ABSTRACT

Title of dissertation: A TANNAKIAN DESCRIPTION FOR PARAHORIC BRUHAT-TITS GROUP SCHEMES

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Doctor of Philosophy, 2010

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Let K be a field which is complete with respect to a discrete valuation and let $\mathcal{O}$ be the ring of integers in K. We study the Bruhat-Tits building $\widetilde{\mathscr{B}}(G)$ and the parahoric Bruhat-Tits group schemes $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ associated to a connected reductive split linear algebraic group G defined over $\mathcal{O}$.

In order to study these objects we use the theory of Tannakian duality, developed by Saavedra Rivano, which shows how to recover G from its category of finite rank projective representations over $\mathcal{O}$. We also use Moy-Prasad filtrations in order to define lattice chains in any such representation. Using these two tools, we give a Tannakian description to $\widetilde{\mathscr{B}}(G)$.

We also define a functor $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ associated to a facet $\widetilde{\mathfrak{F}} \subset \widetilde{\mathscr{B}}(G)$ in terms of lattice chains in a Tannakian way. We show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is representable by an affine group scheme of finite type, has generic fiber G_K, and satisfies $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) = \mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$ for every unramified Galois extension E of K.

Moreover, we show that there is a canonical morphism from $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ to $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$, which we conjecture to be an isomorphism. We prove that it is an isomorphism when the residue characteristic of K is zero and G is arbitrary, when $G = GL_n$ and K is arbitrary, and when $\widetilde{\mathfrak{F}}$ is the minimal facet containing the origin and G and K are arbitrary.

A TANNAKIAN DESCRIPTION FOR PARAHORIC BRUHAT-TITS GROUP SCHEMES

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2010

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Acknowledgments

I would like to take this opportunity to thank all of the people that have made this thesis possible. First, and foremost, I would like to thank my advisor, Professor Thomas Haines, for all of the time and energy he has put into developing me as a mathematician. He has truly been a fantastic advisor, and I could not have asked for a better thesis problem to work on. Moreover, I greatly appreciate all of the financial support he has provided me with, which has allowed me to focus on my research and to attend several conferences.

I would also like to thank the Maryland Mathematics Department as a whole for all of the support, both financial and otherwise, that they have given me throughout my time here. I am especially grateful for the NSF VIGRE fellowship that was awarded to me for my first four years here. I owe a special thank you to Professor Jeffrey Adams for all of the knowledge he has imparted to me over the years through the wonderful classes he taught and the various discussions we have had. I also owe a debt of gratitude to Professor Niranjan Ramachandran for pointing out an important reference which is used in the proof of Proposition 4.2.13.

Graduate school would have been much less enjoyable without the help and friendship of the many people I have met here. In particular, I would like to thank Ted Clifford, Walter Ray-Dulany, Sean Rostami, Chris Zorn, Nick Long, Moshe Adrian, and Ben Lauser for all of the fun times and interesting discussions we have had over the years. Finally, I would like to thank my parents, Kevin and Betty Wilson, and my fiancée, Beth McLaughlin, for always being there, and for providing all of the love and support I could ever want or need.

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Chapter 1

Introduction and Summary of Results

The notion of a building was introduced by Jacques Tits in the 1950's to provide a geometric tool for the study of semi-simple Lie groups over general fields in a systematic way. See [T2], [Br], and [Rou]. These buildings are simplicial complexes made up of apartments which are spheres, and hence they are called spherical buildings.

In the 1960's, François Bruhat and Jacques Tits ([BT1]) defined another building associated to a semi-simple Lie group, or more generally a reductive group, over a field equipped with a non-archimedean discrete valuation. These buildings are (poly)-simplicial complexes whose apartments are affine spaces, and they are called affine buildings or Bruhat-Tits buildings. They are the analogue in the non-archimedean setting of the symmetric space associated to a real reductive Lie group.

Let G be a connected reductive group defined over a non-archimedean field K with ring of integers $\mathcal{O}$, and let $\widetilde{\mathscr{B}}(G(K))$ denote the Bruhat-Tits building associated to G. In their seminal works [BT1] and [BT2], Bruhat and Tits proved many things about the structure of $\widetilde{\mathscr{B}}(G(K))$ and also defined some important objects which relate to $\widetilde{\mathscr{B}}(G(K))$.

At least for our purposes, the most important related object they defined was the parahoric Bruhat-Tits group scheme $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ associated to a (poly)-simplex $\widetilde{\mathfrak{F}}$, called a facet, in $\widetilde{\mathscr{B}}(G)$. It is a smooth connected affine group scheme over $\mathcal{O}$ with generic fiber G such that, at least when G is simply connected, $\mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O})$ is the stabilizer in $G(K)$ of the facet $\widetilde{\mathfrak{F}}$. More generally, the group $\mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O})$ is a subgroup of the fixer of $\widetilde{\mathfrak{F}}$ in $G(K)$ called a parahoric subgroup.

Parahoric subgroups are important objects in the theory of Shimura varieties (see [RZ], [H2]) and also in representation theory. In particular, in [MP1] and [MP2] Allen Moy and Gopal Prasad defined an $\mathbb{R}$-filtration on parahoric subgroups and on the Lie algebra associated to G. This allowed them to define the depth of an admissible representation, which is an important invariant of such representations. Moreover, in the depth zero case they related the representation theory of G to the well understood representation theory of an associated finite group.

Thus it is important to understand the parahoric Bruhat-Tits group schemes, which are somewhat mysterious objects in general. Bruhat and Tits themselves did significant work on understanding these objects in [BT2], [BT3], and [BT4]. In the latter two papers, they described the Bruhat-Tits building of a classical group G in terms of graded lattice chains on the standard representation V of G. Under this identification, the group schemes $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ can be realized as stabilizers of these lattice chains in V. Moreover, using these concrete descriptions, the Moy-Prasad filtrations on a parahoric subgroup can be described in terms of lattice chains. See [Yu].

More recently, in [GY1] and [GY2] Wee Teck Gan and Jiu-Kang Yu described the Bruhat-Tits buildings and group schemes associated to the exceptional groups G_2, F_4, and E_6 in terms of a set of norms on a representation V, or equivalently a set of graded lattice chains in V. In each of these cases they choose one representation of the particular group which remains fixed throughout their descriptions.

The goal of this thesis is to give a uniform lattice chain description to all of the Bruhat-Tits buildings and group schemes associated to the split classical groups and the exceptional groups. In all of the above cases, a particular representation is chosen through which these objects are described. However, the choice made depends greatly on the case you are in. In order to get a more uniform description it seemed natural to use the entire category of representations associated to G, and for this we need the theory of Tannakian duality.

In the 1930's, Pontryagin showed in [Pon] that every locally compact abelian group can be recovered from its character group. This is known as Pontryagin duality. The idea of recovering a group from its representations has been generalized in many different ways. In the late 1930's, Tannaka showed in [Tan] that every compact Lie group can be recovered from its category of representations, and so results of this kind have been called a Tannakian duality.

Now let G denote a split connected reductive group defined over $\mathcal{O}$, and let $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ denote the category of finite rank projective $\mathcal{O}$-representations of G. The generalization of Tannakian duality that is important for us was given by Saavedra Rivano in [Saa], and it applies to G and the category $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. We use this version of Tannakian duality to define a group functor $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ that depends on lattice chains in every representation $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. It is a candidate for a general lattice chain description of $\mathscr{G}_{\widetilde{\mathfrak{F}}}$.

1.1 Outline of Thesis

In order to define and understand a functor which depends on the category of representations $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, we first need to study this category. This takes up the bulk of chapter 2. In the first section, we set the standard notation that will hold throughout this thesis. In Sections 2.2.1 and 2.2.2, we review the facts that every group G as above has a faithful representation in $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, and that any faithful representation is a tensor generator for the category of representations. We use these results several times throughout the thesis. Their main use is to prove that the functor $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is of finite type.

In Section 2.2.3, we review a specific kind of representation, called a Weyl module, which we use in chapter 3. Chapter 2 concludes with a section which discusses Tannakian duality in our context, which is essential to the rest of the thesis.

In chapter 3 we begin to apply Tannakian ideas to Bruhat-Tits theory. We first define, for any $x \in \widetilde{\mathscr{B}}(G), r \in \mathbb{R}$, a Moy-Prasad filtration $V_{x,r}$ of an arbitrary representation V in $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. In this way, we obtain lattice chains in an arbitrary V. We then prove some basic facts about how subgroups of the group G act on these filtrations and give some characterizations of the building in terms of these filtrations. In particular, we show

Proposition. *(Proposition 3.7.1) The map $x \to \{V_{x,r}\}$ is a bijective map from $\widetilde{\mathscr{B}}(G)$ to the set of Moy-Prasad filtrations on the category $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$.*

Theorem 3.7.2 is similar, but it has a more Tannakian flavor.

Chapter 4 is the heart of the thesis. It begins with a characterization of $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ and a review of a few of its basic properties. We then give the definition of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ (see Definition 4.1.1) and proceed to show that it shares many properties with $\mathscr{G}_{\widetilde{\mathfrak{F}}}$. In chapter 5 we present proofs of the various cases in which we know $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$. Across these 2 chapters we prove the following:

Theorem. *Let E be an unramified Galois extension of K, and let k be the residue field associated to K.*

(a) *(Theorem 4.2.1) The group functor* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *is represented by an affine $\mathcal{O}$-group scheme of finite type such that the generic fiber of* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *is G_K, and* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) = \mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$*. Moreover, there is a unique homomorphism ϕ of group schemes defined over $\mathcal{O}$,*

$$\phi : \mathscr{G}_{\widetilde{\mathfrak{F}}} \to \mathrm{Aut}_{\widetilde{\mathfrak{F}}},$$

which is the identity on the generic fiber and on $\mathcal{O}_E$-points.

(b) *(Chapter 5) In the following cases,* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$*:*

1. *if G is arbitrary and char$(k) = 0$,*
2. *if $G = GL_n$ and K is arbitrary, and*
3. *if $\widetilde{\mathfrak{F}}$ is the minimal facet containing the origin and G, K are arbitrary.*

Note that in order to show $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ in general, it is sufficient to show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is smooth. In chapter 6 we relate $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ to $\mathrm{Aut}_{\widetilde{\mathfrak{G}}}$ where $\widetilde{\mathfrak{F}}$ and $\widetilde{\mathfrak{G}}$ are related facets, and we study the Iwahori case.

In Section 5.3.1, where we show $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ in the case of $G = GL_n$, we review the description of $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ in terms of lattice chains in the standard representation, as it is presented in [H2]. For the proof that this lattice chain description is smooth, Haines refers the reader to Rapaport's and Zink's book [RZ]. In appendix A, we give a detailed account of the arguments presented in [RZ] which are used to show smoothness in this case. We also go over some more background material related to representability and smoothness for the sake of completeness.

Chapter 2

Preliminaries

2.1 Notation

Let K be a field which is complete with respect to a discrete valuation ν, let $\mathcal{O}$ be the ring of integers in K, let $\pi \in \mathcal{O}$ be a uniformizer, and let $k = \mathcal{O}/\pi\mathcal{O}$ be the residue field. Assume that k is a perfect field.

Let G be a connected reductive split linear algebraic group defined over $\mathcal{O}$, and let T be a split maximal torus. Let $X^*(T)$, $X_*(T)$ denote the characters and cocharacters of T respectively. We let $\langle \cdot, \cdot \rangle$ denote the extension of the perfect pairing between $X^*(T)$ and $X_*(T)$ to a perfect pairing between $X^*(T) \otimes \mathbb{R}$ and $X_*(T) \otimes \mathbb{R}$.

Let Φ denote the roots of G with respect to T and let $\Phi^\vee$ denote the corresponding coroots. Let G_{der} denote the derived group of G and let $T_{\text{der}} = T \cap G_{\text{der}}$.

2.1.1 Lattice Chains

Lattice chains will play a major role throughout this thesis, so we will set some standard notation now. Let V be an $\mathcal{O}$-module.

Definition 2.1.1. *An $\mathcal{O}$-lattice in $V \otimes K$ is an $\mathcal{O}$-submodule Λ such that Λ is finitely generated and contains a K-basis for $V \otimes K$. Moreover, an $\mathcal{O}$-lattice chain is a set $\{\Lambda_i\}_{i \in \mathbb{R}}$ of $\mathcal{O}$-lattices Λ_i such that*

1. *if* $i \leq j$ *then* $\Lambda_j \subset \Lambda_i$,

2. $\pi\Lambda_i = \Lambda_{i+1}$ *for every* i.

You might think that there could be some strange $\mathcal{O}$-submodule which satisfies these properties, but in fact they all have a nice form which is described in the next lemma. A proof of this easy fact can be found in [Gar] following Proposition 18.4.

Lemma 2.1.2. *Every* $\mathcal{O}$*-lattice* Λ *in a* n*-dimensional vector space* $V \otimes K$ *is of the form*

$$\Lambda = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_n$$

for some K*-basis* $e_1, e_2, \ldots, e_n$ *of* $V \otimes K$.

Any $\mathcal{O}$-lattice chain is determined by the lattices indexed by r to $r+1$ for any given r, and there can only be finitely many distinct lattices occurring between Λ_r and Λ_{r+1} since $V \otimes K$ is finite dimensional. Thus any lattice chain is determined by a finite set of lattices.

2.1.2 Buildings Notation

Let $\mathscr{B}(G)$ represent the Bruhat-Tits building of $G_{\mathrm{der}}(K)$. Let $\widetilde{\mathscr{B}}(G)$ represent the Bruhat-Tits building of $G(K)$ which we will call the enlarged building. Recall that $\widetilde{\mathscr{B}}(G)$ can be decomposed as

$$\widetilde{\mathscr{B}}(G) \cong \mathscr{B}(G) \times \mathscr{B}(Z(G)^\circ),$$

where $Z(G)^\circ$ denotes the identity component of the center $Z(G)$ of G.

We identify the apartment $\widetilde{\mathcal{A}}$ in $\widetilde{\mathscr{B}}(G)$ corresponding to T with $X_*(T) \otimes \mathbb{R}$ by choosing an origin. The projection $\mathcal{A}$ of $\widetilde{\mathcal{A}}$ onto $\mathscr{B}(G)$ is an apartment which is identified with $X^*(T_{\text{der}}) \otimes \mathbb{R}$. By the above decomposition of $\widetilde{\mathscr{B}}(G)$, we see that

$$\widetilde{\mathcal{A}} \cong \mathcal{A} \times \mathcal{A}(Z(G)^\circ) \cong (X_*(T_{\text{der}}) \otimes \mathbb{R}) \times \mathbb{R}^l$$

where $\mathcal{A}(Z(G)^\circ)$ denotes an apartment of $\mathscr{B}(Z(G)^\circ)$ and l is the dimension of $Z(G)^\circ$.

Let B be a Borel Subgroup containing T. This choice of B corresponds to a choice of positive roots $\Phi^+ \subset \Phi$ and also to a fixed Weyl chamber $\widetilde{C}$ in $X_*(T) \otimes \mathbb{R}$. Let C be the projection of $\widetilde{C}$ onto the apartment $\mathcal{A}$ in $\mathscr{B}(G)$. Let $X^+(T) \subset X^*(T)$ denote the set of B-dominant characters, i.e. the set of $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for every $\alpha \in (\Phi^\vee)^+$. Similarly define the set of B-dominant cocharacters $X_+(T)$. So we have that $\widetilde{C}$ is identified with the $\mathbb{R}_{\geq 0}$-span of $X_+(T)$, which we will denote by $X_+(T)_{\mathbb{R}_{\geq 0}}$.

Let $\widetilde{\mathfrak{A}}$ be the unique alcove in $\widetilde{C}$ which contains the origin in its closure $\overline{\widetilde{\mathfrak{A}}}$, and let $\mathfrak{A}$ be the projection of $\widetilde{\mathfrak{A}}$ onto $\mathscr{B}(G)$. Let $I \subset G(K)$ denote the Iwahori subgroup which is the $\mathcal{O}$-points of the Iwahori Bruhat-Tits group scheme corresponding to $\widetilde{\mathfrak{A}}$.

The extended affine Weyl group of $G(K)$ is defined to be

$$\widetilde{W} := \text{Norm}_{G(K)}(T(K))/T(\mathcal{O}).$$

The action of G on $\widetilde{\mathscr{B}}(G)$ passes to an action of $\widetilde{W}$ on $\widetilde{\mathscr{B}}(G)$, and $\widetilde{W}$ stabilizes the apartment $\widetilde{\mathcal{A}}$. The finite Weyl group $W := \text{Norm}_{G(K)}(T(K))/T(K)$ associated to $G(K)$ can naturally be identified with the subgroup $\text{Norm}_{G(\mathcal{O})}(T(\mathcal{O}))/T(\mathcal{O})$ of $\widetilde{W}$. Thus we can decompose $\widetilde{W}$ into translations and reflections in $\widetilde{\mathcal{A}} = X_*(T) \otimes \mathbb{R}$ as follows:

$$\widetilde{W} = T(K)/T(\mathcal{O}) \rtimes W = X_*(T) \rtimes W$$

Here we identify $X_*(T)$ with $T(K)/T(\mathcal{O})$ by the isomorphism $\lambda \to \lambda(\pi^{-1})$. We make this identification so that the image of I under the projection map proj : $G(\mathcal{O}) \to G(k)$ is B, our fixed Borel subgroup[1]. We denote the translation in $\widetilde{W}$ associated to $\lambda \in X_*(T)$ by t_λ.

We will at times need to consider $\widetilde{W}$ as a subset of $G(K)$. As just mentioned above, for every $\lambda \in X_*(T)$, we identify t_λ with $\lambda(\pi^{-1})$. For every $w \in W$ we fix once and for all a lift in $\mathrm{Norm}_{G(O)}(T(O))$, which we will also denote by w. For any $\lambda \in X^*(T)$, $\mu \in X_*(T)$, let $(w\lambda)(t) = \lambda(w^{-1}tw)$ and let $(w\mu)(a) = w\mu(a)w^{-1}$ for $t \in T$ and $a \in \mathbb{G}_m$. Thus we have that $\langle w\lambda, w\mu\rangle = \langle \lambda, \mu\rangle$, i.e. that $\langle, \cdot, \cdot\rangle$ is a W-invariant bilinear form.

The set of affine roots Ψ of G, which is given by

$$\Psi = \{\alpha + n | \alpha \in \Phi, n \in \mathbb{Z}\},$$

can be viewed as a set of $\mathbb{R}$-valued affine linear transformations of $\widetilde{\mathcal{A}}$. The group $\widetilde{W}$ acts transitively on Ψ and also on the set of alcoves in $\widetilde{\mathscr{B}}(G)$. Let $\Omega \subset \widetilde{W}$ be the stabilizer of $\widetilde{\mathfrak{A}}$. Then we have

$$\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$$

where W_{aff}, the affine Weyl group, is the Coxeter group generated by the reflections through the walls of $\widetilde{\mathfrak{A}}$, or equivalently $\mathfrak{A}$. Recall that a simple affine reflection is of

[1]If the identification $\lambda \to \lambda(\pi)$ is made instead, then since we chose $\widetilde{\mathfrak{A}} \subset \widetilde{C}$, we would be forced to have $\mathrm{proj}(I) = B^*$ where B^* is the unique Borel subgroup containing T which is opposite to B.

the form $t_{\alpha_0^\vee} s_{\alpha_0^\vee}$ where α_0 is a B-highest root.

2.1.3 Categorical Notation

Let $\mathcal{O}$-**Alg**, **Sets**, **Grps** denote the categories of commutative $\mathcal{O}$-algebras, sets, and groups respectively. Let M be an $\mathcal{O}$-module and let M_a denote the $\mathcal{O}$-functor defined by $M_a(R) = M \otimes_{\mathcal{O}} R$ for any $\mathcal{O}$-algebra R.

Definition 2.1.3. *Let M be a module over $\mathcal{O}$. We call M a G-module if there is an $\mathcal{O}$-functor morphism*

$$\psi : G \times M_a \to M_a$$

such that each $G(R)$ acts on $M_a(R)$ through R-linear maps. We call the induced map $\rho : G \to GL(M)$ an $\mathcal{O}$-representation of G.

Let $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ denote the category of $\mathcal{O}$-representations of G whose corresponding module is finite rank and projective. Note that all such modules are actually free since $\mathcal{O}$ is a discrete valuation ring.

Let $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. For $\lambda \in X^*(T)$, define the λ-weight space V^λ as

$$V^\lambda = \{v \in V | t(v \otimes 1) = v \otimes \lambda(t) \text{ for all } t \in T(R) \text{ and for all } \mathcal{O} - \text{algebras } R\}.$$

Since $T \subset G$ is a split torus, it is a diagonalizable group (see [Jan] I.2.5). Thus we can write V as a direct sum of weight spaces,

$$V = \bigoplus_{\lambda \in X^*(T)} V^\lambda,$$

as can be seen, for example, in [Jan] I.2.11. We say that λ is a weight of V if V^λ is not the empty set.

2.2 Representations of G

In the study of algebraic groups over a field, it is a well known result that all such groups can be viewed as closed subgroups of GL_n, for some n. In other words, there exists a faithful representation of any algebraic group over a field into some general linear group. Let H be an algebraic group over a field F. Here, as expected, a faithful representation means a functor morphism from H to GL_n which is a one-to-one group homomorphism on R-points for every F-algebra R. In this context, i.e. for affine group schemes over a field, this notion of faithful representation is equivalent to saying there is a surjective Hopf algebra map from $F[GL_n]$ to $F[H]$ (see [Wat] §15.3).

However, when considering affine group schemes over $\mathcal{O}$, this is no longer the case. The more useful property for our purposes is having a surjective map from $\mathcal{O}[GL_n]$ to $\mathcal{O}[G]$. Hence, we will follow [BT2] §1.4.5 and call a map from G to GL_n a faithful representation if the corresponding Hopf algebra map from $\mathcal{O}[GL_n]$ to $\mathcal{O}[G]$ is surjective. When we restrict to flat affine group schemes over $\mathcal{O}$, it turns out that being algebraic is equivalent to the existence of a faithful representation. We will go through this argument now.

2.2.1 Existence of Faithful Representations

For Sections 2.2.1 and 2.2.2, we will work more generally and allow G to be any flat affine group scheme over $\mathcal{O}$. Most of the results in these two sections can be found in Chapter 3 of [Wat], although we had to adjust some of the proofs slightly since he works over a field. However, we make no claims that these results are original.

Recall that there is a bijective correspondence between representations of G and $\mathcal{O}[G]$-comodules, so we can study the representations of G by looking at the comodules of $\mathcal{O}[G]$.

Definition 2.2.1. *If M is a comodule of G, then N is called a subcomodule if $\Delta_M(N) \subset N \otimes \mathcal{O}[G]$.*

Note that the intersection of subcomodules is again a subcomodule. In particular, if M is a comodule, then for any subset $S \subset M$ there exists a smallest subcomodule containing S, which we will call the comodule generated by S.

Lemma 2.2.2. *([Wed], § 5.7) Let (V, ρ) be an $\mathcal{O}[G]$-comodule, let v be an element of V, and let $U \subset V$ be the $\mathcal{O}[G]$-subcomodule generated by v. Then U is finitely generated as an $\mathcal{O}$-module.*

Proof. Let

$$\rho(v) = \sum_{i}^{r} u_i \otimes a_i$$

where $u_i \in U$ and $a_i \in \mathcal{O}[G]$ and let U' be the $\mathcal{O}$-module generated by the u_i. We will show that $U' = U$. Clearly $U' \subset U$ so we just need to show $U \subset U'$. Let $E := \rho^{-1}(U' \otimes \mathcal{O}[G]) \subset V$. We know that $\mathrm{Id}_V = (\mathrm{Id}_V \otimes \epsilon_G) \circ \rho$ since V is a comodule, so

$$\mathrm{Id}_V(E) = (\mathrm{Id}_V \otimes \epsilon_G) \circ \rho(E) = (\mathrm{Id}_V \otimes \epsilon_G)(U' \otimes \mathcal{O}[G]) \subset U'.$$

Thus we see that $E \subset U'$. We have $v \in E$ by definition, so it suffices to show that E is a subcomodule of V, i.e. that $\rho(E) \subset E \otimes \mathcal{O}[G]$. Now, again since V is a comodule, we have that $(\rho \otimes \mathrm{Id}_{\mathcal{O}[G]}) \circ \rho = (\mathrm{Id}_V \otimes \Delta_G) \circ \rho$. Thus

$$(\rho \otimes \mathrm{Id}_{\mathcal{O}[G]}) \circ \rho(E) = (\mathrm{Id}_V \otimes \Delta_G) \circ \rho(E) \subset (\mathrm{Id}_V \otimes \Delta_G)(U' \otimes \mathcal{O}[G]) \subset U' \otimes \mathcal{O}[G] \otimes \mathcal{O}[G].$$

Since G is flat, we know that $\mathcal{O}[G]$ is flat and hence have $E \otimes \mathcal{O}[G] = (\rho \otimes \mathrm{Id}_{\mathcal{O}[G]})^{-1}(U' \otimes \mathcal{O}[G] \otimes \mathcal{O}[G])$. Thus E is a subcomodule and we have $U = U'$. $\square$

Corollary 2.2.3. *Let (V, ρ) be an $\mathcal{O}[G]$-comodule. Any finite subset of V is contained in a sub-comodule of V which is finitely generated over $\mathcal{O}$.*

Corollary 2.2.4. *Any $\mathcal{O}$-representation of a flat affine group scheme G defined over $\mathcal{O}$ is a directed union of finite rank subrepresentations of G.*

This is an immediate corollary of the above proposition since the proposition implies that every comodule is a directed union of finite rank subcomodules and there is a bijective correspondence between representations and comodules.

