ABSTRACT

Title of dissertation: RADIATION REACTION AND SELF-FORCE IN CURVED SPACETIME IN A FIELD THEORY APPROACH

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Doctor of Philosophy, 2007

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This dissertation, in three parts, presents self-consistent descriptions for the motion of relativistic particles and compact objects in an arbitrary curved spacetime from a field theory approach and depicts the quantum and stochastic (part I), semiclassical (parts I and II), and completely classical regimes (part III).

In the semiclassical limit of an open quantum system description, the particle acquires a stochastic component in its dynamics. The interrelated roles of noise, decoherence, fluctuations and dissipation are prominently manifested from a stochastic field theory viewpoint and highlighted with our derivations of Langevin equations for the particle in curved space, which are useful for studying influences imparted by a stochastic source. We also derive non-local and history-dependent equations for radiation reaction and self-force in a curved spacetime when the stochastic sources are negligible.

When the scales of the mass and the field are very different, as for an astrophysical mass or compact object, the stochastic features of the system are strongly

suppressed and the stochastic description yields a (semiclassical) effective field theory. The appropriate expansion parameter μ is the ratio formed by the size of the compact object and the background curvature scale. Within an effective field theory framework we derive the second order self-force and the leading order contributions to the equations of motion from spin-orbit and spin-spin interactions on a compact object. The finite size of the compact body affects its motion at $O(\mu^4)$ and the self-force at $O(\mu^5)$. These results are useful for constructing more accurate templates that the space-based interferometer LISA will need for parameter estimation.

Within a purely classical setting we introduce a new framework that describes fully relativistic gravitating binary systems, possibly with comparable masses, and allows for the background geometry to dynamically respond with the motions and influences of the compact objects and gravitational waves. The approach self-consistently incorporates mutual action and backreaction on every component in the total system. We derive the equations of motion and identify the parameter regimes where this new approach is applicable with the aim of establishing a common framework applicable to the detection ranges of both LIGO and LISA interferometers.

Radiation Reaction and Self-Force in Curved Spacetime in a Field Theory Approach

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2007

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Dedication

To Karrie,

for her unwavering support, patience,

strength and encouragement through

this long and taxing

journey...

...and to Maynard, the dog.

Acknowledgments

There are many people to whom I owe much gratitude for help in making this thesis possible.

It has been a great honor and experience to work with my advisor, Bei-Lok Hu, over the past nearly seven years. His guidance throughout my graduate career has provided an invaluable source of knowledge and wisdom that will be put to good use for years to come. His ability to cut through to the heart of the matter and to see clearly, without distraction, the underlying important issues, concepts and problems is one that I have always felt strongly impressed by and hope to one day imitate successfully to the same degree of quality.

My best friend and wife, Karrie, has been more patient, supportive and loving than a person should have to be. I would not have been able to finish this work without her constant encouragement and faith in me. I cannot thank her enough for putting up with all of the long hours and late nights that I spent writing this dissertation.

Getting through graduate school would have been much more difficult without great friends including Willie Merrell and Matt Reames, who were with me in the trenches during the first two years of grad school, and Mike Ricci, Nick Cummings, Patrick Hughes, Chad Mitchell, Tanja Horn, Sanjiv Shresta, Dave Mattingly, Jesse Stone and Jennifer Hall.

I thank Professors Rick Greene, for giving me the opportunity to work in his lab, Betsy Beise, who gave me a lot of practical and hands-on knowledge in experimental physics, and Xiangdong Ji, who spent the time teaching me quantum field theory while I was an undergraduate. During my experiences with them I have met some good friends including Tanja Horn, Kenneth Gustaffson, Damon Spayde and Silviu Corvu as well as friendly researchers including, Vera Smolyaninova, Patrick Fournier, Amlan Biswas, Josh Higgins, and Jonathan Osborne.

I also want to thank Luis Orozco, Andris Skuja, Bob Anderson and Bob Dorfman for time spent outside of the classroom answering my questions and discussing with me about physics.

There are many people whom I have met at Maryland who want nothing to do with physics including Lorraine DeSalvo, David Watson, Reka Montfort, Rachel Sprecher, Dusty Aeiker, Nick Hammer, Carole Cuaresma and the one who enticed me with her charm, good looks and a bowl of candy on her desk, Karrie.

I am deeply indebted to Jane Hessing for all of her work and help over the years with submitting forms, dealing with deadlines and retro-actively handling many things due to my forgetfulness. I also thank Linda Ohara for many free lunches at the Dairy in exchange for talking to perspective graduate students.

Lastly, but certainly not least, I thank my family for helping me to this point in my life and career, for putting up with the many nights working while I lived at their house, and for their constant encouragement and support.

I apologize for those people whom I haven't included on these two pages or whom I have simply forgotten. I am indebted and grateful to these many people and cannot begin to repay them for their time, efforts and friendships.

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Chapter 1

Introduction and overview

The operation of a network of ground-based gravitational wave interferometers (LIGO [1], VIRGO, GEO600, etc.) and the proposal of a space-based interferometer LISA (Laser Interferometer Space Antenna) [2] to probe the properties and interactions of strongly gravitating systems has generated a growing theoretical interest in the gravitational two-body problem. Due to the complexity of the problem there are two limits that admit (quasi-)analytical approximation techniques. The first, appropriate for the kinds of binary systems that LIGO is expected to observe, uses the post-Newtonian (PN) formalism, which assumes that the two bodies, possibly spinning, are weakly gravitating sources moving at slow velocities under their mutual gravitational influences. Recently, the equations of motion for the two bodies and the radiation these emit have been computed using the PN formalism to $O(v^6)$ or 3PN order (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein).

The second limit of interest is the case where one of the bodies is considerably more massive than the other as occurs when a small black hole or neutron star orbits a supermassive black hole. In this context, the small compact object can be approximated reasonably well by a point particle. The motion of the particle perturbs the background metric (i.e., the metric of the supermassive black hole in isolation) which generates metric perturbations that cross the event horizon of the large hole and propagate far away to a detector. These perturbations also react on the particle causing it to slowly spiral in toward the large black hole.

The back-reaction of the emitted radiation on the particle results from two possible types of interactions with the gravitational wave. The first is a reactive force describing the recoil on the particle as it emits the radiation. In particular, this interaction is purely local. The second results from the interaction of the particle with previously emitted radiation that back-scatters off of the background curvature and later interacts with the particle at a different time and position. This is an intrinsically non-local process. The effects of both kinds of interactions with the emitted metric perturbations manifest on the particle as *self-force* and is responsible for the slow in-spiral to the supermassive black hole. The equations of motion for the particle moving on a general vacuum background spacetime were derived within the last ten years by Mino, Sasaki and Tanaka [20] and Quinn and Wald [21]. We refer to this equation throughout the remainder as the MSTQW equation.

This work is divided roughly into three parts. In the first part we derive the equations of motion for a small "particle" (e.g., an atom, a molecule, a piece of dust, etc.) moving through an arbitrary curved background. In particular we consider the motion of a scalar and electric point charge as well as a small mass, each separately interacting with their respective scalar, electromagnetic and gravitational fields. We describe the motion of the particle using a quantum mechanical worldline while the field is taken to be linear and quantized. This first principles approach allows for the particle to be described as an open quantum system upon integrating out (a form of coarse-graining) the quantum field. If the worldline can be sufficiently decohered then the particle will evolve dynamically within a semiclassical limit. In this regime we recover the well-known radiation reaction equations of Abraham, Lorenz and Dirac for the scalar and electric point charges but generalized to motions in a curved spacetime [22, 23]. For the gravitational case we recover the MSTQW self-force equations, which are devoid of any manifestly local radiation reaction forces [20, 21].

Despite the strong degree of decoherence, the ongoing particle-field interactions allow for the coarse-grained quantum field fluctuations to manifest as noise via the appearance of a classical stochastic forcing term in the particle equations of motion. The particle equations of motion are now extended to the form of a Langevin equation, which can depict dissipative dynamics and accommodate stochastic sources. This suggests that observables involving the worldline coordinates must be calculated using stochastic correlation functions to average over these fluctuations. The correlations of the noise provide information about the state of the quantum field, which is particularly important if the state of the quantum environment is unknown [24]. The noise is also intimately related to the decoherence of the particle worldlines that defines this stochastic semiclassical limit in the first place.

This Langevin equation can also be used for stochastic sources, of classical origin, introduced phenomenologically to model an environment. We show that such noise can cause the particle to undergo a stochastic-averaged secular motion in a manner similar to the velocity drifts encountered by a charged particle moving through an inhomogeneous external electromagnetic field [25].

In the second part of this work, we introduce an effective field theory (EFT) approach for studying the extreme mass ratio inspirals (EMRI), which are expected to be detected with the LISA gravitational wave interferometer [2]. The EFT approach replaces the compact object with effective point particle degrees of freedom. This effective particle is constructed to be sufficiently robust to capture all finite size effects that result from tidally induced moments, spin and intrinsic multipole moments describing the perturbations of the compact object away from its equilibrium configuration in isolation. This is the first of two effective theories.

In the second EFT we couple this effective particle to the quantized metric perturbations off a given background spacetime, which we simply call gravitons throughout. Integrating out the gravitons yields an effective action given perturbatively in powers of μ, which is defined as the ratio of the size of the compact object to the background curvature length scale. At each order in μ we can assemble Feynman diagrams describing the relevant interactions and terms that must be calculated to construct the full self-force at that order. In fact, there is in principle no obstacle to compute the self-force to any order desired.

The EFT comes with two powerful advantages. On the one hand, even though the dynamics of the short and long distance scales are cleanly separated we nevertheless are able to deduce the role of finite size effects, how they influence the motion of the effective particle (i.e., the compact object) and at what order in μ this occurs. On the other hand, being a legitimate quantum field theory, there is a plethora of well-established methods for regularizing the divergences that ultimately appear in a theory of point particles and fields. In this regard, using a mixture of distributional methods and dimensional regularization we are able to render the theory finite in an efficient and well-defined manner. Furthermore, our choice of regularization scheme implies that only logarithmic divergences have observable consequences, which implies the existence of classical renormalization group scaling for the parameters of the effective particle couplings describing the induced and intrinsic moments of the compact object [26].

Spin is easily accomodated within our formalism as it represents just another set of operators on the worldline of the effective particle. As such, we are able to determine the leading order spin-orbit and spin-spin contributions to the self-force for both a maximally rotating compact object as well as a co-rotating body.

The third part of this work introduces a new approach for studying gravitationally bound systems. For concreteness we consider two bodies. The first is a compact object (a neutron star or a black hole) with mass m and the second is a black hole with mass M. We assume that the first mass is smaller than the second $m < M$ but not necessarily much smaller. We introduce a formalism in which the smaller body (described as an effective point particle as in the EFT approach), the metric perturbations and the background black hole metric evolve self-consistently with each other. Because of this self-consistency all three variables affect each other through their mutual backreaction and may provide a way to apply the methods used in studying the EMRI scenario to a post-Newtonian system, namely, to a binary system with comparable masses. Since this formalism does not *a priori* rely on a slow motion approximation (even though this may be necessary in practical calculations) nor a flat background then our approach may also be useful for studying intermediate mass ratio inspirals (IMRI) using numerical techniques[1]. While this approach is still developing we present the basic philosophy and equations of motion, at least formally.

We turn now to a more thorough overview of each of these three parts.

1.1 Stochastic field theory of a particle and quantum fields in curved space

There are many approaches for deriving the self-force on a (possibly charged) massive point particle. The first derivation of the electromagnetic self-force is given by DeWitt and Brehme [23, 27] who use the conservation of stress-energy, both of the field and the charge, across a worldtube placed around the particle worldline to derive the self-force on the charge. See [28] for a comparison and criticism of several other derivations of self-forces given in the literature.

Most approaches study self-force on a classical particle due to a classical field, with perhaps the notable exception of [20] who do not use a point particle treatment. However, it is believed that all known classical fields, including the electromagnetic field and the gravitational field (in particular, the metric perturbations about a background space) possess a fundamentally quantum nature. The most famous example of this is provided by the resoundingly successful theory of quantum electrodynamics describing the quantized electromagnetic field interacting with electrons (and positrons). If an elementary particle, an atom, a molecule, a piece of dust, etc.,

[1] We thank Alessandra Buonanno for pointing this out to us.

which we collectively refer to as a "particle" despite the appearance of a small finite size, is interacting with a fundamentally quantum field then the question arises as to the circumstances under which the intrinsic fluctuations of the quantum field affect the motion of the particle in the spacetime. One may also wonder how the quantum field fluctuations manifest themselves to the particle.

Such questions are best answered with an approach that starts from first principles by treating the field and the particle as quantum objects. Specifically, in a first principles approach the field is described using the theory of quantum fields and allows for the occurrence of nontrivial quantum field processes that can affect the motion of the particle. The quantum mechanical particle, on the other hand, is treated as following a worldline that is free to move with relativistic speeds. By describing the particle quantum mechanically we must ignore those worldlines in which the particle number is not constant at any point in the particle's history [29]. This is a physically reasonable requirement given that the energies involved for the vacuum to spontaneously create an atom or a piece of dust, say, is very high by most standards. Furthermore, the relativistic interactions of such "particles" may cause a transformation to other objects, such as in the electron-positon annihilation reaction $e^- + e^+ \to \gamma$, but only at energies and momenta of order the particle's rest mass. Therefore, so long as one is interested in the motion of a well-defined and localized particle at an energy scale below its rest mass then a quantum mechanical description of the particle worldline should suffice.

There are two advantages to using an approach that begins from first principles. First of all, since quantum theory is the fundamental framework by which a system can be studied, a first principles approach begins at the most fundamental level. Hence, all known physical particle and field interactions can be captured within the framework and may contribute to the overall dynamics of the particle-field system. Second, if the particle admits a semiclassical limit then a first principles approach ought to be able to not only produce that limit but also give the conditions under which the semiclassical limit is well-defined. This gives information about the viability of using a fully classical description of particles and fields versus the semiclassical particle limit of a description that is derived self-consistently from a quantum-based treatment.

In Chapters 2 and 3 we implement a first principles approach using the influence functional of Feynman and Vernon [30] to describe the evolution and interactions between a quantum mechanical particle worldline $z^\alpha(\lambda)$ and a massless, linear quantum field Φ. In particular, we study the particle-field interactions within the open quantum system paradigm in which one subsystem, the quantum field, acts as a large environment that couples to another subsystem, the particle degrees of freedom, that is relatively small and easily influenced by interactions with the environment. As we are interested in the dynamics of the particle and are not necessarily concerned with computing field observables here, we may integrate out, or coarse-grain, the quantum field so that we are left with full information about the particle worldline only. Coarse-graining provides a way to self-consistently evolve the particle with the field so that all quantum processes of both the particle and the field are accounted for.

Dissipation in an open quantum system depends crucially on how one introduces coarse-graining into the total particle-field system. For example, if we coarse-grain the modes of the quantum field (in flat spacetime) with energies higher than the Planck mass $k > m_{pl}$, say, then the system of interest consists of the particle degrees of freedom and those modes of the field for which $k < m_{pl}$. In this example, the system will not manifest dissipation. However, by coarse-graining all of the field degrees of freedom, as we are doing in this work, our system will consist solely of the particle variables. For this coarse-graining, the system may manifest dissipation through processes relating to, for example, radiation reaction and self-force. The Poincare recurrence time, which is the time it takes for energy initially lost by the system to be returned, is practically infinite when the environment contains $\gtrsim 20$ degrees of freedom [31]. The field possesses an infinite number of degrees of freedom implying that the energy dissipated by the system will be redistributed to the environment variables and never (fully) return to the system. As we will elucidate shortly, the appearance of dissipation is also intimately connected with noise and decoherence.

The open quantum system paradigm naturally allows for a statistical interpretation for the particle's motion. Near the semiclassical limit, where the concept of a particle is sufficiently well-defined from a field theory perspective, the fluctuations of the coarse-grained quantum field manifest as noise in the form of a classical stochastic force on the particle. This stochastic force, in turn, induces fluctuations about the average worldline, which is the semiclassical trajectory, so that the particle acquires a stochastic component to its dynamics. This new result for a particle in a curved spacetime is provided in Chapter 3 and extends previous work done in flat spacetime [29, 32, 33]. Provided that these induced fluctuations are small the particle remains approximately within the semiclassical limit. Nevertheless, it is more accurate to refer to this regime of the particle's evolution as the stochastic semiclassical limit.

This feature is reminiscent of quantum Brownian motion in which a massive oscillator is coupled to many oscillators having much smaller masses. Upon coarse-graining the small oscillators one finds that the large oscillator undergoes a stochastic evolution due to its interactions with the small quantum fluctuations of the coarse-grained oscillators [34, 35, 36, 37, 38].

We also demonstrate that the noise (i.e., the classical stochastic force) is intimately related to the fluctuations of the coarse-grained quantum field. As such, the stochastic correlations of the noise provide information about the state of the fluctuating quantum field. Interestingly, using more sophisticated formalisms than what is presented in this work, one can develop a BBGKY hierarchy of stochastic correlation functions that relate to certain quantum correlation functions of the quantum field [39, 40, 24]. In this way, one can probe the quantum information of the environment by measuring the stochastic correlations of system variables and observables. For example, the dispersion of a small mass moving in flat spacetime is given by the stochastic correlation function $\langle \tilde{z}(\tau)\tilde{z}(\tau)\rangle_{stoch}$, which is related to the quantum two-point function of the graviton $\langle \hat{h}_{\alpha\beta}(\bar{z}(\tau))\hat{h}_{\gamma\delta}(\bar{z}(\tau))\rangle_{qm}$ evaluated along the average worldline $\bar{z}(\tau)$

Using the open quantum system paradigm and the influence functional approach we demonstrate the relationship between the decoherence of the quantum particle variables and the fluctuations of the quantum field. The fluctuations of the quantum field essentially generate the stochastic classical force on the particle that acts as a source of noise for the worldline coordinates. Decoherence is related to the suppression of off-diagonal elements of the (reduced) density matrix for the particle interacting with the coarse-grained quantum field. This, in turn, is related to the magnitude of the influence functional F, which for an electric point charge e coupled to the electromagnetic field in a curved space, for example, is

$$\begin{aligned} |F[z,z']| &= \exp\Big\{ -\frac{e^2}{4\hbar}\int d\tau d\tau' \,\big[u^\alpha(\tau)-u'^\alpha(\tau)\big] D^H_{\alpha\beta}\big(z^\alpha(\tau), z'^\alpha(\tau')\big) \\ &\qquad \times \big[u^\alpha(\tau')-u'^\alpha(\tau')\big]\Big\} \end{aligned} \tag{1.1}$$

where z and z' represent two fine-grained worldline histories. The quantity $D^H_{\alpha\beta}$ is the symmetric quantum two-point function of the electromagnetic field evaluated in the initial state of the field. This quantity describes the fluctuations of the quantum field. Qualitatively speaking, if D^H is large in magnitude then the histories have to be chosen so that $z' \approx z$. In this way the velocity difference of the two histories may be small enough to guarantee that the magnitude of the influence functional, and hence the reduced density matrix for the particle, is $O(1)$ and not small. Therefore, the decoherent worldline histories are the dominant ones, but their identification depends upon the quantum field fluctuations, given by $D^H_{\alpha\beta}(x,x')$.

The worldline influence functional formalism, which is used throughout Chapters 2 and 3, allows for a self-consistent description of the interplay between dissipation, noise, decoherence and correlations. As such, when there exists a stochastic semiclassical limit for the particle one may ask under what conditions is such a limit valid and when do higher order quantum effects, from loop corrections say, become important. We address some of these issues and deduce that a stochastic semiclassical limit is well-defined if there is a minimal amount of additional coarse-graining for the worldline fine-grained histories (via smearing over the scale of the particle's Compton wavelength) and only the leading order fluctuations are taken into account. By allowing for nonlinear stochastic corrections from higher order terms in the fluctuation coordinate it seems that a stochastic semiclassical limit is no longer well-defined.

The noise on the particle dynamics is derived from the quantum field fluctuations using the influence functional formalism and depends on how we introduce the coarse-graining, the particle-field coupling, etc. For this reason, stochastic correlation functions of worldline quantities contain information about the quantum state of the field and its correlations. Nevertheless, it is often the case in reality that the source of noise is simply stipulated and put into the particle equations of motion by hand. This added noise, or stochastic force, could have a classical origin, (e.g. high temperature thermal fluctuations of a bath) or it could have no single identifiable origin. Furthermore, one needs to stipulate a two-point correlation function for the stochastic force, called the noise kernel, in order to compute stochastic correlation functions for particle observables that depend on the stochastic worldline. Therefore, if the environment is quantum and this noise is simply added then there is no guarantee for self-consistent backreaction, no guarantee for the existence of a fluctuation-dissipation relation, nor an ability to extract information about the actual state of the environment from the assumed noise kernel.

With these issues in mind we nevertheless introduce a source of noise by adding a classical stochastic force to the classical particle equations of motion. We find that expanding to second order in the coordinate fluctuations about some background trajectory and performing a stochastic average of the resulting expansion implies that the motion of the particle undergoes a secular drifting motion away from the classical trajectory. This effect is particularly pronounced in the presence of a non-homogeneous external field coupled to the particle. This drifting is encountered frequently in plasma physics where the time-averaged Larmor motion results in a net velocity drift if the charge is moving in an inhomogeneous external magnetic field [25, 41].

While we focus mostly in Chapters 2 and 3 on the semiclassical and stochastic semiclassical limits, respectively, for the particle's motion there is, in principle, no obstacle to considering the leading order quantum loop corrections. This issue has been raised in [42, 43] where they compute the contributions to geodesics from one-loop quantum graviton corrections. However, their approach is not completely self-consistent since there is no radiation reaction or self-force taken into account. Since these effects occur classically then it is likely that they will dominate quantum corrections for most considerations. In our approach we can nevertheless obtain the one-loop quantum field corrections to the semiclassical equations of motion, which do incorporate classical radiative effects.

Another advantage of our open quantum system approach is that we can also incorporate the effects coming from the finite extent of the "particle" in a self-consistent manner using effective field theory techniques to replace the small, but extended, body by an effective point particle. This effective particle contains worldline operators that account for moments that are induced by external forces on the particle. While such effects are necessarily small they can nevertheless be accounted for in a systematic and self-consistent manner within the influence functional formalism.

In Chapter 3 we compute the flux of gravitational waves passing through an ideal interferometer, say, from a particle undergoing stochastic fluctuations far away from the detector. Interestingly, we find that the interferometer measures the quantum fluctuations of the metric perturbations, but only locally. That is, the detector does not measure any information about stochastic sources that are far away, namely from the fluctuating particle. Rather, we show that only fluctuations in the local gravitational field are measured within our level of approximation; higher order quantum corrections are likely to contain information about the source of gravitational waves. Regardless, the ability to measure even the quantum fluctuations of the local gravitational field is non-existent with current gravitational wave interferometers and will probably continue to be so for the next generation of detectors, at least.

The stochastic field theory approach developed in Chapters 2 and 3 is rich with physical concepts that span quantum, statistical and classical domains and is a powerful tool for studying the effects of noise, dissipation, fluctuations and decoherence of a quantum system.

1.2 Effective field theory approach for the motion of a compact object in a curved space

The quantization of general relativity is a widely famous problem that has proved troublesome because of its status as a non-renormalizable quantum field theory. Namely, the theory breaks down when energies near the Planck scale and higher are probed. The inability to renormalize the theory at each order in perturbation theory does not spell the end for the quantization of gravity, however. All experiments to date measure processes and interactions occurring with energies far below the Planck scale. These experimental energies set the scale at which one should make predictions with a theory. Therefore, quantizing general relativity, in particular, can be done in a self-consistent manner provided that the energies being probed and the predictions being made are *below* the Planck scale.

A framework that allows for determining the small influences that quantum gravitational corrections have on the leading order classical processes is provided by effective field theory[2]. In Chapters 4 and 5 we treat the naively non-renormalizable quantum field theory of metric perturbations on a given background as an effective field theory to describe the motion of a compact object in an arbitrary curved background spacetime. We have in mind that the background is provided by a supermassive Kerr black hole as we wish to apply this formalism to the case of extreme mass ratio inspirals.

In the scenario we consider here, the compact object is much smaller than the

[2]See [44, 45, 46, 47, 48] for excellent introductions to the subject.

length scale of the background curvature. As such, the small body can be described as if it was a point particle moving through the spacetime, thereby ignoring any effects the size of the body might have on its own motion. In fact, this approach is taken in almost all derivations of the scalar, electromagnetic and gravitational self-forces [49, 50, 22, 23, 27, 20, 51, 52, 53]. Correspondingly, most of these approaches have calculated the leading order self-force in an expansion of the particle's charge or mass.

We utilize the EFT framework to go beyond the familiar leading order MSTQW self-force by calculating the self-force to any desired order in μ. Our goal in computing higher order contributions to the self-force is three-fold. First, the self-force equation through first order is believed to be suitable for the *detection* of gravitational waves with the LISA interferometer [54, 55, 56, 57, 58]. However, for parameter estimation one needs to calculate the second order contributions so that the generated templates will describe the detected gravitational waveforms with suitable precision, which is less than about a quarter of a cycle [59, 60, 61]. Using a two-time expansion, also referred to as an adiabatic approximation, to describe the slow (secular) inspiral of the compact object, the authors of [56, 62] observe that the time-averaged part of the second order self-force is necessary for constructing the LISA measurement templates and for extracting source parameters (mass, spin, etc.) with the claimed fractional accuracy of $\sim 10^{-4}$ [54]. Second, calculating the second order self-force provides concrete estimates for the error in using the first order self-force alone. Likewise, calculating the third order self-force provides concrete estimates for the error in using the self-force through second order alone, etc. Third, we wish to obtain the self-force equations *and* the configurations of the metric perturbations[3] at a sufficiently high order in μ that we can begin to overlap and compare with post-Newtonian results, upon expanding our results in the relative velocity of the binary. Since there is in principle no obstacle to computing higher order self-force then we feel that this should be an attainable goal, at least for certain values of mass, relative velocity and orbital separation.

We briefly describe the effective field theory approach here. We recognize two scales in the scenario of EMRI's. The first is the size of the compact object and the second is the curvature length scale of the background geometry. The ratio of these two largely dissimilar lengths forms a quantity μ that we will use as an expansion parameter for the perturbation theory as well as a parameter that indicates the scaling of each kind of particle-field interaction. The EFT approach begins by "integrating out" the "small" distance features of the system, which occur at the scale of the compact object. In practice this is done by replacing the compact object by an effective point particle description. As such, the effective particle contains many worldline operators (called non-minimal couplings) that account for the effects of induced moments, spin and intrinsic multipole moments of the compact object. The coefficients of these non-minimal couplings can be determined through a matching procedure in which a preferably gauge or coordinate invariant observable is calculated in the effective theory and matched to the long wavelength limit of

[3] These are essentially the graviton one-point functions. We will not discuss how to calculate the emitted radiation or its power/flux in this work. However, see Chapter 6 for some ideas on doing so within our EFT framework.

the corresponding observable in the full theory. Here the full theory describes the compact object interacting with external influences, e.g. as in Compton graviton scattering where the compact object is subject to interactions with the incoming gravitational waves, but otherwise isolated from the supermassive black hole.

The next step is to couple the effective particle to quantized metric perturbations (gravitons) off the background spacetime. By integrating out the gravitons and considering only classical interactions between the particle and the field, as well as graviton self-field interactions, we obtain the effective action that generates the equations of motion for the particle, i.e. the compact object. The effective action can be expressed in the language of Feynman diagrams, which is an indispensable tool in effective field theories, and the self-force is simply read off from the resulting equations of motion.

The effective field theory approach possesses a unique advantage in that the behavior of small scale perturbations, such as tidal deformations of the compact object, are separated from yet consistently incorporated systematically into the dynamics of the long wavelength, or large distance, sector of the theory where the compact object is treated as an effective particle that couples to (radiating) metric perturbations. We will demonstrate this systematic inclusion of finite size effects, which is the first time this has been done within the EMRI scenario. We also determine when finite size effects from tidally induced moments first affect the behavior of the particle's motion. This allows us to state and prove for the first time and Effacement Principle for EMRIs. While such corrections are known to be small the techniques of our effective field theory approach allow us to determine how small these are and if these are somehow enhanced.

Replacing the dynamics of the compact object by an effective point particle description comes with certain important consequences. Perhaps the most well-known of these is the appearance of divergences that stem from the inclusion of arbitrarily high frequency modes in the quantum field theory that interact with a point-like object. The divergent part of a Feynman diagram, which appears in the effective action for the particle dynamics and involves the free-field propagator, is a quasi-local contribution. In a curved spacetime the finite remainder is non-local in time (i.e., history-dependent) and must be isolated from the local divergent part; using techniques from distribution theory we are able to do so. This leads us to evaluate the divergent part of the diagram and so we need to choose a particular regularization scheme. With the EFT being a quantum theory there is a vast array of methods and techniques that regularize these ultraviolet divergences. Of these, dimensional regularization [63] naturally fits within the effective field theory paradigm. As we discuss in Chapter 4 the use of dimensional regularization (or for that matter any so-called mass-independent regularization scheme [46]) provides an efficient means for not only regularizing the singular integrals in the effective action but also for determining which Feynman graphs are important at a particular order in μ.

While we use dimensional regularization to render the theory finite we need a representation of the divergent propagator $D(x, x')$ to do so. We use the momentum space representation originally developed by Bunch and Parker for a scalar field in curved spacetime [64]. At a particular point, x', the authors associate a tangent space and solve the field equations for the Green's function iteratively in powers of the distance from the origin as expressed in Riemann normal coordinates. Unfortunately, this approach is rather cumbersome for higher spin fields.

We give a novel derivation of Bunch and Parker's result in Appendix D using a diagrammatic approach familiar from perturbative quantum field theory. We demonstrate that the use of Feynman diagrams to calculate the terms in the momentum space representation of the propagator is more efficient than that given in [64] and leads to a particularly useful identity that eliminates some of the diagrams that naively appear. This is particularly useful when considering higher spin tensor fields including the graviton propagator on a background. Despite the increased efficiency the calculations are somewhat involved. Nevertheless, we compute the leading order contribution to the quasi-local structure of the graviton propagator, which arises from the non-trivial curvature of the background spacetime[4]. To our knowledge, this has not been given in the literature using momentum space techniques.

Since we know the relevant momentum space divergent structure of a scalar field in a curved background we introduce a nonlinear scalar field model. This toy theory is related to general relativity and can be used to calculate the second order self-force on a non-spinning particle. We find that because our EFT formalism is based on the Closed-Time-Path (CTP), or in-in, formalism that the self-force is manifestly causal. This is to be contrasted with the in-out formalism used in [26], which is more suitable for describing scattering processes than initial value problems.

[4]Other regularization methods including adiabatic regularization [65], point-splitting regularization [66], Hadamard's ansatz [67], spacetime dimensional regularization [68], etc. have been developed for quantized metric perturbations.

In fact, using the in-out formalism to calculate the second order self-force in curved spacetime gives rise to acausal equations of motion for the particle.

The effective field theory approach benefits greatly from its ability to include spin, and other intrinsic multipole moments of the compact object, as a non-minimal coupling on the effective point particle worldline. We include spin in our formalism following the approaches of [69] for a relativistic top in flat spacetime and its generalization to curved spaces in [70]. The effects of spin have been included in a self-force calculation by [51] only through leading order. These authors recover the familiar equations of spin precession first derived by Papapetrou [71] but are augmented by the familiar MSTQW self-force.

Using the EFT formalism we also recover the Papapetrou equation to leading order. However, by knowing how the spin interactions scale with μ we can easily construct the subleading spin interactions. In particular, for a maximally rotating body we deduce for the first time that the leading order spin-orbit interaction is a second order contribution to the self-force while the leading order spin-spin interaction appears at third order in μ. These statements demonstrate some of the power and flexibility of the EFT approach: if one is interested in determining the effect of a particular kind of interaction, say the leading order spin-spin diagram, then all one has to do is construct and compute these relevant Feynman diagrams. That is, we don't have to compute all of the second order contributions and all of the third order contributions to pick out the leading order spin-spin interaction. We simply write down the appropriate diagram and calculate.

For a co-rotating spinning body where its spin angular velocity is approximately equal to the orbital angular velocity, we also show that the leading order spin-orbit contribution is suppressed to third order. Likewise the leading order spin-spin interaction is suppressed to fourth order.

Within the nonlinear scalar field model we introduced above, we calculate the leading order spin-orbit and spin-spin contributions to the self-force. Surprisingly, these diagrams actually appear at fourth and seventh orders, respectively, because of the particular way that the spin and the scalar field couple to each other.

1.3 Self-consistent backreaction approach

As mentioned earlier in this Introduction there are two important limits of the gravitational binary system that admit the use of analytic approximation techniques. The first utilizes the slow motion and weak gravitational fields of the binary constituents to devise a perturbation theory based on their relative velocity. This method, called the post-Newtonian (PN) approximation, is perhaps the most studied approach of the two given its lengthy history, starting from the famous work of Einstein, Infeld and Hoffman [72], and the large number of researchers. As result, this approximation has successfully determined the PN potentials that the (non-spinning) bodies mutually experience through 3PN (or through $O(v^6)$ beyond the Newtonian potential contribution) and the radiation reaction through 3.5PN order. While most recent successes using PN methods have come from augmenting the PN expansions with resummation techniques [73, 74, 75] the formalism still relies on the relative velocity being much smaller than c and the fields experienced by each body being weak.

The second regime applicable for analytic approximation techniques is the extreme mass ratio inspiral. In this scenario, a small compact object moves in a bounded orbit in the background provided by a supermassive black hole. Here, the expansion parameter is the mass ratio of the two objects m/M, which for the detectable frequency bandwidth for LISA is taken between about 10^{-5} to 10^{-7}. For the detection of gravitational waves from such a system one needs to know the leading order corrections to the motion of the small compact object, which is usually treated as a point particle. The $O(m/M)$ correction is the MSTQW self-force [20, 21]. However, for parameter estimation a second order calculation of the self-force is necessary to precisely determine the orbital parameters associated with the binary [60, 61]. While this method can describe the relativistic motion of the compact object it relies heavily on the dissimilarity between the values of the two masses.

In Chapter 6 we introduce a new formalism with the hope of taking some of the best features of the EMRI approximation methods and applying them to binary systems with comparable masses. In particular, we begin developing a formalism with the hope that it can describe binary systems with comparable masses for the constituents, say $m/M \sim 10^{-1} - 10^{-2}$, while still allowing for the masses to move with relativistic speeds. We do not wish to invoke a slow motion assumption but would prefer the system to evolve relativistically. In practical calculations, we may need to invoke an extra assumption(s), such as slow motion, but we stress that our formalism does not *a priori* require a slow motion assumption. Nor does it require that both objects experience weak gravitational fields. Being based on techniques valid for a general curved spacetime we allow for a non-trivial background for the system to evolve on.

We choose the less massive of the two objects to be represented using effective point particle degrees of freedom as we discuss and implement in Chapters 4 and 5. This may allow for the inclusion of some finite size effects into this formalism. Then, by decomposing the full spacetime metric into a background and its perturbations we introduce a formalism with the following properties. First, it is fully relativistic. Second, the effective point particle (i.e., the smaller compact object) moves in a non-trivial curved background. Third, we elevate the background metric from its usual status as a dormant field that is given for all time (e.g. Schwarzschild or Kerr backgrounds in the EMRI scenario) to one that is fully dynamical and interacts with both the particle and the metric perturbations on the background. In this way the equations of motion for all three quantities are dynamical and experience backreaction from each other.

While such self-consistent equations of motion are quite difficult to solve because of the mutual backreaction we hope to apply these to binaries having comparable masses and relativistic velocities. This formalism may then provide a way to describe systems that fall into the gap provided by the somewhat orthogonal limits of the PN and EMRI binary systems. While we may not be able to describe accurately the equal mass case, if our new approach can describe binaries with mass ratios of order $10^{-1} - 10^{-2}$ then we will have succeeded in our mind's eye.

One of several difficult questions we have to answer is: How well can this self-consistent backreaction formalism describe the behavior of the binary system when the objects are in the final stages before merger? In other words, how close can the two bodies be before the formalism breaks down? We begin to answer this question in Chapter 6 with a crude estimate in terms of a plausible expansion parameter that carries some information about the breakdown of the theory.

Recently, remarkable progress has been made [76, 77, 78, 79] for studying equal mass binaries with numerical techniques. Using new gauge conditions and methods for tracking the motion of black hole punctures these authors are able to compute approximately one orbit of inspiral, to carry the numerical calculation through the plunge, merger and ringdown phases, and to track the gravitational radiation emitted by the system. While these methods show promise for equal mass binary systems there are difficulties evolving intermediate and extreme mass ratio binaries with sufficient resolution given the current available computing power. However, for the EMRI scenario [80] evolves the metric perturbations in the Lorenz gauge and calculates waveforms for a test mass (i.e., in the absence of self-force) following a circular geodesic in the background spacetime of a supermassive Schwarzschild black hole. Given that one aim for the self-consistent backreaction approach is to describe binaries with comparable masses (i.e., with mass ratios of $10^{-1} - 10^{-2}$) our new formalism may provide an approximate analytical framework to numerically evolve, with sufficient resolution, the inspiral and plunge phases for IMRIs.

1.4 New results from this thesis work

Using the stochastic field theory approach we derive semiclassical particle equations of motion for a charged particle and a point mass. These equations are given in (2.121) for a scalar charge, (2.149) for an electric charge and (2.176) for a point mass. These equations are the familiar Abraham-Lorenz-Dirac (generalized to curved spacetime) and Mino-Sasaki-Tanaka-Quinn-Wald equations, respectively. In the stochastic semiclassical limit we derive corresponding Langevin equations given by (3.21), (3.35) and (3.51).

In flat spacetime we compute the noise kernel for gravitons in the vacuum state and find a $(\tau - \tau')^{-4}$ dependence in (3.63). In (3.67) we also find that the particle follows a geodesic of an effective stochastic background geometry $\eta_{\alpha\beta} + \kappa \xi_{\alpha\beta}$. When the particle acquires a stochastic component to its dynamics in a curved background we show in (3.72) that the flux of gravitational radiation emitted by the particle and measured with a detector far away contains the usual (classical) gravitational wave flux plus a purely local flux representing the quantum graviton fluctuations in the detector. As such, the purely local flux carries no information about the stochastic motion of the particle; higher order quantum corrections will likely contain information about the source.

In many practical circumstances the particle stochastic dynamics is treated phenomenologically with a noise term put in by hand to account for the particle's interactions with fluctuations in the environment variables. We find in Section 3.3 that this may not yield self-consistent equations of motion or fluctuation-dissipation relations. As opposed to deriving the noise kernel, which we do with the influence functional formalism, one needs to specify the noise kernel befitting the environment being modeled. The Langevin equations with this source of phenomenological noise are given formally in (3.74) and (3.75). Despite these cautions we find that the (phenomenological) noise induces a slowly varying force in the presence of external fields that results from averaging the (fast) stochastic particle fluctuations. The equations of motion for the coordinate fluctuations, the noise-induced drifting force and the (guiding center) background trajectory is given for an electric charge in flat spacetime in (3.78), (3.80) and (3.81), respectively. The effect is analogous to the drifts of an electric charge across external field lines due to the time-averaged (rapid) Larmor oscillations, which is encountered frequently in plasma physics. Most of these results have been given before in [49] and [50].

In the second part of this dissertation we use the effective field theory approach to derive the MSTQW equations of self-force (4.114) in Section 4.4. We also derive, for the first time, an Effacement Principle for EMRIs in Section 4.5 and show that the internal structure of a black hole and a neutron star affect the particle dynamics at $O(\mu^4)$ by causing deviations from the point particle motion that are due to tidally induced moments from gravitational interactions with the central supermassive black hole. The self-force is affected by these tidal distortions at $O(\mu^5)$. For a white dwarf we find that the order at which finite size effects will affect its dynamics depends on the distance from the supermassive black hole. The white dwarf may become tidally disrupted at an orbital distance much further than the innermost stable circular orbit. Newtonian estimates for the tidal disruption suggest that it is a second order process, $O(\mu^2)$. This work is unpublished but will be found in [81].

We give the diagrams relevant for a second order self-force calculation in Fig.(5.1). For gravitational self-force the tensor index manipulations are rather involved so we focus on a toy model describing a nonlinear scalar field on a fixed vacuum background geometry. This model is constructed to have the same power counting rules, Feynman diagrams and Effacement Principle as the gravitational problem. We calculate the second order self-force for this scalar model in (5.51). We expect the gravitational second order self-force to have a similar form but with an additional term given in (5.53). The Feynman diagrams relevant for a third order self-force calculation are shown in Fig.(5.2). These results are in preparation and will shortly be found in [82].

The self-force on spinning compact objects is described in Section 5.2. For a maximally rotating body we find that the Papapetrou-Dixon spin precession enters at $O(\mu)$ along with the MSTQW self-force. We also deduce that the leading order spin-orbit and spin-spin contributions to the self-force occur at $O(\mu^2)$ and $O(\mu^3)$, respectively. These effects enter at one higher power of μ for a corotating compact object. For the nonlinear scalar model used earlier, we calculate the leading order spin-orbit and spin-spin contributions to the self-force and find that these are suppressed to $O(\mu^4)$ and $O(\mu^7)$, respectively, due to the particular spin-field interactions used in this example. We do not anticipate this suppression for the gravitational case. See the forthcoming paper [83] for these calculations and results.

In the third part of this work we detail our motivations for introducing the self-consistent backreaction approach in Chapter 6. We treat the compact object with the lesser mass as an effective point particle. The larger compact object, which we take to be a black hole, is described by the dynamical background geometry. We deduce the equations of motion for the gravitational waves in (6.27), for the background geometry in (6.29) and for the effective particle in (6.31). We develop a crude estimate in Section 6.2.2 for the validity of the self-consistent backreaction approach, which indicates that the self-consistent backreaction equations may be valid near the plunge phase for a mass ratio of about 0.1. These results are as yet unpublished and will be found in [84].

1.5 Notations and conventions

We collect here the notations and unit conventions that we use throughout this work. We work with spacetime metrics that have mostly positive signature $(-,+,+,+)$ and use the conventions of Misner, Thorne and Wheeler [85] for the curvature tensors. We will frequently use the notation that an unprimed (primed) index refers to that component of a tensor field or coordinate evaluated at the point x (x') or worldline parameter value λ (λ'), as appropriate. For example, the graviton propagator is denoted $D_{\alpha\beta\gamma'\delta'}(x,x')$ since it transforms as a rank-2 tensor at both x and x'; it is in fact a bitensor [86]. Another example is the 4-velocity at parameter value λ', which is denoted as $\dot{z}^{\alpha'}(\lambda')$, or simply as $\dot{z}^{\alpha'}$.

A semicolon denotes a covariant derivative that is compatible with the background metric $g_{\mu\nu}$ and a comma denotes the usual partial (coordinate) derivative.

A covariant derivative with respect to the worldline parameter λ is given by

$$\frac{DF}{d\lambda} = \dot{x}^{\alpha}(\lambda)F_{;\alpha} \tag{1.2}$$

where F denotes the pullback of an arbitrary tensor field onto the worldline. Likewise, the usual λ-derivative of F is

$$\frac{dF}{d\lambda} = \dot{x}^{\alpha}(\lambda)F_{,\alpha}. \tag{1.3}$$

Other notations that appear less commonly throughout the remainder will be explained as they are introduced.

In Chapters 2, 3 and 6, we use units where $c = G = 1$ and retain $\hbar$ explicitly in our expressions. In these units, time and length have units of (mass). In Chapters 4 and 5, we use different units where $c = \hbar = 1$ and express Newton's constant G in terms of the Planck mass,

$$G = \frac{1}{32\pi m_{pl}^2}. \tag{1.4}$$

In these units, time and length have units of 1/(mass).

Unless otherwise specified, Greek indices run from 0 to $d-1$ where d is the number of spacetime dimensions. Latin indices with a caret, which run from 0 to $d-1$, represent the component of a tensor field evaluated in a quasi-local coordinate system such as the Riemann normal coordinates. It should be clear from the context at what point the tensor component is evaluated in the quasi-local coordinates. In Section 5.2 capital Latin indices denote the components of a frame field, or tetrad, on a worldline. These indices also run from 0 to $d-1$.

See Appendix A for the definitions and conventions of the propagators and quantum two-point functions that we encounter throughout this work. However, we essentially follow the definitions given in [66] upon changing to the mostly-positive signature convention.

In Appendix D we refer to the quasi-local expansion, given in powers of the ratio of the displacement from the origin to the background curvature length scale, as an adiabatic expansion since this is essentially an expansion in derivatives of the background metric. We use the notation $O(\partial^n)$ to denote the n^{th} adiabatic order of such an expansion.

Chapter 2

The nonequilibrium dynamics of particles and quantum fields in curved space: Semiclassical limit

2.1 Introduction and overview

In this chapter we consider the dynamics of particles and linear quantum fields interacting in a curved spacetime. We consider the particle to be significantly less massive than the mass scale associated with the background curvature of the space it moves in. As such, we allow for the particle's mass to be so small that quantum effects are no longer negligible but provide small corrections to the particle's motion. Under this assumption we may consider the particle to consist of a small extended object. Such bodies include massive elementary particles, atoms, molecules and nanoscale or possibly micron sized objects. We collectively refer to such microscopic bodies as "particles" in this chapter even though these may possess a small but finite extension. As we will show in Chapter 4 the effect that the finite size of an object has on its motion is relevant beginning at fourth order in the ratio of the object's size to the background curvature length scale. Therefore, we will ignore the structure that the "particle" may possess throughout this and the next Chapter.

We begin from first principles and treat the particle as a quantum mechanical object that may move with relativistic speeds. We take the field that the particle interacts with to be a linear quantum field on a curved background spacetime. We wish to study the influence that the quantized field has on the particle. It is therefore useful to consider the particle degrees of freedom, viz. the coordinates describing its trajectory through space, as an open quantum system. In this viewpoint, the dynamics and fluctuations of the field are integrated out through a form of coarse-graining. There results a self-consistent description of the particle evolving under the influence of the coarse-grained quantum field fluctuations.

In the open quantum system paradigm the particle degrees of freedom, or variables, are regarded as the system of interest whereas the (massless[1]) linear quantum field is understood to be a "large" environment. Together, the particle plus field subsystems constitute what we call the total system. We will reserve the word "system" to refer to the particle degrees of freedom.

In systems that can be treated as open there often exists a scale that naturally splits the total system (particle and field) into a system (particle) and an environment (field). Often this division comes from the mass or length scales intrinsic to the total system. Typically, a system has only a few degrees of freedom while the environment has considerably more. In fact, if the environment is described by more than roughly 20 degrees of freedom then the duration of Poincare's recurrence time indicates that energy starting in the system will be transferred to the environment with a vanishingly small probability that it will ever return to the system [31]. Therefore, a system interacting with a large environment will typically undergo dissipation as energy is lost to the many variables composing the environment. We

[1]We consider only massless fields here but the extension to massive fields is straightforward.

expect the same consequences in the particle-field total system.

In our considerations, the massless field contains only massless modes while the particle possesses an intrinsic non-zero rest mass[2]. We are therefore justified in regarding the particle variables as the system of interest and the quantum field as the environment. Furthermore, for a (non-spinning) particle the system contains only six degrees of freedom (spatial position and 3-velocity) while the environment contains an infinite number of modes.

Coarse-graining the quantum field fluctuations provides a mechanism for the quantum mechanical particle worldline coordinates to decohere through its interactions with the environment. This process is called environmentally induced decoherence and allows for the particle to evolve within a semiclassical regime. To obtain a well-defined semiclassical limit for the particle one needs to also coarse-grain the quantum mechanical worldline fluctuations as well. Care needs to be given with regard to the relevant scales of the problem as both worldline and field coarse-grainings necessarily occur at definite length or mass scales. The scale at which the fine-grained worldline histories are coarse-grained (e.g. smeared) needs to be chosen large enough to obtain sufficiently decohered particle histories but small enough not to lose the salient features of the semiclassical point-particle dynamics.

By coarse-graining these worldline fluctuations, which occur on length scales near the particle's Compton wavelength $\lambda_C = 1/m$, the particle moves nearly along

[2]Interestingly, for a scalar charge in certain cosmological spacetimes it has been shown [87] that the charge may radiate a percentage of its rest mass to infinity due to the mass non-conserving particle-field interactions that take place.

its expected classical trajectory. However, interactions with an intrinsically fluctuating quantum field can manifest as noise via classical stochastic forces exerted on the particle. As a result, the particle undergoes Brownian-like motion and carries information about the state of the quantum field that is inducing these fluctuations. This information is carried through the correlation of the stochastic forcing terms, which is related to the fluctuations of the quantum field itself through a quantity called the noise kernel, as we will show in Chapter 3.

Using a cohesive and self-consistent formalism powerful enough to incorporate classical, statistical and quantum processes inherent to particle-field systems will be vitally important. For this reason, we start from first principles using the influence functional[3] approach introduced by Feynman and Vernon [30]. The influence functional appears naturally within a convenient density matrix formulation for the nonequilibrium dynamics of this particle-field open quantum system.

This chapter is outlined as follows. In Section 2.2 we derive a path integral representation for the density matrix describing the quantum statistical state of a quantum mechanical relativistic particle and a quantum field in a curved spacetime. In doing so we will introduce the influence functional. In Section 2.3 we introduce the closed-time-path (CTP) generating functional and demonstrate its relation to the influence functional. From this the (coarse-grained) effective action for the particle is related to a loop expansion of the CTP effective action. The semiclassical limit of the particle's dynamics under the influence of the quantum field is explored in

[3]This functional is closely related to the closed-time-path generating functional, which we will discuss below.

Section 2.4.

2.2 The density matrix, coarse-graining and the influence functional

On a constant-time hypersurface Σ_i specified by an initial time t_i for a given coordinate system, the quantum statistical state of the system (particle) and environment (quantum field) is described by a density operator $\hat{\rho}(\Sigma_i)$. We allow for the particle to interact with an arbitrary (bosonic[4]) tensor field Φ_A where A denotes possible tensor indices. For example, when $A = \mu$ the quantity Φ_A describes a vector (spin-1) field and when $A = \mu\nu$ it describes a symmetric tensor (spin-2) field.

To facilitate easier computation it is customary to choose the initial density operator such that the system and the environment are initially uncorrelated

$$\hat{\rho}(\Sigma_i) = \hat{\rho}_S(\Sigma_i) \otimes \hat{\rho}_E(\Sigma_i). \tag{2.1}$$

Physically, this means that all of the field modes of Φ_A have been instantaneously uncorrelated from the particle degrees of freedom through a measurement or some other method for preparing the state of the field and the particle variables at time t_i. However, this is not a physical state since it requires an infinite amount of energy to separate *all* of the field modes from the particle. As a result, the factorized initial state may give rise to significant transient behavior in physical observables at early times. This behavior appears, for example, in models of quantum Brownian motion [37], which describes a harmonic oscillator of mass M interacting with N harmonic oscillators of an environment, each with mass $m \ll M$. A factorized initial state is

[4]We do not consider fermion fields here.

seen to impart transient behavior to the diffusion coefficients of the master equation for the reduced density matrix. This transient behavior originates from the high frequency oscillators re-correlating with the initially uncorrelated system variables. These correlations develop on a time scale $\sim 1/\omega_N$ where ω_N is the highest frequency in the oscillator environment. For times much longer than this the effects from the initial factorized state are (usually) negligible. Other initial states, most notably constructed from the so-called preparation function method [88], allow for the system and environment variables to be somewhat correlated with varying degrees at time t_i. However, transients remain because the measurement or process that puts the system and environment into that initial state is still performed instantaneously [89].

From a physical point of view one cannot perform an instantaneous measurement to prepare the initial state since the uncertainty principle guarantees that the particle cannot be localized and its momentum determined with arbitrary precision during an arbitrarily short duration of time. Physically, there will be a natural minimum scale associated with the localization of the relativistic particle, viz., the Compton wavelength $\lambda_C = \hbar/mc$. However, quantum mechanically a particle is represented by a wavepacket of width Λ centered on the coordinates z^μ. So long as these wavepackets have a width Λ that is larger than the Compton wavelength while simultaneously much smaller than any other scale in the problem (e.g., the background curvature scale of the spacetime) then a point-particle representation works well for scales much larger than Λ. We will assume this kind of construction for the particle for all time.

Keeping these issues in mind, we will assume the factorized form for the initial state (2.1) throughout this chapter. We will also describe the initial state of the environment $\hat{\rho}_E$ by a Gaussian functional of the initial field configurations Φ_{Ai}. In practice, there may be radiating modes of the field present prior to the measurement/preparation of the state at the initial time. These modes may persist beyond the initial time and affect the particle's motion. To ignore such unwanted features it is physically reasonable to consider the initial time arbitrarily far in the past so that such contributions to the initial state and the particle-field dynamics can be safely ignored.

The action for the (closed) total system consists of an action describing the free evolutions of the system of interest, that of a relativistic point-particle of mass m, the environment, that of a linear quantum field (to be coarse-grained), and their mutual interaction

$$S_{tot}[z^\mu(\lambda), \Phi_A] = S_{pp}[z^\mu(\lambda)] + S_E[\Phi_A] + S_{int}[z^\mu(\lambda), \Phi_A]. \tag{2.2}$$

The system action is given by the usual action for a relativistic point-particle

$$S_{pp}[z] = -m \int d\lambda \left(-g_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta \right)^{-1/2} \tag{2.3}$$

where $\dot{z}^\alpha$ is the particle's 4-velocity and λ, which is not necessarily the particle's proper time, parameterizes the worldline coordinates. The action for the linear quantum field is given by

$$S_E[\Phi_A] = \frac{1}{2} \int d^4x \, g^{1/2} \, g^{\mu\nu} \Phi_{A;\mu} K^{AB} \Phi_{B;\nu} \tag{2.4}$$

where K^{AB} is a tensor that describes the appropriate kinetic term for the given field. For example, if Φ_A is a scalar field then $K^{AB} = K = 1$ and for a symmetric rank-2 tensor field $K^{AB} = K^{\alpha\beta\gamma\delta}$, the form of which depends on the specific choice of gauge. The interaction term is linear in the field variable

$$S_{int}[z, \Phi_A] = \int d^4x\, g^{1/2}\, j^A(x; z)\Phi_A(x) \tag{2.5}$$

where the current density $j_A(x; z)$ is a functional of the worldline coordinates whose specific form will depend upon the field chosen and the type of interaction under consideration. For example, if Φ_A is a scalar field ϕ then we may choose $j^A = j$ to describe a monopole coupling. However, the current density may also represent a derivative interaction if $j \sim J^{\alpha}_{\ ;\alpha}$ for some vector current J^α then

$$S_{int}[z, \phi] = -\int d^4x\, g^{1/2} J^\alpha \phi_{;\alpha}. \tag{2.6}$$

In this way the particle-field interaction in the form given in (2.5) can describe many types of couplings so long as these are linear in the field variable.

Generally, the quantum field may have gauge degrees of freedom with the notable exception of the scalar field. Because of this it is necessary to include a gauge-fixing term into the action to ensure that a well-defined and unique propagator exists for the field. If the gauge choice is implemented through the constraint

$$G_B(\Phi_A) \approx 0 \tag{2.7}$$

for some appropriately chosen function G_B (the $\approx$ denotes weak equality in the sense of Dirac [90]) then the procedure of Faddeev and Popov [91] amounts to introducing the following gauge-fixing term to the field action

$$S_{gf} = -\alpha \int d^4x\, g^{1/2}\, G_B G^B. \tag{2.8}$$

Here, α is some constant that can be chosen rather arbitrarily.

We will be dealing with tree-level fields exclusively in this Chapter and the next so there is no need to keep track of the ghost fields that would normally appear in the field action since ghosts first enter at one-loop. In the remainder of this section we assume that the function G_B is approximately linear in the field so that the gauge-fixing action S_{gf} is quadratic. Any nonlinear term that might appear in G_B we assume to be small and negligible within the context of the approximations used below.

Following the work of [29], the density operator (2.1) is unitarily evolved from the initial time t_i to some later time t_f by the time-evolution operator

$$\hat{U}_{tot}(t_f, t_i) = \exp\left\{ -\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, \hat{H}_{tot}[z, \Phi_A]\right\} \tag{2.9}$$

for the system plus environment as specified by the total Hamiltonian H_{tot}. In this way, the density operator on a constant-time hypersurface at coordinate time $t_f > t_i$ is given by

$$\hat{\rho}(\Sigma_f) = \hat{U}_{tot}(t_f, t_i)\hat{\rho}(\Sigma_i)\hat{U}^{\dagger}_{tot}(t_f, t_i). \tag{2.10}$$

In general, even though the initial state of the total system is factorized the state at time t_f is not because of the correlations that dynamically develop among the mutually interacting particle and field variables.

We can specify the states of the particle and the field as eigenstates of the Schrodinger operators $\hat{z}$ and $\hat{\Phi}$ so that

$$\hat{z}|z\rangle = z|z\rangle, \tag{2.11}$$

$$\hat{\Phi}|\Phi\rangle = \Phi|\Phi\rangle. \tag{2.12}$$

Then choosing the direct product states

$$|z\Phi\rangle \equiv |z\rangle \otimes |\Phi\rangle \tag{2.13}$$

as a basis of the Hilbert space for the total system $\mathcal{H}_{tot} = \mathcal{H}_{pp} \otimes \mathcal{H}_E$ one can show that the transition amplitude $\langle z\Phi; t_f | z\Phi; t_i\rangle$ has the following path integral representation

$$\begin{aligned}
\langle z\Phi; t_f | z\Phi; t_i\rangle &= \langle z_f \Phi_f | \hat{U}_{tot}(t_f) \hat{U}^{\dagger}_{tot}(t_i) | z_i \Phi_i\rangle && (2.14)\\
&= \int_{z_i}^{z_f} \mathcal{D}z \int_{\Phi_i}^{\Phi_f} \mathcal{D}\Phi \, \exp\left\{ \frac{i}{\hbar} S_{tot}[z,\Phi] \right\} && (2.15)\\
&\equiv K(z_f, \Phi_f, t_f; z_i, \Phi_i, t_i) && (2.16)
\end{aligned}$$

where K is the amplitude of the time-evolution operator $\hat{U}_{tot}$.

At time t_f the density matrix follows from the matrix elements of the density operator (2.10)

$$\begin{aligned}
\rho(z_f, \Phi_f; z'_f, \Phi'_f; t_f) &= \int dz_i \int dz'_i \int D\Phi_i \int D\Phi'_i \, K(z_f, \Phi_f, t_f; z_i, \Phi_i, t_i)\\
&\quad \times \rho(z_i, \Phi_i; z'_i, \Phi'_i; t_i) K^*(z'_f, \Phi'_f, t_f; z'_i, \Phi'_i, t_i) \qquad (2.17)
\end{aligned}$$

where a $*$ denotes complex conjugation. The integral $\int D\Phi_i$ denotes the path integral over all field configurations on the constant-time hypersurface Σ_i. This notation is to be contrasted with $\int \mathcal{D}\Phi$, which is a path integral over the field configurations in the bulk $4d$ spacetime. Using the definition for K we find the path integral representation for the density matrix

$$\begin{aligned}\rho(z_f, \Phi_f; z'_f, \Phi'_f; t_f) &= \int dz_i \int dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z'_f} \mathcal{D}z' \, \rho_S(z_i, z'_i; t_i) \\ &\quad \int D\Phi_i \int D\Phi'_i \int_{\Phi_i}^{\Phi_f} \mathcal{D}\Phi \int_{\Phi'_i}^{\Phi'_f} \mathcal{D}\Phi' \, \rho_E(\Phi_i, \Phi'_i; t_i) \\ &\quad \times \exp\Bigg\{ \frac{i}{\hbar}\big(S_{pp}[z] - S_{pp}[z'] + S_E[\Phi] - S_E[\Phi'] \\ &\qquad\qquad + S_{int}[z, \Phi] - S_{int}[z', \Phi']\big) \Bigg\}. \qquad (2.18)\end{aligned}$$

To isolate the influence of the field on the dynamics of the particle we trace over the field configurations on the hypersurface Σ_f. Tracing out the field variables is a form of coarse-graining and the resulting partial trace of the density matrix in (2.18) is called the *reduced density matrix* for the particle,

$$\rho_{red}(z_f, z'_f; t_f) \equiv \int d\Phi_f \, \rho(z_f, \Phi_f; z'_f, \Phi'_f = \Phi_f; t_f), \qquad (2.19)$$

which can be written using (2.18) in the following form.

$$\begin{aligned}\rho_{red}(z_f, z'_f; t_f) &= \int dz_i \int dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z'_f} \mathcal{D}z' \, \rho_S(z_i, z'_i; t_i) \\ &\quad \times \exp\Bigg\{ \frac{i}{\hbar}\big(S_{pp}[z] - S_{pp}[z']\big) \Bigg\} F[z, z']. \qquad (2.20)\end{aligned}$$

The functional $F[z, z']$ is the *influence functional* and is defined as

$$\begin{aligned}F[z, z'] &\equiv \int D\Phi_f \int D\Phi_i \int D\Phi'_i \int_{\Phi_i}^{\Phi_f} \mathcal{D}\Phi \int_{\Phi'_i}^{\Phi_f} \mathcal{D}\Phi' \, \rho_E(\Phi_i, \Phi'_i; t_i) \\ &\quad \times \exp\Bigg\{ \frac{i}{\hbar}\big(S_E[\Phi] - S_E[\Phi'] + S_{int}[z, \Phi] - S_{int}[z', \Phi']\big) \Bigg\}. \qquad (2.21)\end{aligned}$$

The influence functional contains all the information of the field's influence on the particle and treats the worldline histories z and z' as fixed. In fact, it is not difficult to show that $F[z, z']$ is an ensemble average,

$$F[z, z'] = \sum_{\Phi, \Phi'} \rho_E(\Phi, \Phi'; t_i) \, {}_{z'}\big\langle \Phi'; t_f \big| \Phi; t_f \big\rangle_z \tag{2.22}$$

where the state

$$\big|\Phi; t_f\big\rangle_z = T \exp\left\{ \frac{i}{\hbar} \, \hat{\Phi}_A \cdot j^A[z] \right\} |\Phi\rangle \tag{2.23}$$

is evolved from time t_i to t_f in the presence of the fixed worldline $z^\alpha(\lambda)$. The influence functional therefore describes the quantum and statistical information contained in the quantum field environment and may be interpreted as the ensemble average of the overlap at time t_f for the field configuration Φ that evolved in the presence of a particle worldline $z^\alpha(\lambda)$ with the field configuration Φ' that evolved with the worldline $z'^\alpha(\lambda)$.

We remark that when the two worldline histories z and z' are equal that the influence functional is simply 1. To see this, we observe that

$${}_{z'}\big\langle \Phi'; t_f \big| \Phi; t_f \big\rangle_z \Big|_{z'=z} = \big\langle \Phi' \big| \Phi \big\rangle \tag{2.24}$$

since the time-evolution operators in the states (2.22) cancel each other. The ensemble average in (2.22) then becomes

$$\begin{aligned} F[z, z' = z] &= \sum_{\Phi, \Phi'} \big\langle \Phi \big| \hat{\rho}_E(\Sigma_i) \big| \Phi' \big\rangle \big\langle \Phi' \big| \Phi \big\rangle && (2.25) \\ &= \sum_{\Phi} \big\langle \Phi \big| \hat{\rho}_E(\Sigma_i) \big| \Phi \big\rangle, && (2.26) \end{aligned}$$

which is just the trace of the density matrix and equals 1 as claimed. Interestingly, for field configurations with a non-vanishing renormalized free energy, such as for a field constrained by boundaries as in the Casimir effect, the influence functional no longer satisfies this condition [92].

The linearity of the field action and the Gaussian structure of the initial state of the field $\hat{\rho}_E(\Sigma_i)$ allows for the influence functional to be integrated exactly. The standard methods used to evaluate the influence functional in flat spacetime involve decomposing the field in terms of the modes that suitably respect the symmetries of the problem. Namely, the modes used in such a decomposition are the eigenmodes of the Killing vector field that describes the isometries of the system [66]. However, in an arbitrary curved spacetime there does not exist a unique mode decomposition for the field since there are generally no Killing fields to generate such eigenmodes. Furthermore, the modes may not be explicitly calculable, which presents a serious drawback for obtaining an explicit expression for the influence functional. We will therefore need a different method to evaluate the path integrals appearing in (2.21).

For this purpose it will be convenient to write the influence functional using operator language. In the interaction picture, we can write the influence functional as

$$F[z, z'] = \mathrm{Tr}_\Phi \left[\left(T \exp\left\{ \frac{i}{\hbar} \hat{\Phi}_A \cdot j^A \right\} \right) \hat{\rho}_E(\Sigma_i) \left(T^* \exp\left\{ -\frac{i}{\hbar} \hat{\Phi}'_A \cdot j'^A \right\} \right) \right] \quad (2.27)$$

where Tr_Φ denotes the trace over the field variables and j^A denotes the current density in the interaction action S_{int}. We use the notation that a $\cdot$ denotes spacetime integration so that for two functions (possibly tensor-valued) we have

$$A \cdot B = \int d^4x \, g^{1/2} \, A(x) B(x). \quad (2.28)$$

The T and T^* denote time-ordering and anti-time-ordering operations on the field operators. From (2.27) we can compute the influence action and expand it in powers of the current j, which we assume to be proportional to the coupling constant of the particle-field interactions (e.g., charge e). We find that the influence functional can be calculated exactly after resumming this coupling constant expansion and is given by

$$F[z,z'] = \exp\left\{ \frac{i}{\hbar}\langle\hat{\Phi}^A\rangle \cdot j_A^- - \frac{1}{4\hbar} j_A^- \cdot G_H^{AB'} \cdot j_{B'}^- + \frac{i}{\hbar} j_A^- \cdot G_{ret}^{AB'} \cdot j_{B'}^+ \right\}. \quad (2.29)$$

It will be convenient to define the influence action S_{inf} as the logarithm of the influence functional,

$$S_{inf}[z,z'] = -i\hbar \ln F[z,z'] \quad (2.30)$$

and evaluates to, using (2.29),

$$S_{inf}[z,z'] = \langle\hat{\Phi}^A\rangle \cdot j_A^- + \frac{i}{4} j_A^- \cdot G_H^{AB'} \cdot j_{B'}^- + j_A^- \cdot G_{ret}^{AB'} \cdot j_{B'}^+. \quad (2.31)$$

The currents j_A^- and j_A^+ are the difference and average, respectively, of j_A and j'_A,

$$\begin{aligned} j_A^- &= j_A[z] - j_A[z'] && (2.32) \\ j_A^+ &= \frac{j_A[z] + j_A[z']}{2}. && (2.33) \end{aligned}$$

The functions G_H^{AB} and G_{ret}^{AB} are the Hadamard two-point function and the retarded propagator, respectively. These are given in terms of correlations of the field with respect to the initial state of the field $\hat{\rho}(\Sigma_i)$

$$\begin{aligned} G_H^{AB'}(x,x') &= \left\langle \{\hat{\Phi}^A(x) - \langle\hat{\Phi}^A(x)\rangle, \hat{\Phi}^{B'}(x') - \langle\hat{\Phi}^{B'}(x')\rangle\}\right\rangle && (2.34) \\ &= \left\langle \{\hat{\Phi}^A(x), \hat{\Phi}^{B'}(x')\}\right\rangle - \left\langle\hat{\Phi}^A(x)\right\rangle\left\langle\hat{\Phi}^{B'}(x')\right\rangle && (2.35) \\ G_{ret}^{AB'}(x,x') &= i\theta_+(x, \Sigma_{x'})\left\langle[\hat{\Phi}^A(x), \hat{\Phi}^{B'}(x')]\right\rangle. && (2.36) \end{aligned}$$

The curly braces in the Hadamard function denote the anti-commutator and the square brackets in the retarded propagator signify the commutator of the field operators. These two-point functions are evaluated using the interaction picture for the fields since we calculated the influence functional in this representation. The expectation values are computed using the initial state of field on the hypersurface Σ_i so that

$$\langle \hat{\mathcal{O}} \rangle = \mathrm{Tr}_\Phi \left[\hat{\rho}_E \hat{\mathcal{O}} \right], \tag{2.37}$$

which describes the quantum statistical expectation value of the operator $\hat{\mathcal{O}}$.

The retarded propagator in (2.36) is proportional to the step function distribution $\theta_+(x, \Sigma_{x'})$, which is defined to be 1 for all points in the future of x' and zero otherwise where the constant time hypersurface $\Sigma_{x'}$ contains the point x'. This distribution is a generalization to curved spacetime of the usual step function $\theta(t - t')$. We will discuss other distributions below, however, the reader is referred to [53] for a pedagogical introduction to certain distributions in curved space.

For a Gaussian initial state of the field possessing a vanishing expectation value we can set to zero the one-point function appearing in the exponential of the influence functional in (2.29) and in the definition of the Hadamard two-point function in (2.35). If the field's expectation value is non-zero then there exists a nontrivial configuration for the field so that even in vacuum there persists a field structure on the spacetime, which some references call a classical field configuration [93, 94]. We remark here that the expectation value of a field operator does not represent a classical field as is sometimes implied. Achieving the classical limit of a quantum theory involves processes requiring the quantum variables of the system of interest to decohere. We will discuss more about decoherence as it applies to open quantum systems below.

If the initial state of the field $\hat{\rho}_E(\Sigma_i)$ contains non-Gaussian contributions, one would have many additional terms involving higher powers of the coupling appearing in (2.29). Likewise, for nonlinear particle-field interactions (e.g. $j^A(z) \cdot \Phi_A^n$) there will appear more terms in the influence functional. In such cases, assuming that the particle-field coupling constant is small and that non-Gaussianities in the initial state of the field are also small or zero then we can still use (2.29) as a leading order approximation.

The influence functional (2.29) represents the influence that the quantum field has on the evolution of the quantum mechanical worldline histories z and z' of the particle. Therefore, $F[z, z']$ contains all of the information regarding the content of the environment, its dynamics and its interactions with the system. As such, by computing the influence functional one may describe the environmental effects on the system in a self-consistent manner. We will show below that the influence action (2.30) introduces an effective forcing term on the particle that accounts for the particle-field couplings and the dynamical processes that evolve the coarse-grained quantum field.

