

## ABSTRACT

Title of dissertation:       NONEQUILIBRIUM QUANTUM  
                                  FLUCTUATION FORCES  
                                  Ryan Orson Behunin, Doctor of Philosophy, 2010

Dissertation directed by:  Professor Bei-Lok Hu  
                                  Department of Physics

We study all known and as yet unknown forces between neutral atoms and neutral atoms and surfaces. The forces arise from mutual influences mediated by an attending electromagnetic field and not from direct interaction. We allow as dynamical variables the center of mass motion of the atom (or surface Chapter 5), its internal degrees of freedom, modeled as a three dimensional harmonic oscillator (the internal degrees of freedom of the surface in chapter 4), and the quantum field treated relativistically.

We adopt the methods of nonequilibrium quantum field theory (NEqQFT) to study the problem of fluctuation forces beginning from first principles. NEqQFT provides a fully dynamical description of systems far from equilibrium having the advantage of being the synthesis of quantum field theory and nonequilibrium statistical mechanics. The integration of these two paradigms is necessary for a complete study of fluctuation forces; quantum field theory for providing effects such as retardation and quantum field fluctuations, and nonequilibrium statistical mechanics for treating processes involving quantum dissipation and noises. By embarking from

first principles we avoid wrong or only partially correct results from inconsistent theories that can be generated from assumptions made at lower levels of accuracy.

In thermodynamic equilibrium we reproduce all the effects and forces known in the last century, such as Casimir-Polder– between neutral atoms, Lifshitz– between an atom and a surface and Casimir between surfaces (and the generalization of these forces to nonequilibrium stationary-states). More noteworthy is the discovery of the existence of a new type of interatomic force which we call the ‘entanglement force’, originating from the quantum correlations of the internal degrees of freedom of entangled atoms.

Fluctuation phenomena associated with quantum fields is a new frontier of future research in atom-field interaction. With NEqQFT we have derived Langevin equations which account for fluctuations of an atom’s trajectory about its semi-classical value. These quantum field-induced perturbations of the atom’s position could lead to measurable results such as the damping of the center-of-mass oscillations of a trapped Bose-Einstein condensate near a surface or backaction cooling of moving mirror by radiative pressure and quantum viscosity discussed respectively in Chapter 3 and 5 of this thesis.

The methods introduced in this thesis for treating atom-field interactions or mirror-field interactions go beyond previous work by providing a fully dynamical description of these forces valid for arbitrary atom and surface motion, indeed the inclusion of self-consistent backactions are necessary for the study of phenomena such as quantum decoherence and entanglement dynamics, including non-Markovian processes which invariably will appear when backaction is taken into consideration

(especially for strong fields, low temperatures, or fast response).

# NONEQUILIBRIUM QUANTUM FLUCTUATION FORCES

by

Ryan Orson Behunin

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Advisory Committee:  
Professor Bei-Lok Hu, Chair/Advisor  
Professor Dieter Brill  
Professor Peter Shawhan  
Professor Victor Yakovenko  
Professor Christopher Jarzynski

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## List of Abbreviations

NEqQFT	Nonequilibrium Quantum Field Theory
CP	Casimir-Polder
AMO	Atomic Molecular Optical
IDF	Internal Degrees of Freedom
MQED	Macroscopic Quantum Electrodynamics
QFEXT	Quantum Field Theory Under the Influence of External Conditions
FDR	Fluctuation Dissipation Relation
IF	Influence Functional
S.E.T.	Self-Energy Terms
ORIF	Oscillator-Reduced Influence Functional
HO	Harmonic Oscillator
DOF	Degree(s) of Freedom
PF	Polariton Fluctuations
PW	Propagating Waves
EW	Evanescent Waves
BC	Barton and Calogeracos, or Boundary Conditions

# Chapter 1

## Introduction

Fluctuation forces are ubiquitous. Two canonical examples are the attractive force between neutral atoms— the London, or Casimir-Polder (CP) force [1, 2], resulting from quantum dipole moment fluctuations, and the Casimir effect [3] which, in the simplest case, is the attraction of ideal conducting plates in vacuum due to quantum fluctuations of the electromagnetic field. These forces give rise to physical phenomena ranging from intermolecular binding and phase transitions to frequency shifts, hysteretic behavior and bistability in the frequency response of micromechanical torsional oscillators [4].

It is an open frontier to understand the nonequilibrium aspects of fluctuation forces when the system we study is fully dynamical (changing with time), out of thermodynamic equilibrium (could be steady-state), or can possess significant fluctuations. With the advances of sophisticated and highly controllable experiments in atomic, molecular and optical (AMO) physics made possible by; the expanded capability of ultrafast, high intensity lasers, high-precision manipulation techniques applied to cold atoms in optical lattices (see, e.g., the experiments and the theoretical analysis of [5]) or cavities (with the capability of tracking atoms in real time [6]), or nanoelectromechanical systems we are entering an era where the nonequilibrium behavior of fluctuation forces can be addressed experimentally and where the

traditional theories will soon become inadequate.

Anticipating this demand this thesis will begin an exploration into nonequilibrium fluctuation forces, first with a study of fully dynamical interatomic interactions where the known forces, London and CP, are generalized for arbitrary atomic motion and initial states. In our thorough exploration of these effects we identify a new force between atoms when their internal degrees of freedom are entangled. Next, we study the forces between atoms and surfaces beginning with a phenomenological description for the surface implemented through boundary conditions on the quantum field and later by modeling the surface as a collection of microscopic degrees of freedom that interact with the field. Lastly, we examine the surface-surface interaction to understand the cooling of a mirror by radiation pressure and vacuum viscosity.

## 1.1 Atom-Atom Forces

It is a remarkable fact that the dominant force between neutral atoms arises from quantum fluctuations. Indeed, at zero temperature for distances larger than the wavelength associated with an atom's first optical resonance (far-field) the interaction originates from zero point fluctuations of the electromagnetic field. The last decade saw intensified research in this area of fluctuation-generated forces between atoms and between an atom and a conducting or dielectric surface. This was brought about by advances in the high-precision capability in the manipulation of trapped atoms in cavities and optical lattices [7, 8], superconductivity experiments

[9] and the design and operation of nanoelectromechanical devices [10], amongst others. These set-ups which hold the promise of ushering in a new era of quantum engineering also made possible a wide range of theoretical inquiries, such as measurements of non-Newtonian forces [11, 12, 13], the utilization of experimental systems out of thermal equilibrium [14], and the investigation of quantum field theoretical effects in the laboratory without the need for accelerators.

In our research program which began with two earlier papers [15, 16] we apply the methods of nonequilibrium quantum field theory [17] within the open quantum system conceptual framework to the study of fluctuation forces. This method can provide a fully dynamical description of (non-stationary) systems far from equilibrium under the influence of various environments, or acted upon by different noises, going beyond the traditional textbook mean field or linear response treatments. This theory is ostensibly very different from the usual approaches researchers in AMO physics are familiar with in the treatment of atomic-optical systems, and it may at first sight appear to be too cumbersome or complicated to be necessary. As this thesis will hopefully illustrate a small initial investment into this new method can pay bountifully.

The method we introduce has the advantage of being the amalgamation of both quantum field theory and nonequilibrium statistical mechanics, the former is required for quantum field (customarily referred to as retardation, but there are more involved) effects, the latter for treating processes involving quantum dissipation and noises. Not only can this method reproduce all the effects and forces known in the last century as detailed below [1, 2, 18, 19], it can also deal with

phenomena and processes more recently brought to central attention from quantum foundational and information processing issues, such as quantum decoherence and entanglement dynamics, including non-Markovian processes (those carrying memories) which invariably will appear when back-action is taken into consideration. Since this method can treat quantum backaction and feedback in a self-consistent manner, it is uniquely adept to quantum control considerations [20].

In Chap. 2 we consider an assembly of  $n$  neutral atoms (labeled by  $a = 1, \dots, n$ ) and model the internal degrees of freedom (idf) of the  $a$ th atom by a three dimensional harmonic oscillator with coordinates  $\vec{Q}_a$ , (thus describing the atom's spontaneous and stimulated emissions and absorption while interacting with a field). (The perspective gained from the case of two atoms provides all of the necessary insight to understand the forces that arise between atoms and surfaces as well.) The atoms interact with an electromagnetic field (from near-field Coulomb force to far-field radiation) with vector potential  $A^\mu$  through a dipole interaction, but not directly with one another. The force between them arises through field-mediated mutual influences. The non-relativistic trajectory of the  $a$ th atom is described by  $\vec{z}_a$  which, unlike in most previous treatments, is a dynamic variable (not prescribed) determined self-consistently by a negotiation amongst all the other variables ( $\vec{Q}_a, A^\mu$ ). Our interest in this paper is primarily focused on the center of mass motion of each atom and not on the microscopic details of the other variables. The open quantum systems [21] approach can efficiently isolate the desired information about the atom's trajectory through a succession of coarse-graining procedures, as detailed below, which take into account the overall effects (backaction) of the remaining

variables. Using the influence functional method we can incorporate the effects of the microscopic physics of the field and the atom's idf into an effective equation of motion for the atom's trajectory from which the force between the two atoms can be extracted by appealing to Newton's Second Law.

In this setup the dipole moment of the atom modeled by an oscillator is not permanent but only instantaneously non-vanishing. The uneven distribution of charge in the atom comes from two effects. First, semi-classically speaking, the magnitude and direction of the nucleus-electron separation (which is proportional to the dipole moment of the atom) will unpredictably vary in time even in the absence of quantum fields, and second, in the presence of quantum fields the atom is polarized by electric field fluctuations. Dipole moment fluctuations are the primary source of interaction among neutral atoms and are responsible for two types of forces arising from distinctly different physical origin.

### 1.1.1 Intrinsic Fluctuation Force

In the quantum-field conception of a neutral atom the electronic wavefunction surrounding the nucleus has a fluctuating component, modeled in our approach by a quantum mechanical harmonic oscillator. As a whole the atom will always remain neutral. However, in time *intrinsic fluctuations* of the oscillator, due to its quantum nature, lead to an uneven local distribution of charge in an otherwise (globally) neutral atom which gives rise to an instantaneous dipole moment that couples to the attending electromagnetic field. Radiation traveling away from the



first (fluctuating) atom carries information about the orientation of its dipole (at the time of emission in the past) which eventually reaches and polarizes the second atom Figure 1.1. The second atom's response to the field leads it also to produce a time-varying electric field that travels back to the first atom and is correlated with the activities of the fluctuating atom's idf, leading to nonvanishing interaction energy. One can think of the second atom as a transponder which receives a signal from the fluctuating atom and then rebroadcasts it. In this analogy the fluctuating atom will receive a signal reflected from the transponder atom which encodes its own history.

This is easily conceptualized if we consider the atoms to be so close that the light transit time between them is much shorter than all other characteristic time scales governing the dynamics. In such a case the retarded electric field is well approximated by the the electrostatic field. Thus, an intrinsic fluctuation of the idf of one atom will source a static dipole electric field seen by the second atom. The second atom is polarized by this external field leading it too to source a dipole field felt by the fluctuating atom. This process leads to an energetically favorable arrangement of the two atom's dipole moments which gives rise to the attractive force between them.

This type of force due to intrinsic fluctuations in the neutral atoms' dipole moments contain two well-known forces: 1) the van der Waals force, usually used to describe all interactions between neutral atoms and molecules categorically, and 2) the London force which arises from the Coulombic interaction between atoms without permanent multipole moments and without the consideration of retardation

# Intrinsic Dipole Forces

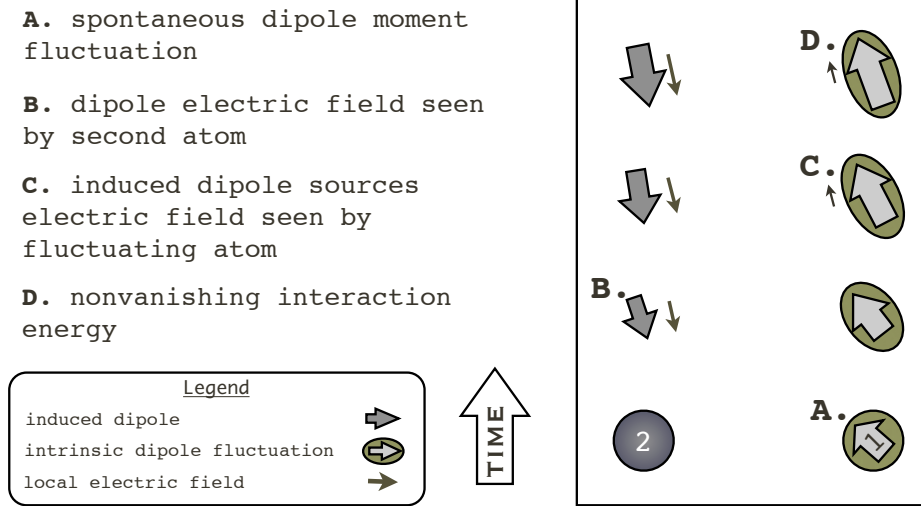


Figure 1.1: The physics of the intrinsic fluctuation force can be described through three steps, fluctuation, induction, and correlation described heuristically in the figure above. **(A)** The dipole moment of atom 1 spontaneously *fluctuates* leading it to source a dipole electric field seen by atom 2 at **(B)**. As atom 2 is polarized (*induction*) by the local electric field it will source a dipole field seen by atom 1 which *correlates* with its instantaneous dipole moment leading to a non-vanishing interaction energy between the two atoms **(C)**.

effects (as Casimir-Polder force does). We refer to forces of this type as **intrinsic fluctuation forces**.

## 1.1.2 Induced Dipole Force

It goes without saying that the quantum field itself possesses intrinsic fluctuations. Any instantaneously generated local electric field will *induce* non-vanishing

dipole moments in both atoms. We classify the interaction of dipole moments induced by the fluctuations of the quantum field as **induced dipole forces**. We suggest making a clean separation between forces arising from intrinsic (before) and induced (here) fluctuations of the dipole moment of a neutral atom because the physical processes produce quite distinct results, as shown in later sections.

The physical origin of this component of the force is the spatial correlation of field fluctuations. Any given field fluctuation will induce correlated dipole moments for the two atoms, much like a long wavelength water wave on the ocean will raise and lower two nearby buoys in phase Figure 1.2. The excitation of the dipoles by the field will lead to radiation that contains information about the emitter. When the radiation from one atom reaches the other the correlation between the induced motion of each dipole moment at the time of emission, and subsequent communication of that motion via radiation leads to a nonvanishing interaction energy.

A well known force of this nature is that of Casimir and Polder [2] who included considerations of the quantum nature of the field. This CP force (there is also the CP force between an atom and a mirror which is treated in Chapter 3) is a generalization of the London description including retardation corrections as well as effects of field quantization – quantization being what imbues the field with its own intrinsic fluctuations.

# Induced Dipole Forces

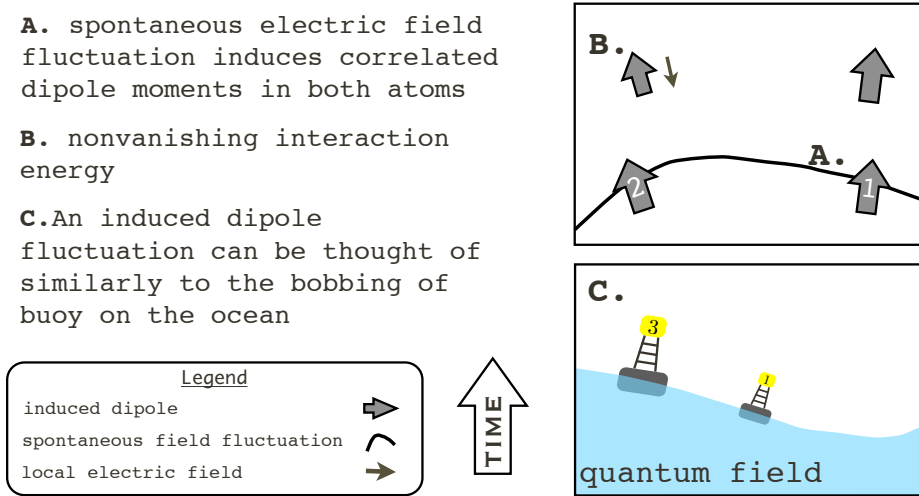


Figure 1.2: The physics of the induced fluctuation force is described through the same steps as the intrinsic fluctuation force, fluctuation, induction, and correlation described in the figure above. At point (A) a long wavelength field *fluctuation* (wavelength  $>$  atom spacing) *induces* correlated dipole moments in both atoms. A dipole field is generated by each atom and *correlates* with its partner's instantaneous dipole moment (B) leading to a non-vanishing interaction energy between the two atoms. The induced dipole force arises from the spatial correlation of field fluctuations over large distances similar to the way nearby buoys on the ocean rise in fall nearly in phase as long-wavelength water waves pass by (C).

### 1.1.2.1 Coarse-graining and Back-action

For a description of the forces between two atoms we need to know only the averaged effect of the quantum field and the oscillator's idf on the atom's trajectory, their details are not of great concern in this quest. Imagine the transition amplitude for the *total* system to evolve from some initial state,  $|\vec{z}_{in}, \varphi_{in}\rangle$  to some final state  $|\vec{z}_{out}, \varphi_{out}\rangle$  in time  $T$  where  $\vec{z}$  labels the atom's position and  $\varphi$  is a collective label of the state of all the remaining (environment) variables in the total system. Our primary interest is the time development of the atom's center of mass for which the field and its interaction with the atom's idf plays a central role through processes like dissipation and radiation reaction see Figure 1.3. For a given final position of the center of mass there can be many consistent final field and oscillator states, likely unobservable. Summing over all final environment states compatible with the atom's motion is *necessary* when we are ignorant of the final state of the environment whether we choose to ignore those details or they are not measurable. This leads to an effective transition amplitude for the trajectory of the atom *alone* where all environmental effects on the trajectory have been taken into account. Carrying out this process of *coarse – graining* where the final field and oscillator states are traced over leads to an effective action that self-consistently accounts for all back-action of the field and the atom's idf on the atom's trajectory. The equation of motion for the atom, and thus the atom-atom force can be obtained through a variation of this action [15, 16].

As we shall show the present formulation goes beyond previous work in that

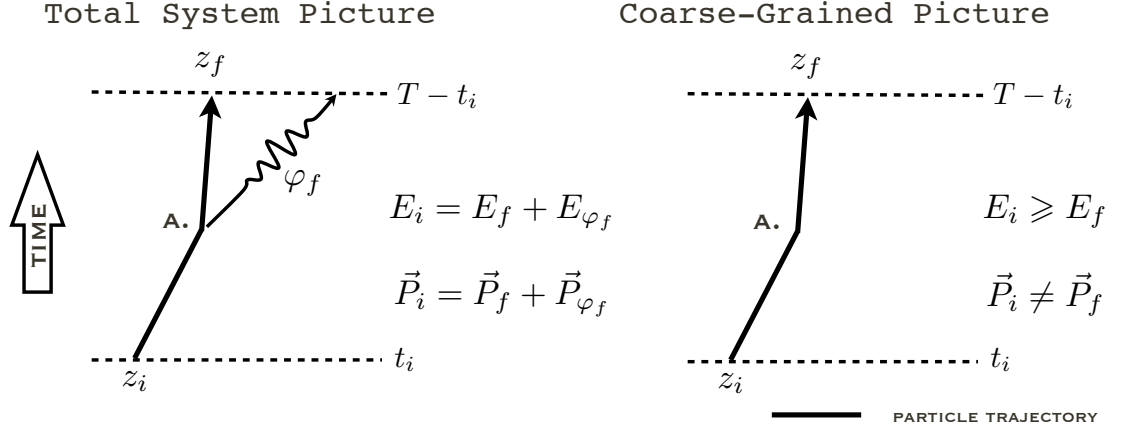


Figure 1.3: The figure heuristically illustrates the effects of coarse-graining. *On the left:* Depiction of a hypothetical momentum and energy conserving microscopic process. A charged particle is moving from initial position  $z_i$  with constant momentum  $\vec{P}_i$  and energy  $E_i$  when at point **(A)** it emits a photon, labeled  $\varphi_f$ . The energy and momentum for the particle subsequently changes to  $E_f$  and  $\vec{P}_f$ , respectively. The remaining energy and momentum is accounted for in the created photon's energy  $E_{\varphi_f} = E_i - E_f$  and momentum  $\vec{P}_{\varphi_f} = \vec{P}_i - \vec{P}_f$ . *On the right:* If we're ignorant of the details of the environment, in this case the photon field, the illustrated microscopic process exhibits *dissipation* ( $E_f < E_i$ ) and *noise* (the particle appears to be kicked at point **A**).

we can derive the forces between two atoms for fully dynamical and under nonequilibrium conditions. When the spacing between the atoms is held fixed we recover the well-known London and CP forces. For the case where the atoms and field are not in thermal equilibrium we find the novel far field scaling for the induced dipole force diminishing as  $1/z^3$  rather than  $1/z^8$  in agreement with [22], and for the case when the two atom's are entangled we find a novel near field scaling that enters at second order in perturbation theory as  $q^2/z^2$  as opposed to the standard  $q^4/z^7$  where  $z$  quantifies the interatomic distance and  $q$  the electronic charge.

## 1.2 Atom-Surface Force

The atom-surface force between a neutral atom and a mirror [2] or a dielectric surface [18, 23] has drawn renewed attention of theorists [19, 24, 25, 26] because of real possibilities of detection [14, 27, 28, 29]. At short distances ( $<100$  nm) CP-Lifshitz type forces dominate the interaction between neutral bodies making them a relevant or even essential factor in the design of micromechanical devices, traps for cold atoms and in precision measurements for the detection of deviations from known forces.

There are two major approaches for the study of the surface-atom force, quantum field theory under the influence of external conditions, and macroscopic quantum electrodynamics (MQED). Quantum field theory under the influence of external conditions (QFEXT) [30] is particularly adept at describing ideal systems, such as those composed of ideal conductors. For such a case it is unnecessary to model the

microscopic physics of the elements constituting the surface, and it is sufficient to simply apply boundary conditions to the quantum field modes. This is the approach we take in Chapter 2.

*Relation to intrinsic and induced dipole force: QFEXT*

When the geometry of the surface has a high degree of symmetry the electromagnetic field can be most easily derived by use of the method of images. We can turn this calculational tool around to derive insight about the physical origin of the interaction of an atom with an idealized surface by forgetting about the surface and treating the image as if it were real. The key distinction between the atom-atom force lies in the fact that the dynamics of the image are determined entirely by the atom itself. By appropriately prescribing the dynamics of the image the heuristic description for atom-atom forces follows here.

### 1.2.1 Non-Ideal Surfaces

The state-of-the-art as far as we can discern from the literature to treat surfaces composed of general materials is MQED which describes electromagnetic field fluctuations in a lossy medium characterized by a complex permittivity. The field fluctuations in MQED are generated by driving the classical Maxwell's equations for the electric field by a stochastic polarization. In order for the system to remain in thermodynamic equilibrium the energy absorbed by the dielectric medium is compensated for by a stochastic forcing term which is added by hand in a manner consistent with the fluctuation dissipation relation (FDR) [31]. Previous works



employing MQED [19, 25, 32, 33] (to name a few) have been skillfully employed to study surface-atom forces for a plethora of experimental setups. The key limitation of this technique is that it requires stationarity, or at least local thermodynamic equilibrium in order to apply the FDR. The method we use reproduces these earlier results for systems under stationary nonequilibrium conditions, but is capable of treating fully nonequilibrium conditions, with arbitrary atomic motion and including full back-action from the field and the medium in a self-consistent manner.

A nonequilibrium generalization of MQED has been successfully used to describe surface-atom forces when the field and surface are not in global thermodynamic equilibrium yet held stationary. It relies crucially on the validity of a ‘local source hypothesis’, which assumes no spatial correlation of the fluctuating polarization that drives the field. This *ad hoc* hypothesis is equivalent to ignoring interactions among the micro constituents of the dielectric medium. When the temperature of the body is much higher than the interaction energy among the medium’s micro elements we believe that the ‘local source hypothesis’ should be an excellent approximation, and that the dissipation can safely be assumed to be local. But at low temperatures or when the coherence length of fluctuations in the medium become large this approximation is expected to break down and new techniques are needed to probe the fully nonequilibrium regimes.

A key challenge in our endeavor to describe atom-surface forces for general materials is to understand the effects of dissipation on the field. Introducing a complex permittivity to the theory by hand violates energy conservation, and so a normal mode decomposition of the field cannot be performed. A first step was taken

by [34, 35] who considered quantization of the field in a dielectric half-space for the case of a real and frequency-independent permittivity. For a real permittivity there is no dissipation and the field can readily be quantized but the material response violates the Kramers-Kronig relations [36, 37] and engenders acausal response.

Our ultimate aim is a first-principles description of the atom-surface force derived from microphysics. We will account for the dissipation of the field by beginning with an action that describes the microscopic physics of the dielectric medium, field, and atom where at the micro-level the *total system* is energy conserving. Along these lines we adopt a strategy similar to Huttner and Barnett [38] who modeled the micro-elements of a dielectric material by a continuous lattice of harmonic oscillators (from now on we'll refer to the matter field as the dielectric, or the medium as well) coupled to the electromagnetic field. In order to provide absorption over a broad range of frequencies each oscillator comprising the dielectric was coupled to a reservoir (though in our formulation a reservoir introduced specifically for this purpose is not necessary) which provides dissipation and noise for the matter. For such a model composed of field + dielectric + reservoir the problem can be solved exactly by Fano's diagonalization [39] when the coupling between respective components is bilinear. In distinction to this work, we are not particularly interested in the microscopic details of the dielectric but only need to capture the averaged effect of the medium upon the remaining degrees of freedom. By invoking the concepts and techniques for open quantum systems we *coarse-grain* the medium by tracing over the dielectric variables leading to a complex permittivity which accounts for the dielectric's response to the field, and the absorption and emission of energy.

In addition, the fluctuations of the micro-elements of the dielectric manifest as a stochastic polarization that drives the field, and when the system is in equilibrium serve to balance the dissipative losses. From this microscopic viewpoint we see that a complex permittivity, quantifying material response to an external field, *cannot* be added to the theory freely unless the fluctuations of the same degrees of freedom are accounted for. It can be shown that at this level the semi-classical equations of motion for the field under the influence of the medium (4.20) take the exact same form as in MQED theory when the medium is assumed to be in a thermal state. The microscopic approach we take offers a unique vantage point to see that the stochastic polarization driving the field put in by hand in MQED without field quantization actually arises from what is equivalent to the media's fluctuations. One could also interpret the field fluctuations in Lifshitz's theory as being *induced* by a fictitious matter field.

The approach we adopt is similar to that of [40] where a path integral formulation was used to derive an effective action describing the medium influenced dynamics of the electromagnetic field. For the specific case of a dielectric half-space our results can be compared with [41] who generalized the results of Carniglia and Mandel to frequency dependent and lossy permittivities. This was done by beginning from a microscopic formulation where the dielectric + field dynamics was calculated exactly using the Wiener-Hopf method [42] and a sum over diagrams. We go beyond these results by considering the specific case of the atom-surface force, and formulate the problem within nonequilibrium quantum field theory.

When the atom is held fixed our nonequilibrium formulation recovers the well-

known CP force for systems in thermodynamic equilibrium and reproduces the recently reported force for stationary systems out of global thermodynamic equilibrium. More importantly, (for the ideal conducting case) it gives a first-principles derivation of a Langevin equation which describes the atom's stochastic motion. Our result for the dispersion of an atomic cloud could motivate experiments designed to measure its change in shape as a function of its distance from the mirror or could predict the line broadening in the frequency response of the center of mass motion of a BEC trapped in a harmonic well.

### 1.2.2 Mirror Cooling by Quantum Field Backaction

The attainment of quantum mechanical states for macroscopic objects is a current topic of intense theoretical [44, 45, 46, 48, 47] and experimental research [49, 50, 51, 52, 53, 54] driven by the potential for testing quantum mechanics at the macroscopic level [55, 56], for probing physics near the quantum to classical transition [57], and by the possibility of quantum limited measurement in a variety of systems [58, 59]. A major obstacle has been to overcome the limitations of standard cryogenic techniques which are incapable of cooling larger (a few  $\mu\text{m}$ ) devices to within the quantum regime. The experimental demand for new cooling techniques has been met by, in part, advances in laser cooling relying on radiation pressure, either through a passive interaction with the electromagnetic field, backaction cooling, or cold-damping involving an active feedback [46, 47]. These techniques have been applied to cool micromechanical systems nearly to the quantum regime and

are being applied to larger systems with cooling factors as large as 40,000 for a one gram mirror [60].

Theoretically, the state-of-the-art for describing the quantum mechanics of micromechanical oscillators interacting with the cavity field is provided by linearized quantum Langevin equations [47] which have proved to show good agreement with experiment [52]. However, as experiment becomes more sophisticated a new and more complete set of tools will be needed. To preempt this demand we present a new formulation of the mirror cooling problem based on nonequilibrium quantum field theory [17] capable of a fully dynamical description of mirror motion where all backaction from the environment, here the quantum field, is taken into account self-consistently. Our formulation goes beyond previous work by providing a complete description of the mirror's interaction with the field (that is all modes are allowed to interact with the mirror and the rotating-wave approximation is not taken *ab initio*) including backaction and noise under generally nonequilibrium conditions.

Beginning from a microscopic model we study the interaction of the mechanical degrees of freedom,  $z$ , of a partially reflecting moving mirror with a quantum field,  $\phi$ . To account for the boundary conditions on the field we use a variant of the Barton and Calogeracos model [61, 62] for a partially transmitting mirror. The boundary conditions of the field on the mirror are implemented through the use of a constraining field,  $\psi$ , which couples to the field linearly along the mirror's worldline and results in partial transmission of incident radiation when the coupling constant is finite. Our primary concern is the dynamics of the mirror as influenced by the field and not the microscopic details of the field's state. For this purpose

we adopt the methods of open quantum systems [21] and treat the field as an *environment* which we coarse-grain to recover an effective action describing mirror motion with the averaged effect of the field on the mirror trajectory taken into account. Mirror trajectories representing a self-consistent negotiation amongst all involved parties  $(\phi, \psi, z)$  satisfy the resulting equations of motion. And although the quantum field variables no longer appear explicitly in the effective action for the trajectory, the mirror is allowed to exchange energy and momentum with the quantum field— leading to dissipation and noise. The microscopic theory we employ is energy conserving and so from this first principles derivation we can identify the dissipative effects on the mirror motion with a transfer of energy from the mechanical motion of the mirror to the field— cooling.

## Chapter 2

### Atom-Atom Forces

In this chapter we formulate the description of interatomic forces within the framework of nonequilibrium quantum field theory beginning with a description of the microscopic physics of all involved parties.

#### 2.1 The Model: Atom-Atom Forces

To capture the salient features of atom dynamics yet maintain calculational simplicity the  $a$ th atom's internal degrees of freedom are modeled by a (3-dimensional) harmonic oscillator with natural frequency  $\Omega_a$ <sup>1</sup>, and move on a trajectory  $\vec{z}(t)$  in a quantum field  $A^\mu$ , the electromagnetic vector potential, in the presence of other atoms (here) and surfaces (later). The dynamics of the system is determined self-consistently by allowing these variables  $(\vec{Q}_a, A^\mu, \vec{z})$  to negotiate amongst themselves. Even for a stationary atom it is necessary in the set up of the problem to assume its position  $\vec{z}$  to be a dynamical variable so its resultant trajectory comes from the mutual interactions with the other two variables.

The entire system composed of  $n$  atoms, their internal degrees of freedom, and the quantum field is described through the following action

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<sup>1</sup>For the exploration of these effects we shall thus use the word atom and oscillator interchangeably.

$$S[\vec{Q}_a, \vec{z}_a, A^\mu] = \sum_a^n (S_Q[\vec{Q}_a] + S_Z[\vec{z}_a] + S_{int}[\vec{Q}_a, \vec{z}_a, A^\mu]) + S_E[A^\mu]. \quad (2.1)$$

The action describing the internal degrees of freedom for the atoms is approximated by the dynamics of a harmonic oscillator giving

$$S_Q[\vec{Q}_a] = \frac{\mu_a}{2} \int d\lambda [\dot{\vec{Q}}_a(\lambda)^2 - \Omega_a^2 \vec{Q}_a(\lambda)^2] \quad (2.2)$$

where  $\mu_a$  is the  $a$ th oscillator's reduced mass and  $\lambda$  its worldline parameter,  $\Omega_a$  being its natural frequency. The electromagnetic field action is given by

$$S_E[A^\mu] = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} \quad (2.3)$$

(the subscript  $E$  stands for the electric field) where  $A^\mu$  is the four-vector potential and  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength tensor. The action describing the nonrelativistic motion of the atom's center of mass is

$$S_Z[\vec{z}_a] = \int d\lambda \left[ \frac{M_a}{2} \dot{\vec{z}}_a^2(\lambda) - V[\vec{z}_a] \right] \quad (2.4)$$

where  $M_a$  is the atom's total mass and  $V[\vec{z}_a]$  is an external potential.

In the dipole approximation, the potential energy for an atom interacting with the photon field takes the form  $-q \vec{Q} \cdot \vec{E}[\vec{z}]$ , where  $q\vec{Q}$  is the atom's instantaneous dipole moment and  $\vec{E}$  is the electric field leading to the interaction action  $S_{int}[\vec{Q}_a, \vec{z}_a, A_\mu] = q_a \int d\lambda Q_a^i(\lambda) E_i[z_a^\mu(\lambda)]$ . Above,  $q_a$ , quantifies the coupling of the  $a$ th atom to the field. [Greek indices will refer to spacetime components of a four-vector, zero referring to time, and Roman indices refer to spatial components where



we will exclusively use the letters  $\{i, j, k\}$  to avoid confusion with the letter  $a$  used to label atoms. Contraction of four-vectors is undertaken with the Minkowski metric with  $(-, +, +, +)$  signature, and the Einstein summation convention is used throughout.

### 2.1.0.1 Worldline Influence Functional

Assume that at time  $t_{in}$  the quantum statistical state of the oscillators, trajectory and field is described by a density operator  $\hat{\rho}(t_{in})$ . This state is unitarily evolved from the initial time  $t_{in}$  to a later time  $t > t_{in}$ , and can be expressed in terms of path integrals by considering matrix elements in an appropriate basis.

