\}$, there exists $t, s < 2$ such that $B^s \in \mathcal{A}_t$. Furthermore, $\mathcal{A}_t$ has independence dimension $\leq 1$ with respect to $A^t$ for both $t < 2$ (replacing $B$ with $B^0$ does not change the independence dimension). We claim that $\mathcal{A}_t$ satisfies the hypotheses of Claim 1, hence showing that there is a linear order $<_t$ on $A^t$ so that, for all $B \in \mathcal{A}_t$, $B$ is a $<_t$-convex subset of $A^t$.

To see this, choose any $B_0, B_1 \in \mathcal{A}_t$ and suppose that $B_0 \cap B_1 \neq \emptyset$, $(B_0 - B_1) \neq \emptyset$, and $(B_1 - B_0) \neq \emptyset$. Then, since $A^t \neq X$ and $B_0, B_1 \subseteq A^t$, we have that $B_0 \cup B_1 \neq X$. These four conditions together contradict the fact that $\mathrm{ID}_X(\mathcal{A}) \leq 1$.

Now, as before, let $<$ be the global ordering on $X$ that extends $<_0$ and $<_1$ so that $A^0 < A^1$. Then, we claim that this ordering on $X$ satisfies the desired condition. To see this, fix any $B \in \mathcal{A}$. Then, for some $t, s < 2$, $B^s \in \mathcal{A}_t$ (hence $B^s \subseteq A^t$). Therefore, $B^s$ is a $<_t$-convex subset of $A^t$. As $<$ extends $<_t$ and $B^s \subseteq A^t$, $B^s$ is a $<$-convex subset of $X$. That is, we have shown that all elements of $\mathcal{A}$ are either $<$-convex subsets of $X$ or the compliment of $<$-convex subsets of $X$. This concludes the proof of the lemma. $\square$

*Proof of Proposition 4.2.3.* Fix any set $X$ and fix $\mathcal{A} \subseteq \mathcal{P}(X)$ with independence dimension $\leq 1$. Consider the language $L$ which consists of unary predicate symbols

$P_B$ for each $B \in \mathcal{A}$ and let $M = (X; B)_{B \in \mathcal{A}}$ be the natural $L$-structure. Let $L' = L \cup \{<\}$, where $<$ is a binary relation symbol. By compactness and Lemma 4.2.4, there exists $N \succeq M$ such that $N$ is naturally an $L'$-structure with each $P_B(N)$ a $<$-convex subset of $N$ or the compliment of a $<$-convex subset of $N$. However, being a convex subset is closed under linear subspace. Thus, if we consider $< |_{X \times X}$, we get a linear ordering on $X$ with the desired result. $\square$

We are now ready to prove Theorem 4.2.2, showing that a structure with a VC-minimal theory is convexly orderable.

*Proof of Theorem 4.2.2.* Fix $T$ a complete theory, and $M \models T$. Suppose $T$ is VC-minimal and let $\Psi = \{\psi_i(x; \overline{z}_i) : i \in I\}$ be a VC-minimal instantiable family witnessing this fact. Fix a formula $\varphi(x; \overline{y})$. Then, by compactness, there exists a number $K < \omega$ such that, for all $\overline{b} \in M^{\lg(\overline{y})}$, $\varphi(x; \overline{b})$ is a boolean combination of at most $K$ instances of formulas from $\Psi$. Let $\mathcal{A} = \{\psi_i(M; \overline{c}_i) : i \in I, \overline{c}_i \in M^{\lg(\overline{z}_i)}\}$. By VC-minimality, the independence dimension of $\mathcal{A}$ is $\leq 1$. Thus, by Proposition 4.2.3, there exists a linear order $<$ on $M$ so that all elements of $\mathcal{A}$ are either $<$-convex or the compliment of a $<$-convex set. Therefore, for any $\overline{b} \in M^{\lg(\overline{y})}$, since $\varphi(M; \overline{b})$ is a boolean combination of at most $K$ elements from $\mathcal{A}$, it is a union of at most $K + 1$ $<$-convex subsets of $M$. $\square$

Reducts of convexly orderable structures are convexly orderable. One interesting open question we get from this is the following: Does being the reduct of a structure with a VC-minimal theory characterize convexly orderability? We can relate theories with convexly orderable structures to notions of VC-density and dp-minimality?

**Proposition 4.2.5.** *If $T$ is any complete theory all of whose models are convexly orderable, then all formulas of the form $\varphi(x;\overline{y})$ (with $x$ a singleton) have VC-density $\leq 1$.*

*Proof.* Fix $M \models T$, so $M$ is convexly orderable, and fix $<$ an ordering on $M$ witnessing this fact. By definition, there exists $K < \omega$ such that, for all $\overline{b} \in M^{\lg(\overline{y})}$, $\varphi(M;\overline{b})$ is a union of at most $K$ $<$-convex subsets of $M$. Therefore, there are at most $2K$ "endpoints" (i.e., the truth value of $\varphi(x;\overline{b})$ alternates at most $2K$ times). So if we take any finite $B \subseteq M^{\lg(\overline{y})}$, there are at most $2K \cdot |B|$ "endpoints" from all the $\varphi(M;\overline{b})$ as we range over all $\overline{b} \in B$. Therefore, we get that $|S_{\varphi}(B)| \leq 2K \cdot |B| + 1$, hence $\varphi$ has VC-density $\leq 1$. □

The proof of Proposition 3.2 in [6] yields: If $T$ is a theory such that all formulas of the form $\varphi(x;\overline{y})$ have VC-density $\leq 1$, then $T$ is dp-minimal. Therefore, we get the following corollary:

**Corollary 4.2.6.** *If $T$ is any complete theory all of whose models are convexly orderable, then $T$ is dp-minimal.*

Convexly orderability does not characterize dp-minimality. Consider the following example taken from Proposition 3.7 in [6]:

**Example 4.2.7.** Let $L = \{P_n : n < \omega_1\}$ for $P_n$ unary predicates. For any $I, J \subset \omega_1$

finite and disjoint, let

$$\sigma_{I,J} = \exists x \left( \bigwedge_{i \in I} P_i(x) \wedge \bigwedge_{j \in J} \neg P_j(x) \right)$$

and let $T = \{\sigma_{I,J} : \text{ for all such } I, J\}$. As shown in [6], $T$ is complete, has quantifier elimination, and is dp-minimal. However, we claim that $T$ has no convexly orderable structure. Suppose, by way of contradiction, that $M \models T$ is convexly orderable, witnessed by $<$. By pigeon-hole principal, there exists $N < \omega$ and $I \subseteq \omega_1$ with $|I| = \aleph_1$ such that $P_i(M)$ is the union of at most $N$ $<$-convex subsets of $M$ for all $i \in I$. Hence, for each $i \in I$, $P_i(M)$ has at most $2N$ "endpoints." Therefore, for any $k < \omega$ and any finite $I_0 \subseteq I$ with $|I_0| = k$, there are, at most, $2Nk + 1$ $\Delta_{I_0}$-types over $\emptyset$ (where $\Delta_{I_0} = \{P_i(x) : i \in I_0\}$). However, by $M \models \sigma_{I_1, I_0 - I_1}$ for all $I_1 \subseteq I_0$, there are $2^k$ $\Delta_{I_0}$-types over $\emptyset$. Hence $2^k \leq 2Nk + 1$ for all $k < \omega$, a contradiction.