2.3 The CTP generating functional and the coarse-grained effective action

The influence functional of Feynman and Vernon [30] is closely related to the closed-time-path (CTP) formalism of Schwinger and Keldysh [95, 96]. The former is useful for describing the processes and interactions of systems that occur within a finite duration of time and is often self-consistently implemented for a quantum system that is open to external and dynamical influences. The latter utilizes the so-called "in-in" boundary conditions, which specifies the initial state in the infinite past $t \to -\infty$, to compute n-point correlation functions of a quantum field. We demonstrate in this section that despite these differences the influence action is equivalent to the effective action obtained from the CTP generating functional by coarse-graining (or integrating out) the quantum field variables. To show this, however, we need to define and discuss the CTP generating functional in so much as it applies to our particle-field total system. We refer the reader to [29, 97] and references therein for an excellent discussion of the CTP formalism.

The CTP, or "in-in," generating functional of Schwinger and Keldysh is defined for our particle-field total system in the following way

$$Z[J_z, J'_z, J_\Phi, J'_\Phi] \equiv \sum_{z,z'} \sum_{\Phi,\Phi'} \rho(z, \Phi, z', \Phi'; t_i) \; {}_{J'_z, J'_\Phi}\langle z'\Phi'; t_f \big| z\Phi; t_f \rangle_{J_z, J_\Phi}. \tag{2.38}$$

The CTP generating functional represents the ensemble average at time t_f over the configurations of the field and particle variables in the presence of the external current densities J_z and J_Φ, which couple bilinearly to the particle worldine and the field, respectively. It represents the functional obtained by evolving the state forward in time with currents J_z, J_Φ to some state at time t_f, evolving that state backward in time to t_i in the presence of different currents J'_z, J'_Φ and finally summing over all of the unknown states at t_f. If the currents are equal across both paths in time then the generating functional equals 1. Observe that (2.38) and (2.22) have similar structures in that they both represent ensemble averages.

The CTP generating functional has the path integral representation

$$
\begin{aligned}
Z[J_z, J'_z, J_\Phi, J'_\Phi] \;=\; & \int dz_f \int dz_i \int dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z_f} \mathcal{D}z' \\
& \int D\Phi_f \int D\Phi_i \int D\Phi'_i \int_{\Phi_i}^{\Phi_f} \mathcal{D}\Phi \int_{\Phi'_i}^{\Phi_f} \mathcal{D}\Phi' \, \rho(z_i, \Phi_i, z'_i, \Phi'_i; t_i) \\
& \times \exp\bigg\{ \frac{i}{\hbar} \Big(S_{pp}[z] - S_{pp}[z'] + S_E[\Phi] - S_E[\Phi'] \\
& \quad + S_{int}[z, \Phi] - S_{int}[z', \Phi'] + \int d\lambda \, (J_z^\alpha z_\alpha - J_z'^\alpha z'_\alpha) \\
& \quad + J_\Phi^A \cdot \Phi_A - J_\Phi'^A \cdot \Phi'_A \Big) \bigg\},
\end{aligned}
\tag{2.39}
$$

which follows from similar steps used in deriving the path integral form of the density matrix and influence functional in (2.18) and (2.21), respectively. Correlation functions of the particle worldline coordinates evaluated with respect to the state $\hat{\rho}(\Sigma_i)$ of the particle-field total system can be calculated from derivatives of the generating functional

$$
\big\langle \bar{T} \hat{z}_a(\lambda_1) \cdots \hat{z}_b(\lambda_n) \big\rangle_{J_z, J'_z, J_\Phi, J'_\Phi} \equiv \frac{1}{Z} \frac{\delta^n Z}{\delta i J_z^a(\lambda_1) \cdots \delta i J_z^b(\lambda_n)}. \tag{2.40}
$$

The superscripts a and b label the unprimed and primed worldline coordinates. There is a similar expression for correlation functions of the quantum field where the functional derivatives are taken with respect to J_Φ^a instead. The CTP time-ordering operator $\bar{T}$ is defined so that unprimed operators are time-ordered, primed operators are anti-time-ordered and all primed operators are ordered to the left of unprimed operators.

If the initial state of the particle-field total system is factorizable as in (2.1) we find that the CTP generating functional simplifies

$$
\begin{aligned}
Z[J_z, J'_z, J_\Phi, J'_\Phi] \;=\; & \int dz_f \int dz_i \int dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z_f} \mathcal{D}z' \, \rho_S(z_i, z'_i; t_i) \\
& \times \exp\left\{ \frac{i}{\hbar}\Big(S_{pp}[z] - S_{pp}[z'] + S_{inf}[z, z'; J_\Phi, J'_\Phi] \Big) \right. \\
& \left. \qquad + \frac{i}{\hbar} \int d\lambda \, (J_z^\alpha z_\alpha - J_z'^\alpha z'_\alpha) \right\} \qquad (2.41)
\end{aligned}
$$

where the influence action $S_{inf}[z, z'; J_\Phi, J'_\Phi]$, given by the logarithm of the influence functional in (2.21) or (2.29), contains all of the information about the dynamics of the field and its interactions with the particle. For the remainder we will not need to concern ourselves with the external field currents J_Φ^a so we will set them to zero so that the influence action reads $S_{inf}[z, z']$. Expressed solely in terms of particle variables Z will be called the coarse-grained generating functional,

$$
\begin{aligned}
Z_{cg}[J_z, J'_z] \;=\; & \int dz_f \int dz_i \int dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z_f} \mathcal{D}z' \, \rho_S(z_i, z'_i; t_i) \\
& \times \exp\left\{ \frac{i}{\hbar}\Big(S_{pp}[z] - S_{pp}[z'] + S_{inf}[z, z'] + \int d\lambda \, (J_z^\alpha z_\alpha - J_z'^\alpha z'_\alpha) \Big) \right\}.
\end{aligned}
\qquad (2.42)
$$

We find it convenient to simply the notation and define

$$
\int_{CTP} \mathcal{D}z^a \, (\cdots) \equiv \int dz_f \int dz_i \int dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z_f} \mathcal{D}z' \, (\cdots) \qquad (2.43)
$$

as the particle integrations satisfying the "in-in" boundary conditions. Then,

$$Z_{cg}[J_z, J'_z] = \int_{CTP} \mathcal{D}z^a \, \rho_S(z_i, z'_i; t_i) \exp\left\{ \frac{i}{\hbar}\Big(S_{pp}[z] - S_{pp}[z'] + S_{inf}[z, z'] \Big) + \frac{i}{\hbar} \int d\lambda \, (J_z^\alpha z_\alpha - J_z'^\alpha z'_\alpha) \right\}. \tag{2.44}$$

Using the coarse-grained generating functional we can calculate CTP time-ordered n-point correlation functions of the quantum mechanical particle worldline coordinates as in (2.40). To compute *connected* correlation functions (i.e., cumulants of the worldline operators) we can use the logarithm of Z,

$$W_{cg}[J_z, J'_z] \equiv -i\hbar \ln Z_{cg}[J_z, J'_z]. \tag{2.45}$$

From the definition of the worldline correlation functions (2.40) we remark that the expectation value of the worldline coordinate operator is given by the functional derivative,

$$\big\langle \hat{z}^a \big\rangle_{J_z, J'_z} = \frac{\delta W_{cg}[J_z, J'_z]}{\delta J_z^a}. \tag{2.46}$$

Given this we can now compute the coarse-grained effective action by calculating the Legendre transform of W_{cg}. We find

$$\Gamma_{cg}[\langle \hat{z} \rangle, \langle \hat{z}' \rangle] \equiv W_{cg}[J_z, J'_z] - \int d\lambda \left(J_z^\alpha \big\langle \hat{z}_\alpha \big\rangle - J_z'^\alpha \big\langle \hat{z}'_\alpha \big\rangle \right) \tag{2.47}$$

and it generates one-particle-irreducible (1PI) correlation functions of the particle worldline variables subjected to interactions with the coarse-grained quantum field.

The equations of motion for the expectation values in (2.46) can be found from the variation

$$\frac{\delta \Gamma_{cg}}{\delta \big\langle \hat{z}^a \big\rangle} = J_z^a \tag{2.48}$$

where a indicates if the quantity is unprimed or primed. Using (2.44) and (2.45) we substitute these equations of motion (2.48) into (2.47) giving

$$\begin{aligned}\Gamma_{cg}[\langle\hat{z}\rangle,\langle\hat{z}'\rangle] &= -i\hbar \ln \int_{CTP} \mathcal{D}z^a \, \rho_S(z_i, z_i'; t_i) \exp\left\{ \frac{i}{\hbar}\Big(S_{pp}[z] - S_{pp}[z'] + S_{inf}[z,z'] \Big) \right. \\ &\left. + \frac{i}{\hbar} \int d\lambda \left[\frac{\delta\Gamma_{cg}}{\delta\langle\hat{z}^\alpha\rangle} \big(z^\alpha - \langle\hat{z}^\alpha\rangle\big) - \frac{\delta\Gamma_{cg}}{\delta\langle\hat{z}'^\alpha\rangle} \big(z'^\alpha - \langle\hat{z}'^\alpha\rangle\big)\right] \right\}, \qquad (2.49)\end{aligned}$$

which is an integrodifferential equation for Γ_{cg}. Let us define the fluctuation of the worldline coordinate away from the expectation value as

$$\tilde{z}^a \equiv z^a - \langle\hat{z}^a\rangle. \qquad (2.50)$$

The integration measure $\mathcal{D}z^a$ remains unchanged under this shift

$$\mathcal{D}z^a = \mathcal{D}\tilde{z}^a \qquad (2.51)$$

so that the coarse-grained effective action can be written as a CTP path integral over the worldline fluctuations away from the average

$$\begin{aligned}\Gamma_{cg}[\langle\hat{z}\rangle,\langle\hat{z}'\rangle] &= -i\hbar \ln \int_{CTP} \mathcal{D}\tilde{z}^a \, \rho_S(z_i, z_i'; t_i) \\ &\times \exp\left\{ \frac{i}{\hbar}\Big(S_{pp}\big[\langle\hat{z}\rangle + \tilde{z}\big] - S_{pp}\big[\langle\hat{z}'\rangle + \tilde{z}'\big] + S_{inf}\big[\langle\hat{z}\rangle + \tilde{z}, \langle\hat{z}'\rangle + \tilde{z}'\big] \Big) \right. \\ &\left. + \frac{i}{\hbar} \int d\lambda \left(\frac{\delta\Gamma_{cg}}{\delta\langle\hat{z}^\alpha\rangle} \tilde{z}^\alpha - \frac{\delta\Gamma_{cg}}{\delta\langle\hat{z}'^\alpha\rangle} \tilde{z}'^\alpha \right) \right\}. \qquad (2.52)\end{aligned}$$

Formally expanding in powers of the fluctuation coordinate $\tilde{z}^a$ we find that the integrodifferential equation can be perturbatively solved via a loop expansion. We find

$$\begin{aligned}\Gamma_{cg}[\langle\hat{z}\rangle,\langle\hat{z}'\rangle] &= S_{pp}\big[\langle\hat{z}\rangle\big] - S_{pp}\big[\langle\hat{z}'\rangle\big] + S_{inf}\big[\langle\hat{z}\rangle,\langle\hat{z}'\rangle\big] + (\text{particle}-\text{loop terms}) \\ &\equiv S_{cgea}[\langle\hat{z}\rangle,\langle\hat{z}'\rangle] + (\text{particle}-\text{loop terms}) \qquad (2.53)\end{aligned}$$

where the particle-loop terms denote quantities of $O(\hbar)$. These higher order terms can be derived by observing that the formal expansion of the integrand of Γ_{cg} in powers of $\tilde{z}^a$ results in a nonlinear interacting theory. One can construct Feynman rules and diagrams for the "interactions" using the usual techniques of a perturbative quantum theory. The first non-trivial subleading contribution in this series is the term quadratic in $\tilde{z}^a$, which gives the propagator at one-loop for the fluctuations of the particle worldline about the expected trajectory. We remark that such a propagator includes a contribution from the second functional derivative of the influence action S_{inf} and therefore is *not* the free propagator for the worldline fluctuations. Rather this term includes the effects of the coarse-grained field on the particle and in this sense one may call the resulting propagator "dressed" by the field.

The leading order terms in (2.53) collectively define the coarse-grained effective action S_{cgea}. It describes the evolution of the expectation value under the influence of the coarse-grained quantum field. The coarse-grained effective action is tree-level in the particle variables and so it might be tempting to interpret S_{cgea} as giving rise to the classical equations of motion for the particle coordinates. However, we need to ensure that the contributions from the $O(\hbar)$ particle-loop corrections are small. In the next section we will show that this smallness is related to the suppression of off-diagonal matrix elements of the reduced density matrix for the particle variables.

2.4 semiclassical limit

In this section we discuss how the particle's semiclassical limit is identified from the first principles approach provided by the influence functional and the coarse-grained effective action. We derive the semiclassical worldline equations of motion and demonstrate the existence of a divergence in the force on the particle that results from the well known ultraviolet pathologies inherent to the (coarse-grained) quantum field. The removal of this divergence is necessary to obtain well-behaved solutions. We end this section by deriving the self-force equations of motion on scalar and electric charges as well as on a small mass. We turn now to obtaining the semiclassical limit.

The reduced density matrix (2.20) depends upon the influence functional in a linear manner. That is, as a functional, the reduced density matrix is linear in F. For the particle-field total system that we have been considering all along the influence functional is integrated to give a closed form expression in (2.29). We remark that the influence functional contains both a real and an imaginary part. Since both the retarded propagator and the Hadamard two-point function are real then the complex norm of F is

$$\big|F[z,z']\big| = \exp\bigg\{ -\frac{1}{4\hbar}\, j^-_A \cdot G_H^{AB'} \cdot j^-_B \bigg\}, \tag{2.54}$$

which depends on the quantum field only through its fluctuations in the given initial state as indicated by the Hadmard function defined in (2.34) and (2.35). Furthermore, this a Gaussian functional in j^-_A so that it is peaked around some configuration. Given that $j^- = j[z] - j[z']$ is very small when z' approaches z we would expect that $|F|$ is peaked about $z' = z$. This is true, however, in the limit $z' \to z$ the Hadamard function diverges on account of the usual pathologies associated with the ultraviolet behavior of products of quantum fields. For example, in flat spacetime, the Hadamard function of a scalar field in the vacuum state is

$$G_H(x, x') = \frac{1}{4\pi^2} \frac{1}{\eta_{\alpha\beta}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)} = \frac{1}{8\pi^2 \sigma(x, x')}, \tag{2.55}$$

which diverges like $1/(t - t')^2$ when $\mathbf{x}' = \mathbf{x}$. Therefore, a regularization must be given to $G_H^{AB'}$ in order to make sense of $|F|$ and the reduced density matrix.

Up until now the particle worldlines have been regarded as fine-grained histories through space and time. By fine-grained histories we mean the set of all possible trajectories that contribute to the worldline path integrals encountered in this section[5]. However, such worldlines can never be regarded as classical paths in space since their quantum mechanical fluctuations can be very large indeed. Therefore, constructing semiclassical worldlines involves coarse-graining the particle worldlines themselves. Conceptually, the most convenient way to do this is to smear the trajectories over an appropriate length scale. For example, this scale might be the Compton wavelength of the particle or the particle's deBroglie wavelength if the momentum is approximately known. Regardless, we denote this scale by λ_{cg}.

After smearing the quantum mechanical fluctuations of the worldlines we see that the Hadamard function in (2.54) inherits this smearing and can be made finite and well-behaved since z' can only equal z within an uncertainty of order λ_{cg}. In this way, $|F|$ takes on its largest values for two worldline histories that lie within

[5]Actually, only those fine-grained histories that are consistent with a single particle interpretation are summed over in the worldline path integrals for the reasons mentioned in Section 1.1.

λ_{cg} of each other. Otherwise, the magnitude of the influence functional falls off rapidly. The reduced density matrix is therefore dominated by those worldlines that lie within λ_{cg} of each other for all times after t_i. When the initial state of the particle-field total system and the interactions and scales involved are such that the off-diagonal[6] elements of the reduced density matrix are suppressed, we say that the particle worldline coordinates have decohered. The decoherence results from interactions with the quantum fluctuations of the coarse-grained field since $|F|$ depends on the field solely through the Hadamard function (2.34).

Decoherence can be quantified by the complex norm of the influence functional, which is a factor that appears in the coarse-grained effective action in (2.52). If the worldlines have achieved a sufficient amount of decoherence then the expectation values of the two (coarse-grained, smeared) particle histories are approximately equal $\langle \hat{z} \rangle \approx \langle \hat{z}' \rangle$, with an uncertainty of order λ_{cg}, and the particle fluctuations $\tilde{z}^a$ are suppressed. Under such circumstances the subleading particle-loop corrections in (2.53) are small indeed and the coarse-grained effective action Γ_{cg} gives rise to a semiclassical limit for the expectation value of the decohered particle worldline.

In the semiclassical limit we can obtain the equations of motion for the particle worldline from the coarse-grained effective action in (2.53),

$$0 = \left. \frac{\delta \Gamma_{cg}}{\delta \langle \hat{z} \rangle} \right|_{z'=z} = \left. \frac{\delta S_{cgea}}{\delta \langle \hat{z} \rangle} \right|_{z'=z} \tag{2.56}$$

upon ignoring the very small loop corrections. (See (2.48) with $J_z^a = 0$.) We remark that the solution to (2.56) makes the leading order phases in (2.20) and

[6]By off-diagonal we mean those trajectories for which z and z' do not lie within about λ_{cg} of each other for all times.

(2.52) stationary. The equations of motion obtained from the coarse-grained effective action are causal and real since Γ_{cg} describes quantum systems with an initial value condition [98]. The causality and reality will be demonstrated with an explicit calculation of the semiclassical particle equations of motion.

Below, it will be convenient to simplify the notation and use a bar over a semiclassical quantity, such as the worldline coordinates

$$\bar{z}^\mu(\lambda) \equiv \big\langle \hat{z}^\mu(\lambda) \big\rangle, \tag{2.57}$$

and over functions and tensors that are evaluated along the semiclassical trajectory.

Using the integrated result for the influence functional in (2.29) it is straightforward to show that the semiclassical equations in (2.56) are

$$m\bar{a}^\mu(\tau) = \vec{w}^{\mu A}[\bar{z}] \int_{\tau_i}^{\tau_f} d\tau'\, G^{ret}_{AB'}\big(\bar{z}^\alpha(\tau), \bar{z}^{\alpha'}(\tau')\big) j^{B'}[\bar{z}^{\alpha'}(\tau')] \tag{2.58}$$

where we parameterize the worldline by the particle's proper time in the last step of the calculation. As claimed, the equations of motion are explicitly real and causal, due to the presence of the retarded field propagator. We observe that the field manifests itself on the particle through the retarded propagator, which is a state-independent two-point function.

We will find it useful to employ the more compact notation that we introduced earlier so that the semiclassical equations of motion are written as

$$\begin{aligned} m\bar{a}^\mu(\tau) &= \vec{w}^{\mu A}[\bar{z}^\alpha] \int_{\tau_i}^{\tau_f} d\tau'\, G^{ret}_{AB'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) j^{B'}[\bar{z}^{\alpha'}] &(2.59)\\ &= \vec{\bar{w}}^{\mu A} \int_{\tau_i}^{\tau_f} d\tau'\, \bar{G}^{ret}_{AB'} \bar{j}^{B'}. &(2.60) \end{aligned}$$

The acceleration 4-vector is defined by

$$\bar{a}^\mu(\tau) = \frac{D\dot{\bar{z}}^\mu}{d\tau} = \dot{\bar{z}}^\alpha \nabla_\alpha \dot{\bar{z}}^\mu \tag{2.61}$$

and $\dot{\bar{z}}^\alpha(\tau)$ is the 4-velocity of the semiclassical worldline at proper time τ. The operator $\vec{w}^{\mu A}$ is usually proportional to a covariant derivative and is defined by the functional derivative of the current j^A,

$$\vec{w}^{\mu A} f(z) \equiv \frac{\delta}{\delta z_\mu(\tau)} \int d^4x \, g^{1/2} j^A(x; z) f(x) \tag{2.62}$$

where $f(x)$ is an arbitrary C^1 function.

The curvature of the background spacetime allows for a force to act on the particle that depends on the precise history the particle has taken up to that point in time, which can be seen in (2.58). This history-dependent, or non-Markovian, feature comes from the backscattering of the field off the curvature. The backreaction of the emitted radiation on the particle at proper time τ results from two types of interactions with the field. The first is a reactive force describing the recoil on the particle as it emits the radiation. This is the familiar radiation reaction of Abraham, Lorenz and Dirac; notably, it is a purely local interaction $\sim \delta(\tau - \tau')$. The second results from the interaction of the particle with previously emitted radiation that backscatters off of the background curvature and interacts with the particle at a later time τ and different position. This is an intrinsically non-local process in time and space. The effects of both kinds of interactions with the coarse-grained quantum field manifest on the particle as *self-force* and is responsible for the dissipation of the particle's mechanical energy and momentum.

2.4.1 Hadamard expansion of the retarded propagator

The retarded propagator for a quantum field Φ_A is difficult to construct in an arbitrary curved spacetime. A mode expansion for the field and the retarded propagator can be constructed if the spacetime possesses a sufficient number of isometries. For example, the retarded propagator has been calculated exactly for a scalar field in de Sitter and certain Friedmann-Robertson-Walker (FRW) cosmologies [87] by utilizing the conformally flat nature of these spacetimes. However, it will be much more useful for our purposes to know the general structure and form of the retarded propagator in an arbitrary background spacetime as this will enable us to determine the structure of the self-force on the particle, which will be necessary for its regularization. Throughout this subsection we will borrow from the presentation in [53] to which we refer the reader to for more details.

We use Hadamard's construction [99] for the retarded propagator $G^{ret}_{AB'}(x,x')$, which is valid only within the normal convex neighborhood $\mathcal{N}$ of x. (The normal convex neighborhood $\mathcal{N}(x)$ is the set of all points in the spacetime that can be connected to x by a unique geodesic.) In curved spacetime, Hadamard's construction allows for the Green's function to be written in the following way

$$G^{ret}_{AB'}(x,x') = U_{AB'}(x,x')\Delta^{1/2}(x,x')\delta_+\big(\sigma(x,x')\big) + V_{AB'}(x,x')\theta_+\big(-\sigma(x,x')\big). \tag{2.63}$$

We remark that there are two types of contributions to the propagator. The first describes propagation along the null cone based at x' and is called the *direct* part of the Green's function. The second, called the *tail* part, describes propagation within the null cone as a result of the wavefronts bending, or backscattering, off of the background curvature. In a flat spacetime this backscattering does not occur as there is no curvature to bend the wavefronts. Therefore, only the direct part contributes to the familiar flat space propagator.

The bitensors $U_{AB'}$ and $V_{AB'}$ are smooth functions at x and x'. The tensor $U_{AB'}$ involves products of the bi-tensor $g_{\alpha}{}^{\beta'}(x,x')$ that parallel transports a tensor at x' to x. For example, if Φ_A describes a vector field A_μ then

$$U_{\alpha\beta'}(x,x') = g_{\alpha\beta'}(x,x') \tag{2.64}$$

and for a scalar field, $U(x,x') = 1$. We will see more examples of its form when we construct the retarded Green's function for other fields below. The tensor $V_{AB'}$ satisfies the appropriate homogeneous field equations that are subject to characteristic data provided along the forward lightcone at x'. This tensor field is quite difficult to calculate for generic spacetimes so throughout the remainder we consider this object formally. As such we refer the reader to [53] for details that have been omitted here.

The factor $\Delta^{1/2}(x,x')$ in the direct part of the propagator is the square root of the van Vleck determinant,

$$\Delta(x,x') = -\frac{\det\big(-\sigma_{\alpha\beta'}(x,x')\big)}{g^{1/2}(x)g^{1/2}(x')}. \tag{2.65}$$

Within the normal convex neighborhood this biscalar is well-defined. However, at the boundary of $\mathcal{N}(x)$ the van Vleck determinant diverges due to the appearance of caustics (the intersection of two or more geodesics that each connect back to the point at x).

The distributions $\delta_+(\sigma)$ and $\theta_+(-\sigma)$ are defined as $\theta_+(x, \Sigma_{x'})$, which we recall is the generalization of the usual step function to curved spacetime, multiplying $\delta(\sigma)$ and $\theta(-\sigma)$, respectively. The biscalar $\sigma = \sigma(x, x')$ is Synge's world function defined along the (unique) geodesic linking x and x'. Numerically, the world function equals half of the squared geodesic distance between the two points. Further, σ is negative, positive and zero for time-like, space-like and light-like separated points, respectively. For a lightcone centered on x', which lies on the space-like hypersurface $\Sigma_{x'}$, it follows that $\delta_+(\sigma)$ has support only along the forward lightcone while $\theta_+(-\sigma)$ equals one in the causal future of x' (the interior of the forward lightcone) and vanishes everywhere else. Fig.(2.1) schematically shows the regions of support for these distributions.

Given the Hadamard construction of the propagator we need to determine how it can be used in (2.58). The self-force can be written as two contributions. The first results from the propagation of the field in the normal convex neighborhood $\mathcal{N}(\bar{z}^\alpha)$ of $\bar{z}^\alpha(\tau)$. The second comes from propagation in the spacetime complementary to $\mathcal{N}(\bar{z}^\alpha)$. If $\tau_<$ is the proper time at which the worldline enters $\mathcal{N}(\bar{z}^\alpha)$ and $\tau_>$ is the proper time that it leaves then we may write the self-force as

$$\vec{\bar{w}}^{\mu A} \int_{\tau_i}^{\tau_f} d\tau' \, \bar{G}^{ret}_{AB'} \bar{j}^{B'} = \vec{\bar{w}}^{\mu A} \left[\int_{\tau_i}^{\tau_<} d\tau' + \int_{\tau_>}^{\tau_f} d\tau' + \int_{\tau_<}^{\tau_>} d\tau' \right] \bar{G}^{ret}_{AB'} \bar{j}^{B'} \qquad (2.66)$$

See Fig.(2.2) for a depiction of these regions and points in the spacetime. Observe that the last term is a worldline integral of the retarded propagator that is entirely within the normal convex neighborhood of $\bar{z}^\alpha(\tau)$.

We observe that $\tau < \tau_>$ since the retarded propagator in the normal convex

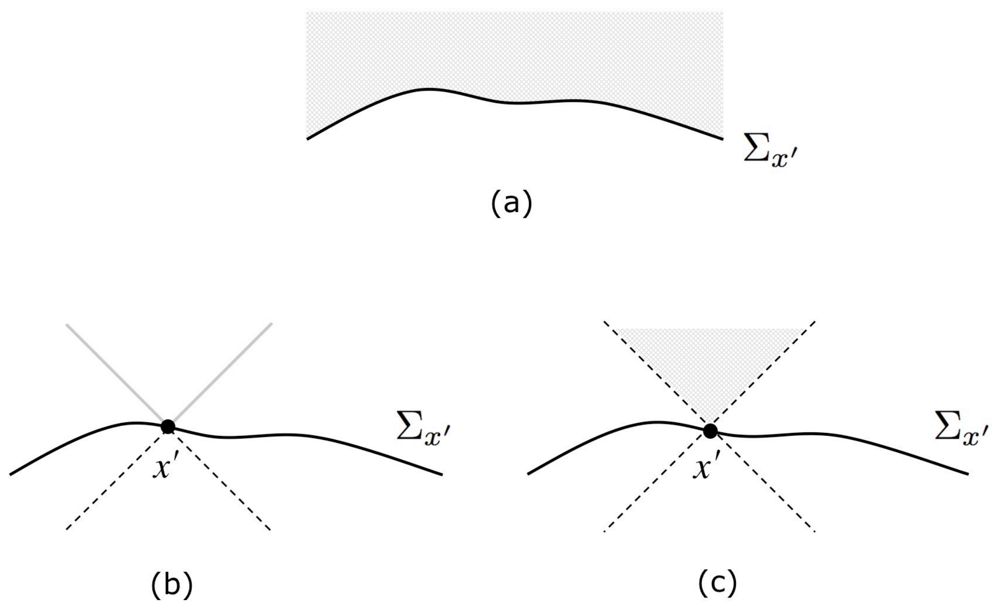

Figure 2.1: The distributions used in Hadamard's construction of the retarded propagator. The grey regions or lines denote a non-zero value for the distribution and the dotted lines form the null cone at x'. The space-like hypersurface $\Sigma_{x'}$ contains the point x'. (a) The generalized step function $\theta_+(x, \Sigma_{x'})$ equals 1 in the future of $\Sigma_{x'}$. (b) The delta function $\delta_+(\sigma(x, x'))$ receives support on the forward lightcone. (c) The step function $\theta_+(-\sigma(x, x'))$ equals one inside the forward lightcone.

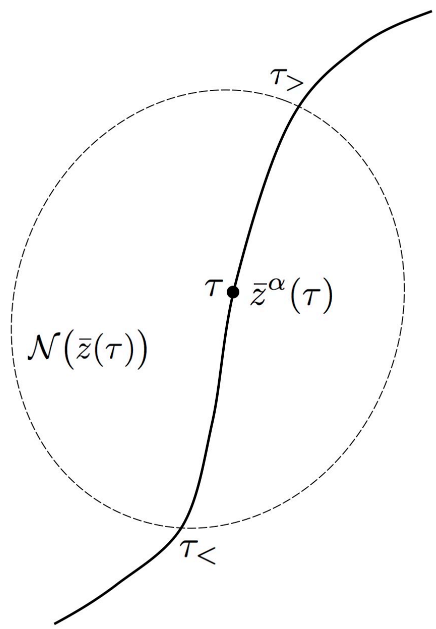

Figure 2.2: The normal convex neighborhood $\mathcal{N}$ of a point $\bar{z}^\alpha(\tau)$ on the semiclassical worldline. The boundary $\partial\mathcal{N}$ of $\mathcal{N}$ is given by the dashed line.

neighborhood (2.63) is proportional to $\theta_+(\bar{z}(\tau), \Sigma_{\bar{z}(\tau')})$. We may therefore write the self-force as

$$\vec{\bar{w}}^{\mu A} \int_{\tau_i}^{\tau_f} d\tau' \, \bar{G}^{ret}_{AB'} \bar{j}^{B'} = \vec{\bar{w}}^{\mu A} \left[\int_{\tau_i}^{\tau_<} d\tau' + \int_{\tau_<}^{\tau} d\tau' \right] \bar{G}^{ret}_{AB'} \bar{j}^{B'}. \tag{2.67}$$

Here, using the Hadamard construction for the propagator in the normal convex neighborhood, we find that

$$\begin{aligned} m\, \bar{a}^{\mu}(\tau) \; = \; \vec{\bar{w}}^{\mu A} \int_{\tau_<}^{\tau} d\tau' \Big[& U_{AB'}\big(\bar{z}^{\alpha}, \bar{z}^{\alpha\prime}\big) \Delta^{1/2}\big(\bar{z}^{\alpha}, \bar{z}^{\alpha\prime}\big) \delta\big(\sigma(\bar{z}^{\alpha}, \bar{z}^{\alpha\prime})\big) \\ & + V_{AB'}\big(\bar{z}^{\alpha}, \bar{z}^{\alpha\prime}\big) \Big] + \vec{\bar{w}}^{\mu A} \int_{\tau_i}^{\tau_<} d\tau' \, \bar{G}^{ret}_{AB'} \bar{j}^{B'} \end{aligned} \tag{2.68}$$

where we have used the fact that for two time-like separated points (e.g., on a worldline) the generalized step function equals one and the world function σ is negative in our conventions so that

$$\theta\big(-\sigma(\bar{z}^{\alpha}, \bar{z}^{\alpha'})\big) = +1. \tag{2.69}$$

We remark that the contribution to the self-force in (2.68) from the direct part of the propagator is divergent since

$$\delta\big(\sigma(\bar{z}^{\alpha}, \bar{z}^{\alpha'})\big) = \delta(\tau - \tau') \left| \frac{d\sigma}{d\tau} \right|^{-1} \sim \frac{\delta(\tau - \tau')}{\tau - \tau'}. \tag{2.70}$$

The divergence comes from the infinitely high frequency modes of the field interacting with a point-like object. To make sense of the self-force will require a method of regularization to render the divergence finite and possibly the renormalization of appropriate coupling constants in the theory.

2.4.2 Quasi-local expansion of the self-force

The presence of the $\delta(\sigma)$ term in the direct part of the retarded propagator indicates a divergence in the self-force when the two points τ and τ' coincide under the integral in (2.68). This is the usual divergence that results from considering interactions between a point-particle and arbitrarily large high frequency modes of the quantum field. To regularize this divergence we use the prescription originally developed in [32, 33] for particle-field interactions in flat spacetime. Its extension to curved spacetimes is given in [49, 50].

We introduce a mass (or momentum) scale Λ for the field such that for particle energies much lower than Λ we expect the semiclassical equations of motion (2.68) to accurately describe the particle's behavior. The divergence is regularized by smearing the direct part of the propagator through the following replacement

$$\delta\big(\sigma(x^\alpha, x'^\alpha)\big) \to \delta_\Lambda(\sigma) \equiv \theta(-\sigma)\sqrt{\frac{8}{\pi}}\,\Lambda^2 e^{-2\Lambda^4\sigma^2}. \tag{2.71}$$

The usual delta function $\delta(\sigma)$ is recovered in the limit that Λ tends to infinity. The function $\delta_\Lambda(\sigma)$ is smooth, satisfies

$$\int_{-\infty}^{\infty} d\sigma\, \delta_\Lambda(\sigma) = 1 \tag{2.72}$$

and approximates $\delta(\sigma)$ well only if $\Lambda^2\sigma \gg 1$. This inequality will not hold if σ is strictly zero so we will assume that σ is small and approaching zero while maintaining Λ^2 significantly larger than $1/\sigma$. For time-like separated points, e.g., points on a particle worldline, $\theta(-\sigma) = 1$.

Since Λ serves to provide a minimum resolution for the particle-field interaction comprising the self-force then the sharp step function $\theta_+(x, \Sigma_{x'})$ should be replaced by the smooth function $\theta_\Lambda(-s)$, defined by

$$\theta_+(\bar{z}^\alpha, \Sigma_{\bar{z}^{\alpha'}}) \to \theta_\Lambda(-s) \equiv \frac{2^{1/4}\sqrt{\pi}}{\Gamma(1/4)} \int_{-\infty}^{-s} ds' \, \delta_\Lambda(\sigma(s)) \tag{2.73}$$

when x and x' are on a particle worldline. For such a pair of points the dominant contribution to $\delta(\sigma)$ comes from those points that are nearly coincident so that the proper time difference $s \equiv \tau' - \tau$ is small but still much larger than the resolution scale $1/\Lambda$. Using results obtained later in this subsection it is easy to show that for large Λ

$$\theta_\Lambda(-s) \sim \theta(-s) + \frac{1}{2^{1/4}\Gamma(1/4)} \frac{\text{sgn}(s)}{s^3\Lambda^3} \, e^{-\Lambda^4 s^4/2} = \theta(-s) + O\big((s\Lambda)^{-3}\big) \tag{2.74}$$

and can be approximated by the first term quite well. We will therefore replace $\theta_+(\bar{z}^\alpha, \Sigma_{\bar{z}'^\alpha})$ by $\theta(\tau - \tau')$ in what follows[7].

The small values of σ that we need to consider enables us to perform a quasi-local expansion in which the self-force is expanded near coincidence for those values of proper time τ' that are near τ. In particular, the condition that $\sigma \approx 0$ implies $\tau' \approx \tau$ so that, after smearing the direct part of the propagator, we may expand the self-force in powers of $\sigma^\mu \equiv \sigma^{;\mu}(\bar{z}^\alpha, \bar{z}^{\alpha'})$, which measures the displacement along the unique geodesic connecting $\bar{z}(\tau)$ and $\bar{z}(\tau')$. We will need the quasi-local expansion of several geometric quantities. The first part of the expansion requires expanding these in powers of σ^μ. The second part translates these expansions into a power series of the proper time difference $s = \tau' - \tau$. We begin with the first part following [100, 101] and [53].

[7]We have already implicitly used this result in writing down the tail part in (2.68).

The expansion of the square root of the van Vleck determinant is

$$\Delta^{1/2}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) = 1 + \frac{1}{12}\,\bar{R}_{\alpha\beta}\,\sigma^\alpha\,\sigma^\beta - \frac{1}{24}\,\bar{R}_{\alpha\beta;\gamma}\,\sigma^\alpha\,\sigma^\beta\,\sigma^\gamma + \cdots \tag{2.75}$$

and its covariant derivative is

$$\nabla_\nu \Delta^{1/2}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) = \frac{1}{6}\,\bar{R}_{\alpha\beta}\,\sigma^\alpha{}_{;\nu}\,\sigma^\beta - \frac{1}{24}\,\big(2\bar{R}_{\nu\alpha;\beta} - \bar{R}_{\alpha\beta;\nu}\big)\,\sigma^\alpha\,\sigma^\beta + \cdots \tag{2.76}$$

where σ^μ is a shorthand here for $\sigma^{;\mu}(\bar{z}^\alpha, \bar{z}^{\alpha'})$ and $\bar{R} \equiv R(\bar{z})$, etc. We also deduce that the velocity 4-vector at τ' parallel propagated to $\bar{z}(\tau)$ has the following expansion

$$g_{\alpha\beta'}\bar{u}^{\beta'} \;=\; -\sigma_{\alpha\beta'}\bar{u}^{\beta'} - \frac{1}{6}\,\sigma^{\;\beta}_{\beta'}\,\bar{R}_{\alpha\gamma\beta\delta}\sigma^\gamma\sigma^\delta\bar{u}^{\beta'} + \cdots \tag{2.77}$$

$$g_{\alpha\gamma';\beta}\bar{u}^{\gamma'} \;=\; \frac{1}{2}\bar{u}^{\gamma'}\,g^{\;\epsilon}_{\gamma'}\bar{R}_{\alpha\epsilon\beta\delta}\sigma^\delta + \cdots \tag{2.78}$$

The second part of the quasi-local expansion entails expanding the above series in terms of the proper time difference $s = \tau' - \tau$. This requires having the quasi-local expansion of the covariant derivative of Synge's world function σ^μ, which we now derive.

The world function for a geodesic parameterized by λ is defined as

$$\sigma\big(x(\lambda), x(\lambda')\big) = -\Delta\lambda \int_{\lambda'}^{\lambda} d\lambda''\, g_{\alpha\beta}\frac{dx}{d\lambda''}\frac{dx}{d\lambda''} \tag{2.79}$$

where $\Delta\lambda = \lambda' - \lambda$. Using (C.8) its covariant derivative can be expressed as

$$\sigma^\mu\big(x(\lambda), x(\lambda')\big) = -\Delta\lambda\frac{dx^\mu}{d\lambda} \tag{2.80}$$

where $dx^\mu/d\lambda$ is the tangent vector to the geodesic at parameter value λ. Let us identify the two points $x(\lambda)$, $x(\lambda')$ of this geodesic with the two points $\bar{z}(\tau)$, $\bar{z}(\tau')$ on the semiclassical worldline,

$$\bar{z}^\mu(\tau) - \bar{z}^\mu(\tau') = x^\mu(\lambda) - x^\mu(\lambda')\;; \tag{2.81}$$

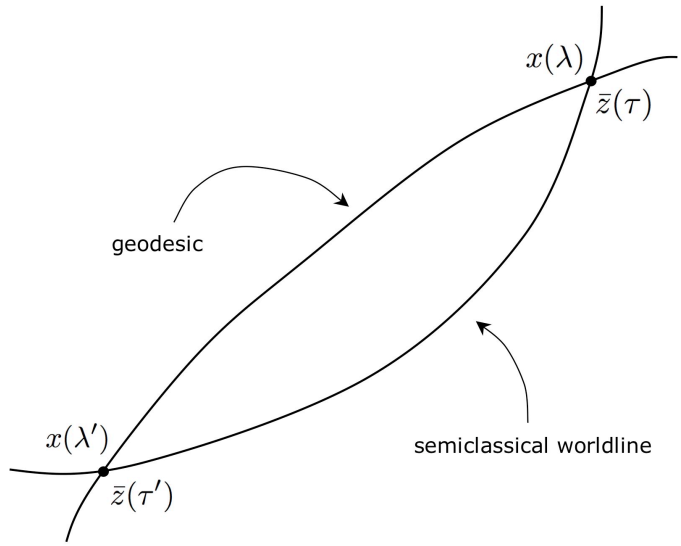

Figure 2.3: The intersection of a spacetime geodesic and the semiclassical worldline at two points.

see Fig.(2.3). Expanding the left side in terms of s gives

$$\bar{z}^\mu(\tau) - \bar{z}^\mu(\tau') = -s\dot{\bar{z}}^\mu(\tau) - \frac{s^2}{2}\ddot{\bar{z}}^\mu(\tau) - \frac{s^3}{6}\dddot{\bar{z}}^\mu(\tau) + \cdots \tag{2.82}$$

where $\dot{\bar{z}}^\mu = d\bar{z}^\mu/d\tau$. Similarly, the right side has the following expansion in terms of $\Delta\lambda$

$$x^\mu(\lambda) - x^\mu(\lambda') = -\Delta\lambda\,\dot{x}^\mu(\lambda) - \frac{(\Delta\lambda)^2}{2}\ddot{x}^\mu(\lambda) - \frac{(\Delta\lambda)^3}{6}\dddot{x}^\mu(\lambda) + \cdots \tag{2.83}$$

where $\dot{x}^\mu = dx^\mu/d\lambda$. Using (2.80) and the geodesic equation

$$\ddot{x}^\mu(\lambda) + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{2.84}$$

gives

$$x^\mu(\lambda) - x^\mu(\lambda') = \sigma^\mu + \frac{1}{2}\Gamma^\mu_{\alpha\beta}\sigma^\alpha\sigma^\beta + \frac{1}{6}\left(-\Gamma^\mu_{\alpha\beta,\gamma} + 2\Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\gamma}\right)\sigma^\alpha\sigma^\beta\sigma^\gamma + \cdots. \tag{2.85}$$

Equating the two coordinate differences (2.82) and (2.85) and rearranging terms we find the quasi-local expansion of σ^μ in terms of s,

$$\sigma^\mu\big(\bar{z}(\tau), \bar{z}(\tau')\big) = -s\bar{u}^\mu(\tau) - \frac{s^2}{2!}\,\bar{a}^\mu(\tau) - \frac{s^3}{3!}\,\frac{D\bar{a}^\mu}{d\tau} + \cdots \tag{2.86}$$

$$= -\sum_{n=1}^{\infty} \frac{s^n}{n!}\left(\frac{D}{d\tau}\right)^{n-1} \bar{u}^\mu(\tau). \tag{2.87}$$

Using the identity $2\sigma = \sigma_\mu \sigma^\mu$ we also find the quasi-local expansion of the world function

$$\sigma(\bar{z}^\alpha, \bar{z}^{\alpha'}) = -\frac{s^2}{2} - \frac{s^4}{24}\,a^2 + \cdots \tag{2.88}$$

where $\bar{a}^2 = \bar{a}_\mu \bar{a}^\mu$ is the norm of the particle's semiclassical 4-acceleration.

The smeared delta function $\delta_\Lambda(\sigma)$ is

$$\delta_\Lambda(\sigma) = \sqrt{\frac{8}{\pi}}\,\Lambda^2 e^{-\Lambda^4 s^4/2} + \cdots \tag{2.89}$$

and its covariant derivative is

$$\nabla_\mu \delta_\Lambda(\sigma) = \sigma_\mu \left(\frac{\partial\sigma}{\partial s}\right)^{-1} \frac{\partial \delta_\Lambda}{\partial s} \tag{2.90}$$

$$= \left(\bar{u}_\mu + \frac{s}{2}\,\bar{a}_\mu + \frac{s^2}{6}\,\bar{w}_\mu{}^\nu \frac{D\bar{a}_\nu}{d\tau} + \cdots\right) \frac{\partial\delta_\Lambda}{\partial s} \tag{2.91}$$

where $\bar{w}_\mu{}^\nu = g_\mu{}^\nu + \bar{u}_\mu \bar{u}^\nu$ projects vectors onto a direction orthogonal to the semiclassical 4-velocity.

These expansions in the proper time difference s can be used to expand the quantities in (2.75)-(2.78). We find for the square root of the van Vleck determinant

$$\Delta^{1/2}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) = 1 + \frac{s^2}{12}\,\bar{R}_{\alpha\beta}\bar{u}^\alpha \bar{u}^\beta + O(s^3) \tag{2.92}$$

and its covariant derivative is

$$\nabla_\mu \Delta^{1/2}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) = -\frac{s}{6}\,\bar{R}_{\mu\alpha}\bar{u}^\alpha + O(s^2). \tag{2.93}$$

The parallel transported 4-velocities in (2.77) and (2.78) are

$$g_{\alpha\beta'}\bar{u}^{\beta'} = \bar{u}_\alpha + s\,\bar{a}_\alpha + \frac{s^2}{2}\frac{D\bar{a}_\alpha}{d\tau} + O(s^3) \tag{2.94}$$

$$g_{\alpha\gamma';\beta}\bar{u}^{\gamma'} = -\frac{s}{2}\,R_{\alpha\gamma\beta\delta}\bar{u}^\gamma\bar{u}^\delta + O(s^2), \tag{2.95}$$

respectively.

In the next section, we obtain the equations of motion describing the self-force on a particle interacting separately with a scalar, electromagnetic and gravitational waves in a curved background spacetime. We use the Hadamard construction of the retarded propagator and the quasi-local expansions developed in this section to regularize the ultraviolet divergence in the self-force.

2.4.3 Scalar field

Consider a massless scalar field $\Phi_A = \phi(x)$ propagating in a curved background spacetime with metric $g_{\mu\nu}$. The action describing this field is

$$S[\phi] = \frac{1}{2}\int d^4x\, g^{1/2}\Big(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \xi R\phi^2\Big) \tag{2.96}$$

where ξ is a constant that couples the field to the background curvature. When $\xi = 0$ the field is said to be minimally coupled and when

$$\xi = \frac{1}{4}\frac{d-2}{d-1} \tag{2.97}$$

the field is said to be conformally coupled. The particle-field interaction is taken to be in the form of a monopole coupling

$$S_{int}[z,\phi] = q\int d\tau\,\phi\big(z(\tau)\big) \tag{2.98}$$

$$= \int d^4x\, g^{1/2} j(x;z)\phi(x) \tag{2.99}$$

where the current density is defined as

$$j(x;z) = q \int d\tau \, \frac{\delta^4\big(x - z(\tau)\big)}{g^{1/2}}. \tag{2.100}$$

The variation of the coarse-grained effective action (2.53) gives the semiclassical equations of motion for the particle (2.58). We find for this particular example,

$$m_o \bar{a}^\mu(\tau) = q\, \vec{w}^\mu[\bar{z}^\alpha] \int_{\tau_i}^{\tau_f} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \tag{2.101}$$

where the operator $\vec{w}^\mu$ is defined using (2.62)

$$\vec{w}^\mu[\bar{z}^\alpha] = q\Big(a^\mu + w^{\mu\nu}[\bar{z}^\alpha]\nabla_\nu\Big) \tag{2.102}$$

and

$$w^{\mu\nu}[\bar{z}^\alpha] = g^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu \tag{2.103}$$

projects vectors onto a direction orthogonal to the semiclassical 4-velocity. Using these expressions we can write the equations of motion as

$$\left[m_o - q^2 \int_{\tau_i}^{\tau_f} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\right] \bar{a}^\mu(\tau) = q^2 w^{\mu\nu}[\bar{z}^\alpha] \nabla_\nu \int_{\tau_i}^{\tau_f} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big). \tag{2.104}$$

The Hadamard construction for the retarded propagator

$$D_{ret}(x,x') = \Delta^{1/2}(x,x')\delta_+\big(\sigma(x,x')\big) + V(x,x')\theta_+\big(-\sigma(x,x')\big) \tag{2.105}$$

and (2.68) allow for the equations of motion to be written as

$$\begin{aligned} &\left[m_o - q^2 \int_{\tau_<}^{\tau} d\tau' \Big(\bar{\Delta}^{1/2}\delta(\bar{\sigma}) + \bar{V}\Big) - q^2 \int_{\tau_i}^{\tau_<} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\right] \bar{a}^\mu(\tau) \\ &\quad = q^2 w^{\mu\nu}[\bar{z}^\alpha] \nabla_\nu \left[\int_{\tau_<}^{\tau} d\tau' \Big(\bar{\Delta}^{1/2}\delta(\bar{\sigma}) + \bar{V}\Big) + \int_{\tau_i}^{\tau_<} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\right]. \end{aligned} \tag{2.106}$$

The integrals over the direct part of the propagator are divergent, which we regularize by smearing $\delta(\sigma)$ via the replacement in (2.71). Focusing on the contributions from the direct part we see that there are two such diverging terms. The first comes from the self-force

$$I_1^\mu \equiv q^2 w^{\mu\nu}[\bar{z}^\alpha] \nabla_\nu \int_{\tau_<}^{\tau} d\tau' \, \bar{\Delta}^{1/2} \delta_\Lambda(\bar{\sigma}). \tag{2.107}$$

and the second comes from the integral of the retarded propagator, i.e. the retarded field $\phi_{ret} = D_{ret} \cdot j$,

$$I_2 \equiv -q^2 \int_{\tau_<}^{\tau} d\tau' \, \bar{\Delta}^{1/2} \delta_\Lambda(\bar{\sigma}). \tag{2.108}$$

Passing the covariant derivative through the first integral we find that

$$I_1^\mu = q^2 w^{\mu\nu}[\bar{z}^\alpha] \left(\big[\bar{\Delta}^{1/2} \delta_\Lambda(\bar{\sigma})\big] \big[\nabla_\nu (\tau - \tau')\big] + \int_{\tau_<}^{\tau} d\tau' \, \nabla_\nu \big(\bar{\Delta}^{1/2} \delta_\Lambda(\bar{\sigma})\big) \right) \tag{2.109}$$

where the $[\cdots]$ denotes the coincidence limit[8] of the quantity inside the brackets. It should be clear from context when we are referring to a coincidence limit and when the square brackets are simply delimiters. From (2.87) it follows that

$$\big[\nabla_\nu (\tau - \tau')\big] = -\big[\nabla_\nu (\bar{\sigma}^\alpha \bar{u}_\alpha)\big] = -\bar{u}_\nu \tag{2.110}$$

thereby implying that

$$w^{\mu\nu}[\bar{z}^\alpha] \big[\nabla_\nu (\tau - \tau')\big] = 0. \tag{2.111}$$

By writing (2.109) as

$$I_1^\mu = q^2 w^{\mu\nu}[\bar{z}^\alpha] \int_{\tau_<}^{\tau} d\tau' \, \Big\{ (\nabla_\nu \bar{\Delta}^{1/2}) \delta_\Lambda(\bar{\sigma}) + \bar{\Delta}^{1/2} \nabla_\nu \delta_\Lambda(\bar{\sigma}) \Big\} \tag{2.112}$$

[8] The coincidence limit is the limit $\tau' \to \tau$ along the unique geodesic connecting $\bar{z}(\tau')$ to $\bar{z}(\tau)$.

and using the quasi-local expansions developed in Section 2.4.2 we find that

$$I_1^\mu = \frac{q^2}{2}\, g_{(1)}(r)\bar{a}^\mu + \frac{q^2}{3}\, g_{(2)}(r)\, w^{\mu\nu}[\bar{z}^\alpha]\frac{D\bar{a}_\nu}{d\tau} + \frac{q^2}{6}\, c_{(1)}(r)\, w^{\mu\nu}[\bar{z}^\alpha]\bar{R}_{\nu\alpha}\bar{u}^\alpha + O(\Lambda^{-1}) \tag{2.113}$$

where $r \equiv \tau - \tau_<$ is the elapsed proper time since the intersection of the particle worldline with the normal convex neighborhood of $\bar{z}(\tau)$.

The r-dependent coefficients are defined by

$$\begin{aligned} c_{(n)}(r) &= -\int_{\tau_<}^{\tau} d\tau' \, \frac{s^n}{n!}\, \delta_\Lambda\big(\sigma(s)\big) \\ &= (-1)^{n+1}\frac{2^{(n-1)/4}}{\pi^{1/2} n!}\, \Lambda^{1-n}\, \gamma\left(\frac{1+n}{4}, \frac{r^4\Lambda^4}{2}\right), \end{aligned} \tag{2.114}$$

which is divergent for $n = 0$, and

$$\begin{aligned} g_{(n)}(r) &= \int_{\tau_<}^{\tau} d\tau' \, \frac{s^n}{n!}\, \frac{\partial\delta_\Lambda\big(\sigma(s)\big)}{\partial s} \\ &= (-1)^{n}\frac{2^{(n+6)/4}}{\pi^{1/2} n!}\, \Lambda^{2-n}\, \gamma\left(1+\frac{n}{4}, \frac{r^4\Lambda^4}{2}\right), \end{aligned} \tag{2.115}$$

which diverges for $n = 0, 1$. In these expressions,

$$\gamma(a,b) \equiv \Gamma(a) - \Gamma(a,b) \tag{2.116}$$

where $\Gamma(a,b)$ is the incomplete gamma function. As Λ or the elapsed proper time r goes to infinity it follows that $\gamma(a, r^4\Lambda^4/2) \to \Gamma(a)$. These coefficients are normalized so that both $c_{(1)}(r)$ and $g_{(2)}(r)$ approach 1 in this limit. On an elapsed time scale $r \gtrsim 1/\Lambda$ the coefficients for any n equal approximately their limiting value. We remark also that for each n these coefficients vanish at $\tau = \tau_<$ when $r = 0$ since $\Gamma(a,0) = \Gamma(a)$. More will be said below concerning these properties and their

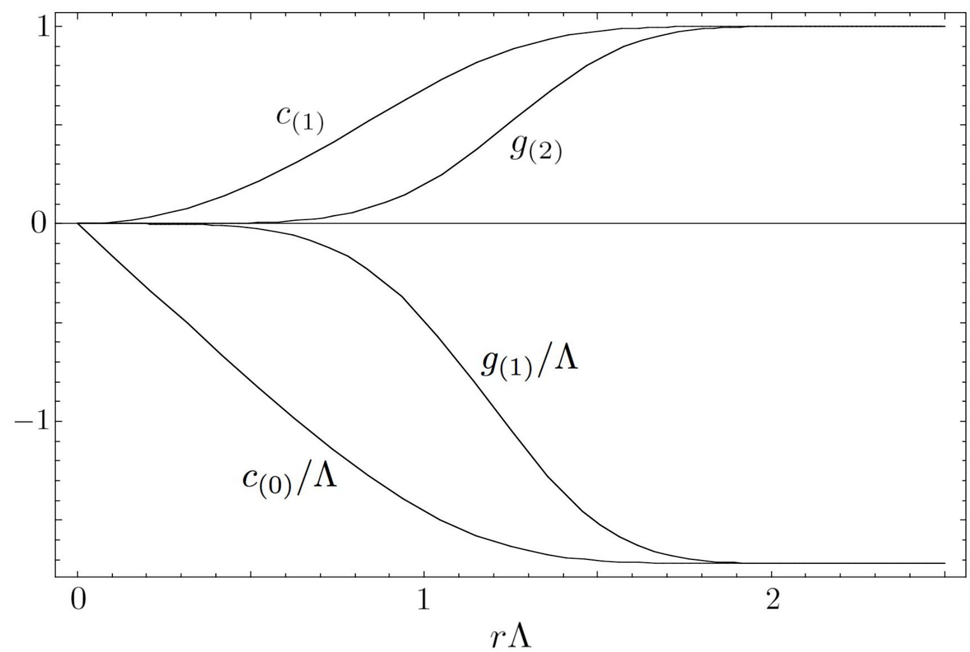

Figure 2.4: Time-dependence of the first few coefficients appearing in (2.114) and (2.115). The functions $c_{(0)}$ and $g_{(1)}$ have been divided through by Λ so that they can be displayed on the same plot with $c_{(1)}$ and $g_{(2)}$.

implication for the validity of the quasi-local expansion. See Fig.(2.4) for a plot of the first few coefficients.

The terms in (2.113) proportional to inverse powers of Λ are irrelevant in the sense that these vanish when the limit $\Lambda \to \infty$ is taken. We will therefore ignore such terms since they can be made arbitrarily small with a sufficiently large value for Λ. Only those terms that diverge as a positive power of Λ and are marginal $\sim \Lambda^0$ will be relevant for obtaining finite semiclassical equations of motion.

The second divergence (2.108), which appears in the effective mass of the particle, can be found following similar steps. There is no derivative that operates on the integral so we may expand the integrand directly using the quasi-local expansions to find that

$$I_2 = q^2 c_{(0)}(r) \tag{2.117}$$

plus irrelevant terms proportional to inverse powers of Λ that do not give a contribution as $\Lambda \to \infty$.

Taken together (2.113) and (2.117) imply that the mass of the particle is shifted by an infinite amount

$$\delta m \equiv -q^2 c_{(0)}(r) + \frac{q^2}{2}\, g_{(1)}(r). \tag{2.118}$$

Absorbing this divergence into the bare mass m_o renders the particle's mass finite so that the renormalized mass

$$m \equiv m_o - \delta m \tag{2.119}$$

is time-dependent, which is typical of renormalization in initial value problems.

However, given that the quasi-local expansion is valid for $r\Lambda \gg 1$ we see that the renormalized mass is effectively constant within the normal convex neighborhood of $\bar{z}(\tau)$.

The semiclassical particle equations of motion can now be written as

$$\begin{aligned}\left[m - q^2 \int_{\tau_<}^{\tau} d\tau' \, \bar{V} - q^2 \int_{\tau_i}^{\tau_<} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\right] \bar{a}^\mu(\tau) \\ = \frac{q^2}{3}\, g_{(2)}(r)\, w^{\mu\nu}[\bar{z}^\alpha] \frac{D\bar{a}_\nu}{d\tau} + \frac{q^2}{6}\, c_{(1)}(r)\, w^{\mu\nu}[\bar{z}^\alpha] \bar{R}_{\nu\alpha}\bar{u}^\alpha \\ + q^2 w^{\mu\nu}[\bar{z}^\alpha] \left[\int_{\tau_<}^{\tau} d\tau' \, \bar{V}_{;\nu} + \int_{\tau_i}^{\tau_<} d\tau' \, D^{ret}_{;\nu}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\right] \end{aligned} \tag{2.120}$$

where we have used (2.111) in the last line. Equivalently, we write this in a more useful form that does not depend on the decomposition of the propagator into contributions inside and outside of the normal convex neighborhood of $\bar{z}(\tau)$. Namely,

$$\begin{aligned}\left[m - q^2 \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D_{ret}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\right] \bar{a}^\mu(\tau) \\ = \frac{q^2}{3}\, g_{(2)}(r)\, w^{\mu\nu}[\bar{z}^\alpha] \frac{D\bar{a}_\nu}{d\tau} + \frac{q^2}{6}\, c_{(1)}(r)\, w^{\mu\nu}[\bar{z}^\alpha] \bar{R}_{\nu\alpha}\bar{u}^\alpha \\ + q^2 w^{\mu\nu}[\bar{z}^\alpha] \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D^{ret}_{;\nu}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \end{aligned} \tag{2.121}$$

where the integrals of the retarded propagator are evaluated in the limit that $\tau' \to \tau$ so as to avoid the divergence that renormalizes the particle's mass. Eq.(2.121) is the main result of this section, which was first derived using axiomatic techniques by Quinn [22]. We remark that the time derivative of the 4-acceleration contributes to the local radiation reaction while the tail part of the retarded propagator (the contribution of which is integrated up until an infinitesimal before the divergence is encountered) accounts for the non-local interactions with the radiated field emitted at some proper time prior to τ. There also appears a local conservative force proportional to $\bar{R}_{\alpha\beta}\bar{u}^\beta$ that vanishes in a vacuum background spacetime.

We remark that the inertia of the point charge is time-dependent

$$\frac{dm_{eff}}{d\tau} = q^2 \bar{u}^\mu \lim_{\epsilon \to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D^{ret}_{;\mu}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \tag{2.122}$$

and depends on the entire past history of the particle's interaction with the scalar field. This feature is observed in [87] wherein the time-dependence of the particle's effective mass is explicitly calculated for the particle and scalar field in de Sitter and certain FRW cosmologies. The monopole particle-field coupling that we have considered in this section allows for the particle's rest mass to be transferred to and from the field since there is no symmetry that guarantees that the particle's mass is a conserved quantity in time. In fact, we can see this from the action for the particle-field total system since

$$S[z, \phi] = \int d\tau \Big[- m_o + q\phi\big(z(\tau)\big)\Big] + S[\phi] \tag{2.123}$$

where the (divergent) effective mass is clearly given by

$$m_{eff}(\tau) = m_o - q\phi\big(z(\tau)\big), \tag{2.124}$$

which agrees with (2.121) after renormalizing the particle's bare mass by the divergence from the direct part of the field.

Detweiler and Whiting [52] also obtain the finite equations of motion in (2.121) by decomposing the covariant derivative of the retarded field $\phi_{ret} = D_{ret} \cdot j$ into a singular piece ϕ^S_μ, containing the diverging contribution that renormalizes the mass, and a regular piece ϕ^R_μ, which contributes to the self-force and is regular on the worldline. A more detailed discussion of their technique is given in [53]. From the previous section we can construct these quantities within our regularization scheme to find that for $\Lambda \to \infty$

$$\phi^S_\mu = -q\sqrt{\frac{8}{\pi}}\,\Lambda^2\,\bar{u}_\mu + q\left(\frac{1}{2}\,g_{(1)}(r) - c_{(0)}(r)\right)\bar{a}_\mu \tag{2.125}$$

$$\phi^R_\mu = -q\,\frac{1-6\xi}{12}\,R(\bar{z})\,\bar{u}_\mu + \frac{q}{3}\,\frac{D\bar{a}_\mu}{d\tau} + \frac{q}{6}\,R_{\mu\alpha}(\bar{z})\,\bar{u}^\alpha + \phi^{tail}_\mu(\bar{z}) \tag{2.126}$$

where

$$\phi^{tail}_\mu = q\lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\,D^{ret}_{;\mu}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \tag{2.127}$$

is the (history-dependent) tail part of the scalar field. Notice that the first term of the singular part does not contribute to renormalizing any physical parameters (at the level of the equations of motion) since ϕ^S_μ is projected onto a direction orthogonal to $\bar{u}_\mu$. Likewise, the first term of the regular part does not contribute to the self-force on the particle nor is the regular part equal to the contribution from the tail, in general.

In a flat spacetime, the tail term in (2.121) vanishes since there is no curvature available to bend the wavefronts of the past-emitted radiation back onto the particle at the present time. As such, (2.121) reduces to the familiar Abraham-Lorenz-Dirac equation for a scalar charged particle interacting with a scalar field

$$m\,\bar{a}^\mu = \frac{q^2}{3}\,w^{\mu\nu}[\bar{z}^\alpha]\,\frac{d\bar{a}_\nu}{d\tau}. \tag{2.128}$$

This equation was derived in the open quantum system formalism in [33].

2.4.4 Electromagnetic field

In this section we will study the dynamics of a point charge e interacting with a quantum vector field $\Phi_A(x) = A_\mu(x)$ in a curved background spacetime. The actions describing the dynamics of the field (environment) and its interaction with the particle (system) are

$$\begin{aligned} S[A_\mu] &= -\frac{1}{4}\int d^4x\, g^{1/2} F_{\mu\nu}F^{\mu\nu} \\ S_{int}[z, A_\mu] &= \int d^4x\, g^{1/2} j^\mu(x;z) A_\mu(x) \end{aligned} \tag{2.129}$$

where the current density is

$$j^\mu(x;z) = e\int d\tau\, \frac{\delta^4(x-z)}{g^{1/2}}\, g^\mu{}_\alpha(x,z)\, u^\alpha \tag{2.130}$$

and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the antisymmetric field strength tensor. Electromagnetism is a gauge theory so we must choose a gauge in order to construct a well-defined retarded propagator. We choose the Lorentz gauge so that the gauge-fixing action is

$$S_{gf}[A_\mu] = -\frac{1}{2}\int d^4x\, g^{1/2} G^2 \tag{2.131}$$

where $G = \nabla_\mu A^\mu$ is the gauge-fixing function.

If the particle worldline is sufficiently decohered and if the particle's own quantum mechanical fluctuations are smeared over a sufficiently small length scale then the semiclassical equations of motion for the particle (2.58) follow by varying the coarse-grained effective action (2.53)

$$m_o \bar{a}^\mu = e\, \vec{w}^{\mu\alpha}[\bar{z}] \int_{\tau_i}^{\tau_f} d\tau'\, D^{ret}_{\alpha\beta'}\left(\bar{z}^\alpha, \bar{z}^{\alpha'}\right) \bar{u}^{\beta'}(\tau') \tag{2.132}$$

where the operator $\vec{w}^{\mu\alpha}$ is derived from the functional derivative of the current density j^α integrated over a suitable test function

$$\frac{\delta}{\delta z_\mu(\tau)} \int d^4x\, g^{1/2} j^\alpha(x;z) f(x) = \vec{w}^{\mu\alpha}[z]\, f(z) \tag{2.133}$$

and equals

$$\vec{w}^{\mu\alpha}[z] = -e\, w^{\mu\alpha\beta}[z]\nabla_\beta = -2e\, g^{\mu[\alpha}(z)\, u^{\beta]}\nabla_\beta \tag{2.134}$$

where $T^{[ab]} = \frac{1}{2}(T^{ab} - T^{ba})$.

Hadamard's construction (2.63) implies the following form for the retarded propagator of the vector field in the Lorentz gauge

$$D^{ret}_{\alpha\beta'}(x,x') \;=\; g_{\alpha\beta'}(x,x')\Delta^{1/2}(x,x')\delta_+\big(\sigma(x,x')\big) + V_{\alpha\beta'}(x,x')\theta_+\big(-\sigma(x,x')\big) \tag{2.135}$$

where $g_{\alpha\beta'}(x,x')$ is the bi-vector that parallel transports tensors along the unique geodesic connecting x and x'. This representation for the propagator in the normal convex neighborhood of x allows us to write the semiclassical equations of motion (2.132) as

$$\begin{aligned} m_0\bar{a}^\mu(\tau) \;=\; & -e^2\, w^{\mu\alpha\beta}[\bar{z}^\alpha]\nabla_\beta \bigg[\int_{\tau_<}^{\tau} d\tau' \left(g_{\alpha\gamma'}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\delta(\bar{\sigma}) + \bar{V}_{\alpha\gamma'}\bar{u}^{\gamma'} \right) \\ & + \int_{\tau_i}^{\tau_<} d\tau'\, D^{ret}_{\alpha\gamma'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\bar{u}^{\gamma'} \bigg] \end{aligned} \tag{2.136}$$

The direct part of the retarded propagator gives rise to a divergence when the two points are light-like separated. The self-force in (2.136) requires the retarded Green's function to be evaluated along the particle's trajectory, which is time-like, so that the only contribution to the self-force from the direct part occurs at coincidence, when the two points are equal.

Introduce a regulator Λ through the replacement (2.71) such that for particle energies much lower than this scale an effective description of the particle dynamics can be given without recourse to information about the high energy physics that is being ignored. The regularized direct part of the self-force is

$$I_3^\mu \equiv -e^2 w^{\mu\alpha\beta}[\bar{z}^\alpha]\nabla_\beta \int_{\tau_<}^{\tau} d\tau'\, g_{\alpha\gamma'}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma}) \tag{2.137}$$

Passing the covariant derivative through the integral gives

$$\begin{aligned} I_3^\mu &= -e^2 w^{\mu\alpha\beta}[\bar{z}^\alpha]\bigg(\Big[g_{\alpha\gamma'}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma})\Big]\Big[\nabla_\alpha(\tau-\tau')\Big] \\ &\qquad + \int_{\tau_<}^{\tau} d\tau'\nabla_\alpha\Big(g_{\alpha\gamma'}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma})\Big)\bigg) \end{aligned} \tag{2.138}$$

From (2.110) and the coincident limits $\tau' \to \tau$ (equivalently, letting $s \to 0$ in the quasi-local expansions in Section 2.4.2) of the quantities in the first term of I_3^μ we find that

$$-e^2 w^{\mu\alpha\beta}[\bar{z}^\alpha]\Big[g_{\alpha\gamma'}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma})\Big]\Big[\nabla_\alpha(\tau-\tau')\Big] = e^2 w^{\mu\alpha\beta}\bar{u}_\alpha\bar{u}_\beta = 0. \tag{2.139}$$

This gives for the divergent part of the self-force

$$\begin{aligned} I_3^\mu &= -e^2 w^{\mu\alpha\beta}[\bar{z}^\alpha]\int_{\tau_<}^{\tau} d\tau'\Big\{g_{\alpha\gamma';\alpha}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma}) + g_{\alpha\gamma'}\bar{u}^{\gamma'}\big(\nabla_\alpha\bar{\Delta}^{1/2}\big)\delta_\Lambda(\bar{\sigma}) \\ &\qquad + g_{\alpha\gamma'}\bar{u}^{\gamma'}\bar{\Delta}^{1/2}\nabla_\alpha\delta_\Lambda(\bar{\sigma})\Big\} \end{aligned} \tag{2.140}$$

and can be integrated using the quasi-local expansions developed in Section 2.4.2. We find that

$$I_3^\mu = e^2 g_{(1)}(r)\,\bar{a}^\mu + \frac{2e^2}{3}\,g_{(2)}(r)\,w^{\mu\alpha}[\bar{z}^\alpha]\frac{D\bar{a}_\alpha}{d\tau} - \frac{e^2}{6}\,c_{(1)}(r)\,w^{\mu\alpha}[\bar{z}^\alpha]\bar{R}_{\alpha\beta}\bar{u}^\beta \tag{2.141}$$

where the time-dependent coefficients $g_{(1)}$, $g_{(2)}$ and $c_{(1)}$ are defined in (2.114) and (2.115) and we have dropped irrelevant terms that vanish in the limit $\Lambda \to \infty$.

