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Mobile Robot Navigation Using Potential Functions

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MOBILE ROBOT NAVIGATION USING POTENTIAL FUNCTIONS

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ABSTRACT

Title of Thesis: Mobile Robot Navigation Using Potential Functions

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This thesis presents a method to construct a smooth obstacle free path for a mobile robot on which to navigate. The first step is to assign potential functions to each obstacle and the goal. Then, the gradient system constructed by the gradient of the sum of the above functions generates the desired path. The construction is analytically proven to produce obstacle free paths to the goal for an environment whose obstacles can be approximated by disks. The procedure does not require complete information on the position of the obstacles beforehand, as long as they can be detected and approximated by disks. The algorithm presented shows a computationally simple way to construct paths and a systematic method to encode the geometric data about the environment into a smooth vector field, which is used for mobile robot navigation.

MOBILE ROBOT NAVIGATION USING POTENTIAL FUNCTIONS

by

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Chapter 1

INTRODUCTION

The problem of obstacle avoidance has been addressed in different contexts and solved with different objectives in mind. For instance, the studies done in [19, 33,32,5] concentrate on the shortest path through a set of polygonal barriers. In each of these studies, given a set of obstacles (which, in general, may be line segments, circles, or polygonal shapes) in the plane, and two distinguished points, X_0 and X^{ob}, they find the shortest path from X_0 to X^{ob} without crossing any of the obstacles.

In the context of the redundant manipulators, the obstacle avoidance problem is addressed to solve the redundancy. In [3,2,1], various methods such as the psuedo inverse method and extended Jacobian method are applied to avoid obstacles using the extra degrees of freedom available in the manipulator. In this way, not only the redundancy in the inverse kinematics problem is solved, but also the manipulator avoids the obstacles.

Another objective in studying the above problem is to determine the complexity of motion planning for multiple independent objects. The problem in this context is defined by J. E. Hopcroft, J. T. Schwartz, and M. Sharir [9] as: Give an initial and final configuration of a set of objects, determine whether there exists a continuous coordinate motion of the objects from the initial to the final configuration during which they do not penetrate either the walls of the enclosing box or each other. They showed even for a simple two dimensional case of coordinate motion planning problem for multiple independent objects is PSPACE-hard.

Another topic of interest is the study of path planning with incomplete information; that is, when an element of uncertainty about the environment is present. V. J. Lumelsky [22] studied this problem and proposed dynamic path planning algorithms, which use sensory information to find a path. The solution to this problem is attractive for application of robotics in the changing environments.

The final approach to obstacle avoidance ,which was originally introduced by O. Khatib [13] is based on the artificial potential field concept. He constructed an artificial potential function $U_{art}(x) = U_{x_d}(x) + U_o(x)$ such that the overall mechanical system subjected to $U_{art}(x)$ is stable and the system drives itself to the desired position while avoiding obstacles. A similar approach without any dynamics in mind is proposed by D. E. Koditschek [18]. He constructs an objective function whose minimum encodes the desired behavior. He shows that the minimum of the constructed function is the global minimum of the function and the value of the function at any point in the work space is lower than its value on the boundary of obstacles. Therefore, the gradient system generated by the above function produces trajectories that avoid obstacles and converge to the minimum point which is the goal.

In this study, we encode the geometric data about the environment into a smooth vector field which is used to generate a smooth obstacle free path to the goal. The idea behind this study originates from the theory of electric and magnetic fields. The robot and obstacles are considered to be positively charged objects whereas the goal is negatively charged. Since objects of the same sign repel and objects of opposite sign attract each other in electricity, the robot avoids the obstacles while being pulled toward the goal, the attractor.

All the obstacles in the environment, given through sensing devices, are approximated by disjoint disks such that the size of the robot is also incorporated in the size of the disks. Therefore, the mobile robot is considered to be a point robot. Then, assigning potential functions with certain spherical symmetry property and range to the obstacles and the goal, a gradient vector field is generated. Next, by assigning a constant weight large enough to the potential function corresponding to the goal, we show that all the trajectories of the gradient system generated by the sum of the above potential functions approach the goal and do not get trapped in any other equilibrium point of the system for all initial points in the work space except a set of points of measure zero.

In Chapter 2, we define a collection of potential functions with five distinguished properties. Then, we produce a collection of potential functions which have the desired properties. In Chapter 3, we show that the finite sum of the above collection of potential functions with appropriate weighting factors generates a gradient system whose equilibria are unstable outside the obstacles provided the magnitude of the weighting factor on the potential function assigned to the goal is large enough. The analytical justifications in this Chapter are a combination of geometric and vectorial analysis. We extensively use the circular symmetry of the potential functions to justify our analysis. In Chapter 4, using the collection of potential functions derived in Chapter 2, we simulate our theoretical results to show their correctness. For a smooth choice of potential functions, it is evident from our construction and analysis that the trajectories of the gradient system are at best smooth (when one weighting factor is chosen for the attractor for all time) or piecewise smooth (when extra obstacles are detected during motion or different weighting factors are used for the attractor). In Chapter 5, we review the thesis and introduce new problems for the future research.

Chapter 2

POTENTIAL FUNCTIONS

The term potential function in the theory of Newtonian potentials refers to a single-valued function of position which is independent of the reference coordinate system. The potential at a given point in space due to a given attracting system (system of gravitating particles, point charges, or magnetic poles) is the work that would be done by the system on a particle (point charge, or magnetic pole) of unit mass (charge or magnet) as it moves along any path from an infinite distance to the point considered. This function for a positive unit point charge, particle, or magnetic pole is given by the inverse of the distance of the unit to the objective point, and denoted by $\frac{1}{r}$. Denoting the potential function by U, the potential of line, surface, and volume distributions are defined by corresponding integrals of the densities times $\frac{1}{r}$. That is,

$$U = \int_c \frac{\lambda}{r} ds \tag{2.1}$$

$$U = \int\int_S \frac{\sigma}{r} dS \tag{2.2}$$

$$U = \int\int\int_V \frac{\gamma}{r} dv \tag{2.3}$$

The potential field is defined by the gradient of the potential function and it takes positive or negative sign depending on whether the elements of like sign attract or repel. In electricity and magnetism, the potential field is given by $-gradU$.

In section one, we will introduce a class of potential functions which have some extra properties. This class of functions will prove useful in our future analysis. In section two, we will derive a potential function with the properties of the functions in section one.

2.1 Desired Collection of Potential Functions

The collection of functions that we will define in this section is a subset of the potential functions defined previously. That is, they vanish at infinity, are single-valued functions, and do not depend on the reference coordinate system. Since the scope of this thesis is limited to the study of mobile robot navigation on a flat surface, we restrict our potential functions to the plane. Formally, we denote any collection of potential functions with the following properties by S_d.

Definition 2.1.1 *S_d is a set of potential functions $f_i : \Re^2 \longrightarrow \Re$ which are twice continuously differentiable and have following properties:*

1. *All the equipotential curves of f_i are circles centered at $X_i \in \Re^2$.*

2. *There exists a constant $r_i \in \Re$ such that*

$$\begin{aligned}\lim_{\|X-X_i\|\to r_i} f_i(X) &= \infty \\ \lim_{\|X-X_i\|\to r_i} \|\nabla f_i(X)\| &= \infty\end{aligned}$$

3. $$\|\nabla f_i(X)\| \geq \frac{1}{\|X - X_i\| - r_i} \tag{2.4}$$

   for all $X \in \{X| \ r_i < \|X - X_i\| < r_i + 1\}$.

4. *If f and g belong to S_d then $\lim_{\|x\|\to\infty} \frac{\|\nabla f(X)\|}{\|\nabla g(X)\|}$ exists and is finite.*

5. $\|\nabla f_i(\hat{X})\| < \|\nabla f_i(\tilde{X})\|$ *if* $\|\hat{X} - X_i\| > \|\tilde{X} - X_i\|$ *for all* $\hat{X}$ *and* $\tilde{X}$ *in* $\Re^2 - \{X \mid \|X - X_i\| \leq r_i\}$.

The properties 1 through 4 imply that all functions in S_d have circular symmetry, their gradient has no zero outside the circle $\|X - X_i\| = r_i$, and becomes arbitrary large close to a fixed circle for a given function. In the next section, we will illustrate a way to construct such a collection of functions.

2.2 Potential of a Homogeneous Circumference

In this section, we will derive the potential due to a positively charged circular wire. The potential function is given by the line integral equation [2.2]. Before carrying out the integral, let us define the variables used in the figure [2.1] and elsewhere in the derivation.

C_i denotes a circular wire with radius r_i and center $X_i = (x_i, y_i)$.

P denotes any point $(x, y) \in D_i^c = \{(x, y) \mid [(x - x_i)^2 + (y - y_i)^2]^{\frac{1}{2}} > r_i\}$.

$d_i = [(x - x_i)^2 + (y - y_i)^2]^{\frac{1}{2}}$ for all $(x, y) \in D_i^c$.

$p_i = d_i + r_i$ which is the maximum distance from P to C_i.

$q_i = d_i - r_i$ which is the minimum distance from P to C_i.

$\lambda_i =$ the charge density on wire C_i.

$U_i(P) =$ the potential at point P due to C_i.

$\vec{r} = (x, y)^T$; $\vec{\acute{r}} = (x_i + r_i \cos\theta, y_i + r_i \sin\theta)^T$.

The potential function U_i at P is given by:

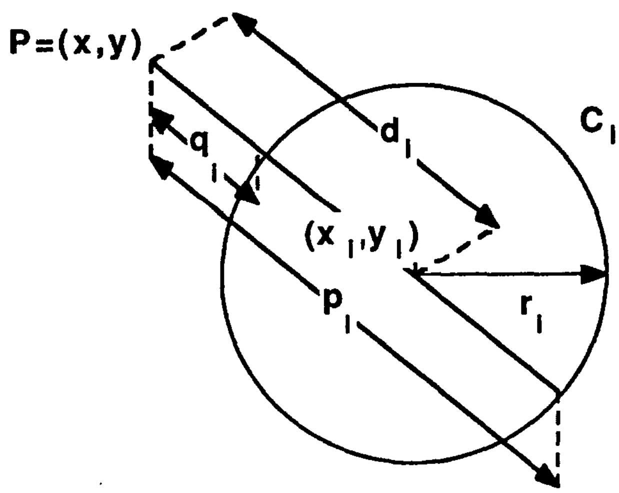

Figure 2.1: The Homogeneous Circumference.

$$U_i(P) = \int_{C_i} \frac{\lambda_i dL'}{\|\vec{r} - \vec{\hat{r}}\|} \qquad (2.5)$$

where $dL' = r_i d\theta$ is an infinitesimal segment of C_i. Assume that the charge density λ_i is constant. Then,

$$U_i(P) = 4\lambda_i r_i V_i(P)$$

where

$$\begin{aligned} V_i(P) &= \frac{1}{4}\int_0^{2\pi} \frac{d\theta}{\|\vec{r} - \vec{\hat{r}}\|} \\ &= \frac{1}{4}\int_0^{2\pi} \frac{d\theta}{[(x - x_i - r_i\cos\theta)^2 + (y - y_i - r_i\sin\theta)^2]^{\frac{1}{2}}} \\ &= \frac{1}{4}\int_0^{2\pi} \frac{d\theta}{[\frac{p_i^2+q_i^2}{2} + \frac{p_i^2-q_i^2}{2}\cos(\theta - \alpha)]^{\frac{1}{2}}} \end{aligned}$$

where

$$\frac{p_i^2 + q_i^2}{2} = d_i^2 + r_i^2$$

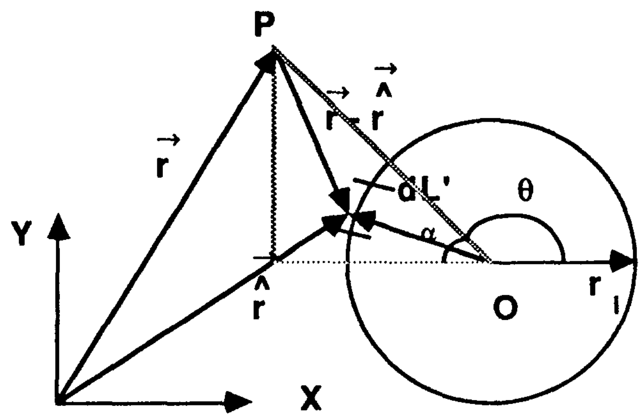

Figure 2.2: Illustrative Diagram for the Potential Function Integral.