We should remark that if we replace $L$ with $L = \{P_n : n < \omega\}$ and build the corresponding theory $T$, then $T$ is actually VC-minimal, hence also has a convexly orderable model. In fact, the following is a VC-instantiable family for $T$:

$$\left\{ \bigwedge_{i<n} P_i(x)^{s(i)} \wedge P_n(x) : n < \omega, s \in {}^n 2 \right\}.$$

## 4.3 Fully VC-Minimal Theories and Weakly VC-Minimal Theories

In this section, we discuss two variants of VC-minimality. The first is full VC-minimality, which is stronger than VC-minimality. We show that weakly o-minimal theories are fully VC-minimal (Proposition 4.3.3) and that fully VC-minimal theories have VC-density one (Proposition 4.3.4). However, we show that not all VC-minimal theories are fully VC-minimal, so this does not prove Conjecture 4.4.1. We discuss this conjecture more in Section 4.4. The second variant we discuss is weakly VC-minimal. We show that VC-minimal theories are weakly VC-minimal, but not conversely. This definition is key to Section 4.5, where we show that all weakly VC-minimal theories satisfy the Kueker Conjecture (Corollary 4.5.6 below).

### 4.3.1 Fully VC-Minimal Theories

**Definition 4.3.1.** Say that $T$ is *fully VC-minimal* if there exists a family of formulas $\Psi$ (each of which has $x$ as a free variable) such that:

(i) For all $\psi_0(x;\overline{y}_0), \psi_1(x;\overline{y}_1) \in \Psi$, $\overline{b}_0 \in \mathfrak{C}^{\lg(\overline{y}_0)}$, and $\overline{b}_1 \in \mathfrak{C}^{\lg(\overline{y}_1)}$, there exists $t(0), t(1) < 2$ such that

$$\models \neg\exists x(\psi_0(x;\overline{b}_0)^{t(0)} \wedge \psi_1(x;\overline{b}_1)^{t(1)}).$$

(ii) For all formulas $\varphi(x;\overline{y})$, there exists $\psi(x;\overline{y})$ a boolean combination of formulas from $\Psi$ with free variables $(x;\overline{y})$ such that $\varphi(x;\overline{y})$ is equivalent to $\psi(x;\overline{y})$.

**Lemma 4.3.2.** *This definition is equivalent to the same definition after replacing (ii) with the following condition:*

*(ii)' For all formulas $\varphi(x;\overline{y})$ and all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, there exists $\psi_0(x;\overline{y}), ..., \psi_{n-1}(x;\overline{y}) \in \Psi$ such that $\varphi(x;\overline{b})$ is equivalent to a boolean combination of the formulas $\psi_i(x;\overline{b})$ for $i < n$.*

First, notice that this only differs from the definition of VC-minimal in that we insist that the instances of $\psi_i$ be exactly $\psi_i(x;\overline{b})$ for the same $\overline{b}$ from $\varphi(x;\overline{b})$. Therefore, it is clear from this lemma that VC-minimal implies fully VC-minimal.

*Proof of Lemma 4.3.2.* Since it is clear that (i) and (ii) implies (i) and (ii)', it suffices to show that (i) and (ii)' implies (i) and (ii). So let $\Psi$ be given satisfying (i) and (ii)'. First, let

$$\Psi' = \Psi \cup \{\delta(\overline{y}) : x \text{ is not a free variable in } \overline{y}\}.$$

Since $x$ acts as a dummy variable in each $\delta(\overline{y}) \in \Psi' - \Psi$, it should be clear that $\Psi'$ still satisfies (i). We claim that $\Psi'$ now satisfies (ii).

Fix any formula $\varphi(x;\overline{y})$. By (ii)' and compactness, there exists finitely many formulas $\gamma_0(x;\overline{y}), ..., \gamma_\ell(x;\overline{y})$ that are each a boolean combination of formulas from $\Psi'$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{b})}$, $\varphi(x;\overline{b})$ is equivalent to $\gamma_i(x;\overline{b})$ for some $i \leq \ell$ (Let

$$\Sigma(\overline{y}) = \{\neg\forall x(\varphi(x;\overline{y}) \leftrightarrow \gamma(x;\overline{y})) : \gamma \text{ a boolean combination of formulas from } \Psi\}$$

and use compactness on this partial type to yield the desired result). Now let

$$\delta_i(\overline{y}) = \forall x(\varphi(x;\overline{y}) \leftrightarrow \gamma_i(x;\overline{y}))$$

for each $i \leq \ell$ and we see that $\varphi(x;\overline{y})$ is equivalent to

$$\bigwedge_{i \leq \ell} \delta_i(\overline{y}) \rightarrow \gamma_i(x;\overline{y}).$$

As $\delta_i(\overline{y}) \in \Psi'$, this is a boolean combination of formulas from $\Psi'$, hence showing (ii). $\square$

**Proposition 4.3.3.** *If $T$ is weakly o-minimal, then $T$ is fully VC-minimal.*

*Proof.* For each formula $\varphi(x;\overline{y})$ from $T$ and each $n < \omega$, we define the formula that gives the $n$th leftward ray carved out by $\varphi(x;\overline{y})$:

$$\psi_{\varphi,n}(x;\overline{y}) = \neg\exists w_0...w_n \left( x = w_n \wedge \bigwedge_{i<n} \big(w_i < w_{i+1} \wedge \varphi(w_i;\overline{y}) \not\leftrightarrow \varphi(w_{i+1};\overline{y})\big) \right).$$

Then, let

$$\Psi = \{\psi_{\varphi,n}(x;\overline{y}) : \varphi(x;\overline{y}) \text{ is any formula, } n < \omega\}.$$

For all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$ and all formulas $\varphi(x;\overline{y})$, $\psi_{\varphi,n}(x;\overline{b})$ is a downward-closed convex set. Therefore, we see that $\Psi$ satisfies (i). By definition of weak o-minimality and $\psi_{\varphi,n}$, it is clear that condition (ii)' holds for any $\varphi(x;\overline{y})$. Therefore, by Lemma 4.3.2, we see that $T$ is fully VC-minimal. $\square$

**Proposition 4.3.4.** *If $T$ is fully VC-minimal, then $T$ has VC-density one.*

*Proof.* We first show that any formula of the form $\varphi(x;\overline{y})$ has UDTFS rank $\leq 1$. By Corollary 3.2.6, this implies $T$ has VC-density one. Fix $\varphi(x;\overline{y})$ any such formula. By condition (ii) of Definition 4.3.1, there exists $\gamma(x;\overline{y})$ a boolean combination of formulas from $\Psi$ with free variables $(x;\overline{y})$ such that $\varphi$ is equivalent to $\gamma$. Say that

$$\gamma(x;\overline{y}) = \bigvee_{\mu \in I} \left( \bigwedge_{i<n} \psi_i(x;\overline{y})^{\mu(i)} \right)$$

for $n < \omega$, $I \subseteq {}^n2$, and $\psi_0, ..., \psi_{n-1} \in \Psi$.

We now generate a uniform algorithm for determining $\varphi$-types using only a single instance, showing $\varphi$ has UDTFS rank $\leq 1$. Fix $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ finite and $a \in \mathfrak{C}$. Choose $i < n$, $t < 2$, and $\overline{b}_0 \in B$ such that

(i) $\models \psi_i(a;\overline{b}_0)^t$, and

(ii) $\psi_i(x;\overline{b}_0)^t$ is $\vdash$-minimal such.