Putting this into the semiclassical equations of motion we find that (2.136) becomes

$$\begin{aligned}\Big[m_o - e^2 g_{(1)}(r)\Big]\bar{a}^\mu(\tau) \;=\; & \frac{2e^2}{3}\, g_{(2)}(r)\, w^{\mu\alpha}[\bar{z}^\alpha]\frac{D\bar{a}_\alpha}{d\tau} - \frac{e^2}{6}\, c_{(1)}(r)\, w^{\mu\alpha}[\bar{z}^\alpha]\bar{R}_{\alpha\beta}\bar{u}^\beta \\ & -e^2\, w^{\mu\alpha\beta}[\bar{z}^\alpha]\nabla_\beta\left[\int_{\tau_<}^{\tau} d\tau'\, \bar{V}_{\alpha\gamma'}\bar{u}^{\gamma'} + \int_{\tau_i}^{\tau_<} d\tau'\, D^{ret}_{\alpha\gamma'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big)\bar{u}^{\gamma'}\right]\end{aligned} \tag{2.142}$$

which shows that the mass of the point charge must be renormalized because of the infinite and time-dependent shift

$$\delta m = e^2 g_{(1)}(r) \tag{2.143}$$

that implies defining

$$m = m_o - \delta m \tag{2.144}$$

as the physical rest mass for the charge.

Passing the covariant derivative through the first integral in (2.136) gives

$$\begin{aligned}-e^2\, w^{\mu\alpha\beta}[\bar{z}^\alpha]\nabla_\beta \int_{\tau_<}^{\tau} d\tau'\, \bar{V}_{\alpha\gamma'}\bar{u}^{\gamma'} \;=\; & -e^2\, w^{\mu\alpha\beta}[\bar{z}^\alpha]\big[\bar{V}_{\alpha\gamma'}\bar{u}^{\gamma'}\big]\big[\nabla_\beta(\tau-\tau')\big] \\ & -e^2\, w^{\mu\alpha\beta}[\bar{z}^\alpha]\int_{\tau_<}^{\tau} d\tau'\, \bar{V}_{\alpha\gamma';\beta}\bar{u}^{\gamma'}\end{aligned} \tag{2.145}$$

where the coincident limit of the smooth function $V_{\alpha\gamma'}$ is given by [53]

$$[V_{\alpha\gamma'}] = -\frac{1}{2}R_{\alpha\gamma} + \frac{1}{12}\, g_{\alpha\gamma}R \tag{2.146}$$

from which we deduce that

$$-e^2\, w^{\mu\alpha\beta}[\bar{z}^\alpha]\big[\bar{V}_{\alpha\gamma'}\bar{u}^{\gamma'}\big]\big[\nabla_\alpha(\tau-\tau')\big] \;=\; \frac{e^2}{2}\, w^{\mu\alpha}[\bar{z}^\alpha]\bar{R}_{\alpha\beta}\bar{u}^\beta \tag{2.147}$$

after also using (2.110). Putting this into the semiclassical equations of motion (2.142) gives

$$m\bar{a}^{\mu}(\tau) = \frac{2e^2}{3}\, g_{(2)}(r)\, w^{\mu\alpha}[\bar{z}^{\alpha}]\frac{D\bar{a}_{\alpha}}{d\tau} + \frac{e^2}{6}\big(3 - c_{(1)}(r)\big) w^{\mu\alpha}[\bar{z}^{\alpha}]\bar{R}_{\alpha\beta}\bar{u}^{\beta}$$
$$-e^2\, w^{\mu\alpha\beta}[\bar{z}^{\alpha}]\left[\int_{\tau_<}^{\tau} d\tau'\, \bar{V}_{\alpha\gamma';\beta}\bar{u}^{\gamma'} + \int_{\tau_i}^{\tau_<} d\tau'\, D^{ret}_{\alpha\gamma';\beta}\big(\bar{z}^{\alpha}, \bar{z}^{\alpha'}\big)\bar{u}^{\gamma'}\right]. \quad (2.148)$$

It is useful to write this in a form that does not depend on the contributions of the propagator that come from propagation in the normal convex neighborhood at $\bar{z}(\tau)$ or outside of this region. To this end we may write

$$m\bar{a}^{\mu}(\tau) = \frac{2e^2}{3}\, g_{(2)}(r)\, w^{\mu\alpha}[\bar{z}^{\alpha}]\frac{D\bar{a}_{\alpha}}{d\tau} + \frac{e^2}{6}\big(3 - c_{(1)}(r)\big) w^{\mu\alpha}[\bar{z}^{\alpha}]\bar{R}_{\alpha\beta}\bar{u}^{\beta}$$
$$-e^2\, w^{\mu\alpha\beta}[\bar{z}^{\alpha}] \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\, D^{ret}_{\alpha\gamma';\beta}\big(\bar{z}^{\alpha}, \bar{z}^{\alpha'}\big)\bar{u}^{\gamma'}. \quad (2.149)$$

This is the main result of this section. As with the scalar field example we see that there is a local radiation reaction term proportional to the derivative of the acceleration as well as a history-dependent contribution to the self-force that depends on the entire past history of the particle and the field. There is also a local conservative forcing term proportional to the Ricci curvature tensor.

In the limit that $\Lambda \to \infty$ we recover the equations of motion first derived in [23] and [27], namely,

$$m\bar{a}^{\mu}(\tau) = \frac{2e^2}{3}\, w^{\mu\alpha}[\bar{z}^{\alpha}]\frac{D\bar{a}_{\alpha}}{d\tau} + \frac{e^2}{3}\, w^{\mu\alpha}[\bar{z}^{\alpha}]\bar{R}_{\alpha\beta}\bar{u}^{\beta}$$
$$-e^2\, w^{\mu\alpha\beta}[\bar{z}^{\alpha}] \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\, D^{ret}_{\alpha\gamma';\beta}\big(\bar{z}^{\alpha}, \bar{z}^{\alpha'}\big)\bar{u}^{\gamma'} \quad (2.150)$$

and is a generalization to curved spacetime of the well-known Abraham-Lorenz-Dirac equations, which in flat spacetime reads

$$m\bar{a}^{\mu}(\tau) = \frac{2e^2}{3}\, w^{\mu\alpha}[\bar{z}^{\alpha}]\frac{D\bar{a}_{\alpha}}{d\tau} \quad (2.151)$$

and describes purely local radiation reaction on the motion of the particle.

2.4.5 Linear metric perturbations

In this section we study the motion of a small point mass m_o moving through a linearized quantum metric perturbation field with

$$\Phi_A(x) = \bar{h}_{\mu\nu}(x) \equiv \kappa h_{\mu\nu}(x) \tag{2.152}$$

in a curved vacuum background spacetime with $R_{\mu\nu} = 0$. The constant $\kappa^2 = 32\pi$ is a convenient normalization for the metric perturbation. The field and interaction actions describing the particle-field dynamics are

$$S[\bar{h}_{\mu\nu}] = \frac{1}{2}\int d^4x\, g^{1/2}\left[2h_{\alpha\beta;\gamma}h^{\alpha\gamma;\beta} - h_{\alpha\beta;\gamma}h^{\alpha\beta;\gamma} - 2h_{;\alpha}\left(h^{\alpha\beta}{}_{;\beta} - \frac{1}{2}\,h^{;\alpha}\right)\right] \tag{2.153}$$

$$S_{int}[z, \bar{h}_{\mu\nu}] = \frac{\kappa}{2}\int d^4x\, g^{1/2} h_{\mu\nu}T^{\mu\nu}(x;z) \tag{2.154}$$

where

$$T^{\mu\nu}(x;z) = m_o \int d\tau\, \frac{\delta^4\big(x - z(\tau)\big)}{g^{1/2}}\, g^{\mu}{}_{\alpha}(x,z)g^{\nu}{}_{\beta}(x,z)u^{\alpha}u^{\beta} \tag{2.155}$$

is the particle's stress-energy tensor. It will be convenient to define the current density $j^{\mu\nu}$ in terms of the stress tensor as

$$j^{\mu\nu}(x;z) = \frac{\kappa}{2}\,T^{\mu\nu}(x;z) \tag{2.156}$$

so that the interaction is given in the standard form

$$S_{int}[z, \bar{h}_{\mu\nu}] = \int d^4x\, g^{1/2} j^{\mu\nu}(x;z)h_{\mu\nu}(x) \tag{2.157}$$

that we have been using throughout this Chapter.

It is assumed that the leading order particle motion describes a geodesic on the background vacuum spacetime so that the acceleration the particle experiences from interactions with the metric perturbations is of the order of the (infinitesimally small) mass m_o. This assumption will be used repeatedly throughout the remainder of this example.

We choose the Lorenz gauge for the trace-reversed metric perturbation $\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h$ using the gauge-fixing action

$$S_{gf}[h_{\mu\nu}] = - \int d^4x \, g^{1/2} G^\mu G_\mu \tag{2.158}$$

where $G_\mu = \nabla^\nu \psi_{\mu\nu}$ is the gauge-fixing function.

If the quantum field fluctuations of the metric perturbations provide a strong enough mechanism for decoherence then varying the coarse-grained effective action (2.53) gives the semiclassical equations of motion for the particle worldline (2.58), which are

$$m_o \bar{a}^\mu(\tau) = \frac{1}{2} \kappa m_o \vec{w}^{\mu\alpha\beta}[\bar{z}^\alpha] \int_{\tau_i}^{\tau_f} d\tau' \, D^{ret}_{\alpha\beta\gamma'\delta'}\left(\bar{z}^\alpha, \bar{z}^{\alpha'}\right) \bar{u}^{\gamma'} \bar{u}^{\delta'} \tag{2.159}$$

where the operator $\vec{w}^{\mu\alpha\beta}$ is computed from the variation of the current density $j^{\mu\nu}$ with respect to the worldline coordinates

$$\frac{\delta}{\delta z_\mu(\tau)} \int d^4x \, g^{1/2} j^{\alpha\beta}(x; z) f(x) = \vec{w}^{\mu\alpha\beta}[z] f(z)$$

where $f(x)$ is a suitable function and

$$\begin{aligned} \vec{w}^{\mu\alpha\beta} &= \kappa m_o w^{\mu\alpha\beta\gamma} \nabla_\gamma = \kappa m_o \left(\frac{1}{2} u^\alpha u^\beta w^{\mu\gamma} - w^{\mu(\alpha} u^{\beta)} u^\gamma \right) \nabla_\gamma & (2.160) \\ &= \frac{1}{2} \kappa m_o \left(-g^{\mu\alpha} u^\beta u^\gamma + g^{\mu\gamma} u^\alpha u^\beta - g^{\mu\beta} u^\alpha u^\gamma - u^\mu u^\alpha u^\beta u^\gamma \right) \nabla_\gamma . & (2.161) \end{aligned}$$

As before, the quantity $w^{\mu\alpha} = g^{\mu\alpha} + u^\mu u^\alpha$ projects vectors in the direction orthogonal to u^α.

It is more convenient to use the retarded Green's function associated with the trace-reversed perturbation $\psi_{\mu\nu}$ rather than the original metric perturbation $h_{\mu\nu}$ itself since Hadamard's construction of the retarded propagator (2.63) can be applied directly to the former. In terms of $\psi_{\mu\nu}$ then, (2.159) becomes

$$m_o \bar{a}^\mu = \frac{1}{2}\kappa m_o \vec{w}^{\mu\alpha\beta}[\bar{z}] \int_{\tau_i}^{\tau_f} d\tau' \Bigg(\check{D}^{ret}_{\alpha\beta\gamma'\delta'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) - \frac{1}{2}\, g_{\alpha\beta} g^{\rho\sigma} \check{D}^{ret}_{\rho\sigma\gamma'\delta'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \Bigg) \bar{u}^{\gamma'} \bar{u}^{\delta'}. \tag{2.162}$$

Using the tensor

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}\Big(g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta} \Big), \tag{2.163}$$

which relates $h_{\alpha\beta}$ and $\psi_{\alpha\beta}$ through

$$h_{\alpha\beta} = P_{\alpha\beta}{}^{\gamma\delta} \psi_{\gamma\delta}, \tag{2.164}$$

we see that this can be written more compactly as

$$m_o \bar{a}^\mu = \frac{1}{2}\kappa m_o P^{\rho\sigma}{}_{\alpha\beta} \vec{w}^{\mu\alpha\beta}[\bar{z}] \int_{\tau_i}^{\tau_f} d\tau'\, \check{D}^{ret}_{\rho\sigma\gamma'\delta'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \bar{u}^{\gamma'} \bar{u}^{\delta'}. \tag{2.165}$$

Hadamard's construction for the retarded propagator (for the trace-reversed metric perturbations) implies [53]

$$\check{D}^{ret\,\gamma'\delta'}_{\alpha\beta}(x, x') = 2 g_{(\alpha}{}^{\gamma'} g_{\beta)}{}^{\delta'} \Delta^{1/2}(x, x') \delta_+\big(\sigma(x, x')\big) + V_{\alpha\beta}{}^{\gamma'\delta'}(x, x') \theta_+\big(-\sigma(x, x')\big) \tag{2.166}$$

while (2.68) allows us to write the semiclassical equations of motion for the small mass m_o in the form

$$m_o \bar{a}^\mu = \frac{1}{2}\kappa^2 m_o^2 P^{\rho\sigma}{}_{\alpha\beta} w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha]\nabla_\gamma \Bigg[\int_{\tau_<}^{\tau} d\tau' \Big(g_{\rho\gamma'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\gamma'} g_{\sigma\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\delta'} \bar{\Delta}^{1/2}\delta(\bar{\sigma}) + \bar{V}_{\rho\sigma\gamma'\delta'}\Big) + \int_{\tau_i}^{\tau_<} d\tau' \, \check{D}^{ret}_{\rho\sigma\gamma'\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'}) \Bigg]. \tag{2.167}$$

As with the other particle-field examples we have so far considered, the direct part of the retarded propagator gives rise to a divergence when the two points are light-like separated. The self-force in (2.167) requires the retarded Green's function to be evaluated along the particle's trajectory, which is time-like, so that the only contribution to the self-force occurs at coincidence when the two points are equal.

Introduce a regulator Λ through the replacement (2.71) such that for particle energies much lower than this scale an effective description of the particle dynamics can be given without recourse to information about the high energy physics that is being ignored. The first term on the right side of (2.167) has the divergent integral,

$$I_4^\mu \equiv \frac{1}{2}\kappa^2 m_o^2 P^{\rho\sigma}_{\alpha\beta} w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha]\nabla_\gamma \int_{\tau_<}^{\tau} d\tau' \, g_{\rho\gamma'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\gamma'} g_{\sigma\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\delta'} \bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma}). \tag{2.168}$$

After passing the derivative through the integral it follows that

$$I_4^\mu = \frac{\kappa^2 m_o^2}{2} P^{\rho\sigma}{}_{\alpha\beta} w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha] \Bigg(\Big[g_{\rho\gamma'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\gamma'} g_{\sigma\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\delta'} \bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma}) \Big] \times \big[\nabla_\gamma(\tau - \tau')\big] + \int_{\tau_<}^{\tau} d\tau' \nabla_\gamma \Big(g_{\rho\gamma'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\gamma'} g_{\sigma\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\bar{u}^{\delta'} \bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma}) \Big) \Bigg) \tag{2.169}$$

while from (2.110) and (2.160) it follows that the first term, which involves the coincident limit, vanishes since

$$w^{\mu\alpha\beta\gamma}\bar{u}_\alpha\bar{u}_\beta\bar{u}_\gamma = 0. \tag{2.170}$$

The quasi-local expansions from Section 2.4.2 imply that

$$I_4^\mu = \frac{1}{2}\kappa^2 m_o^2 w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha]\int_{\tau_<}^{\tau} d\tau'\nabla_\gamma\Big\{g_{\alpha\gamma'}\big(\bar{z}^\alpha,\bar{z}^{\alpha'}\big)\bar{u}^{\gamma'}g_{\beta\delta'}\big(\bar{z}^\alpha,\bar{z}^{\alpha'}\big)\bar{u}^{\delta'}\bar{\Delta}^{1/2}\delta_\Lambda(\bar{\sigma})\Big\} \tag{2.171}$$

vanishes[9] in a vacuum background spacetime (with $R_{\mu\nu} = 0$).

Therefore, from (2.167) the semiclassical particle equations of motion describing the self-force on the small mass

$$\begin{aligned} m_o\bar{a}^\mu(\tau) = \frac{1}{2}\kappa^2 m_o^2 P^{\rho\sigma}{}_{\alpha\beta}w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha]\nabla_\gamma\Bigg[\int_{\tau_<}^{\tau}d\tau'\,\bar{V}_{\rho\sigma\gamma'\delta'} \\ + \int_{\tau_i}^{\tau_<}d\tau'\,\check{D}^{ret}_{\rho\sigma\gamma'\delta'}\big(\bar{z}^\alpha,\bar{z}^{\alpha'}\big)\Bigg]\bar{u}^{\gamma'}\bar{u}^{\delta'} \end{aligned} \tag{2.172}$$

can be written solely in terms of the non-local contributions from the particle-field interactions. Passing the covariant derivative through the first integral gives a term proportional to

$$w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha]\big[\bar{V}_{\alpha\beta\gamma'\delta'}\bar{u}^{\gamma'}\bar{u}^{\delta'}\big]\big[\nabla_\gamma(\tau-\tau')\big] = -w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha]R_{\alpha(\delta|\beta|\epsilon)}\bar{u}_\gamma\bar{u}^\delta\bar{u}^\epsilon, \tag{2.173}$$

which vanishes upon using the coincidence limit of $\bar{V}_{\alpha\beta\gamma'\delta'}$ [53],

$$\big[\bar{V}_{\alpha\beta\gamma'\delta'}\big] = R_{\alpha(\gamma|\beta|\delta)}. \tag{2.174}$$

[9] Actually, there are higher order terms proportional to inverse powers of Λ. However, these will vanish when $\Lambda \to \infty$ and so we ignore their contribution to the particle equations of motion.

Then (2.172) can be written in terms of the propagator for $\psi_{\mu\nu}$ everywhere along the worldline except at coincidence

$$m_o \bar{a}^\mu(\tau) = \frac{1}{2}\kappa^2 m_o^2 P^{\rho\sigma}{}_{\alpha\beta} w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha] \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, \check{D}^{ret}_{\rho\sigma\gamma'\delta'}\big(\bar{z}^\alpha, \bar{z}^{\alpha'}\big) \bar{u}^{\gamma'} \bar{u}^{\delta'}. \qquad (2.175)$$

In terms of the propagator for the $h_{\mu\nu}$ metric perturbations we use (2.164) to show that the self-force on the small mass is given to first order in the metric perturbations by

$$m_o \bar{a}^\mu(\tau) = \frac{1}{2}\kappa^2 m_o^2 w^{\mu\alpha\beta\gamma}[\bar{z}^\alpha] \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D^{ret}_{\alpha\beta\gamma'\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'}) \bar{u}^{\gamma'} \bar{u}^{\delta'}. \qquad (2.176)$$

This is the main result of this section. Eq. (2.176) describes the motion of a small mass that interacts with the metric perturbations generated in the past. In particular, there are no local forcing terms on the particle. The self-force is entirely non-local and depends upon the specific path that the particle has taken through the spacetime.

The equations of motion (2.176) are the self-force equations derived first by Mino, Sasaki and Tanaka [20] using the method of matched asymptotic expansions. Independently, Quinn and Wald [21] used axiomatic techniques to derive the same equations. As a result, (2.176) is called the MSTQW self-force equation.

In this section we have derived the MSTQW self-force as the semiclassical limit of a particle-field total system described from a first principles fully quantum theory. As such, our approach is much richer since there is no need to restrict to the semiclassical limit. We can use the techniques and formalisms developed here to calculate quantum correlation functions of the quantum mechanical worldline coordinates. We may also self-consistently include the effects that the quantum fluctuations of the field have on the motion of the particle. We discuss this latter feature and its implications in the next Chapter.

Chapter 3

The nonequilibrium dynamics of particles and quantum fields in curved space: Stochastic semiclassical limit

In the previous Chapter we derive the equations of motion for a point particle moving through a curved background spacetime and interacting with a linear quantum field from a first principles open quantum system point of view. We assume that environment-induced decoherence is sufficiently provided by the coarse-grained quantum field fluctuations so that only minimal smearing of the particle worldlines (over a scale of order the particle's Compton wavelength, say) allows for a semiclassical description of the particle's evolution.

Nevertheless, even under these assumptions the quantum fluctuations of the field may still influence the classical motion of the particle through the particle-field interactions that are ongoing. This coupling can manifest as noise through the appearance of classical stochastic forces that cause the particle to be perturbed away from its expected semiclassical motion. In this section we demonstrate how such a stochastic semiclassical limit can be obtained from the first principles worldline influence functional formalism developed in Section 2.2.

3.1 The self-force Langevin equations and the noise kernel

In Sections 2.3 and 2.4 we showed that the variation of the coarse-grained effective action in (2.53) gives the semiclassical equations of motion for the particle worldline provided that the quantum field fluctuations have sufficiently decohered[1] the particle worldline histories.

As such, the imaginary part of the influence action, which is proportional to the Hadamard function representing the fluctuations of the coarse-grained quantum field, plays a peripheral role in the semiclassical limit. However, (2.40) and (2.44) imply that the Hadamard function is vital for determining the particle's symmetric two-point correlation function $\langle \{ \hat{z}(\tau), \hat{z}(\tau') \} \rangle$, for example. This quantity represents the fluctuations of the worldline coordinates and contains two types of contributions. The first comes from the quantum mechanical fluctuations intrinsic to the initial state of the particle. The second describes fluctuations that are induced by the particle's interaction with the coarse-grained quantum field fluctuations of the environment. For linear quantum Brownian motion systems the intrinsic fluctuations decay with time and can be viewed as transient fluctuations while the induced fluctuations persist through time [24]. While the particle dynamics in S_{pp} are not linear it is reasonable to expect that the induced fluctuations will be an important contribution so that when the worldline is sufficiently decohered these induced fluctuations manifest as noise in the particle's motion. This scenario is called

[1] From here on whenever we speak of the particle worldline being sufficiently decohered we will implicitly assume that a minimal amount of smearing has been performed to achieve truly decoherent histories of the particle worldline.

the stochastic semiclassical limit to which we now turn.

To see how the stochastic semiclassical limit arises we invoke the relation used by Feynman and Vernon [30],

$$\exp\left\{ -\frac{1}{4\hbar} j_A^- \cdot D_H^{AB'} \cdot j_{B'}^- \right\}$$
$$= N \int \mathcal{D}\xi(x) \, \exp\left\{ -\frac{1}{\hbar} \xi_A \cdot (D_H^{AB'})^{-1} \cdot \xi_{B'} - \frac{i}{\hbar} \xi^A \cdot j_A^- \right\} \quad (3.1)$$

where N is a normalization factor that is independent of the worldline coordinates and $\xi_A(x)$ is an auxiliary field. Essentially, (3.1) is a functional Fourier transform of the exponential of the imaginary part of the influence action.

Now the reduced density matrix (2.20) becomes

$$\rho_r(z_f, z'_f; t_f) \; = \; N \int dz_i \, dz'_i \int_{z_i}^{z_f} \mathcal{D}z \int_{z'_i}^{z'_f} \mathcal{D}z' \, \rho_S(z_i, z'_i; t_i) \int \mathcal{D}\xi_A \, P_\xi[\xi_A]$$
$$\times \exp\left\{ \frac{i}{\hbar} S_{sea}[z, z'; \xi_A] \right\} \quad (3.2)$$

where the *stochastic effective action* (SEA) is defined as

$$S_{sea}[z, z'; \xi_A] = \mathrm{Re}\, S_{cgea}[z, z'] - \xi^A \cdot j_A^- \; . \quad (3.3)$$

The function $\xi_A(x)$ can be interpreted as a classical stochastic, or noise, field [33, 24] with an associated (Gaussian) probability distribution functional

$$P_\xi[\xi_A(x)] = \exp\left\{ -\frac{1}{\hbar} \xi_A \cdot (D_H^{AB'})^{-1} \cdot \xi_{B'} \right\}. \quad (3.4)$$

The fact that this is Gaussian is a direct consequence of the quadratic field action $S[\Phi]$ and the field's linear coupling to the particle current density in $S_{int}[z, \Phi]$. With respect to $P_\xi[\xi_A]$ this implies that ξ_A has zero-mean and its symmetric two-point correlation function is proportional to the Hadamard function encoding the information about the fluctuations in the quantum field

$$\left\langle \{\xi^A(x), \xi^{B'}(x')\} \right\rangle_\xi \;\; = \;\; \hbar\, D_H^{AB'}(x,x') \tag{3.5}$$

where $\langle \ldots \rangle_\xi = N \int \mathcal{D}\xi_A\, P_\xi\, (\ldots)$. In general, for systems with a nonlinear coupling to the environment or for non-Gaussian initial states of the envronment the interpretation of P_ξ as a probability distribution is not always possible. It may take on negative values in which case P_ξ should be interpreted as a pseudoprobability distribution in a similar vein as the Wigner function [24].

Expanding the stochastic effective action around the classical solution $\bar{z}^\mu$ and evaluating the resulting reduced density matrix using the stationary phase approximation[2] results in the following stochastic equations of motion for the worldline fluctuations $\tilde{z}^\mu \equiv z^\mu - \bar{z}^\mu$

$$\int d\tau'\, \tilde{z}^{\alpha'} \frac{\delta^2 \mathrm{Re}\, S_{cgea}}{\delta z^{\alpha'} \delta z^\mu}\bigg|_{z=z'=\bar{z}} = \eta_\mu[\bar{z}^\alpha] \tag{3.6}$$

where the stochastic forcing term is

$$\eta_\mu\big[\bar{z}^\alpha\big] = \vec{w}_\mu{}^A\big[\bar{z}^\alpha\big]\, \xi_A\big(\bar{z}^\alpha\big). \tag{3.7}$$

Eq. (3.6) describes the dynamics of small perturbations $\tilde{z}$ around the semiclassical solution $\bar{z}$ that originate from the classical, stochastic manifestation η_μ of the quantum field fluctuations.

We can obtain a stochastic version of (2.58) if we add (2.56) to the left side of

[2] We can do this since we demonstrated earlier that the semiclassical limit is equivalent to a stationary phase approximation of the reduced density matrix.

(3.6) so that

$$\eta_\mu[z] = \frac{\delta S_{cgea}}{\delta\langle\hat{z}^\mu\rangle}\bigg|_{z'=z} + \int d\tau'\,\tilde{z}^{\alpha'}\frac{\delta^2 \mathrm{Re}\, S_{cgea}}{\delta z^{\alpha'}\delta z^\mu}\bigg|_{z=z'=\bar{z}} \approx \frac{\delta \mathrm{Re}\, S_{cgea}}{\delta z^\mu}\bigg|_{z'=z}, \tag{3.8}$$

which is accurate through first order in an expansion in the fluctuation coordinates $\tilde{z}$. Computing the functional derivative in the last equality gives another form of the stochastic equations of motion in terms of the full worldline z^μ

$$m\, a^\mu = \vec{w}^{\mu A}[z]\int_{\tau_i}^{\tau_f} d\tau'\, D^{ret}_{AB'}(z^\alpha, z^{\alpha'}) j^{B'}[z^{\alpha'}] + \eta^\mu[z]. \tag{3.9}$$

This equation is only valid to *linear* order in the fluctuations $\tilde{z}$ and is the same order as the stochastic force $\eta_\mu[\bar{z}]$ because we are neglecting higher order particle-loop quantum corrections. In practice, (3.9) is expanded to linear order in $\tilde{z}$ and (2.58) is invoked to obtain the particle fluctuation dynamics. We point out that (3.9) is a stochastic equation of motion because observables involving $\tilde{z}$ must be computed via the stochastic correlation functions $\langle \ldots \rangle_\xi$. In fact, both the deterministic and the stochastic components of the self-force can push the particle away from its mean trajectory with respect to a fixed background spacetime.

The stochastic correlation functions of the force η_μ can be evaluated using the ξ_A correlators above. Evaluating these correlation functions along the classical trajectory $\bar{z}$ we find that the mean of the stochastic force is zero and the symmetric two-point function of the stochastic force, which we refer to as the noise kernel, is

$$\big\langle \{\eta_\mu[\bar{z}], \eta_{\nu'}[\bar{z}]\}\big\rangle_\xi = \hbar\, \vec{w}_{(\mu}{}^{A}[\bar{z}]\, \vec{w}_{\nu')}{}^{B'}[\bar{z}]\, D^H_{AB'}(\bar{z}^\alpha, \bar{z}^{\alpha'})\ . \tag{3.10}$$

It follows that the noise term η_μ, which is generally multiplicative and colored, depends on the particle's initial conditions through the semiclassical trajectory and on the field's initial conditions via the initial state used to evaluate the Hadamard function. For most kinds of fields (with the notable exception of the linearized metric perturbation) the operator $\vec{w}^{\mu A}[z]$ enforces the noise kernel to be gauge invariant. For equal proper times $\tau' = \tau$ the Hadamard function diverges implying that a suitable regularization procedure must be used in order to make sense of the noise kernel (3.10) near coincidence.

The noise kernel in (3.10) demonstrates that the stochastic force η_μ is $O(\hbar^{1/2})$ as are the worldline fluctuations $\tilde{z}$. This shows that the Langevin equation (3.6) is at an order between the tree-level and the one-loop $O(\hbar)$ equations of motion and therefore contains information about the lowest order[3] quantum fluctuations of the coarse-grained field, even if the environment is weakly nonlinear. This is the reason why we do not need to include the ghost fields in these considerations since the ghosts first appear at one-loop order and hence provide no contribution to the semiclassical or the stochastic semiclassical dynamics of the particle.

In the next Sections we derive from (3.9) the Langevin equations describing the self-force on a particle interacting separately with a linear scalar, electromagnetic and metric perturbation quantum field in a curved space.

[3] Lowest order in the coupling constant, that is. We consider a linear quantum field (or more generally, a quantum field in the Gaussian approximation), which has no non-trivial vertices and hence no loop contributions to the field two-point functions.

3.1.1 Scalar field

In this subsection we study the effects of the quantum field fluctuations (manifesting as classical stochastic forces) in the stochastic semiclassical limit of a point scalar charge interacting with a scalar field $\phi(x)$.

Varying the stochastic effective action around the classical trajectory $\bar{z}^\mu$ to linear order in the worldline fluctuations $\tilde{z}^\mu$ and performing a stationary phase approximation in the reduced density matrix gives the Langevin equation (3.9) valid for this particle example,

$$m_0 a^\mu(\tau) = \vec{w}^\mu[z] \int_{\tau_i}^{\tau} d\tau' \, D_{ret}(z^\alpha, z^{\alpha'}) + \eta^\mu[z] \tag{3.11}$$

where the stochastic force η^μ is related to the stochastic field $\xi(x)$ through

$$\eta^\mu[z] = \vec{w}^\mu[z]\xi(z) = q\Big(a^\mu + w^{\mu\nu}\nabla_\nu\Big)\xi(z). \tag{3.12}$$

The retarded propagator diverges and must be regularized. The steps used to regularize the semiclassical equations of motion are exactly the same that regularize the divergence here. We may therefore simply write down the regularized and renormalized scalar ALD-Langevin equation using (2.121) and (3.9) to find

$$\begin{aligned} \left[m - q^2 \lim_{\epsilon \to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D_{ret}\big(z^\alpha, z^{\alpha'}\big) \right] a^\mu(\tau) & \\ = \frac{q^2}{3}\, g_{(2)}(r)\, w^{\mu\nu}[z^\alpha] \frac{Da_\nu}{d\tau} + \frac{q^2}{6}\, c_{(1)}(r)\, w^{\mu\nu}[z^\alpha] R_{\nu\alpha} u^\alpha & \\ + q^2 w^{\mu\nu}[z^\alpha] \lim_{\epsilon \to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D^{ret}_{;\nu}\big(z^\alpha, z^{\alpha'}\big) + \eta^\mu[z]. & \end{aligned} \tag{3.13}$$

We must remember that these equations are only valid through linear order in the fluctuations $\tilde{z}$ about the mean worldline coordinates $\bar{z}$ since higher orders correspond to quantum corrections that we are neglecting.

It will be useful to define the effective mass of the particle as

$$m_{eff}(\tau; z] \equiv m - q\phi^{tail}(z^\alpha) \tag{3.14}$$

and to define the self-force 4-vector on the particle as

$$f^\mu(\tau; z] \equiv \frac{q^2}{3}\, g_{(2)}(r)\, w^{\mu\nu}[z^\alpha]\frac{Da_\nu}{d\tau} + \frac{q^2}{6}\, c_{(1)}(r)\, w^{\mu\nu}[z^\alpha]R_{\nu\alpha}u^\alpha + qw^{\mu\nu}[z^\alpha]\phi^{tail}_\nu(z^\alpha). \tag{3.15}$$

In providing these definitions we have also defined the tail part of the retarded field

$$\phi^{tail}(z^\alpha) \equiv q \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau'\, D_{ret}\big(z^\alpha, z^{\alpha'}\big) \tag{3.16}$$

and its coviarant derivative as

$$\phi^{tail}_\nu(z^\alpha) \equiv q \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau'\, D^{ret}_{;\nu}\big(z^\alpha, z^{\alpha'}\big). \tag{3.17}$$

Expanding (3.13) in orders of the fluctuations amounts to doing the same for the time-dependent effective mass (3.14) and the self-force (3.15). Through first order in the particle fluctuations we formally find

$$m_{eff}(\tau; z] = m_{eff}(\tau; \bar{z}] - q \int d\tau'\, \tilde{z}^{\nu'} \left[\frac{\delta}{\delta \bar{z}^{\nu'}}\phi^{tail}(z^\alpha)\right]_{z=\bar{z}} + O(\tilde{z}^2) \tag{3.18}$$

$$f^\mu[z] = f^\mu[\bar{z}] + \int d\tau'\, \tilde{z}^{\nu'} \left[\frac{\delta}{\delta \bar{z}^{\nu'}} f^\mu[z^\alpha]\right]_{z=\bar{z}} + O(\tilde{z}^2). \tag{3.19}$$

Calculating the functional derivative in the mass equation gives

$$m_{eff}(\tau; z] = m_{eff}(\tau; \bar{z}] - q\, \tilde{z}^\nu \phi^{tail}_\nu(\bar{z}) - q^2 \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau'\, \tilde{z}^{\nu'} \nabla_{\nu'} D_{ret}(\bar{z}^\alpha, \bar{z}^{\alpha'}). \tag{3.20}$$

Notice that the linear terms in $\tilde{z}$ vanish in flat spacetime since there is no tail part of the propagator.

Simplifying the self-force fluctuations is slightly more involved. The calculation amounts to performing the variational derivative on f_μ but keeping in mind to expand out the covariant derivatives, which depend on the worldline coordinates. We do not give the result explicitly here since the expressions are rather long and complicated.

Combining the linearized effective mass and self-force into (3.13) and using the fact that $\bar{z}$ satisfies the semiclassical equations of motion (2.121) results in the following equations describing the particle fluctuations about the expected worldline,

$$m_{\mu\alpha}[\bar{z}]\,\ddot{\tilde{z}}^\alpha + \gamma_{\mu\alpha}[\bar{z}]\,\dot{\tilde{z}}^\alpha + \kappa_{\mu\alpha}[\bar{z}]\,\tilde{z}^\alpha - q^2 \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau' \Big(\tilde{z}^\alpha \vec{w}_\mu[\bar{z}^\alpha]\,\bar{D}^{ret}_{;\alpha} + \tilde{z}^{\alpha'}\vec{w}_\mu[\bar{z}^\alpha]\,\bar{D}^{ret}_{;\alpha'}\Big)$$
$$= r_{\mu\alpha}[\bar{z}]\,\dddot{\tilde{z}}^\alpha + \eta_\mu[\bar{z}]. \tag{3.21}$$

Here we use overdots to denote $d/d\tau$. The coefficients $m_{\mu\alpha}$, $\gamma_{\mu\alpha}$ and $\kappa_{\mu\alpha}$ represent a time- and history-dependent inertia for the fluctuations, damping factor and "spring" constant, respectively. Specifically, these are given by the following expressions. The effective inertia is

$$m_{\mu\nu}[\bar{z}] = \Big[m - q\phi^{tail}(\bar{z}^\alpha)\Big]\bar{g}_{\mu\nu} - q^2 g_{(2)}(r)\bar{w}_{\mu\alpha}\bar{\Gamma}^\alpha_{\beta\nu}\bar{u}^\beta \tag{3.22}$$

while the damping factor $\gamma_{\mu\nu}$ is defined as

$$\begin{aligned}\gamma_{\mu\nu}[\bar{z}] = {}& 2\Big[m - q\phi^{tail}(\bar{z}^\alpha)\Big]\bar{g}_{\mu\alpha}\bar{\Gamma}^\alpha_{\beta\nu}\bar{u}^\beta - \frac{2e^2}{3}\,g_{(2)}(r)\bar{u}_{(\mu}\bar{g}^{\beta)}{}_\nu \frac{D\bar{a}_\beta}{d\tau} \\ & -\frac{q^2}{6}c_{(1)}(r)\Big[\bar{w}_\mu{}^\alpha \bar{R}_{\alpha\nu} + 2\bar{u}_{(\mu}\bar{g}^{\alpha)}{}_\nu \bar{R}_{\alpha\beta}\bar{u}^\beta\Big] - \frac{e^2}{3}\,g_{(2)}(r)\bar{w}_{\mu\alpha}\Big[2\bar{\Gamma}^\alpha_{\beta\nu,\gamma}\bar{u}^\beta\bar{u}^\gamma \\ & +2\bar{\Gamma}^\alpha_{\beta\nu}\dot{\bar{u}}^\beta + \bar{\Gamma}^\alpha_{\beta\gamma,\nu}\bar{u}^\beta\bar{u}^\gamma + \bar{\Gamma}^\alpha_{\beta\nu}\bar{a}^\beta + 2\bar{\Gamma}^\alpha_{\beta\gamma}\bar{\Gamma}^\beta_{\delta\nu}\bar{u}^\gamma\bar{u}^\delta\Big] - 2q\bar{u}_{(\mu}\bar{g}^{\alpha)}{}_\nu \bar{\phi}^{tail}_\alpha(\bar{z}^\alpha).\end{aligned} \tag{3.23}$$

The spring constant is given by

$$
\begin{aligned}
\kappa_{\mu\nu}[\bar{z}] &= \Big[m - q\phi^{tail}(\bar{z}^\alpha)\Big]\Big(\bar{g}_{\mu\alpha}\bar{\Gamma}^\alpha_{\beta\gamma,\nu}\bar{u}^\beta\bar{u}^\gamma + \bar{a}^\alpha\bar{g}_{\mu\alpha,\nu}\Big) - \frac{q^2}{6}\, c_{(1)}(r)\bar{w}_\mu{}^\alpha\bar{R}_{\alpha\beta,\nu}\bar{u}^\beta \\
&\quad -\frac{q^2}{3}\, g_{(2)}(r)\bar{w}_{\mu\alpha}\Big[\bar{g}^{\alpha\beta}\bar{g}_{\beta\gamma,\nu}\frac{D\bar{a}^\gamma}{d\tau} + \bar{\Gamma}^\alpha_{\beta\gamma,\delta\nu}\bar{u}^\beta\bar{u}^\gamma\bar{u}^\delta + 2\bar{\Gamma}^\alpha_{\beta\gamma,\nu}\dot{\bar{u}}^\beta\bar{u}^\gamma \\
&\qquad + \bar{\Gamma}^\alpha_{\beta\gamma}\bar{\Gamma}^\beta_{\delta\epsilon,\nu}\bar{u}^\gamma\bar{u}^\delta\bar{u}^\epsilon + \bar{\Gamma}^\alpha_{\beta\gamma,\nu}\bar{u}^\beta\bar{a}^\gamma\Big] \qquad (3.24)
\end{aligned}
$$

and the local radiation reaction term $r_{\mu\alpha}[\bar{z}]$ is

$$r_{\mu\nu}[\bar{z}] = \frac{q^2}{3}\, g_{(2)}(r) w_{\mu\nu}[\bar{z}^\alpha]. \qquad (3.25)$$

The dynamical equation (3.21) for the fluctuations about the semiclassical particle trajectory is the main result of this section. This is a linear integro-differential equation for $\tilde{z}$ with a third derivative term and contains time-dependent coefficients that depend on the non-Markovian behavior of the mean trajectory. Furthermore, because of the integration over past times the last term on the left side of (3.21) depends on the history of the fluctuations as well. Notice that this term vanishes in a flat spacetime so that the fluctuations then obey a third-order differential equation, which is Markovian in the sense that given a mean trajectory $\bar{z}^\mu$ the fluctuations do not depend on their own past history. In fact, the tensor coefficients are

$$
\begin{aligned}
m_{\mu\nu}[\bar{z}] &= m\,\eta_{\mu\nu} && (3.26) \\
\gamma_{\mu\nu}[\bar{z}] &= -\frac{q^2}{3}\left(\bar{u}_\mu\frac{d\bar{a}_\nu}{d\tau} - \eta_{\mu\nu}\bar{a}^2\right) && (3.27) \\
\kappa_{\mu\nu}[\bar{z}] &= 0 && (3.28) \\
r_{\mu\nu}[\bar{z}] &= \frac{q^2}{3}\, g_{(2)}(r) w_{\mu\nu}[\bar{z}^\alpha] && (3.29)
\end{aligned}
$$

in flat spacetime using Lorentzian coordinates and the integral in (3.21) vanishes identically.

The notation in (3.21) has been chosen suggestively since the left side resembles a damped simple harmonic oscillator with time-dependent mass, damping factor, and spring constant. Notice also that $m_{\mu\alpha}[\bar{z}]$ is not diagonal implying that the inertia of the fluctuations is not isotropic. This feature is exhibited in the other coefficients and suggests that the fluctuations of the trajectory in one direction are linked with the fluctuations in the other spacetime directions.

3.1.2 Electromagnetic field

We now study the effects of the quantum field fluctuations (manifesting as classical stochastic forces) in the stochastic semiclassical limit of a point charge interacting with a vector field $A_\mu(x)$.

Varying the stochastic effective action around the classical trajectory $\bar{z}^\mu$ to linear order in the worldline fluctuations $\tilde{z}^\mu$ and performing a stationary phase approximation in the reduced density matrix gives the Langevin equation (3.9) valid for this particle example,

$$m_0 a^\mu(\tau) = \vec{w}^{\mu\alpha}[z] \int_{\tau_i}^{\tau} d\tau' \, D^{ret}_{\alpha\beta'}(z^\alpha, z^{\alpha'}) + \eta^\mu[z] \tag{3.30}$$

where the stochastic force η_μ is related to the stochastic field $\xi^\alpha(x)$ through

$$\eta^\mu[z] = \vec{w}^{\mu\alpha}[z]\xi_\alpha(z) = -e\, w^{\mu\alpha\beta}[z]\nabla_\beta \xi_\alpha(z). \tag{3.31}$$

As before, the retarded propagator diverges and must be regularized. The steps used to regularize the semiclassical equations of motion are exactly the same that regularize the divergence here. We may therefore simply write down the regularized and renormalized ALD-Langevin equation

$$\begin{aligned} ma^\mu(\tau) \;=\; & \frac{2e^2}{3}\, w^{\mu\alpha}[z^\alpha]\frac{Da_\alpha}{d\tau} + \frac{e^2}{3}\, w^{\mu\alpha}[z^\alpha]R_{\alpha\beta}u^\beta \\ & -e^2\, w^{\mu\alpha\beta}[z^\alpha] \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\, D^{ret}_{\alpha\gamma';\beta}\big(z^\alpha, z^{\alpha'}\big)u^{\gamma'} + \eta^\mu[z]. \end{aligned} \tag{3.32}$$

It will be convenient to define the (deterministic part of the) self-force 4-vector as

$$\begin{aligned} f^\mu[z] \;=\; & \frac{2e^2}{3}\, g_{(2)}(r) w^{\mu\alpha}[z]\frac{Da_\alpha}{d\tau} + \frac{e^2}{6}\big(3 - c_{(1)}(r)\big)\, w^{\mu\alpha}[z]R_{\alpha\beta}(z)u^\beta \\ & +e\, w^{\mu\alpha\beta}[z] \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\, D^{ret}_{\alpha\gamma';\beta}\big(z^\alpha, z^{\alpha'}\big)u^{\gamma'}. \end{aligned} \tag{3.33}$$

We remark that these equations are only valid through linear order in the fluctuations $\tilde{z}$ about the semiclassical worldline coordinates $\bar{z}$.

Expanding the self-force in orders of the fluctuations using

$$f^\mu[z] = f^\mu[\bar{z}] + \int d\tau'\, \tilde{z}^{\nu'} \left[\frac{\delta}{\delta z^{\nu'}}\, f^\mu[z^\alpha(\tau)]\right]_{z=\bar{z}} + O(\tilde{z}^2) \tag{3.34}$$

and computing the linearization of those terms in the ALD-Langevin equation (3.32) involving the covariant τ derivatives (e.g. a_μ) gives the following equation for the dynamics of the fluctuations

$$\begin{aligned} & m_{\mu\nu}[\bar{z}]\ddot{\tilde{z}}^\nu + \gamma_{\mu\nu}[\bar{z}]\dot{\tilde{z}}^\nu + \kappa_{\mu\nu}[\bar{z}]\tilde{z}^\nu - e^2 \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau' \Big(\tilde{z}^\sigma \vec{w}_\mu{}^\alpha[\bar{z}^\alpha] \bar{D}^{ret}_{\alpha\gamma';\sigma}\bar{u}^{\gamma'} \\ & + \tilde{z}^{\sigma'} \vec{w}_\mu{}^\alpha[\bar{z}^\alpha]\bar{D}^{ret}_{\alpha\gamma';\sigma'}\bar{u}^{\gamma'} + 2\tilde{z}^\sigma \bar{R}_{\sigma[\mu\alpha]}{}^\lambda \bar{D}^{ret}_{\lambda\gamma'}\bar{u}^\alpha\bar{u}^{\gamma'} + \vec{w}_\mu{}^\alpha[\bar{z}^\alpha]\bar{D}^{ret}_{\alpha\gamma'}\dot{\tilde{z}}^{\gamma'} \Big) \\ & = r_{\mu\nu}[\bar{z}]\dddot{\tilde{z}}^\nu + \eta_\mu[\bar{z}]. \end{aligned} \tag{3.35}$$

Here we use overdots to denote $d/d\tau$. We have also used the semiclassical ALD equation (2.149) in deriving (3.35). The tensor coefficients are given by the following expressions. The effective inertia of the fluctuations is

$$m_{\mu\nu}[\bar{z}] = m\,\bar{g}_{\mu\nu} - 2e^2 g_{(2)}(r)\bar{w}_{\mu\alpha}\bar{\Gamma}^\alpha_{\beta\nu}\bar{u}^\beta, \tag{3.36}$$

the damping factor is

$$\begin{aligned}
\gamma_{\mu\nu}[\bar{z}] &= 2m\,\bar{g}_{\mu\alpha}\bar{\Gamma}^{\alpha}_{\beta\nu}\bar{u}^{\beta} - \frac{4e^2}{3}\,g_{(2)}(r)\bar{u}_{(\mu}\bar{g}^{\beta)}{}_{\nu}\frac{D\bar{a}_{\beta}}{d\tau} - \frac{2e^2}{3}\,g_{(2)}(r)\bar{w}_{\mu\alpha}\Big[2\bar{\Gamma}^{\alpha}_{\beta\nu,\gamma}\bar{u}^{\beta}\bar{u}^{\gamma} \\
&\quad +2\bar{\Gamma}^{\alpha}_{\beta\nu}\dot{\bar{u}}^{\beta} + \bar{\Gamma}^{\alpha}_{\beta\gamma,\nu}\bar{u}^{\beta}\bar{u}^{\gamma} + \bar{\Gamma}^{\alpha}_{\beta\nu}\bar{a}^{\beta} + 2\bar{\Gamma}^{\alpha}_{\beta\gamma}\bar{\Gamma}^{\beta}_{\delta\nu}\bar{u}^{\gamma}\bar{u}^{\delta}\Big] \\
&\quad -\frac{e^2}{6}\left(3 - c_{(1)}(r)\right)\Big[\bar{w}_{\mu}{}^{\alpha}\bar{R}_{\alpha\nu} + 2\bar{u}_{(\mu}\bar{g}^{\alpha)}{}_{\nu}\bar{R}_{\alpha\beta}\bar{u}^{\beta}\Big] - 2e\bar{g}_{\mu}{}^{[\alpha}\bar{g}^{\beta]}{}_{\nu}\bar{A}^{tail}_{\alpha\beta}
\end{aligned} \tag{3.37}$$

where the tail part of the retarded field is given by

$$A_{\alpha}(\bar{z}^{\alpha}) = e\lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\,D^{ret}_{\alpha\gamma'}(\bar{z}^{\alpha},\bar{z}^{\alpha'})\bar{u}^{\gamma'} \tag{3.38}$$

and the tail of the covariant derivative is

$$A_{\alpha\beta}(\bar{z}^{\alpha}) = e\lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\,D^{ret}_{\alpha\gamma';\beta}(\bar{z}^{\alpha},\bar{z}^{\alpha'})\bar{u}^{\gamma'}. \tag{3.39}$$

The spring constant is defined as

$$\begin{aligned}
\kappa_{\mu\nu}[\bar{z}] &= m\Big[\bar{g}_{\mu\alpha}\bar{\Gamma}^{\alpha}_{\beta\gamma,\nu}\bar{u}^{\beta}\bar{u}^{\gamma} + \bar{a}^{\alpha}\bar{g}_{\mu\alpha,\nu}\Big] - \frac{e^2}{6}\left(3 - c_{(1)}(r)\right)\bar{w}_{\mu}{}^{\alpha}\bar{R}_{\alpha\beta,\nu}\bar{u}^{\beta} \\
&\quad -\frac{2e^2}{3}\,g_{(2)}(r)\bar{w}_{\mu\alpha}\Bigg[\bar{g}^{\alpha\beta}\bar{g}_{\beta\gamma,\nu}\frac{D\bar{a}^{\gamma}}{d\tau} + \bar{\Gamma}^{\alpha}_{\beta\gamma,\delta\nu}\bar{u}^{\beta}\bar{u}^{\gamma}\bar{u}^{\delta} + 2\bar{\Gamma}^{\alpha}_{\beta\gamma,\nu}\dot{\bar{u}}^{\beta}\bar{u}^{\gamma} \\
&\quad +\bar{\Gamma}^{\alpha}_{\beta\gamma}\bar{\Gamma}^{\beta}_{\delta\epsilon,\nu}\bar{u}^{\gamma}\bar{u}^{\delta}\bar{u}^{\epsilon} + \bar{\Gamma}^{\alpha}_{\beta\gamma,\nu}\bar{u}^{\beta}\bar{a}^{\gamma}\Bigg]
\end{aligned} \tag{3.40}$$

and the local radiation reaction is given by

$$r_{\mu\nu}[\bar{z}] = \frac{2e^2}{3}\,g_{(2)}(r)w_{\mu\nu}[\bar{z}]. \tag{3.41}$$

Notice that (3.35) is a linear integro-differential equation for $\tilde{z}$ with a third derivative term and contains time-dependent coefficients that depend on the semi-classical trajectory, which possesses non-Markovian features. Furthermore, (3.35) depends on the entire past history of the worldline fluctuations because of the integral over $\tilde{z}$. This integral also contains a term involving $\dot{\tilde{z}}$, which suggest that the dissipation of these fluctuations is a non-Markovian process as well. We remark that the integration is over the tail part and its derivative so that this history dependence disappears in flat spacetime. In fact, the tensor coefficients are

$$
\begin{aligned}
m_{\mu\nu}[\bar{z}] &= m\,\eta_{\mu\nu} && (3.42)\\
\gamma_{\mu\nu}[\bar{z}] &= -\frac{2e^2}{3}\left(\bar{u}_\mu \frac{d\bar{a}_\nu}{d\tau} - \eta_{\mu\nu}\bar{a}^2\right) && (3.43)\\
\kappa_{\mu\nu}[\bar{z}] &= 0 && (3.44)\\
r_{\mu\nu}[\bar{z}] &= \frac{2e^2}{3}\, g_{(2)}(r) w_{\mu\nu}[\bar{z}^\alpha] && (3.45)
\end{aligned}
$$

in flat spacetime using Lorentzian coordinates.

The effective mass $m_{\mu\nu}$ for the fluctuations is not diagonal, generally speaking, implying that the inertia of the fluctuations is not isotropic. This feature is exhibited in the other three tensor coefficients and suggests that the fluctuations in one direction are linked with the fluctuations in the other spacetime directions.

3.1.3 Linear metric perturbations

In this last example we study some attributes and consequences of the quantum field fluctuations of linear metric perturbations manifesting as classical stochastic forces on the motion of a small point mass m_o.

Assuming sufficiently strong decoherence, expanding the stochastic effective action about the semiclassical worldline and performing a stationary phase approximation in the resulting reduced density matrix gives rise to the stochastic semiclassical equations of motion

$$m_o a^\mu(\tau) = \vec{w}^{\mu\alpha\beta}[z] \int_{\tau_i}^{\tau} d\tau' D^{ret}_{\alpha\beta\gamma'\delta'}(z^\alpha, z^{\alpha'}) u^{\gamma'} u^{\delta'} + \eta^\mu[z] \tag{3.46}$$

where the stochastic force η_μ is related to the stochastic field $\xi_{\mu\nu}(x)$ through

$$\eta^\mu[z] = \vec{w}^{\mu\alpha\beta}[z]\, \xi_{\alpha\beta}(z) = -\kappa m_o w^{\mu\alpha\beta\gamma}[z] \nabla_\gamma \xi_{\alpha\beta}(z). \tag{3.47}$$

As before, the retarded propagator diverges and must be regularized. The steps used to regularize the semiclassical equations of motion are exactly the same that regularize the divergence here. We may therefore simply write down the corresponding regularized and renormalized MSTQW-Langevin equation

$$m_o a^\mu = \vec{w}^{\mu\alpha\beta}[z] \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' D^{ret}_{\alpha\beta\gamma'\delta'}(z^\alpha, z^{\alpha'}) u^{\gamma'} u^{\delta'} + \eta^\mu[z] \tag{3.48}$$

where the (deterministic part of the) self-force 4-vector is

$$f^\mu[z] = m_o w^{\mu\alpha\beta\nu}[z] \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' D^{ret}_{\alpha\beta\gamma'\delta';\nu}(z^\alpha, z^{\alpha'}) u^{\gamma'} u^{\delta'}. \tag{3.49}$$

We remark that these stochastic equations are only valid up to linear order in the particle fluctuations $\tilde{z}$ about the semiclassical worldline $\bar{z}$.

Expanding the self-force f^μ in orders of the fluctuations using

$$f^\mu[z] = f^\mu[\bar{z}] + \int d\tau'\, \tilde{z}^{\nu'} \left[\frac{\delta}{\delta z^{\nu'}}\, f^\mu[z^\alpha(\tau)] \right]_{z=\bar{z}} + O(\tilde{z}^2) \tag{3.50}$$

and computing the linearization of those terms involving the covariant τ derivatives (e.g. a^μ) gives the equation of motion for the worldline fluctuations

$$\begin{aligned} m_{\mu\nu}[\bar{z}]\ddot{\tilde{z}}^\nu + \gamma_{\mu\nu}[\bar{z}]\dot{\tilde{z}}^\nu + \kappa_{\mu\nu}[\bar{z}]\tilde{z}^\nu - m_o^2 P_{\alpha\beta}{}^{\delta\epsilon} \lim_{\epsilon\to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \Big(\tilde{z}^\sigma \vec{w}_\mu{}^{\alpha\beta}[\bar{z}^\alpha] \bar{D}^{ret}_{\delta\epsilon\gamma'\delta';\sigma} \bar{u}^{\gamma'} \bar{u}^{\delta'} \\ + \tilde{z}^{\sigma'} \vec{w}_\mu{}^{\alpha\beta}[\bar{z}^\alpha] \bar{D}^{ret}_{\delta\epsilon\gamma'\delta';\sigma'} \bar{u}^{\gamma'} \bar{u}^{\delta'} - 2\tilde{z}^\sigma \bar{R}_{\sigma\gamma}{}^{\lambda}{}_{(\delta} \bar{D}^{ret}_{\epsilon)\lambda\gamma'\delta'} \bar{u}^{\gamma'} \bar{u}^{\delta'} \\ + 2\vec{w}_\mu{}^{\alpha\beta}[\bar{z}^\alpha] \bar{D}^{ret}_{\delta\epsilon\gamma'\delta'} \bar{u}^{(\gamma'} \dot{\tilde{z}}^{\delta')} \Big) = \eta_\mu[\bar{z}]. \end{aligned} \tag{3.51}$$

We have also used the equations of motion for the semiclassical worldline (2.176) in this derivation.

The tensor coefficients $m_{\mu\nu}$, $\gamma_{\mu\nu}$ and $\kappa_{\mu\nu}$ are defined in the following way. The effective inertia for the fluctuations is

$$m_{\mu\nu}[\bar{z}] = m_o \bar{g}_{\mu\nu} \tag{3.52}$$

and the damping factor is

$$\begin{aligned}\gamma_{\mu\nu}[\bar{z}] &= 2m_0 \bar{g}_{\mu\alpha} \bar{\Gamma}^{\alpha}_{\beta\nu} \bar{u}^{\beta} + m_o \bar{h}^{tail}_{\alpha\beta\gamma} \Big[2\bar{u}_{(\mu} \bar{u}^{\alpha} \bar{u}^{\beta} \bar{g}^{\gamma)}{}_{\nu} + \bar{g}_{\mu}{}^{(\alpha} \bar{u}^{\beta)} \bar{g}^{\gamma}{}_{\nu} \\ &\quad - \bar{g}_{\mu}{}^{(\alpha} \bar{u}^{|\beta|} \bar{g}^{\gamma)}{}_{\nu} + \bar{g}_{\mu}{}^{(\alpha} \bar{g}^{\beta)}{}_{\nu} \bar{u}^{\gamma} \Big]\end{aligned} \tag{3.53}$$

where the (history-dependent) tail of the retarded field is

$$h^{tail}_{\alpha\beta}(\bar{z}^{\alpha}) = \kappa m_0 \lim_{\epsilon \to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D^{ret}_{\alpha\beta\gamma'\delta'}(\bar{z}^{\alpha}, \bar{z}^{\alpha'}) \bar{u}^{\gamma'} \bar{u}^{\delta'} \tag{3.54}$$

and the tail of the covariant derivative of the propagator is defined as

$$h^{tail}_{\alpha\beta\gamma}(\bar{z}^{\alpha}) = \kappa m_0 \lim_{\epsilon \to 0} \int_{\tau_i}^{\tau-\epsilon} d\tau' \, D^{ret}_{\alpha\beta\delta'\epsilon';\gamma}(\bar{z}^{\alpha}, \bar{z}^{\alpha'}) \bar{u}^{\delta'} \bar{u}^{\epsilon'}. \tag{3.55}$$

The spring constant is given by

$$\kappa_{\mu\nu}[\bar{z}] = m_o \bar{g}_{\mu\alpha} \bar{\Gamma}^{\alpha}_{\beta\gamma,\nu} \bar{u}^{\beta} \bar{u}^{\gamma}. \tag{3.56}$$

Note that the mass tensor is proportional to the metric indicating that the effective inertia of the fluctuations are isotropic in all of the spacetime directions.

As with the electromagnetic case earlier, (3.51) is a linear differential equation for $\tilde{z}$. The important difference is that the third derivative of $\tilde{z}$ gives no contribution at this order. Hence, only the initial position and velocity of the fluctuations are sufficient to obtain a unique solution. This is unlike the electromagnetic case discussed earlier since one needs to introduce an external force to obtain unambiguous, runaway-free solutions.

In flat spacetime the stochastic semiclassical equations of motion (3.51) are considerably simplified (in Lorentzian coordinates)

$$m_o \ddot{\tilde{z}}_\mu = \eta_\mu[\bar{z}] \tag{3.57}$$

since the tail term vanishes identically. At this order in the mass and in Λ (since the finite terms are proportional to Λ^0) there is no dissipation term appearing in (3.57). This implies that the two-point functions composed of $\dot{\tilde{z}}$ and $\tilde{z}$ could grow unbounded in time in the strict point-particle limit $\Lambda \to \infty$. However, if Λ is large but finite then dissipation effects from the neglected $O(\Lambda^{-1})$ terms could begin to appear on a time scale $\sim \Lambda$. Likewise, dissipation from higher order terms arising from the nonlinearities of the full metric perturbation field equations may begin to appear on a time scale $\sim m_0 \ll \Lambda$.

The structure of the flat spacetime stochastic semiclassical equations of motion in (3.57) provides a simple system to study some effects of the coarse-grained fluctuations of the quantized metric perturbations. For example, we can compute the noise kernel of the stochastic force,

$$\left\langle \{\eta_\mu[\bar{z}^\alpha], \eta_\nu[\bar{z}^{\alpha'}]\} \right\rangle_\xi = \hbar\, \vec{w}_{(\mu}{}^{\alpha\beta}[\bar{z}]\, \vec{w}_{\nu')}{}^{\gamma'\delta'}[\bar{z}^{\alpha'}] D^H_{\alpha\beta\gamma'\delta'}(\bar{z}^\alpha, \bar{z}^{\alpha'}). \tag{3.58}$$

We may write the Hadamard function for the metric perturbations in terms of the Hadamard function for a scalar field in the same vacuum state, for example, using the relation

$$D^H_{\alpha\beta\gamma'\delta'}(\sigma) = P_{\alpha\beta\gamma'\delta'} D^H(\sigma) \tag{3.59}$$

where σ is the world function and $P_{\alpha\beta\gamma'\delta'}$ is defined in (2.163). A particle satisfying the semiclassical MSTQW equations of motion in flat spacetime is a geodesic as can be seen directly from (2.176). Therefore, the world function (2.88) is

$$\sigma(\bar{z}^\alpha, \bar{z}^{\alpha'}) = -\frac{s^2}{2}, \tag{3.60}$$

where $s = \tau' - \tau$, and the covariant derivative of the world function (2.87) is

$$\sigma^\mu(\bar{z}^\alpha, \bar{z}^{\alpha'}) = -s\bar{u}^\mu(\tau). \tag{3.61}$$

The second covariant derivative equals the background metric $\eta_{\mu\nu}$ [53]. Using the expression for the scalar Hadamard function evaluated in the vacuum state [66] (which is a Gaussian state)

$$D^H(\sigma) = \frac{1}{4\pi^2\sigma} \tag{3.62}$$

we find that the noise kernel evaluated along a geodesic of flat spacetime, which is a solution to the semiclassical equations of motion, equals

$$\left\langle \{\eta_\mu[\bar{z}^\alpha], \eta_\nu[\bar{z}^{\alpha'}]\} \right\rangle_\xi \quad = \quad \frac{44}{\pi} \frac{\hbar\, m_o^2}{(\tau - \tau')^4}. \tag{3.63}$$

We remark that there is a tail that falls off as the fourth power of the proper time difference and is divergent when $\tau' = \tau$.

Interestingly, we can write (3.57) as a geodesic equation in a background spacetime possessing a stochastic metric. To show this we observe that the stochastic force

$$\eta_\mu[\bar{z}] = -\kappa m_o w_\mu{}^{\alpha\beta\gamma}[\bar{z}]\nabla_\gamma \xi_{\alpha\beta}(\bar{z}) \tag{3.64}$$

can be written in terms of a stochastic metric perturbation field by defining

$$\tilde{h}_{\alpha\beta}(x) \equiv \kappa\, \xi_{\alpha\beta}(x). \tag{3.65}$$

Using the semiclassical MSTQW equations of motion (2.176) in a flat background, namely $m_o \bar{a}^\mu = 0$, we see that (3.57) can be written as

$$m_o\big(\ddot{\bar{z}}^\mu + \ddot{\tilde{z}}^\mu + w^{\mu\alpha\beta\gamma}[\bar{z}]\, \tilde{h}_{\alpha\beta;\gamma}(\bar{z})\big) = 0 \tag{3.66}$$

which is equivalent to

$$m_o\big(\ddot{z}^\mu + w^{\mu\alpha\beta\gamma}[z]\, \tilde{h}_{\alpha\beta;\gamma}(z)\big) = O(\tilde{z}^2) \tag{3.67}$$

in Lorentzian coordinates where the components of the connection are zero. From (2.161) it follows that the second term in the above equation is the first order correction to the connection components (in an expansion in m_o) of a spacetime with an effectively stochastic metric given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} = \eta_{\mu\nu} + \kappa \xi_{\mu\nu}. \tag{3.68}$$

The stochastic motion of the particle through linear order in the worldline fluctuations is therefore a geodesic in this effectively stochastic spacetime. We remark that we started with quantized linear metric perturbations interacting with a quantum mechanical relativistic point mass. Provided that the particle worldline is decohered through its interaction with coarse-grained quantum fluctuations of $\hat{h}_{\mu\nu}$ we can identify a stochastic semiclassical limit for the particle's motion through the introduction of an auxiliary field $\xi_{\alpha\beta}$ that interacts with the particle and can be interpreted as a stochastic field with an associated probability distribution P_ξ. Up to an overall constant, this stochastic field is effectively a classical stochastic metric perturbation that adds to the flat background metric to generate a total metric given by (3.68).

3.2 Implications for gravitational wave observables

The motion of compact objects (e.g. black holes and neutron stars) in a binary system are candidate sources for detecting the gravitational waves these systems emit. While influences from quantized linear metric perturbations on such a background are expected to be negligibly small we investigate in this Section the flux of gravitational waves emitted by a massive particle moving in a curved background. By calculating the flux of radiation passing through an interferometer, say, we may learn about the stochastic semiclassical limit for the particle's motion.

From the MSTQW-Langevin equation and using (3.47) and the definition of the tail part of the retarded metric perturbations (3.54) we may write (3.48) as

$$m_o a^\mu(\tau) = m_o w^{\mu\alpha\beta\gamma}[z]\nabla_\gamma \left(h^{ret}_{\alpha\beta}(z) + \kappa\xi_{\alpha\beta}(z) \right) \tag{3.69}$$

where we recall that the direct part of the retarded propagator gives no contribution to the self-force at this order in the small mass m_o. From the above form it is tempting to define a (classical) stochastic metric perturbation[4]

$$\tilde{h}_{\alpha\beta}(x) = h^{ret}_{\alpha\beta}(x) + \kappa\xi_{\alpha\beta}(x), \tag{3.70}$$

[4]Note that this is not the same as the stochastic metric perturbation $\kappa\xi_{\alpha\beta}$ defined in (3.65).

which not only interacts with the particle but radiates far away to a gravitational wave detector.

With such an identification, the flux of the emitted gravitational radiation at frequency ω passing through a detector that is far from the particle is

$$F = \frac{\omega^2}{\kappa^2}\left\langle\langle\tilde{h}^{TT}_{\mu\nu}\tilde{h}^{TT\mu\nu}\rangle_\xi\right\rangle \tag{3.71}$$

$$= \frac{\omega^2}{\kappa^2}\left(\left\langle h^{ret,TT}_{\mu\nu}(x)h^{TT\mu\nu}_{ret}(x)\right\rangle + \hbar\,\kappa^2\left\langle D^{TT}_H(x,x)\right\rangle\right) \tag{3.72}$$

and is expressed in the transverse-traceless gauge [102]. The outer brackets denote an average over one period of the gravitational wave's oscillation whereas the inner brackets $\langle\ldots\rangle_\xi$ denote the stochastic average. The effect of the coarse-grained quantum field fluctuations is to impart a small quantum correction to the emitted flux of radiation. We remark that the coincidence limit of the Hadamard function appears so that the flux is formally divergent. In order to have a well-defined flux one needs to regularize D_H to obtain a finite result. In a certain sense, detection of this $O(\hbar)$ correction would provide a direct observation of perturbative quantum gravity but such a detection is likely hopeless with the current and next-generation gravitational wave interferometers.

While this identification of a stochastic metric perturbation $\tilde{h}_{\mu\nu}$ is appealing we wish to emphasize that the metric perturbation considered in this paper is a quantum variable. The identification of a classical stochastic metric perturbation $\tilde{h}_{\mu\nu}$ is therefore only formal and is suggested from the observation that the retarded field $h^{ret}_{\mu\nu}$ is independent of the state of the quantum field. However, the (c-number) stochastic field $\xi_{\mu\nu}$ contains some information about the state since its two-point function depends upon the quantum state of the metric perturbations

$$\left\langle \{\xi_{\alpha\beta}(x), \xi_{\gamma'\delta'}(x')\} \right\rangle_\xi = \hbar\, D^H_{\alpha\beta\gamma'\delta'}(x, x'). \tag{3.73}$$

For this reason the radiated flux (3.72) receives a small quantum correction.

Nevertheless, we observe that the leading order quantum contribution to the flux is purely *local*. It carries no information at leading order in m_o about the system that generated the metric perturbations $\tilde{h}_{\alpha\beta}$. All of that information is contained in the contribution from the retarded gravitational waves. Therefore, a detector measuring the flux of gravitational waves will detect the usual classical gravitational wave flux plus small corrections from the local quantum fluctuations of the metric perturbations in the region around the detector; the stochastic motion of the particle is not registered by the detector at this order.

3.3 Phenomenological noise and self-consistency

The noise $\eta_\mu[\bar{z}]$ in the Langevin equations describing the stochastic motion of the particle in the previous sections are obtained from coarse-graining the environment comprised of a linear quantum field in a curved spacetime. Our derivations of the equations of motion for the semiclassical and stochastic semiclassical worldlines assume a closed system to begin with. This treatment has the distinct advantage that it can preserve the self-consistency between the system and the environment in considering the effects of backreaction. For example, the noise kernel is intimately related to the quantum fluctuations of the field via the Hadamard function $D^H_{AB'}$. Without a self-consistent treatment of the environment's influence on the particle such a relation could not be unambiguously made. However, in many practical circumstances the stochastic dynamics of a system is treated phenomenologically with a noise term put in by hand to account for these (quantum) fluctuations. Quite generally, for a tensor field $\Phi_A(x)$ this description follows by stipulating the stochastic equations of motion

$$ma_\mu = F_\mu^{ext} + f_\mu[z] + \eta_\mu^+ \tag{3.74}$$

where F_μ^{ext} is some external force and f_μ is the self-force on the particle arising from its (non-local and history-dependent) interaction with the retarded field Φ_A^{ret}.

