"Ballparking" Baseball's Sweet-Spots

Analyzing High-Power Hitting Locations on Various Types of Baseball Bats

Mandeep Bedi
Michael DeWitt
Aashish Gadani

1. Introduction

For decades, baseball has been heralded as “America's favorite pastime.” Watching the two teams taking turns pushing their skills to the limit over the nine innings has entertained generation after generation. Naturally, the game's popularity leads to very stiff competition on an international level; players throughout history have tried to increase their equipment's performance in any way possible (i.e. the saliva-covered "spitball," used until 1920[1]). The balls themselves have gradually become more standardized over time - there is a single, official ball used in all MLB game. Contrast this with the rules regarding bats, which list specifications, but leave some choices to the player regarding exact bat length and width.

The bat's “sweet-spot” is the location on the bat where, when hit by the ball, the least vibrations are felt by the hitter. With less vibrations, the hitter has a significant increase in the amount of control he or she has over the ball. Bats are frequently advertised with the size of the sweet-spot, but often this is hard to verify because of its complexity to test and calculate, and also due to the gray area of what exactly the sweet spot is.

Major League Baseball, like any sports league, worries about some players getting an unfair advantage, whether it be through steroids, knowledge of the other teams strategy, or equipment. The equipment issues often start with the bat, with players trying to modify their bat to get an extra kick on the ball. By “corking,” or drilling a hole down the center of the bat and replacing the wood with cork, players hope the lighter material will allow them to swing the bat faster and send the ball much further.

Along with corking, aluminum bats have been banned by the MLB. Aluminum bats are banned because not only do they pose a danger when thrown by the batter, but they also can create whats known as the “trampoline effect.” When a ball strikes a normal wooden bat, the bat creates transverse waves and vibrates as it hits the ball. However, an aluminum bat is hollow on the inside so the ball can bend the bat in slightly (causing hoop waves), and then when the aluminum pops out, similar to a trampoline, the added force will send the ball much farther. The goals of this research project are to model where the baseball bat sweet spot is, and whether corking or aluminum create a larger sweet spot, or an unfair advantage for the hitter.

The “sweet spot,” although often defined to be the spot where the bat can impart maximum power to the ball, can also be defined as the spot that provides maximum comfort to the batter. This comfort is related to the amount of force and vibrations the batter feels when the ball-bat collision occurs. It is not necessarily true that these spots overlap, however many batters believe that the sweet spot for comfort is the point for optimal power output.

Our analysis consists of an evaluation of the spot where the max exit-velocity can be achieved, where the batter will experience max comfort, and finally whether or not these two spots coincide or remain separated in the bat. Also, the analysis consists of whether or not different types of bats, specifically corked or aluminum, have an effect on these spots.

2. Model and Results

2.1 - Analyzing a Wooden Bat

Assumptions:

• The bat acts as if it is pivoting about the batter's wrists.
• The pivot point is considered to the foot of the bat.
• Based on research we assume the Coefficient of Restitution to be .546 and the typical ball speed to be 3880 cm/s (90 mph/hr).
• The bat only revolves around the pivot axis.
• The bat contacts the ball squarely and at a single point.
• The entire collision takes place on a plane. The bat does not rotate along any other axis than that of the pivot.
• The ball-bat contact time is infinitesimally small.
• The bat can be accurately modeled using a free-free beam vibrational analysis equation.
• Bat and ball have an impact angle of 90 degrees.
• The bat being used is a white ash wood Louisville Slugger #C271.

2.10 - Finding the Impact-Point for Maximum Exit-Velocity

In order to find the spot that causes maximum ball exit-velocity, the first step is to consider a simple torque-based analysis. The motion of the bat swinging can be modeled as that of a beam twisting on a pivot. The pivot point is considered to be the knob of the bat. This is due to the fact that it is close to the point that the batter holds and pivots the bat from. According to the concept of torque, as the bat is swung it has an angular acceleration. The motion of the bat can be described by angular acceleration over time that results in a final angular velocity at the instant before impact with a baseball. This angular velocity is uniform for all points on the bat, but the corresponding linear velocities increases linearly with respect to the distance from the pivot point.

$$v_{\text{bat}} = \omega x$$
$$v_{\text{ball}-\text{corrected}} = v_{\text{ball}} + \omega x$$

ω = angular velocity
v_{ball} = velocity of ball at impact
x = distance from pivot at base of bat
$v_{\text{ball}-\text{corrected}}$ = relative incoming ball speed

At the same time that the bat is being swung, the baseball is being thrown by the pitcher. At the moment of the ball-bat collision, the ball has speed v_0. Typical speeds of v_0 are from 80-90 mph[8]. To model the combined effect of the speed of the bat and the speed of the incoming baseball at the point of collision, we consider the idea from Galilean relativity that the properties of mechanical systems remain unchanged under change of reference frames. Using this, instead of considering the baseball to be incoming at a speed v_o and the bat to be at a speed ωx, we consider the ball to be traveling at $v_0 + \omega x$ and consider the bat to be at rest, since only the relative velocity matters.

A useful concept in considering how the ball rebounds upon impact with the bat is the coefficient of restitution (*COR*). The *COR* is a unitless, empirically-derived parameter that describes the collision behavior of two specific objects. Specifically, in a collision between the two objects where one is considered to be an immovable wall and the other an object about to collide with the wall, the *COR* is the ratio of the post-collision velocity of the object to its own impact velocity:

$$COR = \frac{v_f}{v_0}$$

Given that the baseball is considered to have an initial velocity of $v_0 + \omega x$ the final exit velocity will be:

$$COR\,(v_0 + \omega x)$$

The typical value for the *COR* between an MLB baseball and a Louisville Slugger #C271 is 0.546 ± .032[2]. To maximize the velocity given by this simple torque-based model we consider that the only variable is x, the distance along the bat, and when x is maximized the quantity $COR\left(v_0 + \omega x\right)$ is also maximized. Thus, from the simple model, we see that the best place to hit the ball to maximize its exit velocity and thus distance is with the end of the bat.