To isolate the influence of the field on the dynamics of the atom we coarse-grain over the field variables to construct the field-reduced density matrix,

$\rho_r(\vec{Q}_a, \vec{Q}'_a; \vec{z}_a, \vec{z}'_a; t) = \int dA^\mu \rho(\vec{Q}_a, \vec{Q}'_a; \vec{z}_a, \vec{z}'_a; A^\mu, A^\mu; t)$ . By assuming that the field is initially uncorrelated with the other degrees of freedom the reduced density matrix takes the form,

$$\begin{aligned} \rho_r(\vec{Q}_a, \vec{Q}'_a; \vec{z}_a, \vec{z}'_a; t) &= \prod_a \int d\vec{Q}_{in,a} d\vec{Q}'_{in,a} \int d\vec{z}_{in,a} d\vec{z}'_{in,a} \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}_a \int_{\vec{Q}'_{in,a}}^{\vec{Q}'_a} \mathcal{D}\vec{Q}'_a \\ &\times \int_{\vec{z}_{in,a}}^{\vec{z}_a} \mathcal{D}\vec{z}_a \int_{\vec{z}'_{in,a}}^{\vec{z}'_a} \mathcal{D}\vec{z}'_a e^{i(S_Q[\vec{Q}_a] + S_Z[\vec{z}_a] - S_Q[\vec{Q}'_a] - S_Z[\vec{z}'_a])} \rho_{Qa}(\vec{Q}_{in,a}, \vec{Q}'_{in,a}; t_{in}) \\ &\times \rho_Z(\vec{z}_{in,a}, \vec{z}'_{in,a}; t_{in}) \mathcal{F}[J^{\mu-}, J^{\nu+}] \end{aligned} \quad (2.5)$$

which introduces the influence functional (IF)  $\mathcal{F}[J^{\mu-}, J^{\nu+}]$  [63]. If the initial state of the field is Gaussian in field variables (which includes vacuum and thermal states)

the influence functional can be calculated exactly for the dipole field interaction.

$$\mathcal{F}[J^{\mu-}, J^{\nu+}] = \exp \left\{ i \int d^4y \int d^4y' [J^{\mu-}(y) D_{\mu\nu}^{ret}(y, y') J^{\nu+}(y') + \frac{i}{4} J^{\mu-}(y) D_{\mu\nu}^H(y, y') J^{\nu-}(y')] \right\} \quad (2.6)$$

Here the current density is

$$J_\mu(x) = - \sum_a q_a \int d\lambda \kappa_{i\mu} \delta^4(x^\mu - z_a^\mu(\lambda)) Q_a^i(\lambda), \quad (2.7)$$

$J^+ = (J + J')/2$  and  $J^- = J - J'$  are its semi-sum and difference, respectively, where prime distinguishes histories, and  $\kappa_{i\mu} = \partial_i \eta_{0\mu} - \partial_0 \eta_{i\mu}$  is a differential operator that relates the photon field to the electric field by contraction i.e.  $E^i = \kappa_{i\mu}^i A^\mu$ .  $D_{\mu\nu}^{ret}(y, y')$  and  $D_{\mu\nu}^H(y, y')$  are the retarded Green's function and Hadamard function for the field respectively. They can be expressed in terms of the commutator and anticommutator of field operators.

$$D_{\mu\nu}^{ret}(y, y') = i\theta(t - t') \langle [\hat{A}_\mu(y), \hat{A}_\nu(y')] \rangle = i\theta(t - t') \langle \hat{A}_\mu(y) \hat{A}_\nu(y') - \hat{A}_\mu(y') \hat{A}_\nu(y) \rangle \quad (2.8)$$

$$D_{\mu\nu}^H(y, y') = \langle \{ \hat{A}_\mu(y), \hat{A}_\nu(y') \} \rangle = \langle \hat{A}_\mu(y) \hat{A}_\nu(y') + \hat{A}_\mu(y') \hat{A}_\nu(y) \rangle \quad (2.9)$$

The retarded Green's function for the electromagnetic field describes the classical electromagnetic field sourced by currents and charges. It can be obtained from Maxwell's equations by inverting the wave equation for the vector potential. Its classical nature can be further affirmed by noting that because the field commutator is a c-number it is independent of the quantum state of the field. The Hadamard

function however is a purely quantum object which quantifies the fluctuations of the quantum field and the correlation of those fluctuations through space and time. In the Feynman gauge these kernels can be expressed in terms of the retarded,  $D_{ret}$ , and Hadamard,  $D_H$ , Green's function for a massless scalar field.

$$D_{\mu\nu}^{ret}(x, x') = \eta_{\mu\nu} D_{ret}(x, x') \quad D_{\mu\nu}^H(x, x') = \eta_{\mu\nu} D_H(x, x') \quad (2.10)$$

At zero temperature they take on the explicit form

$$D_{ret}(x, x') = \frac{1}{4\pi} \theta(t - t') \delta(\sigma) \quad D_H(x, x') = -\frac{1}{4\pi^2 \sigma} \quad (2.11)$$

where  $\sigma$  is Synge's worldfunction defined to be half the geodesic distance between the four-vectors  $x$  and  $x'$ ,  $\sigma = (x - x')^2/2$ .

### 2.1.0.2 Oscillator-Reduced Influence Functional

We isolate the net influence that the oscillator's idf  $\vec{Q}_a$  and the field  $A^\mu$  have on the trajectory by tracing over all final oscillator configurations  $\rho_{or}(\vec{z}_a, \vec{z}'_a; t) = \int d\vec{Q}_a \rho_r(\vec{Q}_a, \vec{Q}_a; \vec{z}_a, \vec{z}'_a; t)$  introducing the oscillator-reduced density matrix  $\rho_{or}$ .

$$\begin{aligned} \rho_{or}(\vec{z}, \vec{z}'; t) = & \int d\vec{z}_{in,1} d\vec{z}'_{in,1} \int_{\vec{z}_{in,1}}^{\vec{z}} \mathcal{D}\vec{z}_1 \int_{\vec{z}'_{in,1}}^{\vec{z}'} \mathcal{D}\vec{z}'_1 e^{i(S_Z[\vec{z}_1] - S_Z[\vec{z}'_1])} \\ & \times \rho_Z(\vec{z}_{in,1}, \vec{z}'_{in,1}; t_{in}) \mathcal{F}_Z[\vec{z}_1, \vec{z}'_1] \end{aligned} \quad (2.12)$$

All the effects of the environment are now packaged in the oscillator-reduced IF,  $\mathcal{F}_Z[\vec{z}^-, \vec{z}^+]$ . The development has been simplified by working in the rest frame of the second atom in so doing  $\vec{z}_2$  is no longer treated as a dynamical variable.

$$\begin{aligned} \mathcal{F}_Z[\vec{z}_1^-, \vec{z}_1^+] &= \prod_a \int d\vec{Q}_a d\vec{Q}_{in,a} d\vec{Q}'_{in,a} \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}_a \int_{\vec{Q}'_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}'_a e^{i(S_Q[\vec{Q}_a] - S_Q[\vec{Q}'_a])} \\ &\quad \times \rho_{Q_a}(\vec{Q}_{in,a}, \vec{Q}'_{in,a}; t_{in}) \mathcal{F}[J^{\mu-}, J^{\nu+}] \end{aligned} \quad (2.13)$$

To elucidate our approach we write (3.8) in a more suggestive form

$$\mathcal{F}_Z[\vec{z}_1^+, \vec{z}_1^-] = \mathcal{F} \left[ \vec{z}_a^+, \vec{z}_a^-; -i \frac{\delta}{\delta \vec{j}_a^+}, -i \frac{\delta}{\delta \vec{j}_a^-} \right] \prod_a F_a[\vec{j}_a^+, \vec{j}_a^-] \Big|_{j_a^\pm=0} \quad (2.14)$$

which defines the IF for a three dimensional harmonic oscillator,  $F_a[\vec{j}_a^+, \vec{j}_a^-]$ . To bring  $\mathcal{F}[J^{\mu-}, J^{\nu+}]$  out of the path integrals in (3.8)  $[Q_a^{k\pm}(\lambda)]^n$  is replaced with functional derivatives on the IF for the harmonic oscillators  $\left(-i \frac{\delta}{\delta j_a^{\mp}(\lambda)}\right)^n F_a[\vec{j}_a^+, \vec{j}_a^-] \Big|_{j_a^\pm=0}$ . The explicit form for  $F_a[\vec{j}_a^+, \vec{j}_a^-]$  is

$$\begin{aligned} F_a[\vec{j}_a^+, \vec{j}_a^-] &= \int d\vec{Q}_a \int d\vec{Q}_{in,a} d\vec{Q}'_{in,a} \rho_{Q_a}(\vec{Q}_{in,a}, \vec{Q}'_{in,a}; t_{in}) \\ &\quad \times \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}_a \int_{\vec{Q}'_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}'_a e^{iS_Q[\vec{Q}_a] - iS_Q[\vec{Q}'_a] + i \int d\lambda [\vec{j}_a^+ \cdot \vec{Q}_a^- + \vec{j}_a^- \cdot \vec{Q}_a^+]} \end{aligned} \quad (2.15)$$

In the above the dot product between current and oscillator coordinate i.e.  $\vec{j} \cdot \vec{Q}$  is taken with respect to a three dimensional Euclidean metric. For a Gaussian initial state (2.15) can be evaluated exactly

$$F_a[\vec{j}_a^+, \vec{j}_a^-] = \mathcal{N} \exp \left\{ i \int d\lambda d\lambda' [\vec{j}_a^-(\lambda) \cdot \vec{j}_a^+(\lambda') g_{ret,a}(\lambda, \lambda') + \frac{i}{4} \vec{j}_a^-(\lambda) \cdot \vec{j}_a^-(\lambda') g_{H,a}(\lambda, \lambda')] \right\} \quad (2.16)$$

where  $g_{ret,a}(\lambda, \lambda')$  and  $g_{H,a}(\lambda, \lambda')$  (expressed below at  $T = 0$ ) are the retarded and Hadamard Green's functions for a one dimensional harmonic oscillator with natural frequency  $\Omega_a$ , mass  $\mu_a$ , and  $\mathcal{N}$  is a normalization constant.

$$g_{ret,a}(\lambda, \lambda') = \frac{1}{\mu_a \Omega_a} \theta(\lambda - \lambda') \sin \Omega_a(\lambda - \lambda') \quad g_{H,a}(\lambda, \lambda') = \frac{1}{\mu_a \Omega_a} \cos \Omega_a(\lambda - \lambda') \quad (2.17)$$

### 2.1.0.3 Decoherence and the Semi-Classical Limit

The complex norm of (3.8) at leading order in a  $\vec{z}_1^-$ -expansion follows

$$|\mathcal{F}_Z[\vec{z}_1^-, \vec{z}_1^+]| = \exp \left\{ - \int d\lambda d\lambda' z_1^{i-}(\lambda) N_{ij}(\lambda, \lambda') z_1^{j-}(\lambda') \right\}. \quad (2.18)$$

where  $N_{ij}$  is a symmetric positive definite kernel. Thus, we observe that the off-diagonal elements of the density matrix in (3.9) are strongly suppressed for large values of  $\vec{z}^- = \vec{z} - \vec{z}'$  as is indicative of decoherence of the quantum trajectory [64]. Decoherence of the trajectory due to its interactions with the quantum fluctuations of the environment and the internal degrees of freedom of the atoms permits the existence of a semi-classical limit for the oscillator's path through space. Using a saddle-point approximation to evaluate (2.12) one can show that the semi-classical dynamics is determined from the variation

$$\left. \frac{\delta S_{CGEA}[\vec{z}_1^+, \vec{z}_1^-]}{\delta z_1^{k-}(\tau)} \right|_{z_1^{k-}=0} = 0 \implies M \ddot{z}_k(\tau) = f_k(\tau) \quad (2.19)$$

where the so-called coarse-grained effective action is given by  $S_{CGEA}[z_1^{i+}, z_1^{i-}] = S_Z[\vec{z}_1] - S_Z[\vec{z}'_1] + S_{IF}[\vec{z}_1^+, \vec{z}_1^-]$ , and  $S_{IF}[\vec{z}_1^+, \vec{z}_1^-] = -i \ln \mathcal{F}_Z[\vec{z}_1^-, \vec{z}_1^+]$  defines the influence action. The force acting on the trajectory due to its interactions with the oscillators and field is given by

$$f_k(\tau) = \frac{\delta S_{IF}[\vec{z}_1^+, \vec{z}_1^-]}{\delta z_1^{k-}(\tau)} \Big|_{z_1^{k-}=0}. \quad (2.20)$$

For general atom motion this force contains all known effects, including the Lamb shift, radiation reaction, dissipation, and the atom-atom force.

## 2.2 Nonequilibrium Atom-Atom Force

The suppression of the reduced density matrix for off-diagonal elements justifies an expansion of (2.14) for small values of  $\vec{z}_1^-$ . The linear order term yields the influence force and is represented by an infinite series in powers of the coupling. The local (spatially independent) terms in this expansion lead to the aforementioned Lamb shift, radiation reaction, and dissipation. The atom-atom force can be obtained from this series by extracting the terms that depend upon the spatial separation of the atoms.

To simplify the presentation we rewrite the influence functional for the atom's trajectory as  $\mathcal{F}_Z = \langle e^{iS_{eff}} \rangle_o$  where the form of  $S_{eff} = -i \ln \mathcal{F}$  can be taken from (5.16)

$$S_{eff} = \int d^4x \int d^4x' [J^{\mu-}(x) D_{\mu\nu}^{ret}(x, x') J^{\nu+}(x') + \frac{i}{4} J^{\mu-}(x) D_{\mu\nu}^H(x, x') J^{\nu-}(x')] \quad (2.21)$$

and  $\langle \dots \rangle_o$  is the expectation value with respect to both oscillators under the condition of no interactions.

$$\langle \dots \rangle_o = \prod_a \int d\vec{Q}_a d\vec{Q}_{in,a} d\vec{Q}'_{in,a} \rho_{Q_a}(Q_{in,a}, Q'_{in,a}; t_{in})$$

$$\times \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}_a \int_{\vec{Q}'_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}'_a e^{i(S_Q[\vec{Q}_a] - S_Q[\vec{Q}'_a])}(\dots) \quad (2.22)$$

$S_{eff}$  is a quadratic function of the current density (2.7) which depends on a sum of delta functions with support at each atom's position. Thus, one can see that the cross terms  $S_{cross}$  between the two atom's currents appearing in  $S_{eff}$  will lead to terms that depend upon their spatial separation.

$$S_{cross} = \int d^4x d^4x' [J_1^{\mu-}(x) D_{\mu\nu}^{ret}(x, x') J_2^{\nu+}(x') + J_2^{\mu-}(x) D_{\mu\nu}^{ret}(x, x') J_1^{\nu+}(x') + \frac{i}{2} J_1^{\mu-}(x) D_{\mu\nu}^H(x, x') J_2^{\nu-}(x')] \quad (2.23)$$

$J_a^\mu$  refers to the current density of the  $a$ th atom, with  $a = 1$  referring to the distinguished atom, where all the atom-atom forces we are studying here act upon.  $\langle S_{cross} \rangle_o$ , the expectation value of  $S_{cross}$ , will vanish for initially uncorrelated and Gaussian oscillator states because it is linear in the coordinate of each oscillator. Therefore the leading order contribution to the atom-atom force will be proportional to the square of  $S_{cross}$  at order  $q_1^2 q_2^2$ . Expanding  $\mathcal{F}_Z$  in powers of  $S_{eff}$  we express the IF as

$$\mathcal{F}_Z = e^{iS_{IF}[z^\pm]} \approx 1 + (S.E.T.) - \frac{1}{2} \langle (S_{cross}[\vec{z}_a^\pm, \vec{Q}_a^\pm])^2 \rangle_o \quad (2.24)$$

where S. E. T. stands for *self energy terms*. The leading order linear terms  $\mathcal{O}(q_a^2)$  contain the back-action of the field on the motion of the atom itself only. We refer to them as “self energy terms”, borrowing a terminology from particle physics, and for a stationary atom these effects are unimportant. We focus on the quadratic term which contains not only higher order self energy type effects but also the leading

order contribution to the atom-atom force contained in  $S_{cross}$ .

Note that  $S_{eff}$  contains a term linear in the retarded propagator for the field. It makes sense that the force will manifest as the square of (2.23) because, as described in the introduction, fluctuations in the dipole moment of one atom induce radiation that travels to the other, influences its dynamics, and induces the other atom to radiate. From a diagrammatic viewpoint this process requires two propagators.

The leading order expression for the force can be derived from

$$f_k(\tau) = \frac{\delta S_{IF}[\vec{z}_1^\pm]}{\delta z_1^{k-}} \Big|_{z^{k-}=0} \approx -\frac{i}{2} \frac{\delta}{\delta z_1^{k-}(\tau)} \left\langle (S_{cross}[\vec{z}_a^\pm, \vec{Q}_a^\pm])^2 \right\rangle_o \Big|_{z^{k-}=0} \quad (2.25)$$

where the first equality holds for a stationary trajectory (radiation reaction and dissipation vanish).

Carrying out the variation in (2.25) yields three contributions the first two being

$$f_k^A(\tau) = \frac{1}{2} q_1^2 q_2^2 \int_{\lambda_i}^{\lambda_f} d\lambda \int_V d^4x \int_V d^4y g_{ret,1}(\tau, \lambda) E_{ret}^{ij}(z^\alpha(\lambda), y) \times G_H(x, y) \partial_k(x') E_{ij}^{ret}(x', x) \Big|_{x'=z^\alpha(\tau)} \quad (2.26)$$

$$f_k^B(\tau) = \frac{1}{2} q_1^2 q_2^2 \int_{\lambda_i}^{\lambda_f} d\lambda \int_V d^4x \int_V d^4y g_{H,1}(\tau, \lambda) E_{ret}^{ij}(z^\alpha(\lambda), y) \times G_{ret}(x, y) \partial_k(x') E_{ij}^{ret}(x', x) \Big|_{x'=z^\alpha(\tau)} \quad (2.27)$$

where  $E_{ij}(x, x')$  is the dyadic electric field Green's function

$E_{ij}(x, x') = Tr\{\hat{\rho}_E \hat{E}_i(x) \hat{E}_j(x')\}$  which arises in our formalism through a contrac-



tion of the operator  $\kappa_\mu^i$  with the photon field. Note also that after all functional derivatives are taken  $z_1 \rightarrow z$ ,  $G_{ret}(x, x')$  and  $G_H(x, x')$  are meant to generally represent the retarded and Hadamard functions for whatever the atom interacts with. It should however be noted that this perturbative approach is only valid when the atom interacts with a rarefied body. For example to find the surface-atom force the integration volume  $V$  is taken to be the half space, and the Green's functions describing the physics of the media occupying that region are used. For the specific case of an atom located at  $\vec{z}_2$ ,  $G(x, x') \propto g_2(t, t')\delta^3(\vec{x} - \vec{z}_2)\delta^3(\vec{x}' - \vec{z}_2)$ . More general cases will be considered in a future paper.

The form of (2.26) and (2.27) can be explained by appealing to the heuristic description of the force given in the introduction.  $f^A$  and  $f^B$  arise from the *intrinsic fluctuations* in the dipole moments of the atoms. This can be seen by noting that they contain the atom's Hadamard function i.e. the symmetric two point function for the oscillator degree of freedom. The two retarded electric field Green's functions account for the transfer of information between the two atoms, and the retarded Green's function for the atom characterizes its response to an external field see Fig[2.1].

The third component of the force,  $f^C$  arises from *induced fluctuations* of the atom's dipole moments. The retarded Green's functions for the two oscillators,  $g_{ret,a}$ , characterize their response to a given field fluctuation. The  $k$ th component of the induced dipole moment of the  $a$ th atom can be written as  $d_{ind,a}^k = q_a \int d\lambda' g_{ret,a}(\lambda, \lambda') E^k[z_a^\alpha(\lambda')]$  where  $E^k[z_a^\alpha(\lambda')]$  is the  $k$ th component of the electric field at the position of the atom. The symmetric two-point function of the induced

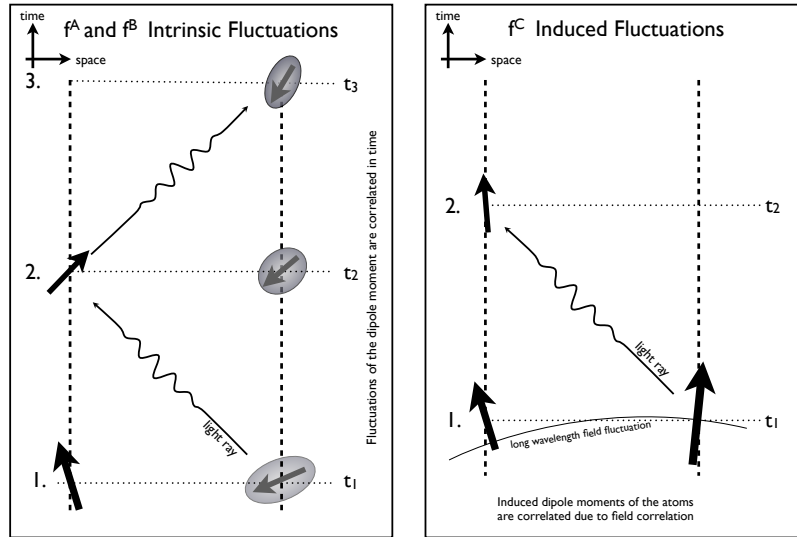


Figure 2.1: The illustrations depict the physical origin of the intrinsic fluctuation and induced dipole forces. *On the left* intrinsic dipole fluctuations (represented by shaded oval); 1. radiate information about their motion, and 2. this radiation induces a correlated dipole moment in the second atom (solid black arrow denotes an induced dipole moment). The induced motion at  $t_2$  leads to radiation that travels back to the fluctuating atom. At  $t_3$  the radiation produced at step 2 will produce a local electric field near the fluctuating atom which carries information about its own fluctuations in the past. The illustration *on the right* depicts the physical origin of the second component of the force arising from field fluctuations and their spatial correlation. Step 1 shows how a field fluctuation induces correlated dipole moments in both atoms. The induced motion of the dipole moments will lead to radiation emitted from both atoms containing information about their motion (only left moving radiation included). At  $t_2$  the radiation generated by the induced motion produces a local electric field around each atom that is correlated with its motion.

dipole moment quantifies its fluctuations

$$\langle \{d_{ind,a}^j(t), d_{ind,b}^k(t')\} \rangle = q_a q_b \int d\lambda d\lambda' g_{ret,a}(t, \lambda) g_{ret,b}(t', \lambda') E_H^{jk}(z_a^\alpha(\lambda), z_b^\alpha(\lambda)). \quad (2.28)$$

The remaining electric field propagator,  $E_{ij}^{ret}$ , carries information about the motion of one atom to the other and accounts for the form of  $f^C$  see Fig[2.1]

$$\begin{aligned} f_k^C(\tau) &= \frac{1}{2} q_1^2 q_2^2 \int_{\lambda_i}^{\lambda_f} d\lambda \int_V d^4x \int_V d^4y g_{ret,1}(\tau, \lambda) \\ &\quad \times [\partial_k(x') E_{ret}^{ij}(x', x) G_{ret}(x, y) E_{ij}^H(z^\alpha(\lambda), y) \\ &\quad + E_{ret}^{ij}(z^\alpha(\tau), x) G_{ret}(x, y) \partial_k(x') E_{ij}^H(x', y)|_{x'=z^\alpha(\lambda)}] \end{aligned} \quad (2.29)$$

where  $\partial_k(x)$  denotes differentiation with respect to  $x^k$ . The previous form of the force is valid for any atomic motion. However, a self consistent treatment would require that the aforementioned ‘self energy’ terms be included in order to account for the back-action of the field on the atom itself.

## 2.2.1 Induced Dipole Force

In this section we calculate the induced dipole force explicitly by plugging in the retarded Green’s function for the second oscillator,  $g_{ret,2}$ , and choosing  $\vec{z}_2$  to be the origin of coordinates.

$$\begin{aligned} f_k^C(\tau) &= \frac{1}{2} q_1^2 q_2^2 \int_{\lambda_i}^{\lambda_f} d\lambda \int dt \int dt' g_{ret,1}(\tau, \lambda) g_{ret,2}(t, t') \\ &\quad \times [E_{ret}^{ij}(z^\alpha(\tau), t', \vec{0}) \partial_k(x') E_{ij}^H(x', t, \vec{0}) \\ &\quad + E_H^{ij}(z^\alpha(\lambda), t', \vec{0}) \partial_k(x') E_{ij}^{ret}(x', t, \vec{0})|_{x'=z^\alpha(\tau)}] \end{aligned} \quad (2.30)$$

The derivatives operating on the various Green's functions can be simplified by employing the equation of motion for the field, the resultant form valid for general atom motion follows.

$$\begin{aligned}
f_k^C &= \frac{q_1^2 q_2^2}{4\pi} \int d\lambda \int dt \int dt' g_{ret,1}(\tau, \lambda) g_{ret,2}(t, t') \theta(\tau, t') \\
&\times \left\{ \delta''' [\sigma(z^\alpha(\tau); t', \vec{0})] \left[ \frac{1}{2} D_H'' [\sigma(z^\alpha(\lambda); t, \vec{0})] \sigma(z^\alpha(\tau); t', \vec{0})_k \left( \sigma(z^\alpha(\lambda); t, \vec{0})_j \right. \right. \right. \\
&\times \sigma(z^\alpha(\lambda); t, \vec{0})^j \sigma(z^\alpha(\tau); t', \vec{0})_i \times \sigma(z^\alpha(\tau); t', \vec{0})^i + [\sigma(z^\alpha(\lambda); t, \vec{0})_i \sigma(z^\alpha(\tau); t', \vec{0})^i]^2 \left. \left. \left. \right) \right. \right. \\
&\quad \left. \left. + \delta'' [\sigma(z^\alpha(\tau); t', \vec{0})] \left[ \frac{1}{2} D_H''' [\sigma(z^\alpha(\lambda); t, \vec{0})] \sigma(z^\alpha(\lambda); t, \vec{0})_k \left( \sigma(z^\alpha(\lambda); t, \vec{0})_j \right. \right. \right. \right. \\
&\times \sigma(z^\alpha(\lambda); t, \vec{0})^j \sigma(z^\alpha(\tau); t', \vec{0})_i \sigma(z^\alpha(\tau); t', \vec{0})^i + [\sigma(z^\alpha(\lambda); t, \vec{0})_i \sigma(z^\alpha(\tau); t', \vec{0})^i]^2 \left. \left. \left. \right) \right. \right. \\
&\quad \left. \left. + D_H'' [\sigma(z^\alpha(\lambda); t, \vec{0})] \left( 6\sigma(z^\alpha(\tau); t', \vec{0})_i \sigma(z^\alpha(\tau); t', \vec{0})^i \sigma(z^\alpha(\lambda); t, \vec{0})_k \right. \right. \right. \\
&\quad \left. \left. \left. + 2\sigma(z^\alpha(\lambda); t, \vec{0})_i \sigma(z^\alpha(\tau); t', \vec{0})^i \sigma(z^\alpha(\tau); t', \vec{0})_k \right) \right] \right. \\
&\quad \left. + \delta' [\sigma(z^\alpha(\tau); t', \vec{0})] \left[ 4 D_H''' [\sigma(z^\alpha(\lambda); t, \vec{0})] \sigma(z^\alpha(\tau); t', \vec{0})_i \sigma(z^\alpha(\tau); t', \vec{0})^i \right. \right. \\
&\quad \left. \left. \times \sigma(z^\alpha(\lambda); t, \vec{0})_k + 20 D_H'' [\sigma(z^\alpha(\lambda); t, \vec{0})] \sigma(z^\alpha(\lambda); t, \vec{0})_k \right] \right\} \quad (2.31)
\end{aligned}$$

Here, primes on functions denote derivatives with respect to  $\sigma$ , and  $\sigma_k = \partial_k \sigma$  denotes differentiation of  $\sigma$  with respect to  $x^k$ .

We can separate (2.31) into 4 terms with differing number of  $\sigma$ -derivatives and specify a static trajectory for the distinguished atom to bring the derivatives outside of the integral i.e.  $d/d\sigma = z^{-1}d/dz$ . To distinguish which Green's function a given  $\sigma$ -derivative acts on we attach a dummy subscript to  $z$  that should not be confused with an atom label. Once all derivatives are taken  $z_1$  and  $z_2$  are set to  $z$ , the separation between the two atoms. The evaluation of the  $t$ -integral can be done

by substituting  $\delta(\sigma(x, x')) = \frac{\delta(t'-t+|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}$ .

$$f_z^{C1}(\tau) = -\frac{q_1^2 q_2^2}{4\pi} z \frac{1}{2} \left[ \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^3 \left( \frac{1}{z_1} \frac{d}{dz_1} \right)^2 + \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^2 \left( \frac{1}{z_1} \frac{d}{dz_1} \right)^3 \right] \\ \times \frac{1}{z_1} \int d\lambda \int dt g_{ret,1}(\tau, \lambda) g_{ret,2}(t, \tau - z_1) D_H[\sigma(z_2^\alpha(\lambda), t, \vec{0})] \left[ 2z^4 \right] \Big|_{z_1=z_2=z} \quad (2.32)$$

$$f_z^{C2}(\tau) = -\frac{q^2 e^2}{4\pi} z \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^2 \left( \frac{1}{z_1} \frac{d}{dz_1} \right)^2 \frac{1}{z_1} \int d\lambda \int dt g_{ret,1}(\tau, \lambda) g_{ret,2}(t, \tau - z_1) \\ \times D_H[\sigma(z_2^\alpha(\lambda), t, \vec{0})] \left[ 8z^2 \right] \Big|_{z_1=z_2=z} \quad (2.33)$$

$$f_z^{C3}(\tau) = -\frac{q^2 e^2}{4\pi} z \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^3 \left( \frac{1}{z_1} \frac{d}{dz_1} \right) \frac{1}{z_1} \int d\lambda \int dt g_{ret,1}(\tau, \lambda) g_{ret,2}(t, \tau - z_1) \\ \times D_H[\sigma(z_2^\alpha(\lambda), t, \vec{0})] \left[ 4z^2 \right] \Big|_{z_1=z_2=z} \quad (2.34)$$

$$f_z^{C4}(\tau) = -\frac{q^2 e^2}{4\pi} z \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^2 \left( \frac{1}{z_1} \frac{d}{dz_1} \right) \frac{1}{z_1} \int d\lambda \int dt g_{ret,1}(\tau, \lambda) g_{ret,2}(t, \tau - z_1) \\ \times D_H[\sigma(z_2^\alpha(\lambda), t, \vec{0})] \left[ 20 \right] \Big|_{z_1=z_2=z} \quad (2.35)$$

We can express the Green's function for the field through a mode sum and subsequently evaluate the  $\lambda$  and  $t$  integrals in the long time limit. The exact field-influenced dynamics of the oscillators will be dissipative, however this dissipative effect does not appear at this order in perturbation theory, but can be modeled phenomenologically by inclusion of an infinitesimal dissipation in the oscillator equation of motion i.e.  $g_{ret,a}(t-t') \rightarrow g_{ret,a}(t-t')e^{-\epsilon(t-t')}$ .

At finite temperature the Hadamard function for the field can be obtained through periodicity in imaginary time, or by taking the trace of symmetrized field operators with respect to a thermal density matrix. Expressing the result as a mode sum we find.

$$f_z^{C1} = -\frac{2}{\pi}z \frac{1}{2} \left[ \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^3 \left( \frac{1}{z_1} \frac{d}{dz_1} \right)^2 + \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^2 \left( \frac{1}{z_1} \frac{d}{dz_1} \right)^3 \right] \\ \times \frac{1}{z_1 z_2} \int_0^\infty d\omega \alpha_1(\omega) \alpha_2(\omega) \coth(\beta\omega/2) \sin \omega z_2 \cos \omega z_1 \left[ 2z^4 \right] \Big|_{z_1=z_2=z} \quad (2.36)$$

$$f_z^{C2} = -\frac{2}{\pi}z \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^2 \left( \frac{1}{z_1} \frac{d}{dz_1} \right)^2 \frac{1}{z_1 z_2} \int_0^\infty d\omega \alpha_1(\omega) \alpha_2(\omega) \\ \times \coth(\beta\omega/2) \sin \omega z_2 \cos \omega z_1 \left[ 8z^2 \right] \Big|_{z_1=z_2=z} \quad (2.37)$$

$$f_z^{C3} = -\frac{2}{\pi}z \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^3 \left( \frac{1}{z_1} \frac{d}{dz_1} \right) \frac{1}{z_1 z_2} \int_0^\infty d\omega \alpha_1(\omega) \alpha_2(\omega) \\ \times \coth(\beta\omega/2) \sin \omega z_2 \cos \omega z_1 \left[ 4z^2 \right] \Big|_{z_1=z_2=z} \quad (2.38)$$

$$f_z^{C4} = -\frac{2}{\pi}z \left( \frac{1}{z_2} \frac{d}{dz_2} \right)^2 \left( \frac{1}{z_1} \frac{d}{dz_1} \right) \frac{1}{z_1 z_2} \int_0^\infty d\omega \alpha_1(\omega) \alpha_2(\omega) \\ \times \coth(\beta\omega/2) \sin \omega z_2 \cos \omega z_1 \left[ 20 \right] \Big|_{z_1=z_2=z} \quad (2.39)$$

As the retarded Green's functions for the atoms characterizes the response of their dipole moments to an external field they play the role of the dynamic polarizability,  $\alpha$ . The result above is expressed in terms of the frequency dependent form,  $\alpha_a(\omega) =$

$q_a^2(4\pi\mu_a(\Omega_a^2 - \omega^2))^{-1}$  , which can be derived from the classical equations of motion (the aforementioned infinitesimal dissipative term kills an imaginary part in the infinite time limit).  $\beta$  is the field's inverse temperature. All the derivatives can be taken and the force can be expressed as an integral over frequency.

$$f_z^C = -\frac{2}{\pi z^7} \int_0^\infty d\omega \alpha_1(\omega)\alpha_2(\omega) \coth(\beta\omega/2) \left[ \omega z(18 - 8z^2\omega^2 + z^4\omega^4) \cos 2\omega z \right. \\ \left. + (-9 + 16\omega^2 z^2 - 3\omega^4 z^4) \sin 2\omega z \right] \quad (2.40)$$

This expression agrees with what can be found in the literature for the CP force in a finite temperature field [66]. In the far-field at zero temperature we recover the well known form [67]

$$f_z^C \approx -\frac{161}{4\pi} \alpha_1(0)\alpha_2(0) \frac{1}{z^8} \quad (2.41)$$

where a UV regulator must be employed to render the frequency integrals finite.

If however we take the dissipation to be zero (as it truly is in our perturbative approach) the force is altered because the polarizability acquires an imaginary part  $\alpha(\omega) \rightarrow (q^2/4\pi\mu\Omega)[\Omega/(\Omega^2 - \omega^2)^{-1} + i(\pi/2)\delta(\omega - \Omega) - i(\pi/2)\delta(\omega + \Omega)]$ . The imaginary term plays an important role when the quantum nature of the dipole moment of the oscillator is accounted for. When such a term is neglected, the contribution to the atom-atom force from  $f^A$  and  $f^B$  dominates in the far field as  $1/z^3$  rather than  $1/z^8$  at  $T = 0$ . We denote this contribution to the force by  $\delta f_z^C$

$$\delta f_z^C = -\frac{q_1^2 q_2^2}{16\pi^2 \mu_1 \mu_2 \Omega_1 \Omega_2 z^7} \int_0^\infty d\omega \coth(\beta\omega/2) \left[ \frac{\Omega_1 \delta(\omega - \Omega_2)}{\Omega_1^2 - \Omega_2^2} + (\Omega_1 \leftrightarrow \Omega_2) \right]$$

$$\times \left[ -9 - 2z^2\omega^2 - z^4\omega^4 + (9 - 16z^2\omega^2 + 3z^4\omega^4) \cos(2\omega z) + z\omega(18 - 8z^2\omega^2 + z^4\omega^4) \sin(2\omega z) \right] \quad (2.42)$$

where it should be noted that all expressions for the force are obtained by symmetrizing with respect to the microscopic variables of each atom. As  $z \rightarrow \infty$  this term possesses the asymptotic scaling,

$$\delta f_z^C \approx \frac{q_1^2 q_2^2}{16\pi^2 \mu_1 \mu_2 \Omega_1 \Omega_2 z^3} \int_0^\infty d\omega \omega^4 \coth(\beta\omega/2) \left[ \frac{\Omega_1 \delta(\omega - \Omega_2)}{\Omega_1^2 - \Omega_2^2} + (\Omega_1 \leftrightarrow \Omega_2) \right] \quad (2.43)$$

but at  $T = 0$  for  $z \rightarrow 0$   $\delta f^C$  is subleading to the dominant  $1/z^7$  near field scaling from the London term.

## 2.2.2 Intrinsic Fluctuation Force

The treatment by London of the atom-atom force can be reproduced by computing the interaction energy of two atoms interacting via the Coulomb potential. The force follows from the negative gradient of the perturbed energy eigenvalues. We obtain an analogous expression for the London force in our formulation but with an additional contribution from retardation effects as we treat the field relativistically.

The contributions to the force from  $f_z^A$  and  $f_z^B$  can be computed in the same way as the contribution  $f_z^C$ , and so we omit the details of that calculation here, and only state the result in the long-time zero temperature limit.

$$f_z^A = -\frac{q_1^2 q_2^2}{16\pi^2 \mu_1 \mu_2 \Omega_1 \Omega_2} \frac{\Omega_1}{\Omega_1^2 - \Omega_2^2} [9 + 2\Omega_2^2 z^2 + \Omega_2^4 z^4] \frac{1}{z^7} \quad (2.44)$$



$f_z^B$  can be obtained from  $f_z^A$  by exchanging  $\Omega_1$  and  $\Omega_2$ . These terms are responsible for the near-field behavior and agree with those derived by London when retardation corrections to the field Green's function are neglected [1].

$$f_z^A + f_z^B \approx f_z^{Lon} = -\frac{9q_1^2 q_2^2}{16\pi^2 \mu_1 \mu_2 \Omega_1 \Omega_2} \frac{1}{\Omega_1 + \Omega_2} \frac{1}{z^7} \quad (2.45)$$

The thermal version of the previous result does not make sense for a single oscillator where temperature is an ill-defined quantity, but does in the case of a gas of atoms. If the gas is sufficiently dilute the force between two collections of trapped atoms can be approximated using the density distribution of the gas and  $f_z$  [25]. The finite temperature form follows where  $\beta_a$  is the inverse temperature of the  $a$ th oscillator (or trapped gas).

$$f_z^A = -\frac{q_1^2 q_2^2}{16\pi^2 \mu_1 \mu_2 \Omega_1 \Omega_2} \frac{\Omega_1}{\Omega_1^2 - \Omega_2^2} [9 + 2\Omega_2^2 z^2 + \Omega_2^4 z^4] \coth(\beta_2 \Omega_2 / 2) \frac{1}{z^7} \quad (2.46)$$

In the far field the leading order behavior reduces to the following form.

$$f_z^A \approx -\frac{q_1^2 q_2^2}{16\pi^2 \mu_1 \mu_2} \frac{\Omega_2^3}{\Omega_1^2 - \Omega_2^2} \coth(\beta_2 \Omega_2 / 2) \frac{1}{z^3} \quad (2.47)$$

Note that when the field and the atoms are in thermal equilibrium this new asymptotic scaling cancels with an equal and opposite contribution contained in  $\delta f_z^C$  and the standard far field scaling  $1/z^8$  is restored. When the atoms and field are out of thermal equilibrium this cancelation no longer occurs and the dominant contribution to the force scales like  $1/z^3$ , where the zero temperature contribution cancels as indicated below:

$$f_z \approx -\frac{q_1^2 q_2^2}{8\pi^2 \mu_1 \mu_2} \frac{\Omega_2^3}{\Omega_1^2 - \Omega_2^2} \left[ \frac{1}{e^{\beta_2 \Omega_2} - 1} - \frac{1}{e^{\beta \Omega_2} - 1} \right] \frac{1}{z^3} + (\Omega_1 \leftrightarrow \Omega_2) \quad (2.48)$$

Our results can be shown to agree with those of Sherkunov [22] who studied the atom-atom force for systems out of thermal equilibrium.