We denote the phenomenological noise η_μ^+ put in by hand by a + superscript to distinguish them from noise *derived* from first-principles considerations, as we have been dealing with until now. This add-on stochastic force could have a classical origin (e.g. high temperature thermal fluctuations of a bath) or it could have no known single identifiable origin. Furthermore, since the η_μ^+ is not derived from an initially closed system it is likely to be inconsistent with the dynamics of the trajectory by failing to satisfy a fluctuation-dissipation relation for worldline displacements around an equilibrium trajectory [37]. In a phenomenological treatment one also needs to *specify* the noise kernel $\langle \eta_\mu^+(\tau)\eta_{\nu'}^+(\tau')\rangle_{\eta^+}$ befitting the model, rather than *deriving* it.

While one may argue that a different choice of noise kernel corresponds to a different initial state for the quantum environment there does not seem to be a clear way to determine that state or to otherwise extract accurate information about the environment one is trying to model. In the self-consistent approach developed in this

Chapter the stochastic two-point functions of the worldline coordinates correspond to quantum two-point functions of the worldline operators so that the stochastic correlations of the worldline contain (most) of the information of the corresponding quantum correlations of $\hat{z}$ [24].

Ignoring these cautionary statements in this Section and the next, we remark that the analysis of the previous sections carry over for a phenomenological source of noise. Given any kind of noise the equations of motion for the fluctuations around the mean trajectory of the particle moving through a (classical) field subjected to the self-force from radiation reaction is given by

$$m_{\mu\nu}[\bar{z}]\ddot{\tilde{z}}^{\nu} + \gamma_{\mu\nu}[\bar{z}]\dot{\tilde{z}}^{\nu} + \kappa_{\mu\nu}[\bar{z}]\tilde{z}^{\nu} + \lim_{\epsilon\to 0}\int_{\tau_i}^{\tau-\epsilon} d\tau'\, K\big[\bar{z}^{\alpha}, \tilde{z}^{\alpha}\big] = r_{\mu\nu}[\bar{z}]\,\dddot{\tilde{z}}^{\nu} + \eta_{\mu}^{+} \tag{3.75}$$

where the kernel K is a functional of the semiclassical worldline and its stochastic fluctuations and the tensor coefficients $m_{\mu\nu}$, etc., are given in the previous Sections for the appropriate field under consideration.

3.4 Secular motions from stochastic fluctuations in external fields

It is interesting to observe that when a source of noise acts as a stochastic force on the particle in the presence of an external field that these classical fluctuations can give rise to a force that drifts the particle away from its semiclassical trajectory. In particular, the stochastic force on the particle causes it to undergo rapid motions that enables the particle to experience different values of the generally inhomogeneous external fields. Averaging over the stochastic fluctuations results in a noise-induced force that depends on the correlations of the stochastic force and the gradients of the external fields. As we will show, the noise-induced force is a second-order effect in terms of the worldline fluctuations. In this discussion we do not need to worry about quantum corrections from higher-order loops in the coarse-grained effective action Γ_{cg}, as the noise here is not necessarily of a quantum origin. Therefore, we can expand (3.74) beyond the linear order in the fluctuations of the particle trajectory.

In order to highlight the essential physics of this noise-induced force we consider the non-relativistic motion of an electrically charged particle moving in an external electromagnetic field in a flat background spacetime. Doing so allows us to focus on this particular issue rather than being distracted by the more complex and subtle technical details that arise in the fully relativistic problem in curved spacetime. We further justify these simplifying assumptions be remarking that the analogous drifting forces commonly encountered in plasma physics are for nonrelativistic charges moving in flat space (e.g., grad-B drift, curvature drift, etc.) [25, 103, 41].

To find the noise-induced force in the electromagnetic case, we begin with the non-relativistic limit of (3.74) describing the motion of an electric point charge moving in an external electromagnetic field and coupled to a phenomenological stochastic force $\eta^+(z)$

$$m\, a_i = e\Big(E_i(z) + \epsilon_{ijk} u_j B_k(z)\Big) + f_i[z] + \eta_i^+(z) \tag{3.76}$$

where the external force $\mathbf{F}^{ext}$ is taken to be the usual Lorentz force. (In this Section, the Latin indices take on the values from 1 to 3 and ϵ_{ijk} is the Levi-Civita totally antisymmetric tensor with $\epsilon_{123} = 1$.) The self-force $f_i = 2e^2/3c^3\,\dot{a}_i$ contains no tail term here since there is no background curvature to backscatter the emitted radiation. Substituting $z = z_0 + \delta z$ in (3.76) and expanding in powers of the fluctuations gives

$$\begin{aligned} m\,(a_{0i} + \delta a_i) \;=\;& eE_i(z_0) + e\delta z^a \partial_a E_i(z_0) + \frac{e}{2}\delta z^a \delta z^b \partial_a \partial_b E_i(z_0) + e\epsilon_{ijk} u_{0j} B_k(z_0) \\ & + e\epsilon_{ijk}\delta u_j B_k(z_0) + e\epsilon_{ijk} u_{0j}\delta z^a \partial_a B_k(z_0) + e\epsilon_{ijk}\delta u_j \delta z^a \partial_a B_k(z_0) \\ & + \frac{e}{2}\epsilon_{ijk} u_{0j}\delta z^a \delta z^b \partial_a \partial_b B_k(z_0) + \frac{2e^2}{3c^3}\dot{a}_{0i} + \frac{2e^2}{3c^3}\dot{\delta a}_i \\ & + \eta_i^+(z_0) + \delta z^a \partial_a \eta_i^+(z_0) + \cdots . \end{aligned} \tag{3.77}$$

We assume that the variations of the external fields occur over distances much larger than $|\delta \mathbf{z}|$. The worldline fluctuations are assumed very fast compared to the averaged motion so we expect that $|\delta \mathbf{u}| \gg |\mathbf{u}|$ and similarly for the accelerations. This allows us to make the approximation

$$m\delta a_i - e\epsilon_{ijk}\delta u^j B^k(z_0) \approx \eta_i^+(z_0) \tag{3.78}$$

where the stochastic force η_i^+ drives the worldline fluctuations. We assume that the typical time scale of the fluctuations Δt is much larger than the time for light to cross the "classical" size of the particle $\sim 2e^2/3mc^2$ so that the radiation reaction term, which is proportional to $\dot{\delta a}$, can be neglected. The equation of motion for z_0 then becomes, after taking the stochastic expectation value of (3.77),

$$\begin{aligned} ma_{0i} \approx\;& e\left(E_i(z_0) + \epsilon_{ijk} u_0^j B^k(z_0)\right) + \frac{2e^2}{3c^3}\dot{a}_{0i} + \frac{e}{2}\left\langle \delta z^a \delta z^b \right\rangle \partial_a \partial_b E_i(z_0) \\ & + e\epsilon_{ijk}\left\langle \delta u^j \delta z^a \right\rangle \partial_a B^k(z_0) + \frac{e}{2}\epsilon_{ijk} u_0^j \left\langle \delta z^a \delta z^b \right\rangle \partial_a \partial_b B^k(z_0) + \left\langle \delta z^a \partial_a \eta_i^+(z_0) \right\rangle . \end{aligned} \tag{3.79}$$

The terms involving the stochastic averages are defined as the noise-induced drift force so that

$$F_i^{drift} = \frac{e}{2}\langle \delta z^a \delta z^b \rangle \, \partial_a \partial_b E_i(z_0) + e\epsilon_{ijk}\langle \delta u^j \delta z^a \rangle \, \partial_a B^k(z_0)$$
$$+\frac{e}{2}\epsilon_{ijk} u_0^j \langle \delta z^a \delta z^b \rangle \, \partial_a \partial_b B^k(z_0) + \langle \delta z^a \partial_a \eta_i^+(z_0) \rangle \quad (3.80)$$

and

$$m\,\mathbf{a}_0 = e\mathbf{E}(z_0) + e\,\mathbf{u}_0 \times \mathbf{B}(z_0) + \frac{2e^2}{3c^3}\dot{\mathbf{a}}_0 + \mathbf{F}^{drift}[z_0]. \quad (3.81)$$

The first three terms of the drift force (3.80) are a result of the variation of the external fields with position and from the curvature of the external electromagnetic field lines. The last term of the drift results from the worldline fluctuation away from z_0 coupling to the variation of the stochastic force. If the stochastic force is independent of position then this term will vanish identically and any contribution to the noise-induced drift will result from variations in the applied electric and magnetic fields. The stochastic fluctuations therefore manifest as a slowly varying force causing the particle to move away from its semiclassical trajectory.

To solve (3.81) we need the solution to (3.78) for the fluctuations in terms of the stochastic force η^+. The solution is easily found to be

$$\delta u_i = \frac{1}{m}\left(K^{-1}\right)_i^{\ a} \int_{t_i}^{t} dt' \, K_{ab} \eta^b(z_0) \quad (3.82)$$

where the homogeneous solution is ignored since we are interested in the effect of the noise on the evolution of the guiding center (or background) trajectory z_0. The integrating factor K_{ab} is

$$
\begin{aligned}
K_{ab} &= \exp\left\{ -\frac{e}{m}\epsilon_{abm}\int_{t_i}^{t} dt'\, B^m(z_0^{\ell'})\right\} \\
&= \delta_{ab} - \frac{e}{m}\epsilon_{abm}\int_{t_i}^{t} dt'\, B^m(z_0^{\ell'}) \\
&\quad + \frac{e^2}{2m^2}\epsilon_{arm}\epsilon^r{}_{bn}\int_{t_i}^{t} dt' \int_{t_i}^{t'} dt''\, B^m(z_0^{\ell'})B^n(z_0^{\ell''}) + \cdots . \qquad (3.83)
\end{aligned}
$$

Integrating δu over time gives the worldline fluctuations δz. The terms in δz not involving the stochastic force are ignored so that the noise-induced force becomes

$$
F_i^{drift} = \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 \int_{t_i}^{t} dt_3 \int_{t_i}^{t_3} dt_4 \Big(F_{imn} + F_{imnp}\partial^p \Big) \Big\langle \eta^{+m}(z_0(t_2))\eta^{+n}(z_0(t_4))\Big\rangle_{\eta^+} \qquad (3.84)
$$

where the kernels F_{imn} and F_{imnp} are given by

$$
\begin{aligned}
F_{imn} &= \frac{e}{2m^2}\partial_a\partial_b\left(E_i(z_0) + \epsilon_{ijk}u_0^j B^k(z_0)\right)(K_1^{-1})^{br}K_{2rm}(K_3^{-1})^{as}K_{4sn} \\
&\quad + \frac{2e}{m^2}\epsilon_{ijk}\partial_a B^k(z_0)\delta(t-t_1)(K_1^{-1})^{jr}K_{2rm}(K_3^{-1})^{as}K_{4sn} \qquad (3.85) \\
F_{imnp} &= \frac{4}{m}\delta(t-t_1)\delta(t_1-t_2)\delta_{im}(K_3^{-1})_p{}^s K_{4sn} \qquad (3.86)
\end{aligned}
$$

and where a subscript on the integrating factor K refers to the designated time, e.g. $K_1 = K(t_1)$, etc. This expression for the drift force (3.84) is then used to solve for the worldline coordinates $\mathbf{z}_0$ in (3.81). This is a difficult task given the nonlinear and non-Markovian behavior of the dynamics. The history-dependent contribution coming from the drift force requires a knowledge of z_0 and the stochastic correlation function for all times in the past as exhibited in (3.84).

This noise-induced force is quite similar to the motion of an electron in an inhomogeneous external magnetic field, for example. If the external field does not change much on the scale of a Larmor radius (the radius at which the electron undergoes a helical motion about a magnetic field line) then the charge's velocity receives a contribution from time-averaging over the rapid Larmor oscillations. Analogously, the rapid Larmor oscillations corresponds to the rapid worldline fluctuations δz and the time-average is similar to the stochastic average.

If the applied fields vary over a distance much larger than the Larmor radius then the usual drifts that occur in plasma physics [25, 41] can still be deduced from (3.81). However, these drifts have been lumped into determining the motion of z_0 in order to isolate the new noise-induced force apart from the usual plasma physics drifts (e.g. the grad-B drift, curvature drift, etc.).

If one replaces the phenomenological noise kernel in (3.84) with that in (3.10) then the background acceleration $\mathbf{a}_0$ will have a contribution proportional to $\hbar$. However, we cannot make this naive replacement because the approximations used in deriving the Langevin equations (3.9) are valid only to linear order in the fluctuations $\tilde{z}$. In particular, $\mathbf{F}^{drift}$ results from expanding the Langevin equation with phenomenological noise (3.76) to *second* order. For the self-consistent derived noise η_μ, we can no longer ignore the effects of loop contributions in the effective action Γ_{cg} that we have so far neglected in the Gaussian approximation of the influence functional and in the (stochastic) semiclassical limit. In fact, a stochastic semiclassical limit for the particle's evolution may no longer exist so that the question of the existence of a drift force becomes one concerning the one-loop quantum corrections to the background motion.

3.5 Similarities with stochastic semiclassical gravity

The features of the particle dynamics seen in the above discussions are typical of nonequilibrium open quantum systems. History-dependent behavior is present in the equations of motion for the system and if a renormalization procedure is required it is usually a time-dependent prescription, as we saw earlier from renormalizing the bare mass of a point charge. Furthermore, the noise kernel is generically non-local in time and is determined by the quantum fluctuations of the coarse-grained environment variables. This formalism does not allow for arbitrary noise kernels since this would destroy the self-consistency between the system and environment evolution. A particular example that contains these features is stochastic semiclassical gravity, which we will briefly describe and compare with below.

Stochastic semiclassical gravity (SSG) is a self-consistent theory of the stochastic dynamics of a classical spacetime containing quantum matter fields. SSG goes beyond semiclassical gravity, for which the geometry is driven by the expectation of the (renormalized) stress tensor, in that the quantum field fluctuations also contribute to the spacetime dynamics through a classical stochastic source. The spacetime is therefore driven by both the quantum expectation value of the renormalized stress tensor and a classical stochastic stress-tensor-like object, ξ_{ab}. For an introduction and review of this subject see [104, 105] and also [106] for a discussion of the domain of validity of SSG.

As an open quantum system, the quantum field fluctuations are coarse-grained using the CTP formalism (see Section 2.3) to study the self-consistent evolution of the (classical) geometry. The quantum fluctuations manifest themselves as stochastic noise thereby imparting a stochastic nature to the spacetime. The resulting Einstein-Langevin equation for the linearized metric perturubations $h_{\alpha\beta}$ is

$$G^{(1)}_{\alpha\beta}[g+h] = \kappa \left\langle \hat{T}^{(1)}_{\alpha\beta}[g+h] \right\rangle_{ren} + \kappa\, \xi_{\alpha\beta}[g]. \tag{3.87}$$

The superscript $^{(1)}$ denotes that those quantities contain all terms to first order in the metric fluctuations $h_{\alpha\beta}$. It should be noted that the counterterms needed to cancel the divergences coming from the stress tensor expectation value have been absorbed into the definition of $\langle \hat{T}^{(1)}_{\alpha\beta} \rangle_{ren}$. The renormalized stress tensor expectation value (evaluated in a Gaussian state) contains an integration over the past history of the metric fluctuations and so the dynamics is generally non-Markovian. This is like what is seen in the self-force Langevin equations (3.9) where the tail term is analogous to the expectation value of the renormalized stress tensor in (3.87). The (covariantly conserved) stochastic source tensor $\xi_{\alpha\beta}$ has zero mean and its correlator is given in terms of the Hadamard function of the stress tensor fluctuations $\hat{t}_{\alpha\beta} = \hat{T}_{\alpha\beta} - \langle \hat{T}_{\alpha\beta} \rangle$

$$\left\langle \{ \xi_{\alpha\beta}(x;g), \xi_{\gamma\delta}(x';g) \} \right\rangle_{\xi} = \hbar \left\langle \{ \hat{t}_{\alpha\beta}(x;g), \hat{t}_{\gamma\delta}(x';g) \} \right\rangle. \tag{3.88}$$

The correlator of the stress tensor fluctuations on the right side does not vanish on a space-like hypersurface. This reflects the fact that the quantum field correlations are themselves non-local. Compare this with the correlator in (3.5) which is also non-local.

SSG also suffers from runaway solutions since the finite contributions to the counterterms needed to cancel the divergences appearing from the expectation value of the stress tensor are quadratic in the curvature. This makes SSG a theory with derivatives higher than two, similar to the radiation reaction terms in the self-force equations derived above, which were of third order in the τ derivatives. One can fix the usual pathologies associated with higher-order derivative theories by reducing the order of the Einstein-Langevin equation through an iterative process to second order thereby yielding well-behaved solutions. However, one needs to be careful to use order reduction at scales that are consistent with the derivation of the Einstein-Langevin equation.

Finally, the symmetrized quantum two-point functions of the metric fluctuations $h_{\alpha\beta}$ can be written in terms of intrinsic fluctuations, representing the dispersion in the initial conditions, and induced fluctuations, encoding the information about the fluctuations of the quantum matter [106]. Just like with the particle motion, one cannot simply use any noise kernel for modeling stochastic metric fluctuations. One needs to do a careful analysis that ensures the self-consistency of the metric and quantum matter dynamics and the existence of fluctuation-dissipation relations.

3.6 The quantum regime and the validity of the quasi-local expansion and order reduction

In most of Chapter 2 and this Chapter we use real-time path integral methods (including the influence functional and the CTP generating functional) and various approximations to obtain the equations of motion for the particle, both for its semi-classical and stochastic semiclassical limits. In this Section, the domain of validity of the quasilocal expansion and the semiclassical treatment will be discussed and compared with the relevant scales for weak and strong radiation damping.

We introduce a regulator Λ for controlling the ultraviolet divergences appearing in the direct part of the self-force such that $\Lambda^2\sigma \gg 1$ with σ small and approaching zero. After expanding σ near coincidence (see Section 2.4.2 for more details) the time scale of the quasi-local expansion $\Delta\tau = |s|$ is governed by $\Delta\tau \gg \Lambda^{-1}$. Recall that for elapsed times larger than $\sim \Lambda^{-1}$ the time-dependent coefficients $c_{(n)}$ and $g_{(n)}$ in (2.114) and (2.115), respectively, rapidly approach their limiting values.

The semiclassical and stochastic semiclassical limits are obtained here by using the Gaussian approximation to compute the reduced density matrix, which amounts to working at the tree-level in both the particle and the field sectors. This implies that $\Delta\tau$ should be much longer than the time scale for creating particle pairs, $\Delta\tau \gg \hbar/m = \lambda_C$ where λ_C is the particle's Compton wavelength.

Another relevant scale appears when trying to find unique, physical solutions to the self-force equations that contain a term with a third derivative of the particle's position. As is well known, this term is responsible for the problematic existence of pre-accelerated, acausal and runaway solutions. These kinds of solutions can be eliminated if the self-force is weak compared to other external forces acting on the particle. In particular, an asymptotic expansion in powers of $r_0 \sim 2e^2/3mc^2$, called the Landau approximation or order-reduction [107], can be employed to obtain physical solutions that require only an initial position and velocity. The Landau approximation converts the ALD equation (of third order) to the so-called Landau-Lifshitz equation (of second order). See [108, 109] for an interesting discussion of these equations and order-reduction. The quantity r_0 is often called the "classical" size of the charge [107].

Using order-reduction, the lowest order solution is found by simply ignoring the self-force so that the radiation damping is assumed weak. The time-scale of the dynamics is then determined mostly by the external force acting on the particle so that if F_μ^{ext} varies on a scale λ_{ext} then $\Delta\tau \sim \lambda_{ext}$. In curved spacetime the self-force will be weak if $r_0 \ll \Delta\tau$ and the scale associated with the spacetime curvature λ_R is much greater than r_0.

For the electron, $\lambda_C = 137 r_0 \gg r_0$ and one might choose to set $\Lambda^{-1} \sim \lambda_C$ to justify ignoring the effects of electron-positron pair production from appearing in the semiclassical particle dynamics. For an ionized atom, for example, its physical size R_0 dwarfs its "classical" size r_0 and Compton wavelength λ_C so that one might choose $\Lambda^{-1} \sim R_0$ in order to ignore any effect resulting from the object's finite spatial extent and describe the object effectively as a point particle. Our approach would need to be augmented if we wished to include the effects of extended charged bodies. One way to do this is to include all possible terms into the point-particle action $S_{pp}[z]$ that are consistent with reparametrization and general coordinate invariance. This approach provides a model-independent way to parametrize the contributions to the dynamics from the body's size. The couplings of these extra terms can then be determined by matching this effective theory to the theory describing the body on microscopic scales. See [26] which takes a similar approach to construct a framework to derive the post-Newtonian equations describing the motions of neutral (spinning) extended bodies interacting gravitationally. See also Chapters 4 and 5 where we use the effective field theory approach to study the self-force problem exclusively in the gravitational context.

Recently, in the context of plasma physics, Koga [110] has investigated the validity of the Landau approximation (and hence the assumption of weak radiation damping) for the classical ALD equation in flat space by numerically integrating the Landau-Lifshitz equation forward in time and, using the final position, velocity, and acceleration from that, integrating the ALD equation backward in time. If the initial position and velocity of the particle differ significantly from the backward-evolved solution of the ALD equation at the initial time then one can assume the Landau approximation has broken down. Koga does this for a counter-propagating electron and ultraintense laser beam (intensity $\sim 10^{22}$ W/cm^2). He finds that the Landau approximation is valid so long as the laser wavelength λ_0 is greater than the Compton wavelength. For λ_0 much smaller than λ_C, he finds disagreement between the solutions of the Landau-Lifshitz and ALD equations. This may imply that the radiation damping is no longer weak but at this scale these equations cannot be fully trusted since quantum effects may become important. It may be interesting to study the effects of strong radiation damping within our formalism or even using the closely related CTP formalism to incorporate the effects of quantum loop corrections to the (quantum) particle dynamics.

Chapter 4

Effective field theory approach for extreme mass ratio inspirals: First order self-force

In Chapters 2 and 3 we derive the leading order scalar, electromagnetic and gravitational self-force on a particle moving in an arbitrary curved background spacetime from first principles. We treat the particle as a quantum mechanical worldline interacting with a linear quantum field. The mass and size of the particle is assumed to be sufficiently small that quantum fluctuations manifest as classical stochastic forces. However, the particle must be heavy and large enough that the particle worldline is decohered from its interactions with the coarse-grained quantum fluctuations of the field.

On the other hand, when the particle has a mass representative of astrophysical objects, which are typically measured in terms of the solar mass, the open quantum system description developed in the previous Chapters yields to an effective field theory description for the classical motion of the particle subsystem [111]. Quantum loop corrections from the field and the intrinsic quantum mechanical worldline fluctuations are very strongly suppressed due to the large separation in the mass scales.

In this Chapter and the next we use the methods of effective field theory (EFT) to derive the self-force on a compact object moving through an arbitrary curved background spacetime. We have in mind that the background is provided by a supermassive black hole such that its curvature length scale $\mathcal{R}$ is much larger than the size of the compact object r_m. In particular, the Schwarzschild and Kerr solutions are appropriate backgrounds for studying the extreme mass ratio inspiral (EMRI) of a small black hole or neutron star when $\mu \equiv r_m/\mathcal{R}$ is small. The smallness of μ implies that it is a good expansion parameter to construct our perturbation theory with. These binary systems are expected to be good candidates for detecting gravitational wave signatures using the space-based gravitational wave interferometer LISA [2]. However, our formalism is general enough to describe the motion of a compact object through an arbitrary background, including those spacetimes sourced by some form of stress-energy and those possessing a cosmological constant.

We begin the Chapter with a brief review of the effective field theory approach for post-Newtonian binary systems introduced in [26] and developed further in [112, 70, 113, 114]. A collection of effective field theories are constructed to describe the motion of two slowly moving compact objects in a flat background. In particular, the compact objects are treated as effective point particles, the worldlines of which carry non-minimal operators describing the moments from companion-induced tidal deformations as well as possible spin degrees of freedom and other intrinsic moments. The use of point particles to source the metric perturbations (or gravitons) about the flat background spacetime implies the appearance of divergences. The EFT approach is a quantum field theory by construction. As a result, there exists a well-established bank of tools and techniques for regularizaing these divergences and renormalizing the parameters and coupling constants of the theory.

Being an effective theory it is renormalizable precisely because the divergences can be absorbed into renormalizing the many coupling constants of the non-minimal world-line operators. The use of dimensional regularization is particular useful in effective field theories because the renormalization group equations are mass-independent for this scheme, thereby allowing for the calculation of the fewest possible Feynman diagrams at any particular order in the (relative) velocity of the binary system [46].

After a brief discussion of the EFT for post-Newtonian binaries, we identify the scales involved in the extreme mass ratio inspiral scenario. In particular, we allow for the compact object to move with relativistic speeds in strong field regions of the background space. This is to be contrasted with the post-Newtonian EFT of [26] wherein the bodies move slowly through a weak gravitational field.

We then construct an effective point particle description for the motion of the compact object. In particular, we introduce all possible terms into the point particle action that are consistent with general coordinate invariance and reparameterization invariance (and invariance under $SO(3)$ rotations for a non-spinning spherically symmetric compact object). In doing so we capture the effects of tidal deformations induced by the background curvature as well as the effects from spin and other intrinsic moments. By implementing a matching procedure using coordinate invariant observables we can match the observables of the effective point particle theory with the long wavelength limit of observables in the full "microscopic" theory to determine the values of the coupling constants of the non-minimal terms. As we show in Section 4.5 this allows us to deduce the order at which finite size effects affect the motion of the compact object through the statement of an Effacement

Principle. To our knowledge this has not been given in the literature before for the EMRI scenario.

We spend a significant amount of time developing the EFT approach in detail in Section 4.4 in order to clearly outline the steps involved in constructing the effective theory. We introduce the CTP, or in-in, generating functional as the foundation for our calculations. Unlike in [26] we do not base our EFT on the in-out formalism. In a flat background spacetime the in-out formalism is acceptable to use since the in- and out-vacua, used to define the vacuum transition amplitude that defines the generating functional, are equivalent up to an irrelevant phase. However, the in-out formalism is constructed to describe scattering processes and not real-time evolution. In the presence of a non-vanishing background curvature, as occurs in the EMRI scenario, this handicap becomes evident as we show in Section 5.1.2. In the in-out approach the equations of motion for the effective particle are not causal. The remedy is to start with the CTP generating functional, which, being an initial value formulation of quantum field theory, guarantees real and causal particle equations of motion [98].

The power counting rules are derived in Section 4.4.2. Power counting is a generalization of dimensional analysis but is crucial for determining how the Feynman rules scale with the expansion parameter μ. The Feynman rules and their scaling with μ are derived in Section 4.4.3. Once the scaling of the Feynman rules are known we determine all of the tree-level Feynman diagrams that appear at a particular order. Those diagrams containing graviton loops are safely ignored. We also assemble the diagrams that include the non-minimal worldline operators describing the finite size of the compact object. Significantly, this lets us determine the order in μ at which finite size effects enter the particle equations of motion. The power counting rules allow for the EFT approach to be an efficient and systematic framework for calculating the self-force to any order in perturbation theory. Furthermore, by knowing how each Feynman diagram scales with μ we can study a particular physical interaction that is of interest by focusing our attention on a single diagram or on a few diagrams without having to calculate every contribution that appears at that order and at lower orders. For example, the leading order spin-spin interaction contributes to the self-force at third order in μ and can be calculated from the appropriate Feynman diagram.

In Section 4.4.4 we propose a method for regularizing the divergences that appear in the effective action. Our approach utilizes a mixture of distributional and momentum space techniques within the context of dimensional regularization. We know from previous work and from Chapter 2 that the finite part of the self-force is generally non-local and history dependent. However, the ultraviolet divergences are quasi-local and independent of the history of the effective point particle's motion. To isolate the quasi-local divergence from the non-local finite part we use the method of Hadamard's *partie finie*, or finite part, from distribution theory. (See Appendix E for a brief review of the definitions and concepts of distribution theory relevant in this work.) Upon isolating the divergence from the non-local, finite remainder we then use the momentum space representation for the propagator to calculate the divergent contributions. Through second order in μ we find that only power divergences appear, which can be immediately set to zero when evaluated using dimensional regularization [63].

To regularize the theory we use the momentum space representation of the propagator in a curved background[1], which was first derived for a scalar field by Bunch and Parker in [64]. Their method is straightforward but not efficient for higher spin fields, including metric perturbations (i.e. gravitons in a curved space). In Appendix D we develop a novel method, which is applicable for any tensor field, for computing the momentum space representation of the Feynman propagator. We also show that the method is sufficiently general to do the same for any quantum two-point function, including the retarded propagator $D_{ret}(x, x')$. Our method makes use of diagrammatic techniques borrowed from perturbative quantum field theory. In Riemann normal coordinates, we expand the field action in terms of the displacement from the point x'. The series can be represented in terms of Feynman diagrams, which allows for an efficient evaluation of each term in the expansion. Furthermore, we prove that some of the diagrams are zero to all orders. This identity is not recognized by Bunch and Parker even though its relation to certain steps made in their calculations is evident.

We then derive the first order self-force equation of MSTQW [20, 21]. While we have already derived this equation in Section 2.4.5 we find it beneficial to show clearly how the effective field theory approach is implemented in detail, including the regularization of divergences, for an actual calculation of the effective action and the self-force. This is particularly useful when calculating the second order self-force

[1]See also the work of [115] who consider a somewhat different approach for a scalar field with a classical background configuration in $\lambda\phi^4$ theory in a curved spacetime.

in the next Chapter.

4.1 Effective field theory approach for post-Newtonian binaries

Before proceeding to construct an effective field theory for extreme mass ratio inspirals we briefly summarize the original work of [26], which introduces effective field theory techniques to the long-studied field of the gravitational problem of binary inspirals.

The aim of [26] is to describe the motion of two slowly moving bodies through a weak gravitational field using effective field theory techniques in order to generate a perturbative expansion in powers of the relative velocity. One of the many benefits of using an effective field theory approach is that the method is systematic and efficient so that there is in principle no obstacle to calculating to any order in the velocity. The obvious intent of such a program is to go beyond the current 3PN calculations[2] and continue to higher orders (e.g., 4PN) to obtain more accurate gravitational waveforms.

The authors in [26] start by replacing the compact objects with effective point particles. These are described by an action consisting of the usual point particle action plus all possible terms that are consistent with general coordinate invariance and reparameterization invariance of the worldline. Then, the in-out generating functional is formed to derive the effective action

$$\exp\left\{iS_{eff}[z]\right\} = \int \mathcal{D}h_{\mu\nu} \exp\left\{iS_{pp}[x, \eta + h] + iS[\eta + h]\right\} \tag{4.1}$$

[2]See [116] for the 2PN potential equations of motion using the EFT approach developed in [26].

where S_{pp} is the effective point particle action and $S[\eta + h]$ is the Einstein-Hilbert action for the spacetime metric.

Before integrating out the metric perturbations the authors observe that it is useful to separate the metric perturbations into potential H and radiation $\bar{h}$ contributions

$$h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu}. \tag{4.2}$$

This is suggested by the fact that the slowly moving bodies see a nearly instantaneous gravitational potential but manage to radiate gravitational waves due to their mutual accelerations. However, this decomposition is actually required in order that the ensuing Feynman diagrams all scale homogeneously with the relative velocity, v. In this way, the perturbative expansion in v is consistent and can be constructed to any order.

Integrating out the potential gravitons using perturbation theory yields a theory of point particles interacting with potentials. The radiation gravitons and the particle worldlines are non-dynamical at this stage and can be treated as external sources. In this effective theory, valid at the orbital scale of the binary, the authors derive the Einstein-Infeld-Hoffman potential [72] as a check of their method.

The last effective theory the authors construct involves integrating out the radiation gravitons. This yields a theory of point particles interacting with gravitational waves. As such, the authors derive the famous power spectrum for quadrupolar gravitational radiation by calculating the first non-vanishing contribution to the imaginary part of the effective action; the real part of the effective action generates equations of motion while the imaginary part is related to the power of the emitted gravitational radiation[3].

With using an effective field theory approach it is not too surprising that some of the parameters of the theory undergo classical renormalization group (RG) scaling. In fact, the appearance of such RG scaling is used by the authors to show that there are no finite size effects up to v^6 order. In their words, "whenever one encounters a log divergent integral at order v^6 in the potential, one may simply set it to zero. Its value cannot affect physical predictions." This therefore resolves the problem of the undetermined regularization parameters that appears from regularizing the singular integrals encountered in the traditional PN expansion techniques.

4.2 Extreme mass ratio inspiral as an EFT

Consider the motion of a compact object (a black hole or a neutron star with a mass m ranging from a few to ~ 100 solar masses) moving through the spacetime of a supermassive black hole (with mass $M \sim 10^{5-7} M_\odot$). We have in mind that the compact object moves in a stationary background provided by the supermassive black hole, such as the Schwarzschild or Kerr spacetimes. Such spacetimes are appropriate for a description of the extreme mass ratio inspiral (EMRI) in which the compact object is bound by the gravitational pull of the supermassive black hole. By emitting gravitational waves the binary system loses energy until the

[3]The appearance of non-local tail terms in the post-Newtonian equations of motion suggest that one may need to use the in-in, or CTP, formalism in order to guarantee causal dynamics in the EFT approach.

compact object eventually plunges into the supermassive black hole. The emission of gravitational radiation from such a system is expected to be detected with the anticipated construction and launch of the LISA space-based interferometer [2].

It is believed that most supermassive black holes lurking in the middle of galaxies, which are thought to host the prime sources of gravitational wave emissions detectable by LISA, are spinning and clean in the sense that most, if not all, of the surrounding material has already fallen into the black hole[4]. Because of this the Kerr background is perhaps the most astrophysically relevant spacetime for the extreme mass ratio inspiral. The Kerr solution is vacuous ($R_{\mu\nu} = 0$), stationary and stable under small perturbations [117] and possesses two Killing fields. The first is time-like ξ^α and describes time-translation invariance everywhere outside of the ergoregion. The second is space-like ψ^α and describes the axial rotation invariance of the spacetime. There are also the Ernst [118] and Preston-Poisson [119] spacetimes to consider. These solutions represent a black hole immersed in an external magnetic field. Astrophysically speaking, the external magnetic fields that a black hole at the center of a galaxy experiences are relatively weak and unlikely to significantly affect the motion of the compact object until a very high order in the perturbation theory.

The length scales involved with the EMRI are two-fold. The smallest scale is set by the size of the compact object itself, denoted r_m. For an astrophysical black hole its size is $r_{bh} = 2G_N m \sim m/m_{pl}^2$ where $m_{pl}^{-2} = 32\pi G_N$ in units where $\hbar = c = 1$ (in this and the next Chapter only)[5]. For a neutron star with a mass $\approx 1.4 M_\odot$ and

[4]Active galactic nuclei are the notable exception.

[5]We follow the conventions of [85] so that the metric has mostly positive signature $(-,+,+,+)$

a radius of ≈ 10 km it follows that $r_{ns} \approx 4.8 G_N m \sim m/m_{pl}^2$. Therefore, it is to be expected that the size of the compact object, be it a black hole or a neutron star, is of the order of its mass.

The second relevant scale is the radius of curvature of the background spacetime, $\mathcal{R}$. We take $\mathcal{R}$ to be related to the following curvature invariant

$$\mathcal{R} = \left(R_{\mu\alpha\nu\beta} R^{\mu\alpha\nu\beta}\right)^{-1/4}. \tag{4.3}$$

The Riemann tensor has units of $(\text{mass})^2$ implying that the units of $\mathcal{R}$ are $(\text{mass})^{-1}$, which is a unit of length, as expected. For a (possibly rotating) stationary supermassive black hole the radius of curvature is

$$\mathcal{R} \sim \sqrt{\frac{m_{pl}^2 r^3}{M}} \tag{4.4}$$

where r is the typical orbital distance for the compact object away from the central black hole. For example, r is the geometric mean of the semi-major and semi-minor axes of a compact object in an inclined elliptical orbit. In an approximately circular orbit r is the orbital radius and for a particle moving faster than the escape velocity r is the impact parameter.

In the strong field regime where $r \sim M/m_{pl}^2$ the curvature scale is also $\sim M/m_{pl}^2$ implying that $r/\mathcal{R} \sim m/M$. Hence, the mass M sets the scale for the long wavelength sector of the effective theory. In what follows, we will denote the large distance scale by $\mathcal{R}$ instead of M to keep our constructions easily applicable to other spacetimes that may not possess a supermassive black hole mass.

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and we follow the Green's functions conventions of [66], after changing their definitions to the mostly positive signature. See Appendix A for more details.

The typical variation in time and space of the background is $\gtrsim \mathcal{R}$. The wavelength λ of radiated metric perturbations from the compact object in a bound orbit is

$$\lambda \sim \sqrt{\frac{m_{pl}^2 r^3}{M}} \sim \mathcal{R}, \tag{4.5}$$

which shows that the wavelength of the gravitational waves does not provide a separate scale independently from $\mathcal{R}$.

This is to be contrasted with the EFT approach for the post-Newtonian (PN) expansion introduced in [26]. As discussed in Section 4.1, there are three effective field theories that can be constructed. The first is the description of the compact object as an effective point particle. The remaining two EFTs rely on the small velocity assumption and allows the metric perturbations to be classified as two types: potential and radiation modes. The slow velocity assumption therefore induces a scale separation that is manifest in the structure of the total metric perturbation about flat space.

Our construction of an EFT does not rely on the slow motion of the bodies nor on the assumption that they move through a weakly curved region of space-time. Quite the contrary, we allow for the compact object to move relativistically through the strong field region of the supermassive black hole background space-time. As a result, the metric perturbations generated by the motion of the compact object cannot be partitioned naturally into an instantaneous potential and radiation contributions.

The expansion parameter we will use to construct an EFT for the EMRI comes from the ratio of the body's size to the curvature scale

$$\frac{r_m}{\mathcal{R}} \sim \frac{m}{m_{pl}^2 \mathcal{R}} \equiv \mu, \tag{4.6}$$

that is, the ratio of the two relevant length scales involved, which is very small. For the EMRI's thought to be detected with the LISA space-based gravitational interferometer μ takes values between 10^{-5} and 10^{-7}, which corresponds to LISA's observable frequency bandwidth. Being so small almost entirely over the dynamical time scales of the inspiral, μ is a good parameter for building a perturbation theory within the context of effective field theory.

Utilizing the dissimilar magnitudes of the compact object's size and the background curvature scale, we can construct two kinds of effective field theories. The first describes the compact object, in isolation from other external sources, as an effective point particle. By allowing for all possible worldline self-interaction terms that are consistent with the symmetries of the theory we can account for the tidal deformations, spin and intrinsic moments that the compact object may experience when it does interact with external sources. The second EFT is valid at scales $\gtrsim \mathcal{R}$ and results from integrating out the metric perturbations (or gravitons as we will often call them in this Chapter). The resulting theory is that of an effective point particle undergoing self-force in the background spacetime, which evolves self-consistently with the emitted gravitational waves. Using a matching procedure we can establish the values of the coupling constants appearing in the effective point particle action.

4.3 EFT of an isolated, compact object

The smallest length scale in the EMRI scenario is the size of the compact object r_m. In applying the EFT formalism we first construct an effective point particle theory for the small mass m. This allows for a point particle description of the compact object's motion through the background spacetime while taking into account any tidally induced moments, or finite size effects, that might affect its motion. This effective point particle description is the first of two effective field theories that we will construct in this Chapter.

In the full theory describing the motion of a neutron star and the dynamics of the spacetime metric it moves in, the total action is given by

$$S_{tot} = S[g] + S_{ns}[g; \rho, p \ldots]. \tag{4.7}$$

The quantities in the neutron star action, $\rho, p, \ldots$, are the appropriate hydrodynamic variables necessary to describe the internal dynamics of the neutron star whatever its equation of state. If the compact object under consideration is a small black hole then there is only the dynamics of vacuum spacetime to consider, which is described entirely by the Einstein-Hilbert action

$$S[g] = 2m_{pl}^2 \int d^4x\, g^{1/2} R \tag{4.8}$$

where R is the Ricci curvature scalar of the spacetime and g is the absolute value of the metric's determinant.

The effective point particle description of the compact object is constructed by "integrating out" the short distance degrees of freedom at the scale $r_m \sim m/m_{pl}^2$.

In doing so we introduce an effective point particle action S_{pp} to describe the motion of the compact object so that the total action becomes

$$S_{tot} = S[g] + S_{pp}[z, g]. \tag{4.9}$$

Here $z^\alpha(\lambda)$ are the coordinates of the particle worldline and λ is its affine parameterization.

Being a description of the extended compact object the effective point particle action should include all possible terms that are consistent with the symmetries of the theory, which are general coordinate invariance and worldline reparameterization invariance. For the discussion here, we will assume that the compact object is perfectly spherical so that it carries no permanent moments. For example, this implies excluding spinning compact objects in our construction, at least for now[6]. Hence, S_{pp} should also be invariant under $SO(3)$ transformations. Regarding these considerations the most general such action is

$$\begin{aligned} S_{pp}[z,g] &= -m \int d\tau + c_R \int d\tau\, R + c_V \int d\tau\, R_{\mu\nu} \dot{z}^\mu \dot{z}^\nu \\ &\quad + c_E \int d\tau\, E_{\mu\nu} E^{\mu\nu} + c_B \int d\tau\, B_{\mu\nu} B^{\mu\nu} + \cdots, \end{aligned} \tag{4.10}$$

which is effectively an expansion in powers of the compact body's radius r_m over the wavelength of the gravitational waves λ. This can be interpreted as a multipole expansion where the multipoles carry information about the induced moments that the background curvature imparts to the compact object. We showed earlier that in the strong field region of a supermassive black hole the wavelength of the metric

[6]We will introduce spin and determine the influence it has on the effective particle's motion in Section 5.2.

perturbations is the same order as the curvature scale of the background spacetime, $\lambda \sim \mathcal{R}$. This implies that the above multipole expansion is equivalently given in powers of μ so that each term has a definite scaling with μ, which we will later confirm.

The tensors $E_{\mu\nu}$ and $B_{\mu\nu}$ are the electric- and magnetic-type tensors of the Weyl curvature, defined as

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}\dot{z}^\alpha\dot{z}^\beta \tag{4.11}$$

$$B_{\mu\nu} = \epsilon_{\mu\alpha\beta\lambda}C^{\alpha\beta}{}_{\nu\rho}\dot{z}^\lambda\dot{z}^\rho \tag{4.12}$$

where $\dot{z}^\alpha$ is the particle's 4-velocity. When contracted with $\dot{z}^\alpha$ these vanish,

$$E_{\mu\nu}\dot{z}^\nu = B_{\mu\nu}\dot{z}^\nu = 0. \tag{4.13}$$

The electric-type tensor is symmetric $E_{\mu\nu} = E_{\nu\mu}$ whereas the magnetic-type tensor is not $B_{\mu\nu} \neq B_{\nu\mu}$.

We will find it beneficial to write (4.10) as an integration over an arbitrary affine parameter λ instead of the proper time τ of the worldline

$$\begin{aligned} S_{pp}[z,g] &= -m\int d\lambda\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{1/2} + c_R\int d\lambda\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{1/2}R \\ &\quad + c_V\int d\lambda\frac{R_{\mu\nu}\dot{z}^\mu\dot{z}^\nu}{\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{1/2}} + c_E\int d\lambda\frac{E_{\mu\nu}E^{\mu\nu}}{\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{3/2}} \\ &\quad + c_B\int d\lambda\frac{B_{\mu\nu}B^{\mu\nu}}{\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{3/2}} + \cdots . \end{aligned} \tag{4.14}$$

Doing so will guarantee that we derive the correct equations of motion for the particle's trajectory. After the relevant calculations and variations have been performed we will then parameterize the worldline with the proper time.

The terms in the effective point particle action (4.10) proportional to the Ricci curvature vanish at leading order in μ. The equations of motion for the full metric

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{pp}_{\mu\nu}[z, g] = O(\mu), \tag{4.15}$$

where $T^{pp}_{\mu\nu}$ is the stress-energy tensor of the effective point particle, can be used to set the c_R and c_V terms to zero to leading order. Equivalently, for the c_R term for example, we can redefine the metric $g_{\mu\nu}$ in terms of a new metric $g'_{\mu\nu}$ through the field redefinition [26]

$$g_{\mu\nu}(x) = g'_{\mu\nu}(x) \left[1 + \frac{\xi}{2m_{pl}^2} \int d\tau \frac{\delta^4(x - z(\tau))}{g'^{1/2}} \right]. \tag{4.16}$$

This conformal transformation implies that the Einstein-Hilbert action is, to linear order in the arbitrary parameter ξ,

$$2m_{pl}^2 \int d^4x \, g^{1/2} R(g) = 2m_{pl}^2 \int d^4x \, g'^{1/2} R(g') + \xi \int d\tau \, R. \tag{4.17}$$

The term in S_{pp} linear in R then appears with the constant $c_R + \xi$, which can be set to zero since ξ is arbitrary. That is, choose $\xi = -c_R$ and the term proportional to the Ricci scalar no longer contributes to S_{pp}. One can find a similar field redefinition to remove the term proportional to c_V. Using the metric field equations or, equivalently, performing a field redefinition of the metric one can remove all occurrences of the Ricci tensor in the effective point particle action. It follows that the non-minimal couplings in S_{pp} contain terms that depend only on the Riemann curvature tensor.

These field redefinitions allow for the effective point particle action (4.10) to be written as

$$S_{pp}[z, g] = -m \int d\tau + c_E \int d\tau \, E_{\mu\nu} E^{\mu\nu} + c_B \int d\tau \, B_{\mu\nu} B^{\mu\nu} + \cdots , \tag{4.18}$$

In a later section we will show that the non-minimal couplings in S_{pp} are entirely negligible for calculating the self-force of MSTQW that the linear metric perturbations impart on the compact object. The first order (MSTQW) self-force is sufficient for computing gravitational waveforms and generating templates for LISA to *detect* gravitational waves from EMRIs[7]. We can confidentally ignore the finite size corrections in deriving the leading order self-force and describe the extended body simply as a point particle to a sufficiently high accuracy. However, for precisely determining the masses, spins, etc. of the binary constituents one must use more accurate higher-order templates, which can be computed by knowing the higher order contributions to the self-force [61, 121, 122]. In Section 4.5 we determine the order in μ that tidally induced moments will affect the motion of the compact body.

In the next section we derive the equations of motion for the compact object using the EFT approach. These equations, which describe the self-force on the mass m, were previously found by Mino, Sasaki and Tanaka [20] using matched asymptotic expansions and independently by Quinn and Wald [21] using axiomatic methods. In principle, we can compute the formal equations of motion to higher orders in μ thereby extending the work of [20] and [21], which we will do in next Chapter through second order in μ.

[7]Actually, this is more than sufficient as recent work suggests that the less accurate "kludge" waveforms may be adequate for the detection phase of at least a certain class of EMRIs [120, 58].

4.4 EFT derivation of MSTQW self-force equation

In the previous Section, we outlined the construction of an effective field theory that replaces the extended compact object by an effective point particle. This allows for tidal deformations to be described and parameterized through the induced multipole moments that are characterized by the curvature-dependent non-minimal terms in S_{pp}. This effective point particle description is valid for distances large compared to the size of the body $r_m \sim m/m_{pl}^2$. This is similar to a multipole moment expansion in which the compact object is treated as a point particle with multipolar operators defined on the particle's worldline. A familiar example is provided by the dipole approximation in electromagnetism in which two charges separated by a distance can be approximated by a single particle with a dipole moment. Radiation with wavelengths much longer than the charge separation interacts with an effective point particle carrying a vector operator on its worldline. In this section we will construct an EFT for the motion of the effective particle by integrating out the metric perturbations at the scale of the radius of curvature $\mathcal{R}$ and find that many such multipolar operators reside on the particle's worldline to describe the extended nature and induced moments of the compact object.

Denote the background metric by $g_{\mu\nu}$ so that the total metric is given by the background plus the perturbations generated by the presence and motion of the compact object

$$g_{\mu\nu}^{\text{full}} = g_{\mu\nu} + \frac{h_{\mu\nu}}{m_{pl}}. \tag{4.19}$$

The metric perturbations $h_{\mu\nu}$ are presumed to be small so that $|h_{\mu\nu}| \ll m_{pl}$. We will occasionally make use of a slight shorthand notation

$$\bar{h}_{\mu\nu} \equiv \frac{h_{\mu\nu}}{m_{pl}} \tag{4.20}$$

for the dimensionless ratio of the metric perturbation to the Planck mass. From (4.9) the total action describing the interactions of the metric perturbations and the particle is given by the sum of the Einstein-Hilbert and effective point particle actions,

$$S_{tot}[g + \bar{h}, z] = S[g + \bar{h}] + S_{pp}[g + \bar{h}, z]. \tag{4.21}$$

We expand the Einstein-Hilbert action to quadratic order in $h_{\mu\nu}$ and find that

$$\begin{aligned} S[g_0 + \bar{h}] \;\; = \;\; & 2m_{pl} \int d^4x\, g^{1/2} \left(\Box h - h^{\alpha\beta}{}_{;\alpha\beta} \right) \\ & + \frac{1}{2} \int d^4x\, g^{1/2} \left(2h_{\alpha\beta;\gamma} h^{\alpha\gamma;\beta} - h_{\alpha\beta;\gamma} h^{\alpha\beta;\gamma} - 2h_{;\alpha} \left(h^{\alpha\beta}{}_{;\beta} - \frac{1}{2}\, h^{;\alpha} \right) \right) \\ & + O(h^3) \end{aligned} \tag{4.22}$$

where the trace of the metric perturbations is $h = g_{\mu\nu} h^{\mu\nu}$. We have also used the fact that the background metric is vacuous. The first term is an integral over a total derivative and can be written in terms of a surface integral

$$\int_V d^4x\, g^{1/2} \left(\Box h - h^{\alpha\beta}{}_{;\alpha\beta} \right) = \int_{\partial V} d\Sigma_\alpha\, g^{1/2} \left(h^{;\alpha} - h^{\alpha\beta}{}_{;\beta} \right), \tag{4.23}$$

which we take to vanish at the boundary of the integration region[8]. The expanded

[8]Strictly speaking there are other boundary terms in the Einstein-Hilbert action that we should include. See [123] for a discussion of these terms and related details. Since these play no essential role here we will not worry about these terms in this work.

Einstein-Hilbert action is then

$$\begin{aligned} S[g+\bar{h}] &= \frac{1}{2}\int d^4x\, g^{1/2}\left(2h_{\alpha\beta;\gamma}h^{\alpha\gamma;\beta} - h_{\alpha\beta;\gamma}h^{\alpha\beta;\gamma} - 2h_{;\alpha}\left(h^{\alpha\beta}{}_{;\beta} - \frac{1}{2}\,h^{;\alpha}\right)\right) \\ &\quad +O(h^3). \end{aligned} \tag{4.24}$$

The remaining contribution at this order is the kinetic term for $h_{\mu\nu}$ and provides the propagator corresponding to some appropriate boundary conditions (e.g., retarded, Feynman, etc.). However, the propagator is ill-defined because of the underlying gauge symmetry of the action, which is expressed as an invariance of the action under infinitesimal coordinate transformations on the background spacetime. General relativity is a gauge theory in this respect and so one must break the gauge symmetry by choosing a particular gauge, or constraint, for the metric perturbations. Below we will use the gauge-fixing procedure developed by Faddeev and Popov [91].

For notational convenience, we write the expansion of the Einstein-Hilbert action in the following way,

$$S[g+\bar{h}] = S^{(2)} + S^{(3)} + \cdots \tag{4.25}$$

where the term $S^{(n)}$ denotes those terms proportional to n powers of $h_{\mu\nu}$. In particular, the $n=2$ term is the kinetic term for the metric perturbations.

We also need to expand the point particle action in powers of $h_{\mu\nu}$. Using (4.14) we find the following expansion

$$\begin{aligned} S_{pp}[z,g+\bar{h}] &= -m\int d\lambda\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{1/2} + \frac{m}{2m_{pl}}\int d\lambda\frac{h_{\alpha\beta}(x)\,\dot{z}^\alpha\dot{z}^\beta}{\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{1/2}} \\ &\quad -\frac{m}{8m_{pl}^2}\int d\lambda\frac{h_{\alpha\beta}(x)h_{\gamma\delta}(x)\,\dot{z}^\alpha\dot{z}^\beta\dot{z}^\gamma\dot{z}^\delta}{\left(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta\right)^{3/2}} + O(h^3). \end{aligned} \tag{4.26}$$

As with the Einstein-Hilbert action, we introduce the following convenient notation

$$S_{pp}[z, g + \bar{h}] = S_{pp}^{(0)} + S_{pp}^{(1)} + S_{pp}^{(2)} + \cdots \tag{4.27}$$

where each term is proportional to the indicated powers of the metric perturbation.

4.4.1 The closed-time-path effective action

The construction of an effective field theory for the motion of the effective point particle in a curved spacetime begins with the CTP, or in-in, generating functional

$$\begin{aligned} Z[j^\mu, j'^\mu, J^{\mu\nu}, J'^{\mu\nu}] \; = \; & \int_{CTP} \mathcal{D}z^a_\mu \int_{CTP} \mathcal{D}h^a_{\mu\nu} \, \exp\bigg\{ iS[g + \bar{h}] - iS[g + \bar{h}'] \\ & + iS_{pp}[z, g + \bar{h}] - iS_{pp}[z', g + \bar{h}'] \\ & + i \int d\lambda (j^\mu z_\mu - j'^\mu z'_\mu) \\ & + i \int d^4x \, g^{1/2} (J^{\mu\nu} h_{\mu\nu} - J'^{\mu\nu} h'_{\mu\nu}) \bigg\}, \end{aligned} \tag{4.28}$$

which is first introduced in Section 2.3. Using the in-in formalism, the particle equations of motion are guaranteed to be real and causal because the CTP generating functional is an initial value formulation of quantum field theory that remains valid in non-trivial backgrounds [98]. On the contrary, the in-out generating functional describes scattering processes via transition amplitudes between states in the far past and future and makes no claim, nor is able, to generate real and causal dynamics in curved backgrounds.

In presenting (4.28) we use the notation of Section 2.3 for the particle coordinates and fields. However, it is much more convenient to relabel the unprimed and primed variables with a lowercase Latin index $a, b, c, ...$ from the beginning of the alphabet. These indices equal 1 and 2 for an unprimed and primed variable, respectively. We introduce the so-called CTP metric c_{ab} that lowers and raises these indices where

$$c_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = c^{ab}. \tag{4.29}$$

For a current "contracted" with a scalar field, for example, the notation implies

$$\begin{aligned} J_a \Phi^a &\equiv c^{ab} J_a \Phi_b & (4.30) \\ &= J_1 \Phi_1 - J_2 \Phi_2 = J^1 \Phi^1 - J^2 \Phi^2 = J\Phi - J'\Phi'. & (4.31) \end{aligned}$$

Lastly, we write the actions involving the unprimed and primed variables as a single term so that the Einstein-Hilbert action, for example, can be written as

$$S[g + \bar{h}^a] \equiv S[g + \bar{h}^1] - S[g + \bar{h}^2] = S[g + \bar{h}] - S[g + \bar{h}']. \tag{4.32}$$

The difference between a power of the field and a CTP index a, b should be clear from context. Having established this new notation we write the CTP generating functional using this more compact and convenient form,

$$\begin{aligned} Z[j^a_\mu, J^a_{\mu\nu}] = \int_{CTP} \mathcal{D} z^a_\mu \int_{CTP} \mathcal{D} h^a_{\mu\nu} \exp \Big\{ & iS[g + \bar{h}^a] + iS_{pp}[z^a, g + \bar{h}^a] \\ & + i \int d\lambda \, j^\mu_a z^a_\mu + i \int d^4x \, g^{1/2} J^{\mu\nu}_a h^a_{\mu\nu} \Big\}. \end{aligned} \tag{4.33}$$

Notice the similarity in appearance to the in-out generating functional.

Calculating derivatives of the generating functional with respect to the external current $J^{\mu\nu}_a$ generates time-ordered correlation functions of the quantum metric perturbations $\hat{h}^a_{\mu\nu}$

$$\big\langle 0, \mathrm{in} | \bar{T} \, \hat{h}^{a_1}_{\mu_1\nu_1}(x_1) \cdots \hat{h}^{a_n}_{\mu_n\nu_n}(x_n) | 0, \mathrm{in} \big\rangle_{\mathrm{full}} = \frac{1}{Z} \frac{\delta^n Z}{\delta i J^{\mu_1\nu_1}_{a_1}(x_1) \cdots \delta i J^{\mu_n\nu_n}_{a_n}} \tag{4.34}$$

where $\bar{T}$ is the CTP time-ordering operator defined in Section 2.3. For example, the full graviton Feynman propagator is calculated from the generating functional by

$$\begin{aligned} iD^F_{\alpha\beta\gamma\delta}(x,x') &= \big\langle 0,\mathrm{in}|T\,\hat{h}^1_{\alpha\beta}(x)\hat{h}^1_{\gamma\delta}(x')|0,\mathrm{in}\big\rangle \\ &= \frac{1}{Z}\,\frac{\delta^2 Z[z^a(\lambda),J^{\mu\nu}_a]}{\delta iJ^{\alpha\beta}_1(x)\,\delta iJ^{\gamma\delta}_1(x')}\bigg|_{J^{\mu\nu}_a=0}. \end{aligned} \tag{4.35}$$

The effective point particle worldline $z^\alpha(\lambda)$ acts as a fixed source in computing these field correlation functions. As a result, (4.34) describes the full correlation functions and includes the effects from (nonlinear) particle-field interactions.

We mentioned earlier that in writing down a well-defined propagator for the metric perturbations we must break the gauge symmetry that is preserved by infinitesimal coordinate transformations on the background spacetime. We follow the approach of Faddeev and Popov [91] and introduce a gauge-fixing action

$$S_{gf} = m^2_{pl}\int d^4x\, g^{1/2} G_\alpha G^\alpha \tag{4.36}$$

that picks the gauge $G_\alpha[h_{\mu\nu}] \approx 0$ for the metric perturbations. The $\approx$ denotes weak equality in the sense of Dirac [90]. As we discuss later we will be dealing with tree-level interactions only so there is no need to introduce ghost fields into the gravitational action.

We choose the Lorenz gauge for the trace-reversed metric perturbations, defined as

$$\psi_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h, \tag{4.37}$$

so that the gauge-fixing function is

$$G_\alpha[h_{\mu\nu}] = \psi_{\alpha\beta}{}^{;\beta} = h_{\alpha\beta}{}^{;\beta} - \frac{1}{2} h_{;\alpha} \approx 0. \tag{4.38}$$

In this gauge, the kinetic term in (4.24) is considerably simplified to

$$S^{(2)} = \frac{1}{2} \int d^4x \, g^{1/2} \left(h_{\alpha\beta;\gamma} h^{\alpha\beta;\gamma} - \frac{1}{2} h_{;\alpha} h^{;\alpha} - 2h^{\alpha\beta} R_{\alpha\ \beta}^{\ \gamma\ \delta} h_{\gamma\delta} \right), \tag{4.39}$$

which applies to both the $h^1_{\alpha\beta}$ and $h^2_{\alpha\beta}$ metric perturbations (equivalently, the unprimed and primed fields, respectively).

The generating functional can now be written as

$$\begin{aligned} Z[j^\mu_a, J^{\mu\nu}_a] \;=\; & e^{iS^{(0)}_{pp}[z^a]} \int_{CTP} \mathcal{D} z^a_\mu \int_{CTP} \mathcal{D} h^a_{\mu\nu} \, \exp \Bigg\{ iS^{(2)}[h^a] + iS^{(1)}_{pp}[z^a, h^a] + iS^{(2)}_{pp}[z^a, h^a] \\ & + \sum_{n=3}^{\infty} \left(iS^{(n)}[h^a] + iS^{(n)}_{pp}[z^a, h^a] \right) \\ & + i \int d\lambda \, j^\mu_a z^a_\mu + i \int d^4x \, g^{1/2} J^{\mu\nu}_a h^a_{\mu\nu} \Bigg\}. \end{aligned} \tag{4.40}$$

We have factored out the lowest order point particle contribution since it is independent of the metric perturbations. If the particle is regarded as a test body then it produces no perturbations about the background metric and its motion is therefore a geodesic of the background spacetime.

Perturbation theory in the CTP formalism is quite similar to that in the in-out formalism. The fact that the metric perturbations couple linearly to the external current $J^{\mu\nu}_a$ implies that every occurrence of the field in (4.40) can be replaced by a functional derivative of the external current,

$$h^a_{\mu\nu}(x) \to -i \frac{\delta}{\delta J^{\mu\nu}_a(x)}. \tag{4.41}$$

This rule allows for the generating functional to be written as

$$\begin{aligned} Z[j^\mu_a, J^{\mu\nu}_a] \;=\; & \int_{CTP} \mathcal{D} z^a_\mu \, \exp \Bigg\{ iS^{(0)}_{pp}[z^a] + i \int d\lambda \, j^\mu_a z^a_\mu \\ & + \sum_{n=1}^{\infty} iS^{(n)}_{pp} \left[z^a, -i \frac{\delta}{\delta J_a} \right] + \sum_{n=3}^{\infty} iS^{(n)} \left[-i \frac{\delta}{\delta J_a} \right] \Bigg\} Z_0[J^{\mu\nu}_a] \end{aligned} \tag{4.42}$$

since the interaction terms can be taken outside of the path integral. The quantity Z_0 is the free field generating functional for the metric perturbations

$$Z_0[J_a^{\mu\nu}] = \int_{CTP} \mathcal{D}h_{\mu\nu}^a \exp\left\{ iS^{(2)}[h^a] + i\int d^4x\, g^{1/2} J_a^{\mu\nu} h_{\mu\nu}^a \right\} \tag{4.43}$$

and is calculated by integrating the quadratic terms to give

$$Z_0[J_a^{\mu\nu}] = \exp\left\{ -\frac{1}{2} J_a^{\alpha\beta} \cdot G^{ab}_{\alpha\beta\gamma'\delta'} \cdot J_b^{\gamma'\delta'} \right\}. \tag{4.44}$$

Upon defining the interaction Lagrangian as

$$\int d^4x\, \mathcal{L}_{int}\left[z^a, -i\frac{\delta}{\delta J_a} \right] \;=\; \sum_{n=1}^{\infty} iS_{pp}^{(n)}\left[z^a, -i\frac{\delta}{\delta J_a} \right] + \sum_{n=3}^{\infty} iS^{(n)}\left[-i\frac{\delta}{\delta J_a} \right] \tag{4.45}$$

we find that the generating functional can be written in the form

$$\begin{aligned} Z[j_a^\mu, J_a^{\mu\nu}] \;=\; & \int_{CTP} \mathcal{D}z_\mu^a \exp\left\{ iS_{pp}^{(0)}[z^a] + i\int d\lambda\, j_a^\mu z_\mu^a + i\int d^4x\, \mathcal{L}_{int}\left[z^a, -i\frac{\delta}{\delta J_a^{\mu\nu}} \right] \right\} \\ & \times \exp\left\{ -\frac{1}{2} J_a^{\alpha\beta} \cdot G^{ab}_{\alpha\beta\gamma'\delta'} \cdot J_b^{\gamma'\delta'} \right\}. \end{aligned} \tag{4.46}$$

Notice that this is expressed as a certain functional derivative operator acting on a Gaussian functional of the external currents $J_a^{\mu\nu}$.

We are interested in this dissertation with calculating the self-force on a compact object and are not concerned with the full correlation functions (4.34) of the metric perturbations generated by the moving mass m. The only correlation functions we will be using in this construction are the free field graviton two-point functions, constructed without reference to the effective point particle worldlines $z^a(\tau)$. Therefore, throughout the remainder we set the external current in (4.46) to zero, $J_a^{\mu\nu} = 0$. Computing the (partial) Legendre transform with respect to the particle current gives the effective action

$$S_{eff} \equiv \Gamma[\langle \hat{z}^\mu_a \rangle] = -i \ln Z[j^\mu_a, J^{\mu\nu}_a = 0] - \int d\lambda \, j^a_\mu \langle \hat{z}^\mu_a \rangle. \tag{4.47}$$

and is the quantity of interest that we calculate in the next Section.

Before continuing to the next Section we remark that our construction, up through (4.46), is fully self-consistent. From (4.46) we can calculate not only the semiclassical equations of motion for the compact object (i.e., the self-force equations) but we can also determine the configuration for the metric perturbations by calculating the (real and causal) equations of motion for the graviton one-point function $\langle \hat{h}^a_{\mu\nu} \rangle$. To see this, we keep the external graviton current $J^{\mu\nu}_a$ arbitrary and perform the full Legendre transform with respect to both particle and field currents. Then the equations of motion for the worldline and the graviton one-point functions are

$$0 = \left. \frac{\delta \Gamma}{\delta \langle \hat{z}^a_\mu \rangle} \right|_{z^1 = z^2, h^1 = h^2, j_a, J_a = 0} \quad \text{and} \quad 0 = \left. \frac{\delta \Gamma}{\delta \langle \hat{h}^a_{\mu\nu} \rangle} \right|_{z^1 = z^2, h^1 = h^2, j_a, J_a = 0} \tag{4.48}$$

Likewise, we could compute the flux of gravitational radiation from the $(0i)$ components of the expectation value of the stress tensor, which is computed from the variation of $-i \ln Z$ with respect to the background metric upon setting $j^\mu_a = 0$. We will reserve ourselves to only study the self-force in this disseration. In future work we will compute the graviton one-point functions and the gravitational wave flux in this manner [81, 82, 83].

4.4.2 Power counting rules

All of the terms following the kinetic term $S^{(2)}$ in (4.40) represent self interactions of the field and various particle-field interactions. Each of these interaction terms may be represented by a Feynman diagram. In turn, these diagrams may be assigned a rule that tells us how to assemble the appropriate diagrams that contribute to the effective action S_{eff} at a specific order in μ. To write down all of the relevant diagrams at a particular order we need to know how each of the interaction terms in (4.40) scale with μ. The scaling rules that we will develop here are called power counting rules and are essentially a generalization of dimensional analysis. We first develop the power counting rules for the parameters of the effective field theory; we ignore for now the non-minimal point particle couplings in S_{pp} (e.g., $c_{R,V}$, $c_{E,B}$, $\ldots$).

As discussed previously, the curvature scale $\mathcal{R}$ describes the length scale of temporal and spatial variations of the curvature in the background spacetime. This implies that each of the spacetime coordinates scale according to

$$x^\mu \sim \mathcal{R}. \tag{4.49}$$

From the kinetic term for the metric perturbations we deduce that if $S^{(2)} \sim 1$ then

$$1 \sim \mathcal{R}^4 h^2 \left(\frac{1}{\mathcal{R}}\right)^2 \sim \mathcal{R}^2 h^2 \tag{4.50}$$

and the metric perturbation scales with $\mathcal{R}$ as

$$h_{\mu\nu} \sim \frac{1}{\mathcal{R}}. \tag{4.51}$$

Table 4.1: Power counting rules

x^μ	$h_{\mu\nu}$	L	m/m_{pl}
$\mathcal{R}$	$1/\mathcal{R}$	$m\mathcal{R}$	$\sqrt{\mu L}$

The particle-field interactions, indicated by the terms $S_{pp}^{(n)}$, contain inverse powers of the Planck mass m_{pl}. To see how these factors of the Planck mass are involved with the power counting we remark that the presumed existence of a rotational Killing field ψ^α in the background spacetime allows for us to define the conserved angular momentum of a geodesic, chosen to lie in the equatorial plane for convenience, as

$$L = m g_{\alpha\beta}\psi^\alpha \dot{x}^\beta = m r^2 \frac{d\phi}{d\lambda} \tag{4.52}$$

where $\dot{x}^\beta$ is the 4-velocity of the geodesic, λ is an affine parameter on the geodesic, and the second equality follows from evaluating L in polar coordinates (e.g. Boyer-Lindquist coordinates for the Kerr background) . The leading order angular momentum of the effective point particle is determined by the leading order motion, which is a geodesic of the background spacetime. Therefore, L is the leading order (conserved) angular momentum and scales as

$$L \sim \frac{mr^2}{\mathcal{R}} \sim m\mathcal{R} \tag{4.53}$$

in the strong field region of the supermassive black hole background.

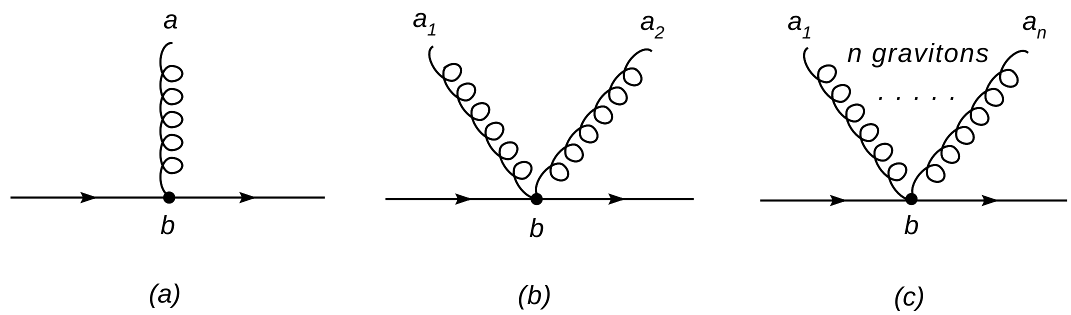

Figure 4.1: Particle-field vertices. Diagram (a) gives the Feynman rule for $iS_{pp}^{(1)}$ and diagram (b) gives the rule for $iS_{pp}^{(2)}$. The last diagram in (c) is the coupling of n gravitons to the particle worldline. The labels $a_1, a_2, \ldots$ and b are CTP indices and take values of 1 and 2.