By condition (i) of Definition 4.3.1, for all $j < n$ and $\overline{b} \in B$, one of the following holds:

(i) $\psi_i(x;\bar{b}_0)^t \vdash \psi_j(x;\bar{b})$;

(ii) $\psi_i(x;\bar{b}_0)^t \vdash \neg\psi_j(x;\bar{b})$;

(iii) $\psi_j(x;\bar{b}) \vdash \psi_i(x;\bar{b}_0)^t$; or

(iv) $\neg\psi_j(x;\bar{b}) \vdash \psi_i(x;\bar{b}_0)^t$.

If (i) holds, then $\models \psi_j(a;\bar{b})$ and if (ii) holds, then $\models \neg\psi_j(a;\bar{b})$, so we may assume (i) and (ii) fails. If (iii) holds, then this contradicts the $\vdash$-minimality of $\psi_i(x;\bar{b}_0)^t$ unless $\models \neg\psi_j(a;\bar{b})$. Similarly, if (iv) holds, then $\models \psi_j(a;\bar{b})$. Thus, $\psi_j(a;\bar{b})$ holds if and only if

$$\begin{aligned}\delta_{i,j,t}(\bar{b};\bar{b}_0) =& \forall x(\psi_i(x;\bar{b}_0)^t \to \psi_j(x;\bar{b})) \vee \\ & [\neg\forall x(\psi_i(x;\bar{b}_0)^t \to \neg\psi_j(x;\bar{b})) \wedge \forall x(\neg\psi_j(x;\bar{b}) \to \psi_i(x;\bar{b}_0)^t)]\end{aligned}$$

holds. Finally, notice that $\gamma(a;\bar{b})$ holds if and only if

$$\delta'_{i,t}(\bar{b};\bar{b}_0) = \bigvee_{\mu\in I}\left(\bigwedge_{j<n}\delta_{i,j,t}(\bar{b};\bar{b}_0)^{\mu(j)}\right)$$

holds. Therefore, we see that the set $\{\delta'_{i,t}(\bar{y};\bar{y}_0) : i < n, t < 2\}$ shows that $\varphi$ has UDTFS rank $\leq 1$. This concludes the proof. $\square$

Since weakly o-minimal theories are fully VC-minimal, Proposition 4.3.4 generalizes the result of Corollary 3.2.7. The problem with full VC-minimality is that, unlike normal VC-minimality, it does not generalize strong minimality. That is, there exists strongly minimal theories that are not fully VC-minimal. It is worse than that; there are strongly minimal theories with a formula $\varphi(x;\bar{y})$ that does not have UDTFS rank $\leq 1$. So the methods of Proposition 4.3.4 cannot be used to yield VC-density one in the case of certain strongly minimal theories.

**Example 4.3.5.** In the theory $\mathrm{ACF}_0$ and the theory $\mathrm{ACVF}_{(0,0)}$, the formula $\varphi(x;\overline{y})$ given by $\varphi(x;y_0,y_1) = [x^2 + y_0x + y_1 = 0]$ has UDTFS rank 2.

*Proof of Example 4.3.5.* Suppose, by means of contradiction, that $\{\psi_\ell(y_0,y_1;z_0,z_1) : \ell < L\}$ is a finite list of formulas witnessing that $\varphi$ has UDTFS rank 1 (so either in the language of $\mathrm{ACF}_0$ or $\mathrm{ACVF}_{(0,0)}$). Notice that $\mathbb{C} \models \mathrm{ACF}_0$ and $\mathbb{C}((t^{\mathbb{Q}})) \models \mathrm{ACVF}_{(0,0)}$, so we work in $\mathbb{C}$ the common subfield of both.

Fix $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ algebraically independent. For any $i < j < 4$, there exists $\sigma_{i,j}$ an automorphism of $\mathbb{C}$ that permutes $\alpha_i$ and $\alpha_j$ and fixes the other two $\alpha_k$'s. This can be extended, in the obvious manner, to an automorphism of $\mathbb{C}((t^{\mathbb{Q}}))$. For each $i < j < 4$, define $\overline{b}_{i,j} = (b^0_{i,j}, b^1_{i,j})$ so that

$$x^2 + b^0_{i,j}x + b^1_{i,j} = (x - \alpha_i)(x - \alpha_j).$$

Let $B = \{\overline{b}_{i,j} : i < j < 4\}$ and let $p(x) = \mathrm{tp}_\varphi(\alpha_0/B)$. Notice that $\sigma_{i,j}(\overline{b}_{i,j}) = \overline{b}_{i,j}$, $\sigma_{i,j}(\overline{b}_{i,k}) = \overline{b}_{j,k}$, and $\sigma_{i,j}(\overline{b}_{k,\ell}) = \overline{b}_{k,\ell}$ for $\{i,j,k,\ell\} = 4$ by definition. Also notice that, for any $i < j < 4$, $\models \varphi(\alpha_0; \overline{b}_{i,j})$ if and only if $i = 0$ (this is the only way for $\alpha_0$ to be a root of $(x - \alpha_i)(x - \alpha_j)$). Now, by the fact that $\{\psi_\ell(y_0,y_1;z_0,z_1) : \ell < L\}$ witnesses UDTFS in $\varphi$, there exists $\ell_0 < L$ and $i_0 < j_0 < 4$ such that the formula

$$\psi_{\ell_0}(\overline{y}; \overline{b}_{i_0,j_0})$$

defines $p$. We now have two cases to consider.

**Case 1.** $i_0 = 0$.

In this case, consider $\sigma = \sigma_{0,j_0}$ and fix $k \in 4 - \{0, j_0\}$. Since $\models \varphi(\alpha_0; \overline{b}_{0,k})$, we get that $\models \psi_{\ell_0}(\overline{b}_{0,k}; \overline{b}_{0,j_0})$. Hence, since $\sigma$ is an automorphism, we see that $\models \psi_{\ell_0}(\sigma(\overline{b}_{0,k}); \sigma(\overline{b}_{0,j_0}))$, i.e., $\models \psi_{\ell_0}(\overline{b}_{j_0,k}; \overline{b}_{0,j_0})$. By definition, we see that $\models \varphi(\alpha_0; \overline{b}_{j_0,k})$, which is clearly a contradiction.

**Case 2.** $i_0 > 0$.

In this case, let $0 < k < 4$ and $k \notin \{i_0, j_0\}$ (there is one such element). Consider now $\sigma = \sigma_{0,k}$, so $\sigma$ still fixes $\overline{b}_{i_0,j_0}$. Now $\models \varphi(\alpha_0; \overline{b}_{0,i_0})$, hence $\models \psi_{\ell_0}(\overline{b}_{0,i_0}; \overline{b}_{i_0,j_0})$. Therefore, $\models \psi_{\ell_0}(\sigma(\overline{b}_{0,i_0}); \sigma(\overline{b}_{i_0,j_0}))$, i.e., $\models \psi_{\ell_0}(\overline{b}_{k,i_0}; \overline{b}_{i_0,j_0})$, hence $\models \varphi(\alpha_0; \overline{b}_{k,i_0})$. This is again a contradiction.