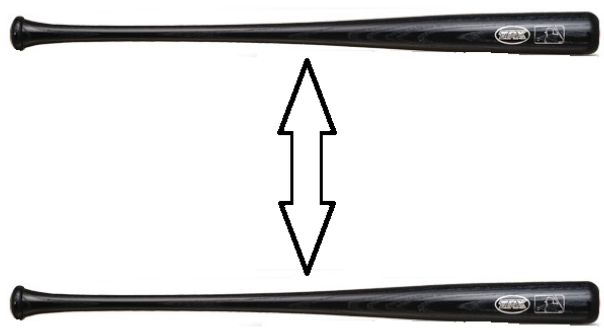
2.11 - Modeling as a Free-Free Beam

To explain the empirical finding that the maximum exit-velocity "sweet spot" is not in fact at the end of the bat, we now consider the additional effect of bat bending mode vibrations that may explain this. Bending modes are harmonics of a certain type that baseball bats and beams in general can undergo in which the material of the bat itself bends, forming standing waves. The nodes of these harmonics are points where the bat does not bend, and antinodes are where maximum bending occurs. From our research we discovered that the ball-bat collision can be modeled by treating the bat as having the vibrational behavior of a uniform beam with ends that are not clamped, also known as a free-free beam[3].

The free-free beam is a physical idealization of a uniformly distributed solid body rod that is capable of vibrating with independent normal modes or harmonics. Treating the bat as a free free beam, the ball striking the bat could be a cause of excitation of these harmonics. The energy transferred as a result of these excitations could be different along the length of the bat. This energy transferred into vibration could detract from the energy of the ball post-collision.

Each harmonic takes the form of a standing wave composed of nodes (where no oscillation occurs), antinodes (where complete oscilation occurs), and a spectrum of oscillation in between. For our model, if the ball strikes a location on the bat where a harmonic's node is located then that harmonic is not excited and the energy is conserved in the baseball. If it strikes any other position, then the extent to which that harmonic is excited is specified by the distance from the harmonic's antinodes--the closer to an antinode the ball strikes, the more energy is imparted to the bat and the less remains with the ball. In our model, we quantify this energy loss from each bending mode by associating a negative contribution to the ball's exit velocity with each harmonic.

Each harmonic consists of evenly spaced, alternating nodes and antinodes where each of the ends is always an antinode. These harmonics are vibrations in the wood of the bat itself, representing bending and deformation in response to the baseball. However the first two harmonics are not included due to the fact that they in fact are simple translational and rotational displacement (See Images 1 and 2), not vibration, thus we did not include them in our analysis[4].



Fundamental harmonic of bending modes (mode 1) in which bat simply experiences translational movement.

Images 1 and 2: These describe the lack of fundamental frequencies 1 and 2.

For all higher harmonics we can model a coefficient of vibrational stimulation (*CVS*) to which the energy imparted into the harmonic is proportional to the number of the harmonic, *n*:

$$CVS = \left| \cos\left(\frac{\pi x(n-1)}{L}\right) \right|$$

This coefficient ranges from [0,1] and achieves a minimum where vibrational stimulation is minimized (nodes), and is maximized where vibrational stimulation is maximized (antinodes). Figure 1 shows a graphical visualization of the pattern of stimulation of the different harmonics given that the ball strikes at different locations on the bat.

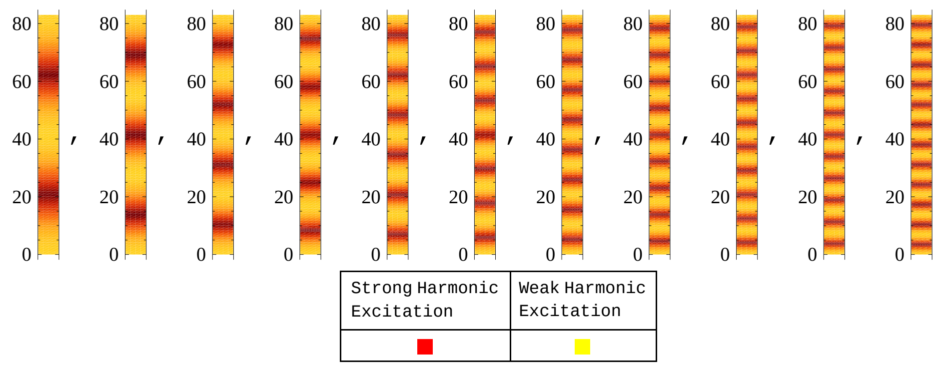


Fig. 1: A series of plots representing the relative excitation of the n^{th} harmonic of the bat's bending mode. The first plot represents the third harmonic (n = 3), and each successive plot is the subsequent harmonic.

In addition, the amount of energy transferred into the harmonics from the impact should depend on the elasticity of the collision. For example, a completely elastic collision would not lose any energy to vibrations. Because the elasticity of the collision is captured by the *COR*, the factor (1 - *COR*) is introduced. The equation also takes into account the incoming ball's energy in order to account for the energy lost in the form of the combined collision velocity given by expression $v_0 + \omega x$. This is because it is reasonable to suspect that the amount of energy lost should be proportional to the amount of energy put into the system.

Because there are an infinite number of harmonics it is reasonable to suspect that the contribution of each subsequent harmonic would be less than the previous one's or else an infinite energy contribution would occur which is clearly unrealistic. To create a dropoff in the energy lost due to each subsequent harmonic we introduce a factor of e^{-n}. This approximately ensures that the energy contributions from the harmonics do not exceed the input energy of the collision. Finally, a constant k is introduced to scale the harmonics, as we expect that the actual value for the amount of energy lost due to the bending modes is proportional to the input factors we have introduced, but may or may not be equal to them. The energy loss due to the bending modes is constant, so the value of k is set to 3 in order to provide a concrete illustration of the effect of vibrational energy loss. In practice, this constant's value would be empirically determined:

$$v_{\text{lost}} = k * (v_0 + \omega x) * \sum_{n=3}^{\infty} \left(e^{-n} (1 - \text{COR}_{\text{wood}}) \left(\left| \cos\left(\frac{x \pi (n-1)}{L} \right) \right| \right) \right)$$

L = length of the bat
n = number of harmonic
k = scaling constant

The loss of energy of each harmonic given in this form manifests itself as a decrease in velocity of the ball post collision. Using this velocity loss we find the sum of the first 48 significant harmonics' contributions to approximate those of the infinite set of possible harmonics, and subtract this total loss from the expected ball return velocity given by the simple torque-based analysis above. Results of this model are shown in Figure 3.