With this estimation of the particle's angular momentum we find that the ratio

$$\frac{m}{m_{pl}} \sim \frac{m}{m_{pl}^2 \mathcal{R}} \frac{m_{pl}}{m} (m\mathcal{R}) \sim \mu \frac{m_{pl}}{m} L \tag{4.54}$$

implies the following scaling

$$\frac{m}{m_{pl}} \sim \sqrt{\mu L}. \tag{4.55}$$

The four scaling laws in (4.49), (4.51), (4.53) and (4.55) determine the power counting rules for identifying the appropriate Feynman diagrams that enter into the evaluation of the effective action. We list these power counting rules in Table (4.1).

Having in hand the power counting rules for the various parameters in the theory we turn our attention to power counting the interactions terms in (4.40). We begin with the diagrams for the two interaction terms $S_{pp}^{(1)}$ and $S_{pp}^{(2)}$ shown in Figs.(4.1a) and (4.1b). The curly line denotes a two-point function D_{ab} of the metric perturbation (i.e. of a graviton). The straight line denotes the effective point

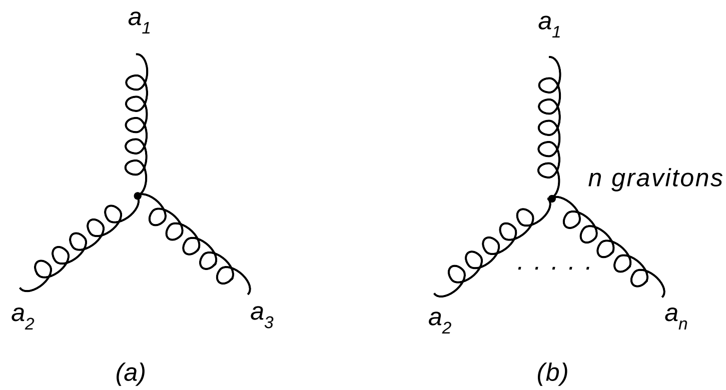

Figure 4.2: Graviton self-interaction vertices. Diagram (a) gives the Feynman rule for $iS^{(3)}$ and diagram (b) gives the rule for the interaction of n gravitons. The a_i labels are CTP indices.

particle. We remark that from the point of view of the gravitons, the particle acts as an external source that couples to the metric perturbations. As such, the straight line does *not* represent the physical propagation of the compact object. However, the straight line does invoke an intuitive picture of the particle-field interactions, which proves to be very useful when calculating the effective action.

The power counting of $S_{pp}^{(1)}$ and $S_{pp}^{(2)}$ is given below

$$\text{Fig. (4.1a)} = i\frac{m}{2m_{pl}}\int d\tau\, h_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \sim \frac{m}{m_{pl}}\,d\tau\, h \sim \sqrt{\mu L} \tag{4.56}$$

$$\text{Fig. (4.1b)} = -i\frac{m}{8m_{pl}^2}\int d\tau\, h_{\alpha\beta}h_{\gamma\delta}\dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma\dot{x}^\delta \sim \frac{m}{m_{pl}^2}\,d\tau\, h^2 \sim \mu. \tag{4.57}$$

The power counting of n gravitons interacting with the effective particle, as shown in Fig.(4.1c), is easily shown to be

$$\text{Fig. (4.1c)} = iS_{pp}^{(n)} \sim \frac{m}{m_{pl}^n}\,d\tau\, h^n \sim \mu\left(\frac{L}{\mu}\right)^{1-\frac{n}{2}}. \tag{4.58}$$

The self-interaction vertices that result from the nonlinearity of the Einstein-Hilbert action are given in Fig.(4.2). The first diagram gives the cubic self-interaction

Table 4.2: Power counting rules for interaction terms

$iS^{(n)}_{pp}$	$iS^{(n)}$
$\mu\left(\frac{L}{\mu}\right)^{1-n/2}$	$\left(\frac{L}{\mu}\right)^{1-n/2}$

term $S^{(3)}$ and scales as

$$\text{Fig. (4.2a)} = iS^{(3)} \sim m_{pl}^2 d^4x\, \partial^2 \frac{h^3}{m_{pl}^3} \sim \sqrt{\frac{\mu}{L}} \tag{4.59}$$

while the second diagram gives the self-interaction of n gravitons,

$$\text{Fig. (4.2b)} = iS^{(n)} \sim m_{pl}^2 d^4x\, \partial^2 \frac{h^n}{m_{pl}^n} \sim \left(\frac{L}{\mu}\right)^{1-\frac{n}{2}}. \tag{4.60}$$

From Table (4.2) we see that the power counting indicates that every type of interaction term involving any number of gravitons scales as L^p where $p \leq 1$.

4.4.3 Feynman rules and calculating the effective action

We now turn to calculating the effective action $S_{eff}[z_a]$ from (4.40). Taking the logarithm of both sides it follows from standard quantum field theory arguments [66, 93, 94] that

$$iS_{eff}[z^a] \;=\; -im\int d\tau_a + \begin{pmatrix} \text{sum of all} \\ \text{connected diagrams} \end{pmatrix}. \tag{4.61}$$

By "connected diagrams" we mean those contiguous diagrams constructed using the Feynman rules for the interaction terms in (4.40). However, we are only interested in those connected diagrams that contribute at the classical level since the quantum corrections due to graviton loops on the motion of an astrophysical body are utterly negligible. In fact, treating the particle as an open quantum system as in Chapters 2 and 3 we find that the influence functional calculated by coarse-graining the (linear) metric perturbations has a magnitude of

$$\big|F[z,z']\big| = \exp\left\{ -\frac{1}{64m_{pl}^2}\, T_-^{\alpha\beta}\cdot D^H_{\alpha\beta\gamma'\delta'}\cdot T_-^{\gamma'\delta'} \right\} \tag{4.62}$$

(See Section 2.4.5 for more details.) The mass of the compact object is typically between $1-100$ solar masses. We deduce that the worldline of a solar mass compact object will fluctuate by

$$\delta z \sim 10^{-35} M \sim 10^{-24}\text{cm} \tag{4.63}$$

where the last estimate assumes a $10^5 M_\odot$ supermassive black hole. (In arriving at this estimate we expand the argument of $|F$ about $z' = z$ and keep the leading order contribution.) Therefore, the effective point particle worldline is truly decoherent[9] since the worldline fluctuations are grossly suppressed. We will simply represent the quantum expectation value of the worldline coordinates $\langle \hat{z}^a \rangle$ by their semiclassical values z^a.

A diagram with ℓ graviton loops scales as $L^{1-\ell}$, in units where $\hbar = 1$. Therefore, those diagrams that scale linearly with L correspond to classical processes and

[9]There will always be non-zero worldline fluctuations, the existence of which will be necessary for computing the semiclassical equations of self-force. However, these fluctuations are so small that only tree-level processes are relevant.

provide the dominant contributions to the effective action so that

$$iS_{eff}[z^a] = -im \int d\tau_a + \left(\begin{array}{c} \text{sum of all } O(L) \\ \\ \text{connected diagrams} \end{array} \right) + \left(\begin{array}{c} \text{higher order graviton} \\ \\ \text{loop corrections} \end{array} \right). \tag{4.64}$$

In this manner we have a systematic method of computing the self-force equations order by order in μ.

The relationship between the connected diagrams, the interaction terms and the power counting is provided by the Feynman rules so that given a diagram at a given order in μ we can translate these into mathematical expressions. The Feynman rules are the following:

1. A vertex is represented by a factor of the particle-field interaction $iS_{pp}^{(n)}$ or the field self-interaction $iS^{(n)}$ as appropriate,

2. Each endpoint and vertex is labeled by a CTP index and can be classified as being of type-1 or type-2. An extra minus sign is associated with each vertex labeled by a 2 (type-2),

3. Include a factor of the graviton two-point function D_{ab} connecting vertices of type a and b,

4. Sum over all CTP indices,

5. Include a symmetry factor.

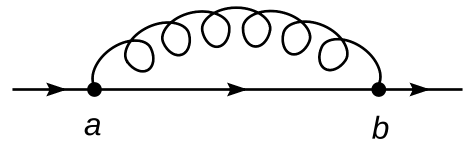

Figure 4.3: The diagram contributing to the first-order self-force of MSTQW.

We will show how these rules are implemented as we continue through the remainder of the Chapter.

To derive the MSTQW self-force equation we only need those diagrams that contribute at $O(\mu L)$. From the Feynman rules for the diagrams in Figs. (4.1) and (4.2) it follows that there is only one such diagram at this order, which is shown in Fig.(4.3). Therefore, the effective action to first order in μ is

$$iS_{eff}[z^a] = -im \int d\tau_a + \text{Fig.}(4.3) + O(\mu^2 L) \tag{4.65}$$

where

$$\text{Fig.}(4.3) \quad = \quad i^2 \left(\frac{1}{2!}\right) \left(\frac{m}{2m_{pl}}\right)^2 \sum_{a,b=1}^{2} \int d\tau \int d\tau' \, \dot{z}_a^\alpha \dot{z}_a^\beta \, D^{ab}_{\alpha\beta\gamma'\delta'}\big(z_a^\alpha, z_b^{\alpha'}\big) \dot{z}_b^{\gamma'} \dot{z}_b^{\delta'}, \tag{4.66}$$

which can be written in terms of the point particle stress tensor (2.155) as

$$\text{Fig.}(4.3) = -\frac{1}{2} \int d\tau \int d\tau' \sum_{a,b=1}^{2} T_a^{\alpha\beta}(z_a^\alpha) D^{ab}_{\alpha\beta\gamma'\delta'}(z_a^\alpha, z_b^{\alpha'}) T^{\gamma'\delta'}(z_b^{\alpha'}). \tag{4.67}$$

Using (B.13) we find that this can be expressed alternatively in terms of the retarded propagator and the Hadamard two-point function

$$\text{Fig.}(4.3) = i \int d\tau \int d\tau' \, T_-^{\alpha\beta} D^{ret}_{\alpha\beta\gamma'\delta'} T_+^{\gamma'\delta'} - \frac{1}{4} \int d\tau \int d\tau' \, T_-^{\alpha\beta} D^{H}_{\alpha\beta\gamma'\delta'} T_-^{\gamma'\delta'}. \tag{4.68}$$

Observe that (4.63) implies the contribution from the real part of iS_{eff}, which is proportional to the Hadamard function, comes from those worldlines with $z_2 \approx z_1$.

It is therefore reasonable to isolate the dominant contribution to the effective action and expand in powers of the difference variable $z_1^\alpha - z_2^\alpha$. Defining the difference and semi-sum (or averaged) coordinates

$$z_-^\alpha = z_1^\alpha - z_2^\alpha \tag{4.69}$$

$$z_+^\alpha = \frac{z_1^\alpha + z_2^\alpha}{2} \tag{4.70}$$

and implementing the identities in Appendix A we find that such an expansion gives

$$\begin{aligned} iS_{eff}[z^a] &= -im \int d\tau\, z_-^\mu g_{\mu\alpha} a_+^\alpha \\ &\quad + \frac{m^2}{2m_{pl}^2} \int d\tau \int d\tau'\, z_-^\mu w_\mu{}^{\alpha\beta\nu}[z_+^\alpha] \nabla_\nu D^{ret}_{\alpha\beta\gamma'\delta'}(z_+^\alpha, z_+^{\alpha'}) \dot{z}_+^{\gamma'} \dot{z}_+^{\delta'} + O(z_-^2) \end{aligned} \tag{4.71}$$

where the 4-acceleration is

$$a_+^\mu(\tau) = \frac{D\dot{z}_+^\mu}{d\tau}, \tag{4.72}$$

τ is the proper time associated with the worldline described by the semi-sum coordinates z_+^α so that

$$g_{\alpha\beta}(z_+)\dot{z}_+^\alpha \dot{z}_+^\beta = -1 \tag{4.73}$$

and the tensor $w^{\mu\alpha\beta\nu}[z]$ is given in (2.160) and (2.161).

We remark that the CTP generating functional and the effective action S_{eff} provide causal dynamics for the effective particle's motion since the retarded propagator is the only two-point function that appears in (4.71). The reason for this stems from the fact that the in-in formalism describes quantum field theory as an initial value problem. This is to be compared to the in-out approach in which the field theory satisfies certain boundary conditions that are more appropriate for scattering then for real time evolution. As such, the in-in formalism is capable of handling non-equilibrium dynamical systems in a manifestly causal way. We will see an explicit example of these different approaches when we calculate the second order self-force equations in the next Chapter. Interestingly, however, the difference in using the in-out versus the in-in formalisms is not made manifest at first order in μ.

Continuing, we observe that the retarded propagator in (4.71) is divergent when $\tau' = \tau$. In order to have a finite and well-behaved force on the compact object from the metric perturbations we will need to regularize this divergence and possibly renormalize the appropriate couplings of the theory.

4.4.4 Regularization of the leading order self-force

The EFT approach is founded in the theory of quantum fields in curved spacetime [66, 124]. The renormalization of divergences in this context has received much attention over the decades and a considerable body of techniques has been developed to remove these divergences in a systematic and self-consistent manner. We therefore find it natural to renormalize the divergence in (4.66) using these methods, even if they are somewhat unfamiliar in classical gravitational problems.

Of these approaches the method of dimensional regularization [63] is particularly useful. This regularization scheme preserves the general coordinate and gauge symmetries of the theory but is also a natural choice to use within an effective field theory framework [44, 46, 47, 48]. The reason for the latter can be seen from the problems that can develop when a simple cut-off regularization is used for the divergent integrals appearing, for example, in the Fermi effective field theory of weak interactions. We refer the reader to [46] for the particular details of this theory. In this EFT the mass of the W-boson M_W is very large compared to the other masses (e.g., quarks) and typical momenta in the problem so that the action describing the low-energy theory is an expansion in powers of $1/M_W$. As a result one finds divergent diagrams at each order that scale like

$$\sim \left(\frac{\Lambda}{M_W} \right)^p \tag{4.74}$$

when using a momentum cut-off Λ to regulate the divergence and p is an integer. Since Λ represents the scale at which high energy physics becomes relevant then it is natural to choose $\Lambda \sim M_W$, which is the scale of the heavy W-bosons. Therefore, all of the (power) divergences at each order contribute at $O(1)$ and the perturbative expansion in the effective field theory breaks down unless if these are resummed [46] in a particular manner. This feature does not occur with dimensional regularization since the dimensional parameter μ_{reg} (which is equivalent to Λ in the above example) never shows up as an explicit power μ_{reg}^p but appears only in logarithms. This is true of any so-called mass-independent renormalization scheme [46].

The smearing prescription developed and implemented in Chapter 2 to regularize the divergent direct part of the retarded propagator cannot be used within our effective field theory because it is a mass-dependent regularization scheme. This is easily seen by looking at the shift in the mass of the electric point charge (2.143) in which the divergence is

$$\frac{\delta m}{m} = e^2 \frac{g_{(1)}(r)}{m} \sim e^2 \frac{\Lambda}{m} \tag{4.75}$$

in the units $G, c = 1$. It is natural to take Λ of order the "classical radius" of the charge $r_o \sim e^2/m$, which defines the length scale at which the vacuum polarization induced by the charge's presence becomes relevant (i.e., pair creation becomes important at this scale and marks roughly the length scale important for quantum electrodynamical processes [107].). Then $\delta m/m$ is a first order contribution thereby causing the perturbative expansion in m to break down since the high-energy physics no longer provides a small correction to low-energy processes. Therefore, the smearing regularization is unsuitable to use within an effective field theory framework.

We will use the dimensional regularization scheme below because of its practical ease and because it allows for the effective field theory to be renormalized in a manner consistent with the associated perturbation series in μ [10]. Because we are applying a quantum field theoretical renormalization scheme to a classical gravitational problem we will provide below a somewhat pedagogical discussion of our steps as they apply to the regularization and renormalization of the effective action.

The renormalization of the retarded propagator $D^{ret}_{\alpha\beta\gamma'\delta'}(x, x')$ happens as follows. The divergent structure of the propagator comes from the inclusion of arbitrarily high frequency modes in the field. We may therefore focus attention on the neighborhood surrounding $x' = x$. There are several approaches one may take, including point-splitting regularization and Hadamard's expansion (see [66, 124] and

[10]The expansion parameter $\mu = r_m/\mathcal{R}$ should not be confused with the dimensional parameter μ_{reg} that appears in dimensional regularization.

references therein), but we will focus on dimensional regularization as it applies to the momentum space representation developed by Bunch and Parker [64].

Our reason for using this approach is two-fold. On the one hand [64] utilizes momentum space techniques that are familiar from flat spacetime interacting quantum field theory. On the other hand, dimensional regularization is a powerful scheme for regularizing and renormalizing, if need be, divergences in a manner that is self-consistent with the effective field theory approach (see the above discussion). We turn now to describing this scheme as we will apply it to regularizing the effective action through first order in μ.

At the point x we may associate a tangent space that is spanned by wave, or momentum, vectors k^{μ}. These momenta provide a representation of the two-point functions via a Fourier transform. Any quasi-local coordinate system constructed around x may be used to generate this Fourier transform. For example, the momentum space representation derived using Fermi normal coordinates gives a different representation compared to using Riemann normal coordinates or retarded coordinates, etc. We find it convenient to use Riemann normal coordinates (RNC) to coordinatize the normal convex neighborhood about the point x, which we will take to be the origin of these coordinates. See Appendix C for a brief review of Riemann normal coordinates and the Taylor series expansion of some relevant tensors.

The spacetime is locally flat around x and so expanding the propagator in powers of the displacement from x, which we denote by $y^{\hat{a}}$, naturally introduces derivatives of the background metric $\partial_{\alpha} g_{\mu\nu}$ and, consequently, the curvature tensors of the background spacetime at x. The spacetime region over which the Fourier transform is valid is presumed small compared to the background curvature scale $\mathcal{R}$. Therefore, the expansion parameter in this Taylor series is $y^{\hat{a}}/\mathcal{R}$. We use the standard terminology that the n^{th} adiabatic order of an expansion, denoted $O(\partial^n)$, refers to the number of derivatives acting on the background metric. For example, the curvature tensors are second adiabatic order $O(\partial^2)$ quantities.

Using this adiabatic expansion we can construct the sought after expansion of the propagator. For example, the momentum space representation for the Feynman propagator of a massless scalar field in d spacetime dimensions is [64]

$$D_F(x,x') = \int \frac{d^dk}{(2\pi)^d}\, e^{ik_{\hat{a}}y^{\hat{a}}} \left[\frac{1}{k^2 - i\epsilon} + \frac{\left(\frac{1}{3}-\xi\right) R}{(k^2-i\epsilon)^2} - \frac{2}{3}\, \frac{R_{\hat{a}\hat{b}}k^{\hat{a}}k^{\hat{b}}}{(k^2-i\epsilon)^3} + O(k^{-5}) \right] \quad (4.76)$$

where $y^{\hat{a}}$ is the displacement of x' from x in RNC and ξ is a constant representing the coupling of the scalar field to the scalar curvature. In $d = 4$ dimensions we see that these terms scale as k^2, k^0 and k^0, respectively, in the high frequency limit. The first term is a power divergence and the last two terms are logarithmically divergent. As we will see later the ability to distinguish between power and logarithmic divergences has a great advantage in renormalizing the self-force at higher orders in μ since all power divergences vanish in dimensional regularization [63].

For our purpose of renormalizing the divergence in the effective action (4.71) we need the momentum space representation for the Feynman propagator of metric perturbations to second adiabatic order[11]. In Appendix D we give a novel derivation of this using diagrammatic techniques of perturbative quantum field theory, which

[11]We are unaware of any such momentum representation for the Feynman propagator of metric perturbations to $O(\partial^2)$ using Riemann normal coordinates or any other normal coordinate system. See however [125].

allows for a more efficient computation of the quasi-local structure of the propagator, especially for higher spin fields.

Let us assume that the divergent part of the retarded propagator is known and is given by the quantity D^{div} where we are temporarily dropping the spacetime indices as well as the ret label. The propagator is divergent in the limit when $x' \to x$ and can be written as the sum of a regular and a divergent contribution [66],

$$D = D^{ren} + D^{div} \tag{4.77}$$

where the finite, renormalized propagator is defined by finite remainder

$$D^{ren} \equiv D - D^{div} = Pf(D) \tag{4.78}$$

where Pf stands for the pseudofunction of the quantity in paranetheses and is well-behaved as a (regular) distribution when $x' = x$. Generically, quantum two-point functions and propagators are regarded as distributions and therefore only make sense when integrated against a test function. Let us therefore define $j(\lambda)$ to be such a testing function so that we can form the (divergent) integral

$$\int_{-\infty}^{\infty} d\lambda' \, D(z(\lambda), z(\lambda'))j(\lambda') \tag{4.79}$$

where we evaluate the propagator on the particle worldline $z^\alpha(\lambda)$. We refer the reader to Appendix E for our notations and definitions regarding distribution theory as well as to the excellent text by Zemanian [126] for further study.

The divergent integral in (4.79) can be written as

$$\int_{-\infty}^{\infty} d\lambda' \, D(z(\lambda), z(\lambda'))j(\lambda') = \lim_{\epsilon \to 0} \Big[I(\epsilon) + H(\epsilon) \Big] \tag{4.80}$$

where $I(\epsilon)$ is the divergent part of the integral and $H(\epsilon)$ is the finite part. These are related to the renormalized and divergent propagators through

$$
\begin{aligned}
\lim_{\epsilon\to 0} I(\epsilon) &= \int_{-\infty}^{\infty} d\lambda'\, D^{div}(z(\lambda), z(\lambda'))j(\lambda') && (4.81)\\
H(0) &= Fp\int_{-\infty}^{\infty} d\lambda'\, D(z(\lambda), z(\lambda'))j(\lambda') && (4.82)\\
&= \int_{-\infty}^{\infty} d\lambda'\, D^{ren}(z(\lambda), z(\lambda'))j(\lambda') && (4.83)
\end{aligned}
$$

where Fp denotes the finite part of the divergent integral, in the sense of Hadamard [99], in (4.79). These relations follow from the fact that the renormalized propagator in (4.78) is a pseudo-function and, by definition, generates the finite part of (4.79).

In general, as we discuss in Appendix E, the divergent part can be written in terms of its power divergent terms as well as powers of logarithms

$$
I(\epsilon) = \sum_{p=1}^{N} \frac{a_p}{\epsilon^p} + \sum_{p=1}^{M} b_p \ln^p \epsilon. \tag{4.84}
$$

as $\epsilon \to 0$. To identify the divergent part of the integral in (4.79) we need to identify a representation for D^{div}. The divergent part of the propagator can be explicitly realized using any suitable representation that allows for a clear separation of the divergent parts from the finite terms. Below, we use a momentum space representation for the graviton propagator initially introduced by Bunch and Parker for a scalar field in [64]. Keeping only those terms that are divergent, and therefore contribute to D^{div} and $I(\epsilon)$, amounts to expanding the propagator D through second adiabatic order when using Riemann normal coordinates. Throwing away all higher adiabatic order terms in the expansion, which are finite as can be verified by power counting the momentum integrals, results in the divergent structure shown in (4.84).

We may therefore write

$$D^{div} = D^{BP}_{(n)} \tag{4.85}$$

where the superscript BP stands for the Bunch-Parker momentum space representation with all of the unnecessary finite terms removed and the subscript (n) signifies the highest adiabatic order kept in the expansion. We remark that if a derivative operates on the propagator then we will need to expand out to one higher adiabatic order to compensate for the extra momentum factor that the derivative implies.

Returning to (4.80), the finite part of the integral is defined via the pseudo-function of the propagator in (4.78) so that

$$\begin{aligned} Fp \int_{-\infty}^{\infty} d\lambda'\, D(z(\lambda), z(\lambda'))j(\lambda') &= \lim_{\epsilon \to 0} \left(\int_{-\infty}^{\lambda-\epsilon} + \int_{\lambda+\epsilon}^{\infty} \right) d\lambda'\, D(z(\lambda), z(\lambda'))j(\lambda') \\ &\quad - \int_{-\infty}^{\infty} d\lambda'\, D^{BP}_{(n)}(z(\lambda), z(\lambda'))j(\lambda'). \end{aligned} \tag{4.86}$$

We may then write the worldline integral of the full propagator D as

$$\begin{aligned} &\int_{-\infty}^{\infty} d\lambda'\, D\big(z(\lambda), z(\lambda')\big) f(\lambda') \\ &\quad = Fp \int_{-\infty}^{\infty} d\lambda'\, D\big(z(\lambda), z(\lambda')\big) j(\lambda') + \int_{-\infty}^{\infty} d\lambda'\, D^{BP}_{(n)}\big(z(\lambda), z(\lambda')\big) j(\lambda'). \end{aligned} \tag{4.87}$$

Using these expressions, we find that the first order self-force in (4.71) is given by

$$\begin{aligned} \text{Fig.}(4.3) &= i \frac{m^2}{2m_{pl}^2} \int_{-\infty}^{\infty} d\tau\, z_-^{\mu}\, w_{\mu}{}^{\alpha\beta\nu}[z_+^{\alpha}] \nabla_{\nu} \Bigg\{ Fp \int_{-\infty}^{\infty} d\tau'\, D_{\alpha\beta\gamma'\delta'}\big(z_+(\tau), z_+(\tau')\big) \dot{z}_+^{\gamma'} \dot{z}_+^{\delta'} \\ &\qquad + \int_{-\infty}^{\infty} d\tau'\, D^{BP}_{(n)\alpha\beta\gamma'\delta'}\big(z_+(\tau), z_+(\tau')\big) \dot{z}_+^{\gamma'} \dot{z}_+^{\delta'} \Bigg\} \end{aligned} \tag{4.88}$$

where we parameterize the worldline by the particle's proper time.

Focus on the divergent contribution that arises from the second term,

$$I_\mu{}^\nu(\tau) = w_\mu{}^{\alpha\beta\nu}[z_+^\alpha] \int_{-\infty}^{\infty} d\tau' \, D^{BP}_{(n)\alpha\beta\gamma'\delta'}\big(z_+(\tau), z_+(\tau')\big) \dot{z}_+^{\gamma'} \dot{z}_+^{\delta'}. \tag{4.89}$$

The particle worldline is a geodesic of the background spacetime at leading order so that the d-velocity at τ' is related to that at proper time τ through

$$\dot{z}_+^{\gamma'}(\tau') = g^{\gamma'}{}_\mu\big(z_+(\tau'), z_+(\tau)\big) \dot{z}_+^\mu(\tau) \tag{4.90}$$

where $g^{\gamma'}{}_\mu$ is the bi-vector of parallel transport, which parallel transports a vector at $z_+(\tau)$ to another point $z_+(\tau')$ along the unique geodesic connecting these points, namely, the leading order worldline of the effective particle's motion. The divergent integral (4.89) can then be written as

$$\begin{aligned} I_\mu{}^\nu(\tau) \;=\; & w_\mu{}^{\alpha\beta\nu}[z_+^\alpha] \dot{z}_+^\rho \dot{z}_+^\sigma \int_{-\infty}^{\infty} d\tau' \, D^{BP}_{(n)\alpha\beta\gamma'\delta'}\big(z_+(\tau), z_+(\tau')\big) \\ & \times g^{\gamma'}{}_\rho\big(z_+(\tau'), z_+(\tau)\big) g^{\delta'}{}_\sigma\big(z_+(\tau'), z_+(\tau)\big). \end{aligned} \tag{4.91}$$

The integrand is now a rank-4 tensor at $z_+(\tau)$ and a scalar at $z_+(\tau')$, which we can simply call

$$\begin{aligned} D^{BP}_{(n)\alpha\beta\gamma\delta}\big(z_+(\tau), z_+(\tau')\big) \;=\; & D^{BP}_{(n)\alpha\beta\gamma'\delta'}\big(z_+(\tau), z_+(\tau')\big) g^{\gamma'}{}_\mu\big(z_+(\tau'), z_+(\tau)\big) \\ & \times g^{\delta'}{}_\nu\big(z_+(\tau'), z_+(\tau)\big). \end{aligned} \tag{4.92}$$

In Appendix D we derive the momentum space representation of the Feynman propagator for metric perturbations. We calculate only those terms that give rise to power or logarithmic divergences in $d = 4$ spacetimes. The divergent part of the propagator $D^{BP}_{(n)}$ in RNC is found from (D.122)

$$
\begin{aligned}
D^{BP}_{(n)\hat{a}\hat{b}\hat{c}\hat{d}}(y) \;=\; & \int_{\mathcal{C}_{ret},k} e^{ik\cdot y} \Bigg\{ \frac{P_{\hat{a}\hat{b}\hat{c}\hat{d}}(\eta)}{k^2+i\epsilon} - \frac{1}{(k^2+i\epsilon)^2}\Big(R_{\hat{a}\hat{c}\hat{b}\hat{d}} + R_{\hat{a}\hat{d}\hat{b}\hat{c}} \Big) \\
& - \frac{1}{3}\frac{k^{\hat{s}}k^{\hat{t}}}{(k^2+i\epsilon)^3} \Bigg[\eta_{\hat{a}\hat{c}} R_{\hat{b}\hat{s}\hat{d}\hat{t}} + \eta_{\hat{a}\hat{d}} R_{\hat{b}\hat{s}\hat{c}\hat{t}} + \eta_{\hat{b}\hat{c}} R_{\hat{a}\hat{s}\hat{d}\hat{t}} + \eta_{\hat{b}\hat{d}} R_{\hat{a}\hat{s}\hat{c}\hat{t}} \\
& + \frac{4}{(d-2)^2}\Big(\eta_{\hat{a}\hat{b}} R_{\hat{c}\hat{s}\hat{d}\hat{t}} + \eta_{\hat{c}\hat{d}} R_{\hat{a}\hat{s}\hat{b}\hat{t}} \Big) \Bigg] + O(\partial^3, k^{-5}) \Bigg\} \qquad (4.93)
\end{aligned}
$$

where we have ignored those terms that fall off as k^{-5} in the integrand since these give finite contributions that we are not interested in. The divergent integral (4.89) is simply

$$
\begin{aligned}
I_{\hat{m}}{}^{\hat{n}}(\tau) &= w_{\hat{m}}{}^{\hat{a}\hat{b}\hat{n}}[z_+] \dot{z}^{\hat{c}} \dot{z}^{\hat{d}} P_{\hat{a}\hat{b}\hat{c}\hat{d}} \int_{-\infty}^{\infty} d\tau' \int_{\mathcal{C}_{ret},k} e^{ik\cdot y} \frac{1}{k^2} \\
&= \frac{1}{2}\frac{d-3}{d-2} w_{\hat{m}}{}^{\hat{n}}[z_+^{\alpha}] \int_{-\infty}^{\infty} d\tau' \int_{\mathcal{C}_{ret},k} e^{ik\cdot y} \frac{1}{k^2} \qquad (4.94)
\end{aligned}
$$

where we have used $\dot{z}^{\hat{b}} \dot{z}^{\hat{c}} \dot{z}^{\hat{d}} R_{\hat{a}\hat{c}\hat{b}\hat{d}} = 0$. The diagram Fig.(4.3) that gives the $O(\mu)$ self-force therefore contains only a simple power divergence that scales as k^2 in the high frequency limit in 4 spacetime dimensions. We regularize this divergence below using dimensional regularization.

In Riemann normal coordinates, $y^{\hat{a}}$ describes the coordinate of point x' relative to the origin at x. Since a geodesic connects these two points we can use the definition of $y^{\hat{a}}$ to find its relation to the 4-velocity at x. From (C.1) and (C.9) we have that

$$
y^{\hat{a}} = -e^{\hat{a}}_{\alpha}\big(z_+(\tau)\big)\sigma^{\alpha}\big(z_+(\tau), z_+(\tau')\big) \qquad (4.95)
$$

$$
= e^{\hat{a}}_{\alpha}\big(z_+(\tau)\big) u^{\alpha}_{+}(\tau)\big(\tau - \tau'\big) \qquad (4.96)
$$

and

$$k \cdot y = k_{\hat{a}} y^{\hat{a}} = k_{\hat{a}} e^{\hat{a}}_{\alpha}\big(z_+(\tau)\big) u^{\alpha}_+(\tau)\big(\tau - \tau'\big) = k_\alpha u^\alpha_+(\tau)\big(\tau - \tau'\big). \tag{4.97}$$

Passing the proper time integral through the momentum integral in (4.94) we find that integrating over τ' gives a delta function that enforces k_α and $u^\alpha_+(\tau)$ to be orthogonal,

$$I_{\hat{m}}{}^{\hat{n}}(\tau) = \frac{1}{2}\frac{d-3}{d-2} w_{\hat{m}}{}^{\hat{n}}[z^\alpha_+] \int_{\mathcal{C}_{ret},k} e^{ik_\alpha u^\alpha_+ \tau} (2\pi)\delta\big(k_\alpha u^\alpha_+\big) \frac{1}{-k_0^2 + \mathbf{k}^2} \tag{4.98}$$

where $\mathbf{k}$ is the $d-1$ dimensional spatial momentum. The condition $k_\alpha u^\alpha_+$ imposes a relationship between k_0 and $\mathbf{k}$,

$$k_0 = \mathbf{k} \cdot \mathbf{v}(t) \tag{4.99}$$

where $\mathbf{v} = \mathbf{u}_+/u^0_+ = d\mathbf{z}/dz^0$ is the particle's $(d-1)$-velocity measured with respect to coordinate time. Writing

$$\delta\big(k_\alpha u^\alpha_+\big) = \frac{1}{u^0_+}\,\delta\big(k_0 - \mathbf{k}\cdot\mathbf{v}\big) \tag{4.100}$$

and integrating over k_0 therefore gives

$$I_{\hat{m}}{}^{\hat{n}}(\tau) = \frac{1}{2}\frac{d-3}{d-2} w_{\hat{m}}{}^{\hat{n}}[z^\alpha_+] \frac{1}{u^0(\tau)} \int_{\mathbf{k}} \frac{1}{(\eta^{ij} - v^i v^j) k_i k_j + i\epsilon} \tag{4.101}$$

where here $i, j = 1, \cdots, d-1$ are spacetime indices for the spatial directions.

To calculate the $d-1$ dimensional $\mathbf{k}$ integral we should first diagonalize the matrix $\eta^{ij} - v^i v^j$ so that the denominator of the momentum integral is over $1/\mathbf{k}\cdot\mathbf{k}$ multiplying a velocity-dependent factor. Without loss of generality we may assume that the $(d-1)$-velocity $\mathbf{v}$ points along one of the coordinate directions, say y^1.

Then $\mathbf{v} = (v, 0, \ldots, 0)$ and the matrix is automatically diagonal

$$\eta^{ij} - v^i v^j = \eta^{ij} - \mathbf{v}^2 \eta^{i1} \eta^{j1}, \tag{4.102}$$

which gives for the denominator of the integrand in (4.101)

$$(\eta^{ij} - v^i v^j) k_i k_j = (1 - \mathbf{v}^2) k_1^2 + k_2^2 + \cdots + k_{d-1}^2. \tag{4.103}$$

By rescaling the k_1 momentum so that $k_1 \to k_1 / \sqrt{1 - \mathbf{v}^2}$ we find that

$$I_{\hat{m}}^{\hat{n}}(\tau) = \frac{2(d-1)}{d-2} \int_{\mathbf{k}} \frac{1}{\mathbf{k}^2 + i\epsilon} \tag{4.104}$$

after recalling that $u^0 = 1/\sqrt{1 - \mathbf{v}^2}$. Since this result is independent of the particle's velocity then this should hold in any local coordinate system about $z^\alpha(\tau)$.

The momentum integral in (4.104) can be integrated by giving a small mass m_g to the graviton so that

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{(\mathbf{k}^2 + m_g^2)^\alpha} = \frac{2\pi^{(d-1)/2}}{(2\pi)^{d-1} \Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty dk \, \frac{k^{d-2}}{(k^2 + m_g^2 + i\epsilon)^\alpha} \tag{4.105}$$

$$= \frac{1}{(4\pi)^{(d-1)/2}} \frac{\Gamma\left(\alpha + \frac{1-d}{2}\right)}{\Gamma(\alpha)} (m_g^2)^{\frac{d-1}{2} - \alpha}, \tag{4.106}$$

for some positive integer α. Strictly speaking, the integral in (4.105) does not converge for $d = 4$. However, by analytically continuing to other values for d we find that the integral converges. In this way, the divergence is renormalized via the analytic continuation and a finite result is obtained upon choosing $d = 4$. For $\alpha = 1$ and $d = 4 - \varepsilon$ the integral is[12]

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\mathbf{k}^2 + m_g^2} = -\frac{m_g}{4\pi} + O(\varepsilon). \tag{4.107}$$

[12]Observe that the difference from four dimensional spacetime $\varepsilon = 4 - d$ is not the same quantity as the $i\epsilon$ that enforces the appropriate boundary conditions on the Feynman propagator.

This procedure to render the initially divergent integral (4.105) finite is called dimensional regularization. This particular regularization scheme has the attractive feature that power divergences, such as the one encountered in (4.104), vanish in the limit that the graviton mass m_g goes to zero. From (4.104) and (4.107) we find that

$$I_\mu{}^\nu(\tau) = 0 \tag{4.108}$$

when evaluated using dimensional regularization. We therefore conclude that the divergent part of the diagram in Fig.(4.3) is zero,

$$\begin{aligned} I(\epsilon) &= i\frac{m^2}{2m_{pl}^2}\int d\tau\, z_-^\mu w_\mu{}^{\alpha\beta\nu}[z_+]\nabla_\nu \int_{-\infty}^{\infty} d\tau'\, D^{BP}_{(n)\alpha\beta\gamma'\delta'}\big(z(\tau), z(\tau')\big)\dot{z}^{\gamma'}\dot{z}^{\delta'} \\ &= 0. \end{aligned} \tag{4.109}$$

The remaining finite part of Fig.(4.3) is

$$\text{Fig.}(4.3) = i\frac{m^2}{2m_{pl}^2}\int d\tau\, z_-^\mu w_\mu{}^{\alpha\beta\nu}[z_+]\nabla_\nu Fp \int_{-\infty}^{\infty} d\tau'\, D^{ret}_{\alpha\beta\gamma'\delta'}\big(z(\tau), z(\tau')\big)\dot{z}^{\gamma'}\dot{z}^{\delta'}. \tag{4.110}$$

Notice that we have not had to renormalize any parameters of the theory at this order since dimensional regularization set the power divergence to zero.

Having regularized the leading order contribution to the self-force diagram in Fig.(4.3) we now compute the equations of motion from (4.110). This is simply found from the variational principles

$$\left.\frac{\delta S_{eff}[z^a]}{\delta z_-^\mu(\tau)}\right|_{z_-=0} = 0 \tag{4.111}$$

and gives

$$m\, a^\mu(\tau) = \frac{m^2}{2m_{pl}^2} w^{\mu\alpha\beta\nu}[z^\alpha] \nabla_\nu Fp \int_{-\infty}^{\infty} d\tau'\, D^{ret}_{\alpha\beta\gamma'\delta'}(z^\alpha, z^{\alpha'})\, \dot{z}^{\gamma'} \dot{z}^{\delta'}. \tag{4.112}$$

Using the definition of Hadamard's finite part of the integral in (4.86) and the fact that the retarded propagator is zero for $\tau' > \tau$ we see that the finite part is given by

$$Fp \int_{-\infty}^{\infty} d\tau'\, D^{ret}_{\alpha\beta\gamma'\delta'}(z^\alpha, z^{\alpha'})\, \dot{z}^{\gamma'} \dot{z}^{\delta'} = \lim_{\epsilon \to 0} \int_{-\infty}^{\tau-\epsilon} d\tau'\, D^{ret}_{\alpha\beta\gamma'\delta'}(z^\alpha, z^{\alpha'})\, \dot{z}^{\gamma'} \dot{z}^{\delta'} \tag{4.113}$$

after recalling that the divergent part of the integral is zero. Inserting this into the equations of motion gives the equation for the self-force on the effective point particle moving in a vacuum background spacetime

$$m\, a^\mu(\tau) = \frac{m^2}{2m_{pl}^2} w^{\mu\alpha\beta\nu}[z^\alpha] \lim_{\epsilon \to 0} \int_{-\infty}^{\tau-\epsilon} d\tau'\, \nabla_\nu D^{ret}_{\alpha\beta\gamma'\delta'}(z^\alpha, z^{\alpha'})\, \dot{z}^{\gamma'} \dot{z}^{\delta'} \tag{4.114}$$

which was originally derived by Mino, Sasaki and Tanaka [20] and by Quinn and Wald [21]. The tensor $w^{\mu\alpha\beta\nu}$ is defined in (2.160) and (2.161).

4.4.5 The procedure for computing the self-force to all orders

In this section we have developed the power counting and the Feynman rules necessary for computing the effective action order by order in μ. Along the way, we encounter the usual ultraviolet divergence in the retarded propagator for the metric perturbations. Utilizing a mixture of distributional methods and momentum space techniques in curved spacetime we regularize the effective action at $O(\mu)$ and find the resulting self-force equation of MSTQW. Before proceeding to the next section, let us formulate a recipe that is applicable for higher order self-force calculations.

The steps required to compute the self-force at any order in μ, say the n^{th} order, are given by the following:

1. Draw all diagrams that appear at $O(\mu^n)$ using the power counting rules in Section 4.4.2,

2. Write down the mathematical expressions that correspond to the Feynman diagrams using the Feynman rules outlined in Section 4.4.3. These expressions are the $O(\mu^n)$ contributions to the effective action,

3. Expand the effective action in powers of the coordinate difference $z_-^\alpha = z_1^\alpha - z_2^\alpha$ using the fact that $z_2^\alpha \approx z_1^\alpha$ for astrophysically relevant binaries. Keep only those contributions through $O(z_-)$. The effective action should be manifestly causal at the end of this step,

4. Distribution theory allows for the $O(\mu^n)$ terms in the effective action to be written in terms of a (generally non-local) finite part, which contributes to the history-dependent self-force, and a (quasi-local) divergent part,

5. Apply dimensional regularization to the momentum space representation of the retarded propagator. All power divergences are zero in this scheme so that logarithmic divergences are the only terms that renormalize the parameters of the theory,

6. Vary the resulting finite part of the effective action with respect to the difference coordinate z_-^α to find the $O(\mu^n)$ contribution to the self-force equation.

In the next Chapter we apply these steps to calculate the self-force at higher orders in μ. While this approach is perhaps overpowering for the linear order calculation of the MSTQW equation we see the efficient handling of divergences using this procedure is significantly beneficial for higher order self-force calculations.

4.5 Effacement Principle for EMRIs

While it is intuitive to expect finite size corrections to be negligibly small whenever computing the linear order self-force one may be concerned with such corrections at higher orders. Specifically, at what order in μ are the tidal deformations of the small body important for computing the self-force? In this section we answer this question using coordinate invariant arguments and demonstrate for the first time, to the best of our knowledge, that such finite size effects from a spherically symmetric compact object moving in a background curved spacetime unambiguously enter the self-force at $O(\mu^5)$ and as deviations from the leading order geodesic motion at $O(\mu^4)$. This is the statement of the Effacement Principle for extreme mass ratio inspirals.

To begin we write down the effective point particle action that includes all possible terms consistent with general coordinate invariance and worldline reparameterization invariance,

$$S_{pp}[z(\tau)] \;=\; -m \int d\tau + c_E \int d\tau \, E_{\mu\nu} E^{\mu\nu} + c_B \int d\tau \, B_{\mu\nu} B^{\mu\nu} + \cdots \quad (4.115)$$

where we have already used a field redefinition to remove those terms involving a Ricci curvature tensor. The terms involving the square of the Riemann curvature (and higher powers) represent the influence of the finite size of the body as it moves through space. This is seen by noting that the equations of motion no longer have vanishing acceleration $ma^\mu \neq 0$ so that the effective point particle does not move along a geodesic of the background spacetime. Such deviation from geodesic motion is typical of tidally distorted bodies and is discussed in more detail in Section 4.3.

The coefficients $c_{E,B}$ are parameters that depend upon the structure of the extended body. We must therefore match the effective point particle theory onto the full theory in order to encode this "microscopic" or "high-energy" structure onto the long wavelength effective theory. The matching procedure involves calculating some (coordinate invariant) observable in both the effective theory and in the full theory[13]. By expanding the observable of the full theory in the long wavelength limit, where the effective theory is applicable, we can simply read off the values of $c_{E,B}$ as well as any other coefficients in (4.115). Instead of preferring a detailed matching calculation we will perform an order of magnitude estimation to determine the scaling behavior of $c_{E,B}$ for a spherically symmetric compact object.

The symmetries of the effective point particle action are shared with observables computed from the full theory in the long wavelength limit. The matching procedure requires calculating these observables in order to fix the coefficients in the effective theory. Below, we will power count the scattering cross-section for graviton Compton scattering shown in Fig.(4.4), which simply represents the scattering of

[13]Strictly speaking, one can use any quantity for the matching but it is simpler to draw coordinate invariant conclusions by matching with a coordinate invariant quantity, such as a scattering amplitude, a cross-section, etc.

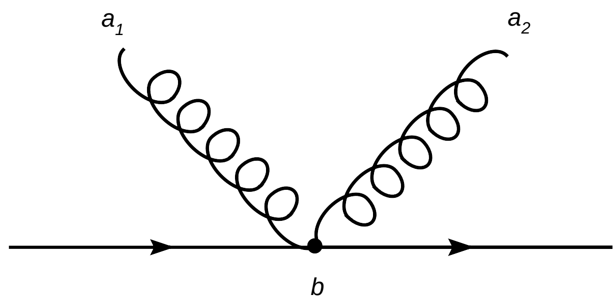

Figure 4.4: Graviton scattering off the background of a static and spherically symmetric extended body (e.g., a Schwarzschild black hole, a non-spinning neutron star, etc.).

metric perturbations in the spacetime generated by the isolated, compact object.

We first compute the cross-section in the effective point particle theory described by S_{pp} in (4.115). The scattering amplitude is computed by expanding the terms in S_{pp} proportional to c_E and c_B to second order in the metric perturbations,

$$\begin{aligned} S_{pp}[z(\tau)] \;=\; & \cdots + c_E \int d\tau \,\left(E^{(0)}_{\mu\nu} E^{(0)\mu\nu} + 2 E^{(1)}_{\mu\nu} E^{(0)\mu\nu} + E^{(1)}_{\mu\nu} E^{(1)\mu\nu} + 2 E^{(2)}_{\mu\nu} E^{(0)\mu\nu} \right) \\ & + c_B \int d\tau \,\left(B^{(0)}_{\mu\nu} B^{(0)\mu\nu} + 2 B^{(1)}_{\mu\nu} B^{(0)\mu\nu} + B^{(1)}_{\mu\nu} B^{(1)\mu\nu} + 2 B^{(2)}_{\mu\nu} B^{(0)\mu\nu} \right) \\ & + \cdots \, . \end{aligned} \tag{4.116}$$

where the superscript denotes the number of metric perturbations appearing in that function so that $B^{(2)}_{\mu\nu}$ is proportional to h^2, for example. From the power counting rules developed in Section 4.4.2 we find that the scattering amplitude associated with Fig.(4.4) scales as

$$i\mathcal{A} \sim \cdots \,\&\, \frac{c_{E,B}}{m_{pl}^2} \left(\frac{1}{\mathcal{R}^2} \right)^2 \,\& \cdots \tag{4.117}$$

where the $1/\mathcal{R}^2$ comes from the two spacetime derivatives in the Riemann tensor.

While the cross-section includes contributions from other terms in the effective particle action it will contain one term proportional to $c_{E,B}^2$,

$$\sigma_{pp} \sim |i\mathcal{A}|^2 \sim \cdots \,\&\, \frac{c_{E,B}^2}{m_{pl}^4}\frac{1}{\mathcal{R}^8} \,\&\, \cdots \tag{4.118}$$

where the pp subscript indicates that this is the cross-section computed in the effective point particle theory and the & is to be read "and a term with the form of."

We turn now to the scattering cross-section in the full theory. A cross-section represents an effective scattering area and the only scale present in the full theory of the isolated compact object is set by the size of the compact object $r_m \sim m/m_{pl}^2$. It follows that

$$\sigma_{full} = r_m^2 f\left(\frac{r_m}{\mathcal{R}}\right) \tag{4.119}$$

where f is a dimensionless function. In the long wavelength limit where $r_m/\mathcal{R} \ll 1$ the cross-section will contain a term proportional to $\mathcal{R}^{-8}$,

$$\sigma_{full} \sim \cdots \,\&\, r_m^2\left(\frac{r_m}{\mathcal{R}}\right)^8 \,\&\, \cdots . \tag{4.120}$$

Since quantities computed in the effective theory ought to match those computed in the long wavelength limit of the full theory we conclude that

$$c_{E,B} \sim m_{pl}^2 r_m^5 \sim \frac{m^5}{m_{pl}^8} \tag{4.121}$$

upon identifying the $\mathcal{R}^{-8}$ terms in both σ_{pp} and σ_{full}.

Using (4.121) we can estimate the order in μ that the non-minimal terms appearing in the effective point particle action (4.115) will affect the motion of

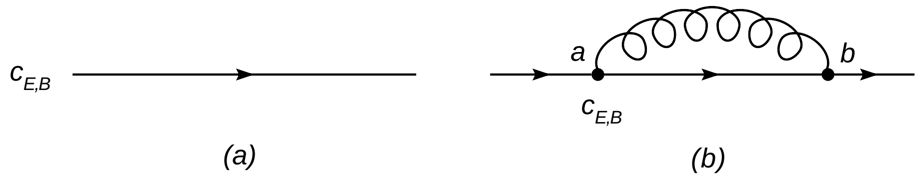

Figure 4.5: Lowest order contributions to (a) deviation from geodesic motion due to the tidal deformations of the compact object and (b) the self-force from the interaction of gravitational radiation with these deformations.

the compact object. The first diagram that the finite size terms (proportional to $c_{E,B}$) will contribute is shown in Fig.(4.5a). This describes the deviation from the leading order geodesic motion experienced by the effective point particle due to the inclusion of the non-minimal couplings to the background spacetime. This diagram scales with μ as

$$\text{Fig. 4.5a} \sim c_{E,B} d\tau \left(\frac{1}{\mathcal{R}^2} \right)^2 \sim \mu^4 L \tag{4.122}$$

and enters at fourth order.

This diagram does not couple to metric perturbations; it persists in the absence of gravitational radiation. As a result, while Fig.(4.5a) will affect the motion of the particle it is not a correction to the self-force. To find the order at which the tidal deformations affect the self-force we power count the diagram in Fig.(4.5b) to find that

$$\text{Fig. 4.5b} \sim c_{E,B} d\tau \left(\frac{1}{\mathcal{R}^2} \right)^2 \frac{h}{m_{pl}} \sqrt{\mu L} \sim \mu^5 L. \tag{4.123}$$

Finite size effects therefore enter the self-force at fifth order in μ. The tidal deformations of the compact object are negligibly small until $O(\mu^4)$ at which point the deviation from geodesic motion dominates the contribution from the self-force.

We recapitulate our results from this Section. We compute for the first time the order in μ at which finite size effects from tidally induced moments enter the dynamics of the compact object. We do so using a coordinate invariant matching procedure that relates the parameters of the effective point particle theory to the full theory describing the tidal deformations of the compact object in isolation. We find that in a vacuous background finite size effects will first enter the dynamics of the compact object's motion at $O(\mu^4)$ in the form of deviation from the leading order geodesic motion. The metric perturbations couple to the compact object's induced moments at $O(\mu^5)$ thereby representing a correction to the self-force on the point particle.

Chapter 5

Effective field theory approach for extreme mass ratio inspirals: Higher order self-force and spin effects

5.1 Second and higher order self-force

The effective field theory approach provides a systematic way to compute the self-force to any order in μ. Having derived the first-order self-force equation of MSTQW it is natural to continue the calculation to second order. But there is an important reason for computing the second order self-force.

It has been argued [61] that the first order self-force does not provide sufficiently accurate gravitational waveforms for determining the source parameters (mass, spin, distance to source, etc.) with the claimed fractional accuracy of $\sim 10^{-4}$ [54] for the LISA mission [2]. The error in using the first order self-force to compute the waveform is about 1 cycle over the roughly 10^5 cycles expected to be observed within the one year that LISA will be operational. While this is sufficient for detection purposes [54, 55, 56, 57, 58] this error will strongly suppress the signal-to-noise ratio used in a coherent matched filter search through the full 10^5 cycles. However, such a coherent search is not computationally possible over the estimated year-long waveform as this will require computing about 10^{40} templates [55]. Nevertheless, breaking up the search over about 3-week intervals and stitching together the resulting templates over the full waveform will allow for a modest estimation of the source parameters [55]. Using these estimations one can then restrict the parameter space and search through increasingly longer and more accurate templates. In this way the source parameters can be determined with LISA's claimed precision. However, in this "measurement" stage, the longer time intervals for a more coherent matched filter search will require more precise knowledge of the gravitational waveform than can be provided by the first order self-force calculations. This is particularly true when the compact object is moving in the deep strong field region of the supermassive black hole background where the phase of the waveform can receive a relatively significant correction compared with the first order predictions [61]. It is even possible that such higher order corrections might affect the detection templates in the last 10^4 cycles or so[1]. Therefore, a second order self-force calculation is absolutely necessary to ensure the construction of sufficiently accurate waveform templates for realizing the desired precision for parameter estimation.

In this section we will discuss some aspects of the calculation for the second order self-force on a compact object. We do not explicitly evaluate the diagrams relevant for such a computation here; this will be done in a separate paper [82]. However, we will determine the second order self-force on a compact object interacting with and sourcing a nonlinear scalar field on the background spacetime. The toy theory that we develop has the same Feynman diagrams (at the topological level) that appears in the gravitational EFT. The nonlinear scalar model can therefore be used as a reliable indicator of what is to occur with the gravitational calculation

[1]We thank Cole Miller for this suggestion.

and may shed some insight on the qualitative features of higher order self-force corrections. For example, both theories share a similar divergent structure that can be renormalized in the same manner with the same consequences. Although the particular form of the (finite) self-force may differ among these two theories we are interested in the mechanics of the regularization and renormalization as they appear in the second order self-force calculation.

5.1.1 Second order Feynman diagrams and renormalization

The diagrams relevant for the second order self-force are found using the power counting rules in Section 4.4.2 to construct all possible connected diagrams that scale as $\mu^2 L$. This is the first step outlined in Section 4.4.5. There are only two such diagrams and these are given in Fig.(5.1). We see that the second order self-force is comprised of two types of interactions.

The first, shown in Fig.(5.1a), is a nonlinear particle-field interaction. It describes the emission of a graviton that is later absorbed by the particle. Upon absorption, another graviton is emitted and absorbed at some later time. This consecutive emission and absorption of gravitons results from the $S_{pp}^{(2)}$ interaction term in (4.40).

The second type of interaction, shown in Fig.(5.1b), comes from the nonlinear structure of the gravitational field equations. A graviton emitted from the particle undergoes a scattering event off of the background curvature thereby producing two gravitons, both of which are reabsorbed by the particle. The graviton scattering

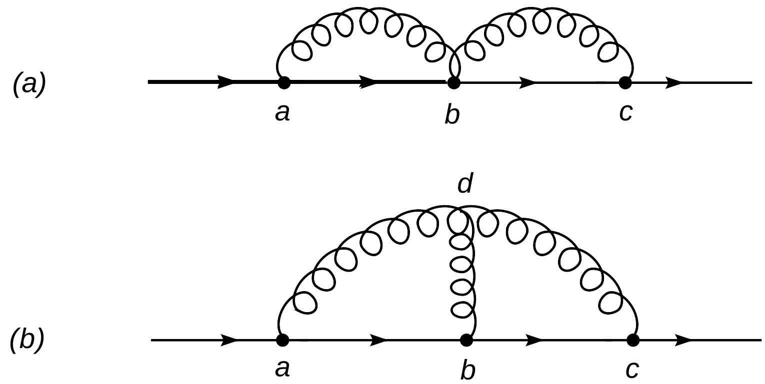

Figure 5.1: Diagrams contributing to the second order self-force. The diagram in (a) describes the leading order nonlinear particle-field interaction while the bottom diagram in (b) results from the nonlinear structure of general relativity. These diagrams are the only two that enter the effective action at $O(\mu^2 L)$.

in the bulk spacetime is a result of the $S^{(3)}$ interaction term in (4.40). Therefore, the nonlinear nature of general relativity first appears in the self-force as a second order effect. We remark that this scattering is *not* the same phenomenon as the backscattering of waves off the background curvature. The latter describes the bending of wavefronts due to the failure of the wave equation to satisfy Huygen's principle in a curved background and appears in the first order self-force equations of motion (4.114) through the appearance of the history-dependent integration over the retarded propagator. This history dependence occurs precisely because of this effect. The former process describing graviton scattering is a legitimate scattering event viewed from a tree-level quantum field theory perspective. Notice that the gravitons connecting the particle worldline to the bulk spacetime event (marked with the CTP label "d" in Fig.(5.1b)) undergo backscattering off of the background curvature.

We will not explicitly write down the terms in the effective action S_{eff} that correspond to these diagrams. The permutation of the indices that appear on the graviton propagators in the diagrams is rather involved. However, we do not need to know the specific forms of these contractions in order to have some qualitative understanding of the divergences that appear here at second order nor to obtain a schematic form for the equations of motion. Therefore, let us schematically write down the contributions to the effective action from Fig.(5.1). The first diagram is

$$\begin{aligned} \text{Fig.(5.1a)} \quad \sim \quad & \left(\frac{m}{m_{pl}}\right)^2 \left(\frac{m}{m_{pl}^2}\right) \sum_{a,b,c,d=1}^{2} (-1)^{a+b+c+d} \int d\tau \int d\tau' \int d\tau'' \\ & \times D_{ab}\big(z(\tau), z(\tau')\big) D_{cd}\big(z(\tau'), z(\tau'')\big) \end{aligned} \tag{5.1}$$

where we are choosing to parameterize the worldline with the particle's proper time in order to make the notation as compact as is usefully possible.

The second contribution is

$$\begin{aligned} \text{Fig.(5.1b)} \quad \sim \quad & \left(\frac{m}{m_{pl}}\right)^3 \left(\frac{1}{m_{pl}}\right) \sum_{a,b,c,d,e,f=1}^{2} (-1)^{a+b+c+d+e+f} \int d\tau \int d\tau' \int d\tau'' \\ & \times \int d^4x \, \nabla D_{ab}\big((x^\mu, z(\tau)\big) \nabla D_{cd}\big(x^\mu, z(\tau')\big) D_{ef}\big(x^\mu, z(\tau'')\big) \end{aligned} \tag{5.2}$$

The covariant derivatives act on the graviton two-point functions with respect to the bulk spacetime coordinate x^μ and not the worldline coordinate $z^\alpha(\tau)$. Both of these diagrams potentially contain high frequency divergences that are typical of particle-field interactions. The benefit of using an effective field theory approach is that we know the theory is renormalizable at length scales much larger than the size of the compact object. This is so because we have included all possible terms in the effective point particle action that are consistent with the general coordinate and worldline reparameterization invariances. Therefore, any divergence that might appear has to be renormalized by either the mass m of the compact object or by any of the infinite number of parameters that are introduced with the non-minimal couplings, such as $c_{R,V}$ or $c_{E,B}$, etc.

While there are only two topologically distinct diagrams at second order we remark that the actual number of diagrams, and corresponding integrals, is larger because of the permutations of the index structure that appears on the graviton two-point functions, $D^{ab}_{\alpha\beta\gamma'\delta'}(x,x')$. Keeping track of these index permutations can be somewhat involved. To avoid obscuring the important issues, including divergences and renormalization, with tedious index shuffling that appears at higher orders in the self-force calculation we will focus on a nonlinear scalar field propagating in a vacuum background spacetime. Since the scalar field possesses no spacetime indices we will find the calculations to be more transparent to the application and interpretation of the divergent structures appearing at second order in the effective action. In the next Section we explicitly calculate the second order corrections to the self-force in such a model.

5.1.2 A scalar field model

We introduce a toy model describing a relativistically moving compact object with mass m interacting with a nonlinear scalar field ϕ propagating on a background vacuum spacetime. We assume that the stress tensor of the scalar field provides a small correction to the background so that the leading order geometry is vacuous.

After integrating out the "microscopic" structure at the scale of the compact object we are left with an effective point particle coupled to the scalar field. The total action of the particle-field system is taken to be

$$S_{tot}[z, \phi] = S_{\phi}[\phi] + S_{pp}[z, \phi] \tag{5.3}$$

where the field action is

$$S_{\phi}[\phi] = \frac{1}{2} \int d^4x \, g^{1/2} g^{\alpha\beta} e^{2\phi/m_{pl}} \phi_{,\alpha} \phi_{,\beta}. \tag{5.4}$$

This action can be obtained from the usual action for a minimally coupled massless scalar field on a curved background by performing the conformal transformation

$$g_{\alpha\beta} \to e^{2\phi/m_{pl}} g_{\alpha\beta} \tag{5.5}$$

to a vacuum spacetime and then rescaling the resulting action. In this way we end up with the theory given in (5.4).

The effective point particle action is given by

$$S_{pp}[z, \phi] = -m \int d\lambda \left(-g_{\alpha\beta} \dot{z}^{\alpha} \dot{z}^{\beta} \right)^{1/2} e^{\phi/m_{pl}} + \cdots \tag{5.6}$$

The structure of the point particle action can also be generated from the conformal transformation (5.5) of the effective point particle action in (4.115). This guarantees the appearance of nonlinear particle-field interactions, which is important for reconstructing the same diagrams, in a topological sense, that appear in the gravitational second order self-force, Fig.(5.1).

We will proceed as before and expand the total action in powers of ϕ. We find

$$S_{tot}[z,\phi] = S_\phi^{(2)} + S_\phi^{(3)} + \sum_{n=4}^{\infty} S_\phi^{(n)} + S_{pp}^{(0)} + S_{pp}^{(1)} + S_{pp}^{(2)} + \sum_{n=3}^{\infty} S_{pp}^{(n)} \tag{5.7}$$

where an integer superscript denotes the number of scalar fields in that term, e.g. $S_\phi^{(n)} \sim \phi^n$. The first term in the expansion of the field action is the kinetic term that gives the propagator on the curved background,

$$S_\phi^{(2)} = \frac{1}{2}\int d^4x\, g^{1/2} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}. \tag{5.8}$$

Since the scalar field is not a gauge field there is no need to worry about gauge-fixing. The propagator is already uniquely defined once global boundary conditions are imposed. The cubic self-interaction term is given by

$$S_\phi^{(3)} = \frac{1}{m_{pl}}\int d^4x\, g^{1/2} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}\, \phi \tag{5.9}$$

and arises from the nonlinear nature of the field. The n^{th} order term in the expansion of $S_\phi[\phi]$ is given by

$$S_\phi^{(n)} = \frac{2^{n-3}}{(n-2)!\, m_{pl}^{n-2}}\int d^4x\, g^{1/2} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}\, \phi^{n-2}. \tag{5.10}$$

The term $S_{pp}^{(0)}$ is the familiar free point particle action

$$S_{pp}^{(0)} = -m \int d\lambda \left(-g_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta\right)^{1/2}. \tag{5.11}$$

We will demonstrate below that the finite size terms in (5.6) first enter the particle dynamics at fourth order. For this reason we ignore the nonminimal terms in this section. The subleading terms of the point particle action describing particle-field interactions are

$$
\begin{aligned}
S_{pp}^{(1)} &= -\frac{m}{m_{pl}} \int d\lambda \left(-g_{\alpha\beta}\dot{z}^{\alpha}\dot{z}^{\beta}\right)^{1/2} \phi \\
S_{pp}^{(2)} &= -\frac{m}{2m_{pl}^2} \int d\lambda \left(-g_{\alpha\beta}\dot{z}^{\alpha}\dot{z}^{\beta}\right)^{1/2} \phi^2. \qquad (5.12)
\end{aligned}
$$

Using these expansions, we construct the effective action in the same way we discussed in Section 4.4.1. We find that the in-in (or CTP) generating functional is

$$
\begin{aligned}
Z[j_a^{\mu}] &= \exp\left\{ iS_{pp}^{(0)}[z^a] + i\int d\lambda\, j_a^{\mu} z_{\mu}^a \right\} \exp\left\{ i\int d^4x\, \mathcal{L}_{int}\left[z^a, -i\frac{\delta}{\delta J_a} \right] \right\} \\
&\quad \times \exp\left\{ -\frac{1}{2} J_a^{\alpha\beta} \cdot D_{\alpha\beta\gamma'\delta'}^{ab} \cdot J_b^{\gamma'\delta'} \right\} \Bigg|_{J_a=0} \qquad (5.13)
\end{aligned}
$$

where the interaction

$$
\int d^4x\, \mathcal{L}_{int}[z_a, \phi_a] = \sum_{n=1}^{\infty} S_{pp}^{(n)}[z_a, \phi_a] + \sum_{n=3}^{\infty} S_{\phi}^{(n)}[\phi_a] \qquad (5.14)
$$

contains all of the information regarding the field self-interactions and the particle-field interactions. The effective action is calculated from the Legendre transform of the generating functional and is equal to

$$
\begin{aligned}
iS_{eff}[z^a] &= -im\int d\lambda \left(-g_{\alpha\beta}\dot{z}^{\alpha}\dot{z}^{\beta}\right)^{1/2} + \begin{pmatrix} \text{sum of all } O(L) \\ \text{connected diagrams} \end{pmatrix} \\
&\quad + \begin{pmatrix} \text{higher order graviton} \\ \text{loop corrections} \end{pmatrix}. \qquad (5.15)
\end{aligned}
$$

For the same reasons discussed earlier we may ignore the negligibly small quantum corrections. We therefore need to determine the tree-level diagrams that appear at second order in μ. To accomplish this we need to power count the interaction terms to determine their scaling with μ and the typical angular momentum of the system L.

The structure of the kinetic term for the field $S^{(2)}_\phi$ is the same as for the metric perturbations $S^{(2)}$ in (4.39). We therefore conclude that if $x^\mu \sim \mathcal{R}$ then

$$\phi \sim \frac{1}{\mathcal{R}}. \tag{5.16}$$

Again, the similar structure also implies that the angular momentum L and the ratio m/m_{pl} scale in the same way as for the gravitational problem, namely,

$$L \sim m\mathcal{R} \tag{5.17}$$

$$\frac{m}{m_{pl}} \sim \sqrt{\mu L}. \tag{5.18}$$

These scalings are identical to those in Section 4.4.2. We therefore conclude that the Feynman diagrams we derived for the gravitational self-force are the same diagrams, topologically speaking, that appear in the nonlinear scalar field model. Furthermore, we can use this equivalence to show that the interaction terms in (5.13) have the same power counting as their gravitational counterparts. It follows that finite size effects first enter the particle dynamics at fourth order in μ in this scalar theory for spherically symmetric black holes and neutron stars.

The first order self-force diagram is given in Fig.(4.3). The effective action at this order is easily shown to be

$$\begin{aligned} iS_{eff}[z^a] &= -im\int d\lambda \left(-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{1/2} + (i^3)\left(\frac{1}{2!}\right)\left(-\frac{m}{m_{pl}}\right)^2 \sum_{a,b=1}^{2}(-1)^{a+b} \\ &\quad \times \int d\lambda \int d\lambda' \Big(-g_{\alpha\beta}\dot{z}^\alpha_a \dot{z}^\beta_a\Big)^{1/2} D_{ab}\big(z^\alpha_a, z^{\alpha'}_b\big)\Big(-g_{\gamma'\delta'}\dot{z}^{\gamma'}_b \dot{z}^{\delta'}_b\Big)^{1/2} + \cdots . \end{aligned} \tag{5.19}$$

Using the methods introduced in Sections 4.4.3 and 4.4.4 we regularize the divergences and obtain a well-defined self-force at $O(\mu L)$, which is given by

$$\text{Fig. (4.3)} \;=\; -\frac{i}{2}\,\frac{m^2}{m_{pl}^2}\int_{-\infty}^{\infty} d\tau\, Fp \int_{-\infty}^{\infty} d\tau'\, z_-^\mu(\tau)\big(a_{+\mu} + w_\mu{}^\nu[z_+^\alpha]\nabla_\nu\big)D^{ret}\big(z_+^\alpha, z_+^{\alpha'}\big). \tag{5.20}$$

The variation with respect to the difference coordinate z_-^μ results in the first order self-force equation

$$\begin{aligned}&\left[1 - \frac{m}{m_{pl}^2}\lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon} d\tau'\, D_{ret}(z^\alpha, z^{\alpha'})\right] a^\mu(\tau)\\ &\qquad = \frac{m}{m_{pl}^2} w^{\mu\nu}[z^\alpha]\lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon} d\tau'\, \nabla_\nu D_{ret}(z^\alpha, z^{\alpha'}).\end{aligned} \tag{5.21}$$

Notice the similarity to the self-force equation for the linear scalar field interacting with a scalar charge in (2.121). We remark in passing that at the first non-trivial order the particle couples to the field via a monopole interaction

$$S_{pp}^{(1)} \sim \int d\tau\, \phi\big(z(\tau)\big). \tag{5.22}$$

As a result, the mass of the particle can be transferred to the radiated field and vice versa. This feature has been demonstrated in [87] for a linear scalar field theory coupled to a scalar charged particle. In fact, defining the time-dependent effective mass as

$$m_{eff}(\tau) \equiv m - \frac{m^2}{m_{pl}^2}\lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon} d\tau'\, D_{ret}\big(z(\tau), z(\tau')\big) \tag{5.23}$$

we find that its rate of change with proper time is

$$\frac{dm_{eff}(\tau)}{d\tau} = -\frac{m^2}{m_{pl}^2}\dot{z}^\nu(\tau)\lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon} d\tau'\, \nabla_\nu D_{ret}\big(z(\tau), z(\tau')\big), \tag{5.24}$$

which is dependent on the entire past history of the particle-field interaction. This time dependence does not occur in the gravitational theory on account of mass conservation.