Lemma 2.2.5. *Let G be an affine group scheme. Suppose (V, ρ) is an $\mathcal{O}[G]$-comodule such that V is a finite rank free $\mathcal{O}$-module. Let $\{v_i\}$ be a basis of size n for V and write $\rho(v_j) = \sum_{i=1}^{n} v_i \otimes a_{ij}$ where $a_{ij} \in \mathcal{O}[G]$. Let $\phi : G \to GL(V) = GL_n(\mathcal{O})$ be the corresponding representation and let $\psi : \mathcal{O}[X_{11}, \ldots, X_{nn}, \frac{1}{\det}] \to \mathcal{O}[G]$ denote the corresponding map between representing algebras. Then*

$$\psi(X_{ij}) = a_{ij}.$$

Proof. First note that (V, Δ), where $\Delta(v_i) = \sum_j v_j \otimes X_{ij}$, is the $GL(V)$-comodule which corresponds to the action of $GL(V)$ on V. By the definition of a representation, we have that the following diagram commutes:

$$\begin{tikzcd} G(R) \times (V \otimes R) \arrow[r, "(\phi, \mathrm{Id})"] \arrow[d, "G - \text{action}"'] & GL(V)(R) \times (V \otimes R) \arrow[d, "GL(V) - \text{action}"'] \\ V \otimes R \arrow[r, "\mathrm{Id}"] & V \otimes R \end{tikzcd}$$

By converting this diagram to a comodule diagram, we see that

$$\begin{tikzcd} \mathcal{O}[G] \otimes V & \mathcal{O}[GL(V)] \otimes V \arrow[l, "(\psi, \mathrm{Id})"'] \\ V \arrow[u, "\rho"] & V \arrow[l, "\mathrm{Id}"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

commutes. Thus we see that $(\psi, \mathrm{Id}) \circ \Delta = \rho \circ \mathrm{Id}$. Evaluating at the basis of V, we see $\psi(\Delta(v_i)) = \psi(\sum_j v_j \otimes X_{ij}) = \sum_j v_j \otimes \psi(X_{ij})$ and $\rho(v_i) = \sum_j v_j \otimes a_{ij}$ and conclude that $\psi(X_{ij}) = a_{ij}$ by matching up coefficients of the basis vectors. $\square$

Recall that G is called algebraic if its representing algebra is finitely generated over $\mathcal{O}$ as an algebra.

Theorem 2.2.6. *Let G be an affine group scheme over $\mathcal{O}$. If G has a faithful finite rank projective $\mathcal{O}$-representation then G is algebraic. If G is flat over $\mathcal{O}$, then the converse holds.*

Proof. If G has a faithful finite rank projective representation V, then by the definition of faithful there is a surjection from the defining algebra of $GL(V)$ onto $\mathcal{O}[G]$ and hence $\mathcal{O}[G]$ is finitely generated over $\mathcal{O}$ as an algebra.

Now assume that G is flat over $\mathcal{O}$ and algebraic. Thus we know that $\mathcal{O}[G]$ is finitely generated as an algebra. Let $(V, \Delta_G|_V)$ be a finite rank subcomodule of

$\mathcal{O}[G]$ containing algebra generators, which we know exists by Proposition 2.2.3. We know that V is a finitely generated submodule of a flat module, since G is flat. Since V is defined over the discrete valuation ring $\mathcal{O}$, it is projective. Moreover, since $\mathcal{O}$ is local, we have that V is free by Proposition B.0.8.

Thus, we obtain a representation $G \to GL(V) = GL_n(\mathcal{O})$ since V is a comodule for G, and also a map from $\mathcal{O}[GL(V)] = \mathcal{O}[X_{11}, \ldots, X_{nn}, \frac{1}{\det}] \to \mathcal{O}[G]$ where n is the dimension of V. Let $\{v_i\}$ be a basis for V and let $\Delta_G(v_i) = \sum_{i,j} v_i \otimes a_{ij}$. We then have that the image of $\mathcal{O}[GL(V)]$ in $\mathcal{O}[G]$ contains the a_{ij} by Lemma 2.2.5. Now, by the relationship between Δ_G and ϵ, we see that

$$v_j = (\epsilon \otimes \mathrm{Id})\Delta_G(v_j) = \sum_{i,j} \epsilon(v_i)a_{ij}.$$

Thus the image will contain V and hence is all of $\mathcal{O}[G]$ since V contains the generators for $\mathcal{O}[G]$. Thus we have that $\mathcal{O}[GL(V)] \to \mathcal{O}[G]$ is surjective and hence we have a faithful representation as desired. $\square$

Note that Theorem 2.2.6 implies the existence of self-dual faithful representations. To see this, note that if V is a faithful representation, then $V \oplus V^\vee$ is a self-dual faithful representation. It is sometimes convenient to look at the representation $V \oplus V^\vee \oplus \mathcal{O}$, where $\mathcal{O}$ denotes the trivial representation, which is also self-dual and faithful.

2.2.2 Tensor Generator

We will now begin the process of showing that if G is a flat algebraic affine group scheme over $\mathcal{O}$ with Hopf algebra A, then any faithful representation $\phi \in$

$\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ is a tensor generator for the category $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. The following two results are slight generalizations of [Wat] § 3.5.

Lemma 2.2.7. *Every finite rank projective representation of G embeds into a finite sum of copies of the regular representation.*

Proof. Let $(V,\ \rho)$ be the comodule which corresponds to a finite rank projective representation of G. So V is isomorphic to $\mathcal{O}^n$ for some n. Let $U = V \otimes_{\mathcal{O}} \mathcal{O}[G]$. We have that U is isomorphic to $\mathcal{O}[G]^n$ and can be made into a comodule by associating to it the comodule map $(\mathrm{Id}\otimes\Delta_G) : U \otimes \mathcal{O}[G] \to U \otimes \mathcal{O}[G] \otimes \mathcal{O}[G]$. Note this actually give an isomorphism between U and the comodule $\mathcal{O}[G]^n$, since the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}^n \otimes \mathcal{O}[G] & \xrightarrow{(\mathrm{Id},\Delta)} & \mathcal{O}^n \otimes \mathcal{O}[G] \otimes \mathcal{O}[G] \\ \cong\big\downarrow & & \cong\big\downarrow \\ \bigoplus_{i=1}^{n} \mathcal{O}[G] & \xrightarrow{\bigoplus \Delta} & \bigoplus_{i=1}^{n} (\mathcal{O}[G] \otimes \mathcal{O}[G]) \end{array}$$

The identity $(\mathrm{Id}\ \otimes\Delta_G) \circ \rho = (\rho\otimes\ \mathrm{Id})\rho$ precisely means that $\rho : V \to U$ is a map of $\mathcal{O}[G]$-comodules. Moreover, it is injective since $v = (\mathrm{Id} \otimes \epsilon)\rho(v)$. $\square$

Theorem 2.2.8. *If G is a flat algebraic affine group scheme over $\mathcal{O}$, then any faithful projective representation ϕ of G is a tensor generator for $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, i.e. every finite rank representation of G can be constructed from ϕ by forming tensor products, direct sums, subrepresentations, quotients, and duals.*

Proof. By Theorem 2.2.6 we know that G has a faithful finite rank projective representation. Let V be any such representation. So we have a closed embedding from $G \to GL_n$ where n is the dimension of V. By Lemma 2.2.7, it is sufficient to construct all finite rank subrepresentations U of $\mathcal{O}[G]^m$ for any m. Any such U can be viewed as a subcomodule of the direct sum of the coordinate projections from $\mathcal{O}[G]^m$ to $\mathcal{O}[G]$. What is meant here by a coordinate projection is the comodule $(\mathcal{O}[G], \rho_i)$ where

$$\rho_i : \mathcal{O}[G] \hookrightarrow \mathcal{O}[G]^m \to \mathcal{O}[G] \to \mathcal{O}[G] \otimes \mathcal{O}[G]$$

where the injection maps into the i-th coordinate, the second map is the projection onto the i-th coordinate, and the third map is Δ_G.

Thus it will suffice for us to just deal with those U in $\mathcal{O}[G]$. The representation V that we started with gives us a Hopf algebra surjection $\psi : B := \mathcal{O}[X_{11}, \ldots, X_{nn}, \frac{1}{\det}] \twoheadrightarrow \mathcal{O}[G]$. Since U is finite rank, we know that U must be contained in the image of some submodule $(1/\det)^r\{f(X_{ij})|\ \deg(f) \leq s\}$ for some $r, s \in \mathbb{N}$. Note that these submodules are B-subcomodules of B, and thus are also $\mathcal{O}[G]$-subcomodules of B with the map $(\mathrm{Id} \otimes \psi) \circ \Delta_{GL(V)}$. It will suffice to construct all of these subcomodules of B.

Let $\{e_i\}$ be the standard basis for $V = \mathcal{O}^n$. The standard representation of GL_n on V has B-comodule structure $\rho(e_i) = \sum e_i \otimes X_{ij}$. For each i, the map $e_j \to X_{ij}$ is a comodule map to B. Thus the submodule of polynomials in X_{ij} homogeneous of degree one is, as a comodule, the sum of n copies of the original representation V. We can construct $\{f \mid f \text{ homogeneous of degree } s\}$ as a quotient of the s-fold tensor product of $\{f \mid f \text{ homogeneous of degree } 1\}$. For $s = n$ this space contains the representation $g \to \det(g)$ of rank one. From that, we can construct its dual $g \to 1/\det(g)$. Summing the homogeneous pieces, we get $\{f|\ \deg(f) \leq s\}$, and tensoring r times with $1/\det(g)$ gives us all of the subcomodules we set out to construct. $\square$

Moreover, for any affine faithfully flat group scheme G defined over $\mathcal{O}$, it is even possible to say whether or not a tensor generator is a faithful representation by examining more closely the types of subrepresentations needed. The following definitions and proposition were given by dos Santos in [San].

Definition 2.2.9. *Let $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. A subrepresentation $U \subset V$ is called splittable if U and V/U are free. A representation X is called a splittable subquotient of V if there is a splittable subrepresentation $U \subset V$ such that X is a G-equivariant quotient of U.*

Now let $V_b^a = \displaystyle\bigoplus_{i=1}^{s} V^{\otimes a_i} \otimes (V^\vee)^{\otimes b_i};\ a = (a_1, \ldots, a_s), b = (b_1, \ldots, b_s) \in \mathbb{N}^s$.

Theorem 2.2.10. *([San], §3.2, Proposition 12) Assume that G is a faithfully flat group scheme defined over $\mathcal{O}$. If V is a faithful representation of G, then any representation U of G is a splittable subquotient of some V_b^a. Conversely, if there exists a representation V such that every representation of G is a splittable subquotient of some V_b^a, then V is faithful.*

Note that our group G, which was defined in Section 2.1, is faithfully flat since it surjects onto $\mathrm{Spec}(\mathcal{O})$. (See Section 13.2 of [Wat].) Moreover, by choosing V to be a self-dual faithful representation of G, Theorem 2.2.10 implies that every representation U can be realized as a splittable subquotient of a direct sum of tensor powers of V. This means that if we start with a self-dual representation, then in order to generate all representations we only need the operations direct sum, tensor product, quotient, and splittable subrepresentation.

2.2.3 Weyl Modules

We will now turn our attention to some particularly nice representations of algebraic groups, which are called Weyl modules. For a more detailed discussion of Weyl Modules, see [Jan] Chapters 2 and 8 of Part II. We are interested in Weyl modules since, among other things, their character structure is well understood and this will allow us to give a nice description to $\widetilde{\mathscr{B}}(G)$ in terms of the representations of G. See Chapter 3 for this description.

We will first describe Weyl modules in the context of a split algebraic group G over a field F. Let B be a Borel subgroup of G and let T be a split maximal torus of G such that $B = TU$ where U is the unipotent radical of B. For $\lambda \in X^*(T)$, denote by F_λ the 1-dimensional character of $B/U = T$ where T acts by λ. The Weyl module, $V(\lambda)$, is a finite dimensional representation associated to λ defined by

$$V(\lambda) = (\mathrm{Ind}_B^G(F_{-\lambda}))^*.$$

It is a fact that

$$V(\lambda) \neq 0 \text{ if and only if } \lambda \in X^+(T).$$

Now assume that $\lambda \in X^+(T)$, i.e. that λ is a B-dominant character. Then $V(\lambda)$ has the following properties:

- $V(\lambda)$ contains a B-maximal vector v_λ of weight λ which generates a B-stable line $L(\lambda) \subset V(\lambda)$.

- For any weight μ of $V(\lambda)$, $\mu \leq \lambda$ in the partial ordering given by B.

- As a G-module, $V(\lambda)$ is generated by $L(\lambda)$.

- Any G-module generated by a B-stable line of weight λ is a quotient of $V(\lambda)$.

It is not true that $V(\lambda)$ is irreducible in general, but this is true if $\mathrm{char}(F) = 0$. Let V_λ be the irreducible representation of highest weight λ. Then by the above properties we have

$$1 \to \mathrm{rad}_G(V(\lambda)) \to V(\lambda) \to V_\lambda \to 1$$

where $\mathrm{rad}_G(V(\lambda))$ is the intersection of all the maximal proper G-submodules of $V(\lambda)$.

Lemma 2.2.11. *Let G be a split smooth algebraic group scheme defined over a field F. There exists a faithful representation V of G which is a finite direct sum of Weyl Modules.*

Proof. First note that it is enough to show this result when $F = \overline{F}$ is algebraically closed since if R is an F-algebra, then $\overline{R} := R \otimes_F \overline{F}$ is an $\overline{F}$-algebra and we have that $R \hookrightarrow \overline{R}$. Thus, by the following diagram

$$\begin{tikzcd}
G(R) \arrow[r] \arrow[d, hook] & GL(V)(R) \arrow[d, hook] \\
G(\overline{R}) \arrow[r] & GL(V)(\overline{R})
\end{tikzcd}$$

it is enough to show $G(\overline{R}) \to GL(V)(\overline{R})$ is injective for every $\overline{F}$-algebra $\overline{R}$.

Now we are in the context of a reduced algebraic group scheme over an algebraically closed field, and hence we are in the standard setting of algebraic groups. First, choose a $\lambda \in X^+(T)$ which is regular dominant, i.e. such that $\langle \lambda, \alpha^\vee \rangle > 0$ for every positive coroot $\alpha^\vee$.

For this choice of λ, we will now show that the stabilizer of the line $L(\lambda) \subset V(\lambda)$ is exactly the Borel subgroup B. By the definition of $L(\lambda)$ it is obvious that B is contained in the stabilizer, so we just have to show the stabilizer is contained in B. Let $g \in G$ such that $gL(\lambda) \subset L(\lambda)$. By the Bruhat decomposition, $g = b_1 w b_2$ for $b_1, b_2 \in B$ and $w \in W$, the finite Weyl group. Thus

$$gL(\lambda) \subset L(\lambda) \text{ if and only if } wL(\lambda) \subset L(\lambda).$$

Recall that if $Y = \bigoplus_{\mu \in X_*(T)} Y^\mu$ is a representation of G, then

$$wY^\mu = Y^{w\mu}.$$

Thus $wL(\lambda) \subset L(\lambda)$ implies $V(\lambda)^\lambda = V(\lambda)^{w\lambda}$, i.e. $\lambda = w\lambda$. Since λ is regular dominant with respect to B, it lies in the interior of the B-dominant Weyl chamber $\widetilde{C}$ and hence we have $w\widetilde{C} = \widetilde{C}$. Thus $w = 1$ and $g \in B$ as desired.

Now consider the character $w_0\lambda$ where $w_0 \in W$ is the longest element. Let B^* denote the opposite Borel of B which contains T. Similar to how there is a B-stable line $L(\lambda) \subset V(\lambda)$ corresponding to the highest weight λ of $V(\lambda)$, there is a B^*-stable line $L(w_0\lambda) \subset V(\lambda)$ corresponding to the lowest weight $w_0\lambda$ of $V(\lambda)$. By essentially the same argument as the one presented above, it can be shown that B^* is the stabilizer of the line $L(w_0\lambda)$. Thus the representation $V(\lambda)$ has kernel contained in $B \cap B^* = T$.

Now choose a finite dimensional faithful representation Y of G and let S be the finite set of weights of Y. For every weight $\chi \in S$, choose a nontrivial Weyl module which has χ in its weight decomposition. To do this, just choose the character in the set $\{w\mu | w \in W\}$ which is B-dominant. For each χ, denote its corresponding B-dominant character by $w_\chi \chi$. Thus we have that

$$V = V(\lambda) \oplus \bigoplus_{\chi \in S} V(w_\chi \chi)$$

is a faithful representation of G. $\square$

All of the group schemes we are considering have actually been shown to exist over $\mathbb{Z}$, and hence over any commutative ring. In this generality it is possible (see [Jan]) to construct, for any $\lambda \in X^+(T)$, a finite rank representation $V(\lambda)_\mathbb{Z}$ such that the above constructed $V(\lambda)$ satisfies

$$V(\lambda) = V(\lambda)_\mathbb{Z} \otimes_\mathbb{Z} F.$$

Now we will return to the setting of this thesis. Define a Weyl module over $\mathcal{O}$ to be the $\mathcal{O}$-representation $V(\lambda)$ given by

$$V(\lambda) = V(\lambda)_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}.$$

The following proposition will come up in the discussion immediately following Theorem 3.7.2. It appears in [HV], where they attribute the proof they give to W. Soergel. The following proof is different, but we do not know if it is original.

Proposition 2.2.12. *There exists a faithful representation of G which is a finite direct sum of Weyl modules over $\mathcal{O}$.*

Proof. By applying Lemma 2.2.11 with $G = G_K$ and $F = K$, and also with $G = G_k$ and $F = k$, we get a finite set of representations which will give us our faithful representation. Let X_K and X_k be the respective finite sets of weights obtained above through each application of the lemma and let $X = X_K \cup X_k$. For every $\lambda \in X$, construct the Weyl module over $\mathcal{O}$ as described above. Let

$$V = \bigoplus_{\lambda \in X} V(\lambda).$$

We want to show that the induced map on Hopf algebras $\phi : \mathcal{O}[GL(V)] \to \mathcal{O}[G]$ is surjective. Note that, by construction, when we base change to K or k, the induced maps ϕ_K and ϕ_k are surjective.

Now consider the $\mathcal{O}$-module $\mathcal{O}[G]/\phi(\mathcal{O}[GL(V)])$, which is a finitely generated algebra over $\mathcal{O}$. Let $M \subset \mathcal{O}[G]/\phi(\mathcal{O}[GL(V)])$ be an $\mathcal{O}$-module generated by algebra generators of $\mathcal{O}[G]/\phi(\mathcal{O}[GL(V)])$.

Note that M is a finitely generated module over a PID, so the standard decomposition theorem applies. Therefore, $M \cong A \oplus B$ where A is free and B is torsion. Now, since ϕ_K and ϕ_k are surjective, we know that

$$M \otimes_{\mathcal{O}} K = 0 \text{ and } M \otimes_{\mathcal{O}} k = M/\pi M = 0.$$

The first equality immediately implies that $A = 0$ and the second equality implies $B = 0$ by Nakayama's Lemma. Thus $M = 0$ and we have that ϕ is surjective as desired. $\square$

2.3 Tannakian Duality

We will finish this introductory chapter by giving a brief discussion of Tannakian duality. In the rest of this chapter, we have seen that there exist faithful representations of G, and moreover that any faithful representation is a tensor generator for $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. It is clear that G determines its category of representations $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, but perhaps surprisingly it is also true that $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ determines G. This is what is meant by the phrase Tannakian duality.

Much of the original work in this area is due to Saavedra ([Saa]), but we will present a result from [Wed]. In [Wed], a Tannakian description is given for $G_{\mathcal{O}} = G$, which yields a similar description for both G_K and G_k.

Let S be a commutative integral domain. S is called a Prüfer ring if the following equivalent conditions hold:

1. every localization of S at a prime ideal is a valuation ring,
2. every finitely generated submodule of a flat S-module is projective.

It is clear that $\mathcal{O}$ satisfies condition 1, and hence is a Prüfer ring.

Theorem 2.3.1. *([Wed], § 5, Corollary 5.20) Let S be a Prüfer ring and let G be a flat affine group scheme over S. Then for any S-algebra R,*

$$G_S(R) = \left\{ (g_V^R)_{V \in \mathbf{Rep_S(G)}} \middle| g_V^R \in GL(V)(R) \text{ and conditions (a), (b), and (c) hold} \right\}$$

where

(a) *If V, $U \in \mathbf{Rep_S(G)}$ and $\phi : V \to U$ is a G-morphism, then $\phi^R \circ g_V^R = g_U^R \circ \phi^R$,* *i.e. the following diagram commutes:*

$$\begin{tikzcd} V \otimes_S R \arrow[r, "\phi^R"] \arrow[d, "g_V^R"] & U \otimes_S R \arrow[d, "g_U^R"] \\ V \otimes_S R \arrow[r, "\phi^R"] & U \otimes_S R \end{tikzcd}$$

(b) $g_S^R = \mathrm{Id}_R$.

(c) *If* V, $U \in \mathbf{Rep_S(G)}$ *then the following diagram commutes:*

$$\begin{tikzcd} (V \otimes_S R) \otimes_R (U \otimes_S R) \arrow[r, "\cong"] \arrow[d, "g_V^R \otimes g_U^R"] & (V \otimes_S U) \otimes_S R \arrow[d, "g_{V\otimes U}^R"] \\ (V \otimes_S R) \otimes_R (U \otimes_S R) \arrow[r, "\cong"] & (V \otimes_S U) \otimes_S R \end{tikzcd}$$

It is easy to see that there is an injective map from $G_S(R)$ to the above set. However, it is much harder to see that the map is surjective. Nevertheless it is true, and in this way we can recover the group from its category of representations.

Chapter 3

A Tannakian Description for Bruhat-Tits Buildings

Before we get to describing the parahoric Bruhat-Tits group schemes, we will first provide a Tannakian description for the Bruhat-Tits Building itself. What we mean by this is that we will describe $\widetilde{\mathscr{B}}(G)$ in terms of the category of representations of G.

In order to do this, we will associate to any representation $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ and any point in $\widetilde{\mathscr{B}}(G)$ a family of filtrations reminiscent of Moy-Prasad filtrations (see [MP1] and [MP2]). We will then show that, in some sense, the data of the building is encoded in these filtrations.

3.1 Moy-Prasad Filtrations Associated to Any $x \in \widetilde{\mathcal{A}}$

Recall that the set of $\mathbb{R}$-valued affine functions on $X^*(T)\otimes\mathbb{R}$ can be described as $X_*(T)\otimes\mathbb{R}\times\mathbb{R}$. We will denote by (x, r) the affine function given by

$$(x, r)(\lambda \otimes s) = r - \langle \lambda \otimes s, x\rangle,$$

for any $\lambda \otimes s \in X^*(T)\otimes\mathbb{R}$. We will first associate a Moy-Prasad filtration to the points in our base apartment $\widetilde{\mathcal{A}} = X_*(T)\otimes\mathbb{R}$.

Definition 3.1.1. *For $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ and (x, r) an $\mathbb{R}$- valued affine function on*

$X^*(T) \otimes \mathbb{R}$, let $V_{x,r}$ be the $\mathcal{O}$-module defined by

$$V_{x,r} = \sum_{\substack{\lambda \in X^*(T), n \in \mathbb{Z} \\ n \geq (x,r)(\lambda)}} V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} = \bigoplus_{\lambda \in X^*(T)} V^\lambda \otimes_{\mathcal{O}} \pi^{n_{\lambda,x,r}} \mathcal{O}$$

where $n_{\lambda,x,r} = \lceil (x,r)(\lambda) \rceil$.

Note that we can view $V_{x,r}$ as an $\mathcal{O}$-submodule of $V \otimes K$. If $\phi : V \to U$ is a G-morphism, then $\phi_K\big|_{V_{x,r}}$ sends $V_{x,r}$ into $U_{x,r}$ since ϕ is an $\mathcal{O}$-morphism and since ϕ must send V^λ into U^λ. Also, note that the functor $V \to V_{x,r}$ is exact, which can easily be seen by a direct examination.

3.2 Moy-Prasad Filtrations of Any Finitely Generated Representation

In this section, we will state how to put a Moy-Prasad filtration on a finitely generated representation of G which is not necessarily projective. This extension of the definition of a Moy-Prasad filtration is not necessary for anything in this thesis, but we record it here anyway because it is interesting in its own right.

Let V be a finitely generated representation of G and let V_t denote the torsion submodule of V. We then have the following exact sequence of representations of G,

$$1 \to V_t \to V \to V/V_t \to 1,$$

where V/V_t is free.

Let (x,r) be an affine function as above. We know that $V/V_t = \bigoplus_{\lambda \in X^*(T)} (V/V_t)^\lambda$, and similarly that $(V/V_t)_{x,r} = \bigoplus_{\lambda \in X^*(T)} (V/V_t)^\lambda \otimes_{\mathcal{O}} \pi^{n_{\lambda,x,r}}\mathcal{O}$. Now define

$$\psi : (V/V_t)_{x,r} \to V/V_t$$

to be the canonical T-equivariant map which sends $(V/V_t)^\lambda \otimes_{\mathcal{O}} \pi^{n_{\lambda,x,r}}\mathcal{O}$ to $(V/V_t)^\lambda$ by $\psi(v \otimes \pi^{n_{\lambda,x,r}}) = v$.

Now define $V_{x,r}$ to be the pullback of the following diagram:

$$\begin{tikzcd} & (V/V_t)_{x,r} \arrow[d, "\psi"] \\ V \arrow[r] & V/V_t. \end{tikzcd}$$

A fairly straightforward diagram chase shows that the functor $V \to V_{x,r}$ is also exact.