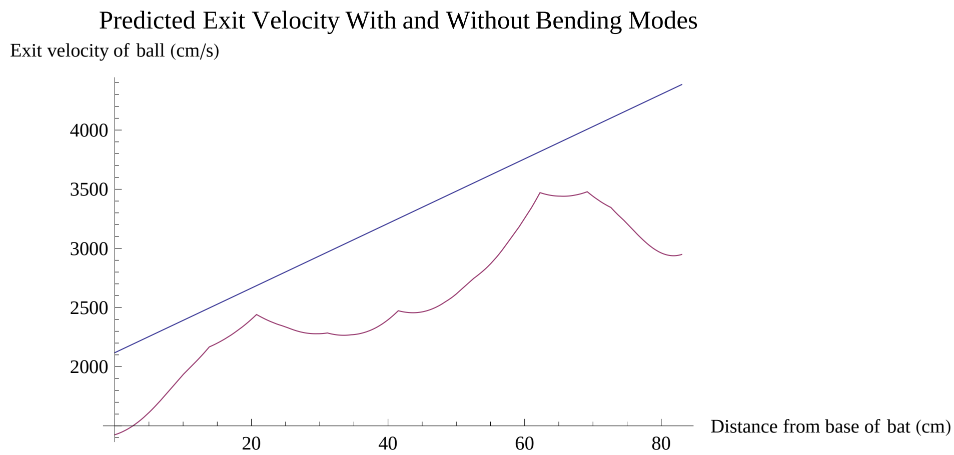


Fig. 2: This is a graph of the ball's return velocity both with and without bending modes. The blue, linear function is the torque-based model, while the jagged, purple function takes into account energy lost through vibration modes.

Looking at Fig. 2, it is clear that the point of maximum velocity of the ball taking into account vibrations is not at the end of the bat as expected but actually occurs towards the middle of the barrel, at approximately 62.25 cm from the knob of the bat, whereas the total length of the bat is 83 cm. This seems to shows that the sweet spot where the batter can acheive maximum exit-velocity is not at the end of the bat. The best area on the bat with which to hit the ball lies within the range

2.12 - Maximizing Comfort Levels

Now that we have found the sweet-spot with regard to exit-velocity, we will consider the spot at which impact is least jarring on the batter's hands. The batter experiences the sensation of the sweet spot when he feels virtually no sting transferred to his wrists/arms from the ball-bat collision. This lack of vibrational feedback can be attributed to the baseball striking the bat in a location where excitation of vibrational harmonics is minimized. Recent studies from MIT have indicated that human hands respond primarily to vibrations to frequencies below 500 Hz[5], indicating that the primary harmonics of interest are those whose frequencies fall below this limit. The canonical expression for the frequency of a free-free beam vibration is given by the following equation:

$$\left(\frac{(2n+1)\pi}{2L}\right)^2 \frac{\sqrt{\frac{(1.22*10^8)\left(\frac{1}{12}\left(.75\pi\int_0^L R^2(x)\,dx\right)^4\right)}{\rho_{wood}*\pi\int_0^L R^2(x)\,dx}}}{2\pi}$$

3	4	5	6	7	8	9	10
229.279	379.013	566.18	790.78	1052.81	1352.28	1689.18	2063.52

Table 1: Frequenceies, in Hz, of the vibrational bending modes of a free-free beam.

Table 1 is a listing of all harmonics between 3 and 10.The third and fourth harmonics are the only ones below the threshold of 500 Hz as given by the MIT research group, thus they are the ones of the most significance--higher harmonics can be ignored. Because the third and fourth harmonics are the most significant bending modes one can expect a sweet spot range to be between the nodes of these two harmonics, around which vibrations are minimized. To find the center of this range, we minimize the total negative velocity contributions from the excitation of the third and fourth harmonics. In Figure 3 we show the sum of these two contributions plotted as a function of x, the distance along the length of the bat. The values of x of interest are on the right side of the graph in Figure 3, since the ball is hit only by the barrel of the bat, and not close to where the hands are placed. Thus the sweet spot in terms of comfort lies between 58cm and 70cm, where the vibrations are minimized. By consider the local minimum in the region of interest (the barrel of the bat), an approximate range of the sweet spot can be identified by marking the boundaries where the jarring effect exceeds half of the difference between the most comfortable point and the point of greatest jarring, local to the barrel.

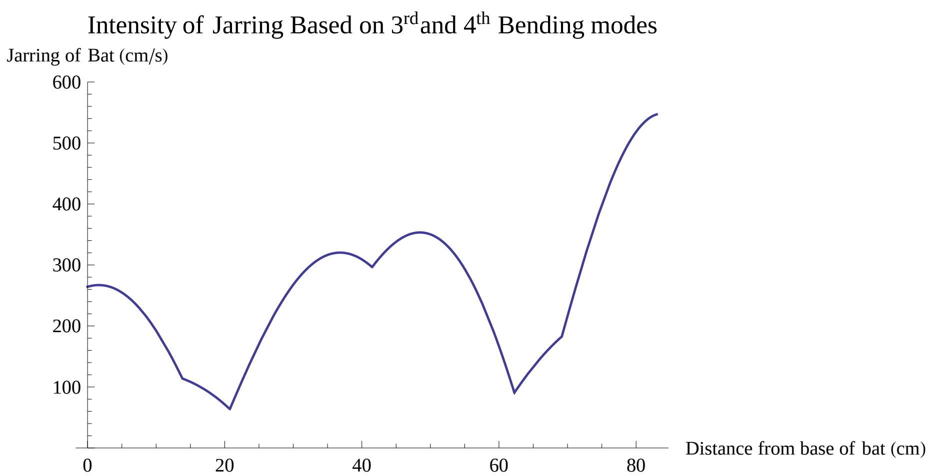


Fig. 3: The bat is jarred least when the ball hits closest to the nodes of the third and fourth beding modes. The sum of these two functions is shown in blue, as a function of distance along the bat.