The diagrams relevant for the second order self-force are given in Fig.(5.1). Their contribution to the effective action is

$$iS_{eff}[z^a(\lambda)] = \cdots + \text{Fig. (5.1a)} + \text{Fig. (5.1b)} + \cdots . \tag{5.25}$$

Using the Feynman rules established in Section 4.4.3 we write down the corresponding expression for the first diagram

$$\begin{aligned}\text{Fig.(5.1a)} \;=\; & (i)^3 \left(\frac{1}{2!}\right)\left(-\frac{m}{m_{pl}}\right)^2\left(-\frac{m}{m_{pl}^2}\right)\sum_{a,b,c=1}^{2}(-1)^{a+b+c+1}\int d\lambda \int d\lambda' \int d\lambda'' \\ & \times j(z_a^\alpha)D_{ab}(z_a^\alpha, z_b^{\alpha'})j(z_b^{\alpha'})D_{bc}(z_b^{\alpha'}, z_c^{\alpha''})j(z_c^{\alpha''}) \end{aligned}\tag{5.26}$$

where the current density is defined by

$$j(z_a^\alpha) = \sqrt{-g_{\alpha\beta}\dot{z}_a^\alpha \dot{z}_a^\alpha}. \tag{5.27}$$

The factor of $(-1)^{a+b+c+1}$ comes from the Feynman rule that one must include a factor of (-1) for each vertex in the diagram that is of type-2 when summing over the CTP indices. We also remark that we are not using the CTP metric to contract the D_{ab} and D_{bc} factors, though we can by writing

$$\sum_{b=1}^{2}(-1)^{b+1}D_{ab}D_{bc} = D_a{}^bD_{bc} = D_{ae}c^{eb}D_{bc} \tag{5.28}$$

where in the last two equalities we are implicitly summing over b using the repeated index summation convention and the CTP metric to contract the two-point functions. All three expressions are equivalent. The second diagram is given by

$$\begin{aligned}\text{Fig.(5.1b)} &= (i)^4 \left(\frac{2}{3!}\right)\left(-\frac{m}{m_{pl}}\right)^3 \left(\frac{1}{m_{pl}}\right) \sum_{a,b,c,d=1}^{2} (-1)^{a+b+c+d} \\ & \int d\lambda \int d\lambda' \int d\lambda'' \int d^4x \, g^{1/2} g^{\alpha\beta} \\ & \times \nabla_\alpha D_{da}(x^\mu, z_a^\alpha) j(z_a^\alpha) \nabla_\beta D_{db}(x^\mu, z_b^{\alpha'}) j(z_b^{\alpha'}) D_{dc}(x^\mu, z_c^{\alpha''}) j(z_c^{\alpha''}).\end{aligned} \tag{5.29}$$

The derivative on the propagator is to be taken with the coordinate x^μ that sits in the bulk background spacetime; the derivative is not to be taken on the spacetime coordinates evaluated along the particle's worldline, $z^\mu(\lambda)$. The factor of 2 in the symmetry factor comes from the two derivatives that can be associated with two of the three graviton lines in Fig.(5.1b).

Let us focus our attention on evaluating the diagram in Fig.(5.1a) first. Following the steps outlined in Section 4.4.5 we expand to linear order in the difference coordinate $z_-^\alpha = z_1^\alpha - z_2^\alpha$ and find

$$\begin{aligned}\text{Fig.(5.1a)} &= -i\frac{m^3}{2m_{pl}^4} \int d\tau \, z_-^\mu \big(a_{+\mu} + w_\mu{}^\nu[z_+^\alpha]\nabla_\nu\big) \int d\tau' \int d\tau'' \\ & \times \Big[D_{ret}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) + D_{ret}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^\alpha, z_+^{\alpha''})\Big]\end{aligned} \tag{5.30}$$

upon parameterizing the worldline by the particle's proper time. We also ignore the higher order corrections in z_-. The retarded propagators possess an ultraviolet divergence when evaluated at coincidence under the integral. Using the distributional methods introduced in Section 4.4.4 we can isolate this quasi-local divergent part and separate it from the non-local and history dependent finite part. The retarded propagator can be written in terms of a divergent and a finite, or renormalized, part

$$D_{ret} = D^{ren} + D^{BP}_{(n)} = Pf(D_{ret}) + D^{BP}_{(n)} \tag{5.31}$$

where the renormalized propagator is defined as

$$D^{ren} \equiv D_{ret} - D^{BP}_{(n)} \tag{5.32}$$

and we are temporarily dropping the spacetime indices in these expressions. We describe the divergent part using the Bunch-Parker momentum space representation of the propagator.

To regularize this divergence we define the divergent integral appearing in (5.30) as

$$I(\tau) \equiv \int d\tau' \int d\tau'' \Big[D_{ret}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) + D_{ret}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^\alpha, z_+^{\alpha''}) \Big]. \tag{5.33}$$

Using (5.31) we expand out the products and write $I(\tau)$ in terms of finite parts Fp (in the sense of Hadamard) and divergent parts so that

$$\begin{aligned} I(\tau) = {} & Fp\!\int d\tau' \, Fp\!\int d\tau'' \Big[D_{ret}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) + D_{ret}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^\alpha, z_+^{\alpha''}) \Big] \\ & + Fp\!\int d\tau' \int d\tau'' \Big[D_{ret}(z_+^\alpha, z_+^{\alpha'}) D^{BP}_{(n)}(z_+^{\alpha'}, z_+^{\alpha''}) + D_{ret}(z_+^\alpha, z_+^{\alpha'}) D^{BP}_{(n)}(z_+^\alpha, z_+^{\alpha''}) \Big] \\ & + \int d\tau' \, Fp\!\int d\tau'' \Big[D^{BP}_{(n)}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) + D^{BP}_{(n)}(z_+^\alpha, z_+^{\alpha'}) D_{ret}(z_+^\alpha, z_+^{\alpha''}) \Big] \\ & + \int d\tau' \int d\tau'' \Big[D^{BP}_{(n)}(z_+^\alpha, z_+^{\alpha'}) D^{BP}_{(n)}(z_+^{\alpha'}, z_+^{\alpha''}) + D^{BP}_{(n)}(z_+^\alpha, z_+^{\alpha'}) D^{BP}_{(n)}(z_+^\alpha, z_+^{\alpha''}) \Big]. \end{aligned} \tag{5.34}$$

Conveniently, all of the divergent terms are zero when evaluating the integrals using dimensional regularization. To demonstrate this, let us evaluate the first term in the last line of (5.34). Power counting the momentum factors in the momentum space representation for the retarded propagator in (D.89) we see that the fourth adiabatic order term $\sim k^{-2}$ and is therefore irrelevant. Hence, the only contribution comes from the leading order term, which scales as k^2 in the ultraviolet limit in $4d$ spacetime dimensions, yielding

$$\int d\tau' \int d\tau'' D^{BP}_{(n)}(z^{\alpha}_{+}, z^{\alpha'}_{+}) D^{BP}_{(n)}(z^{\alpha'}_{+}, z^{\alpha''}_{+})$$
$$= \int d\tau' D^{BP}_{(n)}(z^{\alpha}_{+}, z^{\alpha'}_{+}) \int d\tau'' \int_{\mathcal{C}_{ret},k} e^{ik_{\hat{a}} y^{\hat{a}}(\tau',\tau'')} \frac{1}{k^2} \tag{5.35}$$

where the displacement expressed in Riemann normal coordinates is

$$y^{\hat{a}}(\tau', \tau'') = -e^{\hat{a}}_{\alpha'}(\tau') u^{\alpha'}_{+}(\tau')(\tau' - \tau''). \tag{5.36}$$

We remark that the integral over τ'' and the momentum integral have the same structure as the integral in (4.94) implying that the values of the integrals are the same. Therefore, in dimensional regularization this integral vanishes

$$\int d\tau' \int d\tau'' D^{BP}_{(n)}(z^{\alpha}_{+}, z^{\alpha'}_{+}) D^{BP}_{(n)}(z^{\alpha'}_{+}, z^{\alpha''}_{+}) = 0. \tag{5.37}$$

All of the other divergent terms in (5.34) can be calculated in a similar manner and all give zero. It then follows that (5.34) can be written solely in terms of the Hadamard finite part,

$$I(\tau) \equiv Fp\!\int d\tau' \, Fp\!\int d\tau'' \Big[D_{ret}(z^{\alpha}_{+}, z^{\alpha'}_{+}) D_{ret}(z^{\alpha'}_{+}, z^{\alpha''}_{+}) + D_{ret}(z^{\alpha}_{+}, z^{\alpha'}_{+}) D_{ret}(z^{\alpha}_{+}, z^{\alpha''}_{+}) \Big]. \tag{5.38}$$

Observe that combining distributional methods, which isolate the history-dependent finite part from the quasi-local divergences, with dimensional regularization makes extracting the finite part relatively easy, even at higher orders. Compare this with the approach in [122, 127, 128, 129].

Using these results we find that the contribution to the effective action from Fig.(5.1a) is given by

$$
\begin{aligned}
\text{Fig.(5.1a)} &= -i\frac{m^3}{2m_{pl}^4}\int d\tau\, z_-^\mu\big(a_{+\mu} + w_\mu{}^\nu[z_+^\alpha]\nabla_\nu\big)\, Fp\!\int d\tau'\, Fp\!\int d\tau'' \\
&\quad \times\Big[D_{ret}(z_+^\alpha, z_+^{\alpha'})D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) + D_{ret}(z_+^\alpha, z_+^{\alpha'})D_{ret}(z_+^\alpha, z_+^{\alpha''})\Big]
\end{aligned}
\tag{5.39}
$$

and can be written in the equivalent but alternative form

$$
\begin{aligned}
\text{Fig.(5.1a)} &= -i\frac{m^3}{2m_{pl}^4}\int d\tau\, z_-^\mu\big(a_{+\mu} + w_\mu{}^\nu[z_+^\alpha]\nabla_\nu\big) \\
&\quad \times \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\Bigg[\int_{-\infty}^{\tau-\epsilon} d\tau' \int_{-\infty}^{\tau'-\epsilon'} d\tau''\, D_{ret}(z_+^\alpha, z_+^{\alpha'})D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) \\
&\qquad + \int_{-\infty}^{\tau-\epsilon} d\tau' \int_{-\infty}^{\tau-\epsilon'} d\tau''\, D_{ret}(z_+^\alpha, z_+^{\alpha'})D_{ret}(z_+^\alpha, z_+^{\alpha''})\Bigg]
\end{aligned}
\tag{5.40}
$$

since the divergent part vanishes in dimensional regularization.

Let us now compute the diagram in Fig.(5.1b) from (5.29),

$$
\begin{aligned}
\text{Fig.(5.1b)} &= -\frac{m^3}{3m_{pl}^4}\sum_{a,b,c,d=1}^{2}(-1)^{a+b+c+d}\int d\lambda\int d\lambda'\int d\lambda''\int d^4x\, g^{1/2} g^{\alpha\beta} \\
&\quad \times\nabla_\alpha D_{da}(x^\mu, z_a^\alpha) j(z_a^\alpha)\nabla_\beta D_{db}(x^\mu, z_b^{\alpha'}) j(z_b^{\alpha'}) D_{dc}(x^\mu, z_c^{\alpha''}) j(z_c^{\alpha''}).
\end{aligned}
\tag{5.41}
$$

Normally, we would next expand to linear order in the difference coordinate z_-. However, upon integrating by parts and relabeling the CTP indices and the worldline parameter integration variables we find that

$$\sum_{a,b,c,d=1}^{2} (-1)^{a+b+c+d} \int d\lambda \int d\lambda' \int d\lambda'' \int d^4x\, g^{1/2} g^{\alpha\beta}$$

$$\times \nabla_\alpha D_{da}(x^\mu, z_a^\alpha) j(z_a^\alpha) \nabla_\beta D_{db}(x^\mu, z_b^{\alpha'}) j(z_b^{\alpha'}) D_{dc}(x^\mu, z_c^{\alpha''}) j(z_c^{\alpha''}) \quad (5.42)$$

$$= -\frac{1}{2} \sum_{a,b,c,d=1}^{2} (-1)^{a+b+c+d} \int d\lambda \int d\lambda' \int d\lambda'' \int d^4x\, g^{1/2}$$

$$\times D_{da}(x^\mu, z_a^\alpha) j(z_a^\alpha) \Box D_{db}(x^\mu, z_b^{\alpha'}) j(z_b^{\alpha'}) D_{dc}(x^\mu, z_c^{\alpha''}) j(z_c^{\alpha''}). \quad (5.43)$$

From the wave equation (B.9) satisfied by the two-point functions

$$\Box D_{ab}(x, x') = i c_{ab} g^{-1/2}(x) \delta^d(x - x') \quad (5.44)$$

and noting that the CTP metric is related to the Kronecker delta through

$$c_{ab} = (-1)^{b+1} \delta_{ab} \quad (5.45)$$

we see that (5.42) becomes

$$= -\frac{1}{2} \sum_{a,b,c=1}^{2} (-1)^{a+b+c+1} \int d\lambda \int d\lambda' \int d\lambda''$$

$$\times j(z_a^\alpha) D_{ab}(z_a^\alpha, z_b^{\alpha''}) j(z_b^{\alpha'}) D_{bc}(z_b^{\alpha'}, z_c^{\alpha''}) j(z_c^{\alpha''}) \quad (5.46)$$

since the $\Box D_{db}$ factor is proportional to δ_{db}. This implies that this diagram actually describes the interaction of two gravitons, not three. Furthermore, these two gravitons interact with the particle alone since there is no remaining integration in the bulk spacetime. That is, the diagram no longer contains any graviton scattering off the background curvature. Therefore, the diagrams in Fig.(5.1) are proportional to each other in this nonlinear scalar theory.

Combining our results for the second order contributions to the effective action we find that

$$
\begin{aligned}
iS_{eff}[z^a] \; = \; & \cdots + i\frac{m^3}{3m_{pl}^4}\int d\tau\, z_-^{\mu}\left(a_{+\mu} + w_{\mu}{}^{\nu}[z_+^{\alpha}]\nabla_{\nu}\right) \\
& \times Fp\!\int d\tau'\, Fp\!\int d\tau'' \Big[D_{ret}(z_+^{\alpha}, z_+^{\alpha'})\Big(D_{ret}(z_+^{\alpha'}, z_+^{\alpha''}) + D_{ret}(z_+^{\alpha}, z_+^{\alpha''})\Big)\Big] \\
& + O(\mu^3).
\end{aligned} \tag{5.47}
$$

The contribution of (5.47) to the equations of motion for the effective point particle can be found by varying the effective action with respect to the difference coordinate z_-

$$
\left.\frac{\delta S_{eff}}{\delta z_-^{\mu}(\tau)}\right|_{z_-=0} = 0 \tag{5.48}
$$

which gives

$$
m_{eff}(\tau)a^{\mu}(\tau) = f^{\mu}(\tau) \tag{5.49}
$$

where $m_{eff}(\tau)$ is the time and history-dependent effective mass of the particle and $f^{\mu}(\tau)$ is the self-force on the compact object arising from interactions with gravitons emitted by the particle at some time in the past.

The effective mass has a contribution at first order, see (5.23), and so it is not surprising that m_{eff} receives corrections at second order as well. The total effective mass through second order is given by

$$
\begin{aligned}
m_{eff}(\tau) \; = \; & m - \frac{m^2}{m_{pl}^2}\, Fp\!\int d\tau'\, D_{ret}(z^{\alpha}, z^{\alpha'}) \\
& - \frac{m^3}{3m_{pl}^4}\, Fp\!\int d\tau'\, Fp\!\int d\tau'' \left[D_{ret}(z^{\alpha}, z^{\alpha'})\Big(D_{ret}(z^{\alpha'}, z^{\alpha''}) + D_{ret}(z^{\alpha}, z^{\alpha''})\Big)\right] \\
& + O(\mu^3).
\end{aligned} \tag{5.50}
$$

As mentioned before, an effective mass does not appear in the gravitational case since mass is a conserved quantity. However, in this scalar field model there exists a monopole particle-field coupling that allows for the particle to exchange energy with the scalar field by changing its rest mass.

The self-force on the effective point particle through second order in μ is deduced from (5.47) to be

$$\begin{aligned} f^\mu(\tau) \;=\; & w^{\mu\nu}[z^\alpha]\nabla_\nu \left\{ \frac{m^2}{m_{pl}^2}\, Fp\!\int d\tau'\, D_{ret}(z^\alpha, z^{\alpha'}) \right. \\ & + \frac{m^3}{3m_{pl}^4} Fp\!\int d\tau'\, Fp\!\int d\tau'' \Big[D_{ret}(z^\alpha, z^{\alpha'}) \Big(D_{ret}(z^{\alpha'}, z^{\alpha''}) + D_{ret}(z^\alpha, z^{\alpha''}) \Big) \Big] \\ & \left. + O(\mu^3) \right\}. \end{aligned} \qquad (5.51)$$

We remark that both (5.50) and (5.51) are entirely finite and are made so using well-established regularization techniques from the theory of interacting quantum fields. Furthermore, no parameters of the theory (i.e., the mass, $c_{R,V}$, etc.) are renormalized since all of the divergences encountered so far behave as a power in a cut-off momentum and are thus zero in dimensional regularization.

We are aware of only one other calculation of the second order self-force, which is given by Rosenthal in [122, 127, 128, 129]. Interestingly, to do the calculation, Rosenthal enforces a so-called Fermi gauge for the metric perturbations, which is defined to be the gauge for which the first order self-force is zero. While we are using the Lorenz gauge, it will be fruitful to find the appropriate gauge transformation that relates the Fermi and the Lorenz gauges so that a direct comparison of our (future) results for the gravitational self-force can be made with Rosenthal's.

For the gravitational case, we expect our second order self-force expression to be similar in form to (5.51) but with an additional term that comes from using the wave equation as we did in (5.44). In this case, the wave equation for metric perturbations on a curved background contains the following curvature-dependent term

$$2R_{\alpha\ \beta}^{\ \mu\ \nu} D^{ab}_{\alpha\beta\gamma'\delta'}. \tag{5.52}$$

The second order self-force should therefore contain a term that is schematically given by

$$\begin{aligned} \sim \sum_{a,b,c,d} (-1)^{a+b+c+d} \int d\tau \int d\tau' \int d\tau'' \int d^4x\, g^{1/2} \\ \times D_{da}(x^\mu, z_a^\alpha) D_{db}(x^\mu, z_b^{\alpha'}) D_{dc}(x^\mu, z_c^{\alpha''})\, \mathrm{Riem}(x)(\cdots) \end{aligned} \tag{5.53}$$

where $(\cdots)$ denotes velocity-dependent factors, Riem denotes the Riemann curvature tensor and we ignore the tensor indices. We remark that (5.53) describes the contribution to the self-force from the nonlinear nature of General Relativity as evidenced by the presence of graviton scattering in the bulk curved background geometry. In particular, this term is not proportional to Fig.(5.1a) and should therefore include qualitatively different effects than the self-force for the nonlinear scalar model in (5.51).

5.1.3 Third order self-force Feynman diagrams

Computing the self-force at higher orders in μ proceeds in a manner similar to the second order calculation in the nonlinear scalar toy model we considered in the preceding Section. According to the steps outlined in Section 4.4.5 we need to

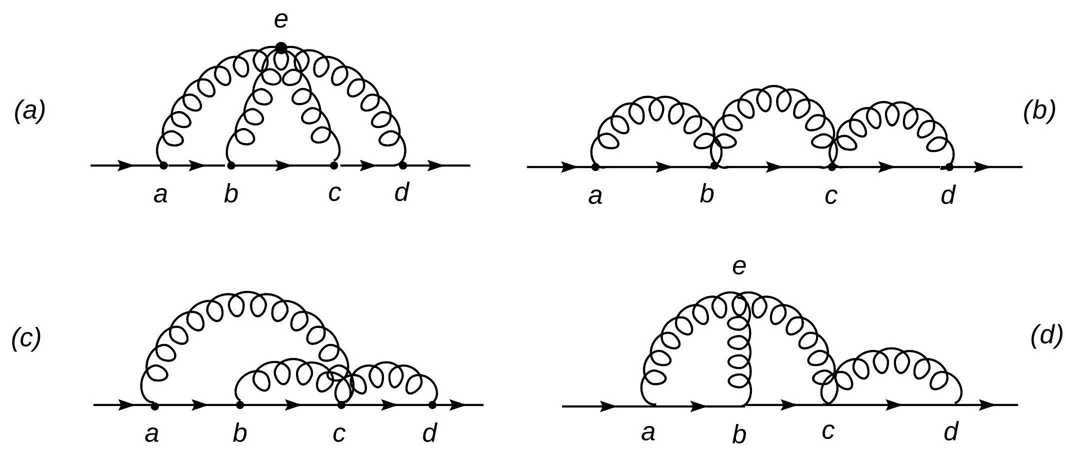

Figure 5.2: The connected diagrams relevant for a calculation of the third order self-force.

construct the distinct Feynman diagrams that appear at the order in μ of interest. Here, we will consider the third order diagrams.

The third order Feynman diagrams that scale as $\mu^3 L$ and contribute to the classical regime of the effective action are the connected, or contiguous, diagrams. We show all such diagrams relevant for a third order calculation of the self-force Fig.(5.2). The calculations of these diagrams will not be given here but are displayed for future reference.

5.2 Self-force on a spinning compact body

In the previous sections we developed an effective field theory approach for determining the equations of motion for a compact object moving through the background vacuum spacetime of a supermassive black hole. The treatment, so far, deals only with spherically symmetric compact objects such as a Schwarzschild black hole. We have not yet considered bodies that possess intrinsic, or permanent, multipole moments. An important example of a permanent moment is the internal angular momentum, or spin, of the compact body. A spinning body fails to retain a spherical shape but instead distorts and deforms according to the centripetal forces experienced by the rotating parts of the body.

The space-based gravitational wave interferometer LISA [2] requires accurate and sufficiently precise gravitational waveforms to construct the templates necessary for the detection of gravitational radiation and the estimation of various parameters associated with the source. Spinning bodies participating in an extreme mass ratio inspiral are thought to constitute the most likely candidates for detection and yet not much is known about the subleading effects of spin on the compact body's motion. However, the self-force on a spinning point particle was first derived by Mino, Sasaki and Tanaka [51] shortly after their seminal work deriving the self-force on a non-spinning particle [20]. Their result describes the usual spin precession in a curved background along with the MSTQW expression for the first order self-force.

In this Section we develop an effective field theory approach that incorporates spin and can be extended to describe permanent multipole moments associated with the compact object. Divergences are encountered as before because we are using an effective point particle to describe the motion of the compact body. Using the methods developed in Section 2.71 we can regularize these divergences and renormalize the coupling constants of the theory if necessary. Our approach is designed to be systematic and easily extendable to higher orders in μ.

The approach and treatment describing spinning particles in flat spacetime was developed using orthonormal basis vectors in [69]. The generalization to spinning particles in a curved spacetime has been given recently in [70] for the purpose of studying binary systems with spinning constituents within an effective field theory approach for the post-Newtonian approximation. We will follow closely the treatments given in [70, 114].

5.2.1 Preliminaries

Introduce an orthonormal basis, or tetrad, e^{μ}_{I} at the point x in a curved spacetime where a Greek index, μ here, labels the spacetime component of a 4-vector and an uppercase Latin index, I here where $I = 0, \ldots, 3$, denotes the frame components. The basis satisfies the following relationships

$$g^{\mu\nu} = e^{\mu}_{I} e^{\nu}_{J} \eta^{IJ} \tag{5.54}$$

$$\eta_{IJ} = e^{\mu}_{I} e^{\nu}_{J} g_{\mu\nu} \tag{5.55}$$

and can be transported to another point in the spacetime in several ways. The particular method appropriate for the description of spinning particles is that of Fermi-Walker transport [85]

$$\dot{e}^{I}_{\mu} \equiv \frac{De^{\mu}_{I}}{d\lambda} = \Omega^{\mu\nu} e_{\nu I} \tag{5.56}$$

where $\Omega^{\mu\nu}$ is an anti-symmetric tensor since

$$\Omega_{\mu\nu} = \eta^{IJ} e_{\mu I} \frac{De_{\nu J}}{d\lambda} \tag{5.57}$$

upon using (5.55). In order to derive the dynamical equations of motion for the particle and the spin degrees of freedom we need to construct an action using the generalized coordinates and velocities of both the particle $(z^\mu, \dot{z}^\mu)$ and of the tetrad $(e^I_\mu, \dot{e}^I_\mu)$.

The action must respect the symmetries of the system. Here those symmetries require the action to be invariant under worldline reparameterizations, general coordinate transformations and locally Lorentz transformations. This last symmetry is required since the tetrad, as 4-vectors, transform as elements of $SO(3,1)$. As such, the Lagrangian is generally given by

$$L = L[z^\mu, \dot{z}^\mu = u^\mu, \Omega^{\mu\nu}] \tag{5.58}$$

where we regard $\Omega^{\mu\nu}$ as an (angular) velocity since it is proportional to $\dot{e}^\mu_I$ in (5.57). Therefore, neglecting parity violating terms the Lagrangian can be a function of only four scalars

$$L = L(a_1, a_2, a_3, a_4) \tag{5.59}$$

where the scalar quantities a_n are

$$a_1 = u^\alpha u_\alpha \tag{5.60}$$

$$a_2 = \Omega^{\alpha\beta}\Omega_{\alpha\beta} \tag{5.61}$$

$$a_3 = u^\alpha \Omega_{\alpha\beta}\Omega^{\beta\gamma}u_\gamma \tag{5.62}$$

$$a_4 = \Omega_{\alpha\beta}\Omega^{\beta\gamma}\Omega_{\gamma\delta}\Omega^{\delta\alpha}. \tag{5.63}$$

The antisymmetric tensor $S^{\mu\nu}$ and the momentum of the particle p^μ can be defined through the variation of the Lagrangian so that [69]

$$\delta L = -p^\mu \delta u_\mu - \frac{1}{2} S^{\mu\nu}\delta\Omega_{\mu\nu} \tag{5.64}$$

where the momenta are given by

$$p^\mu = -\frac{\delta L}{\delta u_\mu} \tag{5.65}$$

$$S^{\mu\nu} = -\frac{\delta L}{\delta \Omega_{\mu\nu}}. \tag{5.66}$$

once an appropriate Lagrangian is specified. Taking into account the Lagrangian's dependence on the a_n we see that [70]

$$p^\mu = -2u^\mu \frac{\partial L}{a_1} - 2\Omega^{\mu\alpha}\Omega_{\alpha\beta}u^\beta \frac{\partial L}{\partial a_3} \tag{5.67}$$

$$S^{\mu\nu} = -4\Omega^{\mu\nu}\frac{\partial L}{\partial a_2} - 2\big(u^\mu\Omega^{\nu\alpha}u_\alpha - u^\nu\Omega^{\mu\alpha}u_\alpha\big)\frac{\partial L}{\partial a_3} + 8\Omega^{\mu\alpha}\Omega_{\alpha\beta}\Omega^{\beta\nu}\frac{\partial L}{\partial a_4}. \tag{5.68}$$

The equations of motion for the momenta can be derived from a variational principle. For the spin equations of motion let us keep $z^\mu(\lambda)$ constant so that $\delta u^\mu = 0$. Then the variation of the action is

$$\delta S = -\int d\lambda\, S^{\mu\nu}\delta\Omega_{\mu\nu}, \tag{5.69}$$

which gives, upon computing the variation of the angular velocity from (5.57),

$$\frac{DS^{\mu\nu}}{d\lambda} = S^{\mu\lambda}\Omega_\lambda{}^\nu - \Omega^\mu{}_\lambda S^{\lambda\nu} \tag{5.70}$$

$$= p^\mu u^\nu - u^\mu p^\nu. \tag{5.71}$$

The last line follows from (5.67) and (5.68). The equations of motion for the particle worldline are found from the variation

$$\delta S = -\int d\tau \Big(p^\mu \delta u_\mu + S^{\alpha\beta}\frac{\partial \Omega_{\alpha\beta}}{\partial x_\mu}\delta x_\mu\Big) \tag{5.72}$$

from which it follows that [70]

$$\frac{Dp^\mu}{d\tau} = -\frac{1}{2}R^\mu{}_{\gamma\alpha\beta}S^{\alpha\beta}u^\gamma \tag{5.73}$$

The equations of motion in (5.70) and (5.73) (or equivalently with (5.71), taken together form the Papapetrou-Dixon equations [71, 130].

As it stands we must add a set of constraints to the Papapetrou-Dixon equations in order to describe the physical degrees of freedom correctly and unambiguously. There are many constraints that can be chosen but we will not discuss the different possibilities here. See, however, [69, 131, 132] for discussions of these different gauges. We impose the covariant constraint

$$S^{\mu\nu}p_\nu \approx 0 \tag{5.74}$$

where the $\approx$ signifies weak equality in the sense of Dirac [90]. This is a second class constraint implying that the number of spin degrees of freedom is reduced from the original 6 to 3, which is expected on physical grounds. Furthermore, this constraint can be imposed from a Lagrangian [69]. The condition, or any other similar kind of constraint, is called a spin supplementary condition or SSC. It follows [70] that these constraints will be preserved by the evolution of the system if

$$p^\mu = mu^\mu - \frac{1}{2m}R_{\beta\nu\rho\sigma}S^{\mu\beta}S^{\rho\sigma}u^\nu \tag{5.75}$$

There is another gauge freedom in the theory coming from the worldline reparameterization. A sensible choice for the worldline parameter is given by $e_0^\mu = u^\mu$. From [70] we see that the spin and the angular velocity are proportional to leading order in the spacetime curvature

$$S^{\mu\nu} = \frac{I}{(-g_{\alpha\beta}u^\alpha u^\beta)^{1/2}}\left(\Omega^{\mu\nu} + \frac{I}{2m}R^{\mu\nu}{}_{\alpha\beta}\Omega^{\alpha\beta} + \cdots\right) \tag{5.76}$$

where I is the moment of inertia. We remark that a SSC different from (5.74) will give rise to a different relationship among the spin angular momentum $S^{\mu\nu}$ and the angular velocity $\Omega^{\mu\nu}$.

In this work we will not study the time evolution of the spin but will instead treat $S^{\mu\nu}$ as being fixed with a given time dependence. See [83] for the case where the spin evolves dynamically with the field and the particle's motion.

For our purposes here we take the point particle action to be

$$S[z] = -m \int d\lambda \, (-g_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta)^{1/2} + \frac{1}{2}\int d\lambda \, S^{IJ}\Omega_{IJ} \tag{5.77}$$

where we are ignoring for now the non-minimal terms describing finite size effects. Following [69, 70] we see that the introduction of the tetrad $e^\mu_I(\lambda)$, which describes the rotating frame of the compact object, allows for the spin to be included covariantly into the particle action. The angular velocity is determined in terms of the tetrad by (5.57) and the spin angular momentum $S^{IJ} = e^I_\mu e^J_\nu S^{\mu\nu}$ is canonically conjugate to Ω_{IJ}.

In order to study the influence that the compact object's spin has on its motion we need to generate the graviton-spin vertices, which describe the interactions of the spinning particle with the metric perturbations. At this point our treatment diverges from that given in [70] since we must expand about a curved background whereas [70] expands about flat spacetime in preparation for a post-Newtonian approximation within the effective field theory framework.

The graviton-spin vertices are calculated by expanding that part of the action (5.77) that depends on spin around the background spacetime. Write the full metric as

$$g^{\rm full}_{\mu\nu} = g_{\mu\nu} + \frac{h_{\mu\nu}}{m_{pl}} \tag{5.78}$$

and expand the full tetrad

$$(e^{\rm full})^J_\mu = e^J_\mu + \frac{1}{2m_{pl}}\, h_{\mu\nu} e^{\nu J} - \frac{1}{8m_{pl}^2}\, h_\mu{}^\sigma h_{\sigma\nu} e^{\nu J} + \cdots \tag{5.79}$$

where e^J_μ is the tetrad in the background spacetime. The expansion of the full tetrad can be derived from (5.55). Using these expansions in (5.77) we find that

$$S = -\int d\tau\, p^\mu u_\mu + \sum_{n=0}^{\infty} S^{(n)}_{spin}[z, S^{IJ}] \tag{5.80}$$

where the first few graviton-spin interaction terms are given by

$$S^{(0)}_{spin} = \frac{1}{2} S^{\mu\nu}\Omega_{\mu\nu} \tag{5.81}$$

$$S^{(1)}_{spin} = \frac{1}{2m_{pl}}\, S^{\mu\nu} h_{\alpha\mu;\nu} u^\alpha \tag{5.82}$$

$$S^{(2)}_{spin} = \frac{1}{4m_{pl}^2}\, S^{\alpha\beta} u^\gamma h_\beta{}^\delta \left(\frac{1}{2} h_{\alpha\delta;\gamma} + h_{\gamma\delta;\alpha} - h_{\gamma\alpha;\delta} \right) \tag{5.83}$$

To determine the order in μ at which these interactions become important we will need to develop the spin power counting rules. Once these are determined we may then draw all of the Feynman diagrams that contribute to the effective action at a given order.

5.2.2 Power counting rules and Feynman diagrams

From the leading order term in (5.76) we see that the spin angular momentum is proportional to the angular velocity

$$S^{\mu\nu} \approx \frac{I}{(-g_{\alpha\beta} u^\alpha u^\beta)^{1/2}}\, \Omega^{\mu\nu} \tag{5.84}$$

where I is the moment of inertia. The compact objects discussed in this work include neutron stars and black holes so that the size of the body is of the order of its mass. For such objects the moment of inertia can be estimated from

$$I \approx m r_m^2 \sim \frac{m^3}{m_{pl}^4}. \tag{5.85}$$

The spin angular momentum therefore scales as

$$S \approx I\Omega \sim \frac{m^3}{m_{pl}^4} \frac{v_{rot}}{r_m} \tag{5.86}$$

where we have dropped the spacetime indices in this expression. The velocity at which the body is rotating about its rotational axis is denoted by v_{rot}.

The magnitude of v_{rot} depends upon the spin of the body itself. If the body is rotating at or near its maximal velocity then $v_{rot} \sim 1$ and

$$S \sim \frac{m^2}{m_{pl}^2} \sim \mu L. \tag{5.87}$$

If the compact object is rotating at a rate that is similar to its orbital velocity so that the body is nearly tidally locked with the large black hole companion then

$$\frac{v_{rot}}{r_m} \sim \frac{v}{r}. \tag{5.88}$$

From this and the relativistic motion of the compact object it follows that $v_{rot} \sim \mu$. Therefore, for the corotating scenario

$$S \sim \mu^2 L. \tag{5.89}$$

Generally, we will assume that the spin angular momentum scales with μ as some power s

$$S \sim \mu^s L \tag{5.90}$$

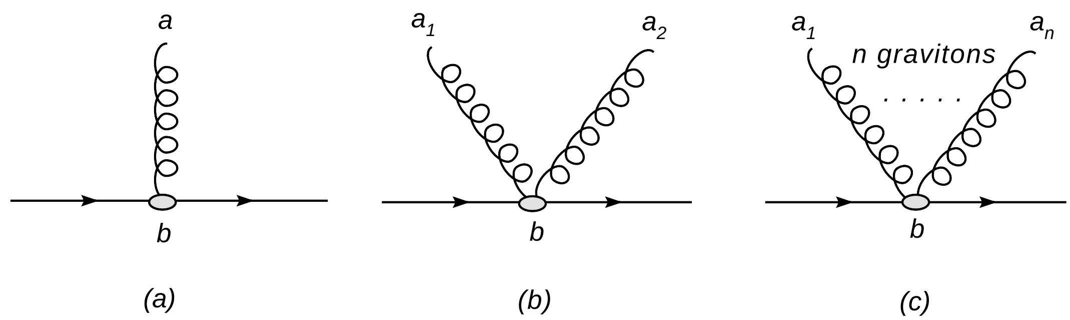

Figure 5.3: The graviton-spin interaction vertices describing the coupling of one, two and n gravitons, respectively, to the spin angular momentum operator. The blob represents an insertion of S^{IJ}.

where $s = 1$ for the maximally rotating case and $s = 2$ for the co-rotating configuration.

With the power counting of the spin complete we can now power count the graviton-spin interaction terms $S^{(n)}_{spin}$. Figs. (5.3a) and (5.3b) show the first two non-trivial vertices. These scale with μ and the orbital angular momentum as

$$\text{Fig.(5.3a)} \quad = \quad iS^{(1)}_{spin} \sim \mu^{s+1/2} L^{1/2} \tag{5.91}$$

$$\text{FIg.(5.3b)} \quad = \quad iS^{(2)}_{spin} \sim \mu^{s+1}. \tag{5.92}$$

Generally, for a vertex describing the interaction of n gravitons with a single spin operator, shown in Fig.(5.3c), we find the scaling rule

$$\text{Fig.(5.3c)} \quad = \quad iS^{(n)}_{spin} \sim \mu^{s+n/2} L^{1-n/2}. \tag{5.93}$$

As before, we observe that these vertices scale as a power of L smaller than or equal to L^1.

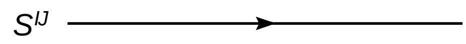

Figure 5.4: The leading order contribution to the particle equations of motion for a maximally rotating spinning body. This diagram is just the usual spin precession described by the Papapetrou-Dixon equations. For a co-rotating compact object this diagram first enters at second order in μ.

5.2.3 Feynman diagrams

In this Section we write down the Feynman diagrams for the first few orders in μ that are relevant for computing the self-force on the effective point particle. The compact object spins at two natural scales, $v_{rot} \sim 1$ (maximal rotation) and $v_{rot} \sim \mu$ (co-rotation). Because the spin effects of a co-rotating body are suppressed relative to the maximally rotating body then the diagrams that we will need to calculate at a given order in μ will be affected by which scenario is under consideration. We will discuss the diagrams relevant for a maximally rotating body first.

The first diagram appears at $O(\mu)$ and is shown in Fig.(5.4). This diagram is the familiar spin precession since

$$S^{(0)}_{spin} = \frac{1}{2} \int d\tau \, S^{\mu\nu} \Omega_{\mu\nu} \sim \mu L \tag{5.94}$$

upon using (5.57) to show that $\Omega_{\mu\nu} \sim 1/\mathcal{R}$. In fact, $S^{(0)}_{spin}$ gives the leading order equations of motion for the spin and is part of the Papapetrou-Dixon equations (5.70) and (5.73), along with the MSTQW self-force.

The inclusion of spin at first order for describing the motion of a particle in a curved spacetime has been given previously by [71, 133] and within the context of

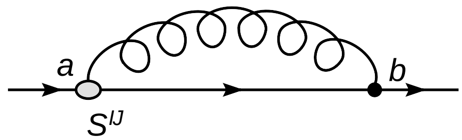

Figure 5.5: The first non-trivial contribution of spin to the self-force on the effective particle appears at second order for a maximally rotating compact object. For a corotating body this same diagram appears at third order.

self-force using the method of matched asymptotic expansions [51]. However, this effect is somewhat trivial since the spin does not interact with the metric perturbations $h_{\mu\nu}$ and therefore only describes the precession of the spinning particle as it moves through the background spacetime.

At second order in μ there is only one diagram and this is shown in Fig.(5.5). Diagramatically, this appears the same as the first order self-force diagram in Fig.(4.3) for a non-spinning particle. However, the insertion of the spin angular momentum operator $S^{IJ} \sim \mu L$ increases the order of the diagram by one. Being a second order diagram we must calculate its contribution to the second order self-force if the compact object is rotating at a maximal speed,

$$\text{Fig.}(5.5) \sim \mu^{s+1} L \tag{5.95}$$

for $s = 1$.

The diagram in Fig.(5.5) represents the leading order spin-orbit contribution to the self-force. This can be seen because the spin angular momentum is influencing the orbital motion of the particle by coupling to a non-spinning vertex operator, specifically $S_{pp}^{(1)}$, via the metric perturbations.

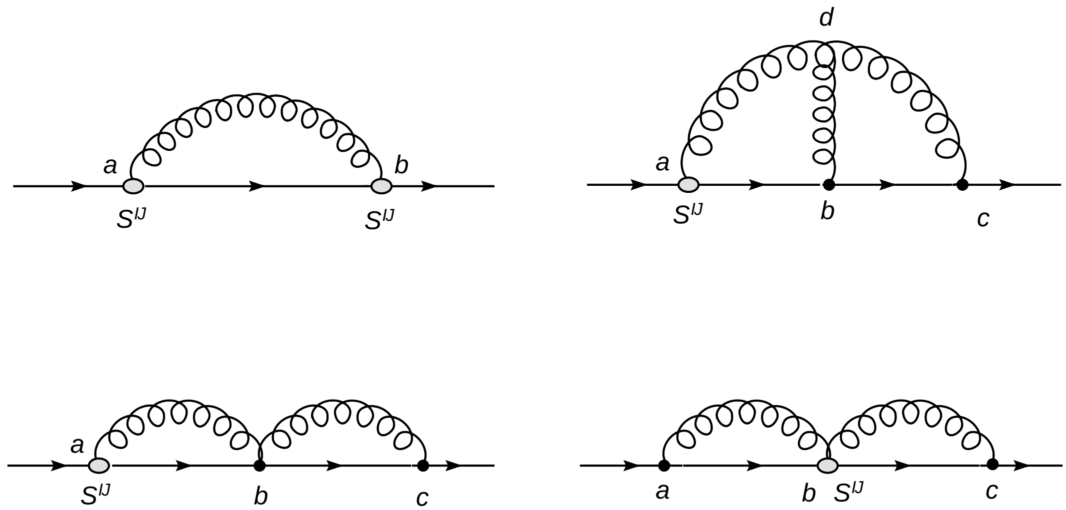

Figure 5.6: The third order diagrams that contribute to the self-force on a maximally rotating compact object. The diagram in (a) represents a spin-spin interaction while the remaining diagrams are subleading spin-orbit corrections. For a co-rotating body (a) appears at fifth order and the remaining diagrams contribute at fourth order.

At third order there appear several spin-orbit diagrams along with a new kind of diagram; see Fig.(5.6). This new diagram, given in Fig.(5.6a) contains only two insertions of the spin angular momentum. We may interpret this as the leading order spin-spin contribution to the self-force. Notice that the spin-orbit diagram in Fig.(5.6b) contains a vertex from graviton scattering in the bulk spacetime.

The co-rotating spinning compact object has diagrams that enter at different orders since the spin angular momentum scales at a higher order than if the body were spinning at a maximal rotational velocity. The first non-trivial diagram is given in Fig.(5.4), which enters at second order in μ. The second diagram is given in Fig.(5.5) and is the leading order spin-orbit interaction, which appears at third order μ. The spin-spin interaction in Fig.(5.6a) is a fifth order contribution and provides a very small correction to the particle's motion. In fact, the spin-spin interaction is dominated by the leading order finite size diagram in Fig.(4.5), which appears at fourth order. The remaining diagrams in Fig.(5.6) are sub-leading spin-orbit interactions.

The EFT approach that we have developed in this Chapter allows us to derive the self-force at higher orders in μ with the inclusion of graviton-spin interactions. Since the approach is systematic at every step there is no obstacle to calculating at higher orders. We have not computed the momentum space representation of the propagator for metric perturbations beyond second adiabatic order in this work. Therefore, unfortunately, we can not compute the higher order diagrams in the gravitational case here but must settle for the nonlinear scalar model instead. We refer the reader to [83] for the calculations and results for the gravitational case.

To evaluate these diagrams we need to determine how the nonlinear scalar field interacts with the spin angular momentum of the compact object. We deduce the scalar-spin interaction vertices in the next Section.

5.2.4 Nonlinear scalar field interacting with a spinning particle

We use the same nonlinear scalar field model introduced in Section 5.1.2. To generate the scalar-spin interaction vertices we utilize the conformal transformation in (5.5)

$$g_{\mu\nu} \to e^{2\phi/m_{pl}} g_{\mu\nu} \tag{5.96}$$

so that (5.55) becomes

$$e^{\mu}_{I} e^{\nu}_{J} \eta^{IJ} = e^{2\phi/m_{pl}} g^{\mu\nu} \tag{5.97}$$

$$= g_{\mu\nu} \left(1 + \frac{2\phi}{m_{pl}} + \frac{2\phi^2}{m_{pl}^2} + \cdots \right). \tag{5.98}$$

Calculating the angular velocity $\Omega_{\mu\nu}$ perturbatively in powers of the field from (5.57) we find that the scalar-spin interactions are

$$S[z, \phi] = -\int d\tau \, p^{\mu} u_{\mu} + \sum_{n=0}^{\infty} S^{(n)}_{spin}[z, S^{IJ}, \phi] \tag{5.99}$$

where the interaction terms are

$$S^{(0)}_{spin} = \frac{1}{2} S^{\mu\nu} \Omega_{\mu\nu} \tag{5.100}$$

$$S^{(1)}_{spin} = \frac{2}{m_{pl}} S^{\mu\nu} u_{\mu} \phi_{;\nu} \tag{5.101}$$

$$S^{(2)}_{spin} = \frac{8}{m_{pl}^2} S^{\mu\nu} u_{\mu} \phi_{;\nu} \phi. \tag{5.102}$$

At this point we observe an interesting feature of spin interactions in this nonlinear scalar theory. From the spin supplementary condition $S^{\mu\nu} p_{\nu} = 0$ we see that

$$S^{\mu\nu} p_{\nu} = 0 \quad \Rightarrow \quad S^{\mu\nu} u_{\nu} = \frac{1}{2m^2} S^{\mu\nu} S_{\nu}{}^{\alpha} S^{\gamma\delta} R_{\alpha\beta\gamma\delta} u^{\beta} \tag{5.103}$$

upon using (5.75). The scalar-spin interaction terms then become

$$S^{(0)}_{spin} = \frac{1}{2} S^{\mu\nu} \Omega_{\mu\nu} \tag{5.104}$$

$$S^{(1)}_{spin} = \frac{1}{m^2 m_{pl}} \mathcal{S}^{\mu}{}_{\beta} u^{\beta} \phi_{;\nu} \tag{5.105}$$

$$S^{(2)}_{spin} = \frac{4}{m^2 m_{pl}^2} \mathcal{S}^{\mu}{}_{\beta} u^{\beta} \phi_{;\nu} \phi \tag{5.106}$$

where we have defined the tensor

$$\mathcal{S}^{\mu}{}_{\beta} \equiv S^{\mu\nu} S_{\nu}{}^{\alpha} S^{\gamma\delta} R_{\alpha\beta\gamma\delta} \tag{5.107}$$

for notational convenience. Notice that all subleading interaction terms are changed in the same way by the SSC. The order at which the subleading spin interactions, $S^{(n)}_{spin}$ with $n > 0$, appear are increased by two so that Fig.(5.4) is actually a *fourth* order diagram for a maximally rotating body. Likewise, the spin-spin interaction in Fig.(5.6) is actually a *seventh* order diagram in this theory. The reason for this comes from the fact that all of the subleading terms in (5.98) are proportional to the background metric. In turn, the spin tensor necessarily contracts with the particle's 4-velocity thereby accounting for this increase in the order of the diagram. We do not anticipate this happening with identical implications in the gravitational case, although this is investigated in [83].

Choosing other SSC's will obviously affect the order at which the spin-orbit and spin-spin diagrams enter the effective action. For example, if we choose

$$S^{\mu\nu} u_\nu = 0 \tag{5.108}$$

then all of the scalar-spin vertices are zero and the only contribution from spin to the particle's motion is via the familiar spin precession. However, the center of mass implied by this SSC describes a particle undergoing rapid helical motions (with a frame-dependent radius) centered on the worldline picked out by (5.74) [130, 134]. The difference between these two SSC's is third order in the spin.

The power counting rules in this theory are the same as in Section 5.2.2 and the interactions have the same structure as their gravitational counterpart using the SSC $S^{\mu\nu} p_\nu = 0$. Therefore, the diagrams generated by the scalar-spin interactions $S^{(n)}_{spin}$ are the same as in the gravitational case. We turn now to computing two of these diagrams in the nonlinear scalar model for the case of a maximally spinning compact object.

5.2.4.1 Leading order spin-orbit interaction

For a maximally rotating body the leading order spin-orbit interaction occurs naively at fourth order in μ and is given by the diagram in Fig.(5.5). The Feynman rules imply that

$$\begin{aligned} \text{Fig.}(5.5) &= (i)^2 \left(\frac{1}{m^2 m_{pl}} \right) \left(-\frac{m}{m_{pl}} \right) \sum_{a,b=1}^{2} (-1)^{a+b} \int d\lambda \int d\lambda' \\ & \left[j(z_a^\alpha) \mathcal{S}^{\alpha'\beta'} u_{b\alpha'} \nabla_{\beta'} D_{ab}(z_a^\alpha, z_b^{\alpha'}) \right. \\ & \left. + \mathcal{S}^{\alpha\beta} u_{a\beta} j(z_b^{\alpha'}) \nabla_\alpha D_{ab}(z_a^\alpha, z_b^{\alpha'}) \right]. \end{aligned} \tag{5.109}$$

This can be simplified by noticing that $D_{ab}(z_a^\alpha, z_b^{\alpha'})$ is symmetric under the interchange of the CTP indices and the λ integrations. Therefore, we may write the diagram as

$$\text{Fig.}(5.5) = \frac{1}{m m_{pl}^2} \sum_{a,b=1}^{2} (-1)^{a+b} \int d\lambda \int d\lambda' \, \mathcal{S}^{\alpha\beta} u_{a\beta} \nabla_\alpha D_{ab}(z_a^\alpha, z_b^{\alpha'}) j(z_b^{\alpha'}). \tag{5.110}$$

Passing the λ-dependent factors through the λ' integral leaves

$$\text{Fig.}(5.5) = \frac{1}{m m_{pl}^2} \sum_{a,b=1}^{2} (-1)^{a+b} \int d\lambda \, \mathcal{S}^{\alpha\beta} u_{a\beta} \nabla_\alpha \int d\lambda' \, D_{ab}(z_a^\alpha, z_b^{\alpha'}) j(z_b^{\alpha'}). \tag{5.111}$$

Next, we expand in powers of the coordinate difference z_- through first order and find that

$$
\begin{aligned}
\text{Fig.}(5.5) \quad = \quad & -\frac{2}{mm_{pl}^2}\int d\tau\, z_-^{\mu}\left[\mathcal{S}^{\alpha\beta}u_{+\beta}\big(a_{+\mu} + w_{\mu}{}^{\nu}[z_+^{\alpha}]\nabla_{\nu}\big) - w_{\mu}{}^{\sigma\nu}[z_+]\nabla_{\nu}\mathcal{S}^{\alpha}{}_{\sigma}\right] \\
& \times \nabla_{\alpha}\int d\tau'\, D_{ret}(z_+^{\alpha}, z_+^{\alpha'}) \qquad (5.112)
\end{aligned}
$$

where

$$
w^{\mu\sigma\nu} = -2g^{\mu[\sigma}u^{\nu]} \qquad (5.113)
$$

satisfies the identity $u^{\mu}w_{\mu}{}^{\sigma\nu} = 0$.

We see that the retarded propagator is acted upon by a covariant derivative so let us focus on the divergent integral

$$
\begin{aligned}
I_{\alpha}(\tau) \quad \equiv \quad & \nabla_{\alpha}\int d\tau'\, D_{ret}(z_+^{\alpha}, z_+^{\alpha'}) \qquad (5.114) \\
= \quad & \int d\tau' \nabla_{\alpha}D_{ret}(z_+^{\alpha}, z_+^{\alpha'}) \qquad (5.115)
\end{aligned}
$$

and apply the methods of Section 4.4.4 to regularize the divergence. Writing the integrand schematically as

$$
\nabla D_{ret} = \nabla D^{ren} + \nabla D^{div} = Pf(\nabla D_{ret}) + \nabla D^{BP}_{(n)} \qquad (5.116)
$$

so as to isolate the non-local finite part of the propagator from the quasi-local divergent part we see that we are led to a similar calculation we performed in Sections 4.4.4 and 5.1.2 to regularize the first order self-force.

We will turn now to calculating the divergent part of the integral $I_{\alpha}(\tau)$. In Riemann normal coordinates the derivative brings down a factor of the momentum from the exponential. Therefore, we need to evaluate the integral

$$I_\alpha(\tau) = i \int d\tau' \int_{ret,k} e^{-ik_\alpha u^\alpha(\tau')(\tau-\tau')} \left[\frac{k_{\hat{a}}}{k^2} + O(k^{-5}) \right] \tag{5.117}$$

where we have used the momentum space representation for the scalar retarded propagator given in (D.89). The $O(k^{-5})$ contribution comes from the fourth adiabatic order term in (D.89). In 4d spacetime the leading order term in the momentum integral scales as k^2 for high frequencies indicating that the integral $I_\alpha(\tau)$ diverges as a power. We expect the integral to vanish when evaluated with dimensional regularization. We sketch this calculation for the sake of being complete.

The proper time integral enforces the orthogonality between the momentum and velocity 4-vectors so that

$$I_\alpha(\tau) = i\big(\delta^0_{\hat{a}} v^i + \delta^i_{\hat{a}}\big) \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{k_i}{(\delta^{jl} - v^j v^l) k_j k_l} \tag{5.118}$$

upon ignoring the ultraviolet finite terms. Diagonalizing the matrix $\delta^{jl} - v^j v^l$ by assuming that $\mathbf{v} = (v, 0, \ldots, 0)$ we find

$$I_\alpha(\tau) = i\big(\delta^0_{\hat{a}} v^i + \delta^i_{\hat{a}}\big) \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{k_i}{(1-v^2)k_1^2 + k_2^2 + \cdots + k_{d-1}^2}. \tag{5.119}$$

From here we know that the integral vanishes since the integrand is odd under $k_i \to -k_i$ thereby implying

$$I_\alpha(\tau) = 0. \tag{5.120}$$

Notice that we did not need to regularize this integral using dimensional regularization to show that it vanishes.

The renormalized contribution from this diagram to the effective action is therefore

$$\text{Fig.(5.5)} \;=\; -\frac{2}{mm_{pl}^2}\int d\tau\, z_-^\mu \Big[\mathcal{S}^{\alpha\beta}u_{+\beta}\big(a_{+\mu}+w_\mu{}^\nu[z_+^\alpha]\nabla_\nu\big)-w_\mu{}^{\sigma\nu}[z_+]\nabla_\nu\mathcal{S}^\alpha{}_\sigma\Big] \times \nabla_\alpha Fp\int d\tau'\, D_{ret}(z_+^\alpha, z_+^{\alpha'}). \tag{5.121}$$

The contribution to the self-force is

$$f^\mu(\tau) \;=\; \cdots - \frac{2}{mm_{pl}^2}\Big[\mathcal{S}^{\alpha\beta}u_\beta w_\mu{}^\nu[z^\alpha]\nabla_\nu - w_\mu{}^{\sigma\nu}[z^\alpha]\nabla_\nu\mathcal{S}^\alpha{}_\sigma\Big] \times Fp\int d\tau'\,\nabla_\alpha D_{ret}(z^\alpha, z^{\alpha'})+\cdots \tag{5.122}$$

and to the effective mass is

$$m_{eff}(\tau) \;=\; \cdots + \frac{2}{mm_{pl}^2}\mathcal{S}^{\alpha\beta}u_\beta Fp\int d\tau'\,\nabla_\alpha D_{ret}(z^\alpha, z^{\alpha'})+\cdots \tag{5.123}$$

where $\mathcal{S}^{\alpha\beta}$ is given in (5.107).

5.2.4.2 Leading order spin-spin interaction

For a maximally rotating body the leading order spin-spin diagram in Fig.(5.6a) appears at $O(\mu^3)$. The Feynman rules indicate that this diagram equals

$$\text{Fig.(5.6a)} \;=\; (i)^2\left(\frac{1}{2!}\right)\left(\frac{2}{m_{pl}}\right)^2\sum_{a,b=1}^{2}(-1)^{a+b}\int d\lambda\int d\lambda' \; u_{a\alpha}\mathcal{S}^{\alpha\beta}\nabla_\beta u_{b\gamma'}\mathcal{S}^{\gamma'\delta'}\nabla_{\gamma'}D_{ab}(z_a^\alpha, z_b^{\alpha'}). \tag{5.124}$$

Summing over the CTP indices and expanding in powers of the difference coordinate z_- results in the following

$$\text{Fig.(5.6a)} \;=\; \frac{4}{m_{pl}^2}\int d\tau z_-^\mu w_\mu{}^{\alpha\nu}[z_+^\alpha]\mathcal{S}_\alpha{}^\beta\nabla_\nu\int d\tau'\,\nabla_\beta\nabla_{\gamma'}D_{ret}(z_+^\alpha, z_+^{\alpha'})S^{\gamma'\delta'}u_{+\delta'}. \tag{5.125}$$

The factor $\mathcal{S}^{\gamma'\delta'}u_{\delta'}$ in the integrand of the τ' integral can be simplified using the bi-vector of parallel propagation so that

$$\mathcal{S}^{\gamma'\delta'}u_{+\delta'}\nabla_\beta\nabla_{\gamma'}D_{ret} \;=\; \mathcal{S}^{\gamma\delta}u_{+\delta}g_\gamma{}^{\gamma'}(z_+^\alpha, z_+^{\alpha'})\nabla_{\gamma'}\nabla_\beta D_{ret} \tag{5.126}$$

$$\;=\; \mathcal{S}^{\gamma\delta}u_{+\delta}\nabla_\beta\nabla_\gamma D_{ret}. \tag{5.127}$$

Then we can define the singular integral as

$$I_{\beta\gamma}(\tau) = \int d\tau'\, \nabla_\beta\nabla_{\gamma'}D_{ret}(z_+^\alpha, z_+^{\alpha'}), \tag{5.128}$$

which captures the divergent part of the spin-spin diagram. In Riemann normal coordinates, the momentum space representation of the propagator in (D.89) gives

$$I_{\hat{b}\hat{c}}(\tau) \;=\; (i)^2 \int d\tau' \int_{\mathcal{C}_{ret}\, k} e^{ik_\alpha u^\alpha(\tau-\tau')} \left[\frac{k_{\hat{b}}k_{\hat{c}}}{k^2} + \frac{1}{15}R_{\hat{l}\hat{k}\hat{m}\hat{n}}R^{\hat{l}\hat{k}\hat{m}\hat{n}}\frac{1}{k^6} - \frac{8}{15}R_{\hat{l}\hat{k}\hat{m}\hat{a}}R^{\hat{l}\hat{k}\hat{m}}{}_{\hat{b}}\frac{k^{\hat{a}}k^{\hat{b}}}{k^8}\right] \tag{5.129}$$

where we include the fourth adiabatic order contribution to the divergent part of the propagator. Integrating over the proper time

$$I_{\hat{b}\hat{c}}(\tau) \;=\; -\frac{1}{u^0(\tau)}\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{k_{\hat{m}}k_{\hat{n}}}{(\delta^{\hat{i}\hat{j}} - v^{\hat{i}}v^{\hat{j}})k_{\hat{i}}k_{\hat{j}}}\left(\delta^0_{\hat{b}}v^{\hat{m}} + \delta^{\hat{m}}_{\hat{b}}\right)\left(\delta^0_{\hat{c}}v^{\hat{n}} + \delta^{\hat{n}}_{\hat{c}}\right) + O(k^{-1}) \tag{5.130}$$

shows that the first term in the ultraviolet sector behaves as k^3 while the last two are actually finite terms that scale as k^{-1}. We therefore ignore the $O(\partial^4)$ contributions to the integral and invoke the familiar result that a power divergent integral in dimensional regularization vanishes. Therefore, none of the parameters in the theory are renormalized by the leading order spin-spin interaction.

The finite contribution to the effective action is then

$$\text{Fig.(5.6a)} \;=\; \frac{4}{m_{pl}^2}\int d\tau\, z_-^\mu w_\mu{}^{\alpha\nu}[z_+^\alpha]\mathcal{S}_\alpha{}^\beta \nabla_\nu\, Fp \int d\tau'\, \nabla_\beta \nabla_{\gamma'} D_{ret}(z_+^\alpha, z_+^{\alpha'}) S^{\gamma'\delta'} u_{+\delta'}. \tag{5.131}$$

The self-force is corrected by the leading order spin-spin interaction to include the term

$$f^\mu(\tau) \;=\; \cdots + \frac{4}{m_{pl}^2}\, w_\mu{}^{\alpha\nu}[z^\alpha]\mathcal{S}_\alpha{}^\beta\, Fp \int d\tau'\, \nabla_\nu \nabla_\beta \nabla_{\gamma'} D_{ret}(z^\alpha, z^{\alpha'}) \mathcal{S}^{\gamma'\delta'} u_{\delta'} + \cdots \tag{5.132}$$

in the particle's equations of motion. We observe that since there is no term proportional to the acceleration then the particle's mass does not get affected by this diagram. This implies that the spin-spin interaction is a mass conserving diagram because of the vector coupling of each vertex $\sim \mathcal{S}^{\alpha\beta} u_\beta$.

The appearance of six spin operators (recall that $\mathcal{S} \sim SSSR$) implies that the leading order spin-spin diagram first contributes at seventh order in μ. Hence, because of the couplings in this scalar model the spin-spin diagram is strongly suppressed relative to those diagrams with spin-orbit couplings, finite size couplings and the usual interactions with spin absent. We do not anticipate this strong suppression for the gravitational case since the spin couples nontrivially to the metric perturbations.

Chapter 6

Self-consistent backreaction approach in gravitating binary systems

In this Chapter we introduce a new approach for studying gravitationally bound binary systems that uses techniques valid in a general curved spacetime. Our aim is to describe the motion of a binary system composed of compact objects with comparable masses that are not restricted to slow motion or weak field approximations. By comparable masses we mean a binary system having a mass ratio of about 10^{-1} to 10^{-2}. Our new approach may be relevant for numerical studies of intermediate mass ratio inspirals (IMRIs). A binary system in this mass range is not described very well with the PN approximation nor with the perturbative techniques developed for the EMRI scenarios[1]. On the one hand, the PN approximation is most useful for comparable masses moving slowly through weak gravitational fields. While the PN formalism can handle extreme mass ratios one must calculate to very high order in the velocity in order to capture the relativistic features of the small compact object as it enters the strong field region of the larger body. On the other hand, the EMRI approximation is capable of handling relativistic speeds and motion in a strongly curved region of the spacetime but only if the masses are very dissimilar.

Before discussing our new approach in detail and to provide it with better context in relation to other methods we first discuss the PN and EMRI perturbation

[1]Throughout the remainder we simply refer to this approach as EMRI perturbation theory.

theory frameworks.

6.1 A brief review of other formalisms

There are two major analytical approaches available for studying the gravitational two-body problem: the post-Newtonian (PN) approximation and the EMRI perturbation theory. We briefly describe these formalisms below and include some discussion about their strengths and weaknesses. We begin with the quadrupole formalism, which is included here for its historical significance as a first step toward the PN expansion.

6.1.1 Quadrupole formalism

The quadrupole-moment formalism was originally developed by Einstein [135, 136] in order to describe the slow motion of (weakly gravitating) Newtonian sources. While it was believed that the quadrupole formalism was valid only for slowly moving bodies with weak internal gravity it was shown much later by [107, 137, 138, 139] that the quadrupole formalism is viable even when the strength of the source's internal gravity is not small, such as for a black hole and a neutron star. Despite this, the slow motion requirement remains necessary. Nevertheless, the quadrupole formalism provides good order-of-magnitude estimates for many sources of gravitational waves [140] but it does not generate waveforms that are accurate enough for detecting gravitational waves in ground-based interferometers. One must augment this formalism with corrections beyond the Newtonian regime that are calculated within the post-Newtonian expansion.

6.1.2 Post-Newtonian approximation

The post-Newtonian (PN) approximation is based upon the assumption that two weakly gravitating objects orbit about each other at nonrelativistic speeds on a flat background[2]. The strict weak field requirement can be lifted by approaching the problem with more sophistication. In particular, the PN expansion can be constructed within an annulus about each compact object where the PN-expanded metric is matched onto the near-field perturbed metric generated by the strongly gravitating compact object [142, 139].

Iteratively solving for the metric perturbations and the motions of the masses yields approximate expressions in powers of the relative velocity for observables of the binary and the gravitational radiation it emits. These observables include the phase of the emitted gravitational radiation, the energy and angular momentum they carry, the innermost stable circular orbit, etc. To date, the metric, the equations of motion for the masses and the gravitational radiation have been computed to order v^6 beyond the Newtonian solution, also denoted as 3PN[3].

For the purpose of detecting and providing accurate estimations for the parameters of the gravitational wave sources, waveform templates used in the analysis of LIGO data are required to be at least as accurate as the 3PN templates [143, 144, 145, 146]. Unfortunately, it is not known how large the higher order PN

[2]See [141] for the original introduction of post-Newtonian corrections to the quadrupole results.

[3]See the references listed at the beginning of Chapter 1 and references therein.

corrections are. The knowledge of these corrections is important to estimate the errors in using the templates.

The PN expansion is valid in the near zone around the source of the gravitational waves. Upon applying the PN approximation far away from the source in the radiation zone one encounters, in the widely used harmonic coordinates, logarithmic divergences because retardation effects cannot be neglected so far from the sources. As a result, the PN metric is matched to a metric describing the propagation of gravitational waves away from the system. This matching is done in a buffer zone between the near and radiation zones. In this way, one can describe the generation and propagation of gravitational waves by slowly moving, weakly gravitating bodies in the PN framework [139, 142]. The metric in the radiation zone is usually calculated using the post-Minkowski (PM) approximation. (Although see [147, 148, 149] who use a mixture of PM methods and multipolar expansions (called the multipolar post-Minkowski approximation) to compute the metric perturbations over all weak-field regions of the spacetime, not just the radiation zone, so long as the sources move slowly.)

The PM approximation entails expanding Einstein's equations in powers of Newton's constant G and solving iteratively for each order of the metric perturbations. In particular, there is no constraint on the velocities of the sources. This method is valid in those regions of spacetime that are weakly gravitating so the PM expansion is not useful for black hole binaries. However, as mentioned earlier, one can use the PM expansion far from such a system to match onto the perturbed metric of the near wave zone computed using the PN expansion.

6.1.3 Extreme mass ratio inspiral perturbation theory

The extreme mass ratio inspiral (EMRI) scenario consists of two bodies with largely dissimilar masses. One body, e.g. a black hole, has a mass M so much larger than the other m that the dominant geometry of the spacetime is determined by the large black hole. Despite having a very much smaller mass than the first, the smaller body nevertheless perturbs the background black hole spacetime. Parts of the metric perturbations radiate away, carrying energy in the form of gravitational waves, and in so doing cause the smaller mass to slowly inspiral toward the large black hole. The mechanism responsible for this inspiral is the self-force the smaller mass experiences as a result of metric perturbations back-scattering off of the background spacetime and encountering m at a later event in its orbit. The EMRI perturbation theory possesses the advantage of treating the relativistic motion of the small body in a strongly curved spacetime. Accurate waveforms can be generated using quasi-analytic techniques. For detecting gravitational waves LISA only requires knowing the self-force and the radiation through first order in the mass of the small body. For parameter estimation the self-force and the gravitational radiation will likely need to be calculated to second order [61].

6.2 Self-consistent backreaction approach

We have two motivations for developing a new formalism. The first comes from a desire to bring into a common framework both types of scenarios that are separately studied within a post-Newtonian scheme and within an extreme mass ratio perturbation theory. By construction, the formalism should also describe binary systems that are not moving slowly, moving in weak field regions, or have equal/dissimilar masses. As such, these binaries fall into a region of parameter space that is not well-described by PN techniques or EMRI perturbation theory. The second motivation comes from our desire to have a fully self-consistent formalism that is able to account for the backreaction on *all* of the dynamical variables. For this reason, we call this new framework a self-consistent backreaction (SCB) approach.

We attempt to develop the SCB approach so that it is valid for comparable mass binary systems. However, we have little expectation of being able to accurately provide a description of equal mass binaries although this is still an open question in our framework. Let the compact object with the lesser mass of the two bodies be given by m. The larger mass, which we take to be a black hole, is denoted by M. We describe the compact object with mass m as an effective point particle, which is discussed in detail in Chapter 4, and the larger black hole by the background geometry. By splitting the spacetime into a background and its perturbations we develop a formalism with the following properties. First, it is a fully relativistic theory; we avoid building into our approach a slow motion or weak field approximation. Second, the effective point particle, representing the motion of the smaller compact object, moves in a general curved spacetime described by the background metric of the larger black hole. Third, and most importantly, we elevate the background metric to a fully dynamical variable. In this way, we allow for all three quantities (effective particle, metric perturbations and background geometry) to interact dynamically with mutual backreaction from each other.

This allowance for dynamical backreaction is a crucial and attractive feature of SCB. It is crucial for the self-consistency of our approach and attractive because we allow for the background to respond to the effective stresses and energies arising from the motion of the compact object, its interactions with the metric perturbations, the propagation of the gravitational waves far away from the system, etc. This is to be contrasted with other approaches, including the PN and EMRI formalism, that choose a fixed background that never deviates from its originally specified form. While this may be convenient for calculations, especially if the fixed background metric possesses some isometries, it is not required for developing a fully self-consistent theory of masses, gravitational waves and background geometry.

While the SCB formalism is difficult to extrude analytical solutions from it may provide a framework useful for studying intermediate mass ratio inspirals with numerical techniques. The IMRI scenario is troublesome to numerically evolve because the relevant time scales are more separated than for the equal mass case. In the latter, the radiation reaction and orbital time scales are approximately the same. In the former, the effects of radiation reaction accrue over a longer time than the orbital period of the binary. Therefore, IMRIs cannot be numerically evolved with sufficient resolution for long enough times to track the inspiral accurately given the currently available computational resources. Since the SCB approach may describe the inspiral and (possibly) plunge phases of IMRIs we are inclined to suggest that our new formalism may provide a sufficiently accurate framework for studying these systems numerically.

We turn now to the technical development of the self-consistent backreaction approach for gravitational binary systems.

6.2.1 Equations of motion in the self-consistent backreaction approach

Let us begin by representing the smaller compact object as an effective point particle theory, a detailed discussion of which is given in Section 4.3. We subsequently introduce the effective point particle action

$$\begin{aligned} S_{pp}[g,z] &= -m \int d\tau + c_R \int d\tau\, R + c_V \int d\tau\, R_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta \\ &\quad + c_E \int d\tau\, E_{\alpha\beta} E^{\alpha\beta} + c_B \int d\tau\, B_{\alpha\beta} B^{\alpha\beta} + \cdots \end{aligned} \tag{6.1}$$

where we recall that the tensors $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are the electric and magnetic parts of the Weyl tensor. For practical calculations it may be easier to replace the E^2 and B^2 terms with

$$\begin{aligned} & c_{R^2} \int d\tau\, R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + c_{R^2 u^2} \int d\tau\, R_{\alpha\beta\gamma\mu} R^{\alpha\beta\gamma}{}_{\nu} u^\mu u^\nu \\ &\quad + c_{R^2 u^4} \int d\tau\, R_{\alpha\mu\beta\nu} R^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma} u^\mu u^\nu u^\rho u^\sigma \end{aligned} \tag{6.2}$$

The price we pay is the introduction of an extra curvature-squared term in the effective particle action. We assume that the smaller compact object is spherically symmetric and does not spin or possess other intrinsic moments, however, there is no obstacle in including these features into the effective point particle action. The metric of the full spacetime is $g_{\mu\nu}$.