3.3 Dual Filtration Associated to a Moy-Prasad Filtration

Let $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, let $V^\vee = \mathrm{Hom}(V, \mathcal{O})$ denote the dual representation, and let $(x, r) \in X_*(T) \otimes \mathbb{R} \times \mathbb{R}$. Associated to the Moy-Prasad filtration $V_{x,r}$, there is a Moy-Prasad filtration $V^\vee_{x,s}$ on the dual representation which has the property of a dual object. Let Λ be the finite set of weights of V. Note that if $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$, then $V^\vee = \bigoplus_{\lambda \in \Lambda} (V^\vee)^{-\lambda}$ where $(V^\vee)^{-\lambda} = \mathrm{Hom}(V^\lambda, \mathcal{O})$.

Choose ϵ_V to be a small positive real number such that

$$\lceil r - \langle \lambda, x \rangle \rceil + \lceil -r - 1 + \epsilon_V - \langle -\lambda, x \rangle \rceil = 0$$

for every weight λ of V. Let $s = -r - 1 + \epsilon_V$. If we think of $\mathcal{O}$ as the trivial representation, then we see that the canonical pairing $V \otimes V^\vee \to \mathcal{O}$ induces a perfect pairing

$$V_{x,r} \otimes V_{x,s}^{\vee} \to \mathcal{O}_{x,0} = \mathcal{O}.$$

Thus we call $V_{x,s}^{\vee}$ the dual filtration of $V_{x,r}$.

3.4 Parahoric Subgroups Stabilize Moy-Prasad Filtrations

When we view $V_{x,r}$ as a submodule of $V \otimes K$, it is natural to ask what subgroups of $G(K)$ fix particular filtrations. In order to determine this, we will analyze the action of the torus and the root subgroups on $V_{x,r}$. First we will recall the definition of an affine root subgroup.

Definition 3.4.1. *For $\psi = \alpha + n \in \Psi$ an affine root of G, define the affine root subgroup $U_\psi \subset G(K)$ to be*

$$U_\psi = \{\chi_\alpha(x) | x \in K,\, \nu(x) \geq n\}$$

where $\chi_\alpha : \mathbb{G}_a \to G$ is the root homomorphism corresponding to α.

Recall that associated to any facet $\widetilde{\mathfrak{F}} \subset \widetilde{\mathscr{B}}(G)$, there is a parahoric subgroup $P_{\widetilde{\mathfrak{F}}} \subset G(K)$ which fixes the facet pointwise. Since there is an obvious parallel between what we are calling Moy-Prasad filtrations and the filtrations defined by Moy and Prasad, one would hope that $P_{\widetilde{\mathfrak{F}}}$ would stabilize the filtrations associated to any $x \in \widetilde{\mathfrak{F}}$, and this is in fact true.

Proposition 3.4.2. *The parahoric subgroup $P_{\widetilde{\mathfrak{F}}}$ stabilizes $V_{x,r}$ for every $x \in \widetilde{\mathfrak{F}}$ and every $r \in \mathbb{R}$.*

By Corollary 4.6.7 of [BT2], we have that

$$P_{\widetilde{\mathfrak{F}}} = \langle T(\mathcal{O}), U_\psi | \psi(x) \geq 0, x \in \widetilde{\mathfrak{F}} \rangle.$$

Thus in order to prove Proposition 3.4.2, it suffices to consider the action of $T(\mathcal{O})$ and U_ψ on the $V_{x,r}$.

We will first look at the action of the torus. Since $\mathcal{O} \hookrightarrow K$ we have $T(\mathcal{O}) \hookrightarrow T(K)$ and hence that every $t \in T(\mathcal{O})$ acts K-linearly on $V_{x,r} \subset V \otimes K$. Thus for any $t \in T(\mathcal{O})$

$$\begin{aligned} t \cdot V_{x,r} &= t \cdot \sum V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} = \sum t V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} \\ &= \sum V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} = V_{x,r} \end{aligned}$$

since V^λ is a T-submodule of V.

Now consider the action of U_ψ on $V_{x,r}$. The following lemma is a generalization of Proposition 27.2 in [Hum].

Lemma 3.4.3. *Let $\rho : G \to GL(V)$ be a representation. Let $\{V^{\lambda_l}\}$ be the weight spaces relative to $\rho(T)$. If α is a root of G and p is an integer, then*

$$\rho(U_{\alpha+p})(V^{\lambda_l} \otimes_{\mathcal{O}} \pi^m \mathcal{O}) \subset \sum_{q \geq 0} V^{\lambda_l + q\alpha} \otimes_{\mathcal{O}} \pi^{m+qp} \mathcal{O}.$$

for every weight λ_j.

Proof. Let n be the rank of V and choose a basis for V by choosing bases for all of the weight spaces of V. Thus $\rho(T)$ is diagonal in $GL(V) = GL_n(\mathcal{O})$ with this basis. Recall that U_α is an $\mathcal{O}$-subgroup scheme of G. Since ρ is defined over $\mathcal{O}$, we then have that

$$(\rho \circ \chi_\alpha)_{ij} : \mathbb{G}_a \to U_\alpha \to \rho(U_\alpha) \to \mathbb{G}_a$$

is a polynomial with coefficients in $\mathcal{O}$, where the sub ij denotes the (i,j)-th coordinate projection. So $\rho(\chi_\alpha(y))_{ij} = c_0 + c_1 y + \cdots + c_l y^l$ for some $c_i \in \mathcal{O}$, $l \in \mathbb{N}$ independent of y. Recall that $\chi_\alpha(\alpha(t)x) = t\chi_\alpha(x)t^{-1}$. So we have

$$\rho(\chi_\alpha(\alpha(t)y))_{ij} = \rho(t\chi_\alpha(y)t^{-1})_{ij} = (\rho(t)\rho(\chi_\alpha(y))\rho(t)^{-1})_{ij} = \rho(t)_{ii}\rho(t)_{jj}^{-1}\rho(\chi_\alpha(y))_{ij}.$$

Thus

$$(1-\rho(t)_{ii}\rho(t)_{jj}^{-1})c_0 + (\alpha(t)-\rho(t)_{ii}\rho(t)_{jj}^{-1})c_1 y + \cdots + (\alpha^l(t)-\rho(t)_{ii}\rho(t)_{jj}^{-1})c_l y^l = 0.$$

Since $\mathcal{O}$ is infinite, we see that all of the coefficients in this polynomial must vanish. Moreover, since α is non-trivial at most one of the $c_k \neq 0$, and for this k we have $\alpha^k(t) = \rho(t)_{ii}\rho(t)_{jj}^{-1}$ and $\rho(\chi_\alpha(y))_{ij} = c_k y^k$.

Let v_l be one of the basis vectors chosen from V^{λ_l}. Now we will examine $\rho(U_{\alpha+p})(v_l \otimes_{\mathcal{O}} \pi^m\sigma)$ where $\sigma \in \mathcal{O}$ and $m \in \mathbb{Z}$. Let $u \in U_{\alpha+p}$ and let u_{ij} denote the (i,j)-th coordinate of $\rho(u)$ in $GL_n(K)$. Then

$$\rho(u)(v_l \otimes_{\mathcal{O}} \pi^m\sigma) = \pi^m\sigma(\rho(u)(v_l \otimes_{\mathcal{O}} 1)) = \pi^m\sigma\left(\sum_{i=0}^{n} v_i \otimes_{\mathcal{O}} u_{il}\right).$$

By our above discussion, if $u_{il} \neq 0$, then $\alpha^{q_{il}}(t) = \rho(t)_{ii}\rho(t)_{ll}^{-1}$ for some $q_{il} \in \mathbb{Z}_{\geq 0}$. However, by the definition of v_l, we know that $\rho(t)_{ll} = \lambda_l(t)$. Thus $\lambda_i(t) = \rho(t)_{ii} = \lambda_j(t)\alpha^{q_{il}}(t)$ where λ_i is the weight corresponding to v_i, which with additive notation says $\lambda_i = \lambda_j + q_{il}\alpha$.

Now, since $u \in U_{\alpha+p}$, we know that $u = \chi_\alpha(\pi^p y)$ for some $y \in \mathcal{O}$. Again, if $u_{il} \neq 0$, then

$$u_{il} = \rho(u)_{il} = \rho(\chi_\alpha(\pi^p y))_{il} = c_{q_{il}}(\pi^p y)^{q_{il}} = \pi^{q_{il}p}c_{q_{il}}y^{q_{il}}$$

where $c_{q_{il}} y^{q_{il}} \in \mathcal{O}$.

Thus

$$\begin{aligned}\rho(u)(v_l \otimes_{\mathcal{O}} \pi^m \sigma) &= \pi^m \sigma \left(\sum_{i=0}^{n} v_i \otimes_{\mathcal{O}} u_{il} \right) \\ &= \sum_{\substack{i=0 \\ u_{il} \neq 0}}^{n} v_i \otimes_{\mathcal{O}} \pi^{m+q_{il}p} \sigma c_{q_{il}} y^{q_{il}} \in \sum_{q \geq 0} V^{\lambda_l + q\alpha} \otimes_{\mathcal{O}} \pi^{m+qp} \mathcal{O}\end{aligned}$$

as desired. $\square$

Note that if $U_{\alpha+p} \subset P_{\widetilde{\mathfrak{F}}}$, then we know $\langle \alpha, x \rangle + p \geq 0$ for every $x \in \mathfrak{F}$. If we keep the notation of the lemma, we see that if $q \geq 0$ and $\langle \lambda, x \rangle + m \geq r$, then $\langle \lambda + q\alpha, x \rangle + m + qp \geq r$ and

$$\rho(U_{\alpha+p})(V^{\lambda} \otimes_{\mathcal{O}} \pi^m \mathcal{O}) \subset \sum_{q \geq 0} V^{\lambda + q\alpha} \otimes_{\mathcal{O}} \pi^{m+qp} \mathcal{O} \subset V_{x,r}.$$

Thus we see that $P_{\widetilde{\mathfrak{F}}} = \langle T(\mathcal{O}),\, U_{\psi} | \psi(x) \geq 0 \rangle$ stabilizes $V_{x,r}$ for every $x \in \widetilde{\mathfrak{F}}$ and $r \in \mathbb{R}$ as desired. This concludes the proof of Proposition 3.4.2. $\square$

3.5 Moy-Prasad Filtrations Associated to Any Point in $\widetilde{\mathscr{B}}(G)$

We now want to associate a Moy-Prasad filtration to any point in the building. For any $y \in \widetilde{\mathscr{B}}(G)$, if $y = gx$ for $g \in G(K)$ and $x \in \widetilde{\mathfrak{A}}$, let

$$V_{y,r} := gV_{x,r}.$$

We must now check that it agrees with the previous definition when they overlap, and moreover that it is well-defined.

Recall that if $y \in \widetilde{\mathcal{A}}$, then $y = \widetilde{w}x$ for some $\widetilde{w} \in \widetilde{W}$ and some $x \in \widetilde{\mathfrak{A}}$. The following lemma shows that this definition agrees with the previous one.

Lemma 3.5.1. *Let $y \in \widetilde{\mathcal{A}}$. If $y = \widetilde{w}x$ for $\widetilde{w} \in \widetilde{W}$ and $x \in \widetilde{\mathfrak{A}}$, then*

$$\widetilde{w}V_{x,r} = V_{y,r}.$$

Proof. Recall that $\widetilde{W} = X_*(T) \rtimes W$, so we have $\widetilde{w} = t_\mu w$ for some $\mu \in X_*(T)$, $w \in W$. Well, for $w \in W$, we have

$$\begin{aligned} w \cdot V_{x,r} = w \cdot \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle \lambda,\, x\rangle + n \leq r}} V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} &= \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle \lambda,\, x\rangle + n \leq r}} V^{(w\lambda)} \otimes_{\mathcal{O}} \pi^n \mathcal{O} \\ &= \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle w^{-1}\lambda,\, x\rangle + n \leq r}} V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} = \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle \lambda,\, wx\rangle + n \leq r}} V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} \\ &= V_{wx,r} \end{aligned}$$

and, for $\mu \in X_*(T)$, we have

$$\begin{aligned} t_\mu \cdot V_{x,r} = \mu(\pi^{-1}) \cdot \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle \lambda,\, x\rangle + n \leq r}} V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} &= \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle \lambda,\, x\rangle + n \leq r}} V^\lambda \otimes_{\mathcal{O}} \pi^{n-\langle\lambda,\mu\rangle} \mathcal{O} \\ &= \sum_{\substack{\lambda \in X^*(T),\, n\in\mathbb{Z} \\ \langle \lambda,\, x\rangle + n + \langle\lambda,\mu\rangle \leq r}} V^\lambda \otimes_{\mathcal{O}} \pi^n \mathcal{O} = V_{x+\mu,r} \\ &= V_{t_\mu x,r} \end{aligned}$$

because of our identification of $X_*(T)$ with $T(K)/T(\mathcal{O})$ by $\mu \to \mu(\pi^{-1})$. Thus

$$\widetilde{w}V_{x,r} = t_\mu w V_{x,r} = t_\mu V_{wx,r} = V_{t_\mu wx,r} = V_{y,r}$$

as desired. ∎

We will now check that the definition is well-defined. Let $y = gx = g'x'$ for $g, g' \in G(K)$ and $x, x' \in \widetilde{\mathfrak{A}}$. Thus $x = g^{-1}g'x'$. Let $h = g^{-1}g'$. In order to show this definition is well defined, we need to show that

$$hV_{x',r} = V_{x,r}.$$

By the Bruhat decomposition we know that $h = i_1 \widetilde{w} i_2$ where $i_1, i_2 \in I$ and $\widetilde{w} \in \widetilde{W}$. Moreover, by Proposition 3.4.2, we know that $I \cdot V_{z,s} = V_{z,s}$ for any $z \in \widetilde{\mathfrak{A}}$, $s \in \mathbb{R}$. Thus it suffices to show

$$\widetilde{w} V_{x',r} = V_{x,r}$$

Note that $i_1 \widetilde{w} i_2 x' = x$ if and only if $\widetilde{w} x' = x$. Thus $V_{\widetilde{w}x',r} = V_{x,r}$, and we have reduced the issue of well-definedness to showing

$$\widetilde{w} V_{x',r} = V_{\widetilde{w}x',r},$$

which is an immediate consequence of Lemma 3.5.1. Thus $V_{y,r}$ is well defined.

3.6 Recovering the Facet Containing $y \in \widetilde{\mathscr{B}}(G)$ From $\{V_{y,r}\}$

It is possible to recover the facet containing y in its interior from the family of filtrations $V_{y,r}$ using Proposition 4.2.3, which implies that $P_{\widetilde{\mathfrak{F}}}$ is the simultaneous stabilizer in $G(K)$ of the filtrations $V_{x,r}$ for $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, $x \in \widetilde{\mathfrak{F}}$, and $r \in \mathbb{R}$. From Proposition 3.4.2 we know that $P_{\widetilde{\mathfrak{F}}}$ stabilizes all of the $V_{x,r}$ and we ask the reader to accept for now that it actually is the simultaneous stabilizer. Given this fact, if $y = gx$ for $x \in \widetilde{\mathfrak{F}}$, then $gP_{\widetilde{\mathfrak{F}}}g^{-1}$ will be the simultaneous stabilizer for the filtrations $V_{y,r}$. By the way that G acts on $\widetilde{\mathscr{B}}(G)$, it is then clear that $y \in g\widetilde{\mathfrak{F}}$.

It is also possible to just recover the type of the facet associated to $V_{x,r}$.

Lemma 3.6.1. *Let $\widetilde{F}_I \subset \widetilde{\mathfrak{A}}$ be a facet of type I and let x be in the interior of $\widetilde{F}_I$. Let $\lambda \in X^+(T)$ and let $V(\lambda)$ be the Weyl module associated to λ. Let I_λ be the set of indices i such that $\langle \lambda, \alpha_i^\vee \rangle = 0$ where $\alpha_i^\vee$ runs over the set of B-simple coroots. Then for $V(\lambda)_{x,\langle \lambda, x\rangle}$,*

$$n_{x,\lambda,\langle \lambda,x\rangle} = 0 \text{ and } n_{x,\mu,\langle \lambda,x\rangle} > 0 \text{ for every weight } \mu \neq \lambda \text{ if and only if } I \subset I_\lambda.$$

Moreover, if $x \neq 0$ then $I = \bigcap_\lambda I_\lambda$ where the intersection runs over those $\lambda \in X^+(T)$ for which we have $n_{x,\lambda,\langle \lambda,x\rangle} = 0$ and $n_{x,\mu,\langle \lambda,x\rangle} > 0$ for every weight $\mu \neq \lambda$ of $V(\lambda)$.

Proof. First note that, by the definition of $V(\lambda)$ and the fact that $x \in \widetilde{\mathfrak{A}}$, $\langle \lambda, x\rangle \geq \langle \mu, x\rangle$ for every weight μ of $V(\lambda)$. Thus the only way for $n_{x,\mu,\langle \lambda,x\rangle} = \lceil \langle \lambda, x\rangle - \langle \mu, x\rangle \rceil$ to be zero is for $\langle \lambda, x\rangle = \langle \mu, x\rangle$. Now, again by the definition of $V(\lambda)$, this happens if and only if $\langle \alpha, x\rangle = 0$ for every root α occurring in the decomposition of $\lambda - \mu$. Recall that s_{α_i} permutes the weights of $V(\lambda)$, and $s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee\rangle \alpha_i$. Thus $s_{\alpha_i}(\lambda) = \lambda$ if and only if $\langle \lambda, \alpha_i^\vee \rangle = 0$.

So we see that $n_{x,\lambda,\langle \lambda,x\rangle} = 0$ and $n_{x,\mu,\langle \lambda,x\rangle} > 0$ for every weight $\mu \neq \lambda$ of $V(\lambda)$ if and only if for every B-simple root α_i such that $\langle \alpha_i, x\rangle = 0$, we have $\langle \lambda, \alpha_i^\vee\rangle = 0$. This establishes the first statement.

The second statement follows easily from the first. Clearly we have $I \subset \bigcap I_\lambda$ from the first statement as long as there exists a λ satisfying the conditions needed to occur in the intersection. In order to find such a λ, choose one which is very nearly along the ray defined by x, which by assumption is non-zero, under the identification between $X^*(T) \otimes \mathbb{R}$ and $X_*(T) \otimes \mathbb{R}$.

In order to show $\bigcap I_\lambda \subset I$, it suffices to show that if $\langle \alpha_i, x\rangle > 0$, then there is a λ that satisfies the conditions needed to occur in the intersection and also satisfies $\langle \lambda, \alpha_i^\vee \rangle = 0$ for exactly those i such that $\langle \alpha_i, x \rangle = 0$. Choose λ to be a positive sum of fundamental weights where ω_i occurs if and only if $i \in I$. □

3.7 Tannakian Description for $\widetilde{\mathscr{B}}(G)$

In order for there to be a nice Tannakian description for $\widetilde{\mathscr{B}}(G)$, the most obvious requirement is that $\widetilde{\mathscr{B}}(G)$ must inject into the set of filtrations $V_{x,r}$. Let $x, y \in \widetilde{\mathscr{B}}(G)$. Without loss of generality, we can assume that x is in the base alcove. Assume that $V_{x,r} = V_{y,r}$ for every $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ and $r \in \mathbb{R}$. We want to show that $x = y$. By the argument above which says that you can recover the facet containing a point z in its interior from the filtrations associated to z, we see that x, y are in the same facet.

Recall that $\widetilde{\mathscr{B}}(G) = \mathscr{B}(G) \times \mathscr{B}(Z(G)^\circ)$. Under this decomposition, let $x = (x', w)$ and $y = (y', z)$. We will first assume that $x' = y'$. Assume that $z \neq w$ for a contradiction. Choose a representation V of G which has a nonzero weight space V^λ corresponding to a character λ such that $\langle \lambda, z \rangle \neq \langle \lambda, w \rangle$. Then it is clear that $V_{x,r} \neq V_{y,r}$ for some r since $n_{x,\lambda,r} = \lceil r - \langle \lambda, x' + w \rangle \rceil$ and $n_{y,\lambda,r} = \lceil r - \langle \lambda, y' + z \rangle \rceil$.

Now assume $x' \neq y'$. Since $x', y' \in \overline{\mathfrak{A}}$, we have $x', y' \in C = X_+(T_{\mathrm{der}})_{\mathbb{R}_{\geq 0}}$. Since $x' \neq y'$, there is some B-positive root α_i such that $\langle \alpha_i, x' \rangle \neq \langle \alpha_i, y' \rangle$. Thus, by taking $V = \mathfrak{g}$, we see that $\mathfrak{g}_{x',r} \neq \mathfrak{g}_{y',r}$ for $r = \max\{\langle \alpha_i, x' \rangle, \langle \alpha_i, y' \rangle\}$.

We will call the set

$$\{(V_{x,r}) \in \prod_{\substack{V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G}) \\ r \in \mathbb{R}}} \mid x \in \widetilde{\mathscr{B}}(G)\}$$

the set of all Moy-Prasad filtrations on the category $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. We have just proved the following statement.

Proposition 3.7.1. *The map $x \to \{V_{x,r}\}$ is a bijective map from $\widetilde{\mathscr{B}}(G)$ to the set of Moy-Prasad filtrations on the category $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$.*

While the above proposition gives a characterization of the points of the Bruhat-Tits building in terms of Moy-Prasad filtrations, it would be nicer to be able to describe these filtrations in some independent way. This seems like a difficult question, but it is at least possible to just check whether or not the filtrations on a single faithful representation are Moy-Prasad, in the sense that they are all defined by a single $x \in \widetilde{\mathscr{B}}(G)$.

In order to see this, we need to think about Moy-Prasad filtrations of a representation V as $\mathcal{O}$-lattices in $V \otimes K$, which form $\mathcal{O}$-lattice chains. Let L be the set of tuples of $\mathcal{O}$-lattice chains where every tuple has exactly one $\mathcal{O}$-lattice chain in $V \otimes K$ for each representation $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$.

Theorem 3.7.2. *The map $x \to \{V_{x,r}\}$ is a bijective map from $\widetilde{\mathscr{B}}(G)$ to the subset of tuples $(\{V_r\}_{r\in\mathbb{R}})$ of L which satisfy the following conditions:*

(A) *if $\phi : V \to U$ is a G-morphism, then for every $r \in \mathbb{R}$ we have that ϕ^K sends V_r into U_r,*

(B) *for every* V *and* U *in* $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ *and for every* $r \in \mathbb{R}$

$$(V \otimes U)_r = \sum_{s+t=r} V_s \otimes U_t,$$

(C) *for every exact sequence* $0 \to U \to V \to Y \to 0$ *in* $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ *and for every* $r \in \mathbb{R}$ *the sequence*

$$0 \to U_r \to V_r \to Y_r \to 0$$

is exact, and

(D) *there exists a faithful representation* V *of* G *and a point* $x \in \widetilde{\mathscr{B}}(G)$ *such that* $V_r = V_{x,r}$ *and* $V_r^\vee = V_{x,r}^\vee$ *for every* $r \in \mathbb{R}$.

Proof. It is clear that the above map is an injection, since Moy-Prasad filtrations satisfy all of the above properties. The surjectivity of the map is then an easy consequence of Theorem 2.2.10 using the conditions (A) - (D).

Let V and x be as in condition (D). Note that condition (A) is necessary for condition (C) to even be possible. Given that condition (A) holds, repeated applications of conditions (B) and (C) give that every $Y \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ which is a splittable subquotient of some direct sum of tensor powers of V and $V^\vee$ must have $Y_r = Y_{x,r}$ for every $r \in \mathbb{R}$. Since every representation in $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ is of this form, we have that the map is surjective. □

Remark 3.7.3. *Note that in the above proof, it was important that every representation was realizable as a splittable subquotient. If we only knew that every representation* $Y \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ *was a subquotient of some direct sum of tensor powers of* V *and* $V^\vee$*, then we would not be able to conclude that* Y_r *must equal* $Y_{x,r}$ *by* *only considering representations in $\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. However, if one instead considers all finitely generated representations of G, and uses the more general definition of a Moy-Prasad filtration given in Section 3.2, then the same characterization holds without the use of splittable subquotients.*

By Proposition 2.2.12, we can take the faithful representation in condition (E) to be a finite direct sum of Weyl modules. This theorem may not give a complete description of what these possible lattice chains look like, but at least when we choose V to be a finite direct sum of Weyl modules we can give some description as to how these lattice chains arise.

For the purpose of visualizing these filtrations, assume that G is semisimple. The set of weights in a Weyl module $V(\lambda)$ are well understood due to the Weyl character formula. See [Jan]. The characters are the set of $\nu \in X^*(T)$ lying in the convex hull of the set $W\lambda$ in $X^*(T) \otimes \mathbb{R}$. We will now try to describe the lattice chains which are Moy-Prasad filtrations corresponding to a point in the apartment $\widetilde{\mathcal{A}}$, since all other such lattice chains are obtained by the action of G.

A lattice chain $\{V_r\}$ is a Moy-Prasad filtration corresponding to a point $x \in \widetilde{\mathcal{A}}$ if there is a family of parallel hyperplanes a fixed distance of 1 apart for which the valuations in the lattice chain corresponding to a weight ν depend on which hyperplanes that weight ν lies between. More concretely, if we fix some $x \in \widetilde{\mathcal{A}}$ and $r \in \mathbb{R}$, then the parallel hyperplanes are the sets H_i where $H_i = \{z \in X^*(T) \otimes \mathbb{R} \big| \ (x,r)(z) = i\}$ for every $i \in \mathbb{Z}$. A weight ν of $V(\lambda)$ has $V(\lambda)^\nu \otimes \pi^i \mathcal{O} \subset V_r$ with i

minimal if

$$i - 1 < (x, r)(\nu) \leq i,$$

which corresponds to ν lying between the hyperplanes H_{i-1} and H_i.

There are many questions still to be addressed in this line of research, such as questions of functoriality. In particular, we would like to describe the relationship between $\widetilde{\mathscr{B}}(G)$ and $\widetilde{\mathscr{B}}(GL(V))$ for any faithful representation V. Also, it should be possible to describe the spherical building associated to G in a similar way by using the building at infinity approach. However, we will now turn from this discussion and move on to the main topic of this thesis.