Since these are the only two cases to consider, we see that no such $\{\psi_\ell(\overline{y}; \overline{z}) : \ell < L\}$ can exist. Hence, the UDTFS rank of $\varphi$ is at least two. It is easy to check that $\varphi$ has UDTFS rank 2. $\square$

By the proof of Proposition 4.3.4, $\mathrm{ACF}_0$ and $\mathrm{ACVF}_{(0,0)}$ cannot possibly be fully VC-minimal. The property of VC-minimality has the advantage of essentially being the fusion of weakly o-minimal and strongly minimal. This is basically why $\mathrm{ACVF}_{(0,0)}$, which has both a weakly o-minimal part (the value group) and a strongly minimal part (the residue field), is VC-minimal. Example 4.3.5 is disappointing because it means that the method of Corollary 3.2.6 cannot be used to show that $\mathrm{ACVF}_{(0,0)}$ has VC-density one. In Section 4.4, we discuss another approach and show that, in particular, strongly minimal theories have VC-density one (Theorem 4.4.4).

### 4.3.2 Weakly VC-Minimal Theories

We now turn our attention to weakly VC-minimal theories.

**Definition 4.3.6.** We say that $T$ is *weakly VC-minimal* if, for all formulas $\varphi(x;\overline{y})$ and all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, there exists formulas $\psi_0(x;\overline{z}_0), ..., \psi_{n-1}(x;\overline{z}_{n-1})$ each with independence dimension $\leq 1$ and $\overline{c}_0, ..., \overline{c}_{n-1}$ from $\mathfrak{C}$ of the appropriate length such that $\varphi(x;\overline{b})$ is a boolean combination of the $\psi_i(x;\overline{c}_i)$'s.

If $T$ is VC-minimal and $\Phi$ is a VC-minimal instantiable family witnessing this, then, in particular, each formula in $\Phi$ has independence dimension $\leq 1$ (by itself). Hence, $T$ is also weakly VC-minimal. To see that this is actually a strictly weaker notion, consider Example 4.2.7 above. This example is not VC-minimal, but it is weakly VC-minimal, since each $\varphi(x;y) = P_i(x)$ has independence dimension $\leq 1$. In fact, weakly VC-minimal theories are not necessarily even dp-minimal.

**Example 4.3.7.** Fix $n > 1$ and let $M = \mathbb{Q}^n$. For each $i < n$, let $<_i$ be a binary relation on $M$ defined by $(x_0, ..., x_{n-1}) <_i (y_0, ..., y_{n-1})$ if and only if $x_i < y_i$. Finally, let $T = \mathrm{Th}(M; <_i)_{i<n}$. In much the same way as one shows it for $\mathrm{Th}(\mathbb{Q}; <)$, we can see that $T$ has quantifier elimination. Therefore, since $x <_i y$ has independence dimension one for all $i < n$, we see that $T$ is weakly VC-minimal. However, $T$ is not dp-minimal, witnessed by the formulas $y <_0 x <_0 z$ and $y <_1 x <_1 z$.

In fact, the dp-rank of $T$ (as defined in [16]) is at least $n$, showing that there are weakly VC-minimal theories with arbitrarily large dp-rank. Consequently, there are weakly VC-minimal theories with arbitrarily large VC-density. One could also use Example 1.3 in [16], as the theory presented there is weakly VC-minimal and not dp-minimal. Clearly weakly VC-minimal theories are dependent. Consider the following modification to Example 4.3.7 showing that dependence does not imply weak VC-minimality:

**Example 4.3.8.** Let $M = \mathbb{Q}^2$ and let $<$ be the binary relation $(x_0, x_1) < (y_0, y_1)$ if and only if $x_0 < y_0$ and $x_1 < y_1$. Let $T^* = \mathrm{Th}(M; <)$ and notice that this is a reduct of the theory from Example 4.3.7 for $n = 2$, as $x < y$ if and only if $x <_0 y \wedge x <_1 y$. Therefore, $T$ is dependent. However, reflection along a line ($\tau_a : M \to M$ via $\tau_a(x)$ is the reflection of $x$ along the line of slope one through $a \in M$) is an automorphism of $(M; <)$. This, coupled with quantifier elimination of $T$ from Example 4.3.7, shows that the only independence dimension $\leq 1$ formula with non-parameter variable $x$ is $x = y$ (use automorphisms to alter an instance of $\varphi$ to overlap independently with the original).

One can also see that $T^*$ is not dp-minimal. Consider $b_i = (i, -i)$ for all $i \in \mathbb{Z}$. Then, for any integers $k < \ell$, the $<$-type

$$p_{k,\ell}(x) = \{x \not< b_i : i < k\} \cup \{x < b_i : k \leq i < \ell\} \cup \{x \not< b_i : \ell \leq i\}$$

is consistent, witnessed by $a = (k - 1/2, -\ell + 1/2) \in \mathbb{Q}^2$. Therefore, if we let $\psi(x; y_0, y_1) = x < y_0 \not\leftrightarrow x < y_1$, then $\psi$ together with $\langle (b_{2i}, b_{2i+1}) : i \in \omega \rangle$ is a TP-pattern as defined in Definition 2.4.4. Therefore, by Proposition 2.4.5, $T^*$ is not dp-minimal. This leads to an interesting open question: Does dp-minimality imply weak VC-minimality?

One should also notice that $<$ has UDTFS in $T^*$. In fact, one can easily see that $T$ from Example 4.3.7 has UDTFS (since UDTFS is closed under boolean combinations). Therefore, since $T$ is not dp-minimal and unstable, this shows that Theorem 2.4.1 is not tight (i.e., there are unstable UDTFS theories that are not dp-minimal).

## 4.4 VC-Density One

The main objective of this section is to develop tools to address the following conjecture, posed by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko in [3]:

**Conjecture 4.4.1** ([3]). *If $T$ is VC-minimal, then $T$ has VC-density one.*

We cannot answer this question yet, but we have several methods to attack this problem. First of all, there is the idea of UDTFS rank and the method of Corollary 3.2.6. In this section, we develop another method that, for example, has the following corollary:

**Theorem 4.4.2.** *If $T$ is strongly minimal, then $T$ has VC-density one.*

This was independently shown by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko in [3]. We begin by discussing a notion of a rank, which is exactly the Morley rank and Morley degree when $T$ is strongly minimal.

**Definition 4.4.3.** Fix $d < \omega$ and let $R$ and $M$ be two functions from parameter-definable formulas to $\mathbb{Z}$. The pair $(R, M)$ is called a *density $d$ rank* on $T$ if the following conditions hold for all parameter-definable formulas, $\varphi(\overline{x})$ and $\psi(\overline{x})$, where $n = \lg(\overline{x})$.

(i) $-1 \leq R(\varphi) \leq n \cdot d$, $1 \leq M(\varphi)$, and $R(\varphi) = -1$ if and only if $\varphi$ is inconsistent.

(ii) If $\psi \vdash \varphi$, then $R(\psi) \leq R(\varphi)$ and, if $R(\psi) = R(\varphi)$, then $M(\psi) \leq M(\varphi)$.

(iii) $R(\varphi \wedge \psi) = R(\varphi)$ or $R(\varphi \wedge \neg\psi) = R(\varphi)$.