$$\frac{p_i^2 - q_i^2}{2} = 2r_i d_i$$

$$tan(\alpha) = \frac{y_i - y}{x_i - x} \tag{2.6}$$

Let $2\psi = \theta - \alpha$ and use half angle formulae for cosine, we get:

$$V_i(P) = \frac{1}{2p_i} \int_{-\frac{\alpha}{2}}^{\pi - \frac{\alpha}{2}} \frac{d\psi}{[1 - k_i^2 \sin^2(\psi)]^{\frac{1}{2}}}$$

where

$$k_i^2 = 1 - \frac{q_i^2}{p_i^2}. \tag{2.7}$$

Using the even symmetry in the integrand and the fact that the integral depends on the limit width not the actual limit points, we have:

$$V_i(P) = \frac{1}{p_i} \int_0^{\frac{\pi}{2}} \frac{d\psi}{[1 - k_i^2 \sin^2(\psi)]^{\frac{1}{2}}} \tag{2.8}$$

where the integral is the elliptic integral of the first kind, $K(k_i)$,

$$K(x) = \frac{\pi}{2}\{1 + (\frac{1}{2})^2 x^2 + \cdots + [\frac{(2n-1)!!}{2^n n!}]^2 x^{2n} + \cdots\} \tag{2.9}$$

and it is defined for $x < 1$. Therefore, the potential function is given by the following equation:

$$U_i(P) = \frac{4\lambda_i r_i}{p_i} K(k_i) \tag{2.10}$$

From equation [2.10], one can obtain the gradient,

$$(U_i)_x(P) = -\frac{4\lambda_i r_i(x - x_i)}{d_i p_i^2}[\frac{2r_i q_i}{p_i^2 k'^2_i k_i^2} E(k_i) + (1 - \frac{2r_i q_i}{p_i^2 k_i^2})K(k_i)], \tag{2.11}$$

$$(U_i)_y(P) = -\frac{4\lambda_i r_i(y - y_i)}{d_i p_i^2}[\frac{2r_i q_i}{p_i^2 k'^2_i k_i^2} E(k_i) + (1 - \frac{2r_i q_i}{p_i^2 k_i^2})K(k_i)]. \tag{2.12}$$

where $E(k_i)$ is the elliptic integral of the second kind.

$$E(k_i) = \int_0^{\frac{\pi}{2}} [1 - k_i^2 \sin^2 \psi]^{\frac{1}{2}} d\psi \tag{2.13}$$

$$E(k_i) = \frac{\pi}{2}\{1 - \frac{1}{2^2}k_i^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2}k_i^4 - \cdots - [\frac{(2n-1)!!}{2^n n!}]^2 \frac{k_i^{2n}}{2n-1} - \cdots\} \tag{2.14}$$

$$k'^2_i = 1 - k_i^2$$

Now, to show that $\{U_1, U_2, \ldots, U_m\}$ for $m > 1$ satisfies the properties of the set of functions , S_d, one has to check for the properties 1 through 4 in Definition 2.1.1. To check for the first one, we have to find out where $U_i(P)$ takes a constant value. Let us try to write everything as a function of d_i.

$$U_i(P) = \frac{4\lambda_i r_i}{d_i + r_i} K[(1 - \frac{(d_i - r_i)^2}{(d_i + r_i)^2})^{\frac{1}{2}}]$$

Since λ_i and r_i are constants, U_i depends only on d_i. Therefore, letting d_i equal to a constant will make U_i a constant.

$$\begin{aligned} d_i &= c \\ (x - x_i)^2 + (y - y_i)^2 &= c^2 \end{aligned}$$

Consequently, the equipotential lines of U_i are circles or annular regions centered at (x_i, y_i). However, since $\nabla U_i(X)$ is nonvanishing everywhere except at the center of the wire C_i, the isopotential regions can not be annular regions.

It is known that the elliptic integral of the first kind converges for $k_i \in [0, 1)$, and diverges for $k_i = 1$. Therefore, to check for the second property, one has to check where $k_i = 1$. Since p_i is always positive, if

$$k_i^2 = 1,$$

then

$$q_i = 0.$$

In other words,

$$\sqrt{(x - x_i)^2 + (y - y_i)^2} = r_i.$$

Therefore, U_i goes to infinity as the point P approaches the wire C_i. To verify that the magnitude of the gradient of the potential function U_i goes to infinity as $\|X - X_i\| \to r_i$, we compute $\|\nabla U_i(X)\|$.

$$\|\nabla U_i(X)\| = \frac{4\lambda_i r_i}{p_i}[\frac{2r_i}{q_i k_i^2}E(k_i) + (1 - \frac{2r_i q_i}{p_i^2 k_i^2})K(k_i)] \tag{2.15}$$

Now, considering the following limits and the fact that $E(k_i)$, as defined by equation [2.13], is positive for $k_i \in [0, 1]$, we have:

$$\begin{aligned} \lim_{\|X - X_i\| \to r_i} k_i &= 1 \\ \lim_{\|X - X_i\| \to r_i} K(k_i) &= \infty \\ \lim_{\|X - X_i\| \to r_i} (1 - \frac{2r_i q_i}{p_i^2 k_i^2}) &= 1 \\ \lim_{\|X - X_i\| \to r_i} \frac{2r_i}{q_i k_i^2} E(k_i) &= \infty. \end{aligned}$$

Therefore,

$$\lim_{\|X - X_i\| \to r_i} \|\nabla U_i(X)\| = \infty.$$

Directly from equation [2.15] and the above limits, we can immediately get the third property; that is,

$$\|\nabla U_i\| \geq \frac{1}{q_i} = \frac{1}{\|X - X_i\| - r_i}.$$

To verify the forth property, we choose two circular wires with distinct centers. First, we compute the norm $\|\nabla U_i(X)\|$ by introducing a new variable $z_i = \frac{q_i}{p_i}$. Therefore,

$$d_i = \frac{1 + z_i}{1 - z_i} r_i$$

$$\|\nabla U_i(X)\| = \frac{\lambda_i(1 - z_i)^2}{r_i} \left| \frac{E(k_i) + z_i K(k_i)}{z_i(1 + z_i)} \right| \tag{2.16}$$

To compute $\lim_{\|X\| \to \infty} \frac{\|\nabla U_i(X)\|}{\|\nabla U_j(X)\|}$, one should recall the following limits:

$$\begin{aligned}
\lim_{\|X\| \to \infty} z_i &= 1 \\
\lim_{\|X\| \to \infty} k_i &= 0 \\
\lim_{\|X\| \to \infty} K(k_i) &= \frac{\pi}{2} \\
\lim_{\|X\| \to \infty} E(k_i) &= \frac{\pi}{2} \\
\lim_{\|X\| \to \infty} \frac{\|\nabla U_i(X)\|}{\|\nabla U_j(X)\|} &= \frac{\lambda_i r_i r_j^2}{\lambda_j r_j r_i^2} \lim_{\|X\| \to \infty} \frac{(1 - z_i)^2}{(1 - z_j)^2} \\
&= \frac{\lambda_i r_i}{\lambda_j r_j} \lim_{\|X\| \to \infty} \frac{d_j^2}{d_i^2} \\
&= \frac{\lambda_i r_i}{\lambda_j r_j}.
\end{aligned}$$

Therefore, the forth property is established. To show the last property, we differentiate the equation [2.16] with respect to z_i and show that it is negative for all $z_i \in (0,1)$.

$$
\begin{aligned}
& \frac{d}{dz_i}\{\frac{\lambda_i(1-z_i)^2(E(k_i)+z_iK(k_i))}{r_i(z_i+z_i^2)}\} \\
= \; & \frac{2\lambda_i(1-z_i)}{r_i(z_i^2+z_i)}[E(k_i)+z_iK(k_i)] - \frac{\lambda_i(1-z_i)^2(2z_i+1)}{r_i(z_i^2+z_i)^2} \times \\
& [E(k_i)+z_iK(k_i)] + \frac{\lambda_i(1-z_i)^2}{r_i(z_i^2+z_i)} \\
& \times[\frac{z_i}{k_i^2}(K(k_i)-E(k_i)) + K(k_i) - z_i^2(\frac{E(k_i)}{k'^2_i k_i^2} - \frac{K(k_i)}{k_i^2})] \\
= \; & -\frac{\lambda_i(1-z_i)}{r_i(z_i^2+z_i)^2}[2z_i^2K(k_i) + (z_i^2+4z_i+1)E(k_i)] \\
< \; & 0.
\end{aligned}
$$

The above inequality follows since everything to the right of the negative sign is positive for $z_i \in (0,1)$. This concludes the verification of the properties of S_d for the set of potential functions U_i.

Chapter 3

MATHEMATICAL MODELING OF THE WORK SPACE

From the theory of electric or magnetic field, we know that the charges or magnets of the same sign repel each other. One can use this property to model the work space of a mobile robot. The idea is that of assigning some charge density of the same sign to the obstacles and the mobile robot. Then, by assigning an opposite sign for the goal, one can make the mobile robot attracted to the goal while avoiding the obstacles by repulsive force. There are two important questions associated with the above construction.

1. How strong should the attractor be to cause the mobile robot to attain the goal?
2. How can the mobile robot avoid possible stable critical points of the potential field?

The answers to the above questions are provided in the sequel. In section one, we define an obstacle and explain how the robot would determine its location. In section two, we construct the potential field and explain why this field would enable the robot to avoid the obstacles. Then, in the following two sections, we answer the above two questions, and show that this field actually generates an obstacle free path to the vicinity of the goal for a special case.

3.1 General Overview of the Problem

Mobile robot navigation is not just a problem of path planning, it is also a problem of obstacle recognition and sensing. It is unrealistic to assume that all obstacles in the operational environment have been identified. There is always an element of uncertainty associated with the environment. To detect such uncertainties, we assume that our robot is capable of exact position sensing of the obstacles and all the obstacles are stationary. Since the mobile robot is navigating on a flat surface, we assume that the obstacles are the projection of three dimensional objects on the plane of motion. Formally, an obstacle is defined as:

Definition 3.1.1 *An obstacle is the smallest disk in the plane of motion that contains the projection of a given object onto the plane.*

Similarly, a robot is defined as:

Definition 3.1.2 *A mobile robot is the smallest disk in the plane of motion that contains the projection of the actual robot onto the plane of motion.*

To further simplify the problem, we reduce the disk shaped robot to a point robot by increasing the size of the obstacles by the radius of the robot. Therefore, from now on, we assume that all the obstacles are the extended ones and the robot is a point robot. Furthermore, any uncertainty in the work space enters in a manner consistent with the assumptions of the given problem. For example, in the problem of a disjoint set of obstacles, an unidentified obstacle can enter only in a disjoint manner. In the following section, we use the collection of functions , S_d, to model the work space.

3.2 Modeling Obstacle Free Path as the Solution of a Gradient System

Consider a set of m (extended) disk shaped obstacles denoted by:

$$D_i^{r_i} = \{X \mid \|X - X_i\| \leq r_i\} \text{ for } i = 1, \ldots, m.$$

The robot is a point robot. Consider a positive number δ, then the work space is defined by:

Definition 3.2.1 *The work space $W(\delta)$ of a mobile robot is a set of points X in $\Re^2$ such that*

$$X \in \{X \mid \|X - X_i\| \geq r_i + \delta\} \textit{ for } i = 1, \ldots, m$$

Let $f_1, f_2, \ldots, f_m$ be functions in S_d such that the center of the equipotential curve of the i^{th} function coincides with the center of the i^{th} obstacle and

$$\lim_{\|X - X_i\| \to r_i} f_i(X) = \infty \text{ for } i = 1, \ldots, m.$$

That is, the radius of the i^{th} obstacle corresponds to the constant in the property 2 of the set S_d. Let $\lambda_1, \ldots, \lambda_m$ be positive constants and λ_0 be a negative number. Also consider $f_0 \in S_d$ with its center at the objective point and the radius r_0 small enough such that the disk (X^{ob}, r_0) does not intersect any obstacles. Note that f_0 is such that $\lim_{\|X - X^{ob}\| \to r_0} f_0(X) = \infty$. The circle (X^{ob}, r_0) is called the attractor because the function f_0 is always weighted with a negative number λ_0. The following gradient system produces trajectories that avoid all obstacles.

Definition 3.2.2 *Let us define the system Σ by :*

$$\dot{X} = -\sum_{i=0}^{m} \lambda_i \nabla f_i(X)$$

$$\begin{aligned} X(0) &= X_0 \\ \lambda_0 &< 0 \\ \lambda_i &> 0 \end{aligned}$$

for all $X_0 \in W$ and $i = 1, \ldots, m$.

The system Σ is a gradient system whose trajectories flow in a direction of decreasing the potential function $V(X) = \sum_{i=0}^{m} \lambda_i f_i(X)$. Since V(X) is infinity at the boundary of the obstacles, starting from any point in W, the system can not generate a trajectory that crosses an obstacle. This can be seen from the fact that at any point in W, $V(.)$ has a lower value than it would have at the boundary of any obstacle. Therefore, it is impossible for a trajectory of the system Σ to cross an obstacle while moving in the direction of decreasing V(X).

In the next section, we show that all the solutions of the system Σ are bounded for $\mid \lambda_0 \mid$ large enough. Furthermore, we show that there is no equilibrium point in W for $\mid \lambda_0 \mid$ large.

3.3 Study of the Critical Points and the Trajectories

To show that all trajectories of Σ are bounded for $\mid \lambda_0 \mid$ large enough, we show that there exists a closed curve around the obstacles and the objective point such that the vector field points inward on that curve.

Lemma 3.3.1 $\sup_{X \in W} \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|}$ *is a finite number.*

Proof: f_i and f_0 belong to S_d; therefore,

$$\lim_{\|X\| \to \infty} \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|} = L < \infty$$

and $\frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|}$ is a continuous function on W. By definition of limit, for all positive numbers ϵ, there exists a number $N_1 > 0$ such that

$$\|X\| > N_1 \Longrightarrow \left| \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|} - L \right| < \epsilon$$

for all $X \in W$. In particular, for $\epsilon = \epsilon_0$ (ϵ_0 fixed positive number), there exists a number $N_2 > 0$ such that

$$\|X\| > N_2 \Longrightarrow \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|} < L + \epsilon_0$$

for all $X \in W$.