Chapter 4

A Tannakian Description for Bruhat-Tits Group Schemes

Let E be an unramified Galois extension of K, and let $\widetilde{\mathscr{B}}(G(E))$ denote the Bruhat-Tits building of $G(E)$. So, based on our previous notation, we have $\widetilde{\mathscr{B}}(G) = \widetilde{\mathscr{B}}(G(K))$. There exists an injective map, see [T1] 2.6,

$$j : \widetilde{\mathscr{B}}(G(K)) \hookrightarrow \widetilde{\mathscr{B}}(G(E))$$

such that its restriction to any apartment is an affine mapping into an apartment, it is $G(K)$-equivariant, and

$$j(\widetilde{\mathscr{B}}(G(K))) = \widetilde{\mathscr{B}}(G(E))^{\mathrm{Gal}(E/K)}.$$

Remark 4.0.4. *Since G is assumed to be split, the apartments in $\widetilde{\mathscr{B}}(G(K))$ and $\widetilde{\mathscr{B}}(G(E))$ are actually the same affine space and hence j sends a facet to a facet. If we identify an apartment $\mathcal{A}$ in $\widetilde{\mathscr{B}}(G(K))$ with $X_*(T) \otimes \mathbb{R}$ and then pass over this identification to the apartment $j(\mathcal{A})$ in $\widetilde{\mathscr{B}}(G(E))$, we have*

$$\langle \lambda, x \rangle = \langle \lambda, j(x) \rangle$$

for any $\lambda \in X^*(T)$.

Let Ω be a non-empty subset of an apartment of $\widetilde{\mathscr{B}}(G)$ whose projection in $\mathscr{B}(G)$ is bounded. We will now give a characterization for the Bruhat-Tits group schemes $\mathscr{G}_\Omega^\circ$ associated to Ω, which are defined in Section 4.6.2 of [BT2]. Similar descriptions can be found in either [T1] or [H2].

Proposition 4.0.5. *The Bruhat-Tits group scheme $\mathscr{G}_\Omega^\circ$ associated to Ω is the unique affine group scheme defined over $\mathcal{O}$ satisfying:*

1. *$\mathscr{G}_\Omega^\circ$ is smooth,*

2. *the generic fiber of $\mathscr{G}_\Omega^\circ$ is G_K, and*

3. *for every unramified Galois extension E of K with ring of integers $\mathcal{O}_E$, we have*

$$\mathscr{G}_\Omega^\circ(\mathcal{O}_E) = \mathrm{Fix}_{G(E)}(j(\Omega)) \cap \mathrm{Ker}(\kappa_G).$$

Here κ_G denotes the Kottwitz homomorphism, which is defined in Section 7 of [Kot]. We will briefly discuss some facts about κ_G in Section 4.2.2. This classification depends upon the results of [HR], which contains the proof of the equality in condition 3.

Remark 4.0.6. *Since we are assuming that G is split, the group scheme $\mathscr{G}_\Omega^\circ$ actually coincides with the group scheme $\mathscr{G}_\Omega$, which is also defined in Section 4.6.2 of [BT2]. This is a consequence of Corollary 2.3.2 of [H3] and the corrigendum to [H3]. Thus we can and will write $\mathscr{G}_\Omega$ for the above group scheme in the remainder of this thesis.*

We will restrict our attention to parahoric Bruhat-Tits group schemes. In other words, we study those $\mathscr{G}_\Omega$ where $\Omega = \widetilde{\mathfrak{F}}$ is a facet in $\widetilde{\mathscr{B}}(G)$. Without loss of generality, we will further assume that the facet $\widetilde{\mathfrak{F}}$ is in the closure $\overline{\widetilde{\mathfrak{A}}}$ of our fixed base alcove $\widetilde{\mathfrak{A}}$.

4.1 Definition of the Functor

The main goal of this thesis is to give a Tannakian description to $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ using lattice chains. In [H2], a description was given to $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ when $G = GL_n$ in terms of lattice chains in the standard representation. We have reproduced this description in Section 5.3 in order to compare it to our functor.

Since there is no standard representation for a general reductive group, it seemed natural to try and use the whole category of representations instead. We will now define the fundamental object of this thesis.

Definition 4.1.1. *Let* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *be the group functor from* $\mathcal{O}$-**Alg** *to* **Grps** *defined as follows:*

$$\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(R) = \left\{ (g^R_{V_{x,r}}) \in \prod_{\substack{V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G}),\\ r \in \mathbb{R}, x \in \mathfrak{F}}} GL(V_{x,r})(R) \middle| \text{ conditions (1) - (4) hold} \right\}$$

(1) *For all pairs of affine functions (x, r), $(y, s) \in \mathfrak{F} \times \mathbb{R}$ on $X^*(T) \otimes \mathbb{R}$ and for every $n \in \mathbb{N}$ such that $V_{x,r+n} \subset V_{y,s} \subset V_{x,r}$ the following diagram commutes:*

$$\begin{tikzcd}
V_{x,r} \otimes_{\mathcal{O}} R \arrow[r, "\cong \cdot \pi^n"] \arrow[d, "g^R_{V_{x,r}}"] & V_{x,r+n} \otimes_{\mathcal{O}} R \arrow[r, "\mathbf{incl} \otimes \mathrm{Id}_R"] \arrow[d, "g^R_{V_{x,r+n}}"] & V_{y,s} \otimes_{\mathcal{O}} R \arrow[r, "\mathbf{incl} \otimes \mathrm{Id}_R"] \arrow[d, "g^R_{V_{y,s}}"] & V_{x,r} \otimes_{\mathcal{O}} R \arrow[d, "g^R_{V_{x,r}}"] \\
V_{x,r} \otimes_{\mathcal{O}} R \arrow[r, "\cong \cdot \pi^n"] & V_{x,r+n} \otimes_{\mathcal{O}} R \arrow[r, "\mathbf{incl} \otimes \mathrm{Id}_R"] & V_{y,s} \otimes_{\mathcal{O}} R \arrow[r, "\mathbf{incl} \otimes \mathrm{Id}_R"] & V_{x,r} \otimes_{\mathcal{O}} R
\end{tikzcd}$$

where **incl** *denotes the inclusion map.*

(2) *For every V, $U \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, every G-morphism $\phi : V \to U$, and every pair of affine functions (x, r), $(y, s) \in \mathfrak{F} \times \mathbb{R}$ such that $\lceil (x, r) \rceil \geq \lceil (y, s) \rceil$ on all of the*

weights common to both V and U, the following diagram commutes

$$\begin{tikzcd}
V_{x,r} \otimes_{\mathcal{O}} R \arrow[r, "\tilde{\phi}^R"] \arrow[d, "g^R_{V_{x,r}}"] & U_{y,s} \otimes_{\mathcal{O}} R \arrow[d, "g^R_{U_{y,s}}"] \\
V_{x,r} \otimes_{\mathcal{O}} R \arrow[r, "\tilde{\phi}^R"] & U_{y,s} \otimes_{\mathcal{O}} R
\end{tikzcd}$$

where $\tilde{\phi}^R(v \otimes \pi^n \otimes a) = \phi^R(v \otimes a) \otimes \pi^n$.

(3) $g^R_{\mathcal{O}_{x,r}} = \mathrm{Id}_R$, *where $\mathcal{O}$ denotes the trivial representation, for all* $x \in \mathfrak{F}$ *and all* $r \in \mathbb{R}$.

(4) *For all* V, $U \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ *and for all triples of affine functions* (x, r), (y, s), $(z, t) \in \mathfrak{F} \times \mathbb{R}$ *such that for every weight λ of V and every weight μ of U* $\lceil (x, r)(\lambda) \rceil + \lceil (y, s)(\mu) \rceil \geq \lceil (z, t)(\lambda + \mu) \rceil$, *the following diagram commutes*

$$\begin{tikzcd}
(V_{x,r} \otimes_{\mathcal{O}} R) \otimes_R (U_{y,s} \otimes_{\mathcal{O}} R) \arrow[r] \arrow[d, "g^R_{V_{x,r}} \otimes g^R_{U_{y,s}}"] & (V \otimes_{\mathcal{O}} U)_{z,t} \otimes_{\mathcal{O}} R \arrow[d, "g^R_{(V \otimes U)_{z,t}}"] \\
(V_{x,r} \otimes_{\mathcal{O}} R) \otimes_R (U_{y,s} \otimes_{\mathcal{O}} R) \arrow[r] & (V \otimes_{\mathcal{O}} U)_{z,t} \otimes_{\mathcal{O}} R.
\end{tikzcd}$$

In the definition of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$, note that x ranges over $\mathfrak{F}$ and not $\widetilde{\mathfrak{F}}$. Here we identify $\mathfrak{F}$ with the subset $\mathfrak{F} \times \{0\} \subset \widetilde{\mathfrak{F}}$ using the isomorphism $\widetilde{\mathcal{A}} \cong \mathcal{A} \times \mathcal{A}(Z(G)^\circ)$ discussed in Section 2.1.2. We define the functor in this way since these are all of the points that are necessary, and since it is not clear if the corresponding functor is finite type when x is allowed to range over all of $\widetilde{\mathfrak{F}}$. Condition (1) generalizes the lattice chain condition given for GL_n in Section 5.3 and the other 3 conditions generalize the standard Tannakian conditions (a) - (c) presented in Section 2.3.

4.2 Comparison of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ with $\mathscr{G}_{\widetilde{\mathfrak{F}}}$

We believe that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ in general and we will now begin to present our evidence.

Theorem 4.2.1. *The group functor* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *is represented by an affine* $\mathcal{O}$*-group scheme of finite type such that*

(GF) *the generic fiber of* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *is* G_K*, and*

(OP) *for every unramified Galois extension* E *of* K *with ring of integers* $\mathcal{O}_E$*, we have*

$$\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) = \mathrm{Fix}_{G(E)}(j(\widetilde{\mathfrak{F}})) \cap \mathrm{Ker}(\kappa_G).$$

Moreover, there is a unique homomorphism ϕ *of group schemes defined over* $\mathcal{O}$*,*

$$\phi : \mathscr{G}_{\widetilde{\mathfrak{F}}} \to \mathrm{Aut}_{\widetilde{\mathfrak{F}}},$$

which is the identity on the generic fiber and on $\mathcal{O}$*-points.*

The proof of this theorem will take quite a few pages and will be broken up into smaller pieces which will be stated as propositions or lemmas. Before we start proving Theorem 4.2.1, we will first list the cases in which we know $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$. The proofs of the following cases can be found in Chapter 5. We know that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ when:

1. G and K are arbitrary and $\widetilde{\mathfrak{F}}$ is the minimal facet containing the origin,

2. G is arbitrary and $\mathrm{char}(k) = 0$, and

3. $G = GL_n$ and K is arbitrary.

The proof of Theorem 4.2.1 will proceed in the following order. We will show (GF), (OP), that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is affine, and finally that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is of finite type. The claim about ϕ follows easily from the other claims and the fact that $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ is étoffé. The meaning of the term étoffé is given in Definition 5.2.1.

4.2.1 Generic Fiber

In order to show the following proposition, we will use the Tannakian description given to G in Section 2.3.

Proposition 4.2.2. *The generic fiber of* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *is equal to the generic fiber of* G.

Proof. Recall that passing to the generic fiber is equivalent to restricting the functor of points to K-algebras. So it will suffice to show that, for any K-algebra R, the functor $R \to \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(R)$ satisfies the Tannakian description of G that is described in Theorem 2.3.1. We will show this by defining mutually inverse maps between the two group functors. Throughout this proof we will refer to the conditions (1) - (4) from Definition 4.1.1 and the conditions (a) - (c) from Theorem 2.3.1.

Let R be a K-algebra and let $(g_V^R) \in G_K(R)$. Since R is a K algebra, we have

$$V_{x,r} \otimes R = V \otimes R = V_{y,s} \otimes R$$

for every pair of affine functions (x, r) and (y, s). By choosing a third affine function (z, t) such that $V_{z,t} \subset V_{x,r}$ and $V_{z,t} \subset V_{y,s}$, we see that any element of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(R)$ has $g^R_{V_{x,r}} = g^R_{V_{z,t}} = g^R_{V_{y,s}}$ by condition (1), since all of the horizontal maps are isomorphisms.

Thus to define a map to $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(R)$, it suffices to specify one map $g^R_{V_{x,r}}$ for each representation V and then check that conditions (1) - (4) hold. We set $g^R_{V_{x,r}} = g^R_V$ for every representation $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. Given this definition, it is clear that condition (1) holds. Moreover, one can see that (a) implies (2), (b) implies (3), and (c) imply (4) just by looking at the diagrams.

Now let $(g^R_{V_{x,r}}) \in \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(R)$. As we noted above, there is a unique map for each representation V, so we just define $g^R_V = g^R_{V_{x,r}}$ for any affine function (x, r). To see that condition (a) is satisfied, let $\phi : V \to U$ be a G-morphism and let $(x, r) \in \mathfrak{F} \times \mathbb{R}$. Then condition (2) applied to ϕ and the pairs (x, r), (x, r) will give condition (a). It is trivial to see that (b) follows from (3).

Finally, let V, $U \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ and choose (x, r), (y, s), and $(z, t) \in \mathfrak{F} \times \mathbb{R}$ such that for every weight λ of V and every weight μ of U $\lceil (x, r)(\lambda) \rceil + \lceil (y, s)(\mu) \rceil \geq \lceil (z, t)(\lambda \otimes \mu) \rceil$. Then condition (c) will follow immediately from condition (4) applied to this triple of affine functions. □

4.2.2 $\mathcal{O}_E$ Points

We will now show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ satisfies (OP) from Theorem 4.2.1. Let E be an unramified Galois extension of K. Since G is split, we can identify the facet $\widetilde{\mathfrak{F}}$ with its image in $\mathscr{B}(G(E))$. As was stated in Remark 4.0.4, by making the proper identifications of apartments, we can assume that $\langle \lambda, x \rangle = \langle \lambda, j(x) \rangle$ for any $\lambda \in X^*(T)$.

Proposition 4.2.3. *With notation as above,* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) = \mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$.

In order to prove the above proposition, we will proceed in the following way. We will show

(i) that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$ is the simultaneous stabilizer in $G(E)$ of $\{V_{x,r}^{\mathcal{O}_E} := V_{x,r} \otimes_{\mathcal{O}} \mathcal{O}_E\}$ for all $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, $x \in \mathfrak{F}$, and $r \in \mathbb{R}$,

(ii) that $P_{\widetilde{\mathfrak{F}}}^E := \mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$ stabilizes all of the $V_{x,r}^{\mathcal{O}_E}$, and

(iii) that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) \subset P_{\widetilde{\mathfrak{F}}}^E$.

First, by Proposition 4.2.2 and the fact that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) \hookrightarrow \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(E)$, we have that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$ is a subgroup of $G(E)$. To see that it is the simultaneous stabilizer of the $\{V_{x,r}^{\mathcal{O}_E}\}$, note that $\mathcal{O}_E$ is flat over $\mathcal{O}$, and hence we can view conditions (1) - (4) as applying to submodules of $V \otimes_{\mathcal{O}} E$. Every $g \in G(E)$ acts on $V \otimes E$ and $V_{x,r}^{\mathcal{O}_E} \subset V \otimes E$ so it is natural to ask whether or not this action preserves $V_{x,r}^{\mathcal{O}_E}$ for every $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ and $r \in \mathbb{R}$. If $g \in G(E)$ does preserve all of the $V_{x,r}^{\mathcal{O}_E}$, then all of the conditions in $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$ must hold for g since all of the $g_{V_{x,r}}^{\mathcal{O}_E}$ are defined by g.

The fact that $P_{\widetilde{\mathfrak{F}}}^E$ stabilizes all of the $V_{x,r}^{\mathcal{O}_E}$ follows easily from Proposition 3.4.2, noting that everything remains true when you pass to an unramified Galois extension E/K. This establishes (i) and (ii), so we will now show (iii).

In order to show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) \subset P_{\widetilde{\mathfrak{F}}}^E$, we will use an alternative description for $P_{\widetilde{\mathfrak{F}}}^E$. Let L denote the completion of the maximal unramified extension of K. In Section 7 of [Kot], Kottwitz defined a surjective homomorphism

$$\kappa_G : G(L) \twoheadrightarrow X^*(Z(\widehat{G}))$$

where $\widehat{G}$ is the dual group of G and $Z(\widehat{G})$ denotes the center of $\widehat{G}$. Using this map, Haines and Rapaport, see [HR], gave a new description of parahoric subgroups of $G(L)$ as

$$P^L_{\widetilde{\mathfrak{G}}} = \mathrm{Fix}_{G(L)}(\widetilde{\mathfrak{G}}) \cap \mathrm{Ker}(\kappa_G)$$

where $\widetilde{\mathfrak{G}}$ is a facet in the building $\widetilde{\mathscr{B}}(G(L))$. Moreover they showed that, when $\widetilde{\mathscr{B}}(G(E))$ is identified with the Galois fixed points in $\widetilde{\mathscr{B}}(G(L))$, this equality descends to the level of E-groups as

$$P^E_{\widetilde{\mathfrak{F}}} = \mathrm{Fix}_{G(E)}(\widetilde{\mathfrak{F}}) \cap \mathrm{Ker}(\kappa_G) \cap G(E).$$

Thus, in order to show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) \subset P^E_{\widetilde{\mathfrak{F}}}$, we need to show that it is contained in $\mathrm{Fix}_{G(E)}(\widetilde{\mathfrak{F}})$ and $\mathrm{Ker}(\kappa_G)$. We will first show containment in the kernel, but before we can do this we need to characterize $\mathrm{Aut}_x(\mathcal{O}_E)$ in the case where G is a split torus.

Lemma 4.2.4. *If $G = T$ is a split torus then $\mathrm{Aut}_x(\mathcal{O}_E) = T(\mathcal{O}_E)$ for every $x \in \mathscr{B}(T)$.*

Proof. To see this, first note that any representation of T can be decomposed in terms of its weight spaces, as we described in Section 2.1.3. Since $T(\mathcal{O}_E)$ preserves all of these weight spaces and acts E-linearly, it is clearly contained in $\mathrm{Aut}_x(\mathcal{O}_E)$. If $t \in T(E)\backslash T(\mathcal{O}_E)$, then for some weight λ, we see that t does not send $V^\lambda \otimes \mathcal{O}_E$ into itself, and consequently that $t \notin \mathrm{Aut}_x(\mathcal{O}_E)$. Thus if $t \in T(E)$ fixes $V^\lambda \otimes \mathcal{O}_E$ for every $\lambda \in X^*(T)$ and every $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, then t must be in $T(\mathcal{O}_E)$. $\square$

Lemma 4.2.5. *With notation as above, $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) \subset \mathrm{Ker}(\kappa_G)$.*

Proof. First assume that G_{der} is simply connected. Let $D = G/G_{\mathrm{der}}$, which is a split torus. Recall that $\widehat{D} = Z(\widehat{G})$ and that κ_G is defined by κ_D as follows:

$$\begin{tikzcd}
G(L) \arrow[r] \arrow[d, "\kappa_G"] & D(L) \arrow[d, "\kappa_D"] \\
X^*(Z(\widehat{G})) \arrow[r, "="] & X^*(\widehat{D}) = X_*(D)
\end{tikzcd}$$

Let $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{D})$ and lift V to a representation $\tilde{V} \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. Let $g \in \mathrm{Aut}_{\tilde{\mathfrak{F}}}(\mathcal{O}_E)$. So g must stabilize $\tilde{V}_{x,r}^{\mathcal{O}_E}$ for every r, and hence $\overline{g}$, the image of g in D, must also stabilize $V_{x,r}^{\mathcal{O}_E}$ for every r. Thus, by Lemma 4.2.4, we have $\overline{g} \in D(\mathcal{O})$. Since $D(\mathcal{O}) \subset \mathrm{Ker}(\kappa_D)$, we obtain that $g \in \mathrm{Ker}(\kappa_G)$ by the above diagram.

Now consider the general case. Choose a z-extension of G. So we have an exact sequence

$$1 \to S \xrightarrow{\phi} \widetilde{G} \to G \to 1$$

where S is a split torus and $\widetilde{G}_{\mathrm{der}}$ is simply connected. Recall that κ_G is defined as the unique map making the following diagram commute

$$\begin{tikzcd}
\widetilde{G}(L) \arrow[r, two heads] \arrow[d, "\kappa_{\widetilde{G}}"] & G(L) \arrow[d, "\kappa_G"] \\
X^*(Z(\widehat{\widetilde{G}})) \arrow[r] & X^*(Z(\widehat{G}))
\end{tikzcd}$$

Now consider the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & S/(S_{\mathrm{der}})(L) \arrow[r, "\overline{\phi}"] \arrow[d, "\kappa_{S/(S_{\mathrm{der}})}"] & \widetilde{G}/\widetilde{G}_{\mathrm{der}}(L) \arrow[r] \arrow[d, "\kappa_{\widetilde{G}/\widetilde{G}_{\mathrm{der}}}"] & G/G_{\mathrm{der}}(L) \arrow[r] \arrow[d, "\kappa_{G/G_{\mathrm{der}}}"] & 1 \\
1 \arrow[r] & X^*(Z(\widehat{S/(S_{\mathrm{der}})})) \arrow[r] & X^*(Z(\widehat{\widetilde{G}/\widetilde{G}_{\mathrm{der}}})) \arrow[r] & X^*(Z(\widehat{G/G_{\mathrm{der}}})) \arrow[r] & 1
\end{tikzcd}$$

Let $g \in \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$. By the same argument as above, its image $\overline{g} \in G/G_{\mathrm{der}}(\mathcal{O}_E)$ is in the kernel of $\kappa_{G/G_{\mathrm{der}}}$. Choose an element $\tilde{g} \in \widetilde{G}(L)$ such that its image $\overline{\tilde{g}} \in \widetilde{G}/\widetilde{G}_{\mathrm{der}}(L)$ maps to $\overline{g}$. We do not necessarily know that $\kappa(\overline{\tilde{g}}) = 0$, but by a diagram chase, one can see that $\kappa(\overline{\phi}(\overline{s}) \cdot \overline{\tilde{g}}) = 0$ for some $s \in S$ and hence that $\kappa_{\widetilde{G}}(\phi(s) \cdot \tilde{g}) = 0$. Thus we have $\kappa_G(g) = 0$ as desired. □

Now we will examine the values the Kottwitz homomorphism takes on the extended affine Weyl group $\widetilde{W} := N_{G(K)}(T(K))/T(\mathcal{O}) = W_{\mathrm{aff}} \rtimes \Omega$ where Ω is the stabilizer of the base alcove. It is easy to see that $W_{\mathrm{aff}} \subset \mathrm{Ker}(\kappa_G)$ by the following diagram

$$\begin{tikzcd}
\widetilde{W_{G_{sc}}} \arrow[r, "="] \arrow[d] & W_{\mathrm{aff}} \arrow[d] \\
G_{sc}(L) \arrow[r] \arrow[d, "\kappa_{\widetilde{G}}"] & G_{\mathrm{der}}(L) \arrow[d, "\kappa_G|_{G_{\mathrm{der}}}"] \\
1 = X^*(Z(\widehat{G_{sc}})) \arrow[r] & X^*(Z(\widehat{G}))
\end{tikzcd}$$

where G_{sc} is the simply connected cover of the derived group of G. Moreover, by Lemma 14 of [HR], Ω is isomorphic to the image of κ_G.

The following lemma will allow us to express $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$ and $P^E_{\widetilde{\mathfrak{F}}}$ in a simple form, which is essential to how we show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) \subset P^E_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$.

Lemma 4.2.6. *Let I_L be an Iwahori subgroup in $G(L)$ and let S be a set of simple reflections generating W_{aff}. Every subgroup $H \subset \mathrm{Ker}(\kappa_G)$ containing I is special. In other words, H can be written as*

$$H = I_L \langle S' \rangle I_L$$

for some subset $S' \subset S$.

Proof. Let $N = \mathrm{Norm}_{G(L)}(T(L)) \cap \mathrm{Ker}(\kappa_G)$. By combining Lemma 17 of [HR] and Proposition 5.2.12 of [BT2], we know that $(\mathrm{Ker}(\kappa_G), I_L, N, S)$ is a double Tits system, and thus that I_L, N form a BN-pair. Moreover, we know the Weyl group associated to this Tits system is W_{aff}.

Recall that a consequence of having a BN-pair is that the Bruhat Decomposition must hold. Thus we have

$$\mathrm{Ker}(\kappa_G) = \coprod_{w \in W_{\mathrm{aff}}} I_L w I_L,$$

and that H can be written as

$$H = \coprod_{w \in W'} I_L w I_L$$

for some $W' \subset W_{\mathrm{aff}}$. Thus we will be done if we can show that if $w \in W'$ has the reduced expression $w = s_1 \dots s_d$ for $s_i \in S$, then $I s_i I \subset H$. This is an easy consequence of the BN-pair axioms, and a proof can be found following Theorem 1 in Chapter 5 Section 2B of [Br]. $\square$

Let I_L be the Iwahori subgroup of $G(L)$ such that $I = I_L \cap G(K)$ and let $I_E = I_L \cap G(E)$. Note that $I_L \subset P^L_{\widetilde{\mathfrak{F}}} \subset \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_L)$ since $\widetilde{\mathfrak{F}}$ was chosen to be in the base alcove. Moreover, $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_L) \subset \mathrm{Ker}(\kappa_G)$ by Lemma 4.2.5. Thus Lemma 4.2.6 applies to both $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_L)$ and $P^L_{\widetilde{\mathfrak{F}}}$, and we have

$$\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_L) = I_L \langle S' \rangle I_L \quad \text{and} \quad P^L_{\widetilde{\mathfrak{F}}} = I_L \langle S'' \rangle I_L$$

for some subsets S', $S'' \subset S$.