(iv) If $R(\varphi) = R(\varphi \wedge \psi)$ and $M(\varphi \wedge \psi) < M(\varphi)$, then $R(\varphi) = R(\varphi \wedge \neg\psi)$.

(v) If $\psi'(\overline{x})$ is a parameter-definable formula, $\psi \vdash \varphi$, $\psi' \vdash \varphi$, and $R(\varphi) = R(\psi) = R(\psi') > R(\psi \wedge \psi')$, then $M(\varphi) \geq M(\psi) + M(\psi')$.

(vi) For all $\emptyset$-definable $\theta(\overline{x};\overline{y})$, there exists $M^* < \omega$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$

$$M(\theta(\overline{x};\overline{b})) \leq M^*.$$

If we let $\varphi(\overline{x};\overline{y})$ be an $\emptyset$-definable formula, we can extend $(R, M)$ a density $d$ rank to finite $\varphi$-types in the following manner: Say that $R(p) = R(\bigwedge p)$ and $M(p) = M(\bigwedge p)$ for any finite $\varphi$-type $p$. Using this, we can count types and get the following result:

**Theorem 4.4.4.** *For any $d < \omega$, if there exists $(R, M)$ a density $d$ rank on $T$, then $T$ has VC-density $d$.*

First, we begin with a lemma on density $d$ ranks for some fixed $d < \omega$.

**Lemma 4.4.5.** *If $(R, M)$ is a density $d$ rank on $T$ and $\varphi(\overline{x};\overline{y})$ is $\emptyset$-definable formula, then the following hold:*

*(i) If $p$ is a $\varphi$-type over a finite set $B$, $B_0 \subseteq B$, and, for all $B_1$ with $B_0 \subset B_1 \subseteq B$ and $|B_1 - B_0| = 1$, we have that $M(p|_{B_1}) = M(p|_{B_0})$ and $R(p|_{B_1}) = R(p|_{B_0})$, then $R(p) = R(p|_{B_0})$ and $M(p) = M(p|_{B_0})$.*

*(ii) There exists $g : (nd+1) \to \omega$ such that, for all finite $\varphi$-types $p$, there exists $p_0 \subseteq p$ with $|p_0| \leq g(R(p))$ such that $R(p_0) = R(p)$.*

*(iii) There exists $M^{**} < \omega$ such that, for all finite $\varphi$-types $p$, $M(p) \leq M^{**}$.*

*Proof.* (i): By induction on $m = |B - B_0|$. If $m = 1$, this follows immediately, so suppose $m > 1$. By induction, we may assume that $M(p|_{B_1}) = M(p|_{B_0})$ and $R(p|_{B_1}) = R(p|_{B_0})$ for all $B_1$ with $B_0 \subset B_1 \subseteq B$ and $|B_1 - B_0| < m$. Fix any two such $B_1, B_2$ so that $B = B_1 \cup B_2$. Let $\theta_0(\overline{x}) = \bigwedge p|_{B_0}$, let $\theta_1(\overline{x}) = \bigwedge p|_{B_1}$, and let $\theta_2(\overline{x}) = \bigwedge p|_{B_2}$. We have that $R(\theta_0) = R(\theta_1) = R(\theta_2)$, $\theta_1 \vdash \theta_0$, and $\theta_2 \vdash \theta_0$. By Definition 4.4.3 (v), if $R(\theta_1 \wedge \theta_2) < R(\theta_0)$, then $M(\theta_0) \geq M(\theta_1) + M(\theta_2)$. This contradicts our assumption that $M(p|_{B_1}) = M(p|_{B_0})$ and $M(p|_{B_2}) = M(p|_{B_0})$. Thus, $R(p) = R(p|_{B_0})$. Furthermore, if $M(p) < M(p|_{B_0})$, then $M(\theta_1 \wedge \theta_2) < M(\theta_1)$. Thus, $R(\theta_1 \wedge \neg\theta_2) = R(\theta_1) = R(\theta_0)$ by Definition 4.4.3 (iv). Thus, $M(\theta_0) \geq M(\theta_2) + M(\theta_1 \wedge \neg\theta_2) \geq M(\theta_2) + 1 = M(\theta_0) + 1$, a contradiction. Therefore, $M(p) = M(p|_{B_0})$.

(ii): By Definition 4.4.3 (vi), for each $\ell < \omega$, there exists $M^*_\ell$ so that, for all $\varphi$-types $p$ with $|p| = \ell$, $M(p) \leq M^*_\ell$ (a $\varphi$-type of size $\ell$ is merely an instance of $\bigwedge_{i<\ell} \varphi(\overline{x}; \overline{y}_i)^{t(i)}$ for some $t \in {}^\ell 2$, so take the maximal corresponding $M^*$ over each $t \in {}^\ell 2$). Define $g(nd) = 0$ and recursively define $g(k) = g(k+1) + M^*_{g(k+1)} + 1$. We claim this $g$ works.

By (reverse) induction on $k = R(p)$. This is trivial if $k = R(\overline{x} = \overline{x}) \leq nd$, as $R(\emptyset) = R(\overline{x} = \overline{x})$, where $\emptyset$ denotes the empty $\varphi$-type. Now assume $k < R(\overline{x} = \overline{x})$. Choose $p' \subseteq p$ with $R(p') > k$ and $R(p')$ is minimal such. By induction, there exists $p_0 \subseteq p'$ with $|p_0| \leq g(k+1)$ and $R(p_0) = R(p')$.

**Case 1.** There exists $\overline{b} \in (\mathrm{dom}(p) - \mathrm{dom}(p_0))$, $R(p_0 \cup p|_{\{\overline{b}\}}) < R(p_0)$.

Then $R(p) = R(p_0 \cup p|_{\{\overline{b}\}})$ by minimal choice of $p'$.

$$\left|p_0 \cup p|_{\{\overline{b}\}}\right| = |p_0| + 1 \leq g(k+1) + 1 \leq g(k).$$

**Case 2.** For all $\overline{b} \in (\mathrm{dom}(p) - \mathrm{dom}(p_0))$, $R(p_0 \cup p|_{\{\overline{b}\}}) = R(p_0)$.

Suppose there exists $\overline{b} \in (\mathrm{dom}(p) - \mathrm{dom}(p_0))$, $R(p_0 \cup p|_{\{\overline{b}\}}) = R(p_0)$ and $M(p_0 \cup p|_{\{\overline{b}\}}) < M(p_0)$. In this case, add $p|_{\{\overline{b}\}}$ to $p_0$, and repeat this at most $M(p_0) \leq M^*_{g(k+1)}$ steps to produce $p_1 \subseteq p$ with $|p_1| \leq |p_0| + M^*_{g(k+1)}$, $R(p_1) = R(p_0)$, and $M(p_1)$ is minimal. If this does not hold, immediately set $p_0 = p_1$. Now suppose that, for all $\overline{b} \in (\mathrm{dom}(p) - \mathrm{dom}(p_1))$, $R(p_1 \cup p|_{\{\overline{b}\}}) = R(p_1)$. By minimality of $M(p_1)$, $M(p_1 \cup p|_{\{\overline{b}\}}) = M(p_1)$. By (i) of this lemma, $R(p) = R(p_1) = R(p_0) = R(p') > k = R(p)$, a contradiction. Thus, there exists some $\overline{b} \in (\mathrm{dom}(p) - \mathrm{dom}(p_1))$ such that $R(p_1 \cup p|_{\{\overline{b}\}}) < R(p_1)$. Hence, $R(p_1 \cup p|_{\{\overline{b}\}}) = R(p)$ by minimality. As in Case 1, we get that

$$\left|p_1 \cup p|_{\{\overline{b}\}}\right| = |p_1| + 1 \leq |p_0| + M^*_{g(k+1)} + 1 \leq g(k+1) + M^*_{g(k+1)} + 1 \leq g(k)$$

as desired.