### 2.3 Entanglement Force

The previous derivation of the atom-atom force assumes that the initial state of the two oscillators is uncorrelated. If however, the two atoms are initially entangled then a new contribution to atom-atom force arises. To our knowledge this force has not been reported in the literature.

We begin by computing the oscillator-reduced IF for two initially entangled atoms.

$$\begin{aligned} \mathcal{F}_Z[\vec{z}_1^-, \vec{z}_1^+] = \prod_a \left[ \int d\vec{Q}_a d\vec{Q}_{in,a} d\vec{Q}'_{in,a} \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}_a \int_{\vec{Q}'_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}'_a e^{i(S_Q[\vec{Q}_a] - S_Q[\vec{Q}'_a])} \right] \\ \times \rho_Q(\vec{Q}_{in,1}, \vec{Q}'_{in,1}, \vec{Q}_{in,2}, \vec{Q}'_{in,2}; t_{in}) \mathcal{F}[J^{\mu-}, J^{\nu+}] \end{aligned} \quad (2.49)$$

Writing (2.49) in a more suggestive form

$$\mathcal{F}_Z[\vec{z}_1^+, \vec{z}_1^-] = \mathcal{F} \left[ \vec{z}_a^+, \vec{z}_a^-; -i \frac{\delta}{\delta \vec{j}_a^+}, -i \frac{\delta}{\delta \vec{j}_a^-} \right] F[\vec{j}_1^+, \vec{j}_1^-, \vec{j}_2^+, \vec{j}_2^-] \Big|_{j_a^\pm=0} \quad (2.50)$$

defines the IF for two entangled harmonic oscillators,  $F[\vec{j}_1^+, \vec{j}_1^-, \vec{j}_2^+, \vec{j}_2^-]$ . To bring  $\mathcal{F}[J^{\mu-}, J^{\nu+}]$  out of the path integrals in (2.49) we replace the oscillator coordinates

with functional derivatives as before.

$$F[\vec{j}_1^+, \vec{j}_1^-, \vec{j}_2^+, \vec{j}_2^-] = \prod_a \int d\vec{Q}_a \int d\vec{Q}_{in,a} d\vec{Q}'_{in,a} \rho_Q(\vec{Q}_{in,1}, \vec{Q}'_{in,1}, \vec{Q}_{in,2}, \vec{Q}'_{in,2}; t_{in}) \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}_a \int_{\vec{Q}_{in,a}}^{\vec{Q}_a} \mathcal{D}\vec{Q}'_a e^{iS_Q[\vec{Q}_a] - iS_Q[\vec{Q}'_a] + i \int d\lambda [\vec{j}_a^+ \cdot \vec{Q}_a^- + \vec{j}_a^- \cdot \vec{Q}_a^+]} \quad (2.51)$$

For the initially entangled squeezed Gaussian state

$$\rho_Q(\vec{Q}_{in,1}, \vec{Q}'_{in,1}, \vec{Q}_{in,2}, \vec{Q}'_{in,2}; t_{in}) = \left(\frac{\beta}{\pi\alpha}\right)^6 \exp \left\{ -\frac{1}{4} \left[ \beta^2 \left( (\vec{Q}_{in,1} + \vec{Q}_{in,2})^2 + (\vec{Q}'_{in,1} + \vec{Q}'_{in,2})^2 \right) + \frac{1}{\alpha^2} \left( (\vec{Q}_{in,1} - \vec{Q}_{in,2})^2 + (\vec{Q}'_{in,1} - \vec{Q}'_{in,2})^2 \right) \right] \right\} \quad (2.52)$$

(2.51) can be evaluated exactly. For this case, like oscillator coordinate components are entangled together with equal magnitude in each direction i.e. the parameters  $\alpha$  and  $\beta$  are common to each component.

The influence functional for two entangled oscillators follows

$$F[\vec{j}_1^+, \vec{j}_1^-, \vec{j}_2^+, \vec{j}_2^-] = \tilde{\mathcal{N}} \exp \left\{ \frac{1}{8} \left( iB_\Delta g_\Delta + iB_\Sigma g_\Sigma - \alpha^2 B_\Delta^2 - \frac{1}{\beta^2} B_\Sigma^2 - \frac{1}{\alpha^2} g_\Delta^2 - \beta^2 g_\Sigma^2 + i\varphi \right) \right\} \quad (2.53)$$

where  $\tilde{\mathcal{N}}$  is a normalization constant and the definitions of  $B_a$ ,  $g_a$  and  $\varphi$  are defined below.

$$B_a = \int dt J_a^-(t) [\cot \Omega T \sin \Omega(t - t_i) + \csc \Omega T \sin \Omega(t_f - t)] \quad (2.54)$$

$$g_a = \frac{1}{\mu\Omega} \int dt J_a^-(t) \sin \Omega(t - t_i) \quad (2.55)$$

$$\begin{aligned} \varphi = \sum_a \left[ \frac{1}{\sin \Omega T} \int dt g_a J'_a(t) \sin \Omega(t_f - t) - \frac{1}{2} \mu \Omega g_a^2 \cot \Omega T \right. \\ \left. + \frac{i}{2} \int dt dt' [J_a(t) J_a(t') g_F^a(t, t') + J'_a(t) J'_a(t') g_D^a(t, t')] \right] \end{aligned} \quad (2.56)$$

The subscript  $\Sigma$  denotes  $C_\Sigma = C_1 + C_2$ , and the subscript  $\Delta$  denotes  $C_\Delta = C_1 - C_2$ . The kernels  $g_F^a$  and  $g_D^a$  are the Feynman and Dyson propagator for the free harmonic oscillator.

The entanglement force comes from the leading order contribution to  $\mathcal{F}_Z$  that depends upon the spatial separation between the atoms. Previously we needed to consider the square of  $S_{cross}$ . However, when the two atoms are entangled there exists nonvanishing cross correlation between their coordinates such that  $\langle S_{cross} \rangle_o \neq 0$ .

So in distinction to the previous section we have

$$\mathcal{F}_Z = e^{iS_{IF}[z^\pm]} \approx 1 + (S.E.T.) + i \langle S_{cross}[\vec{z}_a^\pm, \vec{Q}_a^\pm] \rangle_o + \mathcal{O}(z^{-2}) \quad (2.57)$$

where the force can be derived from

$$f_k^{ent}(\tau) \approx - \frac{\delta}{\delta z_1^{k-}(\tau)} \langle S_{cross}[\vec{z}_a^\pm, \vec{Q}_a^\pm] \rangle_o \Big|_{z^{k-}=0}. \quad (2.58)$$

Expanding  $S_{cross}$  for small  $z_1^{k-}$  we arrive at

$$\begin{aligned} S_{cross} \approx S_o + q_1 q_2 \int d\lambda d\lambda' z_1^{k-}(\lambda) [\partial_k \kappa_i^\mu \kappa_j^{\nu'} D_{\mu\nu}^{ret}(z_1^{\alpha+}(\lambda), z_2^\alpha(\lambda')) Q_1^{i+}(\lambda) Q_2^{j+}(\lambda') \\ + \frac{1}{4} \partial_{k'} \kappa_i^{\mu'} \kappa_j^{\nu} D_{\mu\nu}^{ret}(z_2^\alpha(\lambda), z_1^{\alpha+}(\lambda')) Q_1^{i-}(\lambda) Q_2^{j-}(\lambda') \\ + \frac{i}{2} \partial_k \kappa_i^\mu \kappa_j^{\nu'} D_{\mu\nu}^H(z_1^{\alpha+}(\lambda), z_2^\alpha(\lambda')) Q_1^{i+}(\lambda) Q_2^{j-}(\lambda')] \\ + \mathcal{O}(z^{-2}) \end{aligned} \quad (2.59)$$

where a prime in the index of a derivative operator means differentiation with respect to the second argument. Only one term survives after we take the expectation value, that which contains the cross correlator  $\langle Q_1^{i+}(\lambda)Q_2^{j+}(\lambda') \rangle_o$  which equals

$$\begin{aligned} & \langle Q_1^{i+}(t)Q_2^{j+}(t') \rangle_o = \delta^{ij} g_{ent}(t, t') \\ & = \frac{1}{4} \left( \frac{1}{\alpha^2} - \beta^2 \right) \delta^{ij} \left[ \frac{1}{(\mu\Omega)^2} \sin \Omega(t - t_i) \sin \Omega(t' - t_i) \right. \\ & \quad \left. - \frac{\alpha^2}{4\beta^2} \csc^2 \Omega T S(t) S(t') \right] \end{aligned} \quad (2.60)$$

where  $T = t_f - t_i$  and  $S(t) = \sin \Omega(t - t_f) - \sin \Omega(t - t_i + T)$ . After taking the expectation value and then using (2.58) we obtain the entanglement force.

$$f_k^{ent}(\tau) = -q_1 q_2 \int d\lambda g_{ent}(\tau, \lambda) \partial_k \kappa_i^\mu \kappa^{i\nu'} D_{\mu\nu}^{ret}(z_1^\alpha(\tau), z_2^\alpha(\lambda)) \quad (2.61)$$

All derivatives can be taken on the field's retarded Green's function and simplified using the equation of motion. We then specify the trajectory to be static to arrive at

$$\partial_k \kappa_i^\mu \kappa^{i\nu'} D_{\mu\nu}^{ret}(\sigma) = \delta_{kz} \left[ z^3 \left( \frac{d}{d\sigma} \right)^3 + 5z \left( \frac{d}{d\sigma} \right)^2 \right] D_{ret}(\sigma). \quad (2.62)$$

With a static trajectory specified the  $\sigma$  derivatives can be expressed in terms of  $z$  derivatives and can be factored out of the integral and the retarded Green's function can be expressed as a delta function  $D_{ret}(\sigma(z_1^\alpha(\tau), z_2^\alpha(\lambda))) = \delta(\tau - \lambda - z)/(4\pi z)$ . We work in a coordinate system centered at the  $a = 2$  atom, where various forces act upon (also the origin of the xy-plane), with  $z$  axis along the ray connecting the two atoms at distance  $z$  apart (pointing from atom 2 to atom 1). Thus the distinguished

atom ( $a = 1$ ) is located at  $(0, 0, z)$ . This leads to the explicit expression for the entanglement force.

$$f_z^{ent}(\tau) = \frac{q_1 q_2}{2\pi} \left[ z^3 \left( \frac{1}{z} \frac{d}{dz} \right)^3 + 5z \left( \frac{1}{z} \frac{d}{dz} \right)^2 \right] \frac{1}{z} g_{ent}(\tau, \tau - z) \quad (2.63)$$

In the infinite time limit  $\tau \rightarrow t_f \rightarrow \infty$   $g_{ent}(\tau, \tau - z) \sim \frac{1}{8}(\beta^2 - 1/\alpha^2)(\alpha^2/\beta^2 - 1/(\mu\Omega)^2) \cos \Omega z$ . The force vanishes in the far field but has a well-defined near field limit i.e.  $\Omega z \rightarrow 0$ .

$$f_z^{ent} \sim \frac{q_1 q_2}{32\pi} \left( \beta^2 - \frac{1}{\alpha^2} \right) \left( \frac{\alpha^2}{\beta^2} - \frac{1}{(\mu\Omega)^2} \right) \frac{\Omega^2}{z^2} \quad (2.64)$$

This effect is not only due to entanglement between the two atoms, but it is also due to retardation. For the case considered above where the degree of entanglement between like components of the atom's dipole moments has the same magnitude the interaction energy as described through the Coulomb potential vanishes. Thus, only through the inclusion of relativistic effects does any force manifest.

However, the previous discussion can easily be generalized to the case where the magnitude of the parameters  $\alpha$  and  $\beta$  is not common to all directions. As such the initial state for the oscillators takes the generalized form

$$\begin{aligned} \rho_Q(\vec{Q}_{in,1}, \vec{Q}'_{in,1}, \vec{Q}_{in,2}, \vec{Q}'_{in,2}; t_{in}) = \\ \prod_j \left( \frac{\beta_j}{\pi\alpha_j} \right)^2 \exp \left\{ -\frac{1}{4} \left[ \beta_j^2 \left( (Q_{in,1}^j + Q_{in,2}^j)^2 + (Q_{in,1}^{j'} + Q_{in,2}^{j'})^2 \right) \right. \right. \\ \left. \left. + \frac{1}{\alpha_j^2} \left( (Q_{in,1}^j - Q_{in,2}^j)^2 + (Q_{in,1}^{j'} - Q_{in,2}^{j'})^2 \right) \right] \right\}. \end{aligned} \quad (2.65)$$

The development follows closely that given for the previous case and so we only state the result in the near-field long-time limit,

$$f_z^{ent} \approx -\frac{3}{4\pi} q_1 q_2 \left[ \Delta_x + \Delta_y - 2\Delta_z \right] \frac{1}{z^4} \quad (2.66)$$

where we have used the shorthand  $\Delta_j = \frac{1}{8}(\beta_j^2 - 1/\alpha_j^2)(\alpha_j^2/\beta_j^2 - 1/(\mu\Omega)^2)$ . Note that if the parameters  $\alpha$  and  $\beta$  are equal for all directions (2.66) vanishes. The sign of the force can also be changed by the appropriate choice of the squeeze parameters  $\alpha$  and  $\beta$ .

### 2.3.1 Entanglement Force for Sub- and Superradiant States

In this section we consider the entanglement force for realistic states, working under the stationary and near field approximation. Specifically consider the maximally entangled state between two two-level atoms labeled  $A$  and  $B$   $|\psi_{ENT}\rangle$

$$|\psi_{ENT}\rangle = \frac{1}{\sqrt{2}}(|g\rangle_A |e\rangle_B \pm |e\rangle_A |g\rangle_B), \quad (2.67)$$

where  $|g\rangle$  stands for the ground and  $|e\rangle$  an excited state of the a two-level atom. For our specific case involving two entangled three dimensional harmonic oscillators we approximate the state (2.67) by choosing  $|e\rangle$  to be the state with angular momentum quantum number  $l = 1$  and the azimuthal ( $z$ -component of) angular momentum (magnetic) quantum number  $m = 0$ . In a Fock space representation  $|e\rangle$  can be written as  $|n_x = 0, n_y = 0, n_z = 1\rangle$  where  $n_i$  label the number of quanta for the  $i$ th degree of freedom. For our system we have the state

$$|\psi_{ENT}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |l=1, m=0\rangle_B \pm |l=1, m=0\rangle_A |0\rangle_B). \quad (2.68)$$

One can easily verify the stated properties of (2.68) i.e.  $L^2 |\psi_{ENT}\rangle = 2 |\psi_{ENT}\rangle$  and  $L_z |\psi_{ENT}\rangle = 0$  where  $L^2 = L_A^2 + L_B^2$  is the operator for the Casimir (total angular momentum squared) operator and  $L_z$  is the azimuthal angular momentum. In order to distinguish among the degenerate  $l = 1$  states one could apply an external magnetic field.

In the near field we neglect retarded corrections to the electromagnetic potential between the two atoms. For such a case the interaction between two dipoles separated by  $\vec{R}$  can be written classically as

$$U_{int} = \frac{1}{4\pi\epsilon_0 R^3} [\vec{p}_A \cdot \vec{p}_B - 3(\vec{p}_A \cdot \hat{R})(\vec{p}_B \cdot \hat{R})]. \quad (2.69)$$

If we now consider two quantum dipoles entangled as described by (2.68) we can compute their interaction energy at lowest order in perturbation theory by taking the expectation value of  $U_{int}$ , with respect to (2.68), where the classical dipole moments  $\vec{p}_A$  and  $\vec{p}_B$  are replaced by the dipole moment operators for the two oscillators.

$$\langle U_{int} \rangle = \frac{q^2}{4\pi\epsilon_0 R^3} [\delta_{ij} - 3\hat{R}_i \hat{R}_j] \langle \psi_{ENT} | Q_A^i Q_B^j | \psi_{ENT} \rangle \quad (2.70)$$

The expectation value can be computed,

$$\langle U_{int} \rangle = \pm \frac{q^2}{4\pi\epsilon_0 R^3} [\delta_{ij} - 3\hat{R}_i \hat{R}_j] \frac{\delta^{iz} \delta^{jz}}{2\mu\Omega} \quad (2.71)$$

where we have assumed the two atoms to be the same species.



The result for the interaction energy follows when the atoms are separated in the  $z$ -direction,

$$\langle U_{int} \rangle = \mp \frac{q^2}{4\pi\epsilon_0\mu\Omega R^3} \quad (2.72)$$

and can be obtained when for the case when the atoms are separated in the  $x$  or  $y$ -direction by multiplying by  $-1/2$ . We can obtain the entanglement force by taking minus the spatial gradient of the interaction energy.

$$f^{ent} = \mp \frac{3q^2}{4\pi\epsilon_0\mu\Omega R^4} \quad (2.73)$$

## 2.4 Possibility of Detection

### 2.4.1 Atom and Field Out of Thermal Equilibrium

In this section we compute the relative magnitude for the atom-atom force when the field and atoms are not in thermal equilibrium to the force at zero temperature. We focus our attention on the case where the atom's are in their ground state and the field is in a thermal state of inverse temperature  $\beta$ . Measuring this new asymptotic scaling requires a balance between temperature and the first optical resonance of the atomic species used. For the case when  $\Omega\beta \gg 1$  (2.48) is exponentially suppressed, this would rule out the use of heavier atoms like Rb near room temperature, the only hope is to work in the regime where  $\beta\Omega > 1$ , not only to prevent suppression by the Planck factor but also to prevent the excitation of the atom so that measurement can be done before thermalization.

The relative magnitude of (2.48) to  $f_C$  in the far field shows when this new scaling will dominate. For realistic experiments, atom-atom distance of the order of  $\mu m$ , the high temperature limit is beyond access for the temperatures and atomic species we are considering, so we replace  $\delta f^C$  with its zero temperature form (also the appropriate factors of  $c$  have been restored to ensure that (2.74) is dimensionless).

$$\frac{\delta f^C}{f^C} = -\frac{8\pi}{161c^5} \frac{\Omega_1^2 \Omega_2^5}{\Omega_1^2 - \Omega_2^2} z^5 \frac{1}{e^{\beta\Omega_2} - 1} + (\Omega_1 \leftrightarrow \Omega_2) \quad (2.74)$$

If the atomic species are the same the previous expression reduces to

$$\frac{\delta f^C}{f^C} = \frac{24\pi}{322} \left(\frac{\Omega z}{c}\right)^5 \frac{1}{e^{\beta\Omega} - 1}. \quad (2.75)$$

Tuning  $\Omega$  to hydrogen's first optical resonance ( $\Omega \approx 10eV$ ,  $\Omega \approx 2.4 \times 10^{15} Hz$ , or  $\Omega \approx 116,000K$ ) we find

$$\frac{\delta f^C}{f^C} \approx \frac{24\pi}{322} \left(\frac{8z}{\mu m}\right)^5 \frac{1}{e^{\beta\Omega} - 1}. \quad (2.76)$$

If the atomic species are different we find different behavior. Particularly, when one of the atom's first optical resonance is very large such that  $\beta\Omega \gg 1$  (like Rb near room temperature) the Planck factor for that atom will be strongly suppressed so its contribution to the force can be ignored. In such a case (2.74) takes the form

$$\frac{\delta f^C}{f^C} = -\frac{8\pi}{161} \left(\frac{\Omega z}{c}\right)^5 \frac{1}{e^{\beta\Omega} - 1} = -\frac{8\pi}{161} \left(\frac{8z}{\mu m}\right)^5 \frac{1}{e^{\beta\Omega} - 1} \quad (2.77)$$

where we have a different sign and a slightly different coefficient.

For atoms  $\delta f^C / f^C$  only becomes significantly greater than 1 for large distances and very high temperatures and so is unlikely observable. However, these effects

may play a role in the laboratory for molecules with sub  $eV$  excitation energies. We leave a study of those effects for a later work.

## 2.4.2 Entanglement Force

Now that we have an expression for the entanglement force at short distances we check for regimes in which (2.58) will dominate. To do this we take the ratio of the entanglement force to the near-field van der Waals force. After restoring all physical constants to yield the correct dimensions and allowing both atoms to be the same species we find

$$\frac{f^{ent}}{f_{Lon}} = \frac{4\epsilon_o \mu\Omega^4}{9c^2 q^2} \left( \tilde{\beta}^2 - \frac{1}{\tilde{\alpha}^2} \right) \left( \frac{\tilde{\alpha}^2}{\tilde{\beta}^2} - 1 \right) z^5. \quad (2.78)$$

Above  $\tilde{\beta}^2 = \beta^2/\mu\Omega$  and  $\tilde{\alpha}^2 = \mu\Omega\alpha^2$ .

By tuning the frequency to the first optical resonance of Hydrogen, taking the reduced mass to be the electron mass and  $q$  to be the electronic charge we find

$$\frac{f^{ent}}{f_{Lon}} \approx 8.9 \times \left( \tilde{\beta}^2 - \frac{1}{\tilde{\alpha}^2} \right) \left( \frac{\tilde{\alpha}^2}{\tilde{\beta}^2} - 1 \right) \left( \frac{z}{nm} \right)^5. \quad (2.79)$$

The near field condition requires that the distance between the atoms be much smaller than the wavelength associated with their first optical resonance. For hydrogen this wavelength is  $\lambda = c/\Omega = 122nm$ . So, for the case where the prefactor of (2.79) is order unity and the interatomic distances are in the range of a few nanometers we find the entanglement force dominates over the standard London form.

For atoms separated in the  $z$ -direction the relative magnitude of the entanglement to the London force for sub- or superradiant states gives

$$\frac{f^{ent}}{f^{Lon}} = \pm \frac{8\pi\epsilon_0}{3q^2} \mu\Omega^2 z^3 \approx \pm 15.8 \left( \frac{z}{nm} \right)^3. \quad (2.80)$$

One can see that for separations of a few nanometers, such that the near field condition is still well-satisfied, the entanglement force dominates (at 5 nm by 3 orders of magnitude).

## Chapter 3

### Atom-Surface Forces: Surface as a Boundary Condition on the Quantum Field

In this chapter we begin the study of atom-surface forces focusing on the idealized case when the surface is a perfect reflector. For such a situation the most efficient means for deriving the atom-surface interaction is to apply quantum field theory under the influence of external conditions [30]. In this scheme one accounts for the presence of ideal reflecting bodies by forcing the transverse components of the field to satisfy Dirichlet boundary conditions on the mirror's surface.

#### 3.1 The Atom and Its Trajectory in a Quantum Field with Boundary

As with Chapter 1. we model the internal degrees of freedom of the atom by a (3-dimensional) harmonic oscillator with natural frequency  $\Omega$ . The atom moves on a trajectory  $\vec{z}(t)$  in a quantum field  $A^\mu$ , the electromagnetic vector potential, in the presence of a mirror, here modeled through boundary conditions on the field. The action follows from Chapter 1. with the exception that we reduce the number of atoms to one. The total action is given by

$$S[Q, A^\mu, \vec{z}] = S_Q[\vec{Q}] + S_E[A^\mu] + S_Z[\vec{z}] + S_{int}[Q, A^\mu, \vec{z}] \quad (3.1)$$

(subscript  $E$  stands for the electromagnetic field which serves as an environment)  
with the action for the oscillator given by

$$S_Q[\vec{Q}] = \frac{\mu}{2} \int d\lambda [\dot{\vec{Q}}(\lambda)^2 - \Omega^2 \vec{Q}(\lambda)^2] \quad (3.2)$$

where  $\mu$  is the oscillator's reduced mass and  $\lambda$  parameterizes its worldline. The photon field action is given by

$$S_E[A_c^\mu] = -\frac{1}{4} \int d^4x F_{\mu\nu}^c F_c^{\mu\nu} \quad (3.3)$$

where  $F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c$ ,  $c$  standing for constrained, is the field strength tensor for the constrained electromagnetic field. The action for the motion of the atom's center of mass is

$$S_Z[\vec{z}] = \int d\lambda \left[ \frac{1}{2} M \dot{\vec{z}}^2(\lambda) - V[\vec{z}] \right] \quad (3.4)$$

where  $M$  is the total mass of the atom and  $V[\vec{z}]$  is an external potential.

In the dipole approximation, the Hamiltonian for an atom interacting with the photon field takes the form  $-\vec{d} \cdot \vec{E}$  where  $\vec{d}$  is the dipole moment of the atom and  $\vec{E}$  is the electric field. In this spirit we define the interaction action

$$S_{int}[\vec{Q}, \vec{z}, A_\mu] = q \int d\lambda Q^k(\lambda) E_k[z^\mu(\lambda)]. \quad (3.5)$$

### 3.2 World Line Influence Functional

Assume that at time  $t_{in}$  the quantum statistical state of the oscillator, trajectory and field is described by a density operator  $\hat{\rho}(t_{in})$ . This state is unitarily evolved from the initial time  $t_{in}$  to a later time  $t_f > t_{in}$ , and can be expressed in terms of path integrals by considering matrix elements in an appropriate basis. The overall influence of the field on the dynamics of the atom is obtained by coarse-graining over the field variables resulting in the field-reduced density matrix [64],

$$\begin{aligned} \rho_r(\vec{Q}_f, \vec{Q}'_f; \vec{z}_f, \vec{z}'_f; t_f) = & \int d\vec{Q}_{in} d\vec{Q}'_{in} \int d\vec{z}_{in} d\vec{z}'_{in} \int_{\vec{Q}_{in}}^{\vec{Q}_f} \mathcal{D}\vec{Q} \int_{\vec{Q}'_{in}}^{\vec{Q}'_f} \mathcal{D}\vec{Q}' \int_{\vec{z}_{in}, \vec{z}'_{in}}^{\vec{z}_f, \vec{z}'_f} \mathcal{D}\vec{z} \mathcal{D}\vec{z}' \\ & \times e^{i(S_Q[\vec{Q}] + S_Z[\vec{z}] - S_Q[\vec{Q}'] - S_Z[\vec{z}'])} \rho_Q(\vec{Q}_{in}, \vec{Q}'_{in}; t_{in}) \rho_Z(\vec{z}_{in}, \vec{z}'_{in}; t_{in}) \mathcal{F}[J^{\mu-}, J^{\nu+}], \end{aligned} \quad (3.6)$$

where  $\mathcal{D}k$  is the measure for a path integral over the space of functions. This introduces the IF  $\mathcal{F}[J^{\mu-}, J^{\nu+}]$  [63] for the constrained electromagnetic field.

For the coupling given above and assuming an initially uncorrelated and Gaussian state the influence functional can be calculated exactly and is given by

$$\begin{aligned} \mathcal{F}[J^{\mu-}, J^{\nu+}] = & \exp \left\{ i \int d^4y J^{\mu-}(y) \right. \\ & \left. \times \int d^4y' [D_{\mu\nu}^{ret,c}(y, y') J^{\nu+}(y') + \frac{i}{4} D_{\mu\nu}^{H,c}(y, y') J^{\nu-}(y')] \right\} \end{aligned} \quad (3.7)$$

where the superscript  $c$  denotes that the field's Green's functions satisfy the appropriate boundary conditions on the mirror surface.

To find the combined influence that the oscillator and the field have on the trajectory we continue by coarse graining over oscillator degrees of freedom resulting

in the oscillator-reduced influence functional (ORIF),  $\mathcal{F}_Z[\vec{z}^-, \vec{z}^+]$ .

$$\begin{aligned} \mathcal{F}_Z[\vec{z}^-, \vec{z}^+] &= \int d\vec{Q}_f d\vec{Q}_{in} d\vec{Q}'_{in} \int_{\vec{Q}_{in}, \vec{Q}'_{in}}^{\vec{Q}_f, \vec{Q}'_f} \mathcal{D}\vec{Q} \mathcal{D}\vec{Q}' \\ &\times e^{i(S_Q[\vec{Q}] - S_Q[\vec{Q}'])} \rho_Q(\vec{Q}_{in}, \vec{Q}'_{in}; t_{in}) \mathcal{F}[J^{\mu-}, J^{\nu+}] \end{aligned} \quad (3.8)$$

We cannot trace over the oscillator variables in (3.8) explicitly for arbitrary field boundary conditions, such as in the presence of a mirror. This would require solving the semi-classical equations of motion for the internal degrees of freedom which contain a third time derivative of the oscillator coordinate due to radiation reaction, and generally exhibit non-Markovian behavior due to reflections of the field from surfaces. To proceed we exploit the fact that the coupling of the dipole moment of the oscillator to field is small (scaling as fine structure constant in the influence action) and evaluate the oscillator-reduced IF perturbatively via an expansion in powers of the charge. Writing (3.8) in a more suggestive form  $\mathcal{F}_Z[\vec{z}^+, \vec{z}^{\mu-}] = \exp\{iS_{inf}[z^{\mu+}, z^{\mu-}; -i\frac{\delta}{\delta j_k^+}, -i\frac{\delta}{\delta j_l^-}]\} f_o[\vec{j}^+, \vec{j}^-]|_{j^\pm=0}$ , which defines the influence action,  $S_{inf}[z^{\mu+}, z^{\nu-}; Q_j^-, Q_k^+] = -i \ln \mathcal{F}[J^{\mu+}, J^{\nu-}]$ , and the IF for a three dimensional harmonic oscillator,  $f_o[\vec{j}^+, \vec{j}^-]$  where the exponent has been factored out of the path integral by replacing  $[Q^{k\pm}(\lambda)]^n$  with  $\left(-i\frac{\delta}{\delta j_k^\mp(\lambda)}\right)^n f_o[\vec{j}^+, \vec{j}^-]|_{j^\pm=0}$ . For a Gaussian initial state  $f_o[\vec{j}^+, \vec{j}^-]$  can be evaluated exactly  $f_o[\vec{j}^+, \vec{j}^-] = \mathcal{N} \exp\{i \int d\lambda d\lambda' [\vec{j}^-(\lambda) \cdot \vec{j}^+(\lambda') g_{ret}(\lambda, \lambda') + \frac{i}{4} \vec{j}^-(\lambda) \cdot \vec{j}^-(\lambda') g_H(\lambda, \lambda')]\}$  where  $g_{ret}(\lambda, \lambda')$  and  $g_H(\lambda, \lambda')$  are the retarded and Hadamard Green's functions for a one dimensional harmonic oscillator with natural frequency  $\Omega$ ,  $\mathcal{N}$  is a normalization constant, and the dot product is taken with respect to a 3 dimensional Euclidean



metric.

Expanding (3.8) to lowest order in weak coupling we obtain the influence functional as  $-i \ln \mathcal{F}_Z[\vec{z}^+, \vec{z}^-] \approx S_{inf}[z^{\mu+}, z^{\mu-}; -i \frac{\delta}{\delta j_k^+}, -i \frac{\delta}{\delta j_l^-}] f_o[\vec{j}^+, \vec{j}^-]|_{j^\pm=0}$  from which we get the reduced density matrix describing the center of mass motion as

$$\begin{aligned} \rho_r(\vec{z}_f^+, \vec{z}_f^-; t_f) &= \int d\vec{z}_{in}^+ d\vec{z}_{in}^- \int_{\vec{z}_{in}^+, \vec{z}_{in}^-}^{\vec{z}_f^+, \vec{z}_f^-} \mathcal{D}\vec{z}^+ \mathcal{D}\vec{z}^- \\ &\times e^{i(S_Z[\vec{z}] - S_Z[\vec{z}'])} \rho_Z(\vec{z}_{in}^+, \vec{z}_{in}^-; t_{in}) \mathcal{F}_Z[\vec{z}^+, \vec{z}^-]. \end{aligned} \quad (3.9)$$

### 3.3 Atom's Mean Trajectory

The complex norm of the ORIF,  $|\rho_r| \propto \exp\{-\int d\lambda d\lambda' z^{k-}(\lambda) N_{kj}(\lambda, \lambda') z^{j-}(\lambda')\}$  is non-vanishing and strongly suppressed for large values of the off diagonal elements,  $\vec{z}^- = \vec{z} - \vec{z}'$ , as is indicative of decoherence of the quantum trajectory.  $N_{kj}$  is a symmetric positive definite kernel quantifying the noise in the oscillator and field.

Decoherence of the system due to its interactions with the quantum fluctuations of the environment and oscillator permits the existence of a semi-classical limit for the oscillator's path through space. Using a saddle-point approximation to evaluate (3.9) about its classical solution,  $z_{cl}^k(\lambda) \equiv \bar{z}^k$ , one can show that the semi-classical dynamics is determined from the variation  $\delta S_{CGEA}[z^{k+}, z^{k-}]/\delta z^{j-}(\tau)|_{z^{k-}=0} = 0$  where the so-called coarse grained effective action is given by  $S_{CGEA}[z^{k+}, z^{k-}] = S_Z[\vec{z}] - S_Z[\vec{z}'] - i \ln \mathcal{F}_Z[\vec{z}^+, \vec{z}^-]$ .

Varying  $S_{CGEA}$  with respect to  $\vec{z}^-$  we obtain the mean (semi-classical) equation of motion [65]

$$M\ddot{z}_k(\tau) + \partial_k V[\vec{z}(\tau)] = f_k(\tau) \quad (3.10)$$

where the effective force,  $f_k(\tau)$  (including back-action effects), has the form

$$f_k(\tau) = \frac{q^2}{2} \int_{\lambda_{in}}^{\lambda_f} d\lambda \int_{\lambda_{in}}^{\lambda_f} d\lambda' \delta^{ij} \delta(\lambda - \tau) \partial_k \kappa_i^\alpha \kappa_{j'}^\beta \\ \times \left\{ g_H(\lambda, \lambda') D_{\alpha\beta}^{ret,c}(z^\alpha(\lambda), z^\alpha(\lambda')) + g_{ret}(\lambda, \lambda') D_{\alpha\beta}^{H,c}(z^\alpha(\lambda), z^\alpha(\lambda')) \right\} \quad (3.11)$$

where  $\kappa_j^\mu = \partial_j \eta_0^\mu - \partial_0 \eta_j^\mu$ . Take caution to evaluate the derivatives before the particle trajectory is placed into the various kernels.

The influence or backaction force on the oscillator trajectory describes dissipation and radiation reaction as well as the forces due to constraints on the field. The first two effects must be taken into account when atom motion comes into play. In the following we assume an appropriate form for  $V[\vec{z}]$  so that (3.10) admits static solutions where dissipative effects may be ignored.

### 3.4 Casimir-Polder Force

The placement of an ideal conductor in the  $z = 0$  plane constrains the transverse components of the electric field at all frequencies to vanish there, and will lead to forces on the atom.

More realistic materials could be modeled by imbuing the  $z = 0$  plane with microphysical degrees of freedom (as we'll do in the next chapter), rather than applying a boundary condition on the field. The microscopic elements of the medium will affect the atom's motion through their mutual influence and their fluctuations will induce fluctuations in the field similar to MQED. However, in distinction to MQED

where the fluctuation dissipation relation is used to define the material fluctuations, it is manifest in our approach where the fluctuations and dissipation arise from the microphysical interaction of the medium with the electromagnetic field. They are valid for longer range interactions such as at low temperatures and non-stationary dynamics where MQED's generalization to nonequilibrium conditions may fail.