Let $\overline{N}$ be such that

$$\bigcup_{i=1}^{m} D_i^{r_i+\delta} \subset \{X \mid \|X - X_0\| \leq \overline{N}\}$$

Then, let $N = \max\{\overline{N}, N_2 + \|X_0\|\}$ and

$$Q = W \cap \{X \mid \|X - X_0\| \leq N\}$$

Note that Q is a compact set; therefore, $\frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|}$ has a maximum on Q.

$$\overline{M}_i = \max_{X \in Q} \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|}$$

and on $\{X \mid \|X - X_0\| > N\}$

$$\frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|} < L + \epsilon_0$$

Therefore,

$$\sup_{X \in W} \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|} = \max\{\overline{M}_i, \epsilon_0 + L\} = M_i. \quad \Box$$

Now, we are ready to prove the next theorem whose proof uses the above lemma.

Theorem 3.3.1 *The trajectories $X(t, X_0)$ of the system Σ are bounded for $X_0 \in W(\delta)$ such that $\|X_0 - X^{ob}\| < L < \infty$ and $|\lambda_0| > M \sum_{i=1}^{m} \lambda_i$ where*

$$M = \max_{1 \leq i \leq m} \sup_{X \in W(\delta)} \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|}. \tag{3.1}$$

Proof: Let the compact set R be defined by

$$R = \{X \mid \|X - X^{ob}\| \leq \max\{\|X_{i_{max}} - X^{ob}\| + r_{i_{max}} + \frac{\alpha_0}{2}, \|X^{ob} - X_0\|\}\}$$

where

$$\alpha_0 = \sup\{\alpha \in \Re^+ : D_i^{r_i + \alpha} \cap D_j^{r_j + \alpha} = \emptyset; 0 \leq i \neq j \leq m\}$$

and

$$i_{max} = \arg \max_{1 \leq i \leq m} \{\|X_i - X^{ob}\| + r_i\}.$$

α_0 in the above is the largest positive number by which the disks (obstacles) can be extended without intersecting. i_{max} is the index of the farthest obstacle from the objective point, X^{ob}.

To prove this theorem, we need to show that

$$< -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), \frac{X^{ob} - X}{\|X^{ob} - X\|} > \; > 0$$

for all X on the boundary of R.

$$\begin{aligned} & < -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), \frac{X^{ob} - X}{\|X^{ob} - X\|} > \\ = \;& |\lambda_0| < \nabla f_0(X), \frac{X^{ob} - X}{\|X^{ob} - X\|} > - \sum_{i=1}^{m} \lambda_i < \nabla f_i(X), \frac{X^{ob} - X}{\|X^{ob} - X\|} > \\ = \;& |\lambda_0| \|\nabla f_0(X)\| < \frac{X^{ob} - X}{\|X^{ob} - X\|}, \frac{X^{ob} - X}{\|X^{ob} - X\|} > \end{aligned}$$

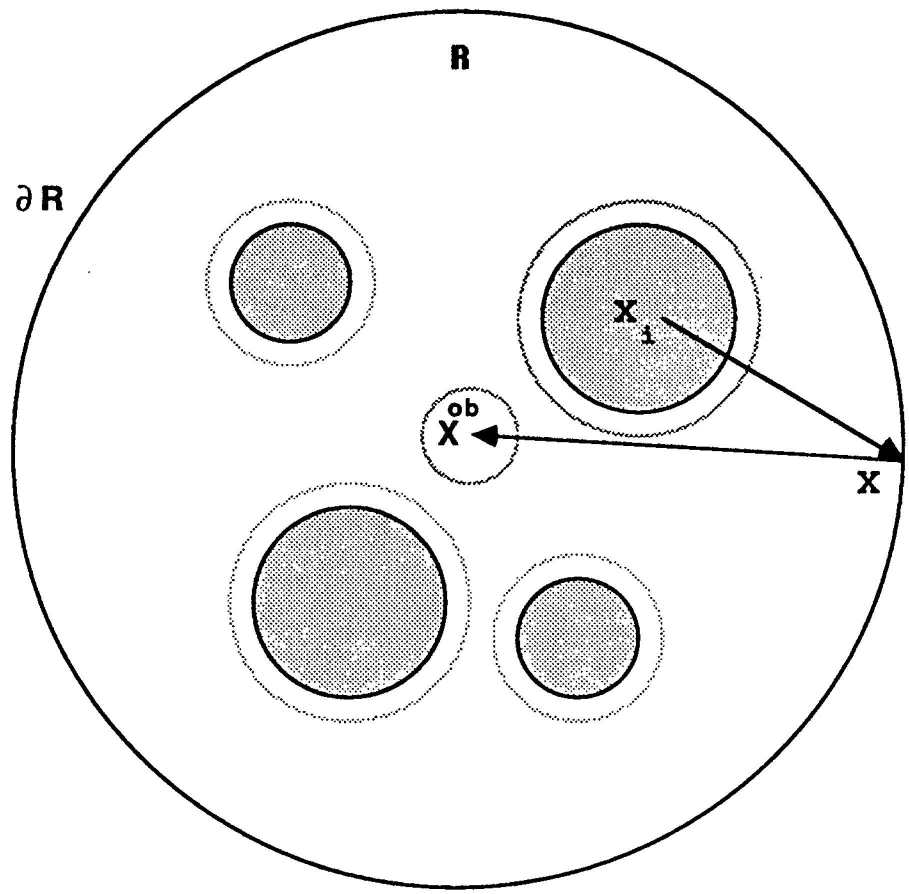

Figure 3.1: An illustrative figure for theorem 3.3.1.

$$+\sum_{i=1}^{m} \lambda_i \|\nabla f_i(X)\| < \frac{X - X_i}{\|X - X_i\|}, \frac{X^{ob} - X}{\|X^{ob} - X\|} >$$

$$\geq \ |\lambda_0| \|\nabla f_0(X)\| - \sum_{i=1}^{m} \lambda_i \|\nabla f_i(X)\|$$

$$= \ |\lambda_0| \|\nabla f_0(X)\| [1 - \sum_{i=1}^{m} \frac{\lambda_i \|\nabla f_i(X)\|}{|\lambda_0| \|\nabla f_0(X)\|}]$$

$$> \ 0$$

where the last inequality holds by the hypothesis. Therefore, all the trajectories of the system Σ starting from any point in $W(\delta)$ stay in the compact set R and they are bounded. □

Assuming that the system Σ has finitely many equilibria, from this theorem and the fact that the trajectories of the system Σ flow in the direction of decreasing $V(X) = \sum_{i=0}^{m} \lambda_i f_i(X)$, we can conclude that only two possibilities exist:

1. The trajectory $X(t, X_0)$ converges to a local minimum of the potential function, and gets trapped there, or

2. the trajectory moves toward the attractor (circle (X^{ob}, r_0)) in which case the obstacle free path from the initial point, X_0, to the vicinity ($r_0 + \delta$ units away from the goal) of the objective point, X^{ob}, is given by $X(t, X_0)$

for all finite norm points $X_0 \in W(\delta)$ except possibly a set of points of measure zero. Remark: The distance $r_0 + \delta$ from the goal, by definition of r_0, does not contain any obstacles. Therefore, once the robot is in that vicinity, we can turn the system off and run a new system such as the one generated by a potential shaped as a quadratic well whose center is at the goal. Of course all this is possible if the system does not have any stable equilibria outside the obstacles.

To be able to say more about the behavior of the system Σ, we show that for the value of $\mid \lambda_0 \mid$ as in the theorem [3.3.1], there are no equilibrium points in $W(\delta)$.

Theorem 3.3.2 *The system Σ does not have any equilibrium points in $W(\delta)$ for*

$$\mid \lambda_0 \mid > M \sum_{i=1}^{m} \lambda_i \text{ for all } X \in W(\delta)$$

and M as in theorem [3.3.1].

Proof: To prove the theorem, need to show that

$$\| \sum_{i=0}^{m} \lambda_i \nabla f_i(X) \| > 0 \text{ for all } X \in W(\delta)$$

$$\begin{aligned}
\|\sum_{i=0}^{m}\lambda_i\nabla f_i(X)\| &= \|\sum_{i=1}^{m}\lambda_i\nabla f_i(X) - |\lambda_0|\nabla f_0(X)\| \\
&\geq |\|\sum_{i=1}^{m}\lambda_i\nabla f_i(X)\| - |\lambda_0|\|\nabla f_0(X)\|| \\
&= |\lambda_0|\|\nabla f_0(X)\| \cdot |\|\sum_{i=1}^{m}\frac{\lambda_i\nabla f_i(X)}{|\lambda_0|\|\nabla f_0(X)\|}\| - 1| \\
&> 0
\end{aligned}$$

because,

$$\|\sum_{i=1}^{m}\frac{\lambda_i\nabla f_i(X)}{|\lambda_0|\|\nabla f_0(X)\|}\| \leq \sum_{i=1}^{m}\frac{\lambda_i\|\nabla f_i(X)\|}{|\lambda_0|\|\nabla f_0(X)\|} < 1$$

for $|\lambda_0|$ as in the hypothesis. Therefore, the assertion follows, and the system can not have any equilibrium points on $W(\delta)$. □

This implies that if there are any equilibrium points, they must belong to the complement of $W(\delta)$. However, since the equilibrium points inside the obstacles can not be reached, the ones that we care about must belong to

$$A = \bigcup_{i=1}^{m}\{X \mid r_i < \|X - X_i\| < r_i + \delta\}$$

In the next section, we see how the results in this section can be applied to a special case to prove that the trajectories of the system Σ are not trapped by any stable equilibria and they flow toward the attractor which is given by the circle (X^{ob}, r_0).

3.4 Special Case Analysis of the Equilibrium Points

In the previous section, we discussed the boundedness of the trajectories of the system Σ and approximated the location of the zeroes of the system. To learn more about the equilibria, one can study their stability. There are two approaches that one can take:

1. Compute the equilibrium points exactly and then using various methods such as Lyapunov analysis or linearization of the system about the equilibrium points, one can identify the stable equilibrium points of the system.

2. To show an equilibrium point is unstable, one needs to show that at any arbitrarily small neighborhood of the critical point, X_c, there exists a point, $\hat{X}$, such that the vector field at $\hat{X}$ pointing away from the equilibrium point. More precisely, the inner product of the vector $X_c - \hat{X}$ and the vector field evaluated at $\hat{X}$ is negative.

In this section, we show that all the equilibria of the system outside the obstacles are unstable if the obstacles are disjoint. This implies that since the system does not have any stable equilibria outside the obstacles, the trajectories do not get trapped. To prove this claim, we use a modified version of the second approach.

In section [3.3], it was shown that all the equilibrium points of the system Σ outside the obstacles belong to a set A which is the union of annular regions of thickness δ about each obstacle. Furthermore, $|\lambda_0|$ depends on the choice of δ. The smaller δ is chosen, the larger $|\lambda_0|$ becomes. This dependence is important in our selection of the point $\hat{X}$ as indicated by the second approach. However, in the case of disjoint obstacles there is a trivial upper bound for δ which is given by the following number.

Definition 3.4.1 *The positive number α_0 is defined as the largest positive number by which the obstacles could be extended without intersecting.*

$$\alpha_0 = \sup\{\alpha \in \Re^+ : D_i^{r_i+\alpha} \cap D_j^{r_j+\alpha} = \emptyset; 0 \leq i \neq j \leq m\}$$

Therefore, δ must be strictly smaller than α_0 so that we can be guaranteed that an equilibrium point in the δ-annular region around a particular obstacle is not in δ-annular region of any other obstacle.

Definition 3.4.2 *An angle is inscribed in an arc if and only if (1) each side contains one end point of the arc and (2) the vertex of the angle is a point of the arc other than an end point (see Wilcox [31]).*

Lemma 3.4.1 *An angle inscribed in a semicircle is a right angle.*

Proof: See Wilcox [31].

Therefore, by this lemma, given two fixed points B and C, the loci of all points A such that

$$< A - B, A - C >= 0$$

is given by the circle, C, with the radius$=\frac{1}{2}\|C-B\|$ and center$=B+\frac{C-B}{2}$.

Lemma 3.4.2 *Consider three points $X_1, X_2, and X_3$ in a plane which are not collinear and $\|X_1 - X_2\| < \|X_1 - X_3\|$. Then, there exists a point $\hat{X}$ arbitrarily close to X_1 such that*

$$\begin{aligned} < X_1 - \hat{X}, X_3 - \hat{X} > &= 0 \\ < \hat{X} - X_2, X_1 - \hat{X} > &> 0 \end{aligned}$$

and $\tilde{X}$ arbitrarily close to X_1 such that

$$\begin{aligned} < X_1 - \tilde{X}, X_3 - \tilde{X} > &= 0 \\ < \tilde{X} - X_2, X_1 - \tilde{X} > &< 0. \end{aligned}$$

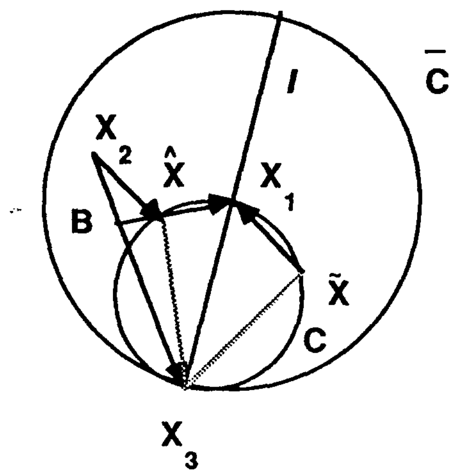

Figure 3.2: The illustrative diagram for lemma 3.4.2.