(iii): Let $M^{**} = \max\{M^*_\ell : \ell \leq g(0)\}$ for $g$ and $M^*_\ell$ from (ii). Fix $p$ any finite $\varphi$-type. By (ii), there exists $p_0 \subseteq p$ with $|p_0| \leq g(R(p)) \leq g(0)$ such that $R(p_0) = R(p)$. Then, $M(p_0) \leq M^*_{|p_0|} \leq M^{**}$ and $M(p) \leq M(p_0)$, so $M(p) \leq M^{**}$. $\square$

We are now ready to prove the main theorem.

*Proof of Theorem 4.4.4.* Fix a formula $\varphi(\overline{x};\overline{y})$, let $n = \lg(\overline{x})$, and let $(R, M)$ be a density $d$ rank. For any finite $\varphi$-type $p$ and any finite $B \supseteq \mathrm{dom}(p)$, define

$$S^p_\varphi(B) = \{q \text{ a } \varphi\text{-type over } B : q \supseteq p\}.$$

So if we let $\emptyset$ denote the empty $\varphi$-type, then $S_\varphi(B) = S^\emptyset_\varphi(B)$. Let $M^{**}$ be as in Lemma 4.4.5 (iii) above, so $M(p) \leq M^{**}$ for all finite $\varphi$-types $p$. We now claim that the following holds for all finite $\varphi$-types $p$ and all $B \supseteq \mathrm{dom}(p)$:

$$|S^p_\varphi(B)| \leq (M^{**})^{R(p)} \cdot M(p) \cdot |B - \mathrm{dom}(p)|^{R(p)}.$$

By applying this to $\emptyset$, we get the desired conclusion (as $R(\emptyset) = R(\overline{x} = \overline{x}) \leq nd$ by Definition 4.4.3 (i)).

The claim follows by induction on $R(p)$ and $M(p)$ and $|B - \mathrm{dom}(p)|$. If $R(p) = 0$, then Definition 4.4.3 (v) yields that there are, at most, $M(p)$ extensions of $p$ to finite $\varphi$-types. Hence, $|S^p_\varphi(B)| \leq M(p)$, as desired.

Let $k = R(p)$, $\ell = M(p)$, and $m = |B - \mathrm{dom}(p)|$. For simplicity of notation, let $L_i = (M^{**})^i$. Suppose $k > 0$. We aim to show that

$$|S^p_\varphi(B)| \leq L_k \cdot \ell \cdot m^k.$$

**Case 1.** There exists $\overline{b} \in (B - \mathrm{dom}(p))$ such that $R(p \cup \{\varphi(\overline{x};\overline{b})^t\}) = k$ for both $t < 2$.

In this case, let $p_0 = p \cup \{\neg\varphi(\overline{x};\overline{b})\}$ and let $p_1 = p \cup \{\varphi(\overline{x};\overline{b})\}$. By Definition 4.4.3 (v), $\ell \geq M(p_0) + M(p_1)$. By induction hypothesis,

$$|S^{p_0}_\varphi(B)| = L_k \cdot M(p_0) \cdot (m-1)^k \text{ and } |S^{p_1}_\varphi(B)| = L_k \cdot M(p_1) \cdot (m-1)^k.$$

Therefore, since $S^p_\varphi(B) = S^{p_0}_\varphi(B) \cup S^{p_1}_\varphi(B)$, we get that

$$|S^p_\varphi(B)| \leq L_k \cdot M(p_0) \cdot (m-1)^k + L_k \cdot M(p_1) \cdot (m-1)^k \leq L_k \cdot \ell \cdot m^k$$

as desired.

**Case 2.** For all $\bar{b} \in (B - \mathrm{dom}(p))$, $R(p \cup \{\varphi(\bar{x};\bar{b})^t\}) < k$ for some $t < 2$.

In this case, let $p_0 = p \cup \{\varphi(\bar{x};\bar{b})^t\}$ and let $p_1 = p \cup \{\varphi(\bar{x};\bar{b})^{1-t}\}$ for the given $t < 2$. By Definition 4.4.3 (iii), $R(p_1) = k$, so $M(p_1) \leq \ell$ by (ii). By the induction hypothesis on $m$,

$$|S^{p_1}_\varphi(B)| \leq L_k \cdot M(p_1) \cdot (m-1)^k \leq L_k \cdot \ell \cdot (m-1)^k.$$

On the other hand, since $R(p_0) < k$, we have that

$$|S^{p_0}_\varphi(B)| \leq L_{R(p_0)} \cdot M(p_0) \cdot (m-1)^{R(p_0)} \leq L_{k-1} \cdot M(p_0) \cdot (m-1)^{k-1}.$$

By Lemma 4.4.5 (iii), $M(p_0) \leq M^{**}$, so we have that

$$|S^{p_0}_\varphi(B)| \leq L_{k-1} \cdot M^{**} \cdot (m-1)^{k-1} = L_k \cdot (m-1)^{k-1}.$$

(as $L_{k-1} = (M^{**})^{k-1}$). Thus,

$$|S^p_\varphi(B)| \leq L_k \cdot \ell \cdot (m-1)^k + L_k \cdot (m-1)^{k-1} \leq L_k \cdot \ell \cdot m^k$$

as desired.

Now, applying this claim to $p = \emptyset$, we get

$$|S_\varphi(B)| \leq (M^{**})^{n+1} \cdot |B|^{nd}$$

for all finite $B$. So $\varphi(\bar{x};\bar{y})$ has VC-density $d \cdot \lg(\bar{x})$. Since $\varphi$ was arbitrary, $T$ has VC-density $d$. $\square$

Notice that we only use the density rank $(R, M)$ relative to $\varphi$ to show that $\varphi$ has VC-density $\leq n$, so if we want to show that $\varphi$ has VC-density $\leq k$ for any $k$, we need only have a density rank relativized to $\varphi$ with maximal rank $k$. For the case when $d = 1$, we get that $T$ having a density one rank implies that $T$ has VC-density one.

One should note that Morley rank and Morley degree satisfy all the axioms of Definition 4.4.3 except possibly (i) (i.e., that $R(\varphi) \leq nd$ for all $\varphi(\overline{x})$ with $n = \lg(\overline{x})$) and (vi) (i.e., a bound on the Morley degree of formulas). Therefore, if we can show that (i) and (vi) hold for $(\mathrm{MR}, \mathrm{Md})$ for some $d < \omega$, then this would imply that $(\mathrm{MR}, \mathrm{Md})$ is a density $d$ rank and $T$ has VC-density $d$. We show this in the following corollary, independently due to Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko in [3]:

**Corollary 4.4.6.** *If $T$ is $\aleph_1$-categorical, then $T$ has VC-density $\mathrm{MR}(x = x)$ (in particular, since such theories are totally transcendental, they have finite VC-density).*

*Proof.* Let $d = \mathrm{MR}(x = x)$ and we aim to show that $(\mathrm{MR}, \mathrm{Md})$ is a density $d$ rank. By Theorem 4.4.4, this suffices. First, all such theories $T$ are superstable, so the Lascar equality holds:

$$U(\overline{a}/B \cup \{\overline{c}\}) + U(\overline{c}/B) = U(\overline{a}, \overline{c}/B).$$

In $\aleph_1$-categorical theories, $U = \mathrm{MR}$, so this holds for Morley rank (see, for example, Corollary 2 of [20]). In particular, we see that MR is additive, so $\mathrm{MR}(\overline{x} = \overline{x}) \leq \lg(\overline{x}) \cdot \mathrm{MR}(x = x) = nd$ for $n = \lg(\overline{x})$. Therefore, condition (i) of Definition 4.4.3 holds for MR.