The ideal conducting boundary condition can be accommodated by appealing to the method of images. Thus to enforce Dirichlet boundary conditions on the transverse components of the electric field on the mirror surface a dipole in the space ( $z > 0$ ) is accompanied by an image on the other side of the mirror correlated on constant time slices. A classical electrostatic treatment for a permanent dipole near a mirror gives the expected  $1/z^4$  dependence when the effects of retardation are ignored. When finite light propagation time and quantum fluctuations of the electromagnetic field are accounted for this attractive force takes a modified form,  $1/z^5$ , in the far field limit where the distance from the mirror is much greater than the period of the oscillator ( $c=1$ ). These effects are known collectively as the Casimir-Polder force.

From the Green's function point of view the field constraint can be satisfied by pairing every Green's function with an image term i.e.  $G(\sigma) \rightarrow G(\sigma) - G(\tilde{\sigma})$  (only for the case of a scalar field) where  $\sigma(x, x')$  is Synge's worldfunction defined to be half the geodesic distance between  $x$  and  $x'$  and  $\tilde{\sigma}(x, x') = \sigma(x, x') + 2zz'$ . The image terms added to the field's Greens function is how the atom knows about the presence of the mirror and so can be isolated as the source of the CP force  $F_k^{CP}$  in our formulation giving

$$F_k^{CP}(\tau) = \frac{q^2}{2} \int_0^{\tau-\lambda_i} ds \delta^{ij} \partial_k \kappa_i^\alpha \kappa_{j'}^\beta \left\{ g_H(s) \tilde{D}_{\alpha\beta}^{ret}(\tilde{\sigma}[z^\alpha(\tau), z^\alpha(\tau-s)]) \right. \\ \left. + g_{ret}(s) \tilde{D}_{\alpha\beta}^H(\tilde{\sigma}[z^\alpha(\tau), z^\alpha(\tau-s)]) \right\} \quad (3.12)$$

where a prime denotes differentiation with respect to the second argument and  $\tilde{D}_{\mu\nu}$  is the image term added to the electromagnetic field's Green's functions. As opposed to a simple scalar field the tensor structure of the image term in the case of the electromagnetic field must be altered in order to accommodate the boundary conditions,  $\tilde{D}_{\alpha\beta}(\tilde{\sigma}) = -(\eta_{\alpha\beta} - 2\hat{z}_\alpha\hat{z}_\beta)G(\tilde{\sigma})$  which can be written here in terms of the Green's functions for a massless scalar and  $\hat{z}_\alpha = (0, 0, 0, 1)$ .

The effective force from the image then takes on the general form

$$F_k^{CP}(\tau) = \frac{q^2}{2m\Omega} \eta^{\mu\nu} \int_0^{\tau-\lambda_i} ds \left[ \tilde{\sigma}_k \tilde{\sigma}_\mu \tilde{\sigma}_{\nu'} \left( \frac{d}{d\tilde{\sigma}} \right)^3 + (\tilde{\sigma}_{\mu\nu'} \tilde{\sigma}_k + \tilde{\sigma}_{\mu k} \tilde{\sigma}_{\nu'} + \tilde{\sigma}_{k\nu'} \tilde{\sigma}_\mu) \left( \frac{d}{d\tilde{\sigma}} \right)^2 \right] \\ \times \left[ \cos \Omega s G_{ret}(\tilde{\sigma}) + \sin \Omega s G_H(\tilde{\sigma}) \right] \quad (3.13)$$

where  $d/d\tilde{\sigma}$  operates only on the Green's functions for the field and  $\tilde{\sigma}_k = \partial_k \tilde{\sigma}$ . To find an explicit expression for the Casimir-Polder force we evaluate (3.13) for a static trajectory,  $z^\mu(\tau) = (\tau, \vec{z})$  and  $\dot{z}^\mu(\tau) = (1, \vec{0})$ . We find an analytic expression for the CP force in the long time limit when the field has dressed the atomic ground state and the field and oscillator were initially in their respective ground states. The CP-force has two contributions  $F_k^{CP}(\tau) = F_k^{CP1}(\tau) + F_k^{CP2}(\tau)$ .

The first arising the *intrinsic fluctuations* of the atom's dipole moment

$$F_k^{CP1}(\tau) = \frac{q^2 \eta_{kz} z}{2^7 \pi m \Omega} \theta(\tau - 2z) \left\{ z^2, \left( \frac{1}{z} \frac{d}{dz} \right)^3 \right\} \frac{\cos 2\Omega z}{z} \quad (3.14)$$

and the second from *induced fluctuations*

$$F_k^{CP2} = -\frac{q^2}{32\pi^2 m \Omega z^5} \eta_{kz} \left[ 8\Omega z + 6(1 - 2\Omega^2 z^2) f(2\Omega z) - 4\Omega z (2\Omega^2 z^2 - 3) g(2\Omega z) \right] \quad (3.15)$$

where  $f(x) = Ci(x) \sin x - Si(x) \cos x$ ,  $Ci(x)$  ( $Si(x)$ ) are the cosine (sine) integral functions,  $g(x) = -(d/dx)f(x)$ , and  $\{A, B\}$  is the anticommutator of  $A$  and  $B$ .

The interpretation for each component can be taken from the kernels contained in each. Here  $F^{CP1}$  is derived from the term containing the retarded Green's function for the field and so arises from classical electrodynamics being sourced by quantum dipoles as made evident by the appearance of the oscillator's Hadamard function which characterizes the atom's dipole moment fluctuations.

Because  $F^{CP2}$  contains the Hadamard function for the field its physical origin can be traced to the quantum fluctuations of the electromagnetic field. For interactions linear in the oscillator coordinate as assumed here, in perturbation theory the quantum amplitude to go from the ground state to any but the first excited state vanishes. Thus the agreement of the present HO results with previous results for two level atoms using energy gradient methods [66] is not surprising.

In the near and far field limits we recover the asymptotic expressions

$$\Omega z \ll 1 \quad F_z^{CP} \approx -\frac{3q^2}{32\pi m \Omega z^4} \quad (3.16)$$

$$\Omega z \gg 1 \quad F_z^{CP} \approx -\frac{3q^2}{8\pi^2 m \Omega^2 z^5} \quad (3.17)$$

where our results agree with the literature if we identify the static polarizability,  $\alpha$ , with  $q^2/4\pi m \Omega^2$ , this form for  $\alpha$  can be argued by examining the static solutions to the classical equations of motion.

### 3.5 Thermal CP force

The form of the CP force in a thermal field can be taken directly from (3.12) with all Green's functions replaced with their appropriate finite temperature version. The assumption of an initially factorized density matrix allows us to independently choose the initial oscillator and field state. Choosing the oscillator and field to be in thermal states of different temperature (with inverse temperatures  $\bar{\beta}, \beta$  respectively) gives rise to two distinct thermal contributions to the CP force.

Because the retarded Green's functions appearing in (3.12) are proportional to the commutator of the field or oscillator position operators (they are c-numbers) they are independent of the quantum state and so will not contribute to the thermal effects. Modifications due to an initially thermal state will arise from the Hadamard functions only. The thermal Hadamard function for the field can be found by imposing a periodicity condition on imaginary time [80]. For a harmonic oscillator the thermal Hadamard function can be calculated directly giving  $g_H^{\bar{\beta}}(\tau, \tau') = \coth(\bar{\beta}\Omega/2) \cos \Omega(\tau - \tau')/m\Omega$ . This gives

$$F_k^{CP\bar{\beta}1} = \coth(\bar{\beta}\Omega/2) F_k^{CP1} \quad (3.18)$$

$$F_k^{CP\beta 2}(\tau) = \frac{q^2}{2m\Omega} \sum_{k=-\infty}^{\infty} P.V. \int_0^{\tau-\lambda_i} ds \sin \Omega s \partial_k \partial^\nu \partial_{\nu'} G_H(t + ik\beta(\vec{z}), \vec{z}, \vec{z}') \quad (3.19)$$

where  $\tilde{z} = (t, x, y, -z)$ . We have included the generalized case of a field state of spatially nonuniform temperature i.e.  $\beta \rightarrow \beta(\vec{x})$  as it is Gaussian in field variables.

In the high temperature, long time and far field limit we find the asymptotic form

$$F_k^{CP\beta 2} \approx -\frac{3q^2\eta_{kz}}{16\pi\beta m\Omega^2} \frac{1}{z^4} = -\frac{3}{4}\eta_{kz} \frac{\alpha}{\beta z^4}. \quad (3.20)$$

The finite temperature force, where the atom and field are defined to have the same temperature, can be evaluated in the long-time limit and expressed in terms of hypergeometric functions and is plotted for  $\Omega = 1$  in Figure 3.5.

### 3.6 Stochastic trajectory

The IF can produce a Langevin equation for the trajectory with deviations from the mean caused by the quantum fluctuations of the internal degrees of freedom of the oscillator and the electromagnetic field. It is given by

$$M\ddot{\tilde{z}}_k + \partial_\alpha \partial_k V[\tilde{z}_i] \tilde{z}^\alpha - \partial_\alpha f_k[\tilde{z}_i] \tilde{z}^\alpha = \xi_k[\tilde{z}_i] \quad (3.21)$$

where  $\xi_k$  is a classical stochastic forcing term responsible for driving the trajectory from its mean value. The key link in identifying a classical stochastic source (noise) from a quantum field is provided by the Feynman-Vernon identity for Gaussian integrals [63]. The two-point function for this classical stochastic source is related

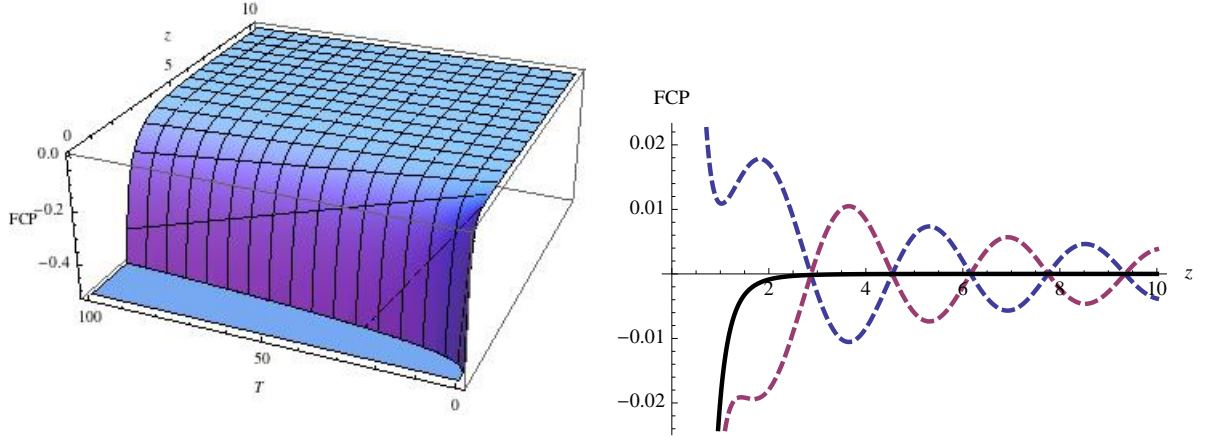


Figure 3.1: Plots of the CP force in units of  $\hbar q^2/m$ , for  $\Omega = 1$ , in the long time limit against perpendicular distance  $z$  (in units  $c/\Omega$ ) of the atom from a plane mirror. *On the right:* The CP force at finite temperature  $T$  (in  $(\hbar\Omega/k_B)K$ ). *On the left:* Plot of the CP force at zero temperature. The dashed blue line represents the contribution to the force from field fluctuations  $F^{CP2}$  while the pink dashed line shows the contribution arising from the intrinsic fluctuations of the atom's dipole moment  $F^{CP1}$ . The sum of these two contributions gives a monotonic attractive force.



to the noise kernel by

$$\langle \{\xi_k[z^\alpha(\lambda)], \xi_j[z^\alpha(\lambda')]\} \rangle = \frac{q^2}{2} \delta^{mn} g_H(\lambda, \lambda') \partial_k \partial_{j'} \kappa_m^\alpha \kappa_n^{\beta'} D_{\alpha\beta}^H[z^\alpha(\lambda), z^\alpha(\lambda')]. \quad (3.22)$$

The Langevin equation enables us to calculate the dispersion of the atom's trajectory,  $\langle \Delta \tilde{z}^2(\tau) \rangle$ , which is defined as the effective distance (squared) from the mean value that an ensemble of stochastic realizations takes. As the noise kernel contains the Hadamard function for the field it is sensitive to the boundary condition at  $z = 0$ . The image term present will make the distribution of noise vary with the distance from the mirror and in turn the dispersion in the atoms' positions as well. This manifests as a fractional change in volume of a gas of noninteracting atoms. If we trap the atoms in a harmonic potential with frequency  $\Omega_k$  in the  $k$ th direction, such that  $|\Omega_k^2 - \Omega^2| \gg q^2/m\Omega^3 M z^6$  then the dissipation can be ignored in the final expression for the dispersion and we can directly compute the far-field long-time limit. The dispersion in the  $z$ -direction is given by

$$\langle \delta \tilde{z}^2 \rangle_\xi \approx -\frac{15q^2}{16\pi^2 m \Omega M^2} \frac{1}{(\tilde{\Omega}_z^2 - \Omega^2)^2} \frac{1}{z^6} \quad (3.23)$$

where  $\tilde{\Omega}_z$  is the trapping potential frequency in the presence of a mirror. The parallel components can be calculated and can be expressed in terms of (3.23) by dividing by  $-15$  and substituting the trap potential frequency for the unperturbed dimension. The expression for the dispersion shows that the presence of the mirror leads to a focusing in the perpendicular direction and a broadening in the parallel directions. Precision measurements (see e.g., [6]) in the shape deformation of an atomic gas near a mirror as a function of atom-mirror spacings would provide a direct check

against our theoretical predictions.

## Chapter 4

### Atom-Surface Force: Dielectric Materials

In the previous chapter we have calculated the atom-surface force under the idealized situation that the surfaces interacting with the atom are perfect reflectors. If we wish to consider the more general cases where; the surfaces can dissipate energy, allow partial transmission of incident radiation, or are at finite temperature we must adopt a more sophisticated description of the surfaces physics. In this chapter we generalize the idealized treatment of surfaces given simply by boundary conditions by modeling the surface as a localized collection of microscopic degrees of freedom interacting with the field.

#### 4.1 Microscopic Model

The action describing the entire system  $S[\vec{z}, \vec{Q}, A^\mu, \vec{P}_\nu]$  is the sum of six terms. Beginning with the free actions pertaining to the four dynamical variables,  $S_Z$  describes the motion  $\vec{z}$  of the atom's center of mass  $M$  under the influence of an external potential  $V$ :

$$S_Z[\vec{z}] = \int d\lambda \left[ \frac{M}{2} \dot{\vec{z}}^2(\lambda) - V[\vec{z}] \right] \quad (4.1)$$

where  $\lambda$  is the atom's worldline parameter. The internal degrees of freedom of the atom are modeled by a three dimensional harmonic oscillator with coordinate  $\vec{Q}$  and

natural frequency  $\Omega$  .

$$S_Q[\vec{Q}] = \frac{\mu}{2} \int d\lambda [\dot{\vec{Q}}^2(\lambda) - \Omega^2 \vec{Q}^2(\lambda)] \quad (4.2)$$

where  $\mu$  is the oscillator's reduced mass. The dynamics of the free photon field are described by  $S_E$  where 'E' stands for the electric field acting as environment, and  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength tensor,  $A_\mu$  being the photon field.

$$S_E[A^\mu] = \frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu} \quad (4.3)$$

The medium may be described by a collection of harmonic oscillators with frequency dependent mass,  $I(\nu)$ , the coordinates of each is the polarization field,  $\vec{P}_\nu$ , with natural frequency,  $\nu$ . The form of  $I(\nu)$  cannot be completely arbitrary, it is required to be even in  $\nu$  in order for the permittivity of medium to satisfy the Kramers-Kronig relation.

$$S_M[\vec{P}_\nu] = \frac{1}{2} \int_V d^4x \int d\nu I(\nu) [\dot{\vec{P}}_\nu^2(x) - \nu^2 \vec{P}_\nu^2(x)] \quad (4.4)$$

We note that our model can be brought into the Huttner and Barnett form by first representing the matter field as a continuum of oscillators with one natural frequency and then coupling the matter field to a reservoir.

The remainder is the two interaction actions. The interaction between the internal degree of freedom (dof) of the atom and the field is

$$S_{int}^{AF}[A^\mu, \vec{Q}, \vec{z}] = \int d\lambda q Q^i(\lambda) E_i(\lambda, \vec{z}(\lambda)) \quad (4.5)$$

where  $q$  represents the electronic charge. For the case of the medium the dipole moment of each oscillator is coupled with the local electric field with a frequency-dependent ‘charge’  $g(\nu)$ ,

$$S_{int}^{PF}[A^\mu, \vec{P}_\nu] = \int_V d^4x \int d\nu g(\nu) P_\nu^i(x) E_i(x) \quad (4.6)$$

where the spatial integration is performed over the volume occupied by the medium  $V$ . It should be noted that we have *not* included interactions among the elements of the dielectric. This is done to produce the local form of the permittivity which appears in the macroscopic Maxwell’s equations.

## 4.2 Field equations in the presence of a medium

In this section we will show how coarse-graining over the medium degrees of freedom leads to a permittivity and a classical stochastic source causing an additional induced component of fluctuations of the field. For now let us forget about the atom and focus entirely upon the field and medium. We begin with the time-dependent density matrix describing the dynamics of the electromagnetic field and the medium.

Consider the time evolution of the density matrix describing the combined (field + medium) system,  $\hat{\rho}(t) = \hat{U}(t, t_i) \hat{\rho}(t_i) \hat{U}^\dagger(t, t_i)$ . By considering matrix elements in an appropriate basis we can express  $\hat{\rho}(t)$  as a product of path integrals

$$\begin{aligned} \rho(A_f^\mu, A_f^{\mu'}, \vec{P}_{\nu f}, \vec{P}'_{\nu f}; t) &= \int_{-\infty}^{\infty} dA_i^\mu \int_{-\infty}^{\infty} dA_{i'}^{\mu'} \int_{A_i^\mu}^{A_f^\mu} \mathcal{D}A^\mu \int_{A_{i'}^{\mu'}}^{A_f^{\mu'}} \mathcal{D}A^{\mu'} \\ \exp\{i(S_E[A^\mu] - S_E[A^{\mu'}])\} &\int_{-\infty}^{\infty} d\vec{P}_i \int_{-\infty}^{\infty} d\vec{P}_{i'} \int_{\vec{P}_{\nu i}}^{\vec{P}_{\nu f}} \mathcal{D}\vec{P}_\nu \int_{\vec{P}'_{\nu i}}^{\vec{P}'_{\nu f}} \mathcal{D}\vec{P}'_\nu \end{aligned}$$

$$\begin{aligned} & \exp\{i(S_M[\vec{P}_\nu] - S_M[\vec{P}'_\nu] + S_{int}^{PF}[A^\mu, \vec{P}_\nu] - S_{int}^{PF}[A^{\mu'}, \vec{P}'_\nu])\} \\ & \times \rho(A_i^\mu, A_i^{\mu'}, \vec{P}_{\nu i}, \vec{P}'_{\nu i}; t_i). \end{aligned} \quad (4.7)$$

By treating the field as an open quantum system we can trace over the medium degrees of freedom leading to a reduced density matrix that accounts for the averaged effects the medium has on the field. Formally this requires calculating the trace of the density matrix over the medium variables  $\rho_r = Tr_{\vec{P}_\nu} \rho$ . If we assume that the initial state for the field and medium is uncorrelated i.e.  $\rho(A_i^\mu, A_i^{\mu'}, \vec{P}_{\nu i}, \vec{P}'_{\nu i}; t_i) = \rho(A_i^\mu, A_i^{\mu'}; t_i) \rho(\vec{P}_{\nu i}, \vec{P}'_{\nu i}; t_i)$  then for linear coupling between the medium and the field and an initially Gaussian medium state the path integrals for the medium can be evaluated exactly yielding the medium-reduced density matrix.

We should point out here that when the medium and the field interact very strongly the assumption that the state is initially factorizable breaks down. We adopt such a state here for illustrative purposes and generalize to consider more realistic initial states later.

$$\begin{aligned} \rho_r(A_f^\mu, A_f^{\mu'}; t) &= \int_{-\infty}^{\infty} dA_i^\mu \int_{-\infty}^{\infty} dA_i^{\mu'} \int_{A_i^\mu}^{A_f^\mu} \mathcal{D}A^\mu \int_{A_i^{\mu'}}^{A_f^{\mu'}} \mathcal{D}A^{\mu'} \\ & \times \exp\{i(S_E[A^\mu] - S_E[A^{\mu'}])\} \rho(A_i^\mu, A_i^{\mu'}; t_i) \mathcal{F}_M[A^\mu, A^{\mu'}] \end{aligned} \quad (4.8)$$

The previous equation introduces the influence functional  $\mathcal{F}_M[A^\mu, A^{\mu'}]$  which accounts for the averaged effect the medium has on the field, and can be expressed in terms of the influence action,  $S_{IF}^M$ , as  $-i \ln \mathcal{F}_M = S_{IF}^M$ .

$$S_{IF}^M[A^\mu, A^{\mu'}] = \int_V d^4x \int_V d^4x' E^{i-}(x) [G_{ij}^{ret}(x, x') E^{j+}(x') + \frac{i}{4} G_{ij}^H(x, x') E^{j-}(x')] \quad (4.9)$$

The superscript  $+$  ( $-$ ) denotes semi-sum (difference) variables defined by  $A^+ = (A + A')/2$ ,  $A^- = A - A'$  where prime distinguishes between the forward and backward histories in the closed-time-path (or Schwinger-Keldysh) sense [74].  $G_{ij}^{ret}$  ( $G_{ij}^H$ ) is the retarded Green's (Hadamard) function for the medium's dynamics. Because the oscillators comprising the medium do not interact and the medium is assumed to be isotropic, the polarization fluctuations within the medium are not spatially correlated and the Green's tensor is proportional to the identity matrix i.e.  $G_{ij}(x, x') = G(t, t')\delta_{ij}\delta^3(\vec{x} - \vec{x}')$ . Under these conditions the influence action simplifies greatly.

$$S_{IF}^M[A^\mu, A^{\mu'}] = \int dt \int dt' \int_V d^3x E^{j-}(t, \vec{x}) [G_{ret}(t, t') E_j^+(t', \vec{x}) + \frac{i}{4} G_H(t, t') E_j^-(t', \vec{x})] \quad (4.10)$$

where the explicit form for  $G_{ret}$  and  $G_H$  are

$$G_{ret}(t, t') = \int_0^\infty d\nu u(\nu) \sin \nu(t - t') \theta(t - t') \quad (4.11)$$

$$G_H(t, t') = \int_0^\infty d\nu u(\nu) \coth(\beta_M \nu / 2) \cos \nu(t - t') \quad (4.12)$$

with the shorthand  $u(\nu) = g^2(\nu)/(\nu I(\nu))$ .

At this level we see that tracing over the medium's degrees of freedom leads to two terms that are nonlocal in time. The term containing the retarded Green's function characterizes the response of the dielectric to the field ( $G_{ret}$  plays the role of the susceptibility), and the one containing the Hadamard function quantifies

the effects that quantum and thermal fluctuations ( $\beta_M$  being the medium's inverse temperature) of the polarization within the solid has upon the field. For the case we've considered where the medium is in thermodynamic equilibrium these kernels obey the fluctuation dissipation relation. This can be seen by taking the Fourier transform of (4.11) and (4.12)

$$G_{ret}(\omega) = \int_0^\infty d\nu u(\nu) \left[ \frac{\nu}{\nu^2 - \omega^2} + \frac{i\pi}{2} (\delta(\omega - \nu) - \delta(\omega + \nu)) \right] \quad (4.13)$$

$$G_H(\omega) = \pi \int_0^\infty d\nu u(\nu) \coth(\beta_M \nu / 2) (\delta(\omega - \nu) + \delta(\omega + \nu)) \quad (4.14)$$

We see that the imaginary part of the susceptibility  $G_{ret}$  is related to the Hadamard function by

$$2 \coth(\beta_M \omega / 2) \text{Im}[G_{ret}(\omega)] = G_H(\omega) \quad (4.15)$$

Also Fourier transform in time diagonalizes the influence action.

To see explicitly how medium fluctuations enter we appeal to a Gaussian path integral identity first suggested by Feynman and Vernon [63]. Note the complex modulus of the influence functional can be written as

$$|\mathcal{F}[A^\mu, A^{\mu'}]| = \int \mathcal{D}\vec{\chi} P[\vec{\chi}] \exp \left\{ i \int d^4x \chi_i(x) E^{i-}(x) \right\} \quad (4.16)$$

where  $P[\vec{\chi}]$  takes the form following, where an irrelevant normalization constant has been ignored.

$$P[\vec{\chi}] = \exp \left\{ - \int d^4x \int d^4x' \chi^j(x) G_H^{-1}(x, x') \chi_j(x') \right\} \quad (4.17)$$



This allows us to rewrite the influence functional in the form.

$$\mathcal{F}[A^\mu, A^{\mu'}] = \int \mathcal{D}\vec{\chi} P[\vec{\chi}] \exp \left\{ i \int d^4x E^{i-}(x) [\chi_i(x) + G_{ij}^{ret}(x, x') E^{j+}(x')] \right\} \quad (4.18)$$

In this process we have replaced the kernel quantifying fluctuations in the medium with the variable  $\chi_j$ . To retrieve the information about the fluctuations it is necessary to integrate over the functional distribution  $P[\vec{\chi}]$ . The kernel  $G_H^{-1}$  is symmetric and positive definite thus rendering  $P[\vec{\chi}]$  positive definite. Therefore we interpret the field  $\chi$  as a stochastic force driving the field with probability distribution described by  $P[\vec{\chi}]$ . Due to the Gaussianity of  $P[\vec{\chi}]$  all of its moments are specified by the mean  $\langle \chi^j \rangle_\chi = 0$  and the variance  $\langle \{ \chi^j(x), \chi^k(x') \} \rangle_\chi = \delta^{jk} G_H(t, t') \delta^3(\vec{x} - \vec{x}')$  where  $\langle \dots \rangle_\chi = \int \mathcal{D}\vec{\chi} P[\vec{\chi}] (\dots)$ .

Now we turn to the stochastic effective action  $S_\chi$  for the field which is the sum of the free-field action and the influence actions after the term quantifying the fluctuations of the environment (the medium in this case) has been replaced with a stochastic force. As we work with the photon field in the path integral a gauge fixing prescription must be adopted. The Fadeev-Popov trick [68] could be employed but the ghost fields introduced do not couple to the electromagnetic field in flat space and only contribute an overall multiplicative constant. Furthermore, the currents in our microscopic model which couple to the photon field are conserved. So we are free to choose any gauge we wish to evaluate the path integral. As a result the Green's functions which appear will be gauge dependent. However, as a consequence of current conservation any gauge choice will give equally valid descriptions of the physical processes. After choosing the temporal gauge ( $A^0 = 0$ ) and taking the

Fourier transform we find

$$S_\chi[A^\mu, A^{\mu'}] = \frac{1}{2\pi} \int d\omega \int d^3x \left[ -(\nabla \times \vec{A}^-(-\omega, \vec{x})) \cdot (\nabla \times \vec{A}^+(\omega, \vec{x})) \right. \\ \left. + \vec{A}^-(-\omega, \vec{x}) \cdot \vec{A}^+(\omega, \vec{x}) \omega^2 (1 + G_{ret}(\omega)) + \vec{E}^-(-\omega, \vec{x}) \cdot \vec{\chi}(\omega, \vec{x}) \right]. \quad (4.19)$$

The stochastic semi-classical equations of motion for the field can be derived from the saddle point condition for the reduced density matrix which gives

$$\nabla \times \nabla \times \vec{A}(\omega, \vec{x}) - \omega^2 (1 + G_{ret}(\omega)) \vec{A}(\omega, \vec{x}) = i\omega \vec{\chi}(\omega, \vec{x}). \quad (4.20)$$

In this form the role of the coarse-grained medium is evident. The permittivity is given by  $\epsilon(\omega, \vec{x}) = 1 + G_{ret}(\omega)$ , and the fluctuations of the medium drive the field through  $\vec{\chi}$ . Note that we've added the explicit position dependence of the permittivity because  $G_{ret}$  only has support within the volume containing the dielectric.

One can see from equation (4.20) a striking similarity with MQED. Indeed if one were to proceed from this point on treating the field semi-classically and choosing all space to filled with a dielectric material, albeit in vacuum this dielectric is fictitious, one would exactly reproduce the predictions of MQED using (4.20) and choosing the dielectric to be in a thermal state. However, it is important to note that in this case the stochastic field  $\chi^j$  represents the fluctuations of the medium only and does not include the intrinsic fluctuations of the field. After the next level of coarse-graining described in the following section the intrinsic quantum fluctuations of the field will enter which is different from the induced fluctuations from interaction with the dielectric medium.

### 4.3 Field-Reduced Density Matrix

In the last section we showed how the medium influences the field, and for the particular case of local fluctuations we found that the stochastic semi-classical action for the field takes the same form as the noninteracting field (with frequency dependent velocity) driven by an external current.

The microscopic details of the field are unimportant in the description of the surface atom force, and in principle are beyond our measuring capabilities. Coarse-graining over the medium-influenced field allows us to incorporate the averaged effect of the field on the atom's trajectory without a specific knowledge of the final field state. This leads to the reduced density matrix for the atom

$$\rho_r(z_f, z'_f, \vec{Q}_f, \vec{Q}'_f; t_f) = \int_{-\infty}^{\infty} dA_f^\mu \left[ \int \right]_{(\vec{z}_f, \vec{z}'_f)} \left[ \int \right]_{(\vec{Q}_f, \vec{Q}'_f)} \left[ \int \right]_{(A_f^\mu, A_f^\mu)} \int \mathcal{D}\vec{\chi} P[\vec{\chi}] e^{i(S_Z[\vec{z}] + S_Q[\vec{Q}] - S_Z[\vec{z}'] - S_Q[\vec{Q}'] + S_{int}^{AF}[\vec{z}, \vec{Q}, A^\mu] - S_{int}^{AF}[\vec{z}', \vec{Q}', A^{\mu'}] + S_\chi[A^\mu, A^{\mu'}])} \quad (4.21)$$

where the first integral traces over the field variables and we've adopted the shorthand

$$\left[ \int \right]_{(Y_f, Y'_f)} = \int_{-\infty}^{\infty} dY_i \int_{-\infty}^{\infty} dY'_i \int_{Y_i}^{Y_f} \mathcal{D}Y \int_{Y'_i}^{Y'_f} \mathcal{D}Y' \rho(Y_i, Y'_i; t_i) \quad (4.22)$$

to keep long expressions compact.

For the initially factorized state we have assumed the path integrals over the field can be evaluated exactly if the initial state is Gaussian yielding the field-reduced influence functional,  $\mathcal{F}_\chi[J^\mu, J^{\mu'}, \chi^j]$  expressed conveniently in terms of the influence

action,  $S_{IF}^E$  through  $-i \ln \mathcal{F}_\chi[J^\mu, J^{\mu'}, \chi^j] = S_{IF}^E[J^\mu, J^{\mu'}, \chi^j]$  (superscript  $E$  stands for the field as ‘environment’).

$$S_{IF}^E[J^\mu, J^{\mu'}, \chi^j] = \frac{1}{2\pi} \int d\omega d^3x d^3x' J^{\mu-}(-\omega, \vec{x}) \times \left[ \tilde{D}_{\mu\nu}^{ret}(\omega, \vec{x}, \vec{x}') (J^{\nu+}(\omega, \vec{x}') - \kappa_i^\nu \chi^i(\omega, \vec{x}')) + \frac{i}{4} \tilde{D}_{\mu\nu}^H(\omega, \vec{x}, \vec{x}') J^{\nu-}(\omega, \vec{x}') \right] \quad (4.23)$$

Above  $\tilde{D}_{\mu\nu}^{ret}$  and  $\tilde{D}_{\mu\nu}^H$  are the retarded Green’s and Hadamard function for the medium-assisted electromagnetic field which result from solving the semi-classical equations of motion sourced by a delta function. As we noted in the previous section those equations take the form of a driven field in the presence of a medium described by a frequency- and spatially- dependent permittivity and so must satisfy the boundary conditions associated with the geometry of the dielectric. The retarded Green’s function for the field satisfy the classical equation of motion for the electromagnetic field in the presence of a dielectric medium

$$\xi^{ab} \xi^{mn} \partial_a \partial_m \tilde{D}_{nk}(\omega, \vec{x}, \vec{x}') - \omega^2 \epsilon(\omega, \vec{x}) \tilde{D}_{ik}(\omega, \vec{x}, \vec{x}') = \delta_{ik} \delta(\vec{x} - \vec{x}') \quad (4.24)$$

where  $\xi_{abc}$  is the Levi-Civita symbol which is completely antisymmetric under permutations of its indices and Roman indices refer to spatial components. The solution to (4.24) gives the particular solution,  $A_j^P$ , to the semiclassical equation of motion (4.20) for the electromagnetic field

$$A_j^P(\omega, \vec{x}) = \int d^3x' \tilde{D}_{jk}(\omega, \vec{x}, \vec{x}') i\omega \chi^k(\omega, \vec{x}'). \quad (4.25)$$

The current density  $J^\mu$  in (4.23) comes from the atom-field interaction and takes the explicit form

$$J^\mu(x) = -q \int d\lambda Q^i(\lambda) \kappa_i^\mu \delta^4(x^\alpha - z^\alpha(\lambda)) \quad (4.26)$$

where the derivative operator  $\kappa_i^\mu = -\partial_0 \eta_i^\mu + \partial_i \eta_0^\mu$  yields the electric field when contracted with the vector potential  $E^i = \kappa_\mu^i A^\mu$  and also enforces current conservation  $\partial_\mu \kappa_i^\mu f(x) = (-\partial_0 \partial_i + \partial_i \partial_0) f(x) = 0$ .

Thus the field-reduced density matrix takes the form

$$\begin{aligned} \rho_r(z_f, z'_f, \vec{Q}_f, \vec{Q}'_f; t_f) &= \left[ \int \right]_{(\vec{z}_f, \vec{z}'_f)} \left[ \int \right]_{(\vec{Q}_f, \vec{Q}'_f)} \int \mathcal{D}\vec{\chi} P[\vec{\chi}] \\ &\times e^{i(S_Z[\vec{z}] + S_Q[\vec{Q}] - S_Z[\vec{z}'] - S_Q[\vec{Q}'])} \mathcal{F}_\chi[J^\mu, J^{\mu'}, \chi^j] \end{aligned} \quad (4.27)$$

#### 4.4 Atom's internal dof (Q)-Reduced Density Matrix

At this point we have the density matrix that describes the dynamics of the atom's trajectory and its internal degrees of freedom under the influence of the medium-assisted field. Since it is only the averaged effect and not the microscopic details of the atom's internal degree of freedom which we need for the description of the force, we can trace over the oscillator variables to obtain the oscillator-reduced density matrix. This quantity characterizes the dynamics of the atom's trajectory determined by its interaction with all remaining parties,

$$\rho_Z(z_f, z'_f; t_f) = \left[ \int \right]_{(\vec{z}_f, \vec{z}'_f)} e^{i(S_Z[\vec{z}] - S_Z[\vec{z}'])} \mathcal{F}_Z[\vec{z}, \vec{z}'] \quad (4.28)$$

The environmental influences are now packaged in the oscillator-reduced influence functional  $\mathcal{F}_Z[\vec{z}, \vec{z}']$  with their back-action accounted for in a self consistent manner. We proceed by evaluating  $\mathcal{F}_Z[\vec{z}, \vec{z}']$  perturbatively to lowest order in the atom-field and medium-field coupling.

To lessen the notational burden we define

$$\langle \dots \rangle_o = \int_{-\infty}^{\infty} d\vec{Q}_f \left[ \int \right]_{(\vec{Q}_f, \vec{Q}_f)} e^{i(S_Q[\vec{Q}] - S_Q[\vec{Q}'])} (\dots) \quad (4.29)$$

which corresponds to the noninteracting time-dependent expectation value with respect to the oscillator's initial state. With this simplification the oscillator-reduced influence functional can be compactly expressed in terms of quantum and stochastic expectation values

$$e^{iS_{IF}[\vec{z}, \vec{z}']} \stackrel{def}{=} \mathcal{F}_Z[\vec{z}, \vec{z}'] = \left\langle \left\langle e^{iS_{IF}^E} \right\rangle_o \right\rangle_\chi \quad (4.30)$$

which introduces the influence action,  $S_{IF}$ .