Proof: Consider the figure [3.2]. Let $\overline{C}$ be the circle centered at X_1 and radius $\|X_1 - X_3\|$ and ℓ be the diameter of $\overline{C}$ which passes through X_3. Then, by the hypothesis, X_2 is inside the circle $\overline{C}$ and off the diameter ℓ. Let, without loss in generality, X_2 be to the left of ℓ as shown in figure [3.2]. By lemma [3.4.1], the circle C with center at $\frac{1}{2}(X_1 + X_3)$ and radius $\frac{1}{2}\|X_1 - X_3\|$ is the loci of all points X such that

$$< X_1 - X, X_3 - X >= 0.$$

Let $\mathcal{N}(X_1)$ be any arbitrarily small neighborhood (open ball) of X_1 that does not intersect the line passing through X_3 and X_2. Then, $C \cap \mathcal{N}(X_1) \neq \emptyset$ since the circle C passes through X_1. The diameter ℓ divides the circle C into two half circles at X_1 and X_3 in C. Let the left and right half circles to the diameter ℓ be called LHC and RHC, respectively. Now, let

$$\tilde{X} \in \mathcal{N}(X_1) \cap RHC \neq \emptyset$$

$$\hat{X} \in \mathcal{N}(X_1) \cap LHC \neq \emptyset$$

It is clear from the definition of C that

$$
\begin{aligned}
< X_1 - \hat{X}, X_3 - \hat{X} > &= 0 \\
< X_1 - \tilde{X}, X_3 - \tilde{X} > &= 0
\end{aligned}
$$

Now, we have to show the inequalities. Without loss in generality, let

$$
\begin{aligned}
X_1 &\triangleq (0,0) \\
X_3 &\triangleq (0,-2) \\
X_2 &= (z_1, z_2)
\end{aligned}
$$

where $z_1 < 0$ by our previous assumption. Let P be any point on the circle C with radius=1 and center=(0,-1). Then,

$$P = (\cos\theta, \sin\theta - 1)$$

and

$$
\begin{aligned}
& < P - X_2, X_1 - P > \\
= & -2 + 2\sin\theta + z_1\cos\theta + z_2\sin\theta - z_2 \\
\triangleq & \ h(\theta).
\end{aligned}
$$

Let $\theta = \phi + \frac{\pi}{2}$, then

$$g(\phi) \triangleq h(\phi + \frac{\pi}{2}) = -2 + 2\cos\phi - z_1\sin\phi + z_2\cos\phi - z_2.$$

Now, since

$$
\begin{aligned}
g(0) &= 0 \quad and \\
\frac{d}{d\phi}g(0) &= [-2\sin\phi - z_1\cos\phi - z_2\sin\phi]_{\phi=0} \\
&= -z_1 \\
&> 0,
\end{aligned}
$$

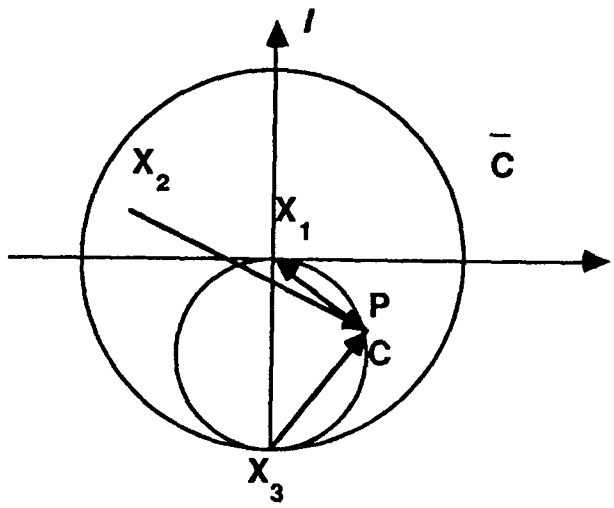

Figure 3.3: The illustrative diagram for the proof of lemma 3.4.2.

for a small $|\phi|$, $g(\phi)$ is positive for $\phi > 0$ and negative for $\phi < 0$. Therefore, arbitrarily close to X_1, there exists points $\hat{X}$ and $\tilde{X}$ such that the inequalities in the lemma hold. □

Lemma 3.4.3 *Consider three disjoint collinear points X_1, X_2, and X_3 such that $X_2 = \lambda X_1 + (1-\lambda)X_3$ for $\lambda \in (0,1)$. Then, there exists a point $\hat{X}$ in any neighborhood of X_1 such that*

$$\begin{aligned} < X_1 - \hat{X}, X_3 - \hat{X} > &= 0 \\ < \hat{X} - X_2, X_1 - \hat{X} > &< 0 \end{aligned}$$

Proof: Let $\hat{X} \in \mathcal{N}(X_1) \cap C$ where C is as in the proof of the lemma [3.4.2] and $\mathcal{N}(X_1)$ is any arbitrary small neighborhood of X_1. Then,

$$< X_1 - \hat{X}, X_3 - \hat{X} >= 0$$

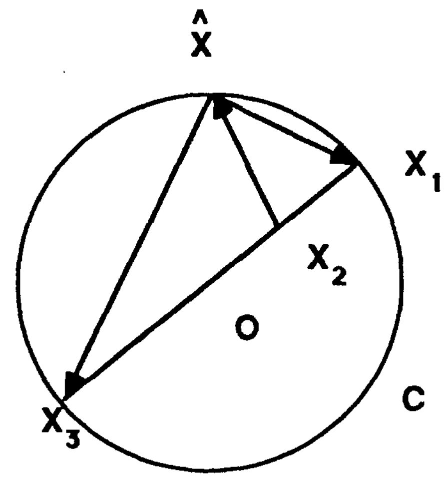

Figure 3.4: The illustrative diagram for the proof of lemma 3.4.3.

follows immediately. To show the inequality, we consider the right triangle $\triangle X_3\hat{X}X_1$ where X_2 belongs to the hypotenuse $\overline{X_1X_3}$. Therefore,

$$\begin{aligned} \angle X_2\hat{X}X_1 &< 90^\circ \\ <\hat{X}-X_2, X_1-\hat{X}> &< 0 \end{aligned}$$

Theorem 3.4.1 *At regular points, the vector field $-gradV(X)$ is perpendicular to the level surface of V.*

Proof: For the proof see M. W. Hirsch and S. Smale [8].

For $V = f_i \in S_d$, the level surfaces are circles, therefore, its gradient is pointing perpendicular to circles centered at X_i; that is,

$$-\nabla f_i(X) = \frac{\|\nabla f_i(X)\|}{\|X - X_i\|}\begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix}$$

This implies that the $-gradf_i$ points radially in a direction of leaving the level curves. This fact is used in the sequel.

Lemma 3.4.4 *Let X_{c_i} be an equilibrium point of the system Σ in U_i, then X^{ob}, X_i, and X_{c_i} are collinear if and only if*

$$-\sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j < \nabla f_j(X_{c_i}), X - X_{c_i} >= 0$$

for X such that

$$< X_i - X^{ob}, X - X_{c_i} >= 0$$

and $X \neq X_{c_i}$.

Proof: The proof is almost trivial. Since X_{c_i} is an equilibrium point of the system Σ, we have

$$-\sum_{i=0}^{m} \lambda_i \nabla f_i(X_{c_i}) = 0$$

$$< -\sum_{i=0}^{m} \lambda_i \nabla f_i(X_{c_i}), X - X_{c_i} > = 0$$

$$|\lambda_0| < \nabla f_0(X_{c_i}), X - X_{c_i} > -\lambda_i < \nabla f_i(X_{c_i}), X - X_{c_i} >$$

$$+ < - \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \nabla f_j(X_{c_i}), X - X_{c_i} > = 0$$

$$|\lambda_0| \|\nabla f_0(X_{c_i})\| < \frac{X^{ob} - X_{c_i}}{\|X^{ob} - X_{c_i}\|}, X - X_{c_i} > +\lambda_i \|\nabla f_i(X_{c_i})\| \times$$

$$< \frac{X_{c_i} - X_i}{\|X_{c_i} - X_i\|}, X - X_{c_i} > - < \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \nabla f_j(X_{c_i}), X - X_{c_i} > = 0$$

However, the first and second terms are identically zero by the hypothesis and collinearity of X^{ob}, X_i, and X_{c_i}.

To prove the lemma in the other direction, we assume the nontrivial case where

$$|\lambda_0|\nabla f_0(X_{c_i}) - \lambda_i \nabla f_i(X_{c_i}) \neq 0.$$

Since X_{c_i} is an equilibrium point of the system Σ and

$$< - \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \nabla f_j(X_{c_i}), X - X_{c_i} >= 0,$$

we have:

$$< |\lambda_0|\nabla f_0(X_{c_i}) - \lambda_i \nabla f_i(X_{c_i}), X - X_{c_i} >= 0$$

for all X such that

$$< X_i - X^{ob}, X - X_{c_i} >= 0.$$

$$\begin{aligned} &< |\lambda_0| \|\nabla f_0(X_{c_i})\| \frac{X^{ob} - X_{c_i}}{\|X^{ob} - X_{c_i}\|} \\ &+\lambda_i \|\nabla f_i(X_{c_i})\| \frac{X_{c_i} - X_i}{\|X_{c_i} - X_i\|}, X - X_{c_i} > \quad = \quad 0 \end{aligned}$$

Let

$$\alpha = |\lambda_0| \|\nabla f_0(X_{c_i})\| / \|X^{ob} - X_{c_i}\|$$

$$\beta = \lambda_i \|\nabla f_i(X_{c_i})\| / \|X_{c_i} - X_i\|,$$

then, considering the nontrivial case that $\alpha \neq \beta$, we have:

$$\begin{aligned} < \alpha X^{ob} - \beta X_i - (\alpha - \beta) X_{c_i}, X - X_{c_i} > &= 0 \\ < \frac{\alpha}{\alpha - \beta} X^{ob} - \frac{\beta}{\alpha - \beta} X_i - X_{c_i}, X - X_{c_i} > &= 0. \end{aligned}$$

However, since

$$\frac{\alpha}{\alpha - \beta} + \frac{-\beta}{\alpha - \beta} = 1,$$

the point

$$X_1 = \frac{\alpha}{\alpha - \beta} X^{ob} + \frac{-\beta}{\alpha - \beta} X_i$$

lies on the line $X^{o\bar{b}}X_i$ and the only way for the equality [3.2] to hold is the collinearity of X_i, X_{c_i}, and X^{ob}.

$$< X_1 - X_{c_i}, X - X_{c_i} >= 0 \tag{3.2}$$

For the case that $\alpha = \beta$, we have:

$$|\lambda_0| \nabla f_0(X_{c_i}) = \lambda_i \nabla f_i(X_{c_i})$$

and, therefore, X_{c_i} belongs to the line $X_i\bar{X}^{ob}$. □

Lemma 3.4.5 *Let $X_{c_i} \in U_i(\delta)$ be an equilibrium point of the system Σ. Then, for*

$$|\lambda_0| > \frac{M_1}{\xi_1} \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j,$$

there exists no equilibrium point, X_{c_i}, in the set

$$B_i(\delta) = \{X \in U_i(\delta) | < X - X_i, X^{ob} - X_i > \; \geq \; 0\}$$

where

$$M_1 = \max_{\substack{1 \leq j \leq m \\ j \neq i}} \sup_{X \in U_i(\delta)} \frac{\|\nabla f_j(X)\|}{\|\nabla f_0(X)\|},$$

and

$$\xi_1 = \inf_{X \in B_i(\delta)} \cos(\angle X X^{ob} X_i).$$

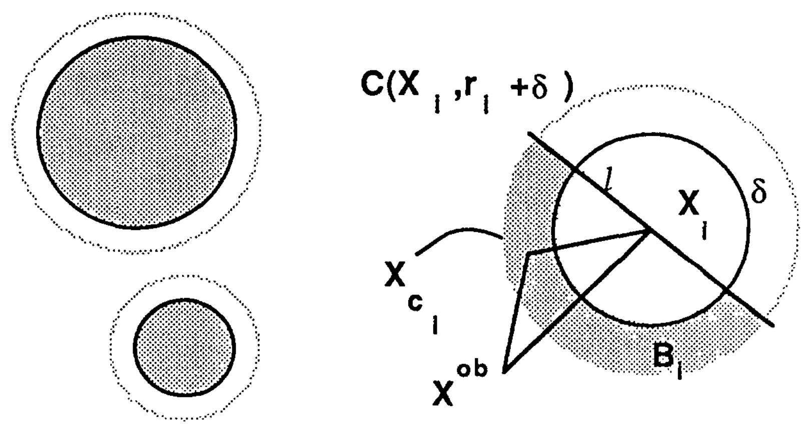

Figure 3.5: An illustrative diagram for the proof of lemma 3.4.5.