We have that $P^E_{\widetilde{\mathfrak{F}}} = P^L_{\widetilde{\mathfrak{F}}} \cap G(E)$ and $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) = \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_L) \cap G(E)$ since $\mathcal{O}_E = \mathcal{O}_L \cap E$. Thus the above decompositions descend, and we have

$$\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E) = I_E\langle S'\rangle I_E \quad \text{and} \quad P^E_{\widetilde{\mathfrak{F}}} = I_E\langle S''\rangle I_E.$$

By the definition of $P^E_{\widetilde{\mathfrak{F}}}$, we know that S'' is the set of elements in S which fix $\widetilde{\mathfrak{F}}$. Thus to show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}) \subset P^E_{\widetilde{\mathfrak{F}}}$, it will suffice to show that $S' \subset S''$, i.e. that every element of S' fixes $\widetilde{\mathfrak{F}}$.

Lemma 4.2.7. *Let $y \in \widetilde{\mathfrak{F}}$. With notation as above, every $s \in S'$ fixes y.*

Proof. Assume, for a contradiction, that $sy \neq y$. Recall that $\widetilde{\mathscr{B}}(G) \cong \mathscr{B}(G) \times \mathscr{B}(Z(G)^\circ)$, and that the action of W_{aff} on $\widetilde{\mathscr{B}}(G)$ is solely on $\mathscr{B}(G)$, i.e. it acts trivially on $\mathscr{B}(Z(G)^\circ)$. Let x be the projection of y onto $\mathscr{B}(G)$.

We now have two cases for s. Namely, either s is a reflection s_{α_i} corresponding to a simple root α_i or it is an affine reflection $s_0 = t_{\alpha_0^\vee} s_{\alpha_0}$ where α_0 is a highest root. First assume that $s = s_{\alpha_i}$. Since $s_{\alpha_i} x \neq x$, we know that $x \neq 0$, where 0 is the origin in the apartment $\mathcal{A} = X_*(T) \otimes \mathbb{R}$. Moreover, since x is B-dominant, we have that $\langle \alpha_i,\, x\rangle > 0$ and hence that

$$\langle s_{\alpha_i}\alpha_i, x\rangle = \langle -\alpha_i, x\rangle < 0.$$

Now consider the adjoint representation $(\mathfrak{g},\, \mathrm{Ad}_G)$ of G. Let $r_i = \langle \alpha_i,\, x\rangle$. Then $\mathfrak{g}^{\alpha_i} \otimes \pi^0\mathcal{O}_E \subset \mathfrak{g}^{\mathcal{O}_E}_{x,r_i}$ but $\mathfrak{g}^{\alpha_i} \otimes \pi^0\mathcal{O}_E \not\subseteq s_{\alpha_i}\mathfrak{g}^{\mathcal{O}_E}_{x,r_i}$ since

$$s_{\alpha_i}\mathfrak{g}_{x,r_i}^{\mathcal{O}_E} = \mathfrak{g}^0 \otimes_{\mathcal{O}} \pi^{\lceil r_i \rceil}\mathcal{O}_E \oplus \sum_{\substack{\alpha\in\Phi, n\in\mathbb{Z} \\ \langle \alpha, x\rangle + n \geq r_i}} \mathfrak{g}^{s_{\alpha_i}\alpha} \otimes_{\mathcal{O}} \pi^n \mathcal{O}_E$$

$$= \mathfrak{g}^0 \otimes_{\mathcal{O}} \pi^{\lceil r_i \rceil}\mathcal{O}_E \oplus \sum_{\substack{\alpha\in\Phi, n\in\mathbb{Z} \\ \langle s_{\alpha_i}\alpha, x\rangle + n \geq r_i}} \mathfrak{g}^{\alpha} \otimes_{\mathcal{O}} \pi^n \mathcal{O}_E.$$

Thus $s_{\alpha_i}\mathfrak{g}_{x,r_i}^{\mathcal{O}_E} \neq \mathfrak{g}_{x,r_i}^{\mathcal{O}_E}$ and $s_{\alpha_i} \notin \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}_E)$.

Now consider $s = t_{\alpha_0^\vee} s_{\alpha_0}$. Since we have assumed $sx \neq x$, we know that $\langle \alpha_0, x\rangle < 1$. We will show that $s\mathfrak{g}_{x,r_0}^{\mathcal{O}_E} \neq \mathfrak{g}_{x,r_0}^{\mathcal{O}_E}$ where $r_0 = 1 - \langle \alpha_0,\, x\rangle > 0$. Note that

$$t_{\alpha_0^\vee} s_{\alpha_0}\mathfrak{g}_{x,r_0}^{\mathcal{O}_E} = t_{\alpha_0^\vee}\left(\mathfrak{g}^0 \otimes_{\mathcal{O}} \pi^{\lceil r_0 \rceil}\mathcal{O}_E \oplus \sum_{\substack{\alpha\in\Phi, n\in\mathbb{Z} \\ \langle s_{\alpha_0}\alpha, x\rangle + n \geq r_0}} \mathfrak{g}^{\alpha} \otimes_{\mathcal{O}} \pi^n \mathcal{O}_E \right)$$

$$= \mathfrak{g}^0 \otimes_{\mathcal{O}} \pi^{\lceil r_0 \rceil}\mathcal{O}_E \oplus \sum_{\substack{\alpha\in\Phi, n\in\mathbb{Z} \\ \langle s_{\alpha_0}\alpha, x\rangle + n \geq r_0}} \mathfrak{g}^{\alpha} \otimes_{\mathcal{O}} \pi^{n - \langle \alpha, \alpha_0^\vee\rangle} \mathcal{O}_E.$$

Here we have $\mathfrak{g}^{\alpha_0} \otimes \pi^{-1}\mathcal{O}_E \subseteq t_{\alpha_0^\vee} s_{\alpha_0}\mathfrak{g}_{x,r_0}^{\mathcal{O}_E}$ but $\mathfrak{g}^{\alpha_0} \otimes \pi^{-1}\mathcal{O}_E \nsubseteq \mathfrak{g}_{x,r_0}^{\mathcal{O}_E}$. □

This concludes the proof of Proposition 4.2.3.

4.2.3 Representable by an Affine Scheme

Let I denote an index set. For each $i \in I$, let X_i be affine $\mathcal{O}$-schemes with respective representing $\mathcal{O}$-algebras A_i and define X to be the $\mathcal{O}$-scheme

$$X = \prod_{i\in I} X_i.$$

Note that there are natural projection maps $\mathrm{proj}_i : X \to X_i$. Moreover, let

$$A := \varinjlim_{J\subset I} \bigotimes_{i\in J} A_i$$

where J ranges over the finite subsets of I. Let $\mathrm{incl}_i : A_i \to A$ be the natural maps.

Lemma 4.2.8. *X is an affine scheme represented by A.*

Proof. Note that X satisfies the following universal property: For any affine scheme Y and any morphisms $q_i : Y \to X_i$ there exists a unique morphism ϕ such that

$$\begin{tikzcd} X \arrow[dr, "\mathrm{proj}_i"'] & & Y \arrow[ll, "\phi"'] \arrow[dl, "q_i"] \\ & X_i & \end{tikzcd}$$

commutes. When viewing all of these objects as $\mathcal{O}$-functors, it is clear that ϕ must send $y \in Y(R)$ to $(q_i(y))_{\{i \in I\}}$ and hence is unique.

Similarly, A satisfies the universal property that if B is an $\mathcal{O}$-algebra and $p_i : A_i \to B$ are algebra maps then there is a unique map ψ such that

$$\begin{tikzcd} A \arrow[rr, "\psi"] & & B \\ & A_i \arrow[ul, "\mathrm{incl}_i"] \arrow[ur, "p_i"'] & \end{tikzcd}$$

commutes. To see this, note that by the universal property of (finite)tensor products, there is a unique map

$$\psi_i : \bigotimes_{i \in J} A_i \to B$$

which sends $\otimes a_i$ to $\prod_{i \in J} p_i(a_i)$ for every finite $J \subset I$. Moreover, for finite subsets $J \subset J' \subset I$ the transition map from $\bigotimes_{i \in J} A_i \to \bigotimes_{i \in J'} A_i$ sends $\bigotimes_{i \in J} a_i$ to $\left(\bigotimes_{i \in J} a_i\right) \otimes \left(\bigotimes_{i \in J' \setminus J} 1\right)$.

Putting these two facts together, we obtain a unique map $\psi : A \to B$ as claimed. From the universal property for A, we can see by Yoneda's Lemma (B.0.15) that the $\mathcal{O}$-functor $\mathrm{Hom}_{\mathcal{O}-alg}(A, \_\_)$ will satisfy the universal property for X listed above, and consequently that X is represented by A and is affine. $\square$

Remark 4.2.9. *Note that if X_i is a group functor for each i, then X will also be a group functor. Moreover, if A_i is flat for every i, then A will also be flat since finite tensor products of flat modules are flat and a direct limit of flat modules is flat.*

Proposition 4.2.10. $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ *is an affine scheme over $\mathcal{O}$.*

Proof. Define H to be the $\mathcal{O}$-group scheme

$$H = \prod_{\substack{V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G}) \\ x \in \mathfrak{F}, r \in \mathbb{R}}} GL(V_{x,r}).$$

Now denote the representing algebra of $GL(V_{x,r})$ by $\mathcal{O}[GL(V_{x,r})]$, let I be the set of triples (V, x, r) where $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, $x \in \mathfrak{F}$, and $r \in \mathbb{R}$. Let

$$A := \varinjlim_{J \subset I} \bigotimes_{i \in J} \mathcal{O}[GL(V_{x,r})]$$

where J ranges over the finite subsets of I. By Lemma 4.2.8, we see that H is represented by A. Moreover, we see that A is flat by Remark 4.2.9. Note that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is clearly an $\mathcal{O}$-subfunctor of H. We want to show that it is a closed subfunctor, i.e. that the conditions (1) - (4) (see Definition 4.1.1) that cut it out are polynomial conditions in A.

Recall that each $V_{x,r}$ is a free module of finite rank and hence is isomorphic to $\mathcal{O}^m$ for some m. Choose such an isomorphism for each $V_{x,r}$ and denote the rank by m_V. Then, for every $V_{x,r}$, we have

$$\mathcal{O}[GL(V_{x,r})] \simeq \mathcal{O}[y_{11}^{V_{x,r}}, \dots, y_{m_V m_V}^{V_{x,r}}]_{(\det)}$$

for some elements $y_{ij}^{V_{x,r}} \in \mathcal{O}[GL(V_{x,r})]$. Thus any element of $H(R)$, and consequently any element of $\mathrm{Aut}_{\tilde{\mathfrak{F}}}(R)$, is given by the images in R of all of the $y_{ij}^{V_{x,r}}$ for each $V_{x,r}$.

First consider condition (2). Let $\phi : V \to U$ be a G-morphism and hence a natural transformation. Since G is defined over $\mathcal{O}$, we also have that ϕ is defined over $\mathcal{O}$. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}^{m_V} \otimes A \arrow[r, "\tilde{\phi}"] \arrow[d, "g^A_{V_{x,r}}"] & \mathcal{O}^{m_U} \otimes A \arrow[d, "g^A_{U_{y,s}}"] \\
\mathcal{O}^{m_V} \otimes A \arrow[r, "\tilde{\phi}"] & \mathcal{O}^{m_U} \otimes A
\end{tikzcd}$$

Note that $\tilde{\phi}$ can be interpreted as a $m_U \times m_V$ matrix, and moreover it will have entries in $\mathcal{O}$ since it is defined over $\mathcal{O}$. Similarly, $g^A_{V_{x,r}}$ and $g^A_{U_{y,s}}$ can be thought of as $m_V \times m_V$ and $m_U \times m_U$ matrices respectively with entries $y_{ij}^{V_{x,r}}$ and $y_{ij}^{U_{y,s}}$, the generators for $\mathcal{O}[GL(V_{x,r})]$ and $\mathcal{O}[GL(U_{y,s})]$ chosen above. Now saying that this diagram commutes is just saying that certain polynomials in the $y_{ij}^{V_{x,r}}$'s with coefficients in $\mathcal{O}$ equal certain other polynomials in the $y_{ij}^{U_{y,s}}$'s with coefficients in $\mathcal{O}$, when regarded in A. Thus condition (2) is clearly a closed condition.

Under the isomorphisms $V_{x,r} \simeq \mathcal{O}^{m_V}$, condition (1) turns into the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}^{m_V} \otimes A \arrow[r, "\mathrm{Id}"] \arrow[d, "g^A_{V_{x,s}}"] & \mathcal{O}^{m_V} \otimes A \arrow[r, "\rho_{s+1,r} \otimes \mathrm{Id}_A"] \arrow[d, "g^A_{V_{x,s+1}}"] & \mathcal{O}^{m_V} \otimes A \arrow[r, "\rho_{r,s} \otimes \mathrm{Id}_A"] \arrow[d, "g^A_{V_{x,r}}"] & \mathcal{O}^{m_V} \otimes A \arrow[d, "g^A_{V_{x,s}}"] \\
\mathcal{O}^{m_V} \otimes A \arrow[r, "\mathrm{Id}"] & \mathcal{O}^{m_V} \otimes A \arrow[r, "\rho_{s+1,r} \otimes \mathrm{Id}_A"] & \mathcal{O}^{m_V} \otimes A \arrow[r, "\rho_{r,s} \otimes \mathrm{Id}_A"] & \mathcal{O}^{m_V} \otimes A
\end{tikzcd}$$

where $\rho_{s+1,r}$ and $\rho_{r,s}$ are maps that multiply by various positive powers of π on the different coordinates such that their composition $\rho_{s,s+1}$ is multiplication by a power of π. It is easy to see here that each square in the above diagram turns into $\mathcal{O}$-polynomial conditions on the generators of $\mathcal{O}[GL(V_{x,r})]$, $\mathcal{O}[GL(V_{x,s})]$, and $\mathcal{O}[GL(V_{x,s+1})]$.

Condition (4) can be analyzed in the same way to see that it is equivalent to placing polynomial conditions on the generators of A. For condition (3), it just says that we set $y = 1$ where $\mathcal{O}[GL(\mathcal{O}_{x,r})] \simeq \mathcal{O}[y, 1/y]$. Thus all of these conditions are closed and hence $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is represented by A', where A' is the quotient $\mathcal{O}$-algebra defined by taking A modulo all of the polynomial equations determined by conditions (1) - (4). $\square$

4.2.4 Finite Type

In order to show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is of finite type, we will first show that a slightly simpler functor is of finite type and prove that there is a closed embedding of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ into this finite type functor. For any $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, define the functor $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(V)$ from $\mathcal{O}$-$\mathbf{Alg}$ to $\mathbf{Grps}$ by

$$\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(V)(R) = \left\{ (g^R_{V_{x,r}}) \in \prod_{r \in \mathbb{R}, x \in \mathfrak{F}} GL(V_{x,r})(R) \middle| \text{ condition (1) holds} \right\}.$$

In order to show $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(V)$ is of finite type, we will need the following easy lemma.

Lemma 4.2.11. *For every bounded subset $\Omega \subset \overline{\mathfrak{A}}$ and for every $\lambda \in X^*(T)$ there exist integers n_L^λ and n_U^λ such that for every $x \in \Omega$ and $r \in [0,1)$,*

$$n_L^\lambda \leq \lceil (x,r)(\lambda) \rceil \leq n_U^\lambda.$$

The above lemma is needed to find another functor, which is of finite type, into which $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(V)$ naturally embeds.

Lemma 4.2.12. *Let $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. The functor $\mathrm{Aut}_{\tilde{\mathfrak{F}}}(V)$ is an affine $\mathcal{O}$-scheme of finite type.*

Proof. Note that there are a finite number of weights in V. In fact, the number of weights in V is bounded by the rank of V, which is assumed to be finite. Applying Lemma 4.2.11 to each of these weights, we get a finite set of lower bounds and upper bounds. Choose the smallest lower bound and call it n_L^V and choose the largest upper bound and call it n_U^V. Note that, by condition (1), every $g^R_{V_{x,r}}$ is determined by a $g^R_{V_{x,s}}$ for some $s \in [0,1)$.

By our choice of n_L^V and n_U^V, we know that for every $x \in \mathfrak{F}$ and every $r \in [0,1)$

$$V_{x,r} = \bigoplus_{\lambda} V^{\lambda} \otimes \pi^{n_{\lambda,x,r}} \mathcal{O}$$

where $n_L^V \leq n_{\lambda,x,r} \leq n_U^V$ for every λ. Thus we see that the set $\{V_{x,r}\}_{x \in \mathfrak{F}, r \in \mathbb{R}}$ is determined by a finite set $\{V_i\}$ of lattices, and consequently that there is a closed embedding

$$\mathrm{Aut}_{\tilde{\mathfrak{F}}}(V) \hookrightarrow \prod_i GL(V_i)$$

since condition (1) is a closed condition. Finally, it is clear that the above product of GL's is of finite type since it is a finite product and GL is of finite type. $\square$

Proposition 4.2.13. *Let V be a faithful self-dual projective representation of G. The canonical morphism*

$$\mathrm{Aut}_{\tilde{\mathfrak{F}}} \to \mathrm{Aut}_{\tilde{\mathfrak{F}}}(V)$$

is a closed embedding, and hence $\mathrm{Aut}_{\tilde{\mathfrak{F}}}$ is an affine $\mathcal{O}$-scheme of finite type.

Proof. We will keep all of the notation established in the proof of Proposition 4.2.10. Let V be a faithful self-dual projective representation of G, which we know exists by Theorem 2.2.6. (If the V produced by Theorem 2.2.6 is not self-dual, then replace it by $V \oplus V^\vee$ which is self-dual.) Let B be the representing algebra for $\prod_{x\in\mathfrak{F}, r\in\mathbb{R}} GL(V_{x,r})$. Note that B is a Hopf subalgebra of A. Let ϕ denote the canonical map from A to A' and let $B' = \phi(B)$.

Let C be the representing algebra for $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(V)$. The map $\phi\big|_B : B \twoheadrightarrow B'$ factors through C and gives a surjection from C to B'. So we have the following commutative diagram:

$$\begin{tikzcd}
 & B \arrow[r, hook] \arrow[dl] \arrow[d, "\phi|_B"] & A \arrow[d, "\phi"] \\
C \arrow[r, two heads] & B' \arrow[r, hook] & A'.
\end{tikzcd}$$

We will show that $B' = A'$, and hence obtain a surjection from C to A' which will give us our desired closed embedding.

As in Proposition 4.2.10, choose bases for every $U_{x,r}$ such that

$$\mathcal{O}[GL(U_{x,r})] \simeq \mathcal{O}[y_{11}^{U_{x,r}}, \ldots, y_{m_U m_U}^{U_{x,r}}]_{(\det)}$$

for some elements $y_{ij}^{U_{x,r}} \in \mathcal{O}[GL(U_{x,r})]$, where m_U denotes the rank of U. We will show that every element in $\phi(\{y_{ij}^{U_{x,r}}, 1/\det(y_{ij}^{U_{x,r}}) \big| \, U \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G}), x \in \mathfrak{F}, r \in \mathbb{R}\})$ can be expressed as a polynomial with coefficients in $\mathcal{O}$ in terms of the elements in $\phi(\{y_{ij}^{V_{x,r}}, 1/\det(y_{ij}^{V_{x,r}}) \big| \, x \in \mathfrak{F}, r \in \mathbb{R}\})$.

In order to show this, we will use Theorem 2.2.10 which gives us that any $U \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ can be expressed in terms of V using the operations direct sum, tensor product, quotient, and splittable subrepresentation. We will use these operations and the conditions of Definition 4.1.1 to show the above claim.

Before we do this, note that choosing a different basis for a $U_{x,r}$ corresponds to choosing different generators $z_{ij}^{U_{x,r}}$ of $\mathcal{O}[GL(U_{x,r})]$ such that

$$(z_{ij}^{U_{x,r}}) = M(y_{ij}^{U_{x,r}})M^{-1}$$

for some $M \in GL(U_{x,r})(\mathcal{O})$. Thus the $z_{ij}^{U_{x,r}}$'s can be expressed as linear polynomials with coefficients in $\mathcal{O}$ in the $y_{ij}^{U_{x,r}}$'s, and we are allowed to change bases freely to show our above claim.

Let us first look at the operation of direct sum. Let $U, X, Y \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ such that $U = X \oplus Y$. By the definition of a Moy-Prasad filtration, we know that $U_{x,r} = X_{x,r} \oplus Y_{x,r}$ for any pair $(x, r) \in \mathfrak{F} \times \mathbb{R}$. Let (x, r) be such a pair. Choose bases for $X_{x,r}$ and $Y_{x,r}$, which gives us a basis for $U_{x,r}$. By condition (2) applied to $X \hookrightarrow U$, $Y \hookrightarrow U$, and (x, r), we know that the polynomials given by the following diagram must vanish under the map ϕ from A to A':

$$\begin{tikzcd}
X_{x,r} \otimes_{\mathcal{O}} A \arrow[r, "\mathrm{Incl}"] \arrow[d, "(y_{ij}^{X_{x,r}})"] & U_{x,r} \otimes_{\mathcal{O}} A \arrow[d, "(y_{ij}^{U_{x,r}})"] & Y_{x,r} \otimes_{\mathcal{O}} A \arrow[l, "\mathrm{Incl}"'] \arrow[d, "(y_{ij}^{Y_{x,r}})"] \\
X_{x,r} \otimes_{\mathcal{O}} A \arrow[r, "\mathrm{Incl}"] & U_{x,r} \otimes_{\mathcal{O}} A & Y_{x,r} \otimes_{\mathcal{O}} A. \arrow[l, "\mathrm{Incl}"']
\end{tikzcd}$$

Thus we see that the $\phi(y_{ij}^{U_{x,r}})$'s can be expressed as polynomials with coefficients in $\mathcal{O}$ in terms of the $\phi(y_{ij}^{X_{x,r}})$'s and $\phi(y_{ij}^{Y_{x,r}})$'s. More concretely, by the choice of bases we made, we see that

$$\phi((y_{ij}^{U_{x,r}})) = \left[\begin{array}{c|c} \phi((y_{ij}^{X_{x,r}})) & 0 \\ \hline 0 & \phi((y_{ij}^{Y_{x,r}})) \end{array}\right],$$

and have $\phi(1/\det((y_{ij}^{U_{x,r}}))) = \phi(1/\det((y_{ij}^{X_{x,r}})))\phi(1/\det((y_{ij}^{Y_{x,r}})))$.

We will next look at the operation of tensor product. Let $X, Y \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, and let $(x,t) \in \mathfrak{F} \times \mathbb{R}$. Choose bases for X and Y by choosing bases for all of the weight spaces inside of X and Y. This gives us a basis for $X \otimes Y$, and moreover, bases for $X_{x,r}$, $Y_{x,s}$, and $(X \otimes Y)_{x,t}$ for every $r, s \in \mathbb{R}$. By condition (4) of definition 4.1.1, we know that if $r, s \in \mathbb{R}$ such that $r + s = t$, then

$$\begin{tikzcd}
(X_{x,r} \otimes_{\mathcal{O}} A) \otimes_A (Y_{x,s} \otimes_{\mathcal{O}} A) \arrow[r] \arrow[d, "g^A_{X_{x,r}} \otimes g^A_{Y_{x,s}}"] & (X \otimes_{\mathcal{O}} Y)_{x,t} \otimes_{\mathcal{O}} A \arrow[d, "g^A_{(X \otimes Y)_{x,t}}"] \\
(X_{x,r} \otimes_{\mathcal{O}} A) \otimes_A (Y_{x,s} \otimes_{\mathcal{O}} A) \arrow[r] & (X \otimes_{\mathcal{O}} Y)_{x,t} \otimes_{\mathcal{O}} A
\end{tikzcd}$$

commutes. Moreover, it is possible to choose finitely many $r_i, s_i \in \mathbb{R}$ with $r_i + s_i = t$ so that

$$\begin{tikzcd}
\bigoplus_{i=1}^{n} (X_{x,r} \otimes_{\mathcal{O}} A) \otimes_A (Y_{x,s} \otimes_{\mathcal{O}} A) \arrow[r, two heads] \arrow[d, "\bigoplus_{i=1}^{n} g^A_{X_{x,r}} \otimes g^A_{Y_{x,s}}"] & (X \otimes_{\mathcal{O}} Y)_{x,t} \otimes_{\mathcal{O}} A \arrow[d, "g^A_{(X \otimes Y)_{x,t}}"] \\
\bigoplus_{i=1}^{n} (X_{x,r} \otimes_{\mathcal{O}} A) \otimes_A (Y_{x,s} \otimes_{\mathcal{O}} A) \arrow[r, two heads] & (X \otimes_{\mathcal{O}} Y)_{x,t} \otimes_{\mathcal{O}} A.
\end{tikzcd}$$

In order to get this surjection, at most one pair (r_i, s_i) is necessary for every weight of $X \otimes Y$. Now to see that the $\phi((y_{ij}^{(X \otimes Y)_{x,t}}))$'s can be expressed as polynomials with coefficients in $\mathcal{O}$ in terms of the $\phi((y_{ij}^{X_{x,r}}))$'s and $\phi((y_{ij}^{Y_{x,s}}))$'s, take every basis vector in $(X \otimes Y)_{x,t}$, lift it arbitrarily to an element in the direct sum, and follow the element around the diagram. We will handle the determinant condition for the tensor operation in the quotient case since the argument is the same.