A saddle point approximation of (4.28) gives the semi-classical equation of motion for the trajectory where the saddle point is determined by the equation,

$$\frac{\delta}{\delta z^{k-}(\tau)} S_{CGEA}[\vec{z}, \vec{z}'] \Big|_{z^{k-}=0} = 0 \Rightarrow M\ddot{z}_k(\tau) + \partial_k V[\vec{z}] = f_k(\tau) \quad (4.31)$$

where the coarse-grained effective action is defined as  $S_{CGEA}[\vec{z}, \vec{z}'] = S_Z[\vec{z}] - S_Z[\vec{z}'] + S_{IF}[\vec{z}, \vec{z}']$ , and the influence force is given by

$$\frac{\delta}{\delta z^{k-}(\tau)} S_{IF}[\vec{z}, \vec{z}'] \Big|_{z^{k-}=0} = f_k(\tau). \quad (4.32)$$

We now wish to evaluate the influence force perturbatively for an expansion

in terms of small atom-field coupling. We begin by expanding both sides of (4.30) because  $S_{IF}^E$  contains terms linear and quadratic in the atom-field coupling we find the following to  $\mathcal{O}(q^3)$ ,

$$1 + iS_{IF} + \mathcal{O}(q^3) \approx 1 + i \left\langle \left\langle S_{IF}^E \right\rangle_o \right\rangle_\chi - \frac{1}{2} \left\langle \left\langle (S_{IF}^E)^2 \right\rangle_o \right\rangle_\chi + \mathcal{O}(q^3). \quad (4.33)$$

Examining the right hand side we break the two contributions down, the term linear in  $S_{IF}^E$  reduces to

$$\begin{aligned} \left\langle \left\langle S_{IF}^E \right\rangle_o \right\rangle_\chi = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int d^3x d^3x' \left[ \left\langle J^{\mu-}(-\omega, \vec{x}) J^{\nu+}(\omega, \vec{x}') \right\rangle_o \tilde{D}_{\mu\nu}^{ret}(\omega, \vec{x}, \vec{x}') \right. \\ \left. + \frac{i}{4} \left\langle J^{\mu-}(-\omega, \vec{x}) J^{\nu-}(\omega, \vec{x}') \right\rangle_o \tilde{D}_{\mu\nu}^H(\omega, \vec{x}, \vec{x}') \right] \end{aligned} \quad (4.34)$$

while the term linear in  $\chi^j$  vanishes. The quadratic term in  $S_{IF}^E$  reduces to

$$\begin{aligned} \left\langle \left\langle (S_{IF}^E)^2 \right\rangle_o \right\rangle_\chi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega \int d^3x d^3x' \int d\omega' d^3y d^3y' \left\langle J^{\mu-}(-\omega, \vec{x}) J^{\alpha-}(-\omega', \vec{y}) \right\rangle_o \\ \times \tilde{D}_{\mu\nu}^{ret}(\omega, \vec{x}, \vec{x}') \tilde{D}_{\alpha\beta}^{ret}(\omega', \vec{y}, \vec{y}') \kappa_i^\nu \kappa_j^\beta \left\langle \chi^i(\omega, \vec{x}') \chi^j(\omega', \vec{y}') \right\rangle_\chi \end{aligned} \quad (4.35)$$

where we've dropped higher order terms that enter at  $\mathcal{O}(q^3)$ . Thus, at leading order the influence force takes the form

$$f_k(\tau) = \frac{\delta}{\delta z^{k-}(\tau)} \left[ \left\langle \left\langle S_{IF}^E \right\rangle_o \right\rangle_\chi + \frac{i}{2} \left\langle \left\langle (S_{IF}^E)^2 \right\rangle_o \right\rangle_\chi \right]. \quad (4.36)$$

The explicit form for (5.53) can be obtained after an integration by parts and evaluating the expectation values of the atom's current density.

$$\begin{aligned}
f_k(\tau) = & \frac{q^2}{2} \int d\lambda \delta^{ij} \left[ g_{ret}(\tau, \lambda) \partial_k(x) \tilde{E}_{ij}^H(x, z^\alpha(\lambda)) \right. \\
& \left. + g_H(\tau, \lambda) \partial_k(x) \tilde{E}_{ij}^{ret}(x, z^\alpha(\lambda)) \right]_{x=z^\alpha(\tau)} \\
& + \frac{q^2}{4\pi} \int d\omega d^3x g_{ret}(\omega) G_H(\omega) \tilde{E}_{ret}^{ij}(\omega, \vec{z}(\lambda), \vec{x}) \partial_k(x') \tilde{E}_{ij}^{ret*}(\omega, \vec{x}', \vec{x}) \Big|_{\vec{x}'=\vec{z}(\tau)} \quad (4.37)
\end{aligned}$$

Above  $g_{ret}(g_H)$  is the retarded (Hadamard) function for the atom's internal degree of freedom and  $\tilde{E}^{ij}$  is the dyadic Green's function for the electric field obtained by contracting  $\kappa_\mu^i$  with each index on the photon field Green's function,  $\tilde{D}_{\mu\nu}$ .

#### 4.4.1 Initial Polariton State

Thus far we have assumed that the initial state of the field and medium are uncorrelated. This turns out to be unsatisfactory. Because the field and medium interact the initial state represented by the product of noninteracting thermal field and medium states is not stationary. To remedy this ill we consider preparing the composite system, medium+field, in a thermal state. This can be done by considering an initial state defined in terms of polaritons, the eigenmodes of the composed system.

Rather than formally diagonalize the Hamiltonian we refer to the treatment given by Suttorp et. al. [75] where the Huttner Barnett model is explicitly diagonalized for inhomogeneous media. Diagonalization of the microscopic model provides the Heisenberg operators for the electric field (translated into the notation of this paper) in terms of the polariton creation and annihilation operators  $C_k^\dagger(\omega, x)$  and  $C_k(\omega, x)$  respectively



$$E^i(x) = \frac{1}{2\pi} \int d^3x' \int_0^\infty d\omega e^{-i\omega t} \tilde{E}_{ret}^{ik}(\omega, \vec{x}, \vec{x}') J_k(\omega, \vec{x}') + h.c. \quad (4.38)$$

where we have a slightly altered definition for  $J_k(\vec{x}, \omega)$  as

$$J_k(\omega, \vec{x}) = \sqrt{\frac{Im[\epsilon(\omega)]}{\pi}} C_k(\omega, \vec{x}). \quad (4.39)$$

The combination of terms in the force (5.16) that arise due to fluctuations of the field, be it quantum or induced by the medium, represent the force due polariton fluctuations. We can combine these terms into a single kernel which can be computed using (4.38)

$$\begin{aligned} \tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{x}') &= \langle \{E_i(\omega, \vec{x}), E_j(\omega, \vec{x}')\} \rangle \\ &\rightarrow \tilde{E}_{ij}^H(\omega, \vec{x}, \vec{x}') + \int d^3y G_H(\omega) \delta_{ik} \tilde{E}_{ret}^{kl}(\omega, \vec{x}, \vec{y}) \tilde{E}_{lj}^{ret*}(\omega, \vec{x}', \vec{y}). \end{aligned} \quad (4.40)$$

Using (4.38) for a spatially varying thermal state i.e.

$$\langle C_i^\dagger(\omega, \vec{x}) C_j(\omega, \vec{x}') \rangle = \delta_{ij} (e^{\beta(\vec{x})\omega} - 1)^{-1} \text{ we find}$$

$$\begin{aligned} \tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{x}') &= 2 \int d^3y Im[\epsilon(\omega, \vec{y})] \delta_{lj} \left[ \tilde{E}_{ik}^{ret}(\omega, \vec{x}, \vec{y}) \tilde{E}_{ret}^{kl*}(\omega, \vec{x}', \vec{y}) (n(\omega, \vec{y}) + 1) \right. \\ &\quad \left. + \tilde{E}_{ik}^{ret*}(\omega, \vec{x}, \vec{y}) \tilde{E}_{ret}^{kl}(\omega, \vec{x}', \vec{y}) n(\omega, \vec{y}) \right] \end{aligned} \quad (4.41)$$

where we note here that general spatially varying thermal states are not easily implemented unless the spatial dependence of the temperature is consistent with the geometry of the problem. A simple example is that of a dielectric half-space

where the quantized field is expressed in terms two sets of creation and annihilation operators corresponding to polaritons in either half-space.

When evaluated at coincident spatial arguments the two terms in (4.41) combine,

$$\tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{x}) = 2 \int d^3y \text{Im}[\epsilon(\omega, \vec{y})] \coth[\beta(\vec{y})\omega/2] \delta^{kl} \tilde{E}_{ik}^{ret}(\omega, \vec{x}, \vec{y}) \tilde{E}_{lj}^{ret*}(\omega, \vec{x}, \vec{y}) \quad (4.42)$$

and if the temperature is constant throughout all space the Green's function identity

$$\int_{space} d^3x \text{Im}[\epsilon(\omega, \vec{x})] \tilde{E}_{ret}^{kj}(\omega, \vec{r}_1, \vec{x}) \tilde{E}_{ij}^{ret*}(\omega, \vec{r}_2, \vec{x}) = \text{Im}[\tilde{E}_i^k(\omega, \vec{r}_1, \vec{r}_2)] \quad (4.43)$$

can be used to evaluate the spatial integration if we assume the Green's functions to vanish at spatial infinity.

$$\tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{x}) = 2 \coth[\beta\omega/2] \text{Im}[\tilde{E}_{ij}^{ret}(\omega, \vec{x}, \vec{x})] \quad (4.44)$$

By the assumption of global thermodynamic equilibrium the previous equation can be viewed as a formal derivation of the fluctuation dissipation relation.

Using the replacement (4.42) the force considered in terms of the initial polariton state can be written in the simplified form below

$$f_k(\tau) = \frac{q^2}{2} \int d\lambda \delta^{ij} [g_{ret}(\tau, \lambda) \partial_k(x) \tilde{E}_{ij}^{Hpol}(x, z^\alpha(\lambda)) + g_H(\tau, \lambda) \partial_k(x) \tilde{E}_{ij}^{ret}(x, z^\alpha(\lambda))] \Big|_{x=z^\alpha(\tau)} \quad (4.45)$$

which is a function of time and depends upon the motion of the atom.

#### 4.4.2 Comparison With Previous Work

For comparison with previous works we note that  $E_{ret}^{ij}(\omega, \vec{x}, \vec{x}')$  and  $q^2 g_{ret}(\omega)$  in our treatment can be identified with the dyadic Green's function for the electric field,  $G^{ij}/(4\pi)$ , and  $4\pi\alpha(\omega)$  the frequency-dependent polarizability used in [33], [19], [25]. By using the fluctuation dissipation relation (4.45) can be brought into the same form as derived by others. After taking the long time limit, Fourier transforming in time, assuming global thermodynamic equilibrium and specifying the atom's trajectory to be static,

$$f_k = \frac{q^2}{4\pi} \int_{-\infty}^{\infty} d\omega \delta^{ij} [g_{ret}(-\omega) \partial_k(x) \tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{z}) + g_H(-\omega) \partial_k(x) \tilde{E}_{ij}^{ret}(\omega, \vec{x}, \vec{z})] \Big|_{\vec{x}=\vec{z}(\tau)}. \quad (4.46)$$

The fluctuation-dissipation relations require

$E_{ij}^{Hpol}(\omega, \vec{z}, \vec{z}) = 2 \coth(\beta\omega/2) \text{Im} E_{ij}^{ret}(\omega, \vec{z}, \vec{z})$ , as was derived in (4.44), and  $g_H(\omega) = 2 \coth(\beta\omega/2) g_{ret}(\omega)$ . With these relations we can rewrite  $f_k$  as

$$f_k = \frac{1}{\pi} \int_0^{\infty} d\omega \delta^{ij} \coth(\beta\omega/2) \text{Im} [\alpha(\omega) \partial_k(x) G_{ij}(\omega, \vec{x}, \vec{z})] \Big|_{\vec{x}=\vec{z}(\tau)} \quad (4.47)$$

reducing it to the form for the surface-atom force as derived from MQED [33].

It is interesting now to consider the specific case of an atom in the vacuum region of a dielectric half-space where the dielectric occupies the region  $z < 0$ . For consideration of the general case we allow the polariton temperature to differ

between the two half-spaces taking on the value  $T_M$  within the medium and the temperature  $T_E$  in the vacuum region. We can accommodate this setup by appealing directly to the Hadamard function for the polaritons. Using (4.42) and (4.43) we write the volume integration over the left half-space,  $V_L$ , at temperature  $T_M$  added to the volume integration over the right half-space,  $V - V_L$ ,  $V$  being the total volume, at temperature  $T_E$ . Splitting the integral over the left half-space from the total volume integral and finally using (4.43) we arrive at the Hadamard function for the field.

$$\begin{aligned} \tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{x}) &= 2 \int_{V_L} d^3y \operatorname{Im}[\epsilon(\omega)] \left[ (\coth[\beta_M \omega/2] - \coth[\beta_E \omega/2]) \right. \\ &\times \delta^{kl} \tilde{E}_{ik}^{ret}(\omega, \vec{x}, \vec{y}) \tilde{E}_{lj}^{ret*}(\omega, \vec{x}, \vec{y}) \left. \right] + 2 \coth[\beta_E \omega/2] \delta^{kl} \operatorname{Im}[\tilde{E}_{ij}^{ret}(\omega, \vec{x}, \vec{x})] \end{aligned} \quad (4.48)$$

The last term in this expression gives back the surface-atom force in thermal equilibrium at temperature  $T_E$ , and the first term gives the correction due to the polaritons being described by different temperatures in the two half-spaces. We denote this correction to the force  $f_k^{neq}$

$$\begin{aligned} f_k^{neq} &= \frac{q^2}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{V_L} d^3y \delta^{ij} g_{ret}(-\omega) \partial_k(x) \operatorname{Im}[\epsilon(\omega)] (\coth[\beta_M \omega/2] - \coth[\beta_E \omega/2]) \\ &\times \tilde{E}_{ret}^{ij}(\omega, \vec{x}, \vec{y}) \tilde{E}_{ij}^{ret*}(\omega, \vec{z}, \vec{y}) \Big|_{\vec{x}=\vec{z}} \end{aligned} \quad (4.49)$$

### 4.4.3 Evanescent and Propagating Waves

To understand the nature of this nonequilibrium correction (4.48) we study the Green's function  $\tilde{E}_{ik}^{ret}(\omega, \vec{x}, \vec{y})$ . In the vacuum half-space  $\tilde{E}_{ik}^{ret}(\omega, \vec{x}, \vec{y})$  is obtained by

solving the classical equations of motion for the field, subject to dielectric boundary conditions in the  $z = 0$  plane, sourced by a delta function. Its form can be taken from the Appendix of [25] and is written below for both spatial arguments lying in the vacuum half-space.

$$\tilde{E}_{ij}^{ret}(\omega, \vec{x}, \vec{x}') = \tilde{E}_{ij}^o + \frac{i\omega^2}{8\pi^2 k_z} \sum_{\sigma=TE, TM} \int d^2 k_{\parallel} e_{\sigma,i}(+) e_{\sigma,j}(-) R_{\sigma} e^{ik_z(z+z')} e^{i\vec{k}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}'_{\parallel})} \quad (4.50)$$

Above  $R_{\sigma}$  are the Fresnel reflection coefficients where the index  $\sigma$  refers to the polarization,  $TE$ , transverse electric and  $TM$  transverse magnetic (implicitly appearing below (4.54) and (4.55)).  $e_{\sigma,i}(\pm)$  are the polarization vectors for the field in the vacuum region,  $e_{TE}(\pm) = \hat{k}_{\parallel} \times \hat{z}$  and  $e_{TM}(\pm) = (k_{\parallel} \hat{z} \mp k_z \hat{k}_{\parallel})/\omega$  where  $\hat{k}_{\parallel}$  and  $\hat{z}$  are unit vectors,  $\hat{z}$  normal to the vacuum dielectric interface and  $\hat{k}_{\parallel}$  being directed parallel to the projection of the wave vector in the plane of the interface, and  $k_z = \sqrt{\omega^2 - k_{\parallel}^2}$  is the z-component of the wavevector.  $\tilde{E}_{ij}^o$  is the free field part of the Green's function and would be present whether the dielectric were there or not. At coincidence it has no spatial dependence, and so we discard it as we require the force to vanish at infinite atom-surface spacing.

Let us now focus on the portion of the surface-atom force in thermal equilibrium that arises from polariton fluctuations (PF), from now on the PF-contribution.

$$\begin{aligned} f_k^{PF}(T) &= \frac{q^2}{4\pi} \int_{-\infty}^{\infty} d\omega \delta^{ij} g_{ret}(-\omega) \partial_k(x) \tilde{E}_{ij}^{Hpol}(\omega, \vec{x}, \vec{z}) \Big|_{\vec{x}=\vec{z}(\tau)} \\ &= 4 \int_0^{\infty} d\omega \operatorname{Re}[\alpha(\omega)] \coth[\beta\omega/2] \delta^{ij} \partial_k(x) \operatorname{Im}[\tilde{E}_{ij}^{ret}(\omega, \vec{x}, \vec{z})] \Big|_{\vec{x}=\vec{z}(\tau)} \end{aligned} \quad (4.51)$$

Tracing over the indices of the Green's function, taking the spatial derivative and setting the two spatial arguments to the position of the atom we can express the PF-contribution to the force.

$$f_k^{PF}(T) = -\frac{1}{2\pi^2} \int_0^\infty d\omega \int d^2k_{\parallel} \operatorname{Re}[\alpha(\omega)] \coth(\beta\omega/2) \times \operatorname{Im} \left\{ \left[ R_{TE} + R_{TM} \left( \frac{k_{\parallel}^2 - k_z^2}{\omega^2} \right) \right] e^{i2k_z z} \right\} \quad (4.52)$$

When both spatial arguments lie in the vacuum half-space  $\tilde{E}_{ik}^{ret}(\omega, \vec{x}, \vec{y})$  decomposes into two contributions. First, field fluctuations in the vacuum region give rise to propagating waves, in particular waves moving in the negative  $z$ -direction will reflect from the dielectric surface giving rise to waves propagating in the positive  $z$ -direction. Second, a field fluctuation within the dielectric producing waves propagating toward the vacuum-dielectric interface will partially transmit into the vacuum region. A consequence of this is that the  $z$ -component of the wave vector in (4.52) is not necessarily real. For values of  $|k_{\parallel}| < \omega$  we see that  $k_z$  is real but becomes pure imaginary for the integration range  $|k_{\parallel}| \in (\omega, \infty)$ . The latter is associated with the propagating solutions to the wave equation in the vacuum and the former with evanescent waves.

We can isolate the influence of the evanescent modes on the force if we restrict the integration range so that the magnitude of the transverse momenta are strictly greater than the wave frequency

$$f_k^{PF,EW} = -\frac{1}{\pi} \int_0^\infty d\omega \int_{\omega}^\infty dk_{\parallel} k_{\parallel} \omega^2 \operatorname{Re}[\alpha(\omega)] \coth(\beta\omega/2)$$

$$\times \text{Im} \left\{ \left[ R_{TE} + R_{TM} \left( \frac{k_{\parallel}^2 - k_z^2}{\omega^2} \right) \right] e^{i2k_z z} \right\} \quad (4.53)$$

where  $EW$  stands for evanescent waves.

In this range the  $z$ -component of the wave vector is pure imaginary,  $k_z \rightarrow i\kappa$ , and this property can be used to simplify the equation for the force by explicitly taking the imaginary part of the integrand inside the curly brackets  $\{..\}$ . We note that

$$\text{Im}[R_{TE}] = \text{Im} \left[ \frac{k_z - k'_z}{k_z + k'_z} \right] = \frac{2\kappa \text{Re}[k'_z]}{|k_z + k'_z|^2} \quad (4.54)$$

$$\text{Im}[R_{TM}] = \text{Im} \left[ \frac{\epsilon(\omega)k_z - k'_z}{\epsilon(\omega)k_z + k'_z} \right] = \frac{2\kappa \text{Re}[\epsilon(\omega)k'_z^*]}{|\epsilon(\omega)k_z + k'_z|^2} = \frac{2\kappa \text{Re}[k'_z](k_{\parallel}^2 + |k'_z|^2)}{\omega^2 |\epsilon(\omega)k_z + k'_z|^2} \quad (4.55)$$

where  $k'_z = \sqrt{\epsilon(\omega)\omega^2 - k_{\parallel}^2}$  is the  $z$ -component of the wave vector in the dielectric and in the last step we have used the identity  $\omega^2 \text{Re}[\epsilon(\omega)k'_z^*] = \text{Re}[k'_z](k_{\parallel}^2 + |k'_z|^2)$ . To bring the force due to evanescent waves into its final form we make the change of variables  $k_{\parallel} = \omega q$  and note that  $\sqrt{2} \text{Re}[k'_z] = \sqrt{\text{Re}[\epsilon(\omega)]\omega^2 - k_{\parallel}^2 + |\epsilon(\omega)\omega^2 - k_{\parallel}^2|}$ .

$$\begin{aligned} f_k^{PF,EW}(T) &= -\frac{\sqrt{2}}{\pi} \int_0^{\infty} d\omega \int_1^{\infty} dq q \omega^4 \text{Re}[\alpha(\omega)] \coth(\beta\omega/2) \\ &\quad \times e^{-2z\omega\sqrt{q^2-1}} \sqrt{q^2-1} \sqrt{\text{Re}[\epsilon(\omega)] - q^2 + |\epsilon(\omega) - q^2|} \\ &\quad \times \left[ \frac{1}{|\sqrt{1-q^2} + \sqrt{\epsilon(\omega) - q^2}|^2} + \frac{(2q^2-1)(q^2 - |\epsilon(\omega) - q^2|)}{|\epsilon(\omega)\sqrt{1-q^2} + \sqrt{\epsilon(\omega) - q^2}|^2} \right] \end{aligned} \quad (4.56)$$

#### 4.4.4 Nonequilibrium Correction

To understand the physical origin of the nonequilibrium correction to the force we begin by considering the convolution of the Green's functions in the expression for  $f_k^{neq}$ . Starting from (4.49) we note that the Green's functions appearing are those with support at one point in the vacuum half-space and the other located within the medium. Using the appendices of the papers [77] and [76] we can evaluate the convolution of the Green's function product.

$$I \stackrel{def}{=} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^0 dz \tilde{E}_{ij}^{ret}(\omega, \vec{r}_1, \vec{x}) \partial_z \tilde{E}_{ret}^{ij*}(\omega, \vec{r}_2, \vec{x}) = \frac{1}{16\pi^2} \int d^2 k_{\parallel} \\ \times \frac{\omega^4}{2|k'_z|^2 \text{Im}[k'_z]} \left[ |T_{TE}|^2 + |T_{TM}|^2 \left( \frac{|k_{\parallel} - k_z|^2}{\omega^2} \right) \left( \frac{|k_{\parallel} - k'_z|^2}{|\epsilon|\omega^2} \right) \right] e^{i\vec{k}_{\parallel} \cdot (\vec{r}_{1\parallel} - \vec{r}_{2\parallel})} \quad (4.57)$$

Above  $T_{TE}$  and  $T_{TM}$  are the Fresnel transmission coefficients for the TE and TM-polarized waves. After making the change of variables  $k_{\parallel} = \omega q$ , plugging in the explicit forms for the Fresnel coefficients, and setting  $\vec{r}_1$  and  $\vec{r}_2$  to be the position of the atom we find the form.

$$I = -\frac{\sqrt{2}}{4\pi} \int_1^{\infty} dq q \frac{\omega^3 \sqrt{q^2 - 1}}{\text{Im}[\epsilon(\omega)]} \sqrt{\text{Re}[\epsilon(\omega)] - q^2 + |\epsilon(\omega) - q^2|} e^{-2z\omega \sqrt{q^2 - 1}} \\ \times \left[ \frac{1}{|\sqrt{1 - q^2} + \sqrt{\epsilon(\omega) - q^2}|^2} + \frac{(2q^2 - 1)(q^2 - |\epsilon(\omega) - q^2|)}{|\epsilon(\omega)\sqrt{1 - q^2} + \sqrt{\epsilon(\omega) - q^2}|^2} \right].$$

Combining this with the expression for the force (4.49) we arrive at

$$f_k^{neq}(T_M, T_E) = -\frac{\sqrt{2}}{\pi} \int_0^{\infty} d\omega \int_1^{\infty} dq q \omega^4 \text{Re}[\alpha(\omega)] (\coth(\beta_M \omega/2) - \coth(\beta_E \omega/2)) \\ \times e^{-2z\omega \sqrt{q^2 - 1}} \sqrt{q^2 - 1} \sqrt{\text{Re}[\epsilon(\omega)] - q^2 + |\epsilon(\omega) - q^2|}$$



$$\times \left[ \frac{1}{|\sqrt{1-q^2} + \sqrt{\epsilon(\omega) - q^2}|^2} + \frac{(2q^2 - 1)(q^2 - |\epsilon(\omega) - q^2|)}{|\epsilon(\omega)\sqrt{1-q^2} + \sqrt{\epsilon(\omega) - q^2}|^2} \right]. \quad (4.58)$$

noting in particular that this form is equivalent to the difference between two evanescent components to the surface-atom force in equilibrium.

$$f_k^{neq}(T_M, T_E) = f_k^{PF,EW}(T_M) - f_k^{PF,EW}(T_E) \quad (4.59)$$

The asymptotic properties of the nonequilibrium contribution to the force have been derived in detail in [33] and so the calculation will not be repeated here. We list the result below for  $z \rightarrow \infty$

$$f_k^{neq}(T_M, T_E) \approx -\frac{\pi}{6} \frac{\epsilon_o + 1}{\sqrt{\epsilon_o - 1}} \frac{(T_M^2 - T_E^2)}{z^3} \quad (4.60)$$

where  $\epsilon_o$  is the static value for the permittivity.

We now have an expression for the surface-atom force for a stationary system when the dielectric medium is out of thermal equilibrium with the field in the vacuum region. By dissecting the Lifshitz force we can now intuitively argue for the form of the force derived previously. First, in thermal equilibrium the Lifshitz force is composed of a propagating and evanescent wave component. The propagating component arises from field fluctuations in the vacuum region and give rise to reflected waves from the dielectric vacuum interface. The evanescent component arises from fluctuations of the field within the dielectric that partially transmit into the vacuum region. By subtracting the evanescent wave component from the equilibrium force, both at temperature  $T_E$ , we describe the effect of propagating waves in the vacuum region. To complete the description we need to add the effect of

field fluctuations within the dielectric at temperature  $T_M$  which is done by adding  $f^{PF,EW}(T_M)$

$$f_k = f_k(T_E) - f_k^{PF,EW}(T_E) + f_k^{PF,EW}(T_M) \quad (4.61)$$

The final form we find here agrees with the results found by others using MQED. However, the stationary case represents only one aspect of the atom-surface interactions derived here. In a future investigation we will employ the full power of our nonequilibrium formulation to describe dynamical processes.

## Chapter 5

### Mirror cooling by quantum field backaction

In this chapter we embark on a study of the interaction between moving surfaces and quantum fields. Describing phenomena ranging from the dynamical Casimir effect to mirror cooling by radiation pressure.

#### 5.1 The Model

To begin we define the microscopic model we will use to study mirror motion under the influence of a quantum field, for simplicity, taken to be a massless scalar field in 1+1 dimensions. The motion of the mirror in the absence of a quantum field is described by a relativistic particle subject to an external potential  $V[z^\mu(\lambda)]$

$$S_Z[z] = \int d\lambda \left[ -M \sqrt{-\dot{z}^\mu(\lambda) \dot{z}_\mu(\lambda)} - V[z^\mu(\lambda)] \right], \quad (5.1)$$

where  $\lambda$  parameterizes the mirror's worldline  $z^\alpha(\lambda)$ . Greek indices refer to spacetime components of a two-vector with 0 referring to the time component, and the dot product between two-vectors is taken with respect to the metric  $\eta_{\mu\nu} = \text{diag}(-1, 1)$ .

The action for the free quantum field has the standard form

$$S_\phi[\phi] = \int d^2x \left[ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + J_{ext}(x) \phi(x) \right] \quad (5.2)$$

where  $J_{ext}(x)$  is an external current that drives the field.

To describe the reflective property of the mirror Dirichlet boundary conditions can be imposed on the field at the mirror's position from the outset. We however take an alternative route, similar to the auxiliary field method, by introducing a constraining field  $\psi$  with 'action'

$$S_\psi[z, \psi] = \frac{1}{2} \mathcal{M} \int d\lambda u(\lambda) \psi^2(\lambda). \quad (5.3)$$

Physically this can be thought of as a model for a mirror composed of oscillators with no inertial resistance to excitation by the field. Also note we have included a factor  $u(\lambda) = \sqrt{-\dot{z}_\mu(\lambda)\dot{z}^\mu(\lambda)}$  in the integrand guaranteeing that the constraining field action is reparameterization invariant and thus able to provide a fully relativistic description.

With the free motion of all parties defined we now specify their interaction

$$S_{int}[z, \phi, \psi] = \gamma \int d\lambda u(\lambda) \psi(\lambda) \phi[z^\mu(\lambda)] = \int d^2x J(x) \phi(x) \quad (5.4)$$

where  $\gamma$  quantifies the coupling of the constraining field to the quantum field. The specific form of this interaction will be justified in the next section, and for convenience we have introduced the alternative form of the interaction written in terms of the current density  $J(x) = \gamma \int d\lambda u(\lambda) \psi(\lambda) \delta^2(x^\alpha - z^\alpha(\lambda))$ . It should be noted that our constraining field approach is identical to the method of Barton and Caloggeracos (BC) [61, 62] for modeling the physics of a partially transmitting mirror, to be shown next, and that we have treated the constraining field independently for technical ease later.

### 5.1.1 Relation to the Barton-Calogheracos (BC) Model

To begin we consider the time evolution of the density matrix for the total system comprising the mirror following a worldline  $z^\mu(\lambda)$  and the quantum field  $\phi$  from some initial state  $\hat{\rho}(t_{in})$  at time  $t_{in}$  to time  $t$ .

$$\hat{\rho}(t) = \hat{U}^\dagger(t, t_{in}) \hat{\rho}(t_{in}) \hat{U}(t, t_{in}) \quad (5.5)$$

Above  $\hat{U}(t, t_{in})$  is the time evolution operator for the total system. By considering matrix elements of  $\hat{\rho}$  in the ‘position’ basis we can write (5.5) in terms of path integrals

$$\begin{aligned} \rho(z_f, z'_f, \phi_f, \phi'_f; t) &= \int dz_{in} \int dz'_{in} \rho_z(z_{in}, z'_{in}) \int d\phi_{in} \int d\phi'_{in} \rho_\phi(\phi_{in}, \phi'_{in}) \\ &\quad \times \int_{z_{in}}^{z_f} \mathcal{D}z \int_{z'_{in}}^{z'_f} \mathcal{D}z' \int \mathcal{D}\psi \int \mathcal{D}\psi' \int_{\phi_{in}}^{\phi_f} \mathcal{D}\phi \int_{\phi'_{in}}^{\phi'_f} \mathcal{D}\phi' \\ &\quad \times \exp \left\{ i \left[ S_Z[z] + S_\phi[\phi] + S_\psi[z, \psi] + S_{int}[z, \phi, \psi] \right. \right. \\ &\quad \left. \left. - S_Z[z'] - S_\phi[\phi'] - S_\psi[z', \psi'] - S_{int}[z', \phi', \psi'] \right] \right\}. \quad (5.6) \end{aligned}$$

At this level one can identify the connection with the BC model for the interaction we’ve chosen. If we isolate one of the constraining field integrals in (5.6) and write out the explicit form of the interaction term we find

$$\int \mathcal{D}\psi \exp \left\{ i \int d\lambda u(\lambda) \left[ \frac{1}{2} \mathcal{M} \psi^2(\lambda) + \gamma \psi(\lambda) \phi[z^\alpha(\lambda)] \right] \right\}. \quad (5.7)$$

Being Gaussian the  $\psi$  path integral can be done leading to

$$\exp \left\{ - \frac{i\gamma^2}{2\mathcal{M}} \int d\lambda u(\lambda) \phi^2[z^\alpha(\lambda)] \right\}, \quad (5.8)$$

where an irrelevant multiplicative factor has been ignored. Combining this term with the free field action we recover the BC model when  $\gamma^2/\mathcal{M} \rightarrow \gamma$ . We can also see the role played by the constraining field by taking the limit  $\mathcal{M} \rightarrow 0$  in (5.7) where the  $\psi$  path integral reduces to a functional delta function which constrains the field to vanish at the position of the mirror

$$\lim_{\mathcal{M} \rightarrow 0} \int \mathcal{D}\psi \exp \left\{ i \int d\lambda u(\lambda) \left[ \frac{1}{2} \mathcal{M} \psi^2(\lambda) + \gamma \psi(\lambda) \phi[z^\alpha(\lambda)] \right] \right\} = \delta\{\phi[z^\alpha(\lambda)]\} \quad (5.9)$$

Finally we note that for finite values of  $\mathcal{M}$  the field will be suppressed but nonvanishing at the location of the mirror giving rise to a partial transmission of incident radiation.

### 5.1.2 Comparison with $NX$ -type coupling

To connect with other approaches used in the literature we illustrate how our model relates to oft-used  $NX$ -type interaction that couples the cavity photon number,  $N = a^\dagger a$ , to the displacement of the mirror from its equilibrium position,  $X$ . For the treatment given in this paper the photon number does not directly couple to the mirror position, to see the connection with  $NX$ -type interaction we first integrate out the constraining field giving the effective self-interaction term for the field

$$S_{self} = -\frac{\gamma^2}{2\mathcal{M}} \int d\lambda u(\lambda) \phi^2[z^\alpha(\lambda)]. \quad (5.10)$$

If we assume the mirror oscillates with a small amplitude compared to the cavity size we can write the mirror trajectory as  $z^\alpha(\tau) = \bar{z}^\alpha(\tau) + \tilde{z}^\alpha(\tau)$  where  $\bar{z}^\alpha(\tau)$  describes a stationary trajectory located at the equilibrium position of the mirror and  $\tilde{z}^\alpha(\tau)$  represents a small amplitude oscillation around this equilibrium. For such an ansatz the self-interaction can be expanded in powers of  $\tilde{z}^\alpha$ . To linear order we find

$$S_{self} \approx -\frac{\gamma^2}{2\mathcal{M}} \int d\lambda u(\lambda) \phi^2[\bar{z}^\alpha(\lambda)] \underbrace{-\frac{\gamma^2}{\mathcal{M}} \int d\lambda u(\lambda) \tilde{z}^\alpha(\lambda) \phi[\bar{z}^\alpha(\lambda)] \partial_\alpha(x) \phi(x)|_{x=\bar{z}^\alpha(\lambda)}}_{S_{self}^{(1)}}. \quad (5.11)$$

The first term accounts for the boundary conditions on the field and will not result in any forces imparted to the mirror and so we focus on the second term.

In the non-relativistic limit  $\tilde{z}^\alpha$  will be purely spatial, this can be argued by demanding that the magnitude of the two-velocity is normalized, that is  $U_\mu U^\mu = -1 = (\bar{U}_\mu + \tilde{U}_\mu)(\bar{U}^\mu + \tilde{U}^\mu)$ . Because  $\bar{U}^\mu$  is the two-velocity of the stationary mirror it has unit time-like component and null spatial component in the lab frame and as such its dot product is  $\bar{U}_\mu \bar{U}^\mu = -1$ . By ignoring second order terms in the velocity, that is  $\tilde{U}_\mu \tilde{U}^\mu \approx 0$ , we determine  $\bar{U}_\mu \tilde{U}^\mu \approx 0$  asserting the supposition. With this the derivative can be written as acting purely on the spatial argument of the field.

Now expanding the second term in terms of plane waves  $S^{(1)}$  can be expressed

in terms of creation and annihilation operators

$$S_{self}^{(1)} = -\frac{\gamma^2}{\mathcal{M}} \sum_k \sum_{k'} \int d\lambda \sqrt{\frac{1}{2\omega_k V}} \sqrt{\frac{1}{2\omega_{k'} V}} \tilde{z}(\lambda) \left[ a_k e^{ik \cdot z(\lambda)} + h.c. \right] (ik') \left[ a_{k'} e^{ik' \cdot z(\lambda)} + h.c. \right]. \quad (5.12)$$

where we've used the fact that the derivative is purely spatial and that  $u(\lambda) \approx 1$  in the non-relativistic limit. Our model considers the interaction of the mirror with all modes of the field, in distinction to most treatments which only consider the interaction of the mirror with only the fundamental mode of the cavity. By collapsing the sum over all frequencies to a single mode and then taking the rotating wave approximation we arrive at

$$S_{self}^{(1)} = -\frac{i\pi\gamma^2}{2V^2\mathcal{M}} \int d\lambda \tilde{z}(\lambda) \left[ a^\dagger a + a a^\dagger \right] \propto \int d\lambda NX \quad (5.13)$$

where we find that for a small amplitude, non-relativistic, and oscillatory mirror trajectory that the interaction employed in this paper reduces to the  $NX$ -type coupling when we choose the mirror to interact with only one mode of the field and take the rotating wave approximation.