Proof: To prove the lemma, we show that

$$< -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), \frac{X^{ob} - X_i}{\|X^{ob} - X_i\|} > \; > \; 0 \tag{3.3}$$

for all $X \in B_i(\delta)$ and $|\lambda_0|$ as in the above.

$$\begin{aligned}
& < -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), \frac{X^{ob} - X_i}{\|X^{ob} - X_i\|} > \\
= \;& |\lambda_0| \|\nabla f_0(X)\| < \frac{X^{ob} - X}{\|X^{ob} - X\|}, \frac{X^{ob} - X_i}{\|X^{ob} - X_i\|} > \\
+ \;& \frac{\lambda_i \|\nabla f_i(X)\|}{\|X - X_i\| \cdot \|X^{ob} - X_i\|} \underbrace{< X - X_i, X^{ob} - X_i >}_{\geq 0} \\
- \;& \sum_{\substack{j = 1 \\ j \neq i}}^{m} \lambda_j \;\; < \nabla f_j(X), \frac{X^{ob} - X_i}{\|X^{ob} - X_i\|} >
\end{aligned}$$

$$\geq \ |\lambda_0| \|\nabla f_0(X)\| \xi_1 - \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \ < \nabla f_j(X), \frac{X^{ob} - X_i}{\|X^{ob} - X_i\|} >$$

$$\geq \ \|\nabla f_0(X)\| \xi_1 [|\lambda_0| - \frac{M_1}{\xi_1} \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j]$$

$$> \ 0.$$

Therefore, the inequality [3.3] is proven and

$$-\sum_{i=0}^{m} \lambda_i \nabla f_i(X) \ \neq \ 0$$

for all $X \in B_i(\delta)$. □

Lemma 3.4.6 *Consider the system,*

$$\dot{X} = |\lambda_0| \nabla f_0(X) - \lambda_i \nabla f_i(X),$$

then, there is only one equilibrium point in $U_i(\delta)$ for $|\lambda_0|$ as before and that is a saddle point which is collinear with X_i and X^{ob}.

Proof: Let

$$|\lambda_0| \nabla f_0(X) - \lambda_i \nabla f_i(X) \ = \ 0,$$

then,

$$|\lambda_0| \cdot \|\nabla f_0(X)\| = \lambda_i \|\nabla f_i(X)\|.$$

However, the above can only happen if the equilibrium point is on the line, ℓ, which passes through X_i and X^{ob}. In addition, since the norms in the radial direction are strictly monotonic functions of distance and $\|X - X_i\| > \|X - X^{ob}\|$ for all $X \in \ell$ which are opposite of X_i from X^{ob}, there exists a unique solution, say X_{c_i}, where the above holds. To show that X_{c_i} is a saddle point, consider the circle $C(\frac{X_{c_i}+X^{ob}}{2}, \frac{\|X^{ob}-X_{c_i}\|}{2})$ and any arbitrarily small neighborhood of X_{c_i}, say $\mathcal{N}(X_{c_i}) \subset U_i(\delta)$. Then, by lemma [3.4.3], we have:

$$
\begin{aligned}
& < |\lambda_0|\nabla f_0(X) - \lambda_i \nabla f_i(X), X_{c_i} - X > \\
= \; & |\lambda_0| \|\nabla f_0(X)\| < \frac{X^{ob} - X}{\|X - X^{ob}\|}, X_{c_i} - X > \\
& + \lambda_i \|\nabla f_i(X)\| < \frac{X - X_i}{\|X - X_i\|}, X_{c_i} - X > \\
= \; & \lambda_i \|\nabla f_i(X)\| < \frac{X - X_i}{\|X - X_i\|}, X_{c_i} - X > \\
< \; & 0
\end{aligned}
$$

for all X in $\mathcal{N}(X_{c_i}) \cap C$. On the other hand, for the points X in $\mathcal{N}(X_{c_i}) \cap \ell - \{X_{c_i}\}$, the above dot product is positive. Therefore, X_{c_i} is a saddle point. □

Theorem 3.4.2 *Let $X_{c_i} \in U_i(\frac{\alpha_0}{2}) = \{X | r_i < \|X - X_i\| < r_i + \frac{\alpha_0}{2}\}$ be any equilibrium point of the system Σ for some $i = 1, \ldots, m$. Suppose X^{ob}, X_i, and X_{c_i} are not collinear. Then, there exists a $\delta > 0$ sufficiently small such that in every neighborhood $\mathcal{N}(X_{c_i})$ of X_{c_i}, there exists $X \in \mathcal{N}(X_{c_i})$ such that*

$$
< -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), X_{c_i} - X >< 0 \tag{3.3}
$$

Proof: Let us first define a few terms and a coordinate system to simplify the proof. Since the potential functions do not depend on the choice of the coordinate system, without loss in generality, we can assume that the coordinate system is attached to the center of the i^{th} obstacle and its y-axis is passing through the objective point, X^{ob}. Let us denote the y-axis of the coordinate system by, ℓ. Since, by the hypothesis, the equilibrium point, X_{c_i}, is not on the line ℓ, assume that it is

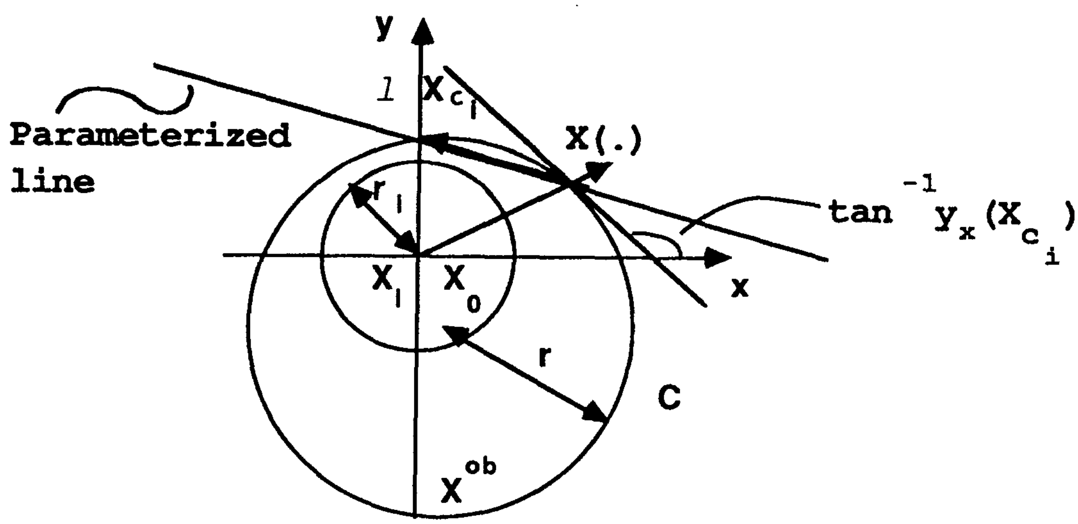

Figure 3.6: An illustrative diagram for the proof of theorem 3.4.2.

to the right of the line ℓ. Now, define the circle C centered at $X_0 = \frac{X_{c_i}+X^{ob}}{2}$ and radius $r = \frac{\|X_{c_i}-X^{ob}\|}{2}$. By lemma [3.4.5], for δ small enough we have,

$$\|X_{c_i} - X_i\| < \|X_{c_i} - X^{ob}\|.$$

Therefore, this fact along with the hypothesis of the theorem satisfies the hypothesis of the lemma [3.4.2]. So, by the lemma [3.4.2], for any neighborhood of X_{c_i} there exists a point $X \in C$ such that

$$< X^{ob} - X, X_{c_i} - X > \;=\; 0 \tag{3.4}$$

$$< X - X_i, X_{c_i} - X > \;<\; 0 \tag{3.5}$$

Let us express the point $X \in C$ as a function of η where η is the slope of the line passing through the point X_{c_i}, as shown in the figure [3.6]. Therefore, $X(\eta) = (x(\eta), y(\eta))^T$ is given by

$$x(\eta) \;=\; \frac{1}{1+\eta^2}\{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0] \pm \{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0]^2$$

$$- \quad (1+\eta^2)[\eta^2 x_{c_i} - 2(y_{c_i} - y_0)x_{c_i}\eta - (x_{c_i} - x_0)^2 + x_0^2]\}^{\frac{1}{2}}\} \qquad (3.6)$$

and

$$y(\eta) \quad = \quad \eta(x(\eta) - x_{c_i}) + y_{c_i} \qquad (3.7)$$

Using our choice of the coordinate system, equation [3.6] can be written as:

$$x(\eta) = \frac{1}{1+\eta^2}\{\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + \frac{x_{c_i}}{2} \pm [(y_{c_i} - y_0)\eta + \frac{x_{c_i}}{2}]\}.$$

To prove the theorem, we expand the left hand side of the inequality [3.3] as follows:

$$\begin{aligned}
& < -\sum_{i=0}^{m} \lambda_i \nabla f_i(X(\eta)), X_{c_i} - X(\eta) > \\
= & \ |\lambda_0| \frac{\|\nabla f_0(X(\eta))\|}{\|X(\eta) - X^{ob}\|} < X^{ob} - X(\eta), X_{c_i} - X(\eta) > \\
+ & \ \lambda_i \frac{\|\nabla f_i(X(\eta))\|}{\|X(\eta) - X_i\|} < X(\eta) - X_i, X_{c_i} - X(\eta) > \\
+ & \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \frac{\|\nabla f_j(X(\eta))\|}{\|X(\eta) - X_j\|} < X(\eta) - X_j, X_{c_i} - X(\eta) > \\
= & \ \lambda_i \frac{\|\nabla f_i(X(\eta))\|}{\|X(\eta) - X_i\|} < X(\eta) - X_i, X_{c_i} - X(\eta) > \\
+ & \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \frac{\|\nabla f_j(X(\eta))\|}{\|X(\eta) - X_j\|} < X(\eta) - X_j, X_{c_i} - X(\eta) >
\end{aligned}$$

where

$$< X^{ob} - X(\eta), X_{c_i} - X(\eta) >= 0.$$

The first term in the above expression is always negative for $\eta < y_x(X_{c_i}) \leq 0$. The second term is a smooth and bounded function of η on the closure of $U_i(\frac{\alpha_0}{2})$, where

$$\alpha_0 = \sup_{\alpha}\{\alpha \in \Re^+ : D_i^{r_i+\alpha} \subset D_j^{r_j+\alpha} = \emptyset; 0 \leq i \neq j \leq m\}$$

and

$$U_i(\delta) = \{X | r_i < \|X - X_i\| < r_i + \delta, \quad \delta > 0\}.$$

Therefore, the derivative of the second term with respect to η is bounded on the closure of $U_i(\frac{\alpha_0}{2})$. If we could show that the derivative of the first term with respect to η (more precisely the right derivative of the first term with respect to η) evaluated at $\eta_0(\delta) = y_x(X_{c_i}(\delta))$ goes to infinity as δ approaches zero, the theorem would be proven. So, we take the derivative of the first term.

$$\begin{aligned}
&\frac{d}{d\eta}\left[\frac{\|\nabla f_i(X(\eta))\|}{\|X(\eta) - X_i\|} < X(\eta) - X_i, X_{c_i} - X(\eta) >\right] \\
= &\frac{d}{d\eta}\left[\frac{\|\nabla f_i(\|X(\eta)\|)\|}{\|X(\eta) - X_i\|}\right] < X(\eta) - X_i, X_{c_i} - X(\eta) > \\
+ &\frac{\|\nabla f_i(X(\eta))\|}{\|X(\eta) - X_i\|}[-2x(\eta)x'(\eta) + x_{c_i}x'(\eta) - 2y(\eta)y'(\eta) + y_{c_i}y'(\eta)].
\end{aligned}$$

Now, we have to evaluate the above derivative for $\eta_0(\delta) = y_x(X_{c_i}(\delta))$ and take the limit as $\delta \to 0$. First, let us recall the following equalities:

$$\begin{aligned}
y(\eta_0(\delta)) &= y_{c_i}(\delta) \\
y_x(X_{c_i}(\delta)) &= -\frac{x_{c_i}(\delta)}{2(y_{c_i}(\delta) - y_0)} \\
x(\eta_0(\delta)) &= x_{c_i}(\delta) \\
x'(\eta) &= \frac{1}{(1+\eta^2)^2}\{2x_{c_i}\eta - 2(y_{c_i} - y_0) + 2(y_{c_i} - y_0)\eta^2\} \\
x'(\eta_0(\delta)) &= -\frac{(y_{c_i} - y_0)^3[3x_{c_i}^2 + 4(y_{c_i} - y_0)^2]}{2[(y_{c_i} - y_0)^2 + \frac{x_{c_i}^2}{4}]^2}
\end{aligned}$$

So, using the above, we get:

$$\frac{d}{d\eta}[\frac{\|\nabla f_i(X(\eta))\|}{\|X(\eta)-X_i\|} < X(\eta)-X_i, X_{c_i}-X(\eta) >]_{\eta=\eta_0(\delta)}$$

$$= -\frac{\|\nabla f_i(X(\eta_0(\delta)))\|}{\|X(\eta_0(\delta))-X_i\|}\cdot[x(\eta_0(\delta))x'(\eta_0(\delta))+y(\eta_0(\delta))\frac{d}{d\eta}y(\eta_0(\delta))].$$