An additional factor that may contribute to the minimization of jarring of the batter's hands at the time of impact is the center of percussion (*COP*). The *COP* is defined as the point of impact along the bat where rotational and translational contributions to movement of the pivot point of the bat cancel each other, resulting in no net motion of the pivot point[6]. The *COP* is given by the following equation:

$$B = \frac{I}{COM\,(M_{\text{bat}})}$$

B = distance of the *COP* to the center of mass

I = moment of inertia around the pivot

M_{bat} = mass of the bat

COM = distance from pivot to the center of mass

To find the center of percussion, we first modeled the shape of the baseball bat, using image analysis the Louisville Slugger #C271, a popular bat. We modeled the radius of the bat as a function of the distance along the length of the bat as a piecewise function. This function was generated from an image found on the website of the Louisville Slugger Co. of the #C271 bat[7]. The bat appears to have four basic sections: the knob, the handle, the barrel, and the end's cap. The bat is represented by a piecewise function to represent the raidus at a given point *x* along the bat's length. The various functions used in this piecewise model were generated by drawing function regressions through coordinates on the surface of the bat; these coordinates were determined by examining the pixel locations of an image of the model #C271 bat and converting their pixel-distances from the midline of the bat into centimeters. The following is a representation of the radius function:

$$R(d) = \begin{cases} (.2 + 4.16667\, d - 1.7299\, d^2) & 0 \le d < L_{\text{knob}} \\ \left(.2 + 0.987937 + \frac{0.812648}{-0.94927 + d}\right) & L_{\text{knob}} \le d < L_{\text{handle}} \\ \left(.2 + \frac{13{*}\text{cf} * 7{*}\text{cf} * \left(e^{(2 * \text{cf}(d - 308{*}\text{cf}))}\right)}{13{*}\text{cf} + 7{*}\text{cf} * \left(e^{(2{*}\,\text{cf}(d-308{*}\text{cf}))-1)}\right)} + 7 * \text{cf}\right) & L_{\text{handle}} \le d < L_{\text{barrel}} \\ (.2 + -0.8126479434242228\,(d - 571 * \text{cf})\text{^}2 + 20 * \text{cf}) & L_{\text{barrel}} \le d \le L_{\text{cap}} \end{cases}$$

$$L_{\text{cap}} = L_{\text{bat}} = \text{the length of the entire bat}$$

$$cf = \text{the conversation factor between pixels and cm}$$

The radius of the knob was modeled as a quadratic function; the handle was modeled as an exponentially decreasing function; the barrel was modeled as a logistic function; the cap was modeled as a quadratic function, similar to the knob. These representations are graphed below, and they are visually very similar to the image of the bat.

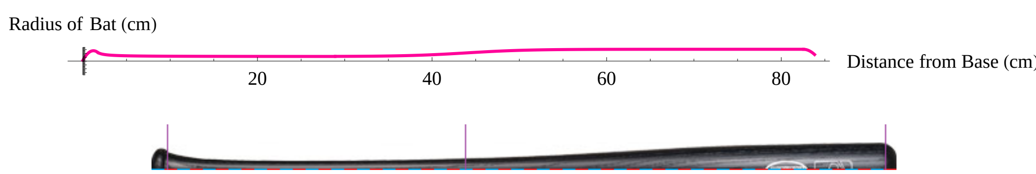


Fig. 4: A side-by-side comparison of the function of the bat's radius and an image of the bat itself (with purple lines indicating the four sections into which the bat was divided).

Given the radius profile of the bat, we can calculate both its moment of inertia and center of mass. The moment of inertia is given by the following:

$$I_{\text{WoodBat}} = \rho_{\text{wood}} \int_0^L \int_{-R(x)}^{R(x)} \int_{-\sqrt{R^2(x) - y\text{^}2}}^{\sqrt{R^2(x) - y\text{^}2}} (x^2 + y^2)\, dz\, dy\, dx$$

$$\frac{I_{\text{Cylinder}}}{\rho} = \int_0^L \int_{-R(x)}^{R(x)} \int_{-\sqrt{R^2(x) - y\text{^}2}}^{\sqrt{R^2(x) - y\text{^}2}} (x^2 + y^2)\, dz\, dy\, dx$$

The center of mass (*COM*) is given by:

$$COM = \frac{\int_0^L \rho\, x\, \pi\, R^2(x)\, dx}{\int_0^L \rho\, \pi\, R^2(x)\, dx}$$

The *COP* was found to be 63.9cm from the *COM*, roughly 36 cm past the end of the bat. Because our model predicts that the *COP* is not along the length of the bat, it likely does not contribute to the sweet spot for comfort. For this reason we suspect that the maximum comfort zone is dictated almost completely by the vibrational dynamics of the third and fourth harmonic bending modes.

2.13 - Two Sweet Spots?

Many MLB batters are convinced that the sweet spot where they feel very little stinging from the ball-bat collision is also the sweet spot from which the ball can achieve its greatest exit velocity. Our results show that the optimal spots on the bat for maximum exit-velocity and comfort are not perfectly overlapping but are very close. The ranges are, respectively, (60cm, 75cm) and (58cm, 70cm). There is an overlap of 15cm, which defines the overall sweet-spot. Together they define an optimal batting range where both comfort and transmitted power are roughly maximized.

2.2 - "Corking" the Wooden Model

Assumptions:

- The batter is always using maximal force to swing so that the torque remains unchanged as the bat mass changes.
- The time spent swinging prior to impact as well as angular acceleration throughout the swing are unchanged.