## 5.2 Field-Reduced Density Matrix

Our ultimate aim is to obtain an equation of motion for the mirror motion under the backaction of a quantum field. For such a purpose it is natural to adopt the open systems conceptual framework [21], deeming the translational degree of freedom of the mirror to be the *system* and the quantum field to be the *environment*. By tracing over the environment variables we can derive the reduced density matrix



for the system describing the effective dynamics of the mirror with the averaged effect of the field taken into account.

It is worth pointing out that to begin coarse-graining we start by integrating out the field and afterward integrate out the constraining field. At this level the field is ignorant of the boundary conditions, and so this coarse-graining order prevents us from having to quantize the field in the presence of an arbitrarily moving boundary. Subsequently integrating out the constraining field is how the boundary conditions are implemented.

To coarse-grain the environment we trace over the final field configurations yielding the reduced density matrix describing the mirror motion

$$\rho_r(z_f, z'_f; t) = \int d\phi_f \rho(z_f, z'_f, \phi_f, \phi_f; t). \quad (5.14)$$

For our model this can be done explicitly when all parties involved are initially uncorrelated. Direct computation of (5.14) gives

$$\begin{aligned} \rho_r(z_f, z'_f; t) = & \int dz_{in} \int dz'_{in} \rho_z(z_{in}, z'_{in}) \int_{z_{in}}^{z_f} \mathcal{D}z \int_{z'_{in}}^{z'_f} \mathcal{D}z \int \mathcal{D}\psi \int \mathcal{D}\psi' \\ & \exp \left\{ i \left[ S_Z[z] + S_\psi[z, \psi] - S_Z[z'] - S_\psi[z', \psi'] + S_{IF}[z, z', \psi, \psi'] \right] \right\} \end{aligned} \quad (5.15)$$

which defines the influence action [63]

$$\begin{aligned} S_{IF}[z, z', \psi, \psi'] = & \int d^2x d^2x' \left[ J^-(x) G_{ret}(x, x') (J^+(x') + J_{ext}(x')) \right. \\ & \left. + \frac{i}{4} J^-(x) G_H(x, x') J^-(x') \right]. \end{aligned} \quad (5.16)$$

Above we have used semi-sum and difference variables  $J^+ = (J + J')/2$  and  $J^- = (J - J')$  where a prime denotes dependence on variables integrated on the reverse time branch in the sense of the closed-time-path formalism [78, 79]. Because we do not treat the external current dynamically it takes the same value on the forward and reverse time branches and so only its semi-sum appears in the influence action. The kernels  $G_{ret}$  and  $G_H$  are the retarded and Hadamard Green's functions [80] respectively defined in terms of the commutator and anticommutator of the field as

$$G_{ret}(x, x') = i\theta(t - t')Tr\{[\phi(x), \phi(x')]_-\hat{\rho}_\phi\} \quad (5.17)$$

$$G_H(x, x') = Tr\{[\phi(x), \phi(x')]_+\hat{\rho}_\phi\} \quad (5.18)$$

where  $\theta(t - t')$  is the Heaviside step function and  $[ , ]_+$  denotes anticommutation.

To implement the boundary condition on the field we need to evaluate the constraining field path integrals whereupon we can represent the reduced density matrix formally as

$$\begin{aligned} \rho_r(z_f, z'_f; t) = & \int dz_{in} \int dz'_{in} \rho_z(z_{in}, z'_{in}) \int_{z_{in}}^{z_f} \mathcal{D}z \int_{z'_{in}}^{z'_f} \mathcal{D}z \\ & \times \exp\left\{i\left[S_Z[z] - S_Z[z'] + \Gamma[z^\pm]\right]\right\} \end{aligned} \quad (5.19)$$

where  $\Gamma[z^\pm]$  captures all the effects of the field, with boundary conditions taken into account, on the motion of the mirror.  $\Gamma[z^\pm]$  will be the primary object of interest for us in determining the backaction, to be computed in the next section.

The saddle points of the reduced density matrix will yield the semi-classical equations of motion for the mirror as given below

$$\begin{aligned} & \frac{\delta}{\delta z^{\alpha^-}} (S_Z[z] - S_Z[z'] + \Gamma[z^\pm])|_{z^-=0} = 0 \\ \Rightarrow M\ddot{z}_\alpha(\tau) + \partial_\alpha V[z^\mu(\tau)] &= \frac{\delta}{\delta z^{\alpha^-}(\tau)} \Gamma[z^\pm] \Big|_{z^{\mu^-}=0}. \end{aligned} \quad (5.20)$$

The left hand side of the last equality is Newton's equation of motion, and on the right, the functional derivative  $\Gamma$  contains the sought after backaction.

Large values of  $z^-$  in the reduced density matrix indicate quantum coherence in the mirror's position. This can be easily ascertained if we assume the mirror to be in pure quantum state where the density matrix elements (in the position basis) become  $\rho(z, z') = \Psi^*(z)\Psi(z')$  where  $\Psi$  is the mirror's wavefunction and  $*$  denotes complex conjugation. For finite values of  $z^- = z - z'$ , that is  $z \neq z'$ , the square of the complex modulus of the density matrix gives the probability for the mirror to occupy two distinct positions  $|\rho(z, z')|^2 = |\Psi(z)|^2|\Psi(z')|^2$ . For the case of a macroscopic mirror we expect there to be a well-defined semi-classical regime and thus  $z^-$  can be treated as a small parameter about which to expand the influence action. Expanding  $\Gamma[z^\pm]$  in successive orders of  $z^-$

$$\Gamma \approx \Gamma^{(0)} + \Gamma^{(1)} + \mathcal{O}((z^-)^2) \quad (5.21)$$

we know from the saddle point condition (5.20) that the field's backaction on the semi-classical mirror trajectory will be obtained from the linear term. Thus we shall focus our attention on computing  $\Gamma^{(1)}$ .

To begin we first expand the scalar current densities to linear order in  $z^-$

$$J^-(x) \approx \gamma \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \left[ \psi^-(\lambda) + \psi^+(\lambda) \left( \frac{U_\mu^+(\lambda) U^{\mu-}(\lambda)}{U^{\nu+}(\lambda) U_\nu^+(\lambda)} - z^{\mu-}(\lambda) \partial_\mu \right) \right] \times \delta^2(x^\alpha - z^\alpha(\lambda)), \quad (5.22)$$

$$J^+(x) \approx \gamma \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \left[ \psi^+(\lambda) + \frac{1}{4} \psi^-(\lambda) \left( \frac{U_\mu^+(\lambda) U^{\mu-}(\lambda)}{U^{\nu+}(\lambda) U_\nu^+(\lambda)} - z^{\mu-}(\lambda) \partial_\mu \right) \right] \times \delta^2(x^\alpha - z^\alpha(\lambda)) \quad (5.23)$$

where  $u^+ = \sqrt{-U^{\mu+} U_\mu^+}$  and  $U^\mu$  is the two-velocity of the mirror. Using (5.22) and (5.23) with (5.16) we can express  $S_{IF}$  to lowest order in  $z^-$ . The zeroth order term becomes

$$S_{IF}^{(0)}[z^\pm, \psi^\pm] = \gamma \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int d^2x \psi^-(\lambda) G_{ret}(z^{\alpha+}(\lambda), x) J_{ext}(x) + \gamma^2 \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int d\lambda' u^+(\lambda') \left\{ \psi^-(\lambda) \psi^+(\lambda') G_{ret}(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) + \frac{i}{4} \psi^-(\lambda) \psi^-(\lambda') G_H(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) \right\} \quad (5.24)$$

and the linear order term

$$S_{IF}^{(1)}[z^\pm, \psi^\pm] = \gamma \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int d^2x \psi^+(\lambda) L(\lambda) G_{ret}(z^{\alpha+}(\lambda), x) J_{ext}(x) + \gamma^2 \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int d\lambda' u^+(\lambda') \left\{ \frac{1}{4} \psi^-(\lambda) \psi^-(\lambda') L(\lambda') G_{ret}(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) + \psi^+(\lambda) L(\lambda) \psi^+(\lambda') G_{ret}(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) + \frac{i}{2} \psi^+(\lambda) L(\lambda) \psi^-(\lambda') G_H(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) \right\} \quad (5.25)$$

where  $L(\lambda) = \frac{U_\mu^+(\lambda)U^{\mu-}(\lambda)}{U^{\nu+}(\lambda)U_\nu^+(\lambda)} + z^{\mu-}(\lambda)\partial_\mu$  is a differential operator. The derivative is taken on the argument that contains the same parameter dependence and afterward the appropriate value for the trajectory is inserted. For example consider  $L(\lambda)$  operating on the function  $H(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda'))$ .

$$L(\lambda)H(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) \stackrel{def}{=} \frac{U_\mu^+(\lambda)U^{\mu-}(\lambda)}{U^{\nu+}(\lambda)U_\nu^+(\lambda)}H(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda')) + z^{\mu-}(\lambda)\partial_\mu H(x, z^{\alpha+}(\lambda'))|_{x=z^{\alpha+}(\lambda)}. \quad (5.26)$$

To ensure relativistic invariance the free action for the constraining field couples to the mirror's position through the magnitude of the the 2-velocity, and so the constraining field's free action will contribute in the small  $z^-$  expansion of the influence action  $\Gamma^{(1)}$ . The zeroth order contribution of  $S_\psi[z, \psi] - S_\psi[z', \psi']$  follows

$$S_\psi^{(0)}[z^\pm, \psi^\pm] = \mathcal{M} \int_{\tau_i}^\tau d\lambda u^+(\lambda)\psi^+(\lambda)\psi^-(\lambda), \quad (5.27)$$

and at linear order we find

$$S_\psi^{(1)}[z^\pm, \psi^\pm] = \frac{\mathcal{M}}{2} \int_{\tau_i}^\tau d\lambda u^+(\lambda) \frac{U_\mu^+(\lambda)U^{\mu-}(\lambda)}{U^{\nu+}(\lambda)U_\nu^+(\lambda)} \left[ \psi^+(\lambda)\psi^+(\lambda) + \frac{1}{4}\psi^-(\lambda)\psi^-(\lambda) \right]. \quad (5.28)$$

### 5.2.1 Influence Action $\Gamma^{(1)}$

In this section we formally represent  $\Gamma^{(1)}$  in terms of modified constraining field expectation values of the influence action  $S_{IF}$ . We begin with the formal definition

for  $\Gamma[z^\pm]$

$$e^{i\Gamma[z^\pm]} = \int \mathcal{D}\psi \int \mathcal{D}\psi' \exp \left\{ i S_{eff}[z^\pm, \psi^\pm] \right\} \quad (5.29)$$

where  $S_{eff}[z^\pm, \psi^\pm] = S_\psi[z, \psi] - S_\psi[z', \psi'] + S_{IF}[z^\pm, \psi^\pm]$ . By expanding  $S_{eff}$  in powers of  $z^-$  inside of the  $\psi$ - path integrals we define the convenient notation

$$e^{i\Gamma[z^\pm]} = \left\langle \exp \left\{ i \left[ S_{eff}^{(1)}[z^\pm, \psi^\pm] + S_{eff}^{(2)}[z^\pm, \psi^\pm] + \dots \right] \right\} \right\rangle_{\psi^\pm} \quad (5.30)$$

where

$$\langle \dots \rangle_{\psi^\pm} = \int \mathcal{D}\psi \int \mathcal{D}\psi' \exp \left\{ i \left[ S_\psi^{(0)}[z^+, \psi^\pm] + S_{IF}^{(0)}[z^+, \psi^\pm] \right] \right\} (\dots) \quad (5.31)$$

can be viewed as providing unnormalized expectation values with respect to the constraining field with the modified free action  $S_{eff}^{(0)}[z^+, \psi^\pm] = S_\psi^{(0)}[z^+, \psi^\pm] + S_{IF}^{(0)}[z^+, \psi^\pm]$ . By expanding both sides of (5.30) to linear order in  $z^-$   $\Gamma^{(1)}$  can be expressed as a modified constraining field expectation value

$$e^{i\Gamma^{(0)}} (1 + i\Gamma^{(1)} + \dots) = \left\langle 1 + i S_{eff}^{(1)} + \dots \right\rangle_{\psi^\pm}. \quad (5.32)$$

Matching order by order on both sides of the equation above we find the normalization constant for (5.31)  $\langle \dots \rangle_{\psi^\pm}$  and the formal expression for  $\Gamma^{(1)}$

$$1 = e^{-i\Gamma^{(0)}} \langle 1 \rangle_{\psi^\pm} \quad \Gamma^{(1)} = e^{-i\Gamma^{(0)}} \left\langle S_{eff}^{(1)} \right\rangle_{\psi^\pm}. \quad (5.33)$$

### 5.3 Constraining Field Generating Functional

In the last section we derived a formal expression for the influence action,  $\Gamma^{(1)}$ , in terms of the modified constraining field expectation values. Here we seek to evaluate (5.33) by use of the closed-time-path generating functional method. We can define the generating functional for  $\psi$  expectation values by adding two external currents to the modified constraining field action i.e.  $S_{eff}^{(0)} \rightarrow S_{eff}^{(0)} + \int(j\psi - j'\psi')$ . The expressions are simplified with the use of semi-sum and difference variables resulting in the formal expression

$$\begin{aligned} Z_\psi[j^\pm] = & \int \mathcal{D}\psi^+ \int \mathcal{D}\psi^- \exp \left\{ i \int_{\tau_i}^{\tau_f} d\lambda u^+(\lambda) \left[ \psi^-(\lambda) \left( \mathcal{M}\psi^+(\lambda) \right. \right. \right. \\ & + \frac{\gamma^2}{2} \int_{\tau_i}^{\tau_f} d\lambda' u^+(\lambda') G_{ret}(\lambda, \lambda') \psi^+(\lambda') + j^+(\lambda) + \gamma \int d^2x G_{ret}(z^{\alpha+}(\lambda), x) J_{ext}(x) \\ & \left. \left. \left. + \psi^+(\lambda) j^-(\lambda) + \frac{i\gamma^2}{4} \psi^-(\lambda) \int_{\tau_i}^{\tau_f} d\lambda' u^+(\lambda') G_H(\lambda, \lambda') \psi^-(\lambda') \right] \right\} \quad (5.34) \end{aligned}$$

from which  $\psi$ -correlation functions are obtained through variation with respect to the external currents  $j^+$  and  $j^-$  i.e.  $-i \frac{\delta}{\delta j^\pm} Z_\psi[j^\pm] |_{j^\pm=0} = e^{i\Gamma^{(0)}} \langle \psi^\mp \rangle_{\psi^\pm}$ , the Jacobian due to the change of variables is 1 and it should be understood that the field's Green's functions are evaluated along the mirror's worldline i.e.  $G(\lambda, \lambda') = G(z^{\alpha+}(\lambda), z^{\alpha+}(\lambda'))$ . By taking the currents to zero and comparing with (5.29) we can see that  $Z_\psi[0] = e^{i\Gamma^{(0)}}$  and so we can interpret (5.33) as the normalized expectation value of  $S_{eff}^{(1)}$  with respect to the modified constraining field action. Because  $S_{eff}$  is quadratic in the fields  $\psi$  the path integration in (5.34) can be performed explicitly.

The variation of the constraining field action with respect to  $\psi^+$  and  $\psi^-$  gives

the semi-classical equations of motion and determines the location of the saddle point. One can think of these equations as describing the dynamics of the constraining that are necessary to pin the field down at the location of the mirror, one could think of  $\psi$  as the magnitude of an eddy current.

$$\mathcal{M}\psi^-(\tau) + \gamma^2 \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) G_{ret}(\lambda, \tau) \psi^-(\lambda) + j^-(\tau) = 0 \quad (5.35)$$

$$\mathcal{M}\psi^+(\tau) + \gamma^2 \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) G_{ret}(\tau, \lambda) \psi^+(\lambda) + \tilde{j}^+ = 0 \quad (5.36)$$

The current  $\tilde{j}^+$  is defined as

$$\begin{aligned} \tilde{j}^+(\tau) = & j^+(\tau) + \gamma \int d^2x G_{ret}(z^{\alpha+}(\tau), x) J_{ext}(x) \\ & + \frac{i\gamma^2}{2} \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) G_H(\tau, \lambda) \psi^-(\lambda). \end{aligned} \quad (5.37)$$

We find from (5.35) and (5.36) that the semi-classical value of the constraining field is governed by the renewal equation with formal solutions

$$\psi^-(\tau) = \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \Sigma(\lambda, \tau) j^-(\lambda) \quad (5.38)$$

and

$$\psi^+(\tau) = \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \Sigma(\tau, \lambda) \tilde{j}^+(\lambda) \quad (5.39)$$

where the kernel  $\Sigma(\lambda, \lambda')$  is the resolvent of the renewal equation which will be explicitly evaluated in later sections case by case. With the use of the semi-classical solutions (5.38) and (5.39) we find the exact form for the generating functional



$$\begin{aligned}
Z_\psi[j^\pm] &= Z_\psi[0] \exp \left\{ i \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) j^-(\lambda) \right. \\
&\times \left[ \gamma \int d\lambda_1 \int d^2x \Sigma(\lambda, \lambda_1) G_{ret}(z^\alpha(\lambda_1), x) J_{ext}(x) + \int_{\tau_i}^{\tau} d\lambda' u^+(\lambda') \left( \Sigma(\lambda, \lambda') j^+(\lambda') \right. \right. \\
&\left. \left. + \frac{i\gamma^2}{4} \int_{\tau_i}^{\tau} d\lambda_1 u^+(\lambda_1) \int_{\tau_i}^{\tau} d\lambda_2 u^+(\lambda_2) \Sigma(\lambda, \lambda_1) G_H(\lambda_1, \lambda_2) \Sigma(\lambda', \lambda_2) j^-(\lambda') \right) \right] \left. \right\} \quad (5.40)
\end{aligned}$$

from which we derive the constraining field expectation values

$$e^{-i\Gamma^{(0)}} \langle \psi^+(\tau) \rangle_{\psi^\pm} = \gamma \int_{\tau_i}^{\tau} d\lambda \int d^2x \Sigma(\tau, \lambda) G_{ret}(z^\alpha(\lambda), x) J_{ext}(x) \quad (5.41)$$

$$e^{-i\Gamma^{(0)}} \langle \psi^+(\lambda) \psi^-(\lambda') \rangle_{\psi^\pm} = -i\Sigma(\lambda, \lambda') \quad e^{-i\Gamma^{(0)}} \langle \psi^-(\lambda) \psi^-(\lambda') \rangle_{\psi^\pm} = 0 \quad (5.42)$$

and

$$\begin{aligned}
e^{-i\Gamma^{(0)}} \langle \psi^+(\lambda) \psi^+(\lambda') \rangle_{\psi^\pm} &= \left( \gamma \int_{\tau_i}^{\tau} d\lambda \int d^2x \Sigma(\tau, \lambda) G_{ret}(z^\alpha(\lambda), x) J_{ext}(x) \right)^2 \\
&+ \frac{\gamma^2}{2} \int_{\tau_i}^{\tau} d\lambda_1 u^+(\lambda_1) \int_{\tau_i}^{\tau} d\lambda_2 u^+(\lambda_2) \Sigma(\lambda, \lambda_1) G_H(\lambda_1, \lambda_2) \Sigma(\lambda', \lambda_2). \quad (5.43)
\end{aligned}$$

Note that because  $Z_\psi[j^\pm]$  is Gaussian all higher order correlation functions can be expressed in terms of (5.41), (5.42) and (5.43).

Using (5.40), (5.33) can be computed to obtain the influence action which we split into four distinct contributions.

$$\Gamma_A^{(1)} = \frac{\gamma^2}{2} \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int_{\tau_i}^{\tau} d\lambda' u^+(\lambda') L(\lambda) G_H(\lambda, \lambda') \Sigma(\lambda, \lambda') \quad (5.44)$$

$$\Gamma_B^{(1)} = \frac{\gamma^4}{2} \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int_{\tau_i}^{\tau} d\lambda' u^+(\lambda') \int_{\tau_i}^{\tau} d\lambda_1 u^+(\lambda_1) \int_{\tau_i}^{\tau} d\lambda_2 u^+(\lambda_2)$$

$$\begin{aligned}
& \times L(\lambda)G_{ret}(\lambda, \lambda')\Sigma(\lambda, \lambda_1) \left[ G_H(\lambda_1, \lambda_2)\Sigma(\lambda', \lambda_2) \right. \\
& \left. + 2 \int d^2x \int d^2x' G_{ret}(z^\alpha(\lambda_1), x)\Sigma(\lambda', \lambda_2)G_{ret}(z^\alpha(\lambda_2), x')J_{ext}(x)J_{ext}(x') \right] \quad (5.45)
\end{aligned}$$

$$\begin{aligned}
\Gamma_C^{(1)} &= \frac{\gamma^4}{4} \mathcal{M} \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int_{\tau_i}^{\tau} d\lambda_1 u^+(\lambda_1) \int_{\tau_i}^{\tau} d\lambda_2 u^+(\lambda_2) \\
& \quad \times \frac{U_\mu^+(\lambda)U^{\mu-}(\lambda)}{U^{\nu+}(\lambda)U_\nu^+(\lambda)} \Sigma(\lambda, \lambda_1) \left[ G_H(\lambda_1, \lambda_2)\Sigma(\lambda, \lambda_2) \right. \\
& \left. + 2 \int d^2x \int d^2x' G_{ret}(z^\alpha(\lambda_1), x)\Sigma(\lambda, \lambda_2)G_{ret}(z^\alpha(\lambda_2), x')J_{ext}(x)J_{ext}(x') \right] \quad (5.46)
\end{aligned}$$

$$\begin{aligned}
\Gamma_D^{(1)} &= \gamma^2 \int_{\tau_i}^{\tau} d\lambda u^+(\lambda) \int_{\tau_i}^{\tau} d\lambda' u^+(\lambda') \int d^2x \int d^2x' L(\lambda)G_{ret}(z^\alpha(\lambda), x) \\
& \quad \times \Sigma(\lambda, \lambda')G_{ret}(z^\alpha(\lambda'), x')J_{ext}(x)J_{ext}(x') \quad (5.47)
\end{aligned}$$

### 5.3.1 Influence Force

The variation of the influence action (5.20) provides the backaction of the constrained quantum field on the mirror motion

$$\frac{\delta\Gamma_A^{(1)}}{\delta z^{\alpha-}(\tau)} = \frac{\gamma^2}{2} \left[ a_\alpha(\tau) + \partial_\alpha + U_\alpha(\tau) \frac{d}{d\tau} \right] \int_{\tau_i}^{\tau} d\tau' G_H(\tau, \tau')\Sigma(\tau, \tau') \quad (5.48)$$

$$\begin{aligned}
\frac{\delta\Gamma_B^{(1)}}{\delta z^{\alpha-}(\tau)} &= \frac{\gamma^4}{2} \left[ a_\alpha(\tau) + \partial_\alpha + U_\alpha(\tau) \frac{d}{d\tau} \right] \int_{\tau_i}^{\tau} d\tau' \int_{\tau_i}^{\tau} d\lambda \int_{\tau_i}^{\tau} d\lambda' G_{ret}(\tau, \tau')\Sigma(\tau, \lambda) \\
& \quad \times \left[ G_H(\lambda, \lambda')\Sigma(\tau', \lambda') + 2 \int d^2x \int d^2x' G_{ret}(z^\alpha(\lambda), x)\Sigma(\tau', \lambda') \right. \\
& \quad \left. \times G_{ret}(z^\alpha(\lambda'), x')J_{ext}(x)J_{ext}(x') \right] \quad (5.49)
\end{aligned}$$

$$\frac{\delta\Gamma_C^{(1)}}{\delta z^{\alpha-}(\tau)} = \frac{\gamma^4}{4} \mathcal{M} \left[ a_\alpha(\tau) + U_\alpha(\tau) \frac{d}{d\tau} \right] \int_{\tau_i}^{\tau} d\lambda \int_{\tau_i}^{\tau} d\lambda' \Sigma(\tau, \lambda)$$

$$\begin{aligned} & \times \left[ G_H(\lambda, \lambda') \Sigma(\tau, \lambda') + \int d^2x \int d^2x' G_{ret}(z^\alpha(\lambda), x) \right. \\ & \left. \times \Sigma(\tau, \lambda') G_{ret}(z^\alpha(\lambda'), x') J_{ext}(x) J_{ext}(x') \right] \end{aligned} \quad (5.50)$$

$$\begin{aligned} \frac{\delta \Gamma_D^{(1)}}{\delta z^{\alpha-}(\tau)} &= \gamma^2 \left[ a_\alpha(\tau) + \partial_\alpha + U_\alpha(\tau) \frac{d}{d\tau} \right] \int_{\tau_i}^\tau d\lambda \int d^2x \int d^2x' G_{ret}(z^\alpha(\tau), x) \\ & \times \Sigma(\tau, \lambda) G_{ret}(z^\alpha(\lambda), x') J_{ext}(x) J_{ext}(x') \end{aligned} \quad (5.51)$$

where we have used  $\frac{\delta}{\delta z^{\alpha-}(\tau)} \int d\lambda L(\lambda) h(\lambda, z^\alpha(\lambda)) = [a_\alpha + \partial_\alpha + U_\alpha \frac{d}{d\tau}] h(\tau, z^\alpha(\tau))$ . By explicitly carrying out the  $\tau$ -derivatives many terms cancel resulting in the final form for the influence force below

$$\begin{aligned} f_\alpha(\tau) &= \delta M(\tau) a_\alpha(\tau) + \frac{\gamma^2}{2} \left[ \eta_\alpha^\beta + U_\alpha(\tau) U^\beta(\tau) \right] \partial_\beta(x) \int_{\tau_i}^\tau d\tau' \left[ G_H(x, z^\alpha(\tau')) \Sigma(\tau, \tau') \right. \\ & \left. + \gamma^2 \int_{\tau_i}^\tau d\lambda \int_{\tau_i}^\tau d\lambda' G_{ret}(x, z^\alpha(\tau')) \Sigma(\tau, \lambda) G_H(z^\alpha(\lambda), z^\alpha(\lambda')) \Sigma(\tau', \lambda') \right] \Big|_{x=z^\alpha(\tau)} \end{aligned} \quad (5.52)$$

$$\begin{aligned} f_\alpha^{ext}(\tau) &= \gamma^2 \left[ \eta_\alpha^\beta + U_\alpha(\tau) U^\beta(\tau) \right] \partial_\beta(x) \int_{\tau_i}^\tau d\tau' \int d^2y \int d^2y' \\ & \times \left[ \left( G_{ret}(x, y) \Sigma(\tau, \tau') G_{ret}(z^\alpha(\tau'), y') \right) \right. \\ & \left. + \gamma^2 \int_{\tau_i}^\tau d\lambda \int_{\tau_i}^\tau d\lambda' G_{ret}(x, z^\alpha(\tau')) \Sigma(\tau, \lambda) G_{ret}(z^\alpha(\lambda), y) \Sigma(\tau', \lambda') \right. \\ & \left. \times G_{ret}(z^\alpha(\lambda'), y') \right) J_{ext}(y) J_{ext}(y') \Big|_{x=z^\alpha(\tau)} \end{aligned} \quad (5.53)$$

which has been separated into the effects due to free field backaction  $f_\alpha(\tau)$  from those arising from the field being driven by an external current  $f_\alpha^{ext}(\tau)$  and three terms

proportional to the acceleration have been lumped into a single mass renormalization  $\delta M(\tau)$ .

## 5.4 Semi-Classical Equations of Motion for a Single Mirror in a Quantum Field

As a concrete example we apply these equations to the case of a single mirror moving in a quantum field without an external current driving the field. The first step is to compute the kernel  $\Sigma(\lambda, \lambda')$  which can be obtained by appealing to the semi-classical equations for the constraining field written below

$$\mathcal{M}\psi(\tau) + \gamma^2 \int_{\tau_i}^{\tau} d\lambda G_{ret}(\tau, \lambda)\psi(\lambda) = -J(\tau) \quad (5.54)$$

where the integration parameter is set to the mirror's proper time. To solve (5.54) we need only substitute the retarded Green's function for the field which can be evaluated directly through the mode sum formula giving

$$G_{ret}(x, x') = \frac{1}{4}\theta(t - t')[sign(\Delta t - \Delta x) + sign(\Delta t + \Delta x)]. \quad (5.55)$$

When  $G_{ret}$  is evaluated along the worldline of the mirror  $|\Delta t| > |\Delta x|$  because the trajectory is timelike leading to the form below

$$G_{ret}(z^\alpha(\tau), z^\alpha(\tau')) = \frac{1}{2}\theta(z^0(\tau) - z^0(\tau')) = \frac{1}{2}\theta(\tau - \tau'). \quad (5.56)$$

For this case the semi-classical equation (5.54) reduces to the Volterra equation of the second kind

$$\psi(\tau) + \epsilon \int_{\tau_i}^{\tau} d\lambda \psi(\lambda) = -J(\tau)/\mathcal{M}. \quad (5.57)$$

with well known solution

$$\psi(\tau) = -\frac{1}{\mathcal{M}} \left[ J(\tau) - \epsilon \int_{\tau_i}^{\tau} d\lambda e^{-\epsilon(\tau-\lambda)} J(\lambda) \right]. \quad (5.58)$$

Writing the solution in the form (5.39) we can identify the resolvent

$$\Sigma(\lambda, \lambda') = -\frac{1}{\mathcal{M}} \left[ \delta(\lambda - \lambda') - \frac{\gamma^2}{2\mathcal{M}} \theta(\lambda - \lambda') \exp \left\{ -\frac{\gamma^2(\lambda - \lambda')}{2\mathcal{M}} \right\} \right]. \quad (5.59)$$

Now we take the spacetime derivatives of the field's Green's functions. The Hadamard function in 1+1 cannot be directly evaluated because of a divergence at low frequency but its derivative can using the mode sum formula giving

$$\partial_{\beta} G_H(z^{\alpha}(\tau), z^{\alpha}(\lambda)) = -\frac{1}{2\pi} \frac{\partial_{\beta} \sigma(z^{\alpha}(\tau), z^{\alpha}(\lambda))}{\sigma(z^{\alpha}(\tau), z^{\alpha}(\lambda))}, \quad (5.60)$$

where  $\sigma(x, x') = (x - x')^2/2$  is Synge's worldfunction. To find the spacetime derivative of  $G_{ret}$  we may use (5.55) which gives

$$\partial_{\beta} G_{ret}(z^{\alpha}(\tau), z^{\alpha}(\lambda)) = -U_{\beta}(\tau) \theta(\tau - \lambda) \delta(\tau - \lambda). \quad (5.61)$$

Because the derivative of the field's retarded Green's function is proportional to the 2-velocity of the mirror we note that it will not contribute to the backaction because it is projected out via the prefactor in (5.53) i.e.  $(\eta^{\mu\nu} + U^{\mu}U^{\nu})U_{\mu} = 0$ .

We now seek to evaluate the back action as the reflectivity of the mirror becomes ideal (i.e.  $\mathcal{M} \rightarrow 0$ ) and when the motion of the mirror is nonrelativistic.

Employing the expression for the influence force (5.53) and (5.59) and making the change of variables  $s = \tau - \lambda$  we find

$$f_\alpha(\tau) = \delta M(\tau) a_\alpha(\tau) - \frac{\gamma^2}{2\mathcal{M}} [\eta_\alpha^\beta + U_\alpha(\tau) U^\beta(\tau)] \int_0^\tau ds \partial_\beta G_H(z^\alpha(\tau), z^\alpha(\tau - s)) \times \left[ \delta(s) - \frac{\gamma^2}{2\mathcal{M}} \exp \left\{ -\frac{\gamma^2 s}{2\mathcal{M}} \right\} \right]. \quad (5.62)$$

The exponential term suppresses the integrand for large values of  $s$  allowing a Taylor expansion of  $\partial_\beta G_H(z^\alpha(\tau), z^\alpha(\tau - s))$  for small  $s$

$$\partial_\beta G_H(z^\alpha(\tau), z^\alpha(\lambda)) \approx \frac{1}{\pi} \left[ \frac{1}{s} U_\beta(\tau) - \frac{1}{2} a_\beta(\tau) + \left( \frac{1}{6} \dot{a}_\beta(\tau) + \frac{a^2}{12} U_\beta(\tau) \right) s + \mathcal{O}(s^2) \right]. \quad (5.63)$$

Because the tensor structure of the prefactor in (5.62) projects out all terms proportional to the 2-velocity of the mirror we can isolate the relevant part of the expansion of  $\partial_\beta G_H$  by dropping all  $U_\beta$ -proportional terms

$$\partial_\beta G_H(z^\alpha(\tau), z^\alpha(\lambda)) \overset{rel}{\approx} \frac{1}{6\pi} \dot{a}_\beta(\tau) s + \mathcal{O}(s^2) \quad (5.64)$$

where the term proportional to the 2-acceleration has been lumped in with  $\delta M a_\alpha$ .

We can now freely evaluate the back action

$$f_\alpha(\tau) \approx \delta M a_\alpha(\tau) + \frac{1}{6\pi} (\eta_\alpha^\beta + U_\alpha(\tau) U^\beta(\tau)) \dot{a}_\beta(\tau) \left[ 1 - \exp \left\{ -\frac{\gamma^2 \tau}{2\mathcal{M}} \right\} \left( 1 + \frac{\gamma^2 \tau}{2\mathcal{M}} \right) \right] + \mathcal{O}(\mathcal{M}/\gamma^2) \quad (5.65)$$

After taking the nonrelativistic limit of (5.20), renormalizing the mirror mass i.e.

$M \rightarrow \tilde{M} = M - \delta M$  and dropping  $\mathcal{O}(\mathcal{M})$  terms we obtain the semi-classical equation of motion

$$\tilde{M}\ddot{z} + V'[z] \approx \frac{1}{6\pi}\dot{a} \quad (5.66)$$

where our expression for the back action agrees with what has been found by others using different methods [81, 82].

#### 5.4.0.1 Connection with the dynamical Casimir effect

Because the microscopic system we study is energy conserving the dissipative forces acting on the mirror implicitly characterize the interaction of the system with the environment. Thus the loss of energy in the system corresponds with the production of real quanta in the environment. The mechanical energy lost from the system can be computed by finding the work done on the system by the environment, equivalent to computing  $\int dt v \cdot F_{diss}$  where here the force is due to the dissipative terms in the equation of motion,  $v$  is the velocity of the mirror, and the force is evaluated along the trajectory of the mirror. The total energy of produced particles in the environment can be computed by summing the energy of each mode  $\hbar\omega$  multiplied by the occupation number of that mode  $N_\omega$

*(mechanical energy lost from system) = (particles produced in environment)*

$$\int_0^{\tau_f} d\tau v(\tau) f_z(\tau) = \int_0^\infty d\omega \hbar\omega N_\omega. \quad (5.67)$$

### 5.4.1 Order-Reduced Equation of Motion

For consideration of the semi-classical mirror motion let us specialize to a harmonic potential, the ideal mirror limit, and consider the motion when driven by an external force,  $F_{ext}$ . After restoring all physical constants the equation of motion reduces to

$$\ddot{z} + \Omega^2 z = \frac{\hbar}{6\pi c^2 \tilde{M}} \dot{a} + F_{ext}/\tilde{M} \quad (5.68)$$

where the back action can be treated as a perturbation when the period of the oscillator is large ( $\hbar\Omega/c^2\tilde{M}) \ll 1$  which should be well satisfied for realistic oscillators as  $\hbar/c^2 \approx 10^{-51} kg s$ .