From the above equalities, we can easily verify the following limits:

$$\begin{aligned}
\lim_{\delta\to 0} x(\eta_0(\delta)) &= 0 \\
\lim_{\delta\to 0} x'(\eta_0(\delta)) &= -2r(\delta=0) \\
\lim_{\delta\to 0} y'(\eta_0(\delta)) &= 0 \\
\lim_{\delta\to 0}(\|X(\eta_0(\delta))\|-r_i) &= 0 \\
\lim_{\delta\to 0}\frac{d}{d\eta}[\|X(\eta)\|-r_i]_{\eta=\eta_0(\delta)} &= 0 \\
\lim_{\delta\to 0}\frac{d}{d\eta}[x(\eta)x'(\eta)+y(\eta)y'(\eta)]_{\eta=\eta_0(\delta)} &= 4r^2(0)-4r(0)r_i.
\end{aligned}$$

Using the above, the L'Hopital's rule, the convention that $\eta_0(\delta) = y_x(X_{c_i}(\delta))$, and the following inequality which comes from the properties of the potential functions,

$$\|\nabla f_i(X)\| \geq \frac{1}{\|X-X_i\|-r_i}. \tag{3.8}$$

We get:

$$\lim_{\delta\to 0} -\frac{\|\nabla f_i(X(\eta_0(\delta)))\|}{\|X(\eta_0(\delta))-X_i\|}[x(\eta_0(\delta))x'(\eta_0(\delta))+y(\eta_0(\delta))y'(\eta_0(\delta))]$$

$$\geq \lim_{\delta\to 0} -\frac{[x(\eta_0(\delta))x'(\eta_0(\delta))+y(\eta_0(\delta))y'(\eta_0(\delta))]}{\|X(\eta_0(\delta))\|(\|X(\eta_0(\delta))\|-r_i)}$$

$$
\begin{aligned}
&= \lim_{\delta\to 0} -\{\frac{d}{d\eta_0(\delta)}[x(\eta_0(\delta))x'(\eta_0(\delta)) + y(\eta_0(\delta))y'(\eta_0(\delta))] \cdot \frac{d}{d\delta}\eta_0(\delta)\} \\
&\times \{\frac{d}{d\eta_0(\delta)}[\|X(\eta_0(\delta))\| \cdot (\|X(\eta_0(\delta))\| - r_i)] \cdot \frac{d}{d\delta}\eta_0(\delta)\}^{-1} \\
&= \lim_{\delta\to 0}\{x'^2(\eta_0(\delta)) + x(\eta_0(\delta))x''(\eta_0(\delta)) + y'^2(\eta_0(\delta)) + y(\eta_0(\delta))y''(\eta_0(\delta))\} \\
&\times \{[2 - r_i\|X(\eta_0(\delta))\|^{-1}][-x(\eta_0(\delta))x'(\eta_0(\delta)) - y(\eta_0(\delta))y'(\eta_0(\delta))]\}^{-1} \\
&= \lim_{\delta\to 0} \frac{4r^2(0) - 4r(0)r_i}{|x(\eta_0(\delta))x'(\eta_0(\delta)) + y(\eta_0(\delta))y'(\eta_0(\delta))|} \\
&= \infty.
\end{aligned}
$$

The absolute value in the denominator of the last fraction comes from the fact that $\|X(\eta) - X_i\| = \|X(\eta)\|$ is a decreasing function of η and, therefore, its derivative

$$\frac{d}{d\eta}\|X(\eta)\| = x(\eta)x'(\eta) + y(\eta)y'(\eta) < 0$$

for all $\eta \le y_x(X_{c_i}) \le 0$. As one can see, in the last limit, the numerator is a positive number and the denominator goes to zero; therefore, the limit is infinity as $\delta \to 0$. Consequently, there exists a δ small enough such that in any arbitrarily small neighborhood of X_{c_i}, $\mathcal{N}(X_{c_i}) \subset U_i(\delta)$,

$$< -\sum_{i=0}^{m} \lambda_i \nabla f_i(X(\eta)), X_{c_i} - X(\eta) >< 0$$

for $\eta < y_x(X_{c_i}(\delta)) \le 0$ such that $X(\eta) \in \mathcal{N}(X_{c_i})$. To compute an appropriate δ see chapter 4. □

Theorem 3.4.3 *Let $X_{c_i}(\delta) \in U_i(\frac{\alpha_0}{2})$ be an equilibrium point of the system Σ. Suppose X^{ob}, X_i, and X_{c_i} are collinear. Then, there exists a $\delta > 0$ sufficiently small such that in every neighborhood $\mathcal{N}(X_{c_i})$ of X_{c_i}, there exists $X \in \mathcal{N}(X_{c_i})$*

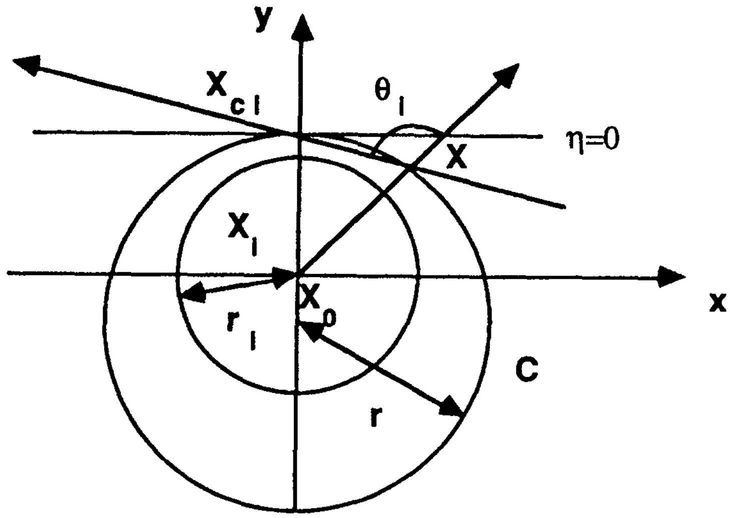

Figure 3.7: An illustrative figure for the theorem 3.4.3.

such that

$$< -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), \frac{X_{c_i} - X}{\|X_{c_i} - X\|} >< 0.$$

Proof: Consider figure [3.7] for the proof. Let the circle C be such that its center is given by $(X^{ob} + X_{c_i})/2$ and its radius is $\|X^{ob} - X_{c_i}\|/2$. Then for any point in $\mathcal{N}(X_{c_i}) \cap C$, we can write the left hand side of the above inequality as follows:

$$\begin{aligned} & < -\sum_{i=0}^{m} \lambda_i \nabla f_i(X), \frac{X_{c_i} - X}{\|X_{c_i} - X\|} > \\ = \quad & |\lambda_0| \|\nabla f_0(X)\| < \frac{X^{ob} - X}{\|X - X^{ob}\|}, \frac{X_{c_i} - X}{\|X_{c_i} - X\|} > \\ + \quad & \lambda_i \|\nabla f_i(X)\| < \frac{X - X_i}{\|X - X_i\|}, \frac{X_{c_i} - X}{\|X_{c_i} - X\|} > \end{aligned}$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \|\nabla f_j(X)\| < \frac{X - X_j}{\|X - X_j\|}, \frac{X_{c_i} - X}{\|X_{c_i} - X\|} >$$

$$= \sum_{j=1}^{m} \lambda_j \|\nabla f_j(X)\| \cos\theta_j(X)$$

$$\triangleq h(X)$$

where

$$\cos\theta_j(X) = \frac{< -\nabla f_j(X), X_{c_i} - X >}{\|\nabla f_j(X)\| \|X_{c_i} - X\|}$$

and

$$< \frac{X^{ob} - X}{\|X - X^{ob}\|}, \frac{X_{c_i} - X}{\|X_{c_i} - X\|} >= 0$$

for all $X \in C$. Singling out the term corresponding to the i^{th} obstacle in the expression for the $h(X)$, we get:

$$h(X) = \lambda_i \|\nabla f_i(X)\| \cos\theta_i(X) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \|\nabla f_j(X)\| \cos\theta_j(X).$$

For X in $\mathcal{N}(X_{c_i}) \cap C$, by the lemma [3.4.3], the first term is always negative. To show that $h(X)$ is negative for at least one point $\hat{X}$ in any arbitrarily small neighborhood of X_{c_i}, we prove that the slope of the directional derivative of $\|\nabla f_i(X)\| \cos\theta_i(X)$ along the direction of the circle C gets arbitrarily large at X_{c_i} when δ goes to zero. Proving this claim and knowing the fact that the second term in h(X) is a smooth function in the closure of $U_i(\frac{\alpha_0}{2})$ and has a bounded derivative for all X in the closure of $U_i(\frac{\alpha_0}{2})$, we can conclude that for δ small enough or $|\lambda_0|$ large enough, there exists a point arbitrarily close to $X_{c_i}(\delta)$ such that the inequality in the theorem holds.

To accomplish the above, we first express the points on the circle C as a function of the slope of the line passing through $X_{c_i}(\delta)$. The line is given by

$$y = \eta(x - x_{c_i}) + y_{c_i}$$

and the point $X(\eta) = (x(\eta), y(\eta))^T$ is the point that the above line intersects the circle C for $\eta \neq 0$. More precisely, $X(\eta)$ is given by:

$$\begin{aligned}
x(\eta) &= \frac{1}{1+\eta^2}\{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0] \pm \{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0]^2 \\
&- (1+\eta^2)[\eta^2 x_{c_i} - 2(y_{c_i} - y_0)x_{c_i} m - (x_{c_i} - x_0)^2 + x_0^2]\}^{\frac{1}{2}}\} \\
y(\eta) &= m(x(\eta) - x_{c_i}) + y_{c_i}
\end{aligned}$$

where $X_0 = (x_0, y_0)^T = (X^{ob} + X_{c_i})/2$.

Now, we have to take the derivative of $\|\nabla f_i(X(\eta))\| \cos\theta_i(X(\eta))$ with respect to η and evaluate it for $\eta_0 = 0$. At this point, it is necessary to point out that since the potential functions do not depend on the choice of the coordinate system, without loss in generality, we can choose the coordinate system with the origin at X_i and its y-axis passing through the objective point, X^{ob}. Then, evaluating the derivative for $\eta = 0$ is equivalent to evaluating the derivative at $X = X_{c_i}$. So, we have:

$$\begin{aligned}
&\frac{d}{d\eta}[\|\nabla f_i(X(\eta))\| \cos\theta_i(X(\eta))] \\
&= \frac{d}{d\eta}[\|\nabla f_i(X(\eta))\|] \cos\theta_i(X(\eta)) - \|\nabla f_i(X(\eta))\| \sin\theta_i(X(\eta)) \frac{d}{d\eta}\theta_i(X(\eta)) \\
&\triangleq g(X(\eta)).
\end{aligned}$$

However, $\theta_i(X(\eta))$ is given by

$$\theta_i(X(\eta)) = \tan^{-1}\eta - \tan^{-1}\frac{y(\eta) - y_i}{x(\eta) - x_i};$$

so, we have:

$$
\begin{aligned}
g(X(\eta)) &= \frac{d}{d\eta}[\|\nabla f_i(X(\eta))\|] \cos\theta_i(X(\eta)) - \|\nabla f_i(X(\eta))\| \sin\theta_i(X(\eta)) \\
&\quad [\frac{1}{1+\eta^2} - \frac{y'(\eta)[x(\eta)-x_i] - x'(\eta)[y(\eta)-y_i]}{\|X(\eta)-X_i\|^2}]
\end{aligned}
$$

where

$$
\begin{aligned}
x'(\eta) &= \frac{1}{(1+\eta^2)^2}\{\{2\eta x_{c_i} - (y_{c_i} - y_0) + \frac{1}{2}\{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0]^2 \\
&- (1+\eta^2)[\eta^2 x_{c_i}^2 - 2(y_{c_i} - y_0)x_{c_i} m - (x_{c_i} - x_0)^2 + x_0^2]\}^{-\frac{1}{2}} \\
&\times \{2[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0](2\eta x_{c_i} - (y_{c_i} - y_0)) \\
&- 2\eta[\eta^2 x_{c_i}^2 - 2(y_{c_i} - y_0)x_{c_i}\eta - (x_{c_i} - x_0)^2 + x_0^2] \\
&- (1+\eta^2)[2\eta x_{c_i}^2 - 2(y_{c_i} - y_0)x_{c_i}]\}\}(1+\eta^2) \\
&- 2\eta\{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0] + \{[\eta^2 x_{c_i} - (y_{c_i} - y_0)\eta + x_0]^2 \\
&- (1+\eta^2)[\eta^2 x_{c_i}^2 - 2(y_{c_i} - y_0)x_{c_i}\eta - (x_{c_i} - x_0)^2 + x_0]\}^{\frac{1}{2}}\}\}
\end{aligned}
$$

and

$$
y'(\eta) = x(\eta) + \eta x'(\eta) - x_{c_i},
$$

or using the collinearity and the assumed coordinate system, we have

$$
x'(\eta) = \frac{2(y_{c_i} - y_0)(\eta^2 - 1)}{(1+\eta^2)^2}.
$$

For δ fixed, the first term in the expression for $g(X(\eta))$ is zero at $\eta = 0$ because the $\cos\theta_i(X(0)) = \cos\frac{\pi}{2} = 0$ and $\frac{d}{d\eta}\|\nabla f_i(X(0))\|$ is a finite number by the properties of the potential function. Therefore,

$$
g(X(0)) = -\|\nabla f_i(X(0))\|[1 - \frac{2r(0)}{y_{c_i}}] \sin\frac{\pi}{2}
$$

where $r = \|X^{ob} - X_{c_i}\|/2$. Now using the bound on $\|\nabla f_i(X)\|$ and taking the limit as $\delta \to 0$, we get:

$$\begin{aligned}
\lim_{\delta \to 0} g(X(0)) &= \lim_{\delta \to 0} -\|\nabla f_i(X(0))\|[1 - \frac{2r(0)}{y_{c_i}}] \\
&\geq \lim_{\delta \to 0} -\frac{[1 - \frac{2r(0)}{y_{c_i}}]}{\|X(0) - X_i\| - r_i} \\
&= \infty
\end{aligned}$$

because the numerator goes to $1 - \frac{2r(0)}{r_i}$ which is strictly negative and bounded away from zero and the denominator goes to zero as $\delta \to 0$. Therefore, as explained before, this proves the theorem. □

Directly, from the last two theorems, one can get the following theorem.