Now consider the quotient operation. Let $X, Y \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$ such that there is a surjective G-morphism $\phi : X \to Y$. Let $(x,r) \in \mathfrak{F} \times \mathbb{R}$ and apply condition (2) of Definition 4.1.1 to X, Y, ϕ, and (x,r). The induced map $\widetilde{\phi} : X_{x,r} \to Y_{x,r}$ is also surjective, and so we have

$$\begin{tikzcd}
X_{x,r} \otimes_{\mathcal{O}} A \arrow[r, two heads, "\tilde{\phi}^A"] \arrow[d, "g^A_{X_{x,r}}"] & Y_{x,r} \otimes_{\mathcal{O}} A \arrow[d, "g^A_{Y_{x,r}}"] \\
X_{x,r} \otimes_{\mathcal{O}} A \arrow[r, two heads, "\tilde{\phi}^A"] & Y_{x,r} \otimes_{\mathcal{O}} A.
\end{tikzcd}$$

Thus, by the same argument as above in the tensor product situation, we see that the $\phi((y_{ij}^{Y_{x,r}}))$'s can be written as $\mathcal{O}$ polynomials in terms of the $\phi((y_{ij}^{X_{x,r}}))$'s. Now notice that in both the quotient and tensor product cases, the range is a projective module. Thus there is a splitting in both situations and we see that image in A' of the matrix D corresponding to the domain can be chosen to be of the form

$$\phi(D) = \left[\begin{array}{c|c} F & * \\ \hline 0 & G \end{array}\right]$$

for some invertible matrices F and G. Note that the matrix corresponding to the range is the matrix G. Thus we see that $1/\det(G) = \det(F)(\phi(1/\det(D))$ is an $\mathcal{O}$-polynomial in the correct terms.

We are now left to consider the case of a splittable subrepresentation. Let X be a splittable subrepresentation of Y. Thus we have that $Y/X \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$, i.e. that Y/X is a finite rank free representation of G. As an $\mathcal{O}$-module, we have $Y = X \oplus Y/X$ and also $Y_{x,r} = X_{x,r} \oplus (Y/X)_{x,r}$ for any (x,r). Choose bases for X and Y/X, which gives us a basis for Y and a basis for every corresponding filtration. By condition (2) applied to $X \hookrightarrow Y$ and any pair (x,r), we see that the same arguments as above show that the $\phi(y_{ij}^{X_{x,r}})$'s can be expressed as polynomials with coefficients in $\mathcal{O}$ in terms of the $\phi(y_{ij}^{Y_{x,r}})$'s.

Finally, we need to consider the determinant for this case. With the choice of bases we made, we see that the matrix $\phi((y_{ij}^{Y_{x,r}}))$ is of the form

$$\phi((y_{ij}^{Y_{x,r}})) = \left[\begin{array}{c|c} \phi((y_{ij}^{X_{x,r}})) & * \\ \hline 0 & G \end{array}\right]$$

for some invertible G, and we can conclude that $1/\det(\phi((y_{ij}^{X_{x,r}}))) = \det(G)(1/\det(\phi((y_{ij}^{Y_{x,r}})))$ is an $\mathcal{O}$-polynomial in the correct terms.

Thus by repeated applications of the operations direct sum, tensor product, quotient, and splittable subrepresentation, we see that every element occurring in $\phi(\{y_{ij}^{U_{x,r}}, 1/\det(y_{ij}^{U_{x,r}}) \big| \; U \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G}), x \in \mathfrak{F}, r \in \mathbb{R}\})$ can be expressed as a polynomial with coefficients in $\mathcal{O}$ in terms of the elements in $\phi(\{y_{ij}^{V_{x,r}}, 1/\det(y_{ij}^{V_{x,r}}) \big| \; x \in \mathfrak{F}, r \in \mathbb{R}\})$. Thus we have shown $B' = A'$ and are done. $\square$

Keep all of the notation from the above proof. It is also true that the generators corresponding to a representation U determine the generators corresponding to $U^\vee$, so it is not actually necessary to take V to be self-dual. From Section 3.3, we have that there is a perfect pairing

$$U_{x,r} \otimes U_{x,s}^\vee \to \mathcal{O}_{x,0} = \mathcal{O}.$$

where $r = -s - 1 + \epsilon_U$. From this pairing, it is clear that $(y_{ij}^{U_{x,s}^\vee}) = ((y_{ij}^{U_{x,r}})^{-1})^T$, and moreover that $\phi(1/\det((y_{ij}^{U_{x,s}^\vee}))) = \phi(\det((y_{ij}^{U_{x,r}}))$.

Note that a representation U being obtained from V in multiple ways from these tensor generator operations is equivalent to a condition on the generators of B'. It just says that two different sets of polynomials in the generators are equal. Thus A' is essentially C cut out by polynomials which say that a tuple in $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(V)$ can be extended uniquely to a tuple in $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$. This concludes the proof of Theorem 4.2.1.

Chapter 5

Special Cases

We will now discuss several special cases where we know that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$.

5.1 $x = 0$

The easiest special case to deal with is when $\widetilde{\mathfrak{F}}$ is the minimal facet containing 0. Note that G satisfies the properties characterizing $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ in Proposition 4.0.5, and so in this case $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ is actually just the group scheme G itself. Moreover, since the only $x \in \mathfrak{F}$ is $x = 0$, we have $\langle \lambda, x \rangle = 0$ for every $\lambda \in X^*(T)$ and $x \in \mathfrak{F}$. So

$$V_{x,r} = \sum_{\substack{\lambda \in X^*(T) \\ n \geq r}} V^{\lambda} \otimes \pi^n \mathcal{O}$$

and hence $V_{x,r}$ is determined by $V_{x,0} = V$ for every $r \in \mathbb{R}$. Thus all of the conditions defining $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ reduce down to the Tannakian conditions describing G given in Theorem 2.3.1, and we have $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = G = \mathscr{G}_{\widetilde{\mathfrak{F}}}$.

5.2 $Char(k) = 0$

For an $\mathcal{O}$-scheme X, let X_K and X_k denote its generic and special fiber respectively.

Definition 5.2.1. *([BT2] §1.7) An $\mathcal{O}$-scheme X is called étoffé if it is flat and if for every $\mathcal{O}$-scheme Y and every K-morphism $\phi : X_K \to Y_K$ such that $\phi(X(\mathcal{O})) \subset$*

$Y(\mathcal{O})$, ϕ can be extended to an $\mathcal{O}$-morphism from X to Y.

Note that if X is étoffé then any such ϕ would extend uniquely since X is flat. In [BT2] §1.7, Bruhat and Tits show that a scheme is étoffé if and only if the following two conditions are satisfied:

(ET1) For every $f \in K[X] \backslash \mathcal{O}[X]$, there exists an $a \in \mathcal{O}$ such that $af \in \mathcal{O}[X]$ and if $b \in K$ with $\nu(b) < \nu(a)$ then $bf \notin \mathcal{O}[X]$.

(ET2) The canonical schematic image of $X(\mathcal{O})$ in $X(k)$ is dense in X_k.

Note that (ET1) is automatically satisfied in our case since our valuation ν is discrete. Also note that (ET2) can only be true if the scheme X_k is reduced since the schematic image of $X(\mathcal{O})$, by definition, is given the reduced induced closed subscheme structure and hence is reduced.

Theorem 5.2.2. *If* $\mathrm{char}(k) = 0$*, then* $\mathrm{Aut}_{\widetilde{\widetilde{\mathfrak{F}}}}$ *is the Bruhat-Tits group scheme* $G_{\widetilde{\widetilde{\mathfrak{F}}}}$ *associated to* $\widetilde{\widetilde{\mathfrak{F}}}$.

Proof. We will first deal with the case where k is algebraically closed. In Corollary 4.6.6 of [BT2], Bruhat and Tits show that $\mathscr{G}_{\widetilde{\widetilde{\mathfrak{F}}}}$ is étoffé. We will show that $\mathrm{Aut}_{\widetilde{\widetilde{\mathfrak{F}}}}$ is also étoffé and then use this property to show that these two schemes are equal. As noted above, in order to show that $\mathrm{Aut}_{\widetilde{\widetilde{\mathfrak{F}}}}$ is étoffé, it will suffice to show that it satisfies (ET2).

Recall that assuming $\mathrm{char}(k) = 0$ implies that K is a function field, and hence that there is a splitting of the natural inclusion of k into $\mathcal{O}$. Putting this together with the fact that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is a functor, we see that

$$\begin{tikzcd}
k \arrow[rr, hook] \arrow[dr, "\mathrm{Id}"'] & & \mathcal{O} \arrow[dl] \\
& k &
\end{tikzcd}$$

commutes and hence that

$$\begin{tikzcd}
\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(k) \arrow[rr, hook] \arrow[dr, "\mathrm{Id}"'] & & \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}) \arrow[dl] \\
& \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(k) &
\end{tikzcd}$$

commutes. Thus we have that the above map from $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}) \to \mathrm{Aut}_{\widetilde{\mathfrak{F}}}(k)$ is surjective. Now, by Cartier's Theorem (B.0.10) which implies that $(\mathrm{Aut}_{\widetilde{\mathfrak{F}}})_k$ is reduced, we have that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(k)$ is dense in $(\mathrm{Aut}_{\widetilde{\mathfrak{F}}})_k$. Combining these two facts, we see that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ satisfies (ET2) and hence is étoffé.

Now, by Propositions 4.2.2 and 4.2.3, we have that $(\mathrm{Aut}_{\widetilde{\mathfrak{F}}})_K = (\mathscr{G}_{\widetilde{\mathfrak{F}}})_K$ and $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}) = \mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O})$. Thus we have unique maps

$$\phi : \mathrm{Aut}_{\widetilde{\mathfrak{F}}} \to \mathscr{G}_{\widetilde{\mathfrak{F}}} \text{ and } \psi : \mathscr{G}_{\widetilde{\mathfrak{F}}} \to \mathrm{Aut}_{\widetilde{\mathfrak{F}}}$$

extending the maps on the generic fibers, since $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ and $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ are étoffé. Now consider

$$\phi \circ \psi : \mathscr{G}_{\widetilde{\mathfrak{F}}} \to \mathscr{G}_{\widetilde{\mathfrak{F}}} \text{ and } \psi \circ \phi : \mathrm{Aut}_{\widetilde{\mathfrak{F}}} \to \mathrm{Aut}_{\widetilde{\mathfrak{F}}}.$$

Note that both of these maps are the identity on their respective generic fibers, so they must be identity maps on the functors since the extension of a map on the generic fiber must be unique. Thus we obtain that both ϕ and ψ are isomorphisms and hence that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ as $\mathcal{O}$-group schemes when $k = \overline{k}$.

Now we will no longer assume that k is algebraically closed. Recall that if $\phi : X \to Y$ is a morphism of affine k-schemes and $\phi_{\overline{k}} : X_{\overline{k}} \to Y_{\overline{k}}$ is an isomorphism, then ϕ is an isomorphism. Thus we have that

$$(\mathrm{Aut}_{\widetilde{\mathfrak{F}}})_k = (\mathscr{G}_{\widetilde{\mathfrak{F}}})_k.$$

Since $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(\mathcal{O}) = \mathscr{G}_{\widetilde{\mathfrak{F}}}(\mathcal{O})$ and $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ is étoffé, we have that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ is étoffé as well and can apply the same argument as above. $\square$

5.3 $G = GL_n$

In [H2], a lattice chain description for $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ is given for $G = GL_n$. We will now give the description occurring in [H2], and use it to prove the following theorem.

Theorem 5.3.1. *If G is GL_n defined over $\mathcal{O}$, then* $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$.

Let G be GL_n, and hence of type A_{n-1}. The standard representation of G, which we will denote by S, is a faithful representation. The lattice chains discussed in [H2], which we will review, all occur in S. We will end up showing that these lattice chains are enough to determine $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$.

Let $S = \mathcal{O}^n$ be the standard representation of G. For every $i \in \{1, \ldots, n\}$, let e_i denote the i-th standard basis vector $(0^{i-1}, 1, 0^{n-i})$, and let $\Lambda_i \subset S$ denote the lattice

$$\Lambda_i = \mathcal{O}e_1 \oplus \ldots \mathcal{O}e_{n-i} \oplus \mathcal{O}\pi e_{n-i+1} \oplus \ldots \mathcal{O}\pi e_n.$$

Moreover, let $\Lambda_0 = \bigoplus_i \mathcal{O}e_i$ and define the lattice chain $\{\Lambda_m\}$ by $\Lambda_m = \pi^k \Lambda_i$ where $m = kn + i$.

We will usually drop the basis vectors e_i and let the position of the term in the direct sum denote which basis vector it corresponds to. For example, if $G = GL_3$ then $\{\Lambda_m\}$ is the following chain:

$$\cdots \subset \Lambda_3 = \pi\mathcal{O}^3 \subset \Lambda_2 = \mathcal{O} \oplus \pi\mathcal{O}^2 \subset \Lambda_1 = \mathcal{O}^2 \oplus \pi\mathcal{O} \subset \Lambda_0 = \mathcal{O}^3 \subset \ldots$$

Definition 5.3.2. *Define the functor $X_{\widetilde{\mathfrak{A}}}$ from $\mathcal{O}$-**Alg** to **Grps** by*

$$X_{\widetilde{\mathfrak{A}}}(R) = \{(g_i^R) \in \prod_{i\in\{0,\ldots,n\}} GL(\Lambda_i)(R) | \text{ condition (a) holds } \}$$

where condition (a) is

(a) *The following diagram commutes*

$$\begin{tikzcd}
\Lambda_0 \otimes R \arrow[r, "\cong \cdot\pi"] \arrow[d, "g_0^R"] & \Lambda_n \otimes R \arrow[r] \arrow[d, "g_n^R"] & \ldots \arrow[r] & \Lambda_1 \otimes R \arrow[r] \arrow[d, "g_1^R"] & \Lambda_0 \otimes R \arrow[d, "g_0^R"] \\
\Lambda_0 \otimes R \arrow[r, "\cong \cdot\pi"] & \Lambda_n \otimes R \arrow[r] & \ldots \arrow[r] & \Lambda_1 \otimes R \arrow[r] & \Lambda_0 \otimes R
\end{tikzcd}$$

For any facet $\widetilde{\mathfrak{F}} \subset \overline{\widetilde{\mathfrak{A}}}$, the functor $X_{\widetilde{\mathfrak{F}}}$ is defined similarly where there is a condition $(a_{\widetilde{\mathfrak{F}}})$ which is just condition (a) with the appropriate lattice chains removed along with possibly adding in one more lattice so that there is the isomorphism $\cong \cdot\pi$ between the second and last terms of the diagram.

In [H2], the argument for why $X_{\widetilde{\mathfrak{A}}} = \mathscr{G}_{\widetilde{\mathfrak{A}}}$ is sketched out. We will go through why $X_{\widetilde{\mathfrak{A}}}$ is smooth in great detail in Appendix A, so for now we will just assume this. The arguments for why $X_{\widetilde{\mathfrak{F}}}$ is smooth are essentially the same and will be left to the interested reader. In this section, we will show that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = X_{\widetilde{\mathfrak{F}}}$ which will suffice to show that $X_{\widetilde{\mathfrak{F}}}$ satisfies the characterization of $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ given in Proposition 4.0.5.

Thus in order to prove Theorem 5.3.2, it will suffice to show $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = X_{\widetilde{\mathfrak{F}}}$. To show this, we will need a lemma whose proof intricately uses information about the type A_{n-1} root system. Let $\omega_i^\vee$, which is given by

$$\omega_i^\vee = e_1 + \cdots + e_i - \frac{i}{n}\sum_{j=1}^{n} e_j,$$

denote the fundamental coweights, let $\alpha_i = e_i - e_{i+1}$ denote the B-simple roots, and let $\widetilde{\alpha} = e_1 - e_n$ denote the longest root.

Lemma 5.3.3. *Let G be GL_n defined over $\mathcal{O}$. Let S denote the standard representation and let λ_i be the character corresponding to e_i. Then in $S_{x,r}$ we have the following for any $x \in \overline{\mathfrak{A}}$ and any $r \in \mathbb{R}$:*

1. $n_{\lambda_1,x,r} \leq n_{\lambda_2,x,r} \leq \cdots \leq n_{\lambda_n,x,r}$,
2. $n_{\lambda_n,x,r} - n_{\lambda_1,x,r} \leq 1$,
3. *if* $\langle \alpha_i, x\rangle = 0$ *then* $n_{\lambda_i,x,r} = n_{\lambda_{i+1},x,r}$,
4. *if* $\langle \widetilde{\alpha}, x\rangle = 1$ *then* $n_{\lambda_1,x,r} = n_{\lambda_n,x,r} - 1$.

Proof. The first claim is a trivial consequence of x being in $X_+(T)$ and the fact that $\lambda_i = \lambda_{i+1} + \alpha_i$. To see the second claim recall that, since $x \in \overline{\mathfrak{A}}$, $x = \sum_i a_i \omega_i^\vee$ where $0 \leq a_i \leq 1$ and $\sum_i a_i \leq 1$. Thus

$$\begin{aligned}
\langle \lambda_1, x\rangle - \langle \lambda_n, x\rangle &= \left(a_1\frac{n-1}{n} + a_2\frac{n-2}{n} + \cdots + a_{n-1}\frac{1}{n}\right) \\
&\quad - \left(a_1\frac{-1}{n} + a_2\frac{-2}{n} + \cdots + a_{n-1}\frac{-(n-1)}{n}\right) \\
&= a_1\left(\frac{n-1}{n} + \frac{1}{n}\right) + a_2\left(\frac{n-2}{n} + \frac{2}{n}\right) + \cdots + a_{n-1}\left(\frac{1}{n} + \frac{n-1}{n}\right) \\
&= a_1 + a_2 + \cdots + a_{n-1} \leq 1.
\end{aligned}$$

Since we know $n_{\lambda_n,x,r} = \lceil r - \langle \lambda_n, x\rangle \rceil$ and $n_{\lambda_1,x,r} = \lceil r - \langle \lambda_1, x\rangle \rceil$, the second claim then easily follows. The third and fourth claims follow immediately from the facts that $\lambda_i = \lambda_{i+1} + \alpha_i$ and $\lambda_1 = \lambda_n + \widetilde{\alpha}$. $\square$

This lemma says that the only lattices that can occur as an $S_{x,r}$ are the standard lattices Λ_i for some $i \in \mathbb{Z}$. It further implies that, when we restrict x to a facet $\mathfrak{F}$ inside $\overline{\mathfrak{A}}$, we get a lattice subchain of the standard lattice chain associated to $\widetilde{\mathfrak{F}}$. Moreover, by choosing an x in general position inside of $\mathfrak{F}$, we get the full standard lattice chain associated to $\widetilde{\mathfrak{F}}$. We are now ready to prove Theorem 5.3.1.

Proof. First, recall that S is a faithful representation for G. Then, by Proposition 4.2.13, we know that there is a closed embedding of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ into $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(S)$. By the above lemma, we have that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(S) = X_{\widetilde{\mathfrak{F}}}$ since condition (1) reduces to condition (a) once we have identified that the only filtrations $S_{x,r}$ that occur are the standard lattices associated to $\widetilde{\mathfrak{F}}$.

Since $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ is étoffé, we have a map from $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ to $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ which is the identity of the generic fiber and on $\mathcal{O}$-points. Moreover, the closed embedding from $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ into $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}(S)$ is clearly the identity on the generic fiber and on $\mathcal{O}$-points. Thus we have

$$\begin{tikzcd} \mathscr{G}_{\widetilde{\mathfrak{F}}} \arrow[r] \arrow[dr, "\mathrm{Id}"'] & \mathrm{Aut}_{\widetilde{\mathfrak{F}}} \arrow[d, hook] \\ & X_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}} \end{tikzcd}$$

since $\mathscr{G}_{\widetilde{\mathfrak{F}}}$ is flat. Let B be the Hopf algebra representing $\mathscr{G}_{\widetilde{\mathfrak{F}}}$. The above diagram then turns into the following Hopf algebra diagram:

$$\begin{tikzcd} B & B/I \arrow[l] \\ & B \arrow[ul, "\mathrm{Id}"] \arrow[u, two heads] \end{tikzcd}$$

Thus we have $I = \{0\}$ and can conclude that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ as desired.

$\square$

Chapter 6

Dependence of $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ on the Facet $\widetilde{\mathfrak{F}}$

6.1 Comparing $\mathrm{Aut}_{\widetilde{\mathfrak{F}}}$ and $\mathrm{Aut}_{\widetilde{\mathfrak{G}}}$

A natural question to ask about these functors is how they relate to one another. In other words, if we take two facets $\widetilde{\mathfrak{G}}$ and $\widetilde{\mathfrak{F}}$ in $\overline{\widetilde{\mathfrak{A}}}$ such that $\widetilde{\mathfrak{G}} \subset \overline{\widetilde{\mathfrak{F}}}$ what can we say about how the functors are related? By looking at results about Bruhat-Tits group schemes, we would expect to find the following:

Proposition 6.1.1. *([Lan], Prop 6.2) Let $\Omega, \Omega' \subset X_*(T) \otimes \mathbb{R}$ be non-empty bounded subsets with $\Omega \subset \Omega'$. Then the identity $G \to G$ extends uniquely to an $\mathcal{O}$-group homomorphism $\mathrm{Res}_{\Omega}^{\Omega'} : \mathscr{G}_{\Omega'} \to \mathscr{G}_{\Omega}$.*

We do indeed find a result very similar to the above proposition which is easily proven and quite concrete. In order to prove our corresponding statement, we need the following lemma.

Lemma 6.1.2. *Let $\mathfrak{G}$ and $\mathfrak{F}$ be facets of the base alcove $\mathfrak{A}$ such that $\mathfrak{G} \subset \overline{\mathfrak{F}}$ and let $V \in \mathbf{Rep}_{\mathcal{O}}(\mathbf{G})$. For every pair $(x,\, r) \in \mathfrak{G} \times \mathbb{R}$ there exists a pair $(y,\, s) \in \mathfrak{F} \times \mathbb{R}$ such that $V_{x,r} = V_{y,s}$.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the list of all distinct weights of V. Let $r_i = r - \langle \lambda_i,\, x\rangle$ and let $n_i = \lceil r_i \rceil$. Let ϵ be a positive real number less than $\frac{\min\{r_i - (n_i - 1)\}}{2}$, let $s = r - \epsilon$, and let $y \in \mathfrak{F}$ such that $|\langle \lambda_i,\, x\rangle - \langle \lambda_i,\, y\rangle| < \epsilon$ for every i. Finally, let $s_i = s - \langle \lambda_i,$

$y\rangle$. We now need to show that $n_i - 1 < s_i \leq n_i$ for every i. By the definition of ϵ and our choice of y and s, we have

$$s_i = s - \langle \lambda_i, y \rangle > s - (\langle \lambda_i, x \rangle + \epsilon) = r - \epsilon - \langle \lambda_i, x \rangle - \epsilon = r_i - 2\epsilon > n_i - 1$$

and

$$n_i \geq r_i = r - \langle \lambda_i, x \rangle + \epsilon - \epsilon = s - (\langle \lambda_i, x \rangle - \epsilon) > s - \langle \lambda_i, y \rangle = s_i.$$

Thus we have that $\lceil (x,\, r)(\lambda_i) \rceil = \lceil (y,\, s)(\lambda_i) \rceil$ for all i and $V_{x,r} = V_{y,s}$ as desired.

$\square$

Proposition 6.1.3. *If $\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{F}}$ are facets of $\widetilde{\mathfrak{A}}$ such that $\widetilde{\mathfrak{G}} \subset \overline{\overline{\widetilde{\mathfrak{F}}}}$, then there is a natural functorial group homomorphism*

$$\mathrm{Aut}_{\widetilde{\mathfrak{F}}} \to \mathrm{Aut}_{\widetilde{\mathfrak{G}}}$$

which extends the identity map from G_K to G_K.

Proof. This proposition is an immediate consequence of Lemma 6.1.2. For any $(x,\, r) \in \mathfrak{G} \times \mathbb{R}$, there is a pair $(y,\, s) \in \mathfrak{F} \times \mathbb{R}$ such that $V_{x,r} = V_{y,s}$, so we define $g^R_{V_{x,r}} = g^R_{V_{y,s}}$. Note that this gives a well-defined element of $\mathrm{Aut}_{\widetilde{\mathfrak{G}}}$ since if there are two pairs $(y,\, s)$ and $(z,\, t) \in \mathfrak{F} \times \mathbb{R}$ such that $V_{y,s} = V_{x,r} = V_{z,t}$ then, by condition (2) of Definition 4.1.1 applied to the identity G-morphism from V to V, we have $g^R_{V_{y,s}} = g^R_{V_{z,t}}$. $\square$

6.2 The Iwahori Case

It is our belief that the Iwahori case is the fundamental case that needs to be solved in order to show $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ in general. We have made some progress towards a proof in this situation, and we will go through our arguments now.

In order to show $\mathrm{Aut}_{\widetilde{\mathfrak{A}}} = \mathscr{G}_{\widetilde{\mathfrak{A}}}$, it will suffice to show that $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}$ is étoffé. The argument for why this is the case is laid out in Section 5.2. Furthermore, in order to show $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}$ is étoffé, we just need to show that the image of $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}(\mathcal{O})$ is dense in $(\mathrm{Aut}_{\widetilde{\mathfrak{A}}})_k$.

Since $\mathscr{G}_{\widetilde{\mathfrak{A}}}$ is étoffé (see [BT2] Corollary 4.6.6), we know that $\mathscr{G}_{\widetilde{\mathfrak{A}}}(\mathcal{O})$ is dense in $(\mathscr{G}_{\widetilde{\mathfrak{A}}})_k$. By Proposition 4.2.3, we know that $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}(\mathcal{O}) = \mathscr{G}_{\widetilde{\mathfrak{A}}}(\mathcal{O})$, and thus it suffices to show $(\mathrm{Aut}_{\widetilde{\mathfrak{A}}})_k = (\mathscr{G}_{\widetilde{\mathfrak{A}}})_k$.