Expanding the trajectory in powers of  $\frac{\hbar}{6\pi c^2 \tilde{M}}$  so that  $z \approx z^{(0)} + z^{(1)} + \dots$  and matching order by order gives

$$\ddot{z}^{(0)} + \Omega^2 z^{(0)} = F_{ext}/\tilde{M} \quad (5.69)$$

$$\ddot{z}^{(1)} + \Omega^2 z^{(1)} = \frac{\hbar}{6\pi c^2 \tilde{M}} \dot{a}^{(0)} \quad (5.70)$$

plus higher order corrections. Taking a time derivative of (5.69) gives  $\dot{a}^{(0)} = \dot{F}_{ext}/\tilde{M} - \Omega^2 z^{(0)}$ . Summing the contributions  $z^{(0)}$  and  $z^{(1)}$  gives the behavior of the trajectory to lowest order where radiation damping effects manifest. In the order-reduction scheme adopted here the back action of the quantum field manifests as a damping factor and a modification of the forcing term.

$$\tilde{M}\ddot{z} + \frac{\hbar}{6\pi c^2}\Omega^2\dot{z} + \tilde{M}\Omega^2 z = F_{ext} + \frac{\hbar}{6\pi c^2\tilde{M}}\dot{F}_{ext} \quad (5.71)$$



The effect of vacuum viscosity is exceptionally weak as can be noted by calculating the damping time scale for a 1 gram mirror with oscillation frequency of 1 Hz

$$(\text{damping time scale}) \approx \frac{6\pi c^2}{\hbar} \frac{1g}{(1Hz)^2} \sim 10^{31} \times (\text{Universe Age}). \quad (5.72)$$

## 5.5 Mirror Motion in a Cavity

In this section we specialize to the case of an optical cavity composed of two mirrors. For ease we will consider one mirror to be fixed in the  $z = 0$  plane and the moving mirror located at  $z > 0$  will close the cavity. For simplicity we will implement Dirichlet boundary conditions at the position of the fixed mirror from the outset and reserve the constraining field method to handle the backaction on the moving mirror see Figure 5.1.

Our ultimate goal is to understand the optomechanical cooling of the oscillatory motion of the moving mirror by radiation pressure and quantum viscosity. For these purposes the calculation can be simplified by considering the case where the amplitude of the oscillatory motion,  $\tilde{z}$  is much smaller than the cavity size  $\bar{z}$ . In such a setup we can expand the action of the combined system for small  $\tilde{z}/\bar{z}$ .

The case of a cavity demands great care if the perfect reflecting limit will be taken. For such a case it is necessary to split the quantum field into two components  $\phi(x) = \phi^<(x)\theta(z(t)-x) + \phi^>(x)\theta(x-z(t))$  corresponding with the field to the left and right of the moving mirror. When the mirror is partially transmitting a single set of modes, continuous across the position of the mirror, describes the physics for all

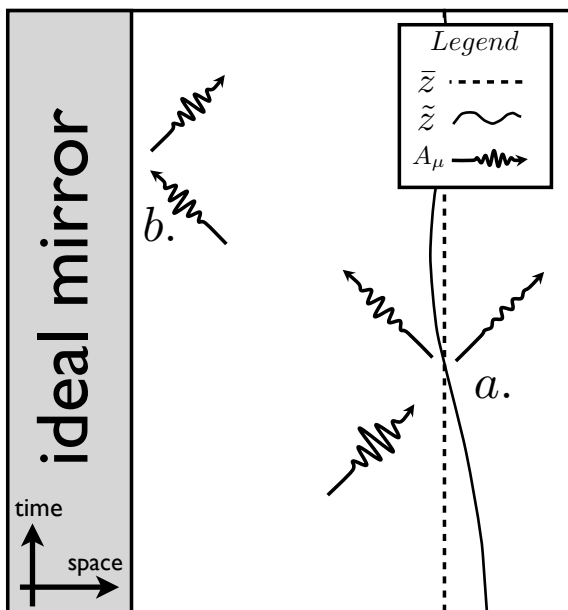


Figure 5.1: Illustration of the setup we employ for study of the backaction on a moving mirror in a cavity. The right hand side of the cavity is closed by a partially reflecting mirror undergoing small ( $|\tilde{z}/\bar{z}| \ll 1$ ) amplitude oscillations. The property of partial transmission of the mirror on the right is illustrated at point  $a.$  where an incident wavepacket is partially reflected and transmitted (note the decrease in amplitude quantifies the reflection and transmission coefficients). This gives a phenomenological description for how dissipation of the cavity field occurs. Because the moving mirror on the right is partially transmitting the quantum field inside the cavity and out interact and are correlated. Ignoring the details of the quantum field outside the cavity formally requires coarse-graining whereupon the field outside plays the role of environment to the cavity. Thus dissipation can be thought of as leaking of the field out of the cavity by partial transmission. On the left the cavity is closed by an immovable ideal mirror, point  $b.$  illustrates how the amplitude of an incident wavepacket is unaltered by reflection.

space and the partitioning of the field trivially reduces to the standard description, that is  $\phi^<(x)\theta(z(t) - x) + \phi^>(x)\theta(x - z(t)) \rightarrow \phi(x)(\theta(z(t) - x) + \theta(x - z(t))) = \phi(x)$ . However, when the ideal reflecting limit is taken the fields  $\phi^<$  and  $\phi^>$  become independent of one another and are described by different Fock spaces. Thus we rewrite the free action for the quantum field as

$$S_\phi[\phi] = -\frac{1}{2} \int d^2x [\partial_\mu \phi^<(x) \partial^\mu \phi^<(x) \theta(z(t) - x) + \partial_\mu \phi^>(x) \partial^\mu \phi^>(x) \theta(x - z(t))] \quad (5.73)$$

to emphasize the potential for spatial dependence and the remainder of the action follows from the introduction.

We now describe the trajectory of the mirror  $z^\alpha(\lambda)$  by one exhibiting small perturbations to a stationary mirror situated at  $z = \bar{z}$  so that  $z^\alpha(\lambda) = \bar{z}^\alpha(\tau) + \tilde{z}^\alpha(\tau)$  where  $\bar{z}^\alpha(\lambda) = (\lambda, \bar{z})$  and  $\tilde{z}/\bar{z} \ll 1$  allowing the action to be expanded in powers of  $\tilde{z}$  as  $S \approx S^{(0)} + S^{(1)} + S^{(2)} \dots$  where  $n$  in  $S^{(n)}$  labels the order. At leading order we have

$$S_\phi^{(0)}[\phi] = \int d^2x \left[ -\frac{1}{2} \partial_\mu \phi^<(x) \partial^\mu \phi^<(x) \theta(\bar{z} - x) - \frac{1}{2} \partial_\mu \phi^>(x) \partial^\mu \phi^>(x) \theta(x - \bar{z}) + J_{ext}(x) \phi(x) \right] \quad (5.74)$$

where we've added an external current  $J_{ext}$  and it is understood that  $\phi(x) = \phi^<(x)\theta(\bar{z} - x) + \phi^>(x)\theta(x - \bar{z})$

$$S_{int}^{(0)}[\phi, \psi] = \gamma \int d\lambda \psi(\lambda) \phi^<[\bar{z}^\alpha(\lambda)] \quad (5.75)$$

$$S_{\psi}^{(0)}[\psi] = \frac{\mathcal{M}}{2} \int d\lambda \psi^2(\lambda). \quad (5.76)$$

At linear order the first effects due to static forces emerge,

$$S_{\phi}^{(1)}[\phi, \tilde{z}] = -\frac{1}{2} \int d\lambda \tilde{z}(\lambda) [\partial_{\mu} \phi^{<}(\lambda, \bar{z}) \partial^{\mu} \phi^{<}(\lambda, \bar{z}) - \partial_{\mu} \phi^{>}(\lambda, \bar{z}) \partial^{\mu} \phi^{>}(\lambda, \bar{z})] \quad (5.77)$$

$$S_{int}^{(1)}[\phi, \psi, \tilde{z}] = -\gamma \int d\lambda \psi(\lambda) [\bar{U}_{\mu} \tilde{U}^{\mu} - \tilde{z}^{\alpha} \partial_{\alpha}] \phi[\bar{z}^{\alpha}(\lambda)] \quad (5.78)$$

$$S_{\psi}^{(1)}[\psi, \tilde{z}] = -\frac{\mathcal{M}}{2} \int d\lambda \bar{U}_{\mu} \tilde{U}^{\mu} \psi^2(\lambda). \quad (5.79)$$

where above we have set  $\bar{U}_{\mu} \bar{U}^{\mu} = -1$  (in principle the variation should be taken before parameterizing the mirror worldline by the proper time, as  $\bar{z}^{\alpha}$  is not a dynamical variable the final result is the same). Quadratic order provides dissipation and optical spring effects [85, 60]

$$S_{\phi}^{(2)}[\phi, \tilde{z}] = \frac{1}{4} \int d\lambda \tilde{z}^2(\lambda) \partial_x [\partial_{\mu} \phi^{<}(\lambda, x) \partial^{\mu} \phi^{<}(\lambda, x) - \partial_{\mu} \phi^{>}(\lambda, x) \partial^{\mu} \phi^{>}(\lambda, x)]_{x=\bar{z}} \quad (5.80)$$

$$S_{int}^{(2)}[\phi, \psi, \tilde{z}] = \gamma \int d\lambda \psi(\lambda) \left[ -\frac{1}{2} w_{\mu\nu} \tilde{U}^{\nu} \tilde{U}^{\mu} - \bar{U}_{\mu} \tilde{U}^{\mu} \tilde{z}^{\alpha} \partial_{\alpha} + \frac{1}{2} \tilde{z}^{\alpha} \tilde{z}^{\beta} \partial_{\alpha} \partial_{\beta} \right] \phi[\bar{z}^{\alpha}(\lambda)] \quad (5.81)$$

$$S_{\psi}^{(2)}[\psi, \tilde{z}] = -\frac{\mathcal{M}}{4} \int d\lambda w_{\mu\nu} \tilde{U}^{\mu} \tilde{U}^{\nu} \psi^2(\lambda). \quad (5.82)$$

where  $w_{\mu\nu} = \eta_{\mu\nu} + \bar{U}_\mu \bar{U}_\nu$  projects out the perpendicular component of a two-vector with respect to the worldline of the stationary mirror.

## 5.5.1 Quantum Field Generating Functional: Ideal Mirror Limit

### 5.5.1.1 Two approaches to coarse-graining

There are two approaches one can take to describe the backaction of a quantum field on the motion of a mirror in a cavity. First is the strategy we adopt in this section where the field is integrated out and then the constraining field. There are two scenarios where this proves to be useful. First, when the trajectory of the mirror is unspecified this approach proves to be most fruitful because it eliminates the need for quantizing the field in a space with an arbitrarily moving boundary. Second, because the field is integrated out first the Green's functions appearing in the influence action are those for the field in the presence of a single perfectly reflecting mirror, and therefore can be easily manipulated in the ideal conducting limit. The second approach, that we use later to derive the full influence action, is to integrate out the constraining field first, and then the quantum field which provides an advantage when the trajectory of the mirror is prescribed. In the approximation we are working the mirror motion is a small perturbation to a stationary trajectory. Under these circumstance the second approach provides a powerful tool for calculating the backaction because the wave equation for the field in the background space of two stationary mirrors can be solved and used to construct a perturbative solution.

### 5.5.1.2 Explicit computation of the quantum field generating functional with approach 1

In this section we wish to derive the forces on a stationary mirror when the ideal mirror limit is taken, and so we proceed with the first approach described above and integrate out the field first to provide the quantum field's generating functional. We then take advantage of the smallness  $\tilde{z}$  to expand the influence action  $S_{IF}[\tilde{z}^\pm, \psi^\pm]$  in powers of  $\tilde{z}$  where the contribution to the static forces can be isolated in the linear term.

The generating functional for the field takes the form

$$\begin{aligned}
Z_\phi[\mathcal{J}^-, \mathcal{J}^+] &= \int d\phi_f \int d\phi_i \int d\phi_{i'} \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{\phi_{i'}}^{\phi_f} \mathcal{D}\phi' \\
&\quad \exp \left\{ S_\phi^{(0)}[\phi] - S_\phi^{(0)}[\phi'] + S_{int}^{(0)}[\phi, \psi] - S_{int}^{(0)}[\phi', \psi'] \right. \\
&\quad \left. + \int d^2x \left[ \left( \mathcal{J}^+(x) + J_{ext}(x) \right) \phi^-(x) + \mathcal{J}^-(x) \phi^+(x) \right] \right\} \rho_\phi(\phi_i, \phi_{i'}) \quad (5.83)
\end{aligned}$$

which generates time dependent expectation values of the field (determined by the dynamics of  $S_\phi^{(0)}[\phi] + S_{int}^{(0)}[\phi, \psi]$ ) by functional differentiation with respect to the currents  $\mathcal{J}^\pm$ . Because we've expanded the action around the stationary trajectory of the moving mirror one notes that the  $Z_\phi$  generates correlation functions for the field in the presence of two stationary mirrors the one at  $z = 0$  perfectly reflecting and the one at  $\bar{z}$  partially transmitting.

Because  $Z_\phi$  is Gaussian it can be evaluated exactly providing

$$\begin{aligned}
-i \ln Z_\phi[\mathcal{J}^-, \mathcal{J}^+] &= \int d^2x \int d^2x' \left[ \mathcal{J}^-(x) + \gamma \int d\lambda \psi^-(\lambda) \delta^2(x^\alpha - \bar{z}^\alpha(\lambda)) \right] \\
&\times \left[ G_{ret}(x, x') \left( \mathcal{J}^+(x') + J_{ext}(x) + \gamma \int d\lambda' \psi^+(\lambda') \delta^2(x^{\alpha'} - \bar{z}^\alpha(\lambda')) \right) \right. \\
&\quad \left. + \frac{i}{4} G_H(x, x') \left( \mathcal{J}^-(x') + \gamma \int d\lambda' \psi^-(\lambda') \delta^2(x^{\alpha'} - \bar{z}^\alpha(\lambda')) \right) \right] \quad (5.84)
\end{aligned}$$

where we note that the Green's functions are composed of the partitioned field modes.

We now define a convenient notation similar to that used in the development of the constraining field generating functional.

$$\begin{aligned}
\langle \dots \rangle_\phi &= \int d\phi_f \int d\phi_i \int d\phi_{i'} \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{\phi_{i'}}^{\phi_f} \mathcal{D}\phi' \exp \left\{ S_\phi^{(0)}[\phi] - S_\phi^{(0)}[\phi'] \right. \\
&\quad \left. + S_{int}^{(0)}[\phi, \psi] - S_{int}^{(0)}[\phi', \psi'] \right\} \rho_\phi(\phi_i, \phi_{i'}) \dots \quad (5.85)
\end{aligned}$$

Using (5.85) we can write the influence action describing the effective dynamics of the mirror trajectory and the constraining field for the field in a cavity

$$S_{IF}[\tilde{z}^\pm, \psi^\pm] = -i \ln \left\langle \exp \left\{ i \left( \underbrace{\Delta S_\phi^{(1)} + \Delta S_\psi^{(1)} + \Delta S_{int}^{(1)}}_{\Delta S^{(1)}} + \dots \right) \right\} \right\rangle_\phi \quad (5.86)$$

where the  $\Delta$  signifies that we've taken the difference between the actions with respect to unprimed and primed variables, for example  $\Delta S_\psi = S_\psi[\psi] - S_\psi[\psi']$ . Expanding both sides of (5.86) in small  $\tilde{z}$  we find

$$S_{IF}^{(0)}[\psi^\pm] = -i \ln \langle 1 \rangle_\phi = -i \ln Z_\phi[0] = \quad (5.87)$$

$$S_{IF}^{(1)}[z^\pm, \psi^\pm] = (Z_\phi[0])^{-1} \langle \Delta S^{(1)} \rangle_\phi \quad (5.88)$$

$$S_{IF}^{(2)}[z^\pm, \psi^\pm] = (Z_\phi[0])^{-1} \left[ \langle \Delta S^{(2)} \rangle_\phi - \frac{i}{2} \langle (\Delta S^{(1)})^2 \rangle_\phi \right] - \frac{i}{2} (Z_\phi[0])^{-2} \langle \Delta S^{(1)} \rangle_\phi^2 \quad (5.89)$$

Using the field generating functional we can compute (5.86) by noting

$$\langle [\phi^\pm]^n \rangle_\phi = (-i)^n \frac{\delta^n}{\delta \mathcal{J}^{\mp n}} Z_\phi[\mathcal{J}^\pm] \Big|_{\mathcal{J}^\pm=0}. \quad (5.90)$$

The zeroth order contribution follows from the generating functional with currents set to zero

$$S_{IF}^{(0)}[\psi^\pm] = \gamma^2 \int d\lambda \int d\lambda' \psi^-(\lambda) \left[ G_{ret}(\bar{z}^\alpha(\lambda), \bar{z}^\alpha(\lambda')) \psi^+(\lambda') + \frac{i}{4} G_H(\bar{z}^\alpha(\lambda), \bar{z}^\alpha(\lambda')) \psi^-(\lambda') \right] + \gamma \int d\lambda \int d^2y \psi^-(\lambda) G_{ret}(\bar{z}^\alpha(\lambda), y^\alpha) J_{ext}(y). \quad (5.91)$$

The linear order contribution can be obtained by taking the appropriate number of functional derivatives

$$S_{IF}^{(1)}[z^\pm, \psi^\pm] \approx -\frac{1}{4} \int d\tau \bar{z}^-(\tau) \partial_\mu(x) \partial^\mu(x') \left[ G_H^<(x, x') - G_H^>(x, x') \right] + 2 \int d^2y \int d^2y' [G_{ret}^<(x, y) G_{ret}^<(x', y') - G_{ret}^>(x, y) G_{ret}^>(x', y')] \times \left( J_{ext}(y) + \gamma \int d\lambda \delta^2(y^\alpha - \bar{z}^\alpha(\lambda)) \psi^+(\lambda) \right)$$



$$\begin{aligned}
& \times \left( J_{ext}(y') + \gamma \int d\lambda' \delta^2(y^\alpha - \bar{z}^\alpha(\lambda')) \psi^+(\lambda') \right) \Big|_{x=x'=\bar{z}(\tau)} \\
& \quad - \gamma^2 \int d\tau \left[ \bar{U}_\mu \tilde{U}^{\mu-}(\tau) - \tilde{z}^{\alpha-}(\tau) \partial_\alpha(x) \right] \\
& \times \left[ \psi^+(\tau) \int d\lambda \left( \psi^+(\lambda) G_{ret}(x, \bar{z}(\lambda)) - \frac{1}{2} \psi^-(\lambda) G_H(x, \bar{z}(\lambda)) \right) \right. \\
& \quad \left. + \frac{1}{4} \psi^-(\tau) \int d\lambda \psi^-(\lambda) G_{ret}(\bar{z}(\lambda), x) \right]_{x=\bar{z}(\tau)} \\
& \quad + \frac{\mathcal{M}}{4} \int d\tau \bar{U}_\mu \tilde{U}^\mu(\tau) \left( \psi^{+2}(\tau) + \frac{1}{4} \psi^{-2}(\tau) \right) \quad (5.92)
\end{aligned}$$

where we have dropped terms that will be irrelevant to the backaction, that is terms independent of  $\tilde{z}^{\alpha-}$ .

## 5.5.2 Forces on a Stationary Mirror

Our study of the quantum field backaction of a mirror in a cavity will begin by deriving all of the forces acting on a stationary mirror in the ideal conducting limit. Unlike the previous case of a single mirror moving in a quantum field there will be nonvanishing force on the cavity mirror even when the mirror is at rest, the so-called Casimir effect [3]. This force is a result of the disturbance of the quantum field by the presence of the second mirror. We do not wish to present this formulation as a replacement for the well-known techniques for computing the Casimir force. Rather we visit this calculation to show how the formalism is applied and to verify the form of the derived forces. The real power of our formulation is that it goes beyond the stationary regime to treat dissipative and reactive forces.

The two necessary ingredients to calculate the stationary forces are the field Green's functions with the appropriate boundary conditions and the resolvent  $\Sigma(\lambda, \lambda')$  which can be obtained using the techniques of the constraining field developed for

the single mirror case. First we list the Green's functions for the field with Dirichlet boundary conditions in the  $z = 0$  plane

$$G_{ret}(t, x; t', x') = \frac{1}{\pi} \theta(t, t') \int \frac{d\omega}{\omega} \sin \omega(t - t') [\cos \omega(x - x') - \cos \omega(x + x')] \quad (5.93)$$

$$G_H(t, x; t', x') = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \cos \omega(t - t') [\cos \omega(x - x') - \cos \omega(x + x')] \quad (5.94)$$

which can easily be obtained from the free space expressions by the method of images i.e.  $G(x, x') = G_o(x, x') - G_o(x; t', -z')$  where  $G$  and  $G_o$  represents the Green's function for the half-space and free space respectively. As mentioned these Green's functions only represent a valid description for the quantum field for finite reflectivity of the movable mirror. To see how the Green's becomes modified in the perfect reflecting limit  $\gamma \rightarrow \infty$  consider the interaction action  $S_{int}$  which is an integral of minus the interaction energy. We demand the total energy of the system to be finite in the perfect mirror limit which translates into the requirement

$$\lim_{\gamma \rightarrow \infty} |S_{int}| < \mathcal{C} \quad (5.95)$$

where  $\mathcal{C}$  is a finite positive constant. By dissecting the interaction energy we see that we can interpret this condition as a requirement for the allowable modes to become discrete inside the cavity. First note that with a perfect mirror at  $z = 0$  we can decompose the field  $\phi(t, x)$  in spatial modes proportional to  $\sin kx$ , that is

$\phi(t, x) \propto \sum_{modes} [A_k e^{-i\omega t} \sin kx + h.c.]$ . Plugging in this mode decomposition we find

$$S_{int} \propto \gamma \sum_{modes} \int d\lambda \psi(\lambda) u(\lambda) [A_k e^{-i\omega t} \sin k\bar{z} + h.c.]. \quad (5.96)$$

The only nontrivial way to satisfy (5.95) is to require  $\sin k\bar{z} = 0$  which forces the previously allowable continuum of modes to collapse to a countable set with eigenfrequencies  $\omega_n = |n|\pi/\bar{z}$ . Thus, for  $\gamma \rightarrow \infty$

$$G_{ret}^<(t, x; t', x') = \frac{1}{2z} \theta(t, t') \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n} \sin \omega_n(t - t') [\cos \omega_n(x - x') - \cos \omega_n(x + x')] \quad (5.97)$$

$$G_H^<(t, x; t', x') = \frac{1}{2z} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n} \cos \omega_n(t - t') [\cos \omega_n(x - x') - \cos \omega_n(x + x')] \quad (5.98)$$

where the limiting procedure  $\frac{1}{\pi} \int d\omega \rightarrow \frac{1}{z} \sum_{\omega_n}$  has been used, and  $G^>$  follows from (5.93) and (5.94)

$$G_{ret}^>(t, x; t', x') = \frac{1}{\pi} \theta(t, t') \int \frac{d\omega}{\omega} \sin \omega(t - t') [\cos \omega(x - x') - \cos \omega(x + x' - 2\bar{z})] \quad (5.99)$$

$$G_H^>(t, x; t', x') = \frac{1}{\pi} \int_0^{\infty} \frac{d\omega}{\omega} \cos \omega(t - t') [\cos \omega(x - x') - \cos \omega(x + x' - 2\bar{z})] \quad (5.100)$$

This should not be surprising. When the perfect reflecting mirror limit is taken the field on either side of the mirror become completely uncorrelated and thus should be described by their own free actions.

### 5.5.3 Solving the Semi-Classical Constraining Field Equations

Following closely the treatment given for the case of a single mirror we expand the influence action in powers of  $\tilde{z}^-$  and evaluate the constraining field path integral perturbatively. The influence action,  $\Gamma^{(1)}$ , describing the dynamics of the mirror under the influence of the field is then given by the modified constraining field expectation value of the influence action at order  $\tilde{z}^-$  (see Section III.A).

In order to calculate the modified constraining field expectation values we need to obtain the resolvent  $\Sigma(\lambda, \lambda')$  which solves the equation

$$\mathcal{M}\psi(\tau) + \gamma^2 \int d\lambda G_{ret}(\tau, \bar{z}; \lambda, \bar{z})\psi(\lambda) + J(\tau) = 0 \quad (5.101)$$

so that  $\psi$  can be expressed as  $\psi(\tau) = \int d\tau' \Sigma(\tau, \tau')J(\tau')$ .

#### 5.5.3.1 Case 1: Ideal Reflecting Limit

As  $\gamma \rightarrow \infty$  the retarded Green's function vanishes on the worldline of the mirror giving the trivial solution

$$\psi(\tau) = -\frac{1}{\mathcal{M}}J(\tau), \quad (5.102)$$

and so for the case of an ideal conducting cavity the correlation functions for the constraining field simplify greatly to become

$$\langle \psi^+(\tau)\psi^-(\tau') \rangle_{\psi^\pm} = -\frac{1}{\mathcal{M}}\delta(\tau - \tau') \quad (5.103)$$

all other two-point functions vanishing.

### 5.5.3.2 Case 2: Partially Transmitting Mirror

Finite reflectivity is necessary to see the effects of cooling, for finite value of  $\gamma$  the Green's functions for the field entering the equations for the influence action are those corresponding with a half-space closed by a perfect mirror (5.93) and (5.94). To find the resolvent we proceed by taking the Fourier transform of (5.101), in the partially transmitting limit

$$\tilde{\psi}(\omega)[\omega + i\epsilon(1 - e^{i2\omega z})] = -\omega J(\omega). \quad (5.104)$$

where we have used  $G_{ret}(\lambda, \bar{z}; \lambda, \bar{z}) = \frac{1}{2}[\theta(\lambda - \lambda') - \theta(\lambda - \lambda' - 2\bar{z})]$  and  $\epsilon = \gamma^2/2\mathcal{M}$ .

Inverting this equation gives

$$\psi^+(\tau) = \int d\tau' \underbrace{\left[ -\frac{1}{2\pi\mathcal{M}} \int_{-\infty}^{\infty} d\omega \frac{\omega e^{-i\omega(\tau-\tau')}}{\omega + i\epsilon(1 - e^{i2\omega z})} \right]}_{\Sigma(\tau, \tau')} J(\tau'). \quad (5.105)$$

The integral representation of the resolvent cannot be evaluated exactly and so we seek to approximate it's form for  $\epsilon \rightarrow \infty$  while retaining the lowest order effect of dissipation. As  $\epsilon$  becomes large we can approximate the poles of the denominator in a perturbative expansion in inverse powers of  $\epsilon$ . The denominator vanishes to  $O(1/\epsilon^2)$  when  $\omega \approx \omega_n - i\Gamma_n$

$$\omega_n = \frac{n\pi}{\bar{z}} \left[ 1 - \frac{1}{2\epsilon\bar{z}} + \frac{1}{(2\epsilon\bar{z})^2} + \dots \right] \quad (5.106)$$

$$\Gamma_n = \frac{\pi^2 n^2}{4\epsilon^2 \bar{z}^3} + \dots \quad (5.107)$$

Using the approximate form for the poles we can evaluate  $\Sigma$  via a contour integration. The exponential factor in the numerator determines which way we can close the integration contour in the complex  $\omega$ -plane, particularly if  $\tau > \tau'$  then an arc at infinity in the lower half  $\omega$ -plane can be used because it vanishes by Jordan's lemma. In this case we can equate the original integral along the real  $\omega$ -axis with the sum of residues enclosed by the contour, here, those arising from all of the poles in the lower half  $\omega$ -plane. Because the imaginary part of all the poles is negative the integral is only non-vanishing when  $\tau > \tau'$  which enforces causality.

$$\Sigma(\tau, \tau') \approx -\frac{1}{\mathcal{M}\epsilon\bar{z}} \frac{\partial}{\partial\tau} \sum_{n=1}^{\infty} \cos \omega_n(\tau - \tau') e^{-\Gamma_n(\tau - \tau')} \theta(\tau - \tau') \quad (5.108)$$

It is crucial to point out here this approximation is valid strictly for  $\epsilon \rightarrow \infty$  and is only used to show how the first dissipative effects enter. As the frequency  $\omega$  approaches  $\epsilon$  the expansion will break down. We justify the use of this expression by the fact that we will be considering the case where the cavity is driven by a laser with frequency close to the fundamental mode. In this case only the lowest mode will be significantly populated in a regime where the poles are well approximated. For the consideration of vacuum viscosity alone this approximation cannot be used and other methods will need to be adopted.

With the resolvent in hand we can take the modified constraining field expectation value of  $S_{IF}^{(1)}$  in the ideal conducting limit to give

$$\begin{aligned}
\Gamma^{(1)}[\tilde{z}^\pm] &= (Z_\psi[0])^{-1} \langle S_{IF}^{(1)}[\tilde{z}^\pm, \psi^\pm] \rangle_{\psi^\pm} = \\
&-\frac{1}{4} \int d\tau \tilde{z}^-(\tau) \partial_\mu(x) \partial^\mu(x') \left[ G_H^<(x, x') - G_H^>(x, x') \right. \\
&\quad \left. + 2 \int d^2y \int d^2y' [G_{ret}^<(x, y) G_{ret}^<(x', y') \right. \\
&\quad \left. - G_{ret}^>(x, y) G_{ret}^>(x', y')] J_{ext}(y) J_{ext}(y') \right]_{x=x'=\bar{z}(\tau)} \\
&-\frac{\gamma^2}{2\mathcal{M}} \int d\tau \left[ \bar{U}_\mu \tilde{U}^{\mu-}(\tau) - \tilde{z}^{\alpha-}(\tau) \partial_\alpha(x) \right] G_H(x, \bar{z}(\tau)) \Big|_{x=\bar{z}(\tau)}, \tag{5.109}
\end{aligned}$$

the leading order contribution to the influence action whose variation with respect to  $z^-$  gives the static forces acting on the mirror.

#### 5.5.4 Casimir Force

In this section we derive the forces on a stationary mirror due to vacuum fluctuations. To isolate the Casimir force we vary the influence action and set the external currents to zero giving

$$\begin{aligned}
f_\alpha^{cas} &= \left[ -\frac{1}{4} \eta_{\alpha z} \partial_\mu(x) \partial^\mu(x') [G_H^<(x, x') - G_H^>(x, x')] \right. \\
&\quad \left. + \frac{\gamma^2}{2\mathcal{M}} \left( \bar{U}_\alpha \frac{d}{d\tau} + \partial_\alpha(x) \right) G_H(x, \bar{z}(\tau)) \right]_{x=x'=\bar{z}(\tau)}. \tag{5.110}
\end{aligned}$$

The second term can be shown to vanish, note the mode structure of the Hadamard function  $G_H \propto \cos \omega(\tau - \tau') [\cos \omega(x - x') - \cos \omega(x + x')]$  so that a single derivative on  $t$  or  $x$  will give either  $\partial_t G_H \propto \sin \omega(\tau - \tau')$  or  $\partial_x G_H \propto [\sin \omega(x - x') - \sin \omega(x + x')]$  which vanish when the Hadamard function is evaluated at coincidence on the worldline of the stationary mirror located at  $\bar{z}$  because  $\omega = n\pi/\bar{z}$ .

Taking the derivatives on the Hadamard function in the first term yields the following form for the Casimir force

$$f_z^{cas} = \frac{1}{4\bar{z}} \sum_{n=-\infty}^{\infty} \omega_n \cos 2\omega_n \bar{z} - \frac{1}{2\pi} \int_0^{\infty} d\omega \omega. \quad (5.111)$$

The sum and the integral in the above expression are individually divergent, to make sense of their difference we need to introduce an ultraviolet regulator to parameterize their divergences, for this purpose we choose an exponential cutoff of the form  $e^{-\lambda\omega}$ . After substituting the explicit expressions for the eigenfrequencies  $\omega_n$  and noting that  $\cos 2\omega_n \bar{z} = 1$  we find

$$f_z^{cas} = \frac{\pi}{2\bar{z}^2} \sum_{n=1}^{\infty} n e^{-n\pi\lambda/\bar{z}} - \frac{1}{2\pi} \int_0^{\infty} d\omega \omega e^{-\lambda\omega} = \frac{\pi}{2\bar{z}^2} \frac{e^{\pi\lambda/\bar{z}}}{(1 - e^{\pi\lambda/\bar{z}})^2} - \frac{1}{2\pi\lambda^2} \quad (5.112)$$

As the regulator is relaxed ( $\lambda \rightarrow 0$ ) we find the well-defined finite expression for the static forces acting on the mirror

$$\begin{aligned} f_z^{cas} &= \lim_{\lambda \rightarrow 0} \left[ \frac{\pi}{2\bar{z}^2} \frac{e^{\pi\lambda/\bar{z}}}{(1 - e^{\pi\lambda/\bar{z}})^2} - \frac{1}{2\pi\lambda^2} \right] \\ &= \frac{\pi}{2\bar{z}^2} \left[ \frac{\bar{z}^2}{\pi^2\lambda^2} - \frac{1}{12} + \frac{\pi^2\lambda^2}{240z^2} + O(\lambda^3) \right] - \frac{1}{2\pi\lambda^2} = -\frac{\pi}{24\bar{z}^2} \end{aligned} \quad (5.113)$$

in agreement with the Casimir force on ideal mirror computed using different methods [83, 84].



### 5.5.5 Radiation Pressure Force: Partially Transmitting Mirror

The radiation pressure force on the movable mirror in the partially transmitting case is relevant to experiment and provides the necessary physical mechanism for mirror cooling to take place—relaxation of the force to a steady-state value. For the partially transmitting case the radiation pressure force reduces to

$$f_z^{ext} = \frac{\gamma^4}{2} \int d\lambda \int d\lambda_1 \int d\lambda_2 \partial_z(x) G_{ret}(x, \bar{z}^\alpha(\lambda)) \Sigma(\tau, \lambda_1) \times \mathcal{F}_{ext}(\lambda_1) \Sigma(\tau, \lambda_2) \mathcal{F}_{ext}(\lambda_2) \Big|_{x^\alpha = \bar{z}^\alpha(\tau)} \quad (5.114)$$

where  $\mathcal{F}_{ext}(\lambda) = \int d^2y G_{ret}(\bar{z}(\lambda), y) J_{ext}(y)$ . For a spatially uniform and sinusoidally time-varying current inside the cavity, that is  $J_{ext}(y) = A \sin \Omega t \theta(\bar{z} - y)$ , we can evaluate  $\mathcal{F}_{ext}$  giving

$$\mathcal{F}_{ext}(\lambda) = -\frac{A}{\Omega^2} \left[ 2 \sin \Omega(\bar{z} - \lambda) - \sin \Omega(2\bar{z} - \lambda) + \sin \Omega(\lambda) \right]. \quad (5.115)$$

The derivative on the retarded Green's function in (5.114) gives a  $\delta$  function in time  $\partial_z(x) G_{ret}(x, \bar{z}^\alpha(\lambda)) \Big|_{x^\alpha = \bar{z}^\alpha(\tau)} = \delta(\tau - \lambda - 2\bar{z})/2$  allowing the integral over  $\lambda$  to be done. In the limit of large but finite  $\gamma$  (where we intend to focus on the fundamental mode) we evaluate the convolution of  $\Sigma$  with  $\mathcal{F}$

$$\begin{aligned} \int d\lambda_1 \Sigma(\tau, \lambda_1) \mathcal{F}_{ext}(\lambda_1) &= \frac{2A}{\Omega^2 \gamma^2 \bar{z}} \sum_{n=1}^{\infty} \left\{ \frac{4e^{-\Gamma_n \tau} \sin^2(\bar{z}\Omega/2)}{(\Gamma_n^2 + \omega_n^2)^2 + 2(\Gamma_n - \omega_n)(\Gamma_n + \omega_n)\Omega^2 + \Omega^4} \right. \\ &\quad \times \left[ \omega_n \Omega \sin \omega_n \tau \left( -(\Gamma_n^2 + \omega_n^2 - \Omega^2) \cos \Omega \bar{z} + 2\Gamma_n \Omega \sin \Omega \bar{z} \right) + \cos \omega_n \tau \right. \\ &\quad \left. \left. \times \left( \Gamma_n \Omega (\Gamma_n^2 + \omega_n^2 + \Omega^2) \cos \Omega \bar{z} + \left( (\Gamma_n^2 + \omega_n^2)^2 + (\Gamma_n - \omega_n)(\Gamma_n + \omega_n)\Omega^2 \right) \sin \Omega \bar{z} \right) \right] \right\} \end{aligned}$$

$$+e^{\Gamma_n \tau} \Omega \left( -\Gamma_n (\Gamma_n^2 + \omega_n^2 + \Omega^2) \cos \Omega(\bar{z} - \tau) + \Omega (\Gamma_n^2 - \omega_n^2 + \Omega^2) \sin \Omega(\bar{z} - \tau) \right) \Big] \Big\}. \quad (5.116)$$

In distinction to the case of an ideal cavity where the external current populates the field modes linearly in time when driven on resonance, the radiation pressure force for the imperfect cavity saturates on a timescale  $T_{s.s.}$  determined by the damping constant  $\Gamma_1 = 1/T_{s.s.}$  to a steady-state value. The other key characteristic is that if the mirror is moved non-adiabatically from one position to another it requires a finite time for the radiation pressure force to build to its new steady-state value.