Theorem 3.4.4 *For a set of disjoint obstacles, the system Σ can have only unstable equilibria outside the obstacles for $|\lambda_0|$ large enough (or equivalently δ small enough).*

Proof: For any equilibrium point outside the obstacles, X_{c_i}, of the system Σ, by theorem [3.3.1], we know that it belongs to an annular region with thickness δ, say U_i. Further more, from the above two theorems, there exist a point $\tilde{X}$ in any arbitrarily small neighborhood of X_{c_i} such that

$$< -\sum_{i=0}^{m} \lambda_i \nabla F_i(\tilde{X}), X_{c_i} - \tilde{X} > \; < \; 0$$

Therefore, by definition, the equilibrium points are either saddle or maximum points and , consequently, unstable.

Given a set of disjoint obstacles, one can model the work space by a gradient system and be guaranteed that all the equilibrium points of the system are unstable. Therefore, all the trajectories flow to the attractor.

Chapter 4

APPLICATIONS

In the last two chapters, we developed a mathematical model for the environment in which the robot navigates. Most of the analysis is done for the worst case situation. Therefore, the performance of an algorithm based on such a model may be improved when the model is applied to a specific problem. In another words, most of the bounds that we have shown can be improved for any given problem. In this chapter, we introduce an algorithm that applies the developed theory to produce an obstacle free path for a mobile robot navigating among a set of disjoint obstacles. Even though the algorithm is consistent with the theory, in the illustrative examples, we use the properties of the potential functions, the information about the environment, and the location of the mobile robot to tighten the bounds given in the theory as much as possible.

4.1 Obstacle Avoidance Algorithm

In this section, we propose an algorithm that generates an obstacle free path from any initial point to any destination point in the work space. The underlined assumption for this algorithm is that all the obstacles in the work space are disjoint. That is, there exists an $\alpha_0 > 0$ such that

$$\alpha_0 = \sup\{\alpha \in \Re^+ : D_i^{r_i+\alpha} \cap D_j^{r_j+\alpha} = \emptyset; 0 \leq i \neq j \leq m\}.$$

For this algorithm to work, we need not know the location of all obstacles in advance as long as any obstacle in the work space can be detected by the sensors and approximated by a disk which does not intersect any other obstacle. It is important to recall that all obstacles are disk shaped and extended with the radius of the robot. The disjoint set of obstacles assumption guarantees the existence of an obstacle free path. The following algorithm generates an obstacle free path.

Algorithm 4.1.1 *Consider that initially m_0 obstacles are detected. For each obstacle a potential function, f_i, with a positive weight, λ_i, is assigned in a preselected manner. The potential functions, f_i, $i = 1, \ldots, m_0$, satisfy the properties of the collection of functions S_d.*

Step–I Input X^{ob} and the location of the known obstacles. Set $k = 0$ and $t_0 = 0$.

Step–II Compute $\alpha_k = \sup\{\alpha \in \Re^+ : D_i^{r_i+\alpha} \cap D_j^{r_j+\alpha} = \emptyset; 0 \leq i \neq j \leq m_k\}$.

Step–III Compute δ_k according to section 3.4 and let it not to be larger than $\frac{\alpha_k}{2}$. Then, we must have:

$$\delta_{k+1} \leq \delta_k$$

Step–IV Compute

$$\begin{aligned} M_k &= \max_{1 \leq i \leq m_k} \sup_{X \in W_k} \frac{\|\nabla f_i(X)\|}{\|\nabla f_0(X)\|} \\ |\lambda_0| &= (M_k + 1) \sum_{i=1}^{m_k} \lambda_i \end{aligned}$$

Step–V Solve the system Σ:

$$\dot{X} = -\sum_{i=0}^{m_k} \lambda_i \nabla f_i(X)$$

$$X_k = X(t_k)$$

for $t_k \leq t < t_{k+1}$ *till one of the following happen at time* t_{k+1}*:*

1. *One or more new obstacles,* n_k*, are detected and* $\|X(t, X_k) - X^{ob}\| > \delta_k + r_0$ *in which case let* $k = k + 1$*,* $m_{k+1} = m_k + n_k$*, and go to step–II.*

2. $\|X(t, X_k) - X^{ob}\| \leq \delta_k + r_0$ *then stop and go to the next step.*

Step–VI Switch from the system Σ *at time* t_{k+1} *to the following system:*

$$\dot{X} = \nabla F_{ob}(X)$$
$$X_{k+1} = X(t_{k+1}, X_k)$$

where $\nabla F_{ob}(X)$ *is any twice continuously differentiable function with an asymptotically stable equilibrium point at* X^{ob} *with the minimum attracting region of* $D_{ob}^{r_0+2\delta_k}$*.*

The algorithm 4.1.1 converges in a finitely many steps for a disjoint set of obstacles since for the $|\lambda_0|$ given by the algorithm, the trajectories of the system Σ are bounded and the system does not have any stable equilibrium points outside the obstacles. Therefore, for finitely many obstacles, a trajectory of the system Σ gets arbitrarily close to the attractor at which time the second system moves the robot to the goal asymptotically.

The function $F_{ob} : \Re^2 \longrightarrow \Re$ can be taken to be a quadratic with its global minimum point at X^{ob}.

$$F_{ob}(X) = (X - X^{ob})^T Q (X - X^{ob})$$

where Q is a positive definite constant matrix. In the next section, we test the algorithm for potential functions generated by the circular charged wire as given in chapter 2.

4.2 Examples for Disjoint Set of Obstacles

To illustrate the performance of the algorithm 4.1.1, we use the homogeneous circumference potential functions introduced in chapter two. We have shown that this collection of functions satisfies the properties of collection of potential functions, S_d. Therefore, everything discussed in chapter 3 holds for this collection of potential functions. To be able to use the results of chapter 3 in a numerically efficient manner, we use the information about the location of the mobile robot and the destination to eliminate certain number of obstacles that do not enter in the domain of motion. In this way, we reduce the magnitude of $|\lambda_0|$ and avoid unnecessarily powerful attraction which may cause a very close encounter with the obstacles. In order to be able to incorporate the current position of the robot and the objective point in eliminating some obstacles, we need to prove a corollary to theorem 3.3.2.

Corollary 4.2.1 *For $|\lambda_0| > M\sum_{i=1}^{m}\lambda_i$ and $X_0 \in W$, all trajectories of the system Σ are confined to the following set:*

$$R = \{X \mid \|X - X^{ob}\| \leq \max\{\|X_{i_{max}} - X^{ob}\| + r_{i_{max}} + \frac{\alpha_0}{2}, \|X^{ob} - X_0\}\} \cap W$$

where

$$i_{max} = \arg\max_{1\leq i\leq m}\{\|X_i - X^{ob}\| + r_i\}.$$

Proof: Follows directly from the proof of the theorem 3.3.2.

Hence, if there exists obstacles such that

$$\|X^{ob} - X_0\| \leq \|X^{ob} - X_j\| - r_j - \delta,$$

we can eliminate the j^{th} potential function (obstacle) by setting its weighting factor to zero. In this manner, we avoid unnecessarily large attracting weight,

$|\lambda_0$. To compute $|\lambda_0|$, we need to compute δ_k first. According to chapter 3, in order to compute δ_k, we need to find a bound on

$$\frac{d}{d\eta}g(X(\eta)) \triangleq \frac{d}{d\eta} \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_j \frac{\|\nabla f_j(X(\eta))\|}{\|X - X_j\|} < X - X_j, X_{c_i} - X >$$

for η such that $X(\eta) \in U_i$. Since the norm of the directional derivative is bounded above by the norm of the Frechet derivative, we analyze the problem by taking the norm of the Frechet derivative of g(.) for all X in the closure of $U_i(\frac{\alpha_0}{2})$.

$$\begin{aligned}
\|\frac{\partial g(X)}{\partial X}\| &= \|\frac{\partial}{\partial X} <- \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_j \nabla f_j(X), X_{c_i} - X > \|_{X=X_{c_i}} \\
&= \| \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_j \frac{\partial}{\partial X}\{\frac{\|\nabla f_j(X)\|}{\|X - X_j\|} \\
&\times \; [(x - x_j)(x_{c_i} - x) + (y - y_j)(y_{c_i} - y)]\}_{X=X_{c_i}}\| \\
&= \| \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_j \frac{\partial \|\nabla f_j(X)\|}{\partial X} \cdot \frac{[(x - x_j)(x_{c_i} - x) + (y - y_j)(y_{c_i} - y)]}{\|X - X_j\|} \\
&+ \; \|\nabla f_j(X)\| \cdot \frac{\partial}{\partial X} \frac{[(x - x_j)(x_{c_i} - x) + (y - y_j)(y_{c_i} - y)]}{\|X - X_j\|} \|_{X=X_{c_i}} \\
&\leq \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_j \|\nabla f_j(X_{c_i})\| \\
&\times \; \|\frac{\partial}{\partial X} \frac{[(x - x_j)(x_{c_i} - x) + (y - y_j)(y_{c_i} - y)]}{\|X - X_j\|} \|_{X=X_{c_i}}
\end{aligned}$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j \|\nabla f_j(X_{c_i})\|$$

$$\leq \max_{\substack{1 \leq j \leq m_k \\ j \neq i}} \sup_{X \in U_i(\frac{\alpha_0}{2})} \|\nabla f_j(X)\| \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j$$

$$= N \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j < \infty$$

Going from the third equality to the inequality, we evaluated the first term in the third equality for $X = X_{c_i}$ which yields a zero and took the norm inside the summation. From there on, the simplification is done by carrying out the partial derivative an evaluating it at $X = X_{c_i}$. The last inequality is introduced because we want the worst situation. From chapter 3, we know that

$$\frac{d}{d\eta}[\lambda_i \frac{\|\nabla f_i(X(\eta))\|}{\|X - X_i\|} < X - X_i, X_{c_i} - X >]$$

goes to infinity as $\delta \to 0$, and the rate of divergence is not slower than $\frac{1}{|x_{c_i}|}$. Therefore, δ must be selected such that

$$|x_{c_i}| \leq \min\{\frac{1}{N \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j}, \frac{\alpha_0}{2}\}.$$

δ can be computed by solving

$$\||\lambda_0|\nabla f_0(X) - \lambda_i \nabla f_i(X)\| = [N \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda_j]^{-1}$$

for all X in $U_i(\frac{\alpha_0}{2})$. Then, using the solution $\hat{X}$ with the largest x coordinate, we can vary $|\lambda_0|$ to make $\hat{x}$ satisfy the above inequality. This procedure may have to be repeated a few times. Once the proper $\hat{X}$ is found, we take δ as

$$\delta_k = |\hat{y} - r_i|.$$

After computing δ_k, $|\lambda_0|$ can be computed as given in the algorithm 4.1.1. Once $|\lambda_0|$ is computed, we can solve the system Σ and find the trajectory that takes the robot to the objective point while avoiding the obstacles. In the sequel, we present two examples where this algorithm is used to generate the paths.

Example 4.2.1 *Consider 36 disjoint obstacles whose locations are given. Use the algorithm 4.1.1 to solve for an obstacle free path from (1,4) to (17,16.5) and from (19.5,13.5) to (11,2).*

A series of programs are written to implement the algorithms given in this chapter. Using those programs, the above problem is solved as you can see in figure [4.1]. To have a clear picture, the obstacles are extended slightly and the charged wire is put at the extended location. In the next example, we illustrate the performance of this algorithm for the case that new obstacles are detected.

Example 4.2.2 *Consider 32 known disjoint obstacles and some more obstacles that are not given in advance. However, assume that such unknown obstacles can only enter the problem in a disjoint manner. Find the path from (6,19) to (17,5).*

As you can see in figure [4.2], the obstacle free path is found and along the way two obstacles are detected which are shown with dashed lines. This concludes our illustrative examples.