In Chapters 4 and 5 the fact that the background spacetime was vacuous allowed for the terms proportional to the Ricci curvature in (6.1) to be set to zero via a field redefinition of the background metric. We will address this issue below after we have derived the equations of motion in the SCB approach. For now, we retain these terms with the proviso that they may eventually be removed using a suitable field redefinition.

The equations of motion for the (free) effective particle moving in a curved background spacetime under no external influences are derived in the usual way by varying the action in (6.1). We find that

$$\big(m - c_R R + \cdots\big) a^\mu(\tau) \quad = \quad c_R w^{\mu\nu}[g, z] R_{;\nu} + \cdots \tag{6.3}$$

where we have included explicitly those terms proportional to c_R in the effective point particle action (6.1). The effective particle does not follow a geodesic, which is not surprising since the induced moments from tidal interactions cause the particle to deviate from geodesic motion. The appearance of a spacetime dependent effective mass

$$m_{eff} = m - c_R R + \cdots \tag{6.4}$$

is interesting. Heuristically speaking, the work required to deform the compact object is stored as potential energy, which then affects the inertia of the particle, as is demonstrated from the appearance of the curvature-dependent terms in m_{eff}. Generally, we can write the free particle equations of motion (6.3) in a more compact way

$$m^{\mu\nu}(g, z) a_\nu(\tau) = f^\mu(g, z; c_R, c_V, \ldots) \tag{6.5}$$

where $m^{\mu\nu}$ is the effective mass of the particle, which is generally space and time dependent, and f^{μ} accounts for all of the forces on the particle arising from the finite size effects of the tidally distorted compact object.

A stress-energy tensor is associated with the effective point particle action (6.1) through

$$T^{pp}_{\mu\nu}(g,z) = \frac{2}{g^{1/2}} \frac{\delta S_{pp}[g,z]}{\delta g_{\mu\nu}} \tag{6.6}$$

The explicit expression for the stress tensor in terms of the usual point particle action $-m\int d\tau$ and the additional infinite number of non-minimal terms in S_{pp} is quite involved. Regardless, the stress tensor in (6.6) describes the stress-energy of the compact object including *all* of the induced moments from tidal interactions with the larger companion black hole.

The full metric $g_{\mu\nu}$ can be separated into a background part[4] $\gamma_{\mu\nu}$ and its perturbations $h_{\mu\nu}$ so that

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu} \tag{6.7}$$

We remark that this decomposition is arbitrary; we are just as free to choose the Kerr metric for $\gamma_{\mu\nu}$ as we are the flat metric $\eta_{\mu\nu}$ or the Ernst metric [118]. As is widely known, calculations and physical insight can be made more efficient and transparent, respectively, for an appropriately chosen background. This decomposition then implies that the Einstein equation

$$G_{\mu\nu}(g) = 8\pi T^{pp}_{\mu\nu}(g,z) \tag{6.8}$$

[4]We work in units where $c = G = 1$ use the same notation as [150] even though it differs slightly from the notation used throughout this dissertation.

can be written as

$$G_{\mu\nu}(\gamma) - 8\pi T^{pp}_{\mu\nu}(\gamma, z) = -\Delta G_{\mu\nu}(\gamma, h) + 8\pi \Delta T^{pp}_{\mu\nu}(\gamma, h, z) \tag{6.9}$$

where the quantities[5] ΔG and ΔT^{pp} contain all of the dependence on $h_{\mu\nu}$ and are not necessarily small with respect to the background Einstein or stress tensors $G(\gamma)$ and $T^{pp}(\gamma)$.

The SCB approach will be self-consistent from the point of view that the quantity $\Delta G - 8\pi\Delta T^{pp}$ is both conserved on the background metric and invariant under coordinate transformations that preserve the structure of the background geometry. The former is straightforward to demonstrate.

Given a solution (γ, z) to (6.9) we know that $G(\gamma)$ and $8\pi T^{pp}(\gamma, z)$ are separately conserved with respect to the background geometry upon using the Bianchi identities for the Einstein tensor and the conservation equation for the effective point particle stress tensor, which gives rise to the particle equations of motion that we have yet to discuss. It therefore follows from (6.9) that the quantity $\Delta G - 8\pi\Delta T^{pp}$ is also conserved with respect to the background geometry whereas ΔG and $8\pi\Delta T^{pp}$ are not separately conserved, in general.

We next demonstrate that $\Delta G - 8\pi\Delta T^{pp}$ is invariant under coordinate transformations that change the perturbed metric but not the background. Since ΔG and ΔT^{pp} are not necessarily small with respect to $G(\gamma)$ and $T^{pp}(\gamma)$ it follows that these coordinate transformations are not necessarily infinitesimal as is usually required for the normal sense of gauge invariance. As such the coordinate transformations,

[5]Throughout the remainder of this Chapter, we will ignore spacetime indices on tensor quantities with impunity. It should be clear from context which objects are scalar, vector, etc.

which were first introduced in [150], are called generalized gauge transformations. We discuss these transformations to prepare for a proof that $\Delta G - 8\pi\Delta T^{pp}$ is gauge invariant in this broader sense.

An arbitrary coordinate transformation can be written in the following way

$$\bar{x}^\mu = x^\mu + \xi^\mu \tag{6.10}$$

where ξ^μ is not necessarily infinitesimal or small. Under this coordinate change the tensor transformation rule for the full metric

$$g_{\mu\nu}(x) = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu} \tag{6.11}$$

can be written as

$$\begin{aligned} \gamma_{\mu\nu}(x) + h_{\mu\nu}(x) &= \gamma_{\mu\nu}(\bar{x}) + \bar{h}_{\mu\nu}(\bar{x}) + \left[\gamma_{\mu\alpha}(\bar{x}) + \bar{h}_{\mu\alpha}(\bar{x})\right]\xi^\alpha{}_{,\nu} \\ &\quad + \left[\gamma_{\alpha\nu}(\bar{x}) + \bar{h}_{\alpha\nu}(\bar{x})\right]\xi^\alpha{}_{,\mu} + \left[\gamma_{\alpha\beta}(\bar{x}) + \bar{h}_{\alpha\beta}(\bar{x})\right]\xi^\alpha{}_{,\mu}\xi^\beta{}_{\nu} \end{aligned} \tag{6.12}$$

where the derivatives of ξ are with respect to x and we have used (6.7). In the limit that ξ^α is small we recover the usual infinitesimal coordinate transformation for the metric perturbations

$$\bar{h}_{\mu\nu}(\bar{x}) \approx h_{\mu\nu}(x) - \gamma_{\mu\nu,\alpha}\xi^\alpha - \gamma_{\mu\alpha}(x)\xi^\alpha{}_{,\nu} - \gamma_{\alpha\nu}(x)\xi^\alpha{}_{,\mu} \tag{6.13}$$

Define the functions

$$A(\gamma, z) = G(\gamma) - 8\pi T^{pp}(\gamma, z) \tag{6.14}$$

$$\Delta A(\gamma, h, z) = \Delta G(\gamma, h) - 8\pi T^{pp}(\gamma, h, z) \tag{6.15}$$

which are convenient quantites to use for proving gauge invariance for some function on the right side of (6.14). The quantity ΔA is said to be gauge invariant under the transformation (6.10) if, when solving for $\bar{h}$ as a function h in (6.12), ΔA satisfies

$$\Delta A(\gamma, \bar{h}, z) = \Delta A(\gamma, h, z) \tag{6.16}$$

We prove that this is true following the derivation given in [150]. Under the generalized gauge transformation (6.12) the background metric does not change so that, defining

$$\Delta A(\gamma, h, z) \equiv \Delta G(\gamma, h) - 8\pi \Delta T^{pp}(\gamma, h, z), \tag{6.17}$$

we have (6.9)

$$A(\gamma(\bar{x}), \bar{z}) - \Delta \bar{A}(\gamma(\bar{x}), \bar{h}(\bar{x}), \bar{z}) = 0 \tag{6.18}$$

Regarding the left side as a function of $\bar{x}^\alpha$ and evaluating at $\bar{x}^\alpha = x^\alpha$ this becomes

$$A(\gamma(x), z) - \Delta \bar{A}(\gamma(x), \bar{h}(x), z) = 0. \tag{6.19}$$

Comparing with (6.9) in the original coordinates it follows that

$$\Delta \bar{A}(\gamma(x), \bar{h}(x), z) = \Delta A(\gamma(x), h(x), z) \tag{6.20}$$

Expressing A in terms of the metric and its derivatives implies that $\Delta \bar{A}$ can be calculated by substituting $\bar{h}$ directly into ΔA so that

$$\Delta \bar{A}(\gamma, \bar{h}, z) = \Delta A(\gamma, \bar{h}, z) \tag{6.21}$$

Comparing with (6.20) gives

$$\Delta A(\gamma, \bar{h}, z) = \Delta A(\gamma, h, z) \tag{6.22}$$

and is the statement of gauge invariance with respect to the background geometry under the transformations in (6.10).

Let us now construct the perturbative expansion of (6.9). If there exists a valid expansion of the Einstein tensor of the form

$$\Delta A = \Delta_1 A + \Delta_2 A + \cdots \tag{6.23}$$

then it is easy to see from (6.8) that the expansion of ΔA through n^{th} order

$$(\Delta A)_{(n)} \equiv \Delta_1 A + \Delta_2 A + \cdots + \Delta_n A \tag{6.24}$$

is conserved with respect to the background geometry. It can also be shown using similar arguments above that $(\Delta A)_{(n)}$ is gauge invariant only through the n^{th} order. This implies, for example, that $\Delta_1 A$ taken by itself is gauge invariant through first order while $\Delta_2 A$ is not gauge invariant at any order. It is the combination $\Delta_1 A + \Delta_2 A$ that is gauge invariant through second order.

In discussing the gauge transformations for which (6.23) is invariant we observe that for any ξ^μ in (6.10) the metric perturbations may change by a large amount. It therefore makes comparing the expansion for ΔA in (6.23) very difficult. Therefore, we should restrict to gauge transformations such that h and $\bar{h}$ are of the same order [150]. This allows for a straightforward comparison of the terms in ΔA and $\Delta \bar{A}$.

The expansion in (6.23) can be generated from an expansion in the metric perturbations

$$h = h^{(1)} + h^{(2)} + \cdots \tag{6.25}$$

from a series of terms that are proportional to a power of an appropriate expansion parameter λ, say; the superscript denotes the power of λ. Expanding (6.8) in powers of the $h^{(n)}$ we find

$$\begin{aligned} G(\gamma) &= 8\pi T^{pp}(\gamma, z) - \Delta_1 G(\gamma, h^{(1)}) - \Delta_2 G(\gamma, h^{(1)}) - \Delta_1 G(\gamma, h^{(2)}) \\ &\quad +8\pi\Delta_1 T^{pp}(\gamma, h^{(1)}, z) + 8\pi\Delta_2 T^{pp}(\gamma, h^{(1)}, z) + 8\pi\Delta_1 T^{pp}(\gamma, h^{(2}, z) \end{aligned} \tag{6.26}$$

where we are using the notation that $\Delta_p G(\gamma, h^{(n)})$ is of order λ^{pn}.

In order to solve (6.9) or (6.26) we need to further specify the decomposition of the full spacetime into a background geometry and its perturbations. We can always specify a fixed metric for the background but for the reasons we mentioned earlier we wish to impart dynamics to the background geometry so that all degrees of freedom in the problem are interacting with each other.

One can further specify the metric decomposition using averaging techniques [151, 152, 153]. However, we do not want to "coarse-grain" any information about the metric perturbations or the background metric, which is what would happen upon averaging over the high frequency modes of the gravitational waves. Fortunately, we do not need to use averaging techniques to describe this system with complete dynamical information[150].

Making a more deliberate choice for the metric decomposition will involve giving some dynamics to the gravitational waves, which we have yet to do. It is natural that metric perturbations satisfy some wave equation on the background geometry, particularly a linear wave equation. Then the solutions to this wave equation should possess propagating modes that reach detectors at null infinity.

Further, we require the wave equation to be gauge invariant to ensure the consistency of the backreaction equation that will ultimately describe the backreaction on the dynamical background geometry. A suitable choice for the (linear) wave equation that is also gauge invariant is

$$\Delta_1 G(\gamma, h^{(1)}) = 8\pi \Delta_1 T^{pp}(\gamma, h^{(1)}) \tag{6.27}$$

Similarly, the second order perturbations can be chosen to solve a gauge invariant wave equation

$$\Delta_1 G(\gamma, h^{(2)}) - 8\pi \Delta_1 T^{pp}(\gamma, h^{(2)}, z) = -\Delta_2 G(\gamma, h^{(1)}) + 8\pi \Delta_2 T^{pp}(\gamma, h^{(1)}, z) \tag{6.28}$$

The Einstein equation in (6.26) then implies the following backreaction equations for the background geometry

$$G(\gamma) \;\; = \;\; 8\pi T^{pp}(\gamma, z), \tag{6.29}$$

which is also gauge invariant through second order.

The equations of motion in (6.27), (6.28) and (6.29) determine the dynamical responses of the gravitational variables due to their respective sources. We also need to develop the particle equations of motion, which we derive from (6.5)

$$\begin{aligned} &ma(\gamma, z) + \Delta m(\gamma, h, z) a(\gamma, z) + m(\gamma, z) \Delta a(\gamma, h, z) + \Delta m(\gamma, h, z) \Delta a(\gamma, h, z) \\ &\qquad = f(\gamma, z) + \Delta f(\gamma, h, z) \end{aligned} \tag{6.30}$$

where we recall that $m\, a = m^{\mu\nu} a_\nu$ involves a contraction of spacetime indices. The expansions in (6.23) and (6.25) imply

$$\begin{aligned} ma(\gamma, z) &= f(\gamma, z) + \Delta_1 f(\gamma, h^{(1)}, z) + \Delta_2 f(\gamma, h^{(1)}, z) + \Delta_1 f(\gamma, h^{(2)}) \\ &\quad -\Delta_1 m(\gamma, h^{(1)}, z) a(\gamma, z) - \Delta_2 m(\gamma, h^{(1)}, z) a(\gamma, z) \\ &\quad -\Delta_1 m(\gamma, h^{(2)}, z) a(\gamma, z) - m(\gamma, z) \Delta_1 a(\gamma, h^{(1)}, z) \\ &\quad -m(\gamma, z) \Delta_2 a(\gamma, h^{(1)}, z) - m(\gamma, z) \Delta_1 a(\gamma, h^{(2)}, z) \\ &\quad -\Delta_1 m(\gamma, h^{(1)}, z) \Delta_1 a(\gamma, h^{(1)}, z) + \cdots \end{aligned} \tag{6.31}$$

The right side of this equation describes several processes: the force on the particle from the tidal deformations due to the background curvature (which is provided in part by the larger companion black hole) $f(\gamma, z)$; the first and second order interactions of these induced moments with the gravitational waves $\sim \Delta_n f^\mu$; the self-force through second order $\sim \Delta_n a^\mu$; and the corrections to the effective mass $\sim \Delta_n m^{\mu\nu}$

The important results of this Section are the equations of motion for the gravitational waves (6.27) and (6.28), the dynamical background geometry (6.29), (**??**) and the effective point particle (6.31). These are the relevant equations of the SCB approach. We observe that these equations describe a rich collection of physical processes that range from self-force, to backreaction on the background geometry and to the accomodation of all possible finite size effects from the compact object.

The equations of motion derived in this Section are relevant for describing the self-consistent dynamics of a compact body, gravitational waves and the background geometry that these evolve in. The SCB formalism is self-consistent in the sense that the effective stress tensor for the gravitational waves is conserved with respect to the background metric and is gauge-invariant with respect to the generalized gauge transformations introduced earlier. The wave equation is also gauge invariant.

6.2.2 Validity of perturbation theory in SCB

In developing the self-consistent background equations of motion given in (6.27), (6.29) and (6.31) in the previous section we assumed the existence of a perturbative expansion for the gravitational waves in (6.25) so that

$$h = h^{(1)} + h^{(2)} + \cdots \tag{6.32}$$

One nice feature of this approach is that we are not restricted to any particular expansion, just those that are compatible with the formalism to ensure that the appropriate quantities remain conserved and gauge invariant. While we have not yet investigated the set of parameters that might be useful for perturbatively solving this theory[6] we assume in this Section that the expansion parameter μ used in Chapters (4) and (5) is acceptable.

We recall that the parameter μ is defined to be the ratio of the size of the small body to a particular curvature invariant relating the curvature length scale to the background Riemann curvature tensors,

$$\mu \equiv \frac{r_m}{\mathcal{R}} = r_m \big(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \big)^{1/4} \tag{6.33}$$

We wish to obtain estimates for the values of μ in relation to the mass ratio m/M

[6] An interesting candidate parameter is the ratio of the reduced mass to the total mass of the binary system. Since this quantity never exceeds 1/4 in the equal mass case it may be useful for describing binary systems with comparable masses.

of the binary constituents and to an invariant measure of their separation L. To do this we will assume that μ is parameterized by m/M and L so that given values for these we can identify a value for μ. To be specific, we are not treating μ as a function of spacetime coordinates even though the Riemann tensor in (6.33) does. The parameter μ is a (constant) number that is given at a particular scale, say the orbital scale of the binary system. This is the same interpretation implicitly assumed for μ in Chapters (4) and (5). We want to estimate below how μ might change with the scale of the orbital dynamics.

In particular, we wish to estimate the maximum value that m/M can take in the second order SCB equations of motion before third order perturbations are needed. We provide a tolerance that will set a reasonable, yet somewhat arbitrary, boundary for our specifications. We require $\mu^3 \lesssim 10^{-3}$ so that the third order corrections to the SCB equations of motion will be $\lesssim 0.1\%$ of the background quantities.

The invariant separation L between the horizon of the larger black hole and the "surface" of the compact object will also be an important quantity to factor into such an estimation. For example, it is not difficult to find a regime for which $m/M = 1$ and $\mu^3 \lesssim 10^{-3}$ for L much larger than the size of either the larger black hole or the smaller compact object. In such a case the scale of the system is for an equal mass binary in the weak field limit of the background geometry and is not very interesting. We are more interested in how small L can be and how large m/M can be and still satisfy our tolerance of $\mu^3 \lesssim 10^{-3}$.

The effective point particle treatment will surely break down when L is of order the size of the compact object r_m. Therefore, we require

$$L \gtrsim r_m \tag{6.34}$$

for all values of the mass ratio m/M.

We can obtain a crude estimate for the case where there is no backreaction from m by considering the fixed geometry of the Schwarzschild solution describing the larger black hole. We will assume that the test mass possesses a fictitious boundary with radius $r_m = 2m$. Let r denote the radial coordinate distance from the center of the large black hole in Schwarzschild coordinates so that it measures the area of concentric spheres centered on the black hole. Define

$$\alpha \equiv \frac{r}{r_M} = \frac{r}{2M} \tag{6.35}$$

$$\beta \equiv \frac{m}{M} = \frac{r_m}{r_M} \tag{6.36}$$

denote the radial coordinate distance from the larger black hole measured in units of its horizon radius and the mass ratio, respectively.

The least proper distance from the horizon of the large black hole, at $r = 2M$, to the edge of the fictitious horizon on the test mass is easily shown to be

$$\frac{L}{r_M} = \frac{1}{r_M}\int_{2M}^{r-2m} dr' \left(1 - \frac{r'}{2M}\right)^{-1/2} \tag{6.37}$$

$$= 2\sqrt{\alpha - \beta - 1} \tag{6.38}$$

and the curvature scale $\mathcal{R}$ for the Schwarzschild background is

$$\frac{r_M}{\mathcal{R}} = 12^{1/4}\alpha^{-3/2} \tag{6.39}$$

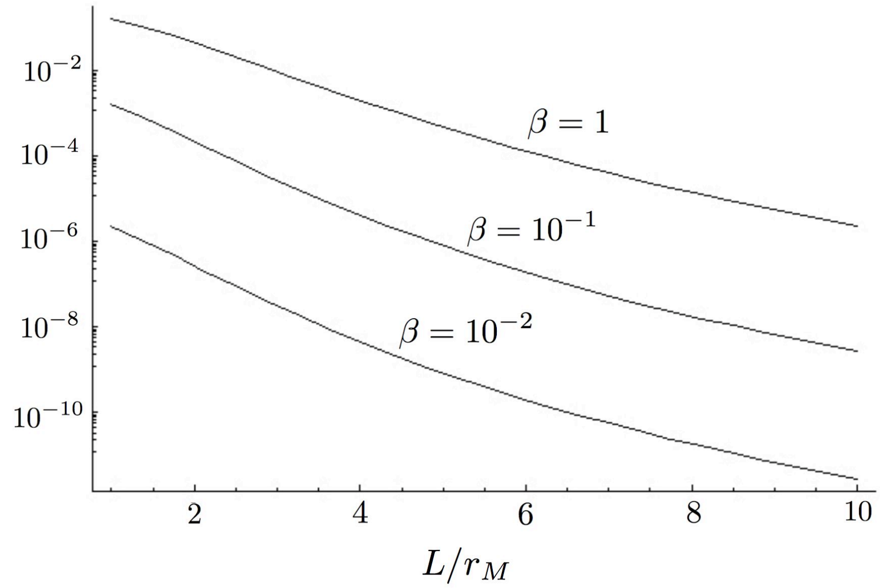

Figure 6.1: A log plot of μ^3 versus L, the shortest radial proper distance between the edge of the horizon of M and the edge of the fictitious horizon given to the test mass.

From the expression for L_{Schw} we solve for α and plug into $\mu = r_m/\mathcal{R}$ to find

$$\mu = (12)^{1/4}\beta\left[\beta + 1 + \frac{1}{4}\left(\frac{L}{r_M}\right)^2\right]^{-3/2} \tag{6.40}$$

We give a log plot of μ^3 versus L in Fig.(6.1). Notice that $\mu^3 \lesssim 10^{-3}$ for the equal mass case implies that the second order SCB equations of motion are valid when $L \gtrsim 5r_M = 10M$. However, when $\beta = 0.1$ we find that $\mu^3 \lesssim 10^{-3}$ when $L \gtrsim 1.25r_M = 3.5M$. This implies that the SCB equations are valid until near the point of plunge and/or merger. However, we have not included backreaction in this estimation. Until we can do so we may assume that our estimations in Fig.(6.1) are valid within about an order of magnitude. Clearly, this will need some improvement.

6.2.3 Further directions for the SCB approach

While there are several formally interesting and attractive features associated with the SCB approach there are still many issues to understand. One of these is the identification of an appropriate expansion parameter(s) to build the perturbation theory in (6.25). We discussed in the previous section using μ as a parameter but there are other choices to consider. One of the more interesting that we wish to investigate further is the parameter formed from the ratio of the reduced mass to the total mass of the binary, which we denote by the symbol ν. A perturbation theory built using ν may be viable for the comparable mass case and even, perhaps, the equal mass case. We find it intriguing that $\nu \leq 1/4$ where the equality holds when the masses are equal. Estimating the mass ratio that corresponds to ν satisfying the tolerance $\nu^3 \lessapprox 10^{-3}$ implies that $m/M \lessapprox 0.13$. Provided that this estimate holds for real perturbative solutions then it would seem to follow that using the SCB approach to second order in ν is applicable for binary systems with comparable masses. However, much work is needed to determine the appropriate way to implement an expansion in ν. For instance, preliminary work with a simplified model, suggests that one obtains more accurate solutions by expanding the theory in ν in the center of mass frame, which may be difficult to identify in the SCB approach.

Our estimates for determining the largest value of the mass ratio m/M that might still render the perturbation theory in SCB applicable (using μ as an expansion parameter) are somewhat crude and leave much room for improvement. To get a better sense of how μ depends on the mass ratio we may be able to use the Misner wormhole initial data [154], which describes the metric of two equal mass black holes with a wormhole topology at the instant when both black holes are stationary, and the Brill-Lindquist initial data [155, 156], which describes the metric of electrically charged black holes and electromagnetic fields at the instant when all the black holes are stationary. The Brill-Lindquist solution is not constrained to the equal mass situation and we may be able to provide stronger constraints on the relationships between μ, m/M and L than with the Misner initial data alone.

Another very important issue to face is solving the equations of motion of SCB for practical calculations, which can be used for obtaining physical predictions, particularly in the study of gravitational wave sources. Since all of the degrees of freedom are interacting non-trivially and non-linearly with other degrees of freedom it is not an easy matter to apply SCB to even the simplest examples. Therefore, we will likely be forced to introduce additional assumptions, including perhaps a weak field assumption or a slow motion for the compact object, that may disturb the internal consistency of SCB.

In most instances where one computes the backreaction on the quantity of interest one makes heavy use of assumed symmetries for the solution. Unfortunately, the general gravitational two-body problem lacks any symmetries that can be exploited. The two most famous examples, perhaps, of the use of symmetries to solve backreaction problems in gravity are those studied by Brill and Hartle [151] and York [157]. Brill and Hartle investigate the spherically symmetric gravitational geon solution and determine the spherically symmetric backreaction on the spacetime from the effective stress energies of the metric perturbations. York determines the backreaction on a spherically symmetric black hole from the expectation value of the stress tensor for a free conformal scalar field in a thermal state. In both cases, spherical symmetry is crucial for obtaining a solution to the backreaction equation. For the general gravitational two-body problem we do not have such isometries to take advantage of and so we will need to address this important issue further to make practical advances using the SBC approach.

Chapter 7

Discussions and future work

In this dissertation we have studied the self-consistent interactions and dynamics of particles and fields on a curved background spacetime using field theoretical approaches and formalisms for the purposes of determining the role and influence of stochastic sources on particle motion, developing an efficient framework to calculate the self-force systematically to all orders in perturbation theory, and providing a means to encompass nearly both domains of LIGO and LISA sources. Before concluding this work we provide some discussions to collect and restate our results, and propose fruitful directions that may be worth pursuing.

7.1 Main results

In this section we present the main results given in this dissertation.

7.1.1 Stochastic field theory approach

In Chapter 2 we introduce the influence functional formalism to describe the interactions and nonequilibrium dynamics of a quantum mechanical particle with a linear quantum field in a curved spacetime. The particle degrees of freedom are treated as an open quantum system that couple to the environment variables of the coarse-grained quantum field. We demonstrate the existence of a semiclassical limit in which we recover the familiar radiation reaction equations of Abraham, Lorenz, and Dirac (ALD), generalized to a curved background, for a scalar and electric charge [22, 23]. We also recover the self-force equations of MSTQW for a small point mass.

Due to the presence of particle-field interactions we find that there exists a stochastic semiclassical limit for the worldline degrees of freedom. In this regime we find that the field manifests as noise through the appearance of classical stochastic forces. These forces, in turn, induce fluctuations in the particle's worldline. As such, we find that the particle satisfies ALD- and MSTQW-Langevin equations in the stochastic semiclassical limit. We show that the two-point correlation function of the stochastic force, called the noise kernel, is intimately related to the quantum fluctuations of the field. We find that the noise is an $O(\hbar^{1/2})$ quantity demonstrating that the stochastic semiclassical limit is between the semiclassical limit and the regime where 1-loop quantum field effects are relevant. We also demonstrate the intricate connection between noise, fluctuations and decoherence of the particle worldline histories.

Within the influence functional approach and the open quantum system paradigm the noise is determined self-consistently with the environment, viz., the coarse-grained quantum field fluctuations. However, one must be somewhat cautious if the noise cannot be derived but is instead specified to model the effect that an environment has on the evolution of the particle worldline. In such a situation, the specified noise may not faithfully represent nonequilibrium state of the environment for all time. Furthermore, such stipulated, or added, noise gives no reliable information concerning the state of the environment. When specifying the noise in this way one must also provide a noise kernel so that the calculation of worldline observables can proceed using stochastic averages.

Having issued these warnings, we consider the effect that second order stochastic fluctuations have on the motion of the particle. We find that the resulting stochastic-averaged equations of motion describe the motion of a background trajectory that evolves self-consistently (through second order) with the induced stochastic worldline fluctuations. This background trajectory drift is not the same as the semiclassical trajectory that we derived. This is so because the averaged contributions from the second order fluctuations are non-zero and therefore contribute to the self-consistent background trajectory. It would be interesting to calculate this effect for a simple scenario of a charged particle moving non-relativistically in flat spacetime, for example, in the presence of a magnetic field with a non-zero gradient. If such a drift motion exists and can be measured then this may provide important information about the fluctuations in the environment.

This drift effect comes from expanding the Langevin equations with the *added* noise to second order in the worldline fluctuations and then performing a stochastic average. However, in the influence functional approach we cannot derive this drift effect since the stochastic semiclassical limit seems to be only well-defined through the first order in the worldline fluctuations. Through second order we may need to include the effects of the intrinsic quantum mechanical worldline fluctuations and possibly the 1-loop quantum field corrections. It would be interesting to investigate further the relationship between the stochastic semi-classical limit, the leading order particle and field quantum corrections, and the noise-induced drifting motion.

7.1.2 Effective field theory approach

In Chapters 4 and 5 we develop an effective field theory approach for systematically deriving the self-force on a compact object. As such, the EFT is a realization of the open quantum system paradigm of Chapter 2 to systems with a large scale separation that renders the induced fluctuations from the coarse-grained quantum field utterly negligible[1]. We replace the compact object by an effective point particle, which is capable of accounting for various finite size effects. The leading order effect first occurs at $O(\mu^4)$ for a non-spinning body. This finite size correction causes a deviation from the background motion that is not caused by interactions with gravitons but is to the torques that develop on the tidally deformed compact object. On the other hand, the self-force is affected by the induced moments of the compact object at $O(\mu^5)$.

We deduce the diagrams relevant for a calculation of the self-force at second, third and fourth orders. At this time, we are unable to compute the Feynman diagrams for the gravitational case since we lack the momentum space representation of the retarded propagator at the appropriate adiabatic order. As a result, we introduce a nonlinear scalar field model (related to general relativity in a specific way) that has the same power counting rules and the same Feynman diagrams, in

[1]See [111] who show that the stochastic nature of the quantum field persists in EFT descriptions but are strongly suppressed until the high energy threshold is approached, at which point a stochastic description of the system becomes important.

a topological sense, as the gravitational case.

Using this scalar model we show that the second order self-force is manifestly real and causal, which is to be expected since we are using the CTP, or in-in, formalism. Furthermore we find that the self-force contains only a power divergence that we can safely set to zero using dimensional regularization. We observe that no parameters in the theory have actually been renormalized at this order. While we have not yet calculated the third order diagrams simple power counting arguments indicate that these diagrams also contain simple power divergences and are therefore trivial. We expect the first non-trivial renormalization to occur at fourth order in the perturbation theory since this is the first order that a logarithmic divergence appears in the effective action. This divergence should renormalize $c_{E,B}$, which are the non-minimal couplings that parameterize the leading order finite size effects from tidally induced moments on the compact object. We expect that these qualitative statements concerning the divergent integrals carry over to the gravitational case provided that the momentum space representations of the scalar and the graviton propagators have a similar structure in a vacuum background spacetime.

Renormalizing the divergent part of the singular integrals requires a representation for the divergent part of the retarded propagator. In Appendix D we introduced a novel approach that utilizes the diagrammatic techniques of perturbation theory to compute the momentum space representation of the graviton propagator on an arbitrary background through second adiabatic order. We also demonstrate the validity of our method by reproducing the original result of Bunch and Parker for a scalar field.

We finally introduce spin into the EFT approach, thereby allowing for a description of the self-force on a rotating compact body. For a black hole and a neutron star the spin angular momentum results in a quadrupole moment for the compact object in the form $Q \propto S^2$. This is our first example of incorporating an *intrinsic* multipole moment into the EFT.

We also determine that for a maximally rotating compact object the leading order spin-orbit interaction appears at second order while the leading order spin-spin interaction is a third order contribution. For a co-rotating body these diagrams are suppressed by one order in μ so that the leading order spin-orbit and spin-spin diagrams enter at third and fourth orders, respectively. Furthermore, the leading order spin diagram describing spin precession becomes a second order effect for a co-rotating body.

We calculate the leading order spin-orbit and spin-spin diagrams using the nonlinear scalar model and find that these contributions are manifestly causal. The spin-orbit diagram contains a trivial power divergence while the spin-spin diagram diverges logarithmically. We demonstrate that this latter divergence renormalizes a coupling constant of a non-minimal spin-dependent worldline operator. The renormalization gives rise to a classical renormalization group equation for that parameter and allows for us to determine how this parameter varies with the regularization scale.

Along the way, we find that the scalar model has a peculiar feature that we believe will not appear in the gravitational case. We find that the leading order spin-orbit and spin-spin diagrams for a maximally rotating object, which naively appear at third and fourth orders, respectively, are suppressed. This suppression manifests upon implementing the spin supplementary conditions but is really a result of the form of the scalar-spin interaction terms. Since these interactions only involve the spin tensor, the 4-velocity and the background metric then only interaction terms involve the factor $S^{\alpha\beta}u_\beta\phi_{;\alpha}$. The imposition of the spin supplementary conditions introduces two extra factors of the spin tensor into each sub-leading scalar-spin vertex. For the gravitational case the metric perturbation $h_{\mu\nu}$ couples non-trivially to the spin tensor and should prevent such a suppression.

7.1.3 Self-consistent backreaction approach

In Chapter 6 we introduce a new approach to describe gravitational binary systems of compact objects. This formalism does not rely *a priori* on the assumptions of slow motion or weak fields as is the case with the PN approximation. This is intentional as our aim is to use techniques borrowed from the extreme mass ratio inspiral scenario and to apply them to systems that are traditionally described using the PN approximation. In particular, we want to describe the relativistic motion of the binaries even as they move in strong field regions of the spacetime. In this way, we hope to describe binary systems that have comparable masses, with mass ratios of the order 10^{-1} to 10^{-2}, say. One important question for this formalism becomes the estimation for when the formalism breaks down. We anticipate that it will breakdown at the point of merger and perhaps somewhat before then. We give loose bounds on the masses and their separation for determining the boundary of applicability of this backreaction approach.

We assume that one of the compact objects is smaller in mass than the other. The smaller body is described using an effective point particle description discussed at length in Chapter 4. By expanding to second order in the metric perturbations we retain some features of the finite size of the smaller compact object. We then find equations of motion describing the mutual interactions and dynamics of the effective point particle, the metric perturbations and the background metric. This last equation is the new feature of our approach since we regard the background metric as no longer stipulated but is allowed to evolve dynamically with the other degrees of freedom in the system. In particular, since the background metric contains information about the larger black hole then allowing for the self-consistent determination of the metric allows for the larger black hole to evolve self-consistently with the mutual backreaction from the smaller compact object and the metric perturbations.

7.2 Further developments and future directions

Here we propose some future directions based on work that we have presented in this dissertation.

7.2.1 Stochastic theory approach

We have been silent about fluctuation-dissipation relations throughout most of our discussion about noise, fluctuations and dissipation in the particle's stochastic semi-classical limit. It is relatively straightforward to deduce (generalized) fluctuation-dissipation relations [158, 29, 159] for particle-field interactions in a flat spacetime. This is because one typically represents the retarded and Hadamard two-point functions using a mode decomposition and then establishes a fluctuation-dissipation relation using the mode structure of the two-point functions. However, in an arbitrary curved spacetime the luxury of a (unique) mode decomposition cannot be afforded. Nor can a mode decomposition generally be expressed analytically. While it is likely that a fluctuation-dissipation relation exists for a particle moving in a quantum field in a curved background it is not clear how to derive the relation. Nevertheless, perhaps using well-known techniques from quantum field theory in curved spacetime [66, 124] we may be able to use an adiabatic expansion for the modes in certain spacetimes to obtain approximate fluctuation-dissipation relations. We feel that this problem is an important one and should be studied further until a consensus can be reached on the construction of fluctuation-dissipation relations for particle-field systems in curved spacetime.

Upon introducing noise by hand into the motion for the particle we find in Chapter 2 that a secular motion develops that is generated by the interactions of the worldline fluctuations with gradients in an external field, be it electromagnetic or gravitational in origin. To gain some insight into these equations it may prove beneficial to study a simple scenario that can be solved analytically, or at least mostly so. For example, the motion of an electric point charge through a non-homogeneous magnetic field in flat spacetime may be a sufficiently simple nontrivial system one could imagine. The solution of such a system would clarify the role of second order stochastic fluctuations and the influence it has on the averaged motion of the particle.

We cannot derive this noise-induced secular motion within the influence functional formalism because the secular motion results from second order fluctuations whereas the stochastic semiclassical limit is well-defined when first order fluctuations act on the particle worldline. Nevertheless, using our first principles approach we can consider the next-to-leading order corrections, which likely result from the intrinsic quantum fluctuations of the particle worldline. This may help to clarify the interpretation of such second order fluctuations. Furthermore, if these next-to-leading order corrections affect the particle motion in a similar manner as the phenomenologically added noise then there may be observable consequences that might be measured in experiments designed to detect the secular, or drifting, motion.

In [42, 43] the authors compute the 1-loop quantum field corrections to a geodesic of a background spacetime. The background is taken to be a solution to the semiclassical Einstein equations in which the quantum expectation value of a quantum field's stress tensor sources the spacetime curvature. In these works the authors seem to ignore the radiative effects of self-force on the small mass. It would be interesting to apply our influence functional approach, which is not restricted to semiclassical or stochastic limits but is valid for describing open *quantum* systems, to a generalization of their problem and compute the 1-loop quantum field corrections to the leading order motion, including self-force effects. Doing so within a background solution of semiclassical gravity would provide a useful comparison with [42, 43]. Furthermore, since the radiative corrections from the self-force on the particle are history-dependent it would be interesting to see if the 1-loop corrections have any potentially observable consequences.

7.2.2 Effective field theory approach

There are several things that we wish to derive and investigate within the effective field theory approach of Chapters 4 and 5. In order to calculate the self-force at second order in μ and higher we will need to have a momentum space representation for the retarded propagator that is carried out to a sufficiently high adiabatic order. In doing so, we will be able to regularize the singular integrals that appear in the effective action. We will also be able to determine which non-minimal couplings in the effective point particle action exhibit a classical renormalization group flow. Furthermore, we can confirm our prediction that finite size effects appear first at fourth order in the equations of motion but at fifth order in the self-force for a non-spinning particle.

It is also useful to calculate the higher order spin contributions to the self-force. In particular, we will have finite and concrete expressions for the leading order spin-orbit and spin-spin interactions in the gravitational self-force. Being a second order contribution for a maximally rotating body, the leading order spin-orbit diagram may be especially important for obtaining templates for LISA that are sufficiently accurate for parameter estimation.

In Chapter 4 we outlined the matching procedure, which relied on using the cross-section for graviton scattering in a curved background. We are not restricted to using the cross-section alone; one can use whatever set of observables that he/she wishes. To obtain the precise numerical coefficients appearing in the matching procedure we need to know the long wavelength expansion of the cross-section. (Of course, this depends upon the identity of the compact object.) We are unaware at this time of any such expressions for the cross-section and are therefore unable to precisely determine the numerical values of the non-minimal coupling constants that appear in the effective point particle action. Such a state of affairs needs to be remedied in order to successfully match the effective theory onto the compact object. See, however, [160, 161] who estimate LIGO's ability to constrain the equation of state for a (polytropic) neutron star by describing its tidal deformations using Love numbers. The $\ell = 2$ Love number is the ratio of the induced quadrupole moment to the perturbing (tidal) gravitational field. As such, the Love numbers should be related to the nonminimal coupling constants $c_{E,B}$, etc. for our effective point particle.

Throughout our discussion in Chapters 4 and 5 we made the explicit assumption that the background spacetime is vacuous, $R_{\mu\nu} = 0$. Lifting this requirement may have some interesting consequences for cosmological scenarios (e.g., de Sitter and Friedmann-Roberson-Walker spacetimes) and semiclassical gravity, in which the expectation value of the stress tensor of a quantum field provides the leading order source of curvature for the spacetime. For example, in a non-vacuous spacetime the non-minimal parameters $c_{R,V}$ appearing in the effective point particle action can no longer be removed. In fact, in the effective point particle action one simply replaces all occurrences of the Ricci tensor by the stress tensor that sources the background curvature. Using the matching calculations it is then easy to estimate that $c_{R,V} \sim m^3/m_{pl}^4$. Finite size effects therefore enter the particle equations of motion at *second* order in μ and in the self-force at third order. Hence, the existence of background stress-energy induces moments on the compact object that enhance their effects on the particle's motion. We feel that further study in this direction will provide some interesting and useful results for studying the motion of extended bodies in cosmology and semiclassical gravity.

Within this work we have only developed the EFT approach in so much as it systematically produces the self-force on a compact object to any order in μ. For a more complete framework, we need to calculate the metric perturbations that the compact object generates as well as the flux of gravitational radiation. Until we calculate these quantities using the CTP formalism our framework will be incomplete. Computing the metric perturbations in our theory is equivalent to calculating the graviton one-point function (i.e., the expectation value of the quantized metric perturbations). It is likely that we can calculate this using the (1PI CTP) effective action $\Gamma[\langle \hat{h}_{\mu\nu} \rangle]$ (which can be derived by introducing a small adjustment from our current presentation in Chapters 4 and 5). By varying Γ with respect to the graviton one-point function we should then be able to obtain the manifestly real and causal equations of motion for the metric perturbations that are also consistent with the particle equations of motion derived in this work. Unfortunately, the 1PI CTP effective action is not robust enough to calculate the flux of radiation emitted to infinity and registered by a gravitational wave interferometer. Since the flux involves the graviton two-point functions it seems likely to us that a 2PI CTP effective action will yield equations of motion for the two-point functions that are real, causal and consistent with the particle and the graviton one-point function equations of motion. With these issues settled our framework will be sufficiently complete and we should be able to compute the gravitational radiation and the emitted flux to any order in μ within an effective field theory approach. We intend to pursue this in an upcoming series of papers [81, 82, 83].

7.2.3 Self-consistent backreaction approach

In Chapter 6 we lay the foundations for a new approach that describes the self-consistent motion of a black hole, represented as a dynamical background geometry, a compact object and the metric perturbations. All of these variables undergo mutual backreaction to ensure the self-consistency of the formalism. A major aim of this approach is to describe binary systems with comparable masses and thereby establish a common framework for the sources expected to be observed with the LISA and LIGO interferometers. As such, this would be useful for studying those gravitational wave sources that are not covered well by either post-Newtonian or EMRI perturbation theory techniques.

There are several important issues to resolve in order to make this framework more user-friendly for practical calculations. The identification of an expansion parameter is important for building a definite perturbation theory. It is interesting that the theory does not seem to pick out a preferred expansion parameter, as happens with the PN and EMRI perturbation theories, but is obliged to describe any perturbative expansion that respects the gauge invariance and conservation of the appropriate quantities, viz. the effective stress energy of the gravitational waves. We would also like to improve the estimates given in Chapter 6 for the validity of the self-consistent backreaction approach for binaries near the plunge and merger phases using the Misner and Brill-Lindquist initial data for including some amount of backreaction into these estimates.

Another very important issue to address is the generation of solutions using the self-consistent backreaction formalism. Due to the mutual backreaction between *all* of the variables in the system it is not surprising that obtaining solutions to even simple examples is difficult. It is likely, however, that one may need to make additional approximations to generate solutions. As such, one may worry about the effect that additional assumptions has on the self-consistency and mutual backreaction that we have built the SCB approach around. Despite this our new formalism may be useful for numerically studying intermediate mass ratios with sufficient resolution. However, more research into the basic framework and the domain of validity of the SCB approach will be necessary to know for sure.

Appendix A

Conventions and definitions relating to the quantum two-point functions

In this Appendix we collect some definitions, identities and relations for the quantum two-point functions that are relevant for this work.

Assume the existence of a massive and real scalar field propagating in a curved spacetime with arbitrary coupling ξ to the background curvature. While we develop here the two-point functions for scalar fields many of the results in this Appendix can be generalized in a straightforward manner to higher spin fields.

The positive and negative frequency Wightman functions are defined as

$$G_+(x,x') = \left\langle \hat{\phi}(x)\hat{\phi}(x') \right\rangle \tag{A.1}$$

$$G_-(x,x') = \left\langle \hat{\phi}(x')\hat{\phi}(x) \right\rangle, \tag{A.2}$$

respectively. The angled brackets represent the quantum expectation value so that

$$\left\langle \hat{O} \right\rangle \equiv \mathrm{Tr}\left[\hat{\rho}(\Sigma_i)\hat{O}\right] \tag{A.3}$$

and $\hat{\rho}(\Sigma_i)$ is the density matrix of the quantum field given on a hypersurface Σ_i at constant coordinate time $x^0 = t_i$.

The Feynman, Dyson, Hadamard and commutator (also known as the Pauli-

Jordan function or the causal function) two-point functions are, respectively,

$$iG_F(x,x') = \left\langle T\,\hat{\phi}(x)\hat{\phi}(x')\right\rangle \tag{A.4}$$

$$iG_D(x,x') = \left\langle T^*\,\hat{\phi}(x)\hat{\phi}(x')\right\rangle \tag{A.5}$$

$$G_H(x,x') = \left\langle \{\hat{\phi}(x),\hat{\phi}(x')\}\right\rangle \tag{A.6}$$

$$iG_C(x,x') = \left\langle [\hat{\phi}(x),\hat{\phi}(x')]\right\rangle \tag{A.7}$$

where T is the time-ordering operator and T^* is the anti-time-ordering operator. The Jordan two-point function is simply the field commutator and is independent of the particular state used to evaluate it. Given the Wightman functions in (A.1) and (A.2) we write the above two-point functions in the form

$$iG_F(x,x') = \theta(t-t')G_+(x,x') + \theta(t'-t)G_-(x,x') \tag{A.8}$$

$$iG_D(x,x') = \theta(t'-t)G_+(x,x') + \theta(t-t')G_-(x,x') \tag{A.9}$$

$$G_H(x,x') = G_+(x,x') + G_-(x,x') \tag{A.10}$$

$$iG_C(x,x') = G_+(x,x') - G_-(x,x'). \tag{A.11}$$

We remark that the Feynman, Dyson and Hadamard functions are not all independent since

$$iG_F + iG_D = G_H = G_+ + G_-. \tag{A.12}$$

From these we may also define the retarded and advanced propagators by

$$G_{ret}(x,x') = -\theta(t-t')G_C(x,x') \tag{A.13}$$

$$G_{adv}(x,x') = \theta(t'-t)G_C(x,x'). \tag{A.14}$$

In terms of the other two-point functions, these propagators satisfy the following identities

$$
\begin{aligned}
-iG_{ret} &= iG_F - G_- = G_+ - iG_D && \text{(A.15)}\\
iG_{adv} &= iG_D - G_- = G_+ - iG_F && \text{(A.16)}
\end{aligned}
$$

and

$$
\begin{aligned}
G_F &= -\frac{1}{2}\big(G_{ret} + G_{adv}\big) - \frac{i}{2}\,G_H && \text{(A.17)}\\
G_D &= +\frac{1}{2}\big(G_{ret} + G_{adv}\big) - \frac{i}{2}\,G_H. && \text{(A.18)}
\end{aligned}
$$

These latter identities may be used to show that

$$
iG_D(x,x') = \big(iG_F(x,x')\big)^* = -iG_F^*(x,x'). \qquad \text{(A.19)}
$$

Under the interchange of x and x' the two-point functions satisfy

$$
\begin{aligned}
G_F(x,x') &= G_F(x',x) && \text{(A.20)}\\
G_D(x,x') &= G_D(x',x) && \text{(A.21)}\\
G_H(x,x') &= G_H(x',x) && \text{(A.22)}\\
G_C(x,x') &= -G_C(x',x) && \text{(A.23)}\\
G_+(x,x') &= G_-(x',x) && \text{(A.24)}\\
G_{ret}(x,x') &= G_{adv}(x',x). && \text{(A.25)}
\end{aligned}
$$

implying that the Feynman and Dyson propagators are symmetric along with the Hadamard function, the commutator is antisymmetric, and the Wightman functions and retarded/advanced propagators are a sort of transpose of each other. Furthermore,

$$G_+, G_-, G_F, G_D \;\in\; C \tag{A.26}$$

$$G_{ret}, G_{adv}, G_H, G_C \;\in\; R \tag{A.27}$$

and so the retarded propagator and the Hadamard function, in particular, are purely real.

Both Wightman functions, the Hadamard function and the commutator satisfy a homogeneous equation

$$\big(-\Box + m^2 + \xi R\big) G^{\pm}_{H,C} = 0 \tag{A.28}$$

while the Feynman and Dyson propagators satisfy

$$\big(-\Box + m^2 + \xi R\big) G_F \;=\; -ig^{-1/2}(x)\delta^d(x - x') \tag{A.29}$$

$$\big(-\Box + m^2 + \xi R\big) G_D \;=\; +ig^{-1/2}(x)\delta^d(x - x') \tag{A.30}$$

and the retarded and advanced propagators are solutions to

$$\big(-\Box + m^2 + \xi R\big) G_{ret,adv} = +g^{-1/2}(x)\delta^d(x - x'). \tag{A.31}$$

Appendix B

The closed-time-path formalism

The closed-time-path (CTP) generating functional (see Sections 2.3 and 4.4.1 for more details) for a real, massive field in a curved spacetime is

$$Z[J_1, J_2] = \int_{CTP} \mathcal{D}\phi_a \, \exp\left\{ \frac{i}{\hbar}\Big(S[\phi_1] - S[\phi_2] + J^a \cdot \phi_a \Big) \right\} \tag{B.1}$$

which can be integrated giving

$$Z[J_1, J_2] = N \exp\left\{ -\frac{1}{2\hbar} J^a \cdot G_{ab} \cdot J^b \right\} \tag{B.2}$$

where N is a constant, the $\cdot$ denotes spacetime integrating so that

$$A \cdot B \equiv \int d^d x \, g^{1/2} A(x) B(x) \tag{B.3}$$

for two (possibly tensor-valued) functions A and B, and the field action is

$$S[\phi] = -\frac{1}{2} \int d^d x \, g^{1/2} \Big(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + m^2 \phi^2 + \xi R \phi^2 \Big) \tag{B.4}$$

for some constant ξ. The CTP indices $a, b = 1, 2$ are raised and lowered with the CTP metric

$$c_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = c^{ab} \tag{B.5}$$

so that for two functions A and B

$$A^a B_a = c^{ab} A_a B_b = A_1 B_1 - A_2 B_2. \tag{B.6}$$

The two-point functions G_{ab} are defined by

$$G_{ab}(x,x') = c_{ac}c_{bd}\, \frac{\delta^2 Z}{\delta i J_c(x) \delta i J_d(x')}\bigg|_{J_a=0} \tag{B.7}$$

so that in terms of the Feynman propagator, etc, these are

$$G_{ab}(x,x') = \begin{pmatrix} iG_F(x,x') & G_-(x,x') \\ \\ G_+(x,x') & iG_D(x,x') \end{pmatrix}. \tag{B.8}$$

The two-point functions satisfy the equation

$$\big(-\Box + m^2 + \xi R \big) G_{ab}(x,x') = -i c_{ab} g^{-1/2}(x) \delta^d(x-x') \tag{B.9}$$

so that the Wightman function are homogeneous solutions of the equations of motion.

Defining the semi-sum and difference currents

$$J_+ = \frac{J_1 + J_2}{2} \tag{B.10}$$

$$J_- = J_1 - J_2 \tag{B.11}$$

to be the average and differences of the currents J_a, respectively, we find using the identities in the previous Appendix that the CTP generating functional can be written in the form

$$Z[J_a] = N \exp\bigg\{ -\frac{1}{2\hbar}\, J^a \cdot G_{ab} \cdot J^b \bigg\} \tag{B.12}$$

$$= N \exp\bigg\{ -\frac{1}{4\hbar}\, J_- \cdot G_H \cdot J_- + i J_- \cdot G_{ret} \cdot J_+ \bigg\}. \tag{B.13}$$

If the field propagates in a *flat* spacetime then we may express these two-point functions using a momentum space representation that is valid everywhere within the space, which is given by

$$G_{ab}(x,x') = \int_{\mathcal{C}_{ab}} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{d^{d-1}k}{(2\pi)^{d-1}}\, e^{-ik^0(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{-(k^0)^2 + \mathbf{k}^2 + m^2}. \quad \text{(B.14)}$$

More compactly, we write this as

$$G_{ab}(x,x') = \int_{\mathcal{C}_{ab}} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{d^{d-1}k}{(2\pi)^{d-1}}\, e^{ik_\alpha(x^\alpha - x'^\alpha)} \frac{1}{k^2 + m^2} \quad \text{(B.15)}$$

upon using the flat metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ to form the contractions. Each of the two-point functions have the same momentum space representation but a different contour that enforces the boundary conditions appropriate to G_{ab}. Fig.(B.1) displays these contours.

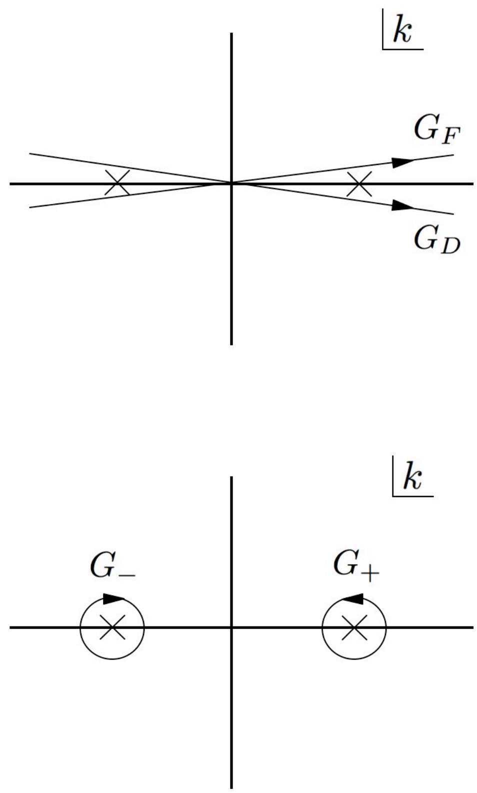

Figure B.1: Contours for the momentum space representation of the in-in two-point functions in flat spacetime for a massive scalar field.

Appendix C

Riemann normal coordinates

Consider a point P' having arbitrary coordinates x'^{α} in the space-time. We will take this point to be fixed and serve as the origin of the Riemann normal coordinates (RNC). Assign a tetrad $e^{\alpha'}_{\hat{a}}(x')$ at this point P'. For any other point P with coordinates x^{α} within the normal convex neighborhood $\mathcal{N}(P')$ of P' (i.e., the set of points that can be connected to P' by a unique geodesic) we define the Riemann normal coordinates of P to be

$$y^{\hat{a}} = -e^{\hat{a}}_{\alpha'}(x')\sigma^{\alpha'}(x,x') \tag{C.1}$$

where $e^{\hat{a}}_{\alpha'} = \eta^{\hat{a}\hat{b}}g_{\alpha'\beta'}e^{\beta'}_{\hat{b}}$ is the tetrad dual to $e^{\alpha'}_{\hat{a}}$. We use a latin index $(\hat{a}, \hat{b}, \ldots = 0, \ldots, d-1)$ to denote tensor components expressed in RNC. For example, the Ricci tensor in Riemann normal coordinates is

$$R_{\hat{a}\hat{b}}(x') = e^{\alpha'}_{\hat{a}}(x')e^{\beta'}_{\hat{b}}(x')R_{\alpha'\beta'}(x'). \tag{C.2}$$

The bi-scalar, $\sigma(x,x')$, appearing in (C.1) is Synge's world function. Numerically, this is equal to half of the squared geodesic distance between P and P' as measured along the unique geodesic connecting these points. The covariant derivative of σ with respect to x^{α} $(x^{\alpha\prime})$ is denoted with an unprimed (primed) Greek index so that

$$\sigma^{\alpha}(x,x') = \sigma^{;\alpha}(x,x') \tag{C.3}$$

$$\sigma^{\alpha'}(x,x') = \sigma^{;\alpha'}(x,x'). \tag{C.4}$$

The world function satisfies important identities that we merely state here (proofs and derivations may be found in [53]),

$$\sigma^{\alpha}\sigma_{\alpha} = \sigma^{\alpha'}\sigma_{\alpha'} = 2\sigma \tag{C.5}$$

$$\sigma^{\alpha}{}_{\mu'}\sigma^{\mu'} = \sigma^{\alpha} \tag{C.6}$$

$$\sigma^{\alpha'}{}_{\mu}\sigma^{\mu} = \sigma^{\alpha'}. \tag{C.7}$$

Geometrically, σ^{α} is proportional to the tangent vector at P along the geodesic connecting P and P'

$$\sigma^{\alpha}(x, x') = (\lambda - \lambda')t^{\alpha}(P) \tag{C.8}$$

and points from P to P'. Similarly, $\sigma^{\alpha'}$ is proportional to the tangent vector at P' along the geodesic connecting P and P' and points from P' to P,

$$\sigma^{\alpha'}(x, x') = -(\lambda - \lambda')t^{\alpha'}(P') \tag{C.9}$$

if λ (λ') is the value of the affine parameter at P (P') and $t^{\alpha'}$ is the unit vector at P'. See Fig.(C.1) at the end of this appendix for a schematic.

Riemann normal coordinates have the useful property that the locally Lorentz invariant quantity $\eta_{\hat{a}\hat{b}}y^{\hat{a}}y^{\hat{b}}$ gives the geodesic distance between the points P and P'.

$$\eta_{\hat{a}\hat{b}}y^{\hat{a}}y^{\hat{b}} = \eta_{\hat{a}\hat{b}}e^{\hat{a}}_{\alpha'}e^{\hat{b}}_{\beta'}\sigma^{\alpha'}\sigma^{\beta'} = g_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} = 2\sigma \tag{C.10}$$

where we have used (C.5) in the last equality.

The transformation from the original coordinates to RNC, and vice versa, can be found by letting $x^{\alpha} \to x^{\alpha} + dx^{\alpha}$. From the definition of $y^{\hat{a}}$ in (C.1),

$$dy^{\hat{a}} = -e^{\hat{a}}_{\alpha'}\sigma^{\alpha'}{}_{\beta}dx^{\beta}. \tag{C.11}$$

Given this we can construct the metric at P in Riemann normal coordinates through the usual transformation rule for tensors,

$$g_{\hat{a}\hat{b}}(y) = \frac{dx^{\alpha}}{dy^{\hat{a}}}\frac{dx^{\beta}}{dy^{\hat{b}}}\, g_{\alpha\beta}(x). \tag{C.12}$$

The metric at P can be written as a Taylor series in $y^{\hat{a}}$. This is equivalent to an expansion in derivatives of the background metric. To see this we write the expansion through $O(y^4)$ of the metric [162, 163, 64],

$$\begin{aligned} g_{\hat{m}\hat{n}} &= \eta_{\hat{m}\hat{n}} - \frac{1}{3}\, R_{\hat{m}\hat{a}\hat{n}\hat{b}} y^{\hat{a}} y^{\hat{b}} - \frac{1}{6}\, R_{\hat{m}\hat{a}\hat{n}\hat{b};\hat{c}} y^{\hat{a}} y^{\hat{b}} y^{\hat{c}} \\ &\quad + \left(-\frac{1}{20}\, R_{\hat{m}\hat{a}\hat{n}\hat{b};\hat{c}\hat{d}} + \frac{2}{45}\, R_{\hat{a}\hat{m}\hat{b}\hat{l}} R^{\hat{l}}{}_{\hat{c}\hat{n}\hat{d}} \right) y^{\hat{a}} y^{\hat{b}} y^{\hat{c}} y^{\hat{d}} + O(y^5). \end{aligned} \tag{C.13}$$

The curvature tensors involve two derivatives of the metric. Therefore, with each power of $y^{\hat{a}}$ there appears a power of $\partial_{\hat{a}} g_{\hat{b}\hat{c}}$. Because of this we will often refer to a series in $y^{\hat{a}}$, such as the metric above, as an *adiabatic expansion*. The validity of the $O(y^n)$ expansion requires a typical component of $|y^{\hat{a}} \partial_{\hat{b}} g_{\hat{c}\hat{d}}|$ to be much smaller than 1. Hence, the scale at which the expansion in RNC is valid is much smaller than the scale at which the metric changes, which is approximately the curvature scale of the background space-time. This supports our vocabulary. We will denote the n^{th} adiabatic order of an expansion by $O(\partial^n)$ to represent the number of derivatives acting on the metric.

We will also collect here the adiabatic expansions of the inverse metric, the metric determinant, its logarithm and the connection components. The inverse metric is easily found from (C.13)

$$
\begin{aligned}
g^{\hat{m}\hat{n}} &= \eta^{\hat{m}\hat{n}} + \frac{1}{3}\,R^{\hat{m}\ \,\hat{n}}_{\ \,\hat{a}\ \,\hat{b}}\,y^{\hat{a}}y^{\hat{b}} + \frac{1}{6}\,R^{\hat{m}\ \,\hat{n}}_{\ \,\hat{a}\ \,\hat{b};\hat{c}}\,y^{\hat{a}}y^{\hat{b}}y^{\hat{c}} \\
&\quad + \left(\frac{1}{20}R^{\hat{m}\ \,\hat{n}}_{\ \,\hat{a}\ \,\hat{b};\hat{c}\hat{d}} - \frac{1}{15}R^{\hat{m}}_{\ \,\hat{a}\hat{b}\hat{l}}R^{\hat{l}\ \,\hat{n}}_{\ \,\hat{c}\ \,\hat{d}}\right) y^{\hat{a}}y^{\hat{b}}y^{\hat{c}}y^{\hat{d}} + O(\partial^5) \qquad \text{(C.14)}
\end{aligned}
$$

The determinant of the metric is

$$
\begin{aligned}
g &= |\det g_{\hat{m}\hat{n}}| \qquad \text{(C.15)} \\
&= 1 - \frac{1}{3}\,R_{\hat{a}\hat{b}}\,y^{\hat{a}}y^{\hat{b}} - \frac{1}{6}\,R_{\hat{a}\hat{b};\hat{c}}\,y^{\hat{a}}y^{\hat{b}}y^{\hat{c}} \\
&\quad + \left(\frac{1}{18}\,R_{\hat{a}\hat{b}}R_{\hat{c}\hat{d}} - \frac{1}{90}\,R_{\hat{l}\hat{a}\hat{b}}{}^{\hat{k}}R^{\hat{l}}{}_{\hat{c}\hat{d}\hat{k}} - \frac{1}{20}R_{\hat{a}\hat{b};\hat{c}\hat{d}}\right) y^{\hat{a}}y^{\hat{b}}y^{\hat{c}}y^{\hat{d}} + O(\partial^5) \qquad \text{(C.16)}
\end{aligned}
$$

and its logarithm is

$$
\begin{aligned}
\ln g &= -\frac{1}{3}\,R_{\hat{a}\hat{b}}\,y^{\hat{a}}y^{\hat{b}} - \frac{1}{6}R_{\hat{a}\hat{b};\hat{c}}\,y^{\hat{a}}y^{\hat{b}}y^{\hat{c}} \\
&\quad - \left(\frac{1}{90}R_{\hat{l}\hat{a}\hat{b}}{}^{\hat{k}}R^{\hat{l}}{}_{\hat{c}\hat{d}\hat{k}} + \frac{1}{20}R_{\hat{a}\hat{b};\hat{c}\hat{d}}\right) y^{\hat{a}}y^{\hat{b}}y^{\hat{c}}y^{\hat{d}} + O(\partial^5). \qquad \text{(C.17)}
\end{aligned}
$$

The components of the connection are

$$
\Gamma^{\hat{a}}_{\hat{b}\hat{c}} = -\frac{2}{3}\,R^{\hat{a}}{}_{(\hat{b}\hat{c})\hat{m}}\,y^{\hat{m}} + O(\partial^3). \qquad \text{(C.18)}
$$

In all of these expansions, the (tensor) coefficients of the $y^{\hat{a}}$ polynomials are evaluated at the origin of the Riemann normal coordinates, which is taken to reside at P'.

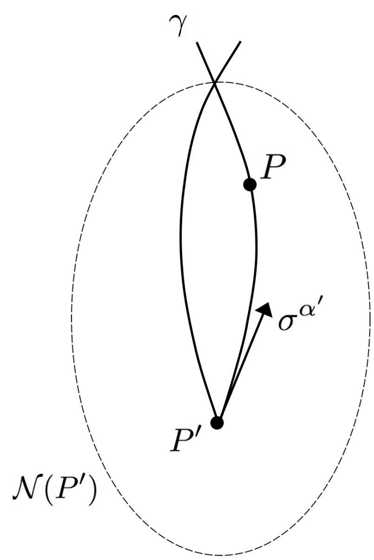

Figure C.1: The normal convex neighborhood $\mathcal{N}(P')$ (dashed oval) of the point P'. Any point P within $\mathcal{N}(P')$ can be connected to P' by a unique geodesic γ. The covariant derivative of Synge's world function $\sigma^{\alpha'}$ is proportional to the tangent vector at P' of the geodesic γ.

Appendix D

Momentum space representation of quantum two-point functions in Riemann normal coordinates

In this Appendix we derive the momentum space representation of the quantum two-point functions for metric perturbations, including the Feynman propagator, in an arbitrary curved space-time using a qausi-local expansion in Riemann normal coordinates. We use a novel method borrowed from perturbative quantum field theory that is based on diagrammatic techniques to streamline the original calculation of Bunch and Parker [64]. We first discuss the role of the state of the field, the local structure of the two-point functions and the relationship between the two-point functions calculated as true expectation values versus matrix elements of a transition amplitude. We then derive the momentum space representation of the two-point functions for a scalar field through fourth adiabatic order and compare our expression for the Feynman propagator with the result of Bunch and Parker to show that our method reproduces their result. We then calculate the momentum space representation of the two-point functions for metric perturbations through second adiabatic order, which is needed to regularize and renormalize the first and higher order diagrams that enter the self-force equations derived in Chapters 4 and 5. In doing so our approach is seen to be relatively efficient for computing the momentum space representation of two-point functions of fields with nontrivial tensor and spin structure.

D.1 The state of the field and the ultraviolet structure of the two-point functions

Throughout this Appendix we will be mostly concerned with the following two-point functions of a quantum field $\hat{\Phi}_A$ where A capital Latin index represent the spacetime indices appropriate to the field under consideration: the Feynman $G^F_{AB'}$ and Dyson $G^D_{AB'}$ propagators and the positive $G^+_{AB'}$ and negative $G^-_{AB'}$ frequency Wightman functions. For simplicity in this discussion let us assume that the field is scalar so that $\Phi_A(x) = \phi(x)$ and the four two-point functions introduced are denoted by G_F, G_D, G_+ and G_-, respectively.

Each of these two-point functions satisfy the equation of motion for a massive scalar field

$$\big(-\Box + m^2 + \xi R\big) G_{ab}(x,x') = -i c_{ab}\, g^{-1/2}(x)\, \delta^d(x-x') \tag{D.1}$$

where the Feynman and Dyson propagators are sourced by a point source of unit strength in d dimensions of spacetime while the Wightman functions satisfy the homogeneous equation. The arbitrary parameter ξ couples the field directly to the background spacetime curvature. We use the CTP indices $a, b = 1, 2$ to denote these two-point functions by G_{ab} as defined in (B.7) and (B.8).

The equations of motion (D.1) do not specify the state(s) used to evaluate these two-point functions. For example, both

$$iG_F(x,x') = \big\langle \text{in}, 0 \big| T\, \hat{\phi}(x)\hat{\phi}(x') \big| 0, \text{in} \big\rangle \tag{D.2}$$

and

$$iG_F(x,x') = \big\langle \text{out}, 0 \big| T\,\hat{\phi}(x)\hat{\phi}(x') \big| 0, \text{in} \big\rangle \tag{D.3}$$

satisfy (D.1) where $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ are the in- and out-vacua defined in the asymptotic past and future, respectively, of the spacetime. There are in fact many solutions to (D.1). To generate the solution with the correct state of interest requires imposing boundary conditions on that solution. For example, in flat spacetime one uses the $i\epsilon$-prescription to obtain the time-ordering of the fields that defines the Feynman propagator. In curved spacetime the issue is more subtle on account of the backscattering of field modes due to the curvature of the space and other global features specific to the state of the field [66, 124].

The reason why both (D.2) and (D.3) satisfy the equations of motion is because (D.1) is a *local* equation and is only concerned with the behavior of the Feynman propagator at x alone. In fact, this holds for any of the two-point functions G_{ab} satisfying (D.1). Therefore, so long as a two-point function G_{ab} is a solution to (D.1) then the particulars of the state(s) used to construct that two-point function are irrelevant from the point of view of the *local* structure of G_{ab}.

We can also show that the local structure of the two-point functions are all the same. To show this we use an approach very similar to the one developed by Bunch and Parker [64] using momentum space techniques for the Feynman propagator in Reimann normal coordinates. See Appendix C for a brief survey of Riemann normal coordinates (RNC).