If we now focus on the specific case where the driving current oscillates at a frequency slightly detuned from the frequency of the fundamental mode of the cavity and drop transient terms we approximate the steady-state radiation pressure force by dropping all but the first term in the sum in (5.116). In the long time limit the resulting force will oscillate around its mean value. Assuming the characteristic time scale for the dynamics of the mirror is much shorter than the light crossing we can safely time-average over the oscillations of the radiation pressure force giving

$$\langle f_z^{ext} \rangle_{T.A.} \stackrel{s.s.}{\approx} \frac{4A^2}{\bar{z}^2} \frac{1}{(\Gamma_1^2 + \omega_1^2)^2 + 2(\Gamma_1 - \omega_1)(\Gamma_1 + \omega_1)\Omega^2 + \Omega^4} \quad (5.117)$$

where *s.s.* stands for steady-state.

### 5.5.5.1 Phenomenological Discussion of Mirror Cooling

To understand how mirror cooling manifests we will now give a phenomenological discussion based on the radiation pressure force (5.117) plotted in Figure 5.2. We begin first by describing the setup. At  $z = 0$  we place an immovable perfect

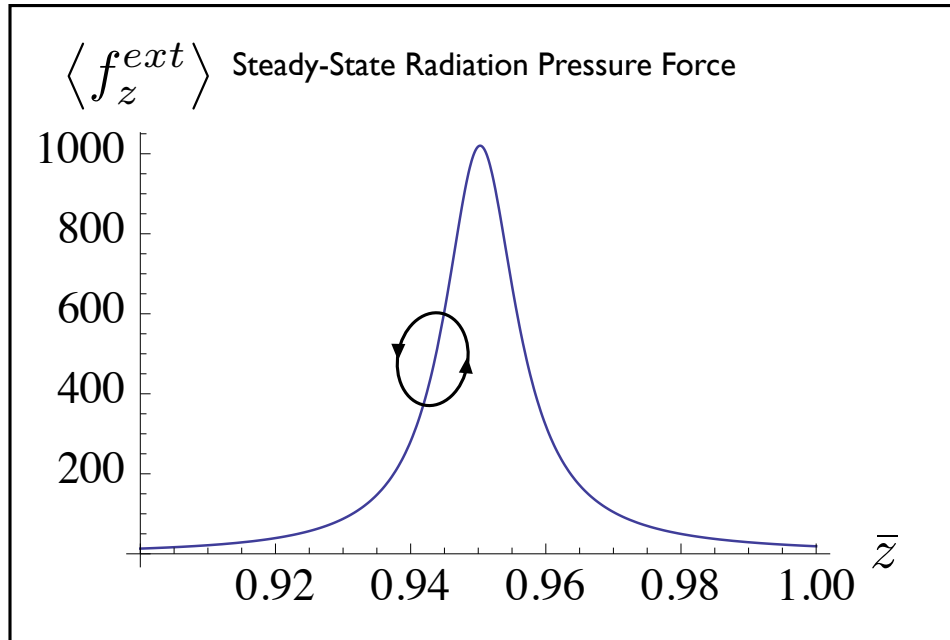


Figure 5.2: The plot above shows the steady state radiation pressure force given in Eqn. (5.117) for  $A = 1$ ,  $\Omega = \pi$ , and  $\epsilon = 10$  (in natural units). The black loop shows the radiation pressure force on a mirror moving non-adiabatically through one oscillation and illustrates how mirror cooling takes place. If the mirror oscillates around an equilibrium position to the left of the resonance, then as the mirror moves to the right the radiation pressure force will increase. However, the actual force on the mirror is less than the steady-state value (as exemplified by the black curve dipping below the blue curve while the mirror is approaching the resonance) because of the finite time required for the radiation pressure force to reach its steady state value. As the mirror begins to recede from the resonance the radiation pressure force is greater than the steady-state value because of the finite ring down rate for the cavity. This process allows for energy to be transferred from the oscillatory motion of the mirror to the cavity field,  $\oint dx F < 0$ .

mirror and place the equilibrium position of a semi-transparent mirror at  $\bar{z}$ . We now choose the frequency of the laser pumping the cavity,  $\Omega$ , to be slightly detuned from the fundamental cavity mode such that the mirror's position is to left of the resonance position, that is  $\bar{z} < \pi/\Omega = z_{res}$ .

In this arrangement as the mirror moves to the right, toward the resonance, the radiation pressure force will become larger, and as the mirror moves to the left the force becomes smaller. For cooling to take place we need the time scale for the dynamics of the mirror to be shorter than the decay time for the cavity. Assuming these criterion are satisfied we reconsider one oscillation of the mirror. As the mirror moves toward the resonance the radiation pressure force will increase, but because it takes less time for the mirror to move to its turning point than it takes for the amplitude of the cavity to field to build up to its steady-state value at the turning point, the radiation pressure force along this piece of the trajectory is less than the steady-state value (as the black curve in Figure 5.2. shows). At the turning point the mirror begins to recede from the resonance and the radiation pressure force will begin to decrease, but the amplitude of the cavity field will decay at the cavity ringdown rate and so the radiation pressure force on the mirror will be greater than the steady-state value along this arc of the trajectory which implies that the cavity field does work on the mirror for each oscillation, that is  $\oint dx F < 0$ , which leads to cooling of the mirror's mechanical motion.

## 5.6 Quantum Field Backaction on a Cavity Mirror

We now understand the mechanism by which mirror cooling occurs—retardation. In this section we aim to go beyond the phenomenological discussion of cooling in the last section and derive the leading order effect of damping due to radiation pressure. To do this we consider the influence action at  $O(\tilde{z}^2)$ . This will give lengthy expressions leading to two effects, modification of the oscillator’s spring constant, the so-called optical spring effects, and dissipation.

To see dissipation we allow the mirror to be partially transmitting. For such a case it pays, computationally, to reformulate the problem slightly. For a single mirror on an unprescribed trajectory the calculations were facilitated by allowing the constraining field to account for the boundary conditions on the mirror. In the case we are considering now the trajectory is prescribed up to a perturbation and so our work is greatly simplified by integrating out the constraining field from the outset. The field modes can then be solved for exactly in the background described by two stationary mirrors, and the solution can then be constructed perturbatively. As stated previously, integrating out the constraining field explicitly leads to a self-interaction of the field,  $\tilde{S}_\phi$ , on the worldline of the mirror so that the total action for the field becomes

$$S_\phi[\phi] = \int d^2x \left[ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + J_{ext}(x) \phi(x) \right. \\ \left. - \underbrace{\frac{\gamma^2}{2\mathcal{M}} \int d\lambda u(\lambda) \phi^2[z^\alpha(\lambda)]}_{\tilde{S}_\phi[\phi]} \right]. \quad (5.118)$$

We can now expand  $\tilde{S}_\phi$  around the stationary trajectory  $\bar{z}^\alpha$  where the first three orders are

$$\tilde{S}_\phi^{(0)}[\phi] = -\frac{\gamma^2}{2\mathcal{M}} \int d\lambda \phi^2[\bar{z}^\alpha(\lambda)] \quad (5.119)$$

$$\tilde{S}_\phi^{(1)}[\phi] = -\frac{\gamma^2}{\mathcal{M}} \int d\lambda \tilde{z}(\lambda) \phi[\bar{z}^\alpha(\lambda)] \partial_x(x) \phi(x) \Big|_{x=\bar{z}^\alpha(\lambda)} \quad (5.120)$$

$$\begin{aligned} \tilde{S}_\phi^{(2)}[\phi] = \frac{\gamma^2}{4\mathcal{M}} \int d\lambda \Big[ & w_{\alpha\beta} \tilde{U}^\alpha(\lambda) \tilde{U}^\alpha(\lambda) + 2\tilde{z}^2(\lambda) [\partial_x(x) \partial_x(x') \\ & + \partial_x^2(x)] \phi(x) \phi(x') \Big]_{x=x'=\bar{z}^\alpha(\lambda)}. \end{aligned} \quad (5.121)$$

Using (5.119) we now construct the influence action describing the averaged effect of the constrained quantum field on the mirror trajectory perturbatively using the generating functional method. To do so we lump  $\tilde{S}_\phi^{(0)}[\phi]$  into the free action for the quantum field which then describes the fields free motion in the cavity. The field modes can be constructed explicitly and so the generating functional can be evaluated

$$\begin{aligned} Z_\phi^{BC}[\mathcal{J}^\pm] = \exp \Big\{ & i \int d^2x d^2x' [\mathcal{J}^-(x) G_{ret}^{BC}(x, x') (\mathcal{J}^+(x') + J_{ext}(x')) \\ & + \frac{i}{4} \mathcal{J}^-(x) G_H^{BC}(x, x') \mathcal{J}^-(x')] \Big\} \end{aligned} \quad (5.122)$$

where the  $G^{BC}$  is a Green's function for the field constructed from field modes obeying the dynamics described by the modified action of the field  $S_\phi + \tilde{S}_\phi^{(0)}$ , and  $BC$  can stand for boundary conditions or Barton and Calogheracos.

The influence action will now be given in terms of field expectation values as given by (5.122) and can be shown to equal

$$\begin{aligned} \Gamma[\tilde{z}^\pm] \approx & (Z_\phi^{BC}[0])^{-1} \langle \Delta \tilde{S}_\phi^{(1)} \rangle_\phi^{BC} + (Z_\phi^{BC}[0])^{-1} \langle \Delta \tilde{S}_\phi^{(2)} \rangle_\phi^{BC} \\ & + \frac{i}{2} \left[ (Z_\phi^{BC}[0])^{-1} \left\langle \left( \Delta \tilde{S}_\phi^{(1)} \right)^2 \right\rangle_\phi^{BC} - (Z_\phi^{BC}[0])^{-2} \left( \langle \Delta \tilde{S}_\phi^{(1)} \rangle_\phi^{BC} \right)^2 \right] \end{aligned} \quad (5.123)$$

to second order in  $\tilde{z}$ .

Each of these expectation values can be evaluated and can be shown to be composed of the following field correlation functions

$$(Z_\phi^{BC}[0])^{-1} \langle \phi^+(x) \phi^-(x') \rangle_\phi^{BC} = -i G_{ret}^{BC}(x, x') \quad (5.124)$$

$$(Z_\phi^{BC}[0])^{-1} \langle \phi^+(x) \phi^+(x') \rangle_\phi^{BC} = \frac{1}{2} G_H^{BC}(x, x') + \mathcal{F}_{ext}(x) \mathcal{F}_{ext}(x') = \mathcal{G}_H(x, x') \quad (5.125)$$

where we've introduced the notation  $\mathcal{G}_H$  and  $\mathcal{F}_{ext}(x) = \int d^2y G_{ret}^{BC}(x, y) J_{ext}(y)$  to keep long expressions compact. There are also four-point functions of the field appearing in  $\langle (\Delta \tilde{S}_\phi^{(1)})^2 \rangle_\phi^{BC}$  that can be expressed in terms of two-point functions

$$\begin{aligned} & (Z_\phi^{BC}[0])^{-1} \langle \phi^+(x) \phi^+(y) \phi^+(z) \phi^-(x') \rangle_\phi^{BC} = \\ & -i G_{ret}^{BC}(x, x') \mathcal{G}_H(y, z) - i G_{ret}^{BC}(y, x') \mathcal{G}_H(z, x) - i G_{ret}^{BC}(z, x') \mathcal{G}_H(x, y). \end{aligned} \quad (5.126)$$

Explicitly evaluating the field expectation values the influence action takes the form below where we've partitioned the terms order by order in powers of  $\tilde{z}$  and dropped terms irrelevant to the backaction.

$$O(\tilde{z}) : \Gamma_1[\tilde{z}^\pm] = -\frac{\gamma^2}{\mathcal{M}} \int d\lambda \tilde{z}^-(\lambda) \partial_x(x) \mathcal{G}_H(x, \bar{z}(\lambda)) \Big|_{x=\bar{z}(\lambda)} \quad (5.127)$$

$$\begin{aligned} O(\tilde{z}^2) : \Gamma_2[\tilde{z}^\pm] = & \frac{\gamma^2}{2\mathcal{M}} \int d\lambda \left[ w_{\alpha\beta} \tilde{U}^{\alpha+}(\lambda) \tilde{U}^{\beta-}(\lambda) + 2\tilde{z}^+(\lambda) \tilde{z}^-(\lambda) [\partial_x(x) \partial_x(x') \right. \\ & \left. + \partial_x^2(x)] \mathcal{G}_H(x, x') \right]_{x=x'=\bar{z}(\lambda)} \\ & + \frac{\gamma^4}{\mathcal{M}^2} \int d\lambda d\lambda' \tilde{z}^+(\lambda) \tilde{z}^-(\lambda') \partial_x(x) \partial_x(y) \left[ \mathcal{G}_H(x, y) G_{ret}^{BC}(x, y) \right]_{x=\bar{z}(\lambda), y=\bar{z}(\lambda')} \end{aligned} \quad (5.128)$$

Variation of  $\Gamma$  provides the backaction on the mirror allowing us to write down the equation of motion

$$\begin{aligned} M(\lambda) \ddot{\tilde{z}}(\lambda) + \eta(\lambda) \dot{\tilde{z}}(\lambda) + M\Omega_{mech}^2 (\tilde{z}(\lambda) - z_{eq}) + M\Delta\Omega_{mech}^2(\lambda) \tilde{z}(\lambda) \\ + \int d\lambda' K(\lambda, \lambda') \tilde{z}(\lambda') = f_{static}[\bar{z}(\lambda)] \end{aligned} \quad (5.129)$$

where  $z_{eq}$  is the equilibrium position of the mirror free of interaction with the field and can be chosen to cancel the forces on a stationary mirror. From here we see several effects come into play. Starting from the left we notice that the field dresses the mirror resulting in a time dependent shift of its mass  $\Delta M(\lambda) = M(\lambda) - M$

$$\Delta M(\lambda) = \frac{\gamma^2}{2\mathcal{M}} \mathcal{G}_H(\bar{z}(\lambda), \bar{z}(\lambda)). \quad (5.130)$$

The field's interaction with the mirror also leads to a time dependent damping (or heating, depending on the sign of  $\eta(\lambda)$ )

$$\eta(\lambda) = -\frac{\gamma^2}{2\mathcal{M}} \frac{d}{d\lambda} \mathcal{G}_H(\bar{z}(\lambda), \bar{z}(\lambda)), \quad (5.131)$$



and to a shift in the natural frequency of the mirror  $\Delta\Omega_{mech}^2(\lambda)$

$$M\Delta\Omega_{mech}^2(\lambda) = -\frac{\gamma^2}{\mathcal{M}}[\partial_x(x)\partial_x(x') + \partial_x^2(x)]\mathcal{G}_H(x, x')\Big|_{x=x'=\bar{z}(\lambda)}. \quad (5.132)$$

The shift in the mechanical frequency is due to two effects the gradient of the Casimir and radiation pressure forces at the position of the mirror. We also find a non-Markovian effect which enters the equation of motion as an integral of the mirror trajectory against a kernel  $K(\lambda, \lambda')$

$$K(\lambda, \lambda') = -\frac{\gamma^4}{\mathcal{M}^2}\partial_x(x)\partial_x(y)\left[\mathcal{G}_H(x, y)G_{ret}^{BC}(x, y)\right]_{x=\bar{z}(\lambda'), y=\bar{z}(\lambda)}. \quad (5.133)$$

Lastly, on the right hand side of equation of motion (5.129) we have the forces on a stationary mirror  $f_{static}[\bar{z}]$  resulting from the Casimir force and radiation pressure.

$$f_{static}[\bar{z}(\lambda)] = -\frac{\gamma^2}{\mathcal{M}}\partial_x(x)\mathcal{G}_H(x, \bar{z}(\lambda))\Big|_{x=\bar{z}(\lambda)} \quad (5.134)$$

We now analyze the structure of the terms that enter the equation of motion for the mirror for the case of radiation pressure. The quantum effects, such as dynamical Casimir effect and vacuum viscosity are interesting yet play a minor role in mirror cooling (for the case of the backaction on a single mirror due to the dynamical Casimir effect the cooling timescale can be the larger than the age of the Universe (5.72) ). For the purposes of this paper, to understand mirror cooling, we focus entirely on the backaction induced by the radiation pressure. To proceed we invoke two approximations that enable us to evaluate the backaction and yet

capture the salient features of the quantum field in the cavity. First, we assume the movable mirror is nearly ideal so that we can approximate the quantum field's mode structure by a discrete set, yet retaining the leading order effects of dissipation

$$G_{ret}^{BC}(x, x') \approx \frac{1}{2\bar{z}} \sum_{modes} \frac{1}{\omega_n} e^{-\Gamma_n(t-t')} \sin \omega_n(t-t') [\cos \omega_n(x-x') - \cos \omega_n(x+x')] \theta(t-t'). \quad (5.135)$$

Second, we assume that the frequency of the pumping laser is slightly detuned from the fundamental resonance of the cavity so that the dominant contribution to the backaction results from the interaction of the mirror with the cavity's fundamental mode. For this approximation to hold it is important to note that mirror velocity must remain small so that scattered radiation cannot populate higher cavity resonances which imposes the constraint  $|\tilde{z}|\Omega_{mech} \ll 1$  on the product of the amplitude and the mechanical frequency of the oscillator.

$$G_{ret}^{BC}(x, x') \approx \frac{1}{\bar{z}\omega_1} e^{-\Gamma_1(t-t')} \sin \omega_1(t-t') [\cos \omega_1(x-x') - \cos \omega_1(x+x')] \theta(t-t'). \quad (5.136)$$

Using this form for the retarded Green's function in the cavity we can plot the behavior of the various coefficients. First, we start with radiation pressure force. Figure 5.3. shows the spatial dependence of radiation pressure at late times and the time dependence for fixed position. We note particularly that the time dependence exhibits oscillations and relaxation. The high frequency behavior corresponds with the sinusoidal pumping of the cavity and longer wavelength oscillation has a period determined by the detuning of the pumping laser from the fundamental resonance

of the cavity. We assume that the dynamics of the mirror occurs on a time scale much longer than the light travel time in the cavity and so we can safely average the radiation pressure force over over a single period of the laser (blue curve at right).

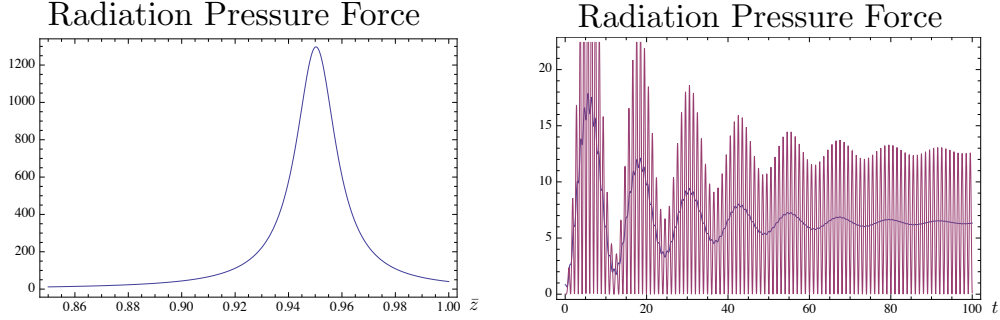


Figure 5.3: Plot of the radiation pressure force. *On the left:* The radiation pressure force as a function of  $\bar{z}$  for  $A = 1$ ,  $\Omega = \pi$ ,  $t = 10,000$ , and  $\epsilon = 10$  at late times the force relaxes to its steady-state value. *On the right:* The radiation pressure force as a function of time for  $z = 0.81$ ,  $A = 1$ ,  $\Omega = \pi$ , and  $\epsilon = 10$ . The pink curve shows the exact behavior of the force and the blue curve shows the force after it has been time averaged over a single period of the pumping laser exhibiting oscillations with frequency determined by the detuning.

From Figure 5.4. we see that the mass shift and the damping constant oscillate in time. The time average of the mass shift takes on a finite value oscillating at a frequency determined by the detuning at early times and relaxes to a constant in the long-time limit. The damping function oscillates symmetrically about the time axis and so its time average with respect to the pumping frequency will yield a negligible contribution.

Lastly, we come to the integral kernel  $K(t, t')$  which we plot in Figure 5.5 as a

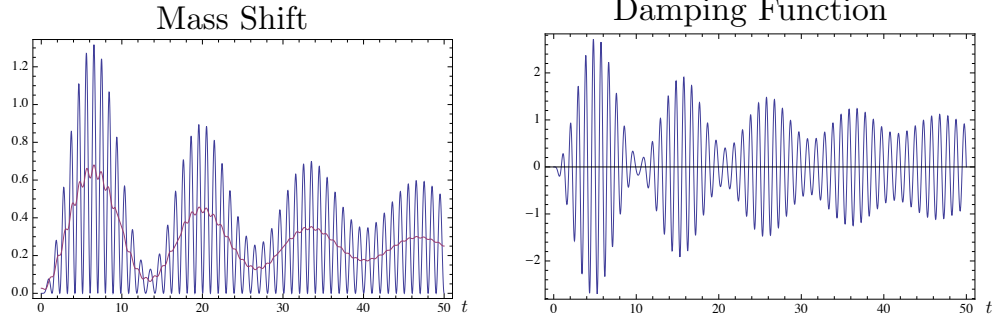


Figure 5.4: Plot of the time dependence of the mass shift,  $\Delta M(t)$  for  $A = 1$ ,  $z = 0.83$ ,  $\Omega = \pi$ , and  $\epsilon = 10$ , and the damping constant  $\eta(t)$  for  $A = 1$ ,  $z = 0.80$ ,  $\Omega = \pi$ , and  $\epsilon = 10$  in the long time limit both will be negligible to the mirror dynamics.

function of  $t'$ . This term will give rise to damping and a modification of the spring constant. We assume that the the period associated with pumping frequency,  $T_{pump}$  is much smaller than time scale for the dynamics of the mirror motion  $T_{pump} \ll T_{dyn}$ , but that the detuning, the difference between the pump and cavity frequency is small such that  $|T_{pump} - T_{cav}| \sim T_{dyn}$ . If we drop transients and rapidly rotating terms (rotating wave approximation) we can approximate the integral kernel  $K(t, t')$

$$K(t, t') \approx -\frac{4\pi}{\bar{z}^2} \langle f_z^{ext} \rangle_{T.A.} e^{-\Gamma_1(t-t')} \sin \Delta(t-t') \theta(t-t') \quad (5.137)$$

where  $\langle f_z^{ext} \rangle_{T.A.}$  is the steady state value for the radiation pressure force (5.117),  $\omega_1$  and  $\Gamma_1$  can be taken from (5.106) and (5.107), and  $\Delta$  is the detuning between the pump frequency and the cavity frequency  $\Delta = \Omega - \omega_1$ . The Fourier transform of the integral kernel  $K(t, t')$  integrated against the mirror trajectory using the approximation (5.137) gives

$$F.T. \left[ \int dt' K(t, t') \tilde{z}(t') \right] \approx \frac{4\pi}{\bar{z}^2} \langle f_z^{ext} \rangle_{T.A.} \frac{\Delta}{\Gamma^2 + \Delta^2 - \omega^2 - 2i\Gamma\omega} \tilde{z}(\omega) \quad (5.138)$$

which has an imaginary component that leads to dissipation giving rise to an effective damping constant

$$\Gamma_{eff}(\omega) = -\frac{4\pi}{M_{s.s.} \bar{z}^2} \langle f_z^{ext} \rangle_{T.A.} \frac{\Delta \Gamma}{(\Gamma^2 + \Delta^2 - \omega^2)^2 + 4\Gamma^2 \omega^2}. \quad (5.139)$$

where we've neglected transient effects in the oscillator's mass by using  $M_{s.s.}$ , *s.s.* stands for steady-state.

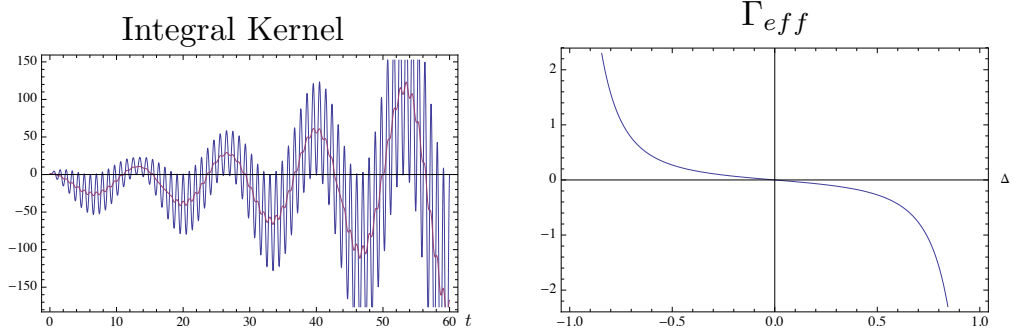


Figure 5.5: *On the left:* Plot of the integral kernel  $K(60, t)$  for  $z = 0.82$ ,  $\Omega = \pi$ , and  $\epsilon = 10$ . The blue curve gives the exact behavior while the pink derived from time-averaging exhibits the behavior of the ansatz (5.137). *On the right:* Plot of the effective damping constant  $\Gamma_{eff}(\omega)$  as a function of detuning  $\Delta$  for  $\omega = 1$  and  $\Gamma = 0.1$  in units of  $\frac{4\pi}{M_{s.s.}} \langle f_z^{ext} \rangle_{T.A.}$ . As the detuning passes through the resonance the effective damping constant changes sign going from cooling to heating.

Our effective damping constant can be directly compared with that derived in [45] (Eqn. 4) using different methods. Our formulation has provided a much more

detailed description of mirror field interactions and only when we ignore transients, other field modes and take the rotating wave approximation do we find our results reduce to the work of others. It will be the focus of a future investigation to study mirror cooling when such approximations are not undertaken.

## Chapter 6

### Discussion

In this thesis we have laid down the theoretical groundwork for the study of dispersion forces under fully nonequilibrium conditions ranging from the forces between atoms to the backaction of a quantum field on a moving mirror.

#### 6.1 Atom-Atom Forces

In Chapter 2. we derived the fully dynamical atom-atom force for general atomic motion and initial states. We have found that a careful treatment of the infinite time limit shows the existence of a novel far field scaling behavior when the atoms and field are not in thermal equilibrium. Seeking the experimental conditions to bring out this effect would require a careful balance between the suitable temperature range versus the right atomic species with the appropriate first optical resonance frequency. For entangled atoms a novel near-field scaling is obtained that dominates the standard London force in certain regimes. This new force could play an important role in quantum computing schemes involving entangled atoms, and could potentially be used as a means for measuring entanglement.

### 6.1.1 A new kind of quantum nondemolition measurement

The entanglement force requires quantum correlation of the internal degrees of freedom between two atoms, whereas the force acts on the center of mass motion. Since the dipole moment and center of mass position operators commute it may be possible to measure the position of the atom without disturbing its internal activities.

If such an experiment is feasible a time series of position measurements could quantify the entanglement force and could be used as a new kind of quantum non-demolition measurement capable of determining, without disturbing, the entanglement between two atoms.

## 6.2 Atom-Surface Force via Boundary Conditions

In Chapter 3. we have derived from first principles the semiclassical and stochastic equations for an atom's motion near a mirror under fully nonequilibrium conditions. Being a quantum field theory derivation based on microphysics, applicable to treat the full dynamics of the atom field system, there is no place for a 'local source hypothesis' which MQED needs to be generalized to nonequilibrium configurations. In the high temperature regime we expect MQED calculations to agree with our model for realistic materials where material fluctuations can be approximated as local. However, for systems out of equilibrium, when the material temperature is lowered we expect long range correlations of fluctuations in the media to produce new phenomena that are beyond the regime of validity of MQED.



### 6.2.1 Failure of the ‘local source hypothesis’

In essence the ‘local source hypothesis’ neglects the interaction among the micro-elements constituting the dielectric medium which is equivalent to describing a dielectric as a lattice of noninteracting oscillators. For a realistic material we expect that this approximation will hold well when the thermal energy of the field is much greater than the interaction energy among micro-elements, or more simply when the permittivity is local (independent of space). However, in materials composed of micro-elements exhibiting long range correlations in their fluctuations (such as near a critical point) the ‘local source hypothesis’ will fail. It remains an open question and interesting future research pursuit to employ our nonequilibrium formulation to see how the physics of atom-surface interactions is modified in such cases.

### 6.2.2 Physical manifestation of atomic trajectory dispersion

The simplest derivation of the CP force is carried out by taking the negative gradient of the expectation value of the interaction energy,  $U_{int}$ , between an atom and the constrained electromagnetic field

$$F_{CP} = -\nabla \langle U_{int} \rangle, \quad (6.1)$$

and as such the force can be regarded as a statistical quantity possessing fluctuations. We captured the statistical effects in Chapter 3 by deriving a Langevin equation to describe the fluctuations of the atom’s trajectory about the semi-classical solution. The Langevin equation provides information about the dispersion of many atomic trajectories which manifests physically in the three following experimental schemes.

### *Opacity measurements of an atom beam*

Imagine a harmonic atom trap situated parallel to, and separated by distance  $L$ , from a finite surface with normal in the  $z$ -direction such that the distance  $L$  is much smaller than the surface's linear dimensions. In this scenario the trap allows for free atom motion along the transverse directions and prevents free motion along the  $z$ -direction.

Next, imagine sending a beam of atoms down the trap with the same initial conditions, as the beam traverses the length of the surface quantum fluctuations of the field will perturb the trajectory of each atom in a way described by (3.21) leading to beam broadening. This broadening will result in a decrease in density directly related to the beam's opacity which can be measured using standard techniques.

### *Size measurement and velocity distribution of a trapped gas of noninteracting atoms*

Alternatively one could consider a trapped gas of noninteracting atoms at rest near a surface. If the atoms do not interact we can view each of the constituents of the gas as one realization of the Langevin equation (3.21). In this case the dispersion of the trajectory is proportional to the size of the 'cloud' of atoms. As the trap is moved perpendicular to the surface the shape of the cloud will be deformed as described by (3.23). A series of measurements of the cloud size by light scattering for various trap surface spacings would yield an experimental test of our predictions.

By considering the dispersion in the momentum, the Langevin equation can be used to derive a velocity distribution function for the atomic cloud. As the trap

is removed the gas will begin to expand freely into the cavity. A time series of density measurements performed via light scattering techniques would provide an experimental means for reconstructing the velocity distribution.

*Frequency response of the center-of-mass oscillations of a Bose-Einstein condensate (BEC) in a harmonic trap*

The experiments of Cornell et. al. [14, 28] measured the Casimir-Polder force by observing the center-of-mass oscillations of a BEC in a magnetic trap. As the trapped BEC is moved relative to a surface the trapping potential is modified by the Casimir-Polder force leading to a spatially dependent change in the center of mass oscillation frequency which relates directly to the gradient of the CP force.

Our formalism has provided a means to quantify the fluctuations in the CP force which can be related to dissipation through the FDR [31]. This can be easily understood by imagining a mass oscillating at the minimum of a harmonic potential. In the absence of a stochastic force the mass will undergo periodic motion indefinitely or in other words the frequency response will have a sharp peak at the resonance frequency of the potential.

As we allow for a stochastic force to impart random kicks to the oscillating mass we'll observe that occasionally the period of oscillation will be less than that determined by the resonance frequency if a kick is imparted parallel to the direction of motion, and sometimes the period will be longer if the random force acts to slow the mass.

This would manifest as a line broadening in the frequency response indicating the presence of dissipation. Repeating the Cornell experiment with the aim of measuring the frequency response of the BEC's motion in the trap or by observing the damping time-scale (to quantify the quality factor) would provide a direct test of our predictions and would provide the first experimental evidence for quantum friction.

### 6.3 General Atom-Surface Forces

In Chapter 4. we have derived the force between an atom and a half-space medium modeled by non-interacting harmonic oscillators. Our aim has been to derive from a microscopic starting point the force between an atom and a slab, including fully dynamical activities such as absorption, radiation and fluctuations. In the end we show that the force is in agreement with the results of previous works employing MQED theory when the long time, stationary limit is taken. We point out that in addition to giving a fully dynamical description for atom motion in the presence of a dielectric material the nonequilibrium quantum field theoretic formulation of this problem is particularly adept at handling fluctuations. It will be the aim of a future study to understand the rich interplay between the propagating and evanescent wave components of the force and their effect upon the fluctuations of the atom's trajectory in space.

## 6.4 Mirror Cooling by Quantum Field Backaction

In Chapter 5 we have described the motion of a partially transmitting mirror in a quantum field. To frame the problem we began with the simplest scenario— a single mirror.

### 6.4.1 Black hole backaction analog

The result that black holes evaporate [87] shakes the very foundations of physical law as we know it. Because the evaporation process allows for the conversion of pure quantum states into mixed, in this case thermal, unitarity is violated— the so-called black hole information loss paradox [88].

A moving mirror with a suitably prescribed trajectory can mimic the radial geometry of a black hole [89]. This connection can be understood by imagining the s-wave scattering of a black hole as it is undergoing gravitational collapse. As an incoming spherical wave closes on the origin of, what will become, the black hole it will undergo gravitational blue-shifting. If the event horizon has not yet formed this wave will collapse through the origin and be converted into an outgoing spherical wave red-shifting as it travels to  $\mathcal{J}^+$  (here  $\mathcal{J}^{+(-)}$  refers to future (past) null infinity). If the collapse is dynamical then there is a net red-shifting of the outgoing s-wave, whereas if the spacetime were stationary the red-shifting would compensate exactly for the blue-shifting and the wave would arrive at  $\mathcal{J}^+$  with the same frequency it had on  $\mathcal{J}^-$ .

The reduction of the s-wave scattering problem to 1+1 spacetime follows from

spherical symmetry where in the reduced space the conversion of an incoming s-wave to outgoing looks like reflection from a mirror. By exploiting the fact that all two dimensional spacetimes [80] are conformal to Minkowski space we can describe the propagation of waves in the reduced space as occurring in flat spacetime where the gradual red and blue-shifting of s-waves in a dynamic gravitational potential is replaced by reflection from a moving mirror. The mirror trajectory therefore will play the role of the radial geometry. A suitable choice for the trajectory of this moving mirror can mimic the effects of the spacetime of a collapsing mass by giving rise to a flux of thermal radiation seen by observers at  $\mathcal{J}^+$ .

However, Hawking's original calculation as well as the moving mirror analogs were done in the test field approximation meaning that the dynamics of the spacetime or that the motion of the mirror is prescribed from the outset. In this approximation the emitted radiation does not back act on the spacetime or the mirror trajectory. In the early stages of collapse, for very large masses, this approximation is well justified but as the evaporation process unfolds the mass of the hole will slowly be shed and a point will be reached when the emitted radiation will have energy of order of the black hole mass. In this regime back action of the radiation will play a significant role in determining the geometry of the spacetime or the motion of the mirror and it is widely believed that these backaction effects can rescue unitarity and quantum mechanics. To date it remains an open problem to describe the self-consistent dynamics of black hole evaporation.

We go beyond the test field approximation because our formulation of the moving mirror problem accounts for the backaction of the quantum field on the

mirror. Although limited, it is our hope that a moving mirror analog of black hole evaporation could shed light on some of these issues ranging from the information loss paradox to the end state problem.

#### 6.4.2 Mirror cooling by quantum field backaction

For the case of a cavity with variable length we have provided the leading order equations of motion for small amplitude oscillations about a stationary trajectory. The equations of motion for the cavity mirror exhibit well known features such as the Casimir effect, optical spring, and radiation pressure damping, but also provide non- Markovian effects which show up as integrals against the mirror trajectory in the equations of motion.

When we ignore transients, modes of the cavity field beyond the fundamental, and take the rotating wave approximation the effective damping constant for the mirror's motion in a cavity under the influence radiation pressure reduces to results derived by others. Our complete formulation of the interaction of a moving mirror with a quantum field provides a much richer picture of the cooling process. As experiments improve and lower temperatures become attainable the effects contained in our formulation (and ignored by others) may play a critical role in the design of future experiments and the attainment of ground state cooling for macroscopic objects.

## 6.5 Conclusion

In conclusion, fluctuation forces play a critical role in atom-field interactions and optomechanics. They are responsible for the dominant force between neutral objects at short distances ( $<100$  nm) making them an essential consideration in AMO physics, nano electro mechanical and micromechanical devices, and null tests of the inverse square law of gravitation at short distances to name a few. Fundamentally, the use of novel techniques like radiation pressure cooling could facilitate observation of the quantum to classical transition in the lab, lead to attainable quantum states for macroscopic objects, and improve noise reduction schemes in LIGO. With the rapid advances in experimental techniques such as the observation of real-time processes there is an urgent need for an upgrade to more sophisticated theories to match the challenge from current and future experimental possibilities. We believe nonequilibrium quantum field theory fulfills this need, its applications to atomic and optical physics is illustrated in this thesis.



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