2.20 - Finding the Corked Bat's Sweet-Spot for Exit-Velocity

In order to investigate how corking affects the max exit-velocity sweet spot and the max comfort sweet spot, we consider how the process deviates from our model of a normal bat. Our first consideration for evaluating the max exit-velocity of a ball is the effect of the corking on the mass of the bat. The process of corking involves the removal of a cylinder of wood 15.4 cm long and with a diameter of 3 cm from the barrel of the bat, and then replacing it with cork[8]. Because cork is less dense than the white ash wood in the bat, the corking results in a net decrease in mass and inertia enabling faster swinging. Many batters believe that corking the bat will give them quicker reaction time, thereby enabling them to hit more precisely on the sweet-spot. However, this faster swing speed does not guarantee that more momentum will be transferred to the ball post-impact. In fact it is possible that swinging faster with a corked bat will actually decrease the momentum transfer. To model the momentum shift due to corking, we consider first the mass of the bat without corking against that with corking. The mass of the corked bat is given by:

$$M_{\text{WoodBat}} - M_{\text{WoodRemoved}} + M_{\text{CorkInserted}} = M_{\text{CorkedBat}}$$

Using the different densities of 0.67 g/cm^3 for white ash wood and 0.25 g/cm^3 for solid cork, the mass of the new bat comes out to be 0.88kg which is approximately 0.5 kg less. This is in agreement with the commonly reported decrease in bat weight due to corking of about 1.5 oz[8].

In order to model the swinging bat, we expect the batter to exert the same amount of torque, swing for the same amount of time, and apply a constant torque throughout the swing independent of bat weight. The torque τ is constant from the begining of the swing until the time of impact (t_{impact}) and the angular velocity at the moment of impact is given by:

$$I_{\text{CorkedBat}} = I_{\text{WoodBat}} - \frac{I_{\text{Cylinder}}}{\rho} (\rho_{\text{wood}} - \rho_{\text{cork}})$$

$$M_{\text{CorkedBat}} = M_{\text{WoodBat}} - (\rho_{\text{wood}} - \rho_{\text{cork}}) \left(\pi R^2(x)\, h\right)$$

Although the momentum gets imparted to the ball upon impact with the bat, the momentum of the bat at the point of impact changes directly as the distance along the bat. For this reason, we do not consider the momentum $M_{\text{bat}}\, v = M_{\text{bat}}\, \omega\, x$ directly but instead consider the quantity $M \cdot \omega$, from which all possible momenta at different points on the bat can be calculated.

In order to determine the effect of corking on the differences in this expression of interest we must determine the following relationship, derived from the expressions for omega and momentum:

$$\frac{M_N}{I_N} \stackrel{?}{=} \frac{M_C}{I_C}$$

The left hand side (*LHS*) of the relationship turns out to be equal to 2.78e-4 m^{-2} and the *RHS* is equal to 2.87e-4 m^{-2}, having a difference of 8e-6. These ratios are extremely close in value to each other, suggesting that corking does not actually make a significant change in the momentum of the bat at the moment of impact with the ball and so does not change the speed of the ball post-collision. Thus our model does not support the idea that corking enhances the sweet spot effect for maximum exit-velocity.

2.21 - Minimizing Jarring in a Corked Bat

To model the effect of corking on the sweet spot for maximum comfort, we again consider the *COP* as being a possible location where ball impact does not transmit force to the pivot point. To find the *COP*, we re-apply the *COP* equation, with different values for the moment of inertia and total mass as given above, as well as the *COM* below:

$$COM = \frac{\int_0^L \rho(x)\, x\, \pi R^2(x)\, dx}{\int_0^L \rho(x)\, \pi R^2(x)\, dx}$$

The distance between the *COM* and *COP* is given by:

$$B = \frac{I}{COM\, (M_{\text{bat}})}$$

This distance, when added to the *COM* distance, places the *COP* 34.4 centimeters beyond the end of the barrel, similar to its location on the uncorked bat. In fact, the difference between where the *COP* is located between the two kinds of bats is 1.8cm, which is an incredibly small margin, indicating that the *COP* doesn't change much at all between the two different kinds of bats. Thus, it would seem that corking does not affect the *COP* as a contributing factor to the creation of a sweet spot where the batter's comfort is maximized. Therefore, the analysis of the third and fourth harmonic bending modes done for the normal wooden bat are assumed to be the defining factor of the maximum comfort range.

2.3 - Analyzing Aluminum Bats

Assumptions:

- The batter is always using maximal force to swing so that the torque remains unchanged as the bat mass changes.
- The time spent swinging prior to impact as well as angular acceleration throughout the swing are unchanged.

2.30 - Analyzing the Aluminum Bat's Sweet-Spot for Exit-Velocity

Bats made of different materials differ in their physical properties and thus have different sweet spot behavior. Specifically, metal bats made of aluminum have very different properties from the legal wooden bats typically made of white ash used in Major League Baseball. Major properties different in the two materials affecting the exit-velocity that the ball travels include changes in the material density, coefficient of restitution, and bat thickness, the last of which is significantly different since the wooden bats are solid pieces of wood whereas metal bats are generally hollow. These features may also significantly shift the *COP*, which may be evidence for a new mechanism for creating a sweet spot maximizing batting comfort.

The sweet spot for maximum exit-velocity can be found by reapplying the techniques from the analysis of a normal wooden bat, but with the additional analysis of the so-called hoop vibrational modes. The hoop modes are a class of vibrations arising only in hollow bats, such as metal bats, in which oscillations are purely radial, with antinodes being points of extreme compression and expansion and nodes being points with no radial dilation[9]. In our model, the hoop modes are taken into account in the form of a well-known phenomenon called the "trampoline effect", which qualitatively can be understood as the redirection of energy stored in the bat from collision with the baseball back into the ball. A normal wooden bat in our model absorbs energy from the incoming ball and stores it in the form of bending mode harmonics oscillating in the bat and into the batter's arm. In the metallic bat, energy from the bending modes is modeled as being transferred into hoop modes, which then put energy back into the ball. In addition, only the fundamental hoop mode is considered, as the rest of the modes do not make any major contribution to redirection of energy.