It is not hard to see that $T$ does not have the finite cover property (see Definition II.4.1 of [22]). Therefore, by Theorem II.4.4 of [22], for any $\theta(\overline{x};\overline{y})$, there exists $M^* < \omega$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, the Morley degree of $\theta(\overline{x};\overline{b})$ is, at most, $M^*$. Therefore, condition (vi) of Definition 4.4.3 holds for Md. Hence, $(\mathrm{MR}, \mathrm{Md})$ is a density $d$ rank, as we aimed to show. $\square$

If $T$ is strongly minimal, then $T$ is $\aleph_1$-categorical and $\mathrm{MR}(x = x) = 1$. Therefore, Theorem 4.4.2 follows as a corollary of Corollary 4.4.6. Notice that all we are using about $\aleph_1$-categorical theories is that the Morley Rank is additive and the Morley degree is bounded. So we can replace the hypotheses with this instead.

## 4.5 Kueker's Conjecture

First we consider a lemma to simplify cases where we have instances of formulas equivalent to instances of formulas from a fixed set.

**Lemma 4.5.1.** *Let $\Psi = \{\psi_i(x;\overline{z}_i) : i \in I\}$ be any set of formulas and let $\varphi(x;\overline{y})$ be a formula. Suppose that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, there exists an $i \in I$ and a $\overline{c} \in \mathfrak{C}^{\lg(\overline{z}_i)}$ so that $\varphi(\mathfrak{C};\overline{b}) = \psi_i(\mathfrak{C};\overline{c})$. Then, there exists a finite $I_0 \subseteq I$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, there exists an $i \in I_0$ and a $\overline{c} \in \mathfrak{C}^{\lg(\overline{z}_i)}$ so that $\varphi(x;\overline{b})$ is equivalent to $\psi_i(x;\overline{c})$.*

*Proof.* Consider the following partial type in $\overline{y}$ over $\emptyset$:

$$\Sigma(\overline{y}) = \{\neg\exists\overline{z}_i\forall x(\varphi(x;\overline{y}) \leftrightarrow \psi_i(x;\overline{z}_i)) : i \in I\}$$

By assumption, $\Sigma$ is inconsistent. Therefore, by compactness, there exists a finite

$I_0 \subseteq I$ so that

$$\Sigma_0(\overline{y}) = \{\neg\exists\overline{z}_i\forall x(\varphi(x;\overline{y}) \leftrightarrow \psi_i(x;\overline{z}_i)) : i \in I_0\}$$

is inconsistent. This yields the desired conclusion. $\square$

This has the following corollary on weak VC-minimality.

**Corollary 4.5.2.** *If $T$ is weakly VC-minimal and $\varphi(x;\overline{y})$ is a formula, then there exists finitely many formulas $\Psi_0 = \{\psi_i(x;\overline{z}_i) : i \in I_0\}$, each a boolean combination of formulas with independence dimension $\leq 1$ (in the variable $x$), such that, for any $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, there exists $i \in I_0$ and $\overline{c} \in \mathfrak{C}^{\lg(\overline{z}_i)}$ such that $\varphi(x;\overline{b})$ is equivalent to $\psi_i(x;\overline{c})$.*

*Proof.* Let $\Psi$ be all boolean combinations of formulas with independence dimension $\leq 1$ in the variable $x$, then use Lemma 4.5.1 to conclude. $\square$

**Lemma 4.5.3.** *If $T$ is weakly VC-minimal and all independence dimension $\leq 1$ formulas of the form $\varphi(x;\overline{y})$ are stable, then $T$ is stable.*

*Proof.* Suppose that each independence dimension $\leq 1$ formula of the form $\psi(x;\overline{y})$ is stable but, by means of contradiction, $T$ is not stable. Then, by sufficiency of a single variable for stability (Theorem 1.2.4 (i)), there exists a formula $\varphi(x;\overline{y})$ that is unstable. By Corollary 4.5.2, if we let $\Theta$ be all boolean combinations of independence dimension $\leq 1$ formulas in the variable $x$, then there exists a finite $\Theta_0 \subseteq \Theta$ such that, for all $\overline{b} \in \mathfrak{C}^{\lg(\overline{y})}$, there exists $\theta(x;\overline{z}) \in \Theta_0$ and $\overline{c} \in \mathfrak{C}^{\lg(\overline{z})}$ such that $\varphi(x;\overline{b})$ is equivalent to $\theta(x;\overline{c})$. Now let $\langle a_i : i < \omega\rangle$ and $\langle \overline{b}_j : j < \omega\rangle$ witness the fact that $\varphi(x;\overline{y})$ has the order property (i.e., $\models \varphi(a_i;\overline{b}_j)$ if and only if $i < j$). By pigeon-hole principal and reindexing, we may assume that there is a single $\theta(x;\overline{z}) \in \Theta_0$

such that, for all $j < \omega$, there exists $\overline{c}_j \in \mathfrak{C}^{\lg(\overline{z})}$ such that $\varphi(x; \overline{b}_j)$ is equivalent to $\theta(x; \overline{c}_j)$. Therefore, $\langle a_i : i < \omega \rangle$ and $\langle \overline{c}_j : j < \omega \rangle$ are a witness to the fact that $\theta$ has the order property. However, $\theta$ is a boolean combination of independence dimension $\leq 1$ formulas in the variable $x$, each of which is stable by hypothesis. Since stability is closed under boolean combinations, this is a contradiction. $\square$

**Theorem 4.5.4.** *If $T$ is weakly VC-minimal, then either $T$ is stable or $T^{\mathrm{eq}}$ defines an infinite linear order.*

*Proof.* Assume $T$ is weakly VC-minimal and unstable. Then, by Lemma 4.5.3, there exists a formula $\varphi(x; \overline{y})$ that has independence dimension $\leq 1$ and is unstable. Since $\varphi$ is unstable, there exists $\langle a_i : i \in \mathbb{Q} \rangle$ and $\langle \overline{b}_j : j \in \mathbb{Q} \rangle$ such that, for all $i, j \in \mathbb{Q}$, $\models \varphi(a_i; \overline{b}_j)$ if and only if $i < j$ (by compactness). We claim that the following is a definable quasi-linear-order in $T$: Let

$$\overline{y}_0 \leq \overline{y}_1 = \big(\forall x(\varphi(x; \overline{y}_0) \to \varphi(x; \overline{y}_1))\big)$$

and let

$$D = \{\overline{b} \in \mathfrak{C}^{\lg(\overline{y})} : \overline{b}_0 \leq \overline{b} \leq \overline{b}_1\}.$$

First, it is clear that $D$ and $\leq$ are definable. Moreover, it is clear that $\leq$ is a quasi-ordering on $D$. To see that $D$ is non-empty, take any $i, j \in \mathbb{Q}$ with $i < j$. Then we get that

(i) $\models \varphi(a_k; \overline{b}_i) \wedge \varphi(a_k; \overline{b}_j)$ for any $k < i$,

(ii) $\models \neg\varphi(a_i; \overline{b}_i) \wedge \varphi(a_i; \overline{b}_j)$, and

(iii) $\models \neg\varphi(a_j; \overline{b}_i) \wedge \neg\varphi(a_j; \overline{b}_j)$.