We express the equation of motion for the two-point functions (D.1) in RNC and rescale G_{ab} so that

$$G_{ab}(x,x') = g^{-1/4}(x)\bar{G}_{ab}(x,x')g^{-1/4}(x') \tag{D.4}$$

to find an equivalent equation for the rescaled two-point functions expressed as an expansion in powers of derivatives of the background metric, which we call an adiabatic expansion. We find through second adiabatic order

$$\big(-\eta^{mn}\partial_m\partial_n + m^2\big)\bar{G}_{ab} + \left(\xi - \frac{1}{6}\right) R\,\bar{G}_{ab} + O(\partial^3) = -ic_{ab}\,\delta^d(y) \tag{D.5}$$

In RNC, the origin of this coordinate system is at x'^α and the point x^α is represented by y^a and the partial derivatives in the above equation are with respect to y. Writing the two-point functions as an adiabatic expansion

$$\bar{G}_{ab}(x,x') = \bar{G}^{(0)}_{ab}(x,x') + \bar{G}^{(1)}_{ab}(x,x') + \bar{G}^{(2)}_{ab}(x,x') + O(\partial^3) \tag{D.6}$$

shows that the leading order term solves the flat space equations of motion

$$\big(-\eta^{mn}\partial_m\partial_n + m^2\big)\bar{G}^{(0)}_{ab} = -ic_{ab}\,\delta^d(y) \tag{D.7}$$

implying the following momentum space representation

$$\bar{G}^{(0)}_{ab}(x,x') = \int_{\mathcal{C}_{ab}} \frac{dk_0}{2\pi} \int_{-\infty}^{\infty} \frac{d^{d-1}k}{(2\pi)^{d-1}}\, e^{ik\cdot y} \frac{-i}{k^2+m^2} \tag{D.8}$$

which agrees with (B.15) and where the contours $\mathcal{C}_{ab}$ are defined in Fig.(B.1).

Let us therefore introduce a momentum space associated with the point x' and introduce a Fourier transform for the full two-point function so that

$$\begin{aligned} \bar{G}_{ab}(x,x') &= \int_{\mathcal{C}_{ab}} \frac{dk_0}{2\pi} \int_{-\infty}^{\infty} \frac{d^{d-1}k}{(2\pi)^{d-1}}\, e^{ik\cdot y}\bar{G}_{ab}(k) && \text{(D.9)} \\ &\equiv \int_{\mathcal{C}_{ab},k} e^{ik\cdot y}\bar{G}_{ab}(k) && \text{(D.10)} \end{aligned}$$

where $\mathcal{C}_{ab}$ denotes the appropriate contour to use for the particular two-point function. Letting

$$\bar{G}_{ab}(k) = \bar{G}^{(0)}_{ab}(k) + \bar{G}^{(1)}_{ab}(k) + \bar{G}^{(2)}_{ab}(k) + O(\partial^3) \tag{D.11}$$

we see that order by order in the derivative of the metric we can solve iteratively for the $\bar{G}^n_{ab}(k)$. At leading order we know that

$$\bar{G}^{(0)}_{ab}(k) = \frac{-i}{k^2 + m^2} \tag{D.12}$$

while at first order $\bar{G}^{(1)}_{ab}(k)$ vanishes and the second order contribution is nontrivial,

$$\bar{G}^{(2)}_{ab}(k) = (-i)^2 \frac{\left(\frac{1}{6} - \xi\right) R}{(k^2 + m^2)^2} \tag{D.13}$$

We remark that neither the zeroeth order nor the second order contributions involve the CTP indices. In fact, this is true at every order in this expansion since the leading order term is independent of a, b. It then follows that all of the two-points functions $\bar{G}_{ab}$ have the same quasi-local structure in a momentum representation. After computing the momentum space two-point functions $\bar{G}^{(n)}_{ab}$ we finally integrate over the momentum as in (D.10) with the contour appropriate for the particular two-point function being calculated. In fact, the contour $\mathcal{C}_{ab}$ is the only object that distinguishes among the two-point functions of this quasi-local momentum space representation.

We therefore conclude that we can obtain the momentum space representation of any of the two-point functions $G_{ab}(x, x')$, computed with any state(s), from the structure of the Feynman propagator in (D.2). The advantage of choosing this particular two-point function to use in our calculations below is that (D.2) is easily calculated from the in-out generating functional, which is sufficiently simple to use in our diagrammatic approach below for deriving those terms in the two-point functions that are relevant for renormalizing the self-force in Chapters 4 and 5.

The arguments presented in this section are not limited to scalar fields and can be extended to higher spin fields, including perturbations of a background metric $h_{\alpha\beta}(x)$, which are relevant for this work.

D.2 Scalar field Feynman propagator

To demonstrate that our diagrammatic technique is viable and correct we develop the momentum space representation of the in-out Feynman propagator and compare with the original result of Bunch and Parker [64].

A massive scalar field propagating in a d-dimensional curved space-time with a metric $g_{\mu\nu}$ can be described by the action

$$S[\phi] = -\frac{1}{2}\int d^d x\, g^{1/2}\big(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + m^2\phi^2 + \xi R\phi^2\big) + \int d^d x\, g^{1/2} J(x)\phi \tag{D.14}$$

where ξ is a constant that couples the field to the background curvature. When $\xi = 0$ the field is said to be minimally coupled and when

$$\xi = \frac{1}{4}\frac{d-2}{d-1} \tag{D.15}$$

the field is said to be conformally coupled. Here, $J(x)$ is an external current that will be used below to generate correlation functions of the scalar field.

It will prove convenient to rescale the fields and external current so that

$$\phi(x) = g^{-1/4}(x)\bar{\phi}(x) \tag{D.16}$$

$$J(x) = g^{-1/4}(x)\bar{J}(x) \tag{D.17}$$

in terms of which the action is

$$\begin{aligned} S[\bar{\phi}] &= -\frac{1}{2}\int d^dx \left\{ g^{\mu\nu}\left[-\frac{1}{4}g^{-1}g_{,\mu}\bar{\phi} + \bar{\phi}_{,\mu}\right]\left[-\frac{1}{4}g^{-1}g_{,\nu}\bar{\phi} + \bar{\phi}_{,\nu}\right] + m^2\bar{\phi}^2 + \xi R\bar{\phi}^2 \right\} \\ &\quad + \int d^dx\, \bar{J}\bar{\phi}. \end{aligned} \tag{D.18}$$

Multiplying out the terms in brackets and noting that

$$g^{-1}g_{,\mu} = (\ln g)_{,\mu} \tag{D.19}$$

we find

$$\begin{aligned} S[\bar{\phi}] &= -\frac{1}{2}\int d^dx \Bigg\{ g^{\mu\nu}\Bigg[\bar{\phi}_{,\mu}\bar{\phi}_{,\nu} + m^2\bar{\phi}^2 + \xi R\bar{\phi}^2 - \frac{1}{4}\bar{\phi}\left[(\ln g)_{,\mu}\bar{\phi}_{,\nu} + (\ln g)_{,\nu}\bar{\phi}_{,\mu}\right] \\ &\quad + \frac{1}{16}(\ln g)_{,\mu}(\ln g)_{,\nu}\bar{\phi}^2\Bigg]\Bigg\} + \int d^dx\, \bar{J}\bar{\phi}. \end{aligned} \tag{D.20}$$

Integrating by parts once and observing the following relations

$$g^{-1}g_{,\mu\nu} = (\ln g)_{,\mu\nu} + (\ln g)_{,\mu}(\ln g)_{,\nu} \tag{D.21}$$

$$g^{-1/4}\left[(g^{1/4})_{,\mu}g^{\mu\nu}\right]_{,\nu} = \frac{1}{4}(\ln g)_{,\mu\nu}g^{\mu\nu} + \frac{1}{4}(\ln g)_{,\mu}g^{\mu\nu}{}_{,\nu} + \frac{1}{16}(\ln g)_{,\mu}(\ln g)_{,\nu}g^{\mu\nu} \tag{D.22}$$

the action is simplified to

$$\begin{aligned} S[\bar{\phi}] &= -\frac{1}{2}\int d^dx \Big\{ g^{\mu\nu}\bar{\phi}_{,\mu}\bar{\phi}_{,\nu} + g^{-1/4}\left[(g^{1/4})_{,\mu}g^{\mu\nu}\right]_{,\nu}\bar{\phi}^2 + m^2\bar{\phi}^2 + \xi R\bar{\phi}^2 \Big\} \\ &\quad + \int d^dx\, \bar{J}\bar{\phi}. \end{aligned} \tag{D.23}$$

We remark that the original form of the kinetic term $g^{1/2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}$ in (D.14) has been converted into two pieces. The first is a kinetic-type term for $\bar{\phi}$ that reads $g^{\mu\nu} \bar{\phi}_{,\mu} \bar{\phi}_{,\nu}$. The second is a curvature-induced effective mass term,

$$g^{-1/4} \left[(g^{1/4})_{,\mu} g^{\mu\nu} \right]_{,\nu} \bar{\phi}^2. \tag{D.24}$$

Categorizing these two kinds of terms will be convenient for determining the nontrivial contributions to the momentum space representation of the quantum two-point functions.

The action in (D.23) is coordinate-invariant. To proceed we choose to work in Riemann normal coordinates within the normal convex neighborhood about a point x'. The adiabatic expansions of some relevant tensors (e.g., the metric, its determinant, etc.) are given in Appendix C. However, we will also need the expansion of the Ricci scalar appearing in (D.23)

$$R(x) = R(x') + R_{;\hat{a}} y^{\hat{a}} + \frac{1}{2} R_{;\hat{a}\hat{b}} y^{\hat{a}} y^{\hat{b}} + O(\partial^5). \tag{D.25}$$

Using these expansions the action (D.23) can be written as an adiabatic expansion,

$$S[\bar{\phi}] = -\frac{1}{2} \int d^d y \left[\eta^{\hat{m}\hat{n}} \bar{\phi}_{,\hat{m}} \bar{\phi}_{,\hat{n}} + m^2 \bar{\phi}^2 \right] + \int d^d y \, \bar{J} \bar{\phi} + S_{int}[\bar{\phi}] \tag{D.26}$$

where the interaction S_{int} contains the sub-leading terms in the expansion

$$S_{int}[\bar{\phi}] = -\frac{1}{2} \int d^d y \sum_{A=2}^{\infty} \left[K_{(A)}^{\hat{m}\hat{n}}(y, x') \bar{\phi}_{,\hat{m}} \bar{\phi}_{,\hat{n}} + M_{(A)}(y, x') \bar{\phi}^2 \right] \tag{D.27}$$

The kernels $K_{(A)}^{\hat{m}\hat{n}}$ and $M_{(A)}$, where the subscript A in parentheses indicates the adiabatic order, represent interactions with respect to the flat space "non-interacting" scalar field theory, which is described by the "free" field action

$$S_0[\bar{\phi}] = -\frac{1}{2} \int d^d y \left[\eta^{\hat{m}\hat{n}} \bar{\phi}_{,\hat{m}} \bar{\phi}_{,\hat{n}} + m^2 \bar{\phi}^2 \right]. \tag{D.28}$$

The interaction terms in S_{int} are quadratic in the field $\bar{\phi}$ and suggest that these perturb the leading order propagator of the "free" theory (associated with $S_0[\bar{\phi}]$). The fact that there are not higher powers of the field results from the original action (D.14) describing a free scalar field in a curved space-time, which can only be quadratic in the field.

Furthermore, the interactions in (D.27) naturally split into two classes. The first describes perturbations of the kinetic part of S_0, being proportional to a product of $\bar{\phi}_{,\hat{m}}\bar{\phi}_{,\hat{n}}$. These terms cannot be transformed into a $\bar{\phi}^2$ type term by integration by parts or other manipulations without introducing terms linear in $\bar{\phi}_{,\hat{m}}$, which are not convenient for our purposes here. The second describes a curvature-induced effective mass (as observed in the nearly flat region about x') that shifts the value of the leading order mass m appearing in the background action S_0.

The interaction kernels for the kinetic part of S_{int} are, through fourth adiabatic order,

$$K_{(2)}^{\hat{m}\hat{n}} = \frac{1}{3}R^{\hat{m}\ \ \hat{n}}_{\ \ \hat{a}\ \ \hat{b}}y^{\hat{a}}y^{\hat{b}} \tag{D.29}$$

$$K_{(3)}^{\hat{m}\hat{n}} = \frac{1}{6}R^{\hat{m}\ \ \hat{n}}_{\ \ \hat{a}\ \ \hat{b};\hat{c}}y^{\hat{a}}y^{\hat{b}}y^{\hat{c}} \tag{D.30}$$

$$K_{(4)}^{\hat{m}\hat{n}} = \left(\frac{1}{20}R^{\hat{m}\ \ \hat{n}}_{\ \ \hat{a}\ \ \hat{b};\hat{c}\hat{d}} - \frac{1}{15}R^{\hat{m}}_{\ \ \hat{a}\hat{b}\hat{l}}R^{\hat{l}\ \ \hat{n}}_{\ \ \hat{c}\ \ \hat{d}}\right)y^{\hat{a}}y^{\hat{b}}y^{\hat{c}}y^{\hat{d}} \tag{D.31}$$

which is found from the adiabatic expansion of the inverse metric,

$$\sum_{A=2}^{\infty} K_{(A)}^{\hat{m}\hat{n}}(y,x') = g^{\hat{m}\hat{n}} - \eta^{\hat{m}\hat{n}}. \tag{D.32}$$

The kernels for the curvature-induced effective mass part of S_{int} are found from expanding

$$\sum_{A=2}^{\infty} M_{(A)}(y,x') = \xi R + g^{-1/4}\left[(g^{1/4})_{,\hat{m}} g^{\hat{m}\hat{n}}\right]_{,\hat{n}} \tag{D.33}$$

which through fourth adiabatic order are

$$M_{(2)}(y,x') = \left(\xi - \frac{1}{6}\right) R \tag{D.34}$$

$$M_{(3)}(y,x') = \xi R_{;\hat{a}} y^{\hat{a}} - \frac{1}{12}\left(R_{;\hat{a}} + 2R_{\hat{a}\hat{b}}{}^{;\hat{b}}\right) y^{\hat{a}} \tag{D.35}$$

$$\begin{aligned} M_{(4)}(y,x') = {} & \frac{1}{2}\xi R_{;\hat{a}\hat{b}} y^{\hat{a}} y^{\hat{b}} + \Big(-\frac{1}{40} R_{;\hat{a}\hat{b}} - \frac{1}{20} R_{\hat{a}\hat{l}}{}^{;\hat{l}}{}_{;\hat{b}} - \frac{1}{20} R_{\hat{a}\hat{l};\hat{b}}{}^{;\hat{l}} - \frac{1}{40} \Box R_{\hat{a}\hat{b}} \\ & + \frac{1}{12} R_{\hat{a}\hat{l}} R^{\hat{l}}{}_{\hat{b}} - \frac{1}{15} R_{\hat{l}\hat{r}} R^{\hat{l}}{}_{\hat{a}}{}^{\hat{r}}{}_{\hat{b}} - \frac{1}{90} R_{\hat{l}\hat{s}\hat{a}}{}^{\hat{r}} R^{\hat{l}}{}_{\hat{b}}{}^{\hat{s}}{}_{\hat{r}} - \frac{1}{90} R_{\hat{l}\hat{s}\hat{a}}{}^{\hat{r}} R^{\hat{l}\hat{s}}{}_{\hat{b}\hat{r}} \Big) y^{\hat{a}} y^{\hat{b}}. \end{aligned} \tag{D.36}$$

These expressions can be simplified by recalling that the Einstein tensor is divergenceless, implying

$$R_{\hat{a}\hat{b}}{}^{;\hat{b}} = \frac{1}{2} R_{;\hat{a}} \tag{D.37}$$

With

$$R_{\hat{a}\hat{l};\hat{b}}{}^{;\hat{l}} = \frac{1}{2} R_{;\hat{a}\hat{b}} - R^{\hat{l}}{}_{\hat{a}}{}^{\hat{s}}{}_{\hat{b}} R_{\hat{l}\hat{s}} + R_{\hat{a}\hat{l}} R^{\hat{l}}{}_{\hat{b}} \tag{D.38}$$

and using the first Bianchi identity $R_{\hat{a}[\hat{b}\hat{c}\hat{d}]} = 0$ to show that

$$R^{\hat{l}}{}_{\hat{b}}{}^{\hat{s}}{}_{\hat{k}} = \frac{1}{2} R^{\hat{l}\hat{s}}{}_{\hat{b}\hat{k}} \tag{D.39}$$

we find simplified expressions for the mass interactions

$$M_2(y,x') = \left(\xi - \frac{1}{6}\right) R \tag{D.40}$$

$$M_3(y,x') = \left(\xi - \frac{1}{6}\right) R_{;\hat{a}} y^{\hat{a}} \tag{D.41}$$

$$\begin{aligned} M_4(y,x') &= \left[\frac{1}{2}\left(\xi - \frac{1}{6}\right) R_{;\hat{a}\hat{b}} + \frac{1}{120} R_{;\hat{a}\hat{b}} - \frac{1}{40}\Box R_{\hat{a}\hat{b}} + \frac{1}{30} R_{\hat{a}\hat{l}} R^{\hat{l}}{}_{\hat{b}} \right. \\ &\quad \left. - \frac{1}{60} R_{\hat{l}\hat{r}} R^{\hat{l}}{}_{\hat{a}}{}^{\hat{r}}{}_{\hat{b}} - \frac{1}{60} R_{\hat{l}\hat{s}\hat{a}}{}^{\hat{r}} R^{\hat{l}\hat{s}}{}_{\hat{b}\hat{r}} \right] y^{\hat{a}} y^{\hat{b}} \\ &\equiv a_{\hat{a}\hat{b}} y^{\hat{a}} y^{\hat{b}} \end{aligned} \tag{D.42}$$

D.2.1 Generating functional

Per the discussion in Section D.1 we need only to find a momentum space representation for the in-out Feynman propagator to determine the ultraviolet behavior of the in-in two-point functions G_{ab}.

We therefore construct the generating functional of in-out correlation functions

$$Z[\bar{J}] = \frac{\langle 0, \mathrm{out}|0, \mathrm{in}\rangle_{\bar{J}}}{\langle 0, \mathrm{out}|0, \mathrm{in}\rangle_{\bar{J}=0}} \tag{D.43}$$

where the vacuum-vacuum persistence amplitude in the presence of the external current $\bar{J}$ is

$$\langle 0, \mathrm{out}|0, \mathrm{in}\rangle_{\bar{J}} = \int \mathcal{D}\bar{\phi}\, \exp\left[i \int d^d x\, \mathcal{L}_{int}\left(\bar{\phi}\right)\right] \exp\left[iS_0[\bar{\phi}] + i\int d^d x \bar{J}\bar{\phi}\right] \tag{D.44}$$

The free action S_0 is given in (D.28) and the interaction Lagrangian can be deduced from (D.27).

In terms of correlations of the scalar field the in-out Feynman propagator is

$$G_F(x,x') = \langle 0, \mathrm{out}|T\, \phi(x)\phi(x')|0, \mathrm{in}\rangle. \tag{D.45}$$

The rescaled field $\bar{\phi}$ has an associated propagator

$$\bar{G}_F(x, x') = \langle 0, \mathrm{out}|T\,\bar{\phi}(x)\bar{\phi}(x')|0, \mathrm{in}\rangle \tag{D.46}$$

that is related to G_F through the rescaling,

$$G_F(x, x') = g^{-1/4}(x)\bar{G}_F(x, x')g^{-1/4}(x'). \tag{D.47}$$

and is implied by (D.16).

The field dependence in $\mathcal{L}_{int}(\bar{\phi})$ can be replaced by derivatives with respect to the external current by expanding the exponential $\exp i\int d^dx\mathcal{L}_{int}$ in powers of the field so that the n^{th} term is

$$\bar{\phi}^n(x)\exp\left[i\int d^dx'\,\bar{J}\bar{\phi}\right] = \left(\frac{\delta}{\delta i\bar{J}(x)}\right)^n \exp\left[i\int d^dx'\,\bar{J}\bar{\phi}\right]. \tag{D.48}$$

This replacement allows for the exponential factor $\exp i\int d^dx\,\mathcal{L}_{int}$ to be pulled out of the path integral. The remaining Gaussian integral is easily evaluated using standard techniques giving

$$Z[\bar{J}] = \exp\left[i\int d^dx\,\mathcal{L}_{int}\left(\frac{\delta}{\delta i\bar{J}(x)}\right)\right]\exp\left[\frac{1}{2}\int d^dud^du'\,\bar{J}(u)\bar{G}_0(u, u')\bar{J}(u')\right] \tag{D.49}$$

where $\bar{G}_0$ is the leading order (flat space-time) Feynman propagator, which has the following momentum-space representation

$$\bar{G}_0(x - x') = \int \frac{d^dk}{(2\pi)^d}\,e^{ik\cdot(x-x')}\frac{-i}{k^2 + m^2 + i\epsilon} \tag{D.50}$$

in d space-time dimensions. The $i\epsilon$ is added to impose the usual Feynman boundary conditions.

The normalized generating functional in (D.43) generates (time-ordered) correlations of the field $\bar{\phi}$ by computing derivatives of $Z[\bar{J}]$ with respect to the current $\bar{J}$. The full propagator, which is actually the free field propagator in a curved space-time, is defined as

$$\bar{G}_F(x,x') = \left.\frac{\delta^2 Z[\bar{J}]}{\delta i\bar{J}(x)\delta i\bar{J}(x')}\right|_{\bar{J}=0}. \tag{D.51}$$

To find the propagator perturbatively about (D.50), we first expand $\exp i\int d^dx\mathcal{L}_{int}$ from (D.49) in adiabatic powers (i.e., in powers of derivatives of the metric) and then compute (D.51) setting $\bar{J}$ to zero at the end of the calculation.

Such a procedure can be tedious, particularly at higher adiabatic orders and for higher spin fields. Nevertheless, the terms in the ensuing expansion give rise to a diagrammatic interpretation. Knowing which diagrams enter at a particular adiabatic order in the expansion allows us to compute their contribution to the full propagator (D.51).

The diagrams relevant for the calculation of $\bar{G}_F(x,x')$ are those that are connected [93]. The (disconnected) vacuum bubble diagrams that might appear from perturbatively evaluating $\langle \mathrm{out}, 0|0, \mathrm{in}\rangle_{\bar{J}}$ are conveniently cancelled by the denominator in (D.43).

D.2.2 Feynman rules

Turn now to the Feynman rules for this diagrammatic approach. Let the leading order, or free, propagator $\bar{G}_0(x,x')$ be represented by a straight line connecting the points x and x' as depicted in Fig.(D.1a). The interaction terms given in (D.27)

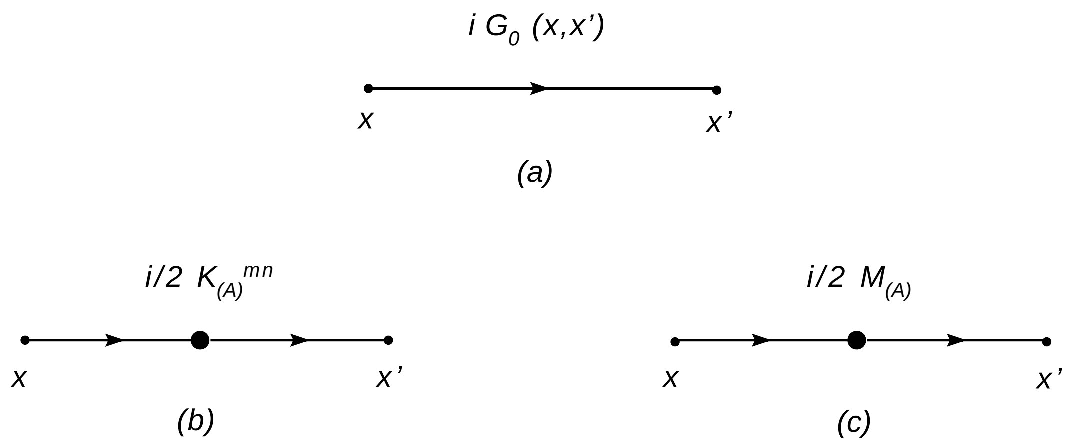

Figure D.1: Feynman rules for computing the free scalar field propagator in a curved space-time. (a) The rule for the leading order (flat space-time) propagator. (b) and (c) show the kinetic and mass vertices that appear in $S_{int}[\bar{\phi}]$.

represent vertices that cause the field to undergo self-interaction and are naturally classified as kinetic $K_{(A)}^{\hat{m}\hat{n}}$ and mass $M_{(A)}$ vertices, as discussed earlier. The kinetic and mass vertices are shown in Fig.(D.1b) and Fig.(D.1c), respectively. We can calculate a given diagram with any appropriate number of vertices and lines using the following Feynman rules:

1. A factor of $-\frac{i}{2}K_{(A)}^{\hat{m}\hat{n}}(y,x')$ for each kinetic vertex.

2. A factor of $-\frac{i}{2}M_{(A)}(y,x')$ for each mass vertex.

3. Spacetime integration for each vertex.

4. A factor of $\bar{G}_0$ if a line connects to a mass vertex.

5. A factor of $\partial_{\hat{a}}\bar{G}_0$ if a line connects to a kinetic vertex (the derivative is with respect to the coordinate integrated in Rule 3.)

6. Symmetry factor S.

To show how these rules are used to compute diagrams let us compute the first few corrections to the free propagator $\bar{G}_0$.

D.2.2.1 Second adiabatic order

The diagrams that contribute to the second adiabatic order correction to the free propagator $\bar{G}_0$ are determined by using the Feynman rules to construct all possible connected diagrams that are $O(\partial^2)$. At this order there are only two such diagrams and these correspond to Fig.(D.1b) and Fig.(D.1c) with $A = 2$. One of these contributions is given by

$$\begin{aligned} V_M(\partial^2) &\equiv \text{ mass vertex at } O(\partial^2) && \text{(D.52)} \\ &= 2\int d^d y\, \bar{G}_0(x,y)\left[-\frac{i}{2}\left(\xi - \frac{1}{6}\right)R(x')\right]\bar{G}_0(y,x'). && \text{(D.53)} \end{aligned}$$

A propagator factor of $\bar{G}_0$ is associated with each external line in the diagram and the vertex gives a factor of $-i/2M_{(2)}$. The overall factor of 2 is a symmetry factor that counts the number of ways to connect the propagator lines to the vertex.

Using the Fourier transform of the leading order propagator

$$\bar{G}_0(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x}\bar{G}_0(k) \equiv \int_k e^{ik\cdot x}\bar{G}_0(k) \qquad \text{(D.54)}$$

where

$$\bar{G}_0(k) = \frac{-i}{k^2 + m^2 - i\epsilon} \qquad \text{(D.55)}$$

we find

$$V_M(\partial^2) = -i\int_k\int_q\int d^d y\, e^{ik\cdot x - iq\cdot x'}e^{-i(k-q)\cdot y}\bar{G}_0(k)\bar{G}_0(q)\left(\xi - \frac{1}{6}\right)R. \qquad \text{(D.56)}$$

Integrating over $y^{\hat{a}}$ gives a delta function $(2\pi)^d \delta^d(k^{\hat{a}} - q^{\hat{a}})$ and imposes momentum conservation through the vertex when $q^{\hat{a}}$ is integrated giving

$$V_M(\partial^2) \;\; = \;\; i \int_k e^{ik\cdot y} \left(\frac{1}{6} - \xi\right) R\, \bar{G}_0^2(k). \tag{D.57}$$

At second adiabatic order, the only other contribution comes from the kinetic vertex, which is given by Fig.(D.1b) with $A = 2$,

$$\begin{aligned} V_K(\partial^2) \;\; &\equiv \;\; \text{kinetic vertex at } O(\partial^2) \\ &= \;\; 2 \int d^d y \, \left(\partial_{\hat{m}} \bar{G}_0(x,y)\right) \left[-\frac{i}{2}\frac{1}{3} R^{\hat{m}\;\;\hat{n}}_{\;\;\hat{a}\;\;\hat{b}}(x') y^{\hat{a}} y^{\hat{b}}\right] \left(\partial_{\hat{n}} \bar{G}_0(y,x')\right). \end{aligned} \tag{D.58}$$

A factor of $\partial_{\hat{a}} \bar{G}_0$ is associated with each external line connected to a kinetic vertex. The partial derivative originates from the derivative acting on the field in (D.27). The kinetic vertex $-i/2K^{\hat{m}\hat{n}}_{(2)}$ connects the propagators and the overall factor of 2 is a symmetry factor.

To calculate this diagram we observe that $x' = 0$ in RNC since it lies at the origin of these coordinates. Furthermore, $\bar{G}_0(y)$ is Lorentz invariant implying that it can be a function of the invariant $y^{\hat{a}} y_{\hat{a}}$ only. A derivative acting on this propagator will therefore be proportional to $y^{\hat{a}}$,

$$\partial_{\hat{n}} \bar{G}_0(y) = \frac{\partial}{\partial y^{\hat{n}}}\, \bar{G}_0(y) \propto y_{\hat{n}}. \tag{D.59}$$

It follows that when this is contracted with the kinetic vertex the entire diagram vanishes because

$$R^{\hat{m}\;\;\hat{n}}_{\;\;\hat{a}\;\;\hat{b}} y^{\hat{a}} y^{\hat{b}} \partial_{\hat{n}} \bar{G}_0(y) \propto R^{\hat{m}\;\;\hat{n}}_{\;\;\hat{a}\;\;\hat{b}} y^{\hat{a}} y^{\hat{b}} y_{\hat{n}} = 0. \tag{D.60}$$

Therefore, $V_K(\partial^2) = 0$.

Adding $V_M(\partial^2)$ and $V_K(\partial^2)$ gives the subleading correction to the flat space propagator

$$V(\partial^2) = V_M(\partial^2) + V_K(\partial^2) \tag{D.61}$$

$$= i \int_k e^{ik\cdot y} \left(\frac{1}{6} - \xi\right) R\, \bar{G}_0^2(k), \tag{D.62}$$

D.2.2.2 Third adiabatic order

At third adiabatic order there are again only two diagrams that contribute. The first comes from the mass vertex with $A = 3$,

$$V_M(\partial^3) = 2 \int d^d y\, \bar{G}_0(x, y) \left[-\frac{i}{2}\left(\xi - \frac{1}{6}\right) R_{;\hat{a}} y^{\hat{a}}\right] \bar{G}_0(y, x'). \tag{D.63}$$

Noting that $\bar{G}_0(y, x') = \bar{G}_0(y)$ (since x' is the origin of RNC) we find from the Fourier transform of the propagator that $y^{\hat{a}}$ becomes a momentum derivative of the propagator,

$$y^{\hat{a}} \bar{G}_0(y) = i \int_q e^{iq\cdot y} \frac{\partial}{\partial q_{\hat{a}}} \bar{G}_0(q). \tag{D.64}$$

Next, integrate $y^{\hat{a}}$ and impose momentum conservation through the vertex by integrating the $q^{\hat{a}}$ momentum to find

$$V_M(\partial^3) = -\int_k e^{ik\cdot y} \left(\frac{1}{6} - \xi\right) R_{;\hat{a}} \bar{G}_0(k) \frac{\partial}{\partial k_{\hat{a}}} \bar{G}_0(k) \tag{D.65}$$

The second contribution comes from the $A = 3$ kinetic vertex and equals

$$V_K(\partial^3) = 2 \int d^d y\, \partial_{\hat{m}} \bar{G}_0(x, y) \left[-\frac{i}{2}\frac{1}{6} R^{\hat{m}}{}_{\hat{a}}{}^{\hat{n}}{}_{\hat{b};\hat{c}} y^{\hat{a}} y^{\hat{b}} y^{\hat{c}}\right] \partial_{\hat{n}} \bar{G}_0(y, x') \tag{D.66}$$

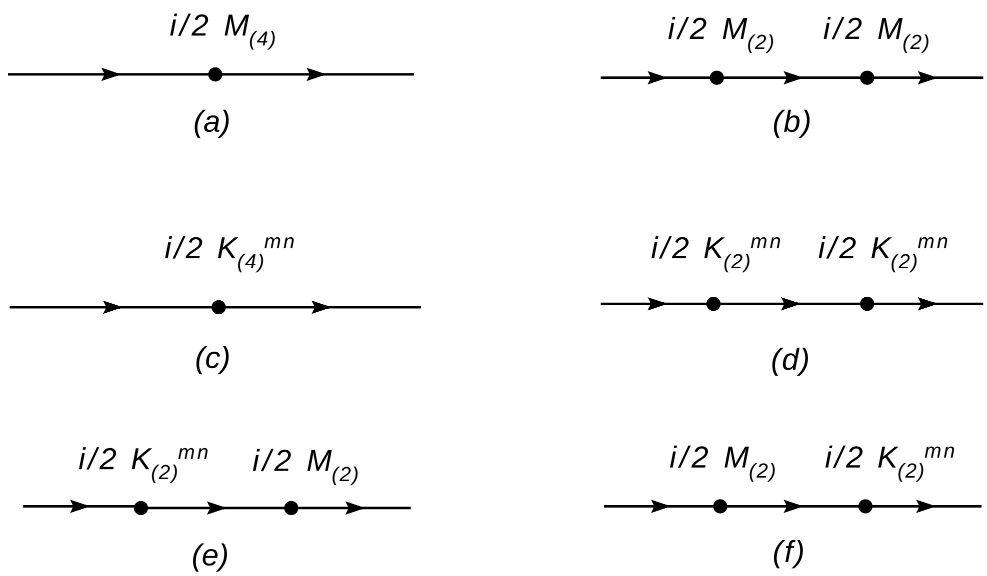

Figure D.2: The six diagrams contributing to the fourth adiabatic order contribution to the propagator.

This also vanishes due to the Lorentz invariance of the leading order propagator,

$$R^{\hat{m}}{}_{\hat{a}}{}^{\hat{n}}{}_{\hat{b};\hat{c}} y^{\hat{a}} y^{\hat{b}} y^{\hat{c}} \partial_{\hat{n}} \bar{G}_0(y, x') \propto R^{\hat{m}}{}_{\hat{a}}{}^{\hat{n}}{}_{\hat{b};\hat{c}} y^{\hat{a}} y^{\hat{b}} y^{\hat{c}} y_{\hat{n}} = 0. \tag{D.67}$$

Therefore, $V_K(\partial^3) = 0$ just as with the second order kinetic vertex.

The total third adiabatic order correction is therefore the sum of these two diagrams

$$V(\partial^3) = V_M(\partial^2) + V_K(\partial^2) \tag{D.68}$$

$$= -\int_k e^{ik\cdot y} \left(\frac{1}{6} - \xi\right) R_{;\hat{a}} \bar{G}_0(k) \frac{\partial}{\partial k_{\hat{a}}} \bar{G}_0(k) \tag{D.69}$$

D.2.2.3 Fourth adiabatic order

At fourth adiabatic order there are six diagrams, as shown in Fig.(D.2). The first comes from $M_{(4)}$ and the second comes from a product of two $M_{(2)}$ vertices.

The calculations proceed similarly as with the previous orders and we find

$$
\begin{aligned}
V_M(\partial^4) &\equiv \text{Fig. D.2a} + \text{Fig. D.2b} && \text{(D.70)} \\
&= i\int_k e^{ik\cdot y}\left\{\left(\frac{1}{6}-\xi\right)^2 R^2\,\bar{G}_0^3(k) + a_{\hat{a}\hat{b}}(x')\,\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\frac{\partial}{\partial k_{\hat{b}}}\bar{G}_0(k)\right\}. && \text{(D.71)}
\end{aligned}
$$

At fourth order, the remaining diagrams in Fig.(D.2) are all zero. These vanish on account of the Lorentz invariance of the leading order propagator. Furthermore, all cross-terms in the $O(\partial^4)$ correction are zero since they are proportional to lower order kinetic vertices. Therefore, only the mass vertices contribute at this order and

$$
\begin{aligned}
V(\partial^4) &= iV_M(\partial^4) && \text{(D.72)} \\
&= i\int_k e^{ik\cdot(x-x')}\left\{\left(\frac{1}{6}-\xi\right)^2 R^2\,\bar{G}_0^3(k) + a_{\hat{a}\hat{b}}(x')\,\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\frac{\partial}{\partial k_{\hat{b}}}\bar{G}_0(k)\right\} && \text{(D.73)}
\end{aligned}
$$

D.2.3 Free field propagator in curved space-time

Putting together the contributions from all of the diagrams through fourth adiabatic order allows us to compute the momentum-space representation of the free field propagator on a background curved space-time. From the definition of the full propagator in (D.51) we have that

$$
i\bar{G}_F(x,x') = i\bar{G}_0(x,x') + iV_M(\partial^2) + iV_M(\partial^3) + iV_M(\partial^4) + O(\partial^5) \quad \text{(D.74)}
$$

are the lowest order non-zero contributions. Multiplying both sides by i and substituting in the expressions for the $iV_M(\partial^n)$ gives, in momentum-space,

$$\begin{aligned}\bar{G}_F(k) &= \bar{G}_0(k) + \left(\frac{1}{6}-\xi\right) R\,\bar{G}_0^2(k) + i\left(\frac{1}{6}-\xi\right) R_{;\hat{a}}\,\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\bar{G}_0(k) \\ &\quad + \left(\frac{1}{6}-\xi\right)^2 R^2\,\bar{G}_0^3(k) + a_{\hat{a}\hat{b}}(x')\,\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\frac{\partial}{\partial k_{\hat{b}}}\bar{G}_0(k) + O(\partial^5)\end{aligned} \quad \text{(D.75)}$$

This agrees with the original derivation by Bunch and Parker [64]. These authors also demonstrate the equivalence of this momentum-space representation to the DeWitt-Schinger proper time representation, which we will not discuss here.

D.2.4 Kinetic vertices do not contribute to the propagator

In the previous section, we give explicit calculations through fourth adiabatic order showing that the diagrams containing any kinetic vertex $K^{\hat{m}\hat{n}}_{(A)}$ do not contribute to the curved space-time free field propagator. We want to show that this is true at all adiabatic orders.

The coordinate transformation from the coordinates x^α to the RNC for the inverse metric is

$$g^{\hat{a}\hat{b}}(y) = \frac{dy^{\hat{a}}}{dx^\alpha}\frac{dy^{\hat{b}}}{dx^\beta}\,g^{\alpha\beta}(x) = \sigma_\alpha{}^{\mu'} e^{\hat{a}}_{\mu'} \sigma_\beta{}^{\nu'} e^{\hat{b}}_{\nu'} g^{\alpha\beta} \quad \text{(D.76)}$$

and the kinetic vertices are defined from (D.32) as

$$\sum_{A=2}^{\infty} K^{\hat{a}\hat{b}}_{(A)}(x', y) = g^{\hat{a}\hat{b}} - \eta^{\hat{a}\hat{b}} \quad \text{(D.77)}$$

All of the diagrams involving the kinetic vertices have the property that there always appear a derivative of the leading order propagator contracted with a $K^{\hat{a}\hat{b}}_{(A)}$. This is clear from the Feynman rules. Since the leading order propagator is Lorentz invariant then its derivative is proportional to $y^{\hat{a}}$ and its contraction with a kinetic vertex is responsible for the vanishing of the first four adiabatic contributions to the full propagator.

To prove that all kinetic vertices vanish to all adiabatic orders we need only to contract the sum of the kinetic vertices (D.77) with $y_{\hat{b}}$ since we know that the kinetic vertices will multiply the derivative of a Lorentz invariant propagator. Let us therefore compute $g^{\hat{a}\hat{b}}y_{\hat{b}}$. We find

$$g^{\hat{a}\hat{b}}y_{\hat{b}} = -\sigma_{\alpha}{}^{\mu'}e^{\hat{a}}_{\mu'}\sigma_{\beta}{}^{\nu'}e^{\hat{b}}_{\nu'}g^{\alpha\beta}e^{\lambda'}_{\hat{b}}\sigma_{\lambda'} = y^{\hat{a}} = \eta^{\hat{a}\hat{b}}y_{\hat{b}} \tag{D.78}$$

where we have used the relations (C.6), (C.7) and the orthonormality of the tetrad

$$e^{\hat{b}}_{\nu'}e^{\lambda'}_{\hat{b}} = g_{\nu'}{}^{\lambda'}. \tag{D.79}$$

It then follows that $\left(g^{\hat{a}\hat{b}} - \eta^{\hat{a}\hat{b}}\right)y_{\hat{b}} = 0$, which implies that the kinetic vertices, when contracted with a derivative of the leading order propagator, will all vanish

$$\sum_{A=2}^{\infty} K^{\hat{a}\hat{b}}_{(A)}(y,x')\partial_{\hat{b}}\bar{G}_0(y) \propto \sum_{A=2}^{\infty} K^{\hat{a}\hat{b}}_{(A)}(y,x')y_{\hat{b}} = 0. \tag{D.80}$$

This is not a proof that each term in the sum is zero when contracted with $y_{\hat{b}}$. Rather, this demonstrates that their sum vanishes, which is enough to show that the kinetic vertices give no contribution to the full propagator for the following reason. All of the kinetic vertices can be collected into a single vertex

$$\mathcal{K}^{\hat{a}\hat{b}}(y,x') \equiv \sum_{A=2}^{\infty} K^{\hat{a}\hat{b}}_{(A)}(y,x') \tag{D.81}$$

We may then compute Feynman diagrams with $\mathcal{K}^{\hat{a}\hat{b}}$ and keep track of the original kinetic vertices at all adiabatic orders through this sum. Because each of the diagrams that contain $\mathcal{K}^{\hat{a}\hat{b}}$ is zero, for the reason just shown, then the kinetic vertices $K_{(A)}^{\hat{a}\hat{b}}$ do not contribute to the free field curved space-time propagator. Therefore, only mass vertices contribute to the subleading terms.

This result can be indirectly seen in the original paper of Bunch and Parker. Their approach involves expanding the equation of motion for the rescaled scalar propagator $\bar{G}$ and then solving this iteratively in powers of derivatives of the background metric (i.e., an adiabatic expansion). In solving the propagator at each adiabatic order Bunch and Parker observe that the Lorentz invariance of the leading order flat space-time propagator causes all terms involving derivatives of $\bar{G}_0$ to cancel completely through fourth adiabatic order. In the language of perturbative quantum field theory these terms correspond to the kinetic vertices and so do not contribute to the full propagator, as we have shown. However, Bunch and Parker do not show that this cancellation happens at all adiabatic orders no do they make the claim. This is one benefit of our approach; at all orders, only the mass vertices contribute to the free field curved space-time propagator.

D.3 Momentum space representation of in-in two-point functions

Having the momentum space representation of the in-out Feynman propagator we recall our arguments from Section D.1. The in-in two-point functions can therefore be written down immediately. Using the notation of (D.10) we find that the rescaled two-point functions (D.4) are

$$\begin{aligned}\bar{G}_{ab}(x,x') \;=\; i\delta_{ab}\int_{\mathcal{C}_{ab},k} e^{ik\cdot y}\Bigg[&\bar{G}_0(k)+\left(\frac{1}{6}-\xi\right)R\,\bar{G}_0^2(k)\\ &+i\left(\frac{1}{6}-\xi\right)R_{;\hat{a}}\,\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\bar{G}_0(k)+\left(\frac{1}{6}-\xi\right)^2 R^2\,\bar{G}_0^3(k)\\ &+a_{\hat{a}\hat{b}}(x')\,\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\frac{\partial}{\partial k_{\hat{b}}}\bar{G}_0(k)+O(\partial^5)\Bigg]\end{aligned} \tag{D.82}$$

where

$$\bar{G}_0(k)=\frac{1}{k^2+m^2+i\epsilon} \tag{D.83}$$

and the contours $\mathcal{C}_{ab}$ are given in Appendix A.

We remark that $\bar{G}_{ab}$ are not the actual two-point functions of the field $\phi(x)$ since we rescaled the field in (D.16). Nevertheless, we can deduce G_{ab} from $\bar{G}_{ab}$ using the relation in (D.47) and the RNC expansion of the metric determinant (C.16)

$$G_{ab}(x,x')=g^{-1/4}(x)\bar{G}_{ab}(x,x') \tag{D.84}$$

where we have also used that $g(x')=1$ in RNC since x' lies at the origin in these coordinates. The metric determinant factor contains powers of the coordinate separation $y^{\hat{a}}$, which can be replaced with

$$y^{\hat{a}}\to -i\frac{\partial}{\partial k_{\hat{a}}} \tag{D.85}$$

Integrating by parts appropriately using the identities

$$\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\bar{G}_0(k) \;=\; \frac{1}{2}\frac{\partial}{\partial k_{\hat{a}}}\bar{G}_0^2(k) \tag{D.86}$$

$$\bar{G}_0(k)\frac{\partial}{\partial k_{\hat{a}}}\frac{\partial}{\partial k_{\hat{b}}}\bar{G}_0(k) \;=\; \frac{1}{3}\frac{\partial}{\partial k_{\hat{a}}}\frac{\partial}{\partial k_{\hat{b}}}\bar{G}_0^2(k)-\frac{2}{3}\eta^{\hat{a}\hat{b}}\bar{G}_0^3(k) \tag{D.87}$$

we determine G_{ab} from the expression for $\bar{G}_{ab}$ given above.

Through third adiabatic order for an arbitrary curved spacetime we find

$$G_{ab}(x,x') = \int_{\mathcal{C}_{ab},k} e^{ik\cdot y}\Bigg[\bar{G}_0(k) + \left(\frac{1}{3}-\xi\right) R\,\bar{G}_0^2(k) - \frac{2}{3}R_{\hat{a}\hat{b}}k^{\hat{a}}k^{\hat{b}}\bar{G}_0^3(k) \\ - i(1-2\xi)R_{;\hat{a}}k^{\hat{a}}\bar{G}_0^3(k) + 2iR_{\hat{a}\hat{b};\hat{c}}k^{\hat{a}}k^{\hat{b}}k^{\hat{c}}\bar{G}_0^4(k) + O(\partial^4)\Bigg] \tag{D.88}$$

and if the background curvature is such that $R_{\mu\nu} = 0$ (i.e. a vacuum spacetime) then the in-in two-point functions have the following momentum space representation through fourth adiabatic order

$$G_{ab}(x,x') = \int_{\mathcal{C}_{ab},k} e^{ik\cdot y}\Bigg[\bar{G}_0(k) + \frac{1}{15}R_{\hat{l}\hat{k}\hat{m}\hat{n}}R^{\hat{l}\hat{k}\hat{m}\hat{n}}\bar{G}_0^3(k) \\ - \frac{8}{15}R_{\hat{l}\hat{k}\hat{m}\hat{a}}R^{\hat{l}\hat{k}\hat{m}}{}_{\hat{b}}k^{\hat{a}}k^{\hat{b}}\bar{G}_0^4(k) + O(\partial^5)\Bigg] \tag{D.89}$$

D.4 Propagator for metric perturbations

We proceed as in the last section and compute, using Riemann normal coordinates, the momentum space representation of the Feynman propagator for metric perturbations in a vacuum background spacetime ($R_{\mu\nu} = 0$). However, we will only derive the propagator to second adiabatic order as this is what is required to regularize the leading order self-force in Chapter 4.

We write the action for the linear metric perturbations (4.24) in an equivalent form

$$S[h_{\mu\nu}] = \frac{1}{2}\int d^dx\, g^{1/2}\left(g^{\mu\nu}P^{\alpha\beta\gamma\delta}(g)h_{\alpha\beta;\mu}h_{\gamma\delta;\nu} - 2R_{(\mu\ \ \nu)}^{\ \ \alpha\ \ \beta}P^{\mu\nu\gamma\delta}(g)h_{\alpha\beta}h_{\gamma\delta}\right) \tag{D.90}$$

which is given in the Lorenz gauge and where

$$P^{\alpha\beta\gamma\delta}(g) = \frac{1}{2}\left(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma} - \frac{2}{d-2}\,g^{\alpha\beta}g^{\gamma\delta}\right), \tag{D.91}$$

which depends on the metric through its inverse. Expanding the covariant derivatives of the metric perturbations using

$$h_{\alpha\beta;\mu} = h_{\alpha\beta,\mu} - \Gamma^{\lambda}_{\mu\alpha}h_{\lambda\beta} - \Gamma^{\lambda}_{\mu\beta}h_{\alpha\lambda}, \tag{D.92}$$

redefining the fields

$$h_{\alpha\beta}(x) = g^{-1/4}(x)\bar{h}_{\alpha\beta}(x) \tag{D.93}$$

and appropriately integrating by parts we find that the action (D.90) can be written in a form that is suitable for an adiabatic expansion in Riemann normal coordinates,

$$\begin{aligned} S[\bar{h}_{\mu\nu}] = \frac{1}{2}\int d^dx\Bigg\{ & g^{\mu\nu}P^{\alpha\beta\gamma\delta}\bar{h}_{\alpha\beta,\mu}\bar{h}_{\gamma\delta,\nu} + g^{-1/4}\left[\left(g^{1/4}\right)_{,\mu}g^{\mu\nu}P^{\alpha\beta\gamma\delta}\right]_{,\nu}\bar{h}_{\alpha\beta}\bar{h}_{\gamma\delta} \\ & - g^{\mu\nu}P^{\alpha\beta\gamma\delta}\Bigg[\Big(-\frac{1}{4}(\ln g)_{,\mu}\bar{h}_{\alpha\beta} + \bar{h}_{\alpha\beta,\mu}\Big)\Big(\Gamma^{\sigma}_{\nu\gamma}\bar{h}_{\sigma\delta} + \Gamma^{\sigma}_{\nu\delta}\bar{h}_{\gamma\sigma}\Big) \\ & \qquad + \Big(-\frac{1}{4}(\ln g)_{,\nu}\bar{h}_{\gamma\delta} + \bar{h}_{\gamma\delta,\nu}\Big)\Big(\Gamma^{\lambda}_{\mu\alpha}\bar{h}_{\lambda\beta} + \Gamma^{\lambda}_{\mu\beta}\bar{h}_{\alpha\lambda}\Big)\Bigg] \\ & + g^{\mu\nu}P^{\alpha\beta\gamma\delta}\Big(\Gamma^{\lambda}_{\mu\alpha}\bar{h}_{\lambda\beta} + \Gamma^{\lambda}_{\mu\beta}\bar{h}_{\alpha\lambda}\Big)\Big(\Gamma^{\sigma}_{\nu\gamma}\bar{h}_{\sigma\delta} + \Gamma^{\sigma}_{\nu\delta}\bar{h}_{\gamma\sigma}\Big) \\ & - 2P^{\mu\nu\gamma\delta}R_{(\mu\ \nu)}^{\ \alpha\ \beta}\bar{h}_{\alpha\beta}\bar{h}_{\gamma\delta}\Bigg\}. \end{aligned} \tag{D.94}$$

We only keep those terms in the action that contribute through second adiabatic order. The connection components are $O(\partial^2)$ quantities implying that the $\Gamma\Gamma$ terms can be ignored. Furthermore, $\partial_m g_{ab}$, $\partial_m g$ and $\partial_m P^{abcd}$ are all second adiabatic order quantities.

Expanding the action using Riemann normal coordinates we find that

$$S[\bar{h}_{\mu\nu}] = S_0[\bar{h}_{\mu\nu}] + S_{int}[\bar{h}_{\mu\nu}] \tag{D.95}$$

where the "non-interacting" theory is described by

$$S_0[\bar{h}_{mn}] = \frac{1}{2}\int d^d y\, \eta^{mn} P^{abcd}(\eta)\bar{h}_{ab,m}\bar{h}_{cd,n} \tag{D.96}$$

and the interaction is given by three kinds of terms,

$$\begin{aligned} S_{int}[\bar{h}_{mn}] \;=\; \frac{1}{2}\int d^d y \sum_{A=2}^{\infty}\Bigg[& K^{mnabcd}_{(A)}(y,x')\bar{h}_{ab,m}\bar{h}_{cd,n} \\ & + \Big(L'^{mabcd}_{(A)}(y,x')\bar{h}_{ab,m}\bar{h}_{cd} + L''^{mabcd}_{(A)}(y,x')\bar{h}_{ab}\bar{h}_{cd,m}\Big) \\ & + M^{abcd}_{(A)}(y,x')\bar{h}_{ab}\bar{h}_{cd}\Bigg] \end{aligned} \tag{D.97}$$

As in the previous section, the interaction terms naturally split into different kinds. The first is a kinetic-type interaction and couples the derivative of the fields. The second is a mass-type interaction and is proportional to $\bar{h}\bar{h}$. The third type of interaction involves a single derivative of the metric perturbation and appears because the graviton is a higher spin field compared to the scalar field of the last section in which such terms do not appear. Let us investigate these three types of contributions to the propagator.

D.4.1 Kinetic vertex

The kinetic vertex can be found from

$$\sum_{A=2}^{\infty} K^{mnabcd}_{(A)}(y,x') = g^{mn}P^{abcd}(g) - \eta^{mn}P^{abcd}(\eta) \tag{D.98}$$

Their contribution to the propagator is simply

$$V^K_{abcd}(\partial^2) = 2\int d^d y \big(\partial_m \bar{G}^0_{abij}(x,y)\big)\left[-\frac{i}{2}K^{mnijkl}_{(2)}(y,x')\right]\big(\partial_n \bar{G}^0_{klcd}(y,x')\big) \tag{D.99}$$

which is found to equal

$$V^K_{abcd}(\partial^2) = iP_{abij}(\eta)K^{mnijkl}_{(2)\quad st}(x')P_{klcd}(\eta)\int_k e^{ik\cdot y}\left[\frac{8}{k^8}\,k_m k_n k^s k^t - \frac{2}{k^6}\left(\eta_n{}^s k_m k^t + \eta_n{}^t k_m k^s + \eta^{st}k_n k_m\right)\right] \tag{D.100}$$

upon using similar manipulations as in the scalar propagator example earlier. Performing the contractions that precede the momentum integral gives our final results for the kinetic vertex

$$V^K_{abcd}(\partial^2) = \frac{i}{3}\int_k e^{ik\cdot y}\frac{k^s k^t}{k^6}\Big[\eta_{ac}R_{bsdt} + \eta_{ad}R_{bsct} + \eta_{bc}R_{asdt} + \eta_{bd}R_{asct} + \frac{4}{(d-2)^2}\Big(\eta_{ab}R_{csdt} + \eta_{cd}R_{asbt}\Big)\Big] \tag{D.101}$$

and includes a dependence on the Riemann curvature.

D.4.2 Mass vertex

Through second adiabatic order the mass vertices are

$$\sum_{A=2}^{\infty} M^{abcd}_{(A)}(y,x') = -\Big(P^{mnab}(\eta)R_{(m\ \ n)}^{\ \ c\ \ d} + P^{mncd}(\eta)R_{(m\ \ n)}^{\ \ a\ \ b}\Big) \tag{D.102}$$

Their contribution to the propagator is trivial and we find that the corresponding diagram is

$$V^M_{abcd}(\partial^2) = 2\int d^dy\,\bar{G}^0_{abij}(x,y)\left[-\frac{i}{2}\,M^{ijkl}_{(2)}(y,x')\right]\bar{G}^0_{klcd}(y,x') \tag{D.103}$$

where the overall factor of 2 is a symmetry factor. We can write the flat space graviton propagator $\bar{G}^0_{abcd}$ in terms of the corresponding propagator for a scalar field $\bar{G}_0$ by noting that (D.96) implies

$$\bar{D}^{abcd}_0(x,x') = P^{abcd}(\eta)\bar{D}_0(x,x') \tag{D.104}$$

Using RNC to construct a momentum space representation for $\bar{G}_0$,

$$\bar{D}_0(x,x') = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot y}}{k^2+i\epsilon} \tag{D.105}$$

$$\equiv \int_k \frac{e^{ik\cdot y}}{k^2+i\epsilon} \tag{D.106}$$

we find that the second adiabatic order mass vertex is

$$V^M_{abcd}(\partial^2) = -iP_{abij}(\eta)M^{ijkl}_{(2)}(x')P_{klcd}(\eta)\int_k e^{ik\cdot y}\frac{1}{(k^2+i\epsilon)^2} \tag{D.107}$$

where the factor multiplying the integral is independent of y^a and is given by

$$P_{abij}(\eta)M^{ijkl}_{(2)}(x')P_{klcd}(\eta) = -\Big(R_{acbd}+R_{adbc}\Big) \tag{D.108}$$

where d is the space-time dimension.

D.4.3 Single-derivative vertices

The self-interaction terms in (D.97) that are linear in a derivative of the metric perturbation are given by

$$\begin{aligned}\sum_{A=2}^{\infty} L'^{mijkl}_{(A)}(y,x') &= \frac{2}{3}\eta^{mn}P^{ijuv}(\eta)\left(R^s{}_{(nu)w}\eta^{(k}_v\eta^{l)}_s + R^s{}_{(nv)w}\eta^{(k}_u\eta^{l)}_s\right)y^w + O(\partial^3)\\ &\equiv L'^{mijkl}_{(2)}{}_w(x')y^w + O(\partial^3).\end{aligned} \tag{D.109}$$

and

$$\begin{aligned}\sum_{A=2}^{\infty} L''^{mijkl}_{(A)}(y,x') &= \frac{2}{3}\eta^{mn}P^{kluv}(\eta)\left(R^s{}_{(nu)w}\eta^{(i}_v\eta^{j)}_s + R^s{}_{(nv)w}\eta^{(i}_u\eta^{j)}_s\right)y^w + O(\partial^3)\\ &\equiv L''^{mijkl}_{(2)}{}_w(x')y^w + O(\partial^3).\end{aligned} \tag{D.110}$$

The first vertex $L'_{(A)}$ contributes the following to the curved space propagator,

$$\begin{aligned} V'^{L}_{abcd}(\partial^2) &= \int d^d y \Big(\partial_m \bar{D}^0_{abij}(x,y)\Big) \left[-\frac{i}{2} L'^{mijkl}_{(2)}{}_{w} y^w\right] \bar{D}^0_{klcd}(y,x') \\ &+ (-i)^2 \int d^d y \, \bar{D}^0_{abij}(x,y) \left[\frac{i}{2} L'^{mijkl}_{(2)}{}_{w} y^w\right] \Big(\partial_m \bar{D}^0_{klcd}(y,x')\Big) \end{aligned} \tag{D.111}$$

We find after an integration by parts that

$$V'^{L}_{abcd}(\partial^2) = \frac{i}{2} P_{abij}(\eta) L'^{mijkl}_{(2)}{}_{m}(x') P_{klcd}(\eta) \int_k e^{ik\cdot y} \frac{1}{(k^2+i\epsilon)^2} \tag{D.112}$$

Similar expressions hold for the $L''_{(2)}$ vertex so that the sum of these two contributions is

$$\begin{aligned} V^{L}_{abcd}(\partial^2) &= V'^{L}_{abcd}(\partial^2) + V''^{L}_{abcd}(\partial^2) &\text{(D.113)} \\ &= \frac{i}{2} P_{abij}(\eta) L^{ijkl}_{(2)}(x') P_{klcd}(\eta) \int_k e^{ik\cdot y} \frac{1}{(k^2+i\epsilon)^2} &\text{(D.114)} \end{aligned}$$

where the vertex $L_{(2)}$ is defined as

$$L^{ijkl}_{(2)}(x') = L'^{mijkl}_{(2)}{}_{m}(x') + L''^{mijkl}_{(2)}{}_{m}(x') \tag{D.115}$$

The factor in front of the momentum space integral evaluates to zero

$$P_{abij}(\eta) L^{ijkl}_{(2)}(x') P_{klcd}(\eta) = 0 \tag{D.116}$$

since it is depends only on the Ricci curvature.

Putting all of these results together we find that the propagator through second adiabatic order is

$$-i\bar{D}_{abcd}(x,x') = -i\bar{D}^0_{abcd} + V^{K}_{abcd}(\partial^2) + V^{M}_{abcd}(\partial^2) + V^{L}_{abcd}(\partial^2) + O(\partial^3) \tag{D.117}$$

where

$$V^K_{abcd}(\partial^2) + V^M_{abcd}(\partial^2) + V^L_{abcd}(\partial^2) = \int_k e^{ik\cdot y}\Bigg\{\frac{i}{(k^2+i\epsilon)^2}\Big(R_{acbd} + R_{adbc}\Big)\Bigg]$$
$$+ \frac{i}{3}\frac{k^s k^t}{(k^2+i\epsilon)^3}\Bigg[\eta_{ac}R_{bsdt} + \eta_{ad}R_{bsct} + \eta_{bc}R_{asdt} + \eta_{bd}R_{asct}$$
$$+ \frac{4}{(d-2)^2}\Big(\eta_{ab}R_{csdt} + \eta_{cd}R_{asbt}\Big)\Bigg]\Bigg\} \qquad \text{(D.118)}$$

Therefore, the momentum space representation of the Feynman propagator for metric perturbations through second adiabatic order is

$$\bar{D}^F_{abcd}(k) = \int_k e^{ik\cdot y}\Bigg\{\frac{P_{abcd}(\eta)}{k^2+i\epsilon} - \frac{1}{(k^2+i\epsilon)^2}\Big(R_{acbd} + R_{adbc}\Big)$$
$$-\frac{1}{3}\frac{k^s k^t}{(k^2+i\epsilon)^3}\Bigg[\eta_{ac}R_{bsdt} + \eta_{ad}R_{bsct} + \eta_{bc}R_{asdt} + \eta_{bd}R_{asct}$$
$$+ \frac{4}{(d-2)^2}\Big(\eta_{ab}R_{csdt} + \eta_{cd}R_{asbt}\Big)\Bigg] + O(\partial^3)\Bigg\} \qquad \text{(D.119)}$$

However, this is the expression for the barred propagator, which is related to the original propagator through (D.93). Realizing that $g^{-1/4}(x') = 1$ when evaluated in Riemann normal coordinates we find that

$$D^F_{abcd}(x,x') = g^{-1/4}(x)\bar{D}^F_{abcd}(x,x') \qquad \text{(D.120)}$$

where, from Appendix C,

$$g^{-1/4}(x) = 1 + \frac{1}{12}R_{mn}y^m y^n + O(\partial^3) = 1 + O(\partial^3) \qquad \text{(D.121)}$$

and so the full propagator for free metric perturbations on a background spacetime does not change,

$$
\begin{aligned}
D_{abcd}(x,x') \;=\; & \int_k e^{ik\cdot y} \Bigg\{ \frac{P_{abcd}(\eta)}{k^2+i\epsilon} - \frac{1}{(k^2+i\epsilon)^2}\Big(R_{acbd}+R_{adbc}\Big) \\
& -\frac{1}{3}\,\frac{k^s k^t}{(k^2+i\epsilon)^3}\bigg[\eta_{ac}R_{bsdt}+\eta_{ad}R_{bsct}+\eta_{bc}R_{asdt}+\eta_{bd}R_{asct} \\
& \qquad + \frac{4}{(d-2)^2}\Big(\eta_{ab}R_{csdt}+\eta_{cd}R_{asbt}\Big)\bigg] + O(\partial^3)\Bigg\} \qquad \text{(D.122)}
\end{aligned}
$$

This is the expression that we use to regularize the leading order self-force for a particle moving in a background vacuum spacetime in Chapter 4. We are unaware that the momentum space representation of the Feynman graviton propagator in (D.122) is given in the literature. The momentum space representation for the in-in two-point functions are obtained by using the appropriate contour $\mathcal{C}_{ab}$ with the k^0 integral.

Appendix E

Distributions, pseudofunctions and Hadamard's finite part

We give a brief review of distribution theory in this Appendix. We present only the basic structure, concepts and definitions that we use in this work. The reader is referred to the excellent text on the subject [126] for more information.

Consider the set of functions ϕ that are infinitely smooth C^∞ and have compact support on any finite interval. These functions, called *testing* or *test functions* form a set $\mathcal{D}$. A *functional* f is a mapping that associates a complex number to every testing function in $\mathcal{D}$. A *distribution* is a linear and continuous functional on the space of test functions $\mathcal{D}$ and is frequently denoted by the symbols $\langle f, \phi \rangle$ and f.

For a locally integrable function $f(t)$ we can associate a natural distribution through the convergent integral

$$\langle f, \phi \rangle \equiv \int_{-\infty}^{\infty} dt\, f(t)\phi(t) \tag{E.1}$$

for some testing function $\phi \in \mathcal{D}$. Notice that we are using the same symbol to denote both the distribution and the function that generates the distribution. This is an example of a regular distribution. All distributions that are not regular are *singular distributions* and will be our main concern in the rest of this Appendix. An example of a singular distribution is the well known delta functional δ. As this is not generated by a locally integrable function $\delta(t)$ (even as the limit of a sequence of locally integrable functions [126]) it must be a singular distribution.

Often, a singular distribution gives rise to a singular integral, which can be written in terms of its divergent and finite parts. For the purposes of clarity and illustration it is best to consider a simple example. Let us compute the integral

$$\left\langle \frac{\theta(t)}{t}, \phi \right\rangle = \int_0^\infty dt\, \frac{\phi(t)}{t} \tag{E.2}$$

for $\phi(t)$ a testing function in $\mathcal{D}$ and $\theta(t)$ the step, or Heaviside, function. This integral is obviously divergent since $1/t$ is not a locally integrable function at the origin. Nevertheless, we may extract the finite part (in the sense of Hadamard [99]) of the integral by isolating the divergences from the finite terms.

To this end we write

$$\phi(t) = \phi(0) + t\,\psi(t) \tag{E.3}$$

where $\psi(t)$ is a continuous function. Putting this into (E.2) and integrating gives

$$\left\langle \frac{\theta(t)}{t}, \phi \right\rangle = \lim_{\epsilon \to 0^+} \left[\phi(0) \log b - \phi(0) \log \epsilon + \int_\epsilon^b dt\, \psi(t) \right] \tag{E.4}$$

where we assume that the testing function $\phi(t)$ vanishes for $t \geq b$ for some real number b. The finite part of (E.2) is defined to be the remainder upon subtracting off the divergent contribution(s). In this case, dropping the $\log \epsilon$ term gives the finite part of the integral

$$Fp \int_0^\infty dt\, \frac{\phi(t)}{t} = \phi(0) \log b + \int_0^\infty dt\, \psi(t) \tag{E.5}$$

where the symbol Fp denotes the finite part of the integral in the sense of Hadamard [99]. Therefore the divergent part of the integral is given by the logarithm $-\phi(0) \log \epsilon$.

A distribution that generates the finite part of the integral is called a *pseudofunction*, which we now calculate for this example. Inserting (E.3) into the finite part (E.5) gives

$$Fp\int_0^\infty dt\,\frac{\phi(t)}{t} = \lim_{\epsilon\to 0^+}\left[\int_\epsilon^\infty dt\,\frac{\phi(t)}{t} + \phi(0)\log\epsilon\right]. \tag{E.6}$$

Since

$$\phi(0) = \int_{-\infty}^\infty dt\,\delta(t)\phi(t) = \langle\delta,\phi\rangle \tag{E.7}$$

it follows that the finite part can be written as an integral of a distribution with a testing function

$$Fp\int_0^\infty dt\,\frac{\phi(t)}{t} = \lim_{\epsilon\to 0^+}\int_\epsilon^\infty dt\left[\frac{1}{t} + \delta(t)\log\epsilon\right]\phi(t), \tag{E.8}$$

which defines the pseudo-function,

$$Pf\frac{\theta(t)}{t} = \frac{\theta(t)}{t} + \delta(t)\log\epsilon. \tag{E.9}$$

Therefore, the finite part of the integral generates a pseudo-function (a regular distribution) that yields a finite value when integrated with a testing function

$$\int_{-\infty}^\infty dt\,Pf\frac{1}{t}\,\phi(t) = Fp\int_{-\infty}^\infty dt\,\frac{\phi(t)}{t}. \tag{E.10}$$

Consider another example [126] and compute the finite part of the integral

$$\int_a^b dt\,\frac{\phi(t)}{(t-a)^k} \tag{E.11}$$

for $a < b$ with a, b real and for k a negative integer. Writing

$$\phi(t) = \phi(a) + (t-a)\phi^{(1)}(a) + \cdots + (t-a)^{k-1}\frac{\phi^{(k-1)}(a)}{(k-1)!} + (t-a)^k\psi(t) \tag{E.12}$$

and inserting into the integral one can show that

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^{b} dt\,\frac{\phi(t)}{(t-a)^k} = \lim_{\epsilon\to 0}\Big[I(\epsilon)+H(\epsilon)\Big] \tag{E.13}$$

where

$$I(\epsilon) = \sum_{n=0}^{k-2}\frac{\phi^{(n)}(a)}{n!(k-1-n)}\,\frac{1}{\epsilon^{k-1-n}} - \frac{\phi^{(k-1)}(a)}{(k-1)!}\,\log\epsilon \tag{E.14}$$

is the divergent part of the integral and

$$H(\epsilon) = \int_{a+\epsilon}^{b} dt\,\psi(t) - \sum_{n=0}^{k-2}\frac{\phi^{(n)}(a)}{n!(k-1-n)(b-a)^{k-1-n}} + \frac{\phi^{k-1}(a)}{(k-1)!}\,\log(b-a) \tag{E.15}$$

is the finite remainder. The finite part of the integral is defined as $H(\epsilon)$ in the limit that $\epsilon\to 0$

$$Fp\int_{a}^{b} dt\,\frac{\phi(t)}{(t-a)^k} = \lim_{\epsilon\to 0} H(\epsilon) = H(0). \tag{E.16}$$

The pseudo-function, which we recall is a regular distribution, generating the finite part can be shown to be

$$\begin{aligned} Pf\frac{\theta(t-a)\theta(b-t)}{(t-a)^k} \quad &= \quad \frac{\phi(t)}{(t-a)^k} - \sum_{n=0}^{k-2}\frac{(-1)^n}{n!(k-1-n)}\,\frac{\delta^{(n)}(t-a)}{\epsilon^{k-1-n}} \\ &\quad + \frac{(-1)^{k-1}}{(k-1)!}\,\delta^{(k-1)}(t-a)\log\epsilon \end{aligned} \tag{E.17}$$

upon following similar steps in our first example. The distribution $\delta^{(n)}(t)$ is the n^{th} derivative of the delta functional and is defined by the distributional identity

$$\big\langle\delta^{(n)},\phi\big\rangle = (-1)^n\big\langle\delta,\phi^{(n)}\big\rangle. \tag{E.18}$$

We remark that the divergent part of the integral $I(\epsilon)$ contains $k-1$ power divergences and one logarithmic divergence. Quite generally, the value that the distribution assigns to a testing function will have a divergent part consisting of both power divergences and powers of logarthmically diverging terms so that

$$I(\epsilon) = \sum_{p=1}^{N} \frac{a_p}{\epsilon^p} + \sum_{p=1}^{M} b_p \log^p \epsilon \tag{E.19}$$

for some appropriate integers N, M. This form for $I(\epsilon)$ is related to the so-called Hadamard's ansatz [99] and appears often in regularizing divergent quantities involving two-point functions of a quantum field in curved spacetime [66, 124].

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