Let $\mathrm{Res}_0^{\widetilde{\mathfrak{A}}} : \mathscr{G}_{\widetilde{\mathfrak{A}}} \to \mathscr{G}_{x=0} = G$ be the map given in Proposition 6.1.1. It is a known fact that, when we pass to the special fiber, the map $(\mathrm{Res}_0^{\widetilde{\mathfrak{A}}})_k$ has image equal to B_k, the special fiber of our fixed Borel subgroup B. Thus one would expect the same thing to be true for the map

$$\phi : \mathrm{Aut}_{\widetilde{\mathfrak{A}}} \to \mathrm{Aut}_{x=0} = G.$$

Lemma 6.2.1. *With notation as above, the image of ϕ_k is B_k.*

Proof. We will first show that the image is contained in B_k. Recall that B_k can be classified as the set of elements in G_k which fix the highest weight line in any Weyl module $V(\lambda)$ corresponding to a regular weight $\lambda \in X^*(T)$. Let n denote the rank of $V(\lambda)$.

Choose a regular B-dominant weight $\lambda \in X^*(T)$ and choose $x \in \mathfrak{A}$ such that $\langle \lambda, x \rangle > \langle \mu, x \rangle$ for every weight $\mu \neq \lambda$ of $V(\lambda)$. Then by choosing $r = \langle \lambda, x \rangle$, we have $V(\lambda)_{x,r} \subset \mathcal{O}^n$ such that $n_{\lambda,x,r} = 0$ and $n_{\mu,x,r} > 0$ for every weight $\mu \neq \lambda$.

By condition (1) applied to $V_{x,r}$ and $V_{y,s} = \mathcal{O}^n$, we have that the diagram

$$\begin{tikzcd}
V(\lambda)_{x,r} \otimes R \arrow[r] \arrow[d, "g^R_{V(\lambda)_{x,r}}"] & \mathcal{O}^n \otimes R \arrow[d, "g^R_{V(\lambda)_{y,s}}"] \\
V(\lambda)_{x,r} \otimes R \arrow[r] & \mathcal{O}^n \otimes R
\end{tikzcd}$$

commutes for every k-algebra R. By following around elements in the highest weight line, namely elements of $V(\lambda)^\lambda$, we see that $g^R_{V(\lambda)_{y,s}}$ stabilizes this line since the horizontal maps are zero on every weight space except $V(\lambda)^\lambda$ and the identity on $V(\lambda)^\lambda$. Thus we have that the image of ϕ_k is contained in B_k since ϕ sends a tuple $(g^R_{V_{x,r}})_{V\in\mathbf{Rep}_{\mathcal{O}}(\mathbf{G}),x\in\mathfrak{F},r\in\mathbb{R}}$ to $(g^R_{V_{y,s}})_{V\in\mathbf{Rep}_{\mathcal{O}}(\mathbf{G})}$ where $V_{y,s}=\mathcal{O}^{n_V}$ and n_V is the rank of V.

We will now show that there is a natural map from B to $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}$. Let $\mathcal{O}[B]$ be the representing algebra of B. It is well-known that $\mathcal{O}[B]$ is flat. By an easy generalization of Lemma 3.4.3, we see that for every $b\in B(\mathcal{O}[B])$ we have

$$b\cdot V_{x,r}\otimes\mathcal{O}[B]\subset V_{x,r}\otimes\mathcal{O}[B].$$

Thus every $b\in B(\mathcal{O}[B])$ simultaneously stabilizes the different filtrations $V_{x,r}\otimes\mathcal{O}[B]$. This gives us a group homomorphism from $B(\mathcal{O}[B])$ to $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}(\mathcal{O}[B])$, which is enough to obtain a group functor map from B to $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}$.

Thus we have the following commutative diagram

$$\begin{tikzcd}
B_k \arrow[r] \arrow[dr, "\mathrm{Id}"'] & (\mathrm{Aut}_{\widetilde{\mathfrak{A}}})_k \arrow[d, "\phi_k"] \\
 & B_k
\end{tikzcd}$$

and can conclude that the image of ϕ_k is B_k as desired. $\square$

We now want to compare these maps $\mathrm{Res}_0^{\widetilde{\mathfrak{A}}}$ and ϕ_k. Since $\mathscr{G}_{\widetilde{\mathfrak{A}}}$ is étoffé, we know that there is a unique homomorphism from $\mathscr{G}_{\widetilde{\mathfrak{A}}}$ to G, and hence the following diagram must commute

$$\begin{tikzcd}
\mathscr{G}_{\widetilde{\mathfrak{A}}} \arrow[r, "\mathrm{Res}_0^{\widetilde{\mathfrak{A}}}"] \arrow[d, "\psi"] & G \arrow[d, "\mathrm{Id}"] \\
\mathrm{Aut}_{\widetilde{\mathfrak{A}}} \arrow[r, "\phi"] & G
\end{tikzcd}$$

where the map ψ from $\mathscr{G}_{\widetilde{\mathfrak{A}}}$ to $\mathrm{Aut}_{\widetilde{\mathfrak{A}}}$ is given by the étoffé property. Let $U = \mathrm{Ker}(\mathrm{Res}_0^{\widetilde{\mathfrak{A}}})$ and let $U' = \mathrm{Ker}(\phi_k)$. The above diagram passes to the special fiber and gives us the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & U \arrow[r, hook] \arrow[d, "\psi_k|_U"] & (\mathscr{G}_{\widetilde{\mathfrak{A}}})_k \arrow[r, "(\mathrm{Res}_0^{\widetilde{\mathfrak{A}}})_k"] \arrow[d, "\psi_k"] & B_k \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\
1 \arrow[r] & U' \arrow[r, hook] & (\mathrm{Aut}_{\widetilde{\mathfrak{A}}})_k \arrow[r, "\phi_k"] & B_k \arrow[r] & 1
\end{tikzcd}$$

Thus we have reduced showing $\mathrm{Aut}_{\widetilde{\mathfrak{A}}} = \mathscr{G}_{\widetilde{\mathfrak{A}}}$ down to showing that $(\psi_k)|_U$ is a bijection.

Conjecture 6.2.2. *With notation as above, $(\psi_k)|_U$ is a bijection.*

This conjecture has resisted proof so far, but we hope to solve it in the near future. Once the Iwahori case is done, we believe that the other cases will follow. However, the details of this generalization process are not completely clear and will have to be worked out in order to prove that $\mathrm{Aut}_{\widetilde{\mathfrak{F}}} = \mathscr{G}_{\widetilde{\mathfrak{F}}}$ in general by this method.

Appendix A

Lattice Chain Description for BT Group Schemes Associated to GL_n

In this appendix, we will prove that the functor $X_{\widetilde{\mathfrak{F}}}$ from Definition 5.3.2 is smooth. In fact, we will present an expanded version of the first part of the appendix to Chapter 3 in Rapaport's and Zink's book [RZ]. In this appendix, they prove that a certain automorphism functor of lattice chains, which is closely related to $X_{\widetilde{\mathfrak{F}}}$, is formally smooth. Using this and a result of Pappas from [Pap], we will be able to conclude that $X_{\widetilde{\mathfrak{F}}}$ is formally smooth and hence smooth.

They work in great generality, but we will not present their results at the full level of abstraction since it is not needed for our current purposes. However, we will still work in a more general setting than we have been throughout the rest of this thesis. In particular, we will work over a commutative ring S instead of over $\mathcal{O}$ unless stated otherwise. Before we get to the results of their appendix, we will give some background material that is helpful in understanding their arguments.

A.1 Background and Examples

Let S be a commutative ring with 1. Let **Rngs**, S-**Alg**, **Sch**, and **Sch**$/S$ respectively be the category of commutative rings with identity, commutative S-algebras, schemes, and schemes defined over S. We will take a very detailed approach to this material and first discuss some basic examples. Let M be a rank n projective

S-module. We first look at the group functor $GL(M)$ from S-**Alg** to **Grps** defined by

$$GL(M)(R) = (\mathrm{End}(M \otimes_S R))^\times$$

and see how it is represented by an affine scheme. Note that we are only concerned with projective finite rank modules over S.

Example A.1.1. ([Jan] Chapter I, 2.2) In order to show $GL(M)$ is representable, we first need to show that $M_a(R) := (M \otimes_S R, +)$ is representable. To see this, we will show that M_a is represented by $S(M^*)$ and hence $M_a(R) = \mathrm{Hom}_{S\text{-}\mathbf{Alg}}(S(M^*), R)$. Recall that $M^{**} = M$ so $M = \mathrm{Hom}_S(M^*, S)$. Thus

$$\begin{aligned} M \otimes_S R &= \mathrm{Hom}_S(M^*, S) \otimes_S R \\ &= \mathrm{Hom}_S(M^*, R) \\ &= \mathrm{Hom}_{S\text{-}\mathbf{Alg}}(T(M^*), R) \\ &= \mathrm{Hom}_{S\text{-}\mathbf{Alg}}(S(M^*), R) \end{aligned}$$

where the third equality is the universal property of $T(M^*)$ and the forth equality is due to R being commutative. We will now show that the functor $R \longrightarrow \mathrm{End}_R(M \otimes_S R)$ can be identified with $(M^* \otimes_S M)_a$. To see this, we will use several facts about the relationships between Hom and tensor products which can be found in [B-A] II §4. Thus

$$\begin{aligned}\mathrm{End}_R(M \otimes_S R) &= \mathrm{Hom}_R(M \otimes_S R, M \otimes_S R)\\ &\simeq \mathrm{Hom}_S(M, \mathrm{Hom}_R(R, M \otimes_S R))\\ &\simeq \mathrm{Hom}_S(M, R \otimes_S M)\\ &\simeq M^* \otimes_S M \otimes_S R\end{aligned}$$

Now all that remains is to show that the open subfunctor $GL(M)$ is cut out by a single equation and hence is representable and affine. There exists a determinant map det defined as follows. If P is a projective module of rank n over S, then $\bigwedge^n P$ is projective of rank 1 and $\mathrm{Hom}_S(\bigwedge^n P, \bigwedge^n P) = (\bigwedge^n P)^* \otimes_S \bigwedge^n P \simeq S$. Then det takes an element $\phi \in \mathrm{End}_R(M \otimes_S R)$ and sends it as follows:

$$\phi \to \bigwedge^n \phi \in \mathrm{Hom}_R\left(\bigwedge^n M \otimes_S R, \bigwedge^n M \otimes_S R\right) \xrightarrow{\cong} x \in (\bigwedge^n M \otimes_S R)^* \otimes_R \bigwedge^n M \otimes_S R \xrightarrow{\cong} \det(\phi) \in R.$$

Hence det is a map from $(M^* \otimes_S M)_a(R) = \mathrm{Hom}_{S\text{-}\mathbf{Alg}}(S((M^* \otimes_S M)^*), R)$ to R. As a consequence of Yoneda's Lemma (B.0.15), we then have that det can be viewed as an element of $S((M^* \otimes_S M)^*)$. Then $GL(M)(R) = \mathrm{Hom}_{S\text{-}\mathbf{Alg}}(S((M^* \otimes_S M)^*)_{(\det)}, R)$ and thus is representable and affine as desired.

Thus we see that $GL(M)$ is represented by an affine scheme. We will now go through the argument for why the Grassmannian is representable by a (projective) scheme, and moreover is smooth, while also recalling the definitions of these various notions.

Definition A.1.2. *Let $V = \mathcal{O}^n$. For any $0 < r < n$, the* <u>*Grassmannian*</u>*, $G(r, n)$, is* *the functor from $\mathcal{O}$-**Alg** to **Sets** given by*

$$G(r,n)(R) = \{\text{locally free } R-\text{modules of rank } r \text{ which are quotients of } V \otimes_{\mathcal{O}} R\}.$$

Given a map of $\mathcal{O}$-algebras $\phi : R \to T$, $G(r,n)(\phi)$ sends an element $\psi : V \otimes_{\mathcal{O}} R \to F$ of $G(r,n)(R)$ to an element of $G(r,n)(T)$ as follows:

$$\begin{tikzcd} V \otimes_{\mathcal{O}} R \arrow[r, "\otimes_R T"] \arrow[d, "\psi"] & V \otimes_{\mathcal{O}} R \otimes_R T \simeq V \otimes_{\mathcal{O}} T \arrow[d, "\widetilde{(\psi, \mathrm{Id})}"] \\ F \arrow[r, "\otimes_R T"] & F \otimes_R T \end{tikzcd}$$

where $\widetilde{(\psi, \mathrm{Id})}$ is the map on the tensor product defined by $(\psi, \mathrm{Id}) : V \otimes_{\mathcal{O}} R \times T \to F \otimes_R T$.

For a definition of what it means for a module to be locally free, see Definition B.0.6. We now define a very similar functor,

$$G'(r,n)(R) := \{ \text{ rank } r \text{ direct summands } F \text{ of } V \otimes_{\mathcal{O}} R\}$$

where $G'(r,n)(\phi)$ sends F to $F \otimes_R T$. It turns out that $G'(r,n)$ is just an alternate formulation of the Grassmannian.

Lemma A.1.3. *With notation as above, $G(r,n) \simeq G'(r,n)$.*

Proof. In order to see this, we will use Theorem B.0.7. An immediate consequence of Theorem B.0.7 is that an R-module M is finitely generated projective of rank r if and only if M is finitely generated and locally free of constant local rank r.

Clearly $G'(r,n)(R) \subset G(r,n)(R)$ for any ring R. Let $F \in G(r,n)(R)$. So F is finitely-generated and locally free of constant rank r. Then F is finitely generated projective of rank r, which gives us that it is a direct summand of $V \otimes_{\mathcal{O}} R$ of rank r and hence $F \in G'(r,n)(R)$ as desired. It is easy to see that this process is natural, i.e. that the following diagram commutes:

$$\begin{tikzcd}
G(r,n)(R) \arrow[r, "G(r,n)(\phi)"] \arrow[d] & G(r,n)(T) \arrow[d] \\
G'(r,n)(R) \arrow[r, "G'(r,n)(\phi)"] & G'(r,n)(T)
\end{tikzcd}$$

Therefore, both of the above notions give the same functor. □

We now recall a criterion for determining whether or not such a functor is representable by a scheme, and also a useful fact about subfunctors of local functors which will be used in Theorem A.3.2.

Theorem A.1.4. *([DG] Chapter I, §1, 4.4 and 6.8) Let F be a functor from $\mathcal{O}$-**Alg** to **Sets**. Then F is representable by a scheme $|F|$ defined over $\mathcal{O}$ if and only if*

1. *F is local, i.e the sequence*

$$X(R) \to \prod_{i=r}^{r} X(R_{f_i}) \rightrightarrows \prod_{1 \le i,j \le r} X(R_{f_i f_j})$$

   is exact for any $\mathcal{O}$-algebra R and any $f_1,\ f_2,\ \ldots,\ f_r \in R$ with $\sum_{i=1}^r Rf_i = R$, and

2. *there exist affine open sub-functors F_i of F such that for every $\mathcal{O}$-algebra L which is also a field, $F(L) = \bigcup F_i(L)$.*

Lemma A.1.5. *([Jan] Chapter I, 1.13) Let X be a local functor and $Y \subset X$ be a local subfunctor. Let $(X_j)_{j \in J}$ be an open covering of X. Then Y is closed in X if and only if $Y \cap X_j$ is closed in X_j for every $j \in J$.*

Note that any closed subfunctor of an affine scheme over $\mathcal{O}$ is again an affine scheme over $\mathcal{O}$. Moreover, if $(Y_j)_{j\in J}$ is a family of closed subfunctors of an $\mathcal{O}$-functor X, then $\cap_{j\in J} Y_j$ is a closed subfunctor of X. See [Jan] for more details. As we mentioned before, the Grassmannian is actually a smooth scheme. Before we show this, we will first recall what it means for a scheme to be smooth.

Definition A.1.6. *([BLR] Chapter 2.2, def. 3) Let X be a scheme over $\mathcal{O}$. Then X is smooth at a point x of X if there exists an open neighborhood U of x and an $\mathcal{O}$-immersion*

$$j : U \hookrightarrow \mathbb{A}^n_{\mathcal{O}}$$

of U into some affine space over $\mathcal{O}$ such that the following conditions hold:

1. *locally at $y := j(x)$, the sheaf of ideals defining $j(U)$ as a subscheme of $\mathbb{A}^n_{\mathcal{O}}$ is generated by $(n-r)$ sections $g_{r+1}, \ldots, g_n$, and*
2. *the differentials $dg_{r+1}(y), \ldots, dg_n(y)$ are linearly independent in $\Omega^1_{\mathbb{A}^n_{\mathcal{O}}/\mathcal{O}} \otimes \mathcal{O}_{X,y}/\mathfrak{m}_y$ where $\mathfrak{m}_y$ is the maximal ideal in the local ring $\mathcal{O}_{X,y}$.*

The scheme X is called smooth if it is smooth at all points.

The $\mathbb{A}^n_{\mathcal{O}}$-module $\Omega^1_{\mathbb{A}^n_{\mathcal{O}}/\mathcal{O}}$ is called the module of relative differential 1-forms of $\mathbb{A}^n_{\mathcal{O}}$ over $\mathcal{O}$. See Chapter 2.1 of [BLR] for more details on this object.

Proposition A.1.7. *The Grassmannian $G(r, n)$ is represented by a smooth scheme defined over $\mathcal{O}$.*

Proof. The representability of $G(r, n)$ follows from Theorem A.1.4. The proof that $G(r, n)$ is local is somewhat time consuming, so we will not go into the details here.

For the full details of this, see [DG] Chapter I, §1, 3.9 - 3.16. In their argument, they show that the open affine sub-functors F_i that cover $G(r,n)$ are represented by $\mathbb{A}_{\mathcal{O}}^{r(n-r)}$. Moreover, in the proof of Theorem A.1.4, they show that the scheme $|F|$ which represents a functor F is covered by the affine schemes that represent the F_i. Since $|G(r,n)|$ is covered by affine spaces, it is clearly smooth. □

It can be difficult to show that a functor is smooth directly. There is an alternate characterization of smoothness given in functor language, called formal smoothness, which is sometimes easier to show.

Definition A.1.8. *A functor X, represented by a scheme $|X|$ over $\mathcal{O}$, is called formally smooth if and only if for every $\mathcal{O}$-algebra R and every nilpotent ideal $I \subset R$ there exists a map from* $\operatorname{Spec}(R)$ *to $|X|$ making the following diagram commute:*

$$\begin{tikzcd}
|X| \arrow[d] & \operatorname{Spec}(R/I) \arrow[l] \arrow[d, "\text{incl}"] \\
\operatorname{Spec}(\mathcal{O}) & \operatorname{Spec}(R) \arrow[l] \arrow[ul, dotted]
\end{tikzcd}$$

In other words, every element of $X(R/I)$ lifts to an element of $X(R)$ such that the lift respects the structure over $\mathcal{O}$.

If we take $X = \operatorname{Spec} A$ to be an affine scheme in the above definition, then we get that formal smoothness is equivalent to there existing a map from A to R making the following diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dotted] & R/I \\ \mathcal{O} \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

The equivalence of formal smoothness and smoothness does not hold in complete generality, but it can be quite useful when the notions do coincide.

Theorem A.1.9. *([BLR] Chapter 2.2, Prop 6) Let X be a scheme which is locally of finite type over $\mathcal{O}$. Then X is smooth over $\mathcal{O}$ if and only if the functor*

$$\mathrm{Mor}_{\mathrm{Sch}}(\mathrm{Spec}(\_\_), X) : \mathcal{O}\text{-}\mathbf{Alg} \to \mathbf{Sets}$$

is formally smooth over $\mathcal{O}$.

An important thing to note is that by examining the proof of Theorem A.1.9, one can see that it is enough to check the formal smoothness condition for noetherian rings in order to show that X is smooth over $\mathcal{O}$.

Remark A.1.10. *As a result of this theorem, we now know that $G(r,n)$ is formally smooth over $\mathcal{O}$, since it is clearly locally of finite type over $\mathcal{O}$. Let I be a nilpotent ideal. A consequence of this is that if we are given two elements $\bar{U}$, $\bar{V} \in G(r,n)(R/I)$, such that there exists an isomorphism $\bar{\alpha} : \bar{U} \to \bar{V}$, then there exists an isomorphism α between their respective lifts U, $V \in G(r,n)(R)$ which is a natural lift of $\bar{\alpha}$. To see this, first note that a defining property of the lift U is that $U/IU = U \otimes_R R/I = \bar{U}$.*

We want to define an isomorphism $\alpha : U \to V$ which lifts a given isomorphism $\bar{\alpha} : U/IU \to V/IV$. To do this, first define a map $\bar{\beta} : (R/I)^n \to (R/I)^n$ that extends

$\bar{\alpha}$, which can be done since U/IU and V/IV are direct summands of $(R/I)^n$. Now lift $\bar{\beta}$ to a map $\beta : R^n \to R^n$ by arbitrarily lifting on a set of generators and then extending linearly. Consider $\beta|_U$ which sends U to $V \oplus IV'$, where $R^n = V \oplus V'$.

Now define $\alpha : U \to V$ to be the composition of the projection $\rho : V \oplus IV' \to V$ with $\beta|_U$. By Corollary B.0.13 we get that $\alpha(U) = V$ since $\bar{\alpha}(U/IU) = (V/IV)$ and $I \subset \mathrm{rad}_{\mathfrak{m}}(R)$. Finally, by Proposition B.0.14 and the fact that U and V are locally free of rank r, we get that α is an isomorphism as desired.

A.2 Definitions

Let V be a finite dimensional K-vector space. We will be considering $\mathcal{O}$-lattice chains inside V. See Definition 2.1.1. Recall that every lattice chain $\mathcal{L} = \{\Lambda_i\}_{i\in\mathbb{R}}$ is actually determined by a finite number of lattices $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{r-1}\}$ such that

$$\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_{r-1} \subsetneq \pi^{-1}\Lambda_0.$$

Moreover, this is a minimal finite set of lattices determining $\mathcal{L}$. If $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{r-1}\}$ is a such a minimal determining set then we say, following [RZ], that the number r is the period of the lattice chain $\mathcal{L}$.

Definition A.2.1. *A chain of S-modules of type $(\mathcal{L})$ is an indexed set of finitely generated S-modules $\{M_i\}_{i\in\mathbb{Z}}$, such that for any $i \in \mathbb{Z}$ there exists a periodicity isomorphism*

$$\theta : M_{i-r} \xrightarrow{\sim} M_i.$$

Moreover there is an S-module homomorphism

$$\rho : M_i \longrightarrow M_{i+1}$$

such that the following conditions are satisfied:

1. *Locally there are isomorphisms of the following S-modules:*

$$M_i \simeq \Lambda_i \otimes_{\mathcal{O}} S, \qquad M_i/\rho(M_{i-1}) \simeq \Lambda_i/\Lambda_{i-1} \otimes_{\mathcal{O}} S.$$

2. *The map $\rho^r : M_{i-r} \longrightarrow M_i$ is equal to $\pi\theta$ and the following diagram*

$$\begin{tikzcd}
M_{i-r} \arrow[r, "\rho"] \arrow[d, "\theta"] & M_{i-r+1} \arrow[d, "\theta"] \\
M_i \arrow[r, "\rho"] & M_{i+1}
\end{tikzcd}$$

   commutes.

First of all, note that we are abusing notation in the above definition. In particular, each θ and ρ should have a subscript which equals the subscript of its domain (or equivalently its range). A trivial example of a chain of S-modules is obtained by tensoring the chain $\mathcal{L}$ by S.

We will denote by $\bar{M}_i$ the $k \otimes S$-module

$$\bar{M}_i = k \otimes M_i = M_i/\pi M_i = M_i/\rho^r(M_{i-r}).$$

Similarly, if N is any $\mathcal{O}$-module, we denote by $\bar{N}$ the $\mathcal{O}$-module $N/\pi N = k \otimes N$.

If $T = \operatorname{Spec} S$ is a scheme over $\mathcal{O}$, then π is called nilpotent on T if under the structure map from $\mathcal{O}$ to S, π gets sent to a nilpotent element in S.

Remark A.2.2. *Note that in [RZ], they only assume that π is locally nilpotent on T. However, when T is assumed to be affine this is actually equivalent to π being nilpotent on T, which we will now show. Clearly if π is nilpotent on $T =$ Spec S, then it is locally nilpotent.*

So assume that π is locally nilpotent. Thus for all $\mathfrak{p} \in$ Spec S, $\pi^{n_\mathfrak{p}} = 0$ in $S_\mathfrak{p}$ for some $n_\mathfrak{p}$. So for every $\mathfrak{p}$ there exists an $f_\mathfrak{p} \in S - \mathfrak{p}$ such that $f_\mathfrak{p}\pi^{n_\mathfrak{p}} = 0$ in S. Then, for all $\mathfrak{q} \in D(f_\mathfrak{p})$, $\pi^{n_\mathfrak{p}} = 0$ in $S_\mathfrak{q}$. Thus, since Spec S is quasi-compact, it is covered by finitely many basic open sets $D(f)$ such that there exists an n_f where

$$\pi^{n_f} = 0 \text{ in } S_\mathfrak{q} \text{ for all } \mathfrak{q} \in \mathrm{D}(f).$$

Thus there exists an N such that $\pi^N = 0$ in $S_\mathfrak{q}$ for every $\mathfrak{q}$ in Spec S, which immediately implies $\pi^N = 0$ in S.

Let $T = \text{Spec } S$ be an affine scheme over $\mathcal{O}$, such that π is nilpotent on T. This means that we are assuming the image of π in S is a nilpotent element. Let $\{M_i\}$ be a chain of S-modules of type $(\mathcal{L})$. We will now define the main object in this appendix.