To begin, our model first takes into account the torque of the batter's swing in exactly the same manner as done in Section 2.10, given by $v_0 - \omega x$. This part of the model describes the maximum velocity that the ball would receive at any point along the length of the bat assuming no vibrations in the bat occur. The bending and hoop modes are then accounted for by using the same analysis as in Section 2.11 for bending modes with the difference that we introduce using a sinusoidal factor that attenuates the negative vibrational contributions along the bat barrel:

$$L(x) = \begin{cases} \left|\cos\left(\frac{2\pi}{2\left(L_f - L_i\right)}\left(x - L_i\right)\right)\right| & L_i \le x \le L_f \\ 1 & x < L_c,\ L_f < x \end{cases}$$

$$v_{\text{ball}} = (v_0 + \omega x) - \left(k * (v_0 + \omega x) * \sum_{n=3}^{\infty} \left(e^{-n}(1 - \text{COR}_{\text{wood}})\left(\left|\cos\left(\frac{x\,\pi\,(n-1)}{L}\right)\right|\right)\right) * L(x)\right)$$

$$L_i = \text{beginning of the barrel}$$
$$L_f = \text{end of the barrel}$$

For all points on the bat between the knob and the beginning of the barrel, the fundamental hoop mode plays a negligible role and does not introduce any modification to the theory of a normal wooden bat. Along the length of the barrel however, the fundamental plays a critical role, and can be modeled as having an antinode at the center of the barrel and two nodes at the ends of the barrel, where the significance of the radial oscillations drops off. The sinusoidal factor introduced in the equation along the barrel roughly models the energy from bending mode harmonics put back into the ball from the fundamental hoop mode, with all bending modes' energies going back into the ball at the antinode ($\cos(x) = 0$) and the normal energy loss for a wooden bat at the nodes ($\cos(x) = 1$).

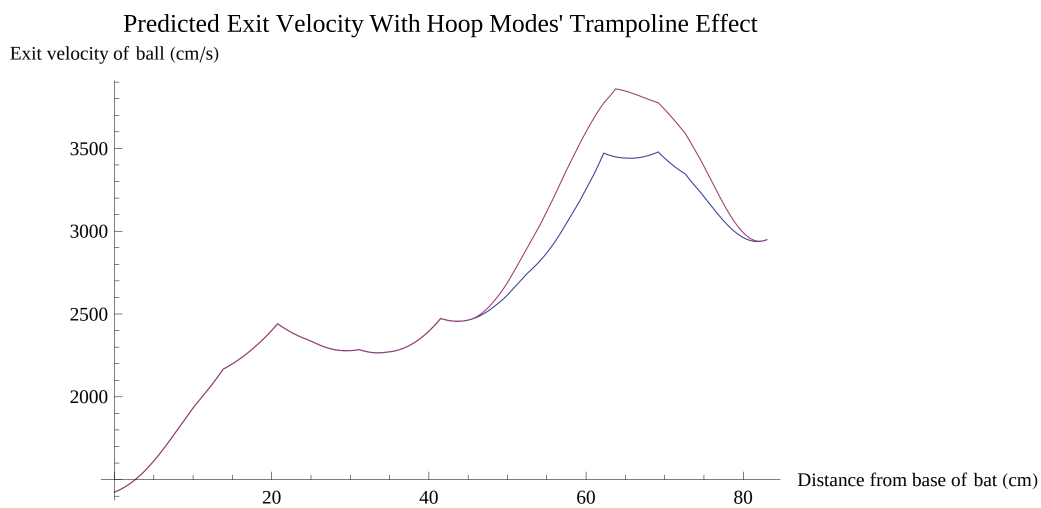


Fig. 5: The blue curve represents the predicted exit-velocities using a normal, wooden bat, while the purple curve shows the exit velocities for an aluminum bat.
This difference is attributed to the fundamental hoop node redirecting energy from the bending nodes back into the ball.

2.31 - Maximizing Comfort with the Aluminum Bat

In considering the maximum comfort to the batter, a plot analogous to Figure 3 is shown below that illustrates the combined velocity loss of the third and fourth harmonics after being attenuated with the fundamental hoop mode. The region where the barrel of the bat is located undergoes a dramatic decrease in velocity loss, indicating that stinging to the hands is also minimized at this point. From this analysis it is clear that the addition of the fundamental hoop mode to the bending modes already existing in a wooden bat greatly reduces the stinging sensation from a normal bat and increases both the range of values along the bat barrel where stinging is minimal, as well as the smallest possible amount of stinging.

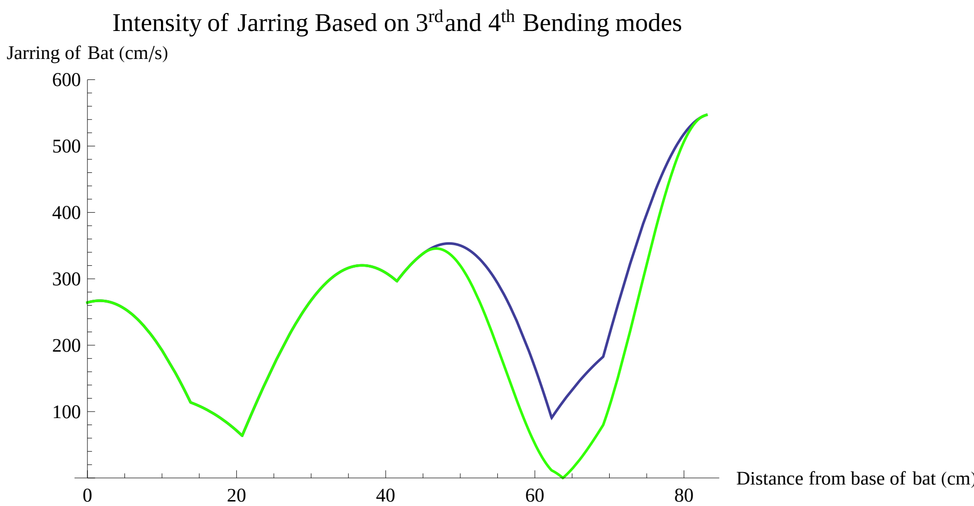


Fig. 6: Plot of the negative velocity contribution due to bending and hoop modes (green curve). The dip at 63cm is lower than the corresponding minimum for the wooden bat (blue curve), indicating the enhanced sweet-spot effect in metal.