By independence dimension $\leq 1$, $\models \forall x(\varphi(x;\bar{b}_i) \to \varphi(x;\bar{b}_j))$. Thus, $\bar{b}_i \leq \bar{b}_j$ if and only if $i \leq j$. In particular, this shows that $\bar{b}_i \in D$ for all $i \in \mathbb{Q}$ with $0 \leq i \leq 1$. It also shows that all such $\bar{b}_i$ are non-$\leq$-equivalent. It remains to show that $\leq$ is a linear quasi-order (i.e., all two elements of $D$ are comparable).

Suppose that $\bar{b}, \bar{b}' \in D$ yet $\bar{b} \not\leq \bar{b}'$ and $\bar{b}' \not\leq \bar{b}$. Then, by definition, $\exists x(\varphi(x;\bar{b}) \wedge \neg\varphi(x;\bar{b}'))$ and $\exists x(\neg\varphi(x;\bar{b}) \wedge \varphi(x;\bar{b}'))$ both hold. As witnessed by $a_{-1}$, for example, $\exists x(\varphi(x;\bar{b}) \wedge \varphi(x;\bar{b}'))$ holds. As witnessed by $a_2$, for example, $\exists x(\neg\varphi(x;\bar{b}) \wedge \neg\varphi(x;\bar{b}'))$ holds. These four statements together contradict the fact that $\varphi$ has independence dimension $\leq 1$.

Therefore, $(D, \leq)$ is a definable quasi-linear-order. Hence, if we take the natural $\emptyset$-definable equivalence relation $\bar{b} E \bar{b}'$ defined by $\bar{b} \leq \bar{b}' \wedge \bar{b}' \leq \bar{b}$, we see that $(D/E, \leq)$ is an infinite $T^{\mathrm{eq}}$-definable linear order. □

We are now ready to discuss the Kueker Conjecture.

**Conjecture 4.5.5** (Kueker's Conjecture). *If $T$ is a theory in a countable language such that every uncountable model of $T$ is $\aleph_0$-saturated, then $T$ is $\aleph_0$-categorical or $\aleph_1$-categorical.*

**Corollary 4.5.6.** *If $T$ is weakly VC-minimal, then $T$ satisfies the Kueker Conjecture.*

*Proof.* By Section 3 of [13], if $T$ is stable, then the Kueker Conjecture holds for $T$. Therefore, we may assume that $T$ is unstable. By Theorem 4.5.4, we have that $T^{\mathrm{eq}}$ has a infinite definable linear order. By Proposition 4.1 of [13], this implies that the Kueker Conjecture holds for $T$. □

As a corollary, if $T$ is VC-minimal, then $T$ has the stable / infinite interpretable linear order dichotomy as in Theorem 4.5.4. Thus, $T$ satisfies the Kueker Conjecture. What other dependent theories have this stable / infinite interpretable linear order dichotomy? For any such theory, the Kueker Conjecture certainly holds by the above argument. One of this author's goals in his future research is to prove the Kueker Conjecture for UDTFS theories. This can either be accomplished by showing UDTFS theories have a stable / infinite interpretable linear order dichotomy or by showing it directly as in the stable case of [13].

# Bibliography


[1] H. Adler. Introduction to theories without the independence property. preprint, June 2008.

[2] H. Adler. Theories controlled by formulas of Vapnik-Chervonenkis codimension 1. preprint, July 2008.

[3] M. Aschenbrenner, A. Dolich, D. Haskell, H.D. MacPherson, and S. Starchenko. VC density in some dependent theories. in preparation.

[4] A. Chernikov and P. Simon. Externally definable sets and dependent pairs. preprint, July 2010.

[5] R. Cluckers, L. Lipshitz, and Z. Robinson. Analytic cell decomposition and analytic motivic integration. *Ann. Sci. Ecole Norm. Sup.* to appear.

[6] A. Dolich, J. Goodrick, and D. Lippel. dp-minimality: Basic facts and examples. preprint, July 2009.

[7] S. Floyd and S. Warmuth. Sample compression, learnability and Vapnik-Chervonekis dimension. *Machine Learning*, 21(3):269–304, 1995.

[8] J. Goodrick. A monotonicity theorem for dp-minimal densely ordered groups. *J. Symbolic Logic*, 75(1):221–238, 2010.

[9] V. Guingona. Dependence and isolated extensions. *Proc. Amer. Math. Soc.* posted on January 21, 2011, PII S 0002-9939(2011)10739-1 (to appear in print).

[10] V. Guingona. On VC-minimal theories and variants. in preparation.

[11] V. Guingona. On uniform definability of types over finite sets. preprint, May 2010.

[12] J. Holly. Canonical forms for definable subsets of algebraically closed and real closed valued fields. *J. Symbolic Logic*, 60:843–860, 1995.

[13] E. Hrushovski. Kueker's conjecture for stable theories. *J. Symbolic Logic*, 54(1):207–220, 1989.

[14] H.R. Johnson. Coherence and uniform denability of types over finite sets. preprint, September 2010.

[15] H.R. Johnson and M.C. Laskowski. Compression schemes, stable definable families, and o-minimal structures. *Discrete Comput. Geom.*, 43:914–926, 2010.

[16] I. Kaplan, A. Onshuus, and A. Usvyatsoc. Additivity of the dp-rank. preprint available on Modnet 251, May 2010.

[17] M.C. Laskowski. Vapnik-Chervonenkis classes of definable sets. *J. London Math. Soc.*, s2-45(2):377–384, 1992.

[18] A. Onshuus and A. Usvyatsov. On dp-minimality, strong dependence, and weight. preprint.

[19] A. Pillay. *Geometric Stability Theory.* Oxford Science Publications, 1996.

[20] B. Poizat. Une théorie $\omega_1$-catégorique à $\alpha_t$ fini. *Groupe d'étude de théories stables*, 8:1–3, 1978.

[21] N. Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145 – 147, 1972.

[22] S. Shelah. *Classification theory and the number of non-isomorphic models.* North-Holland Publishing Company, 1978.

[23] S. Shelah. Classification theory for elementary classes with the dependence property - a modest beginning. *Scientiae Math Japonicae*, 59(2):265–316, 2004.

[24] S. Shelah. Dependent first order theories, continued. *Israel J. Math*, 173:60, 2009.

[25] S. Shelah. Dependent theories and the generic pair conjecture. preprint, September 2009.

[26] S. Shelah. Strongly dependent theories. preprint, February 2009.

[27] P. Simon. On dp-minimal ordered structures. preprint, September 2009.

[28] P. Simon. Distal and non-distal NIP theories. preprint, March 2011.