Definition A.2.3. *Let $Y_{\{M_i\}}$ be the functor from S-**Alg** to **Grps** defined by*

$$Y_{\{M_i\}}(R) := \{((g^R_{M_i}) \in \prod_{i=1}^{r} GL(M_i)(R)| \text{ condition (a) holds } \}$$

where condition (a) is

(a) *The following diagram commutes:*

$$\begin{array}{ccccccc}
M_1 \otimes R & \xrightarrow{\rho_1 \otimes \mathbf{Id}_R} & M_2 \otimes R & \xrightarrow{\rho_2 \otimes \mathbf{Id}_R} & \dots & \xrightarrow{\rho_r \otimes \mathbf{Id}_R} & M_{r+1} \otimes R \\
\Big\downarrow g^R_{M_1} & & \Big\downarrow g^R_{M_2} & & & & \Big\downarrow g^R_{M_3} \\
M_1 \otimes R & \xrightarrow{\rho_1 \otimes \mathbf{Id}_R} & M_2 \otimes R & \xrightarrow{\rho_2 \otimes \mathbf{Id}_R} & \dots & \xrightarrow{\rho_r \otimes \mathbf{Id}_R} & M_{r+1} \otimes R
\end{array}$$

Note that $(X_{\widetilde{\mathfrak{A}}})_S = Y_{\{M_i\}}$ for $\{M_i = \Lambda_i \otimes S\}$ where $\{\Lambda_i\}$ is the standard lattice chain defined in Section 5.3.

A.3 Results

Before we can show that $Y_{\{M_i\}}$ is formally smooth and represented by an affine scheme, we need to prove a lemma about chains of S-modules.

Lemma A.3.1. *For any $k < r$ and $i \in \mathbb{Z}$, the sequence of S-modules*

$$0 \longrightarrow M_i/\rho^k(M_{i-k}) \xrightarrow{\bar{\rho}} M_{i+1}/\rho^{k+1}(M_{i-k}) \xrightarrow{\text{mod } \rho(M_i)} M_{i+1}/\rho(M_i) \longrightarrow 0$$

is split exact where $\bar{\rho}(m + \rho^k(M_{i-k})) = \rho(m) + \rho^{k+1}(M_{i-k})$. Moreover, locally there exist isomorphisms

$$M_i/\rho^k(M_{i-k}) \simeq \Lambda_i/\Lambda_{i-k} \otimes S$$

of S-modules.

Proof. First of all, exactness in the middle and at the right are clear from the definitions of the maps and the fact that $\rho^{k+1}(M_{i-k}) \subset \rho(M_i)$. We are going to first prove the lemma when viewing all of these quotients as $k \otimes S$-modules and then extend the results to the objects as S-modules.

We know that the finitely generated $k \otimes S$-modules $M_{i+1}/\rho(M_i)$ are locally free of constant rank equal to the rank of the k-vector space Λ_{i+1}/Λ_i. Thus, by Theorem B.0.7, we have that the $M_{i+1}/\rho(M_i)$ are projective for every i. The surjection

$$M_{i+1}/\rho^{k+1}(M_{i-k}) \longrightarrow M_{i+1}/\rho(M_i)$$

then splits, with splitting map ϕ and we get surjections

$$M_i/\rho^k(M_{i-k}) \oplus M_{i+1}/\rho(M_i) \xrightarrow{\bar{\rho}\oplus\phi} M_{i+1}/\rho^{k+1}(M_{i-k}).$$

By recursion we see that $M_i/\rho^k(M_{i-k})$ is a quotient of the $k \otimes S$-module

$$F_{i,k} := \bigoplus_{j=i-k+1}^{i} M_j/\rho(M_{j-1})$$

which is locally free with the same rank as the k-vector space Λ_i/Λ_{i-k}. Now, in order to show that $M_i/\rho^k(M_{i-k})$ is a locally free $k \otimes S$-module of the same rank as Λ_i/Λ_{i-k} we will apply descending induction on k. For $k = r$ this follows immediately from the fact that $\rho^r(M_{i-r}) \simeq \pi M_i$ since this implies that locally

$$M_i/\rho^r(M_{i-r}) \simeq M_i/\pi M_i \simeq (\Lambda_i \otimes S)/(\pi\Lambda_i \otimes S) \simeq \Lambda_i/\pi\Lambda_i \otimes S \simeq \Lambda_i/\Lambda_{i-r} \otimes S.$$

This in fact gives us our base case, since the above situation is the same as taking $k = r - 1$ and replacing i with $i + 1$.

Assuming by induction that $M_{i+1}/\rho^{k+1}(M_{i-k})$ is locally free of the given rank, we obtain a surjection of locally free modules of the same rank

$$F_{i,k} \oplus M_{i+1}/\rho(M_i) \longrightarrow M_{i+1}/\rho^{k+1}(M_{i-k})$$

which is then necessarily injective as well by Proposition B.0.14. Hence $F_{i,k} \to M_i/\rho^k(M_{i-k})$ is an isomorphism. The exactness on the left of the sequence as $k \otimes S$-modules is then immediate.

We now want to extend these results to these objects as S-modules. To see this, we just need to understand the action of π on all of these objects. Well, $\pi M_i \simeq \rho^r(M_{i-r}) \subset \rho^k(M_{i-k})$ and $\pi\Lambda_i = \Lambda_{i-r} \subset \Lambda_{i-k}$ for all $k \in \mathbb{Z}$ such that $1 \leq k \leq r$. Thus π acts trivially on all of these objects, so the isomorphisms and the short exact sequences all still hold when we view these objects as S-modules.

$\square$

Theorem A.3.2. *The functor $Y_{\{M_i\}}$ is represented by an affine group scheme over T.*

Proof. For this proof, all tensor products will be over S unless otherwise noted. In order to show that $Y_{\{M_i\}}$ is represented by an affine group scheme, we will use Lemma A.1.5 and the fact that a closed subfunctor of an affine functor is affine. We will assume for now that $r = 2$ and prove the general case once we have dispensed with this one. First of all, note that $Y_{\{M_i\}}$ is a subfunctor of $GL(M_1)\times GL(M_2)$. Moreover, $GL(M_1) \times GL(M_2)$ is represented by $S((M_1 \otimes_S M_1^*)^*)_{(det)} \otimes_S S((M_2 \otimes_S M_2^*)^*)_{(det)}$ as is easily derived from Remark A.1.1 and hence is an affine functor.

We will now show that $Y_{\{M_i\}}$ is a local functor. Let R be an S-algebra and let $f_1, \ldots, f_l \in R$ such that $\sum_{i=1}^{l} Rf_i = R$. Let $(\phi_i, \psi_i)_{i\in\{1, \ldots, l\}} \in \prod_{i=1}^{l} Y_{\{M_i\}}(R_{f_i})$ such that they get sent to the same element in $\prod_{1\leq i,j\leq l} Y_{\{M_i\}}(R_{f_i f_j})$ under the two natural maps. Since $GL(M_1) \times GL(M_2)$ is an affine functor, it is local and hence we can lift the $(\phi_i, \psi_i)_{i\in\{1, \ldots, l\}}$ to the unique element $(\phi, \psi) \in GL(M_1)(R) \times GL(M_2)(R)$. Thus in order to show that $Y_{\{M_i\}}$ is local, it suffices to show (ϕ, ψ) is in $Y_{\{M_i\}}(R)$.

So we want to show

$$\begin{tikzcd} M_1 \otimes R \arrow[r, "\rho \otimes \mathbf{Id}_R"] \arrow[d, "\phi"'] & M_2 \otimes R \arrow[d, "\psi"] \\ M_1 \otimes R \arrow[r, "\rho \otimes \mathbf{Id}_R"] & M_2 \otimes R \end{tikzcd}$$

commutes. Note that $R \hookrightarrow \bigoplus_i R_{f_i}$. Thus the commutativity of the above diagram follows immediately from the fact that $(\phi_i, \psi_i) \in Y_{\{M_i\}}(R_{f_i})$ for every i, i.e. that

$$\begin{tikzcd} M_1 \otimes R_{f_i} \arrow[r, "\rho \otimes \mathbf{Id}_{R_{f_i}}"] \arrow[d, "\phi_i"'] & M_2 \otimes R_{f_i} \arrow[d, "\psi_i"] \\ M_1 \otimes R_{f_i} \arrow[r, "\rho \otimes \mathbf{Id}_{R_{f_i}}"] & M_2 \otimes R_{f_i} \end{tikzcd}$$

commutes for every i. Thus $Y_{\{M_i\}}$ is a local functor.

Now we want to use Lemma A.1.5 to show that $Y_{\{M_i\}}$ is closed. In order to do this, we need to find an open covering of $GL(M_1) \times GL(M_2)$ for which the subfunctor conditions on $Y_{\{M_i\}}$ are easily shown to be closed conditions. First note that since both M_1 and M_2 are projective S-modules of rank n we have, by Theorem B.0.7, that there exist finite families of elements $(f_i)_{i\in I}$ and $(s_j)_{j\in J}$ such that

1. $\sum_{i\in I} f_i = 1$ and $(M_1)_{f_i}$ is a free S_{f_i}-module of rank n, and

2. $\sum_{j\in J} s_j = 1$ and $(M_2)_{s_j}$ is a free S_{s_j}-module of rank n.

Given this, we have that $(f_i s_j)_{i\in I, j\in J}$ is a finite family such that $\sum_{i\in I, j\in J} f_i s_j = 1$ and $(M_1)_{f_i s_j}$, $(M_2)_{f_i s_j}$ are free $S_{f_i s_j}$-modules of rank n. Now let

$$\begin{aligned} U_{i,j} :=\ & \mathrm{Spec}((S((M_1 \otimes M_1^*)^*)_{(det)} \otimes S((M_2 \otimes M_2^*)^*)_{(det)})_{(f_i s_j)}) \\ =\ & \mathrm{Spec}(S(((M_1)_{f_i s_j} \otimes_{S_{f_i s_j}} ((M_1)_{f_i s_j})^*)^*)_{(det)} \otimes_{S_{f_i s_j}} S(((M_2)_{f_i s_j} \otimes_{S_{f_i s_j}} ((M_2)_{f_i s_j})^*)^*)_{(det)}) \end{aligned}$$

for all $i \in I$, $j \in J$. Since $\sum_{i\in I, j\in J} f_i s_j = 1$, we have $\cup_{i\in I,j\in J} U_{i,j} = \mathrm{Spec}(S((M_1 \otimes M_1^*)^*)_{(det)} \otimes S((M_2 \otimes M_2^*)^*)_{(det)})$. Thus by defining

$$X_{i,j}(R) := \mathrm{Hom}_{\mathbf{S-alg}}((S((M_1 \otimes M_1^*)^*)_{(det)} \otimes S((M_2 \otimes M_2^*)^*)_{(det)})_{(f_is_j)}, R)$$

$$= \mathrm{Hom}_{\mathbf{S_{f_is_j}-alg}}((S((M_1)_{f_is_j} \otimes_{S_{f_is_j}} ((M_1)_{f_is_j})^*)^*)_{(det)} \otimes_{S_{f_is_j}} S(((M_2)_{f_is_j} \otimes_{S_{f_is_j}} ((M_2)_{f_is_j})^*)^*)_{(det)}, R)$$

for all $i \in I$, $j \in J$, we have that the $X_{i,j}$ form an open cover of $GL(M_1) \times GL(M_2)$. Note that as an $S_{f_is_j}$-module

$$(S(M_1 \otimes_{S_{f_is_j}} M_1^*)^*)_{(det)} \otimes_{S_{f_is_j}} S((M_2 \otimes_{S_{f_is_j}} M_2^*)^*)_{(det)})_{(f_is_j)} \simeq S_{f_is_j}[GL_n] \otimes_{S_{f_is_j}} S_{f_is_j}[GL_n]$$

where $S_{f_is_j}[GL_n]$ is the Hopf algebra of GL_n over $S_{f_is_j}$. So we have now reduced showing that $Y_{\{M_i\}}$ is closed to the case where the M_i are free for all i. In this case, it is clear that $Y_{\{M_i\}}(R)$ is closed in $GL_n(R) \times GL_n(R)$. To see this, note that saying $(\phi, \psi) \in GL_n(R) \times GL_n(R)$ is in $Y_{\{M_i\}}(R)$ means that

$$\begin{tikzcd} S^n \otimes R \arrow[r, "\rho \otimes \mathbf{Id}_R"] \arrow[d, "\phi"'] & S^n \otimes R \arrow[d, "\psi"] \\ S^n \otimes R \arrow[r, "\rho \otimes \mathbf{Id}_R"] & S^n \otimes R \end{tikzcd}$$

commutes. To see that this is a closed condition, note that any map from $S^n \otimes R \simeq R^n$ to $S^n \otimes R \simeq R^n$ is given by an $n \times n$ matrix with entries in R. Thus the fact that this diagram commutes is just a condition on the entries in the matrices given by ϕ and ψ. This diagram gives general conditions on a pair of matrices which shows that $Y_{\{M_i\}}$ is represented by $(S[GL_n] \otimes S[GL_n])/I$ for an ideal I given by the equations defined by the above diagram.

Thus we have that $Y_{\{M_i\}}$ is a closed subfunctor of $GL(M_1) \times GL(M_2)$ and is represented by an affine group scheme. This concludes the $r = 2$ case. In order to see how this implies the general result, note that the condition placed on $Y_{\{M_i\}}(R)$ as a subfunctor of $\prod_{i=1}^{r} GL(M_i)(R)$ is

$$\begin{tikzcd}
M_1 \otimes R \arrow[r, "\rho_1 \otimes \mathbf{Id}_R"] \arrow[d, "\phi_1"'] & M_2 \otimes R \arrow[r, "\rho_2 \otimes \mathbf{Id}_R"] \arrow[d, "\phi_2"'] & \dots \arrow[r, "\rho_r \otimes \mathbf{Id}_R"] & M_{r+1} \otimes R \arrow[d, "\phi_r"'] \\
M_1 \otimes R \arrow[r, "\rho_1 \otimes \mathbf{Id}_R"] & M_2 \otimes R \arrow[r, "\rho_2 \otimes \mathbf{Id}_R"] & \dots \arrow[r, "\rho_r \otimes \mathbf{Id}_R"] & M_{r+1} \otimes R
\end{tikzcd}$$

Choosing any two of the vertical maps gives a square subcondition which defines a closed subfunctor of $\prod_{i=1}^{r} GL(M_i)(R)$ as in the $r = 2$ case. Thus $Y_{\{M_i\}}$ can be realized as a finite intersection of closed subfunctors and hence is closed itself. Therefore we have shown that $Y_{\{M_i\}}$ is represented by an affine group scheme as desired.

$\square$

An immediate corollary of the above proof is the following.

Corollary A.3.3. *The functor $Y_{\{M_i\}}$ is of finite type over T.*

We are now ready to prove the main result of this appendix.

Theorem A.3.4. *Locally the chain $\{M_i\}$ is isomorphic to $\mathcal{L} \otimes S$. Moreover, the group scheme that represents $Y_{\{M_i\}}$ is smooth.*

Proof. We choose a $k \otimes S = S/\pi S$-linear section of the surjection $\bar{M}_i \to M_i/\rho(M_{i-1})$ and let $\bar{U}_i \subset \bar{M}_i$ be the image of this splitting. We can then lift $\bar{U}_i$ to a direct summand U_i of the S-module M_i such that $U_i \otimes k = \bar{U}_i$.

To see this, first note that since π is nilpotent in S, we have that πS is a nilpotent ideal. Now we can use the formal smoothness of the Grassmannian to lift

$\bar{U}_i$ to a direct summand U_i' of S^n, for some n, of the same rank as $\bar{U}_i$. We then have the following diagram

$$\begin{tikzcd} M_i \arrow[r, two heads] & \bar{M}_i \arrow[d, two heads] \\ U_i' \arrow[r, two heads] & \bar{U}_i \end{tikzcd}$$

Thus we can define a map from M_i to U_i' since M_i is projective. This map is surjective by Corollary B.0.13. Since U_i' is projective, we have a splitting map from U_i' to M_i and can define U_i to be the image of this splitting map. Thus we have that U_i is a direct summand of M_i such that $U_i \otimes k = \bar{U}_i$ as desired.

The U_i may be chosen to be periodic, i.e. such that for each i the morphism θ induces an isomorphism

$$\theta : U_{i-r} \longrightarrow U_i.$$

In order to see this, note that you can just define U_i by choosing a particular i, defining U_j as above for $j \in \{i,\, i+1,\, \ldots,\, i+(r-1)\}$, and then define the rest of the U_i as the images of these particular U_j using the isomorphism θ and the fact that $\rho\theta = \theta\rho$. The last fact is used to show that

$$\theta(M_i/\rho(M_{i-1})) = \theta(M_i)/\rho(\theta(M_{i-1})) = M_{i+r}/\rho(M_{i+r-1})$$

so that the translated U_i are defined as they are supposed to be. The map ρ induces an obvious map

$$\rho' : \bigoplus_{k=r-1}^{0} U_{i-k} \longrightarrow M_i$$

defined by

$$\rho'((u_{i-(r-1)},\, u_{i-(r-2)},\, \ldots,\, u_{i-1},\, u_i)) = (\rho^{r-1}(u_{i-(r-1)}),\, \rho^{r-2}(u_{i-(r-2)}),\, \ldots,\, \rho(u_{i-1}),\, u_i).$$

We claim that this map is surjective. Let

$$N_i := \rho'(\bigoplus_{k=r-1}^{0} U_{i-k}).$$

Thus, by Corollary B.0.13 of Nakayama's Lemma, we only need to verify this modulo $I := \pi S$. Let us denote by M'_{i-k} the image of M_{i-k} in $\bar{M}_i$ for a given i. We then obtain a flag by direct summands

$$0 \subset M'_{i-r+1} \subset \cdots \subset M'_{i-1} \subset \bar{M}_i.$$

using recursion and Lemma A.3.1. To see this, note that the lemma directly implies by choosing "$i = i-1$" and $k = r-1$ that,

$$\bar{M}_i = \rho(M_{i-1})/\rho^r(M_{i-r}) \oplus M_i/\rho(M_{i-1}) = \bar{\rho}(M'_{i-1}) \oplus M_i/\rho(M_{i-1}).$$

We then choose "$i = i-2$" and $k = r-2$ in the lemma to obtain

$$M_{i-1}/\rho^{r-1}(M_{i-1}) = \rho(M_{i-2})/\rho^{r-1}(M_{i-r}) \oplus M_{i-1}/\rho(M_{i-2}) = \bar{\rho}(M'_{i-2}) \oplus M_{i-1}/\rho(M_{i-2})$$

which gives

$$\bar{M}_i = \rho^2(M_{i-2})/\rho^r(M_{i-r}) \oplus \rho(M_{i-1})/\rho^2(M_{i-2}) \oplus M_i/\rho(M_{i-1}).$$

By continuing this process, we obtain

$$\begin{aligned}\bar{M}_i &= \rho^{r-1}(M_{i-(r-1)})/\rho^r(M_{i-r}) \oplus \rho^{r-2}(M_{i-(r-2)})/\rho^{r-1}(M_{i-(r-1)}) \oplus \dots \\ &\oplus \rho(M_{i-1})/\rho^2(M_{i-2}) \oplus M_i/\rho(M_{i-1}) \\ &= \bar{\rho}^{r-1}(\bar{U}_{r-1}) \oplus \bar{\rho}^{r-2}(\bar{U}_{r-2}) \oplus \cdots \oplus \bar{\rho}(\bar{U}_{i-1}) \oplus \bar{U}_i.\end{aligned}$$

Hence the images of $\bar{U}_{i-k}$ in $\bar{M}_i$ define a splitting of the flag. This shows that the above map is surjective mod πS and hence is surjective. Since the map is a surjection of projective modules of the same rank, it is an isomorphism.

In terms of the U_i, the map $M_i \to M_{i+1}$ looks as follows:

$$\bigoplus_{k=r-1}^{0} U_{i-k} \longrightarrow \bigoplus_{r-2}^{k=-1} U_{i-k}.$$

For $k \neq r-1$, this map is the identity on the summand U_{i-k}. On U_{i-r+1}, it induces the map

$$\pi\theta : U_{i-r+1} \longrightarrow U_i.$$

From this we see that any two chains of type $(\mathcal{L})$ are locally isomorphic, since locally the S-module U_i is free of the same rank as the k-vector space Λ_i/Λ_{i-1}.

Definition A.3.5. *The modules $\{U_i\}$ defined above are called a splitting of the chain $\{M_i\}$. They are characterized by the property that each U_i is a direct summand of M_i such that $\bar{U}_i$ maps isomorphically to $M_i/\rho(M_{i-1})$, and the U_i are periodic with respect to θ.*

We will now show that our functor $Y_{\{M_i\}}$ is formally smooth. Let $I \subset S$ be a nilpotent ideal. Let $\{V_i\}$ be a splitting for $\{M_i \otimes_S S/I\}$.

Recall that the Grassmannian is formally smooth, so we can lift $\{V_i\}$ to direct summands $\{U_i\}$ of $\{M_i\}$ that have the same rank. We have a map from U_i to $M_i/\rho(M_{i-1})$ defined by $U_i \hookrightarrow M_i \twoheadrightarrow M_i/\rho(M_{i-1})$. We want to show that the map from $\bar{U}_i$ to $M_i/\rho(M_{i-1})$ is an isomorphism. We will first show that it is surjective. By Corollary B.0.13, it is enough to check that the map is surjective mod I. Well, we know it is true mod I since the map just descends to the isomorphism from $\bar{V}_i$ to $(M_i \otimes S/I)/(\rho(M_{i-1}) \otimes S/I)$. Now we have a surjection between projective modules of the same rank, so it is an isomorphism. It is clear that any such $\{U_i\}$ will be periodic by Remark A.1.10. Thus any set of liftings of the V_i to direct summands U_i of M_i is a splitting for $\{M_i\}$.

The formal smoothness of the $Y_{\{M_i\}}$ is a consequence. Let α be an automorphism of $\{M_i\} \otimes_S S/I$. We find liftings $\{U_i\}$ and $\{U_i'\}$ of the splittings $\{V_i\}$ and $\{\alpha(V_i)\}$ respectively. Then the isomorphism $V_i \to \alpha(V_i)$ lifts to an isomorphism $U_i \to U_i'$ which is periodic again by Remark A.1.10. Thus we obtain a lifting of α to an automorphism of $\{M_i\}$ by patching together the automorphisms of $\{U_i\}$ and recalling that

$$M_i = \rho'(\bigoplus_{k=r-1}^{0} U_{i-k}).$$

This completes the proof of the theorem. □

The fact that $X_{\tilde{\mathfrak{A}}}$ is smooth now follows from the above theorem by a result in [Pap]. In Theorem 2.2 of [Pap], Pappas argues that it is enough to check smoothness in this context at points of the special fiber. The above theorem handles the smoothness of the special fiber, so we see that $X_{\tilde{\mathfrak{A}}}$ is smooth as desired.

Appendix B

Background Results

Let S be a commutative ring with 1.

Definition B.0.6. *An S-module M is* <u>*locally free*</u> *if $M_{\mathfrak{p}}$ is a free $S_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p} \subset S$. We denote the rank of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$ by $r_{\mathfrak{p}}$ and call it the* <u>*local rank*</u> *of M at $\mathfrak{p}$. Moreover, we say an S-module M has* <u>*rank r*</u> *if for every $\mathfrak{p} \subset S$, $r_{\mathfrak{p}} = r$.*

Theorem B.0.7. *([B-CA] II, § 5.2 thm. 1) Let M be an S-module. The following properties are equivalent:*

1. *M is a finitely generated projective module.*
2. *M is a finitely presented module and, for every maximal ideal $\mathfrak{m}$ of S, $M_{\mathfrak{m}}$ is a free $S_{\mathfrak{m}}$-module.*
3. *M is a finitely generated module which is locally free and the function $\mathfrak{p} \to r_{\mathfrak{p}}$ is locally constant on* $\mathrm{Spec}(S)$*.*
4. *There exists a family $(f_i)_{i \in \{1, \ldots, n\}}$ of elements of S, which generate the ideal S, such that for every i, the S_{f_i}-module M_{f_i} is free of finite rank.*

Proposition B.0.8. *([B-CA] II, § 3.2 cor. 2) If M is a finitely generated projective module over a local ring $(S, \mathfrak{m})$ then M is a free S-module.*

Remark B.0.9. *The above proposition is not stated in exactly this way in Bourbaki, but it is an easy consequence of the given reference when you consider theorem B.0.7.*

Theorem B.0.10. *([Wat] 11.4, due to Cartier) Hopf algebras over fields of characteristic zero are reduced.*

Lemma B.0.11. *([B-CA] II, § 3.3 cor. 5) Let $V = S^n$ and let $v_1, \ldots, v_n$ generate V over S. Then $v_1, \ldots, v_n$ form a basis for V over S.*

Proposition B.0.12 (Nakayama's Lemma)**.** *([AM] prop. 2.6) Let M be a finitely generated S-module and let $I \subset \mathrm{rad}_{\mathfrak{m}}(S)$. If $IM = M$ then $M = 0$.*

Corollary B.0.13. *([AM] cor. 2.7) Let M and I be as above and let $N \subset M$ be a submodule. If $M = IM + N$ then $M = N$.*

Proposition B.0.14. *([AM] prop. 3.9) Let $\phi : M \to N$ be an S-module homomorphism. Then the following are equivalent:*

1. *ϕ is injective (surjective)*

2. *$\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (surjective) for every prime ideal $\mathfrak{p}$ of S.*

3. *$\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (surjective) for every maximal ideal $\mathfrak{m}$ of S.*

For any two S-functors X and X', let $\mathrm{Mor}(X, X')$ denote the set of all natural transformations from X to X'.

Lemma B.0.15. *(Yoneda's Lemma, [Jan] I 1.3) For any S-algebra R and any S-functor X, the map $f \to f(R)(\mathrm{Id}_R)$ is a bijection*

$$\mathrm{Mor}(\mathrm{Hom}_{S-\mathbf{alg}}(R, \_\_), X) \xrightarrow{\cong} X(R).$$

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