The location on the bat at which stinging is minimized may also correspond to the new location of the *COP*. For this reason, the *COP* is recalculated:

$$\ldots\text{it} = \rho_{\text{alum}} \int_0^L \int_{-R(x)}^{R(x)} \int_{-\sqrt{R^2(x)-y^{\wedge}2}}^{\sqrt{R^2(x)-y^{\wedge}2}} \left(x^2+y^2\right) dz\, dy\, dx \; - \; \rho_{\text{alum}} \int_0^L \int_{-(R(x)-\delta)}^{R(x)-\delta} \int_{-\sqrt{(R(x)-\delta)-y^{\wedge}2}}^{\sqrt{(R(x)-\delta)-y^{\wedge}2}} \left(x^2+y^2\right) dz$$

$$M_{\text{MetalBat}} = \rho_{\text{alum}} \left(\int_0^L \int_{-R(x)}^{R(x)} \int_{-\sqrt{R^2(x)-y^{\wedge}2}}^{\sqrt{R^2(x)-y^{\wedge}2}} dz\, dy\, dx \; - \; \int_0^L \int_{-(R(x)-\delta)}^{R(x)-\delta} \int_{-\sqrt{(R(x)-\delta)-y^{\wedge}2}}^{\sqrt{(R(x)-\delta)-y^{\wedge}2}} dz\, dy\, dx \right)$$

$$COM_{\text{MetalBat}} = \rho_{\text{alum}} \left(\int_0^L \pi R^2(x)\, x \, dx \; - \; \int_0^L \pi \left(R(x)-\delta\right)^2 dx \right)$$

$$COP_{\text{MetalBat}} = \frac{I_{\text{MetalBat}}}{M_{\text{MetalBat}} * COM_{\text{MetalBat}}}$$

$$\delta = \text{thickness of the aluminum shell (.2 cm)}$$

Using the new physical information for the aluminum bat, the *COP* location came out to be roughly 28cm from the end of the bat. This is closer than the *COP* for either the wooden bat or the corked bat, but is still within a range of 5cm from those. This indicates that the *COP* is not very significant; the sweet-spot is determined by the combined effects of the bending and hoop mode vibrations as described above.

4. Analysis

■ 4.1 - Louisville Slugger #C271 White Ash Bat

The first type of bat for which our model approximates the locations of the sweet spots (maximum exit velocity, minimum sting delivered to the batter) is the Slugger #C271.

The location along the length of the bat for which our model predicted maximum exit velocity was between 59.5cm and 75cm from the pivot of the handle. This value comes from considering the contribution to velocity from a simple torque-based treatment of the bat's swing, resulting in an angular velocity at the moment before impact, as well as from taking into account negative contributions to the velocity of the ball from bending modes absorbing energy from the ball impact. With only the simplistic torque-based component, the predicted exit velocity increases linearly with position on the bat and maximum velocity at the end of the bat. However, this is empirically known not to be the case, and the contributions from the bending modes introduce deviations from the linear model. The general trend of the exit velocity as predicted from both components of the model is still that exit velocity increases with position along the bat, as a batter would intuitively expect, but also has a peak before the end of the bat, in accordance with the empirical result of maximum power transfer at the sweet spot near the middle of the barrel. Thus, it would appear that energy channeled into stimulation of vibratory harmonics is at least a majorly contributing factor to the "sweet spot" effect for maximal power output.

Our analysis of the wooden bat also shows that the optimal range along the bat's length for minimal jarring in the hand is from 58cm to 70cm, centered at 64cm. These values come from consideration of the location of the nodes of the bending modes along the length of the bat where, according to our model in which impact at a node does not excite the associated harmonics, it is expected that vibration stimulation is minimized. In addition, the biological fact that human hands are largely insensitive to vibrations with frequencies greater than 500 Hz or so sets a limit on the number of harmonics whose nodes would define a region where the sensation of jarring due to impact would be minimized. The range around the minimum negative velocity contribution from the most important bending modes spans a length of 12cm, which is a fairly large region, indicating that a hit anywhere near the middle of the bat will cause much less jarring than would be expected outside the range, such as at the end of the bat. For this reason, it would seem that the "sweet spot" effect reported from batters in which the jarring to the hand is minimized near the middle of the barrel is consistent with our model.

Our model also shows that the *COP* doesn't affect the batter's comfort. Our calculations of the *COP* place it 32cm past the end of the bat, where it cannot be struck. For this reason the COP is not a possible location for the sweet spot, and the location of the nodes of the bending modes contribute the most to where the sweet spot is located.

■ 4.2 - Corked Bat

The second type of bat our model considers is the same Louisville Slugger White Ash bat but with a 15.4cm long cylinder (with a radius of 1.5cm) of wood removed from the inside of the barrel of the bat. This hole is filled with a cylinder of cork with the same dimensions as the cavity.

In considering the maximum exit velocity, our model shows that the corking does not significantly change the position of the sweet spot for greatest power output. The difference in M/I for the normal bat and the corked bat is 8.35e-6 m^{-2}, whereas the actual values for $\frac{M}{I}$ are 2.87e-4 m^{-2} and 2.78e-4 m^{-2}, roughly 40 times their difference. Our model also assumes that there are negligible differences in bending mode dynamics between the corked and normal bats, so that negative contributions to the ball's exit velocity follow roughly the same pattern as they do in the normal solid wood bat. Because the difference in M/I, and thus the momenta of the bats, is so small, and because the vibrational dynamics are modeled as being very close to those of the normal bat, we have strong evidence for their being no major shift in the pattern of the sweet spot of the bat at which maximum power output is achieved.

Given that vibrational dynamics have not changed under the operation of corking, the bending modes' nodes have not changed position, and the only other factor that could possibly affect the location of the sweet spot for minimization of sting to the batter is the location of the COP. Because the location of the COP is 30cm past the end of the bat, the influence of the COP has become no more significant than it was in the case of the normal wood bat. Thus, the location on the bat where the batter's comfort is maximized is in nearly the same range in the middle of the barrel as it was for the normal wood bat.

Given that the corked bat shows behavior extremely similar to that of the normal wood bat in where the locations for maximum power output and comfort are located, it is clear that corking the bat actually does not make any significant change in the bat's behavior, and certainly does not enhance the "sweet spot" effect in either of the two definitions for the sweet spot. While the bat may be lighter and thus easier to swing, the increase in velocity gained from the decrease in mass is not great enough to justify corking the bat. While corking may provide other advantage, such as giving the batter extra time to position a swing or more easily control the bat, the sweet spot is not influenced.

Because our model indicates that corking does not change the properties of the sweet spot effect, it does not by itself explain why corking is illegal in MLB. However, because corking allows an increase in bat swing speed, it allows the batter to have more time in judging how to hit the ball and control it. The official statement outlawing corking and other types of bat alterations as given by MLB regulations is: "The uses of attempts to use a bat that in the umpire's judgment has been altered or tampered with in such a way to improve the distance factor or cause an unusual reaction on the baseball." [10] Although corking doesn't seem to affect the sweet spot as such, the other advantages it gives to the batter would allow the batter to more easily aim at the ball, thus improving the "distance factor" that marks the alteration as illegal.

■ 4.3 - Aluminum Bat

The final type of bat analyzed was a standard aluminum bat; this was done in order to determine how making a bat out of a material other than wood might influence the sweet spot effect.

In terms of the location on the bat where the most power is output, the aluminum bat shows a similar pattern to the wood bat in that the peak of power output is in the same range of locations along the bat in both models. However, the models differ in that the peak exit velocity of the ball achieved by the metal bat and the exit velocities caused at locations on the bat surrounding this peak point are much higher than they are in the corresponding region on the wooden bat. In addition, the range of locations on the bat where the exit velocities are near the peak velocity is wider than it was for the normal bat, spanning a length of 17cm as opposed to 12cm. For these reasons, the metal bat seems to greatly enhance the sweet spot effect, both in that its values for the exit velocity of the ball exceed those of the corresponding wood bat near the sweet spot and the range of the sweet spot is also larger.

The sweet spot effect of maximum comfort for the batter is similarly enhanced, as the velocity loss due to vibrations is even less near the sweet spot range for the metal bat and the range for the sweet spot along the length of the bat is increased in the same way as it was for the sweet spot maximizing exit velocity. Using the metal bat thus even further reduces any jarring the batter feels and increases the chance that the batter hits without feeling too much sting.

The sweet spot for maximum comfort to the batter, as with the previous bats, is not affected by the location of the COP. Recalculation of the COP for the aluminum bat yields a location for the COP about 30 cm past the end of the bat, following the pattern of previous bats. Because this location is well past the barrel of the bat where the ball will hit, it is clear that the COP does not play a role in the positioning of the sweet spot and that the nodes of the vibrations in conjunction with the fundamental hoop mode determine the sweet spot range.

Our model does indicate at least one reason for why the use of aluminum bats would be illegal in MLB. The hollow aluminum bat, due primarily to the prominence of the trampoline effect in hollow materials, has an enhanced sweet spot effect for achieving both maximum power output and comfort to the batter as compared to the wood bat. Thus, using an aluminum bat would be illegal to MLB regulations which state that an alteration that "improve[s] the distance factor" is against the rules. In addition, because the aluminum bat is so much more powerful than the normal wood bats, it poses a safety issue as baseballs batted with such bats can travel at alarmingly high speeds and hurt people in the vicinity of the hitter, such as the pitcher[11]. Thus, it follows that the aluminum bat would be illegal, as it is both dangerous for use and gives an unfair advantage to the batter.

4.4 - Conclusion

The goal of this model was to explain the sweet spot effects of maximum exit velocity and batter comfort in a wooden baseball bat, as well as in corked and aluminum bats. Research suggested that a baseball bat had two potential sweet spots, one that would provide the maximum exit velocity of the ball, and one that would provide maximum control. By augmenting a simple torque analysis with a consideration of vibrational bending modes (and hoop modes in the case of the hollow aluminum bat) it was discovered that baseball bats have a "sweet spot range," or a general area that produces increased comfort and velocity throughout a ball-bat collision. It was shown that corking a bat, while allowing the batter to swing faster, does not result in an increased momentum of the bat. However when aluminum bats were considered, the vibrational hoop modes created a "trampoline effect," which caused the ball to be sprung outward with increased velocity after the ball-bat collision as compared to the normal wooden bat.

There are many simplifications in our model that could be improved upon in the future by adding complexity where approximations have been made and providing more careful analysis where rough models are used. One major simplification of the model is that in our consideration of the bending modes of the bat, we assumed that the baseball bat could be accurately modeled by a beam with uniformly distributed mass and rectangular prismatic shape. To improve upon this simplification, future models should account for the non-uniformity of the baseball bat, especially the difference in distribution of the mass between the barrel and the handle, which could influence the positions of the harmonics' nodes and antinodes.

Another simplification we made in the model is the assumption that the collision of the ball and bat is simple with negligible friction, a perfect 90° impact angle between the ball and bat, and no spin of the bat due to the batter's wrists' motion or on the ball from the way the pitcher threw it. The angle of impact would affect the transfer of momentum from the bat into the ball and could be a significant factor. In addition, the rotational energy of the ball and bat could subtract from or add to the ultimate translational energy of the ball post-collision, and this contribution should be accounted for.

A final major simplification of our model that could be improved upon is the treatment of hoop modes in the aluminum bat. Our treatment of the hoop modes is based on the idea that the fundamental mode redirects energy from bending mode vibration back into the ball's exit velocity. While this principle may be a valid basis for our model, one of the implications of our model is that a ball striking at the antinode of the fundamental hoop mode actually loses no energy at all to the bending modes of the bat. While this does capture the essence of our model's hoop mode analysis principle in that the hoop mode redirects energy, we have no direct evidence that all of the bending modes' energy is redirected--it is possible that in reality there is a fraction of energy that is always trapped in the bending modes. Thus, future models must provide a more careful analysis of the hoop modes that either justifies the result that all of the bending modes' energy can go back into the ball or else provides reasoning for the converse.

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