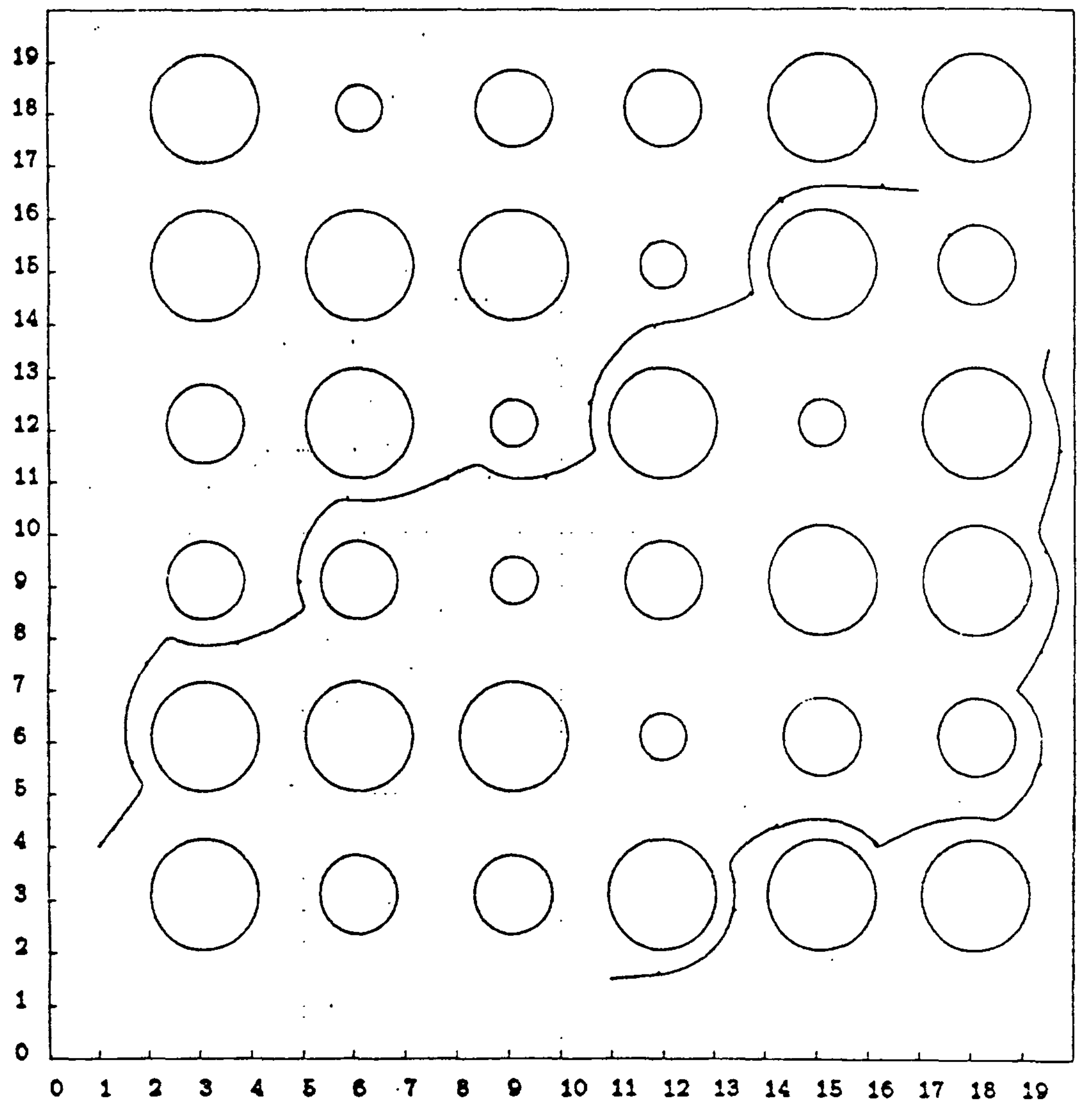

Figure 4.1: Simulation result for example 4.2.1

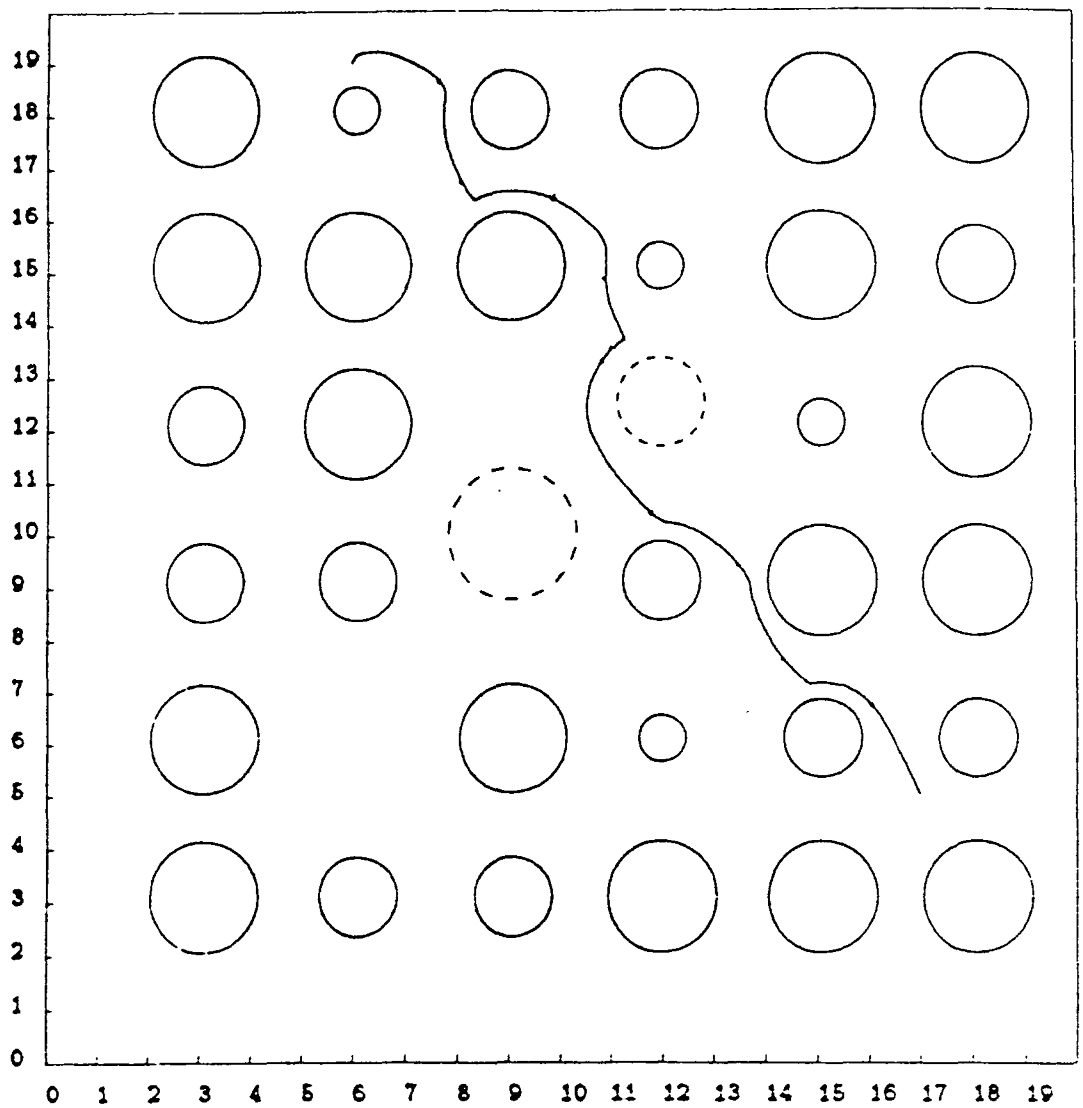

Figure 4.2: Simulation result for example 4.2.2

Chapter 5

CONCLUSION

In this thesis, we developed an algorithm to encode the geometric data about the environment into a smooth vector field which is used to navigate a mobile robot . The vector field is generated by the weighted sum of the gradient of the potential functions about each obstacle plus the attracting field about the goal. The attracting field, in our construction, is a negatively weighted gradient of a potential function with the same properties as the potential function used for the obstacles. The resultant vector field generates a gradient system whose trajectories flow toward the attractor without being trapped in any stable equilibrium points. It is shown that if the weighting factor on the attracting field is large enough, all the equilibria of the gradient system are unstable. Therefore, starting from any initial point in the work space except a set of points with measure zero, all the trajectories of the system flow toward the attractor (goal).

This algorithm is tested for a point robot navigating among a disjoint set of disk shaped obstacles. Due to the manner in which the potential function is constructed, we do not need the exact information on the obstacles prior to the execution of a task. New obstacles which are disk shaped and disjoint can be added to the environment as they are being detected by the sensing devices.

The analysis in this thesis works for a restricted class of problems. To generalize the results obtained, we need to consider obstacles other than disks. One way to do that is to allow intersecting disks in our modeling of the environment. In such a setting, it can be easily shown from the analysis in this thesis that for a sufficiently large weight on the attractor, the extraneous stable equilibria of the system can only appear in the intersected annular regions. Therefore, for a simple problem with not too many disks intersecting, by adjusting the weighting factors on the intersecting disks, one can avoid such stable equilibria for a given goal. In general, we propose to analyze the following control problem:

Consider the control system Σ,

$$
\begin{aligned}
\dot{X} &= -\sum_{i=0}^{m} \lambda_i \nabla f_i(X) \\
\lambda_0 &< 0 \\
\lambda_i &> 0 \qquad i = 1, \ldots, m \\
X_0 &= X(0)
\end{aligned}
$$

and let λ_i's be the control parameters. Then, the problem is to find a set of piecewise constant inputs such that the objective point would be in the reachable set from a given initial point. If a solution to this problem exists, it would depend on the initial and final points.

At this point, we give an illustrative example that shows the possibility of using the weighting factors on the obstacles to construct a smooth path to the goal in an environment with intersected disks. This example is illustrated in two figures. In figure [5.1], we see a zigzag path to the goal which is generated by an ad-hoc change of weighting factors on the obstacles. The algorithm used is as follows:

1. Solve the above differential equation until one of the following happens:

a) Reach the goal;then stop.

b) Reach a stable equilibrium point other than the goal; unevenly adjust the weighting factors on the two closest obstacles till the system exits the sink. Then, go to (1).

Once the goal is reached, we solve the system again for the same initial point and the new weighting factors. The result is shown in figure [5.2]. As one can see, a smooth path is generated without being trapped in any stable equilibria. Even though this algorithm does not work in general, the results obtained illustrate a possibility of exploring the problem proposed earlier and the existence of a solution to that problem.

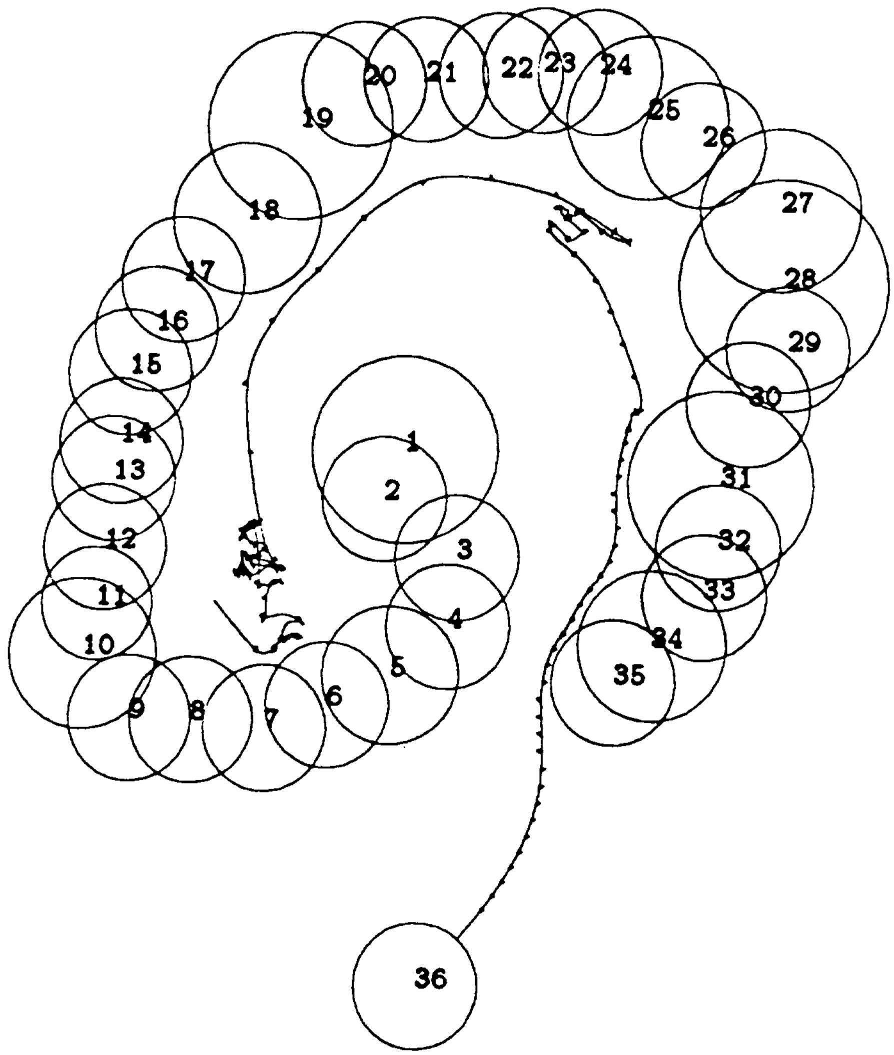

Figure 5.1: Intersected disks step one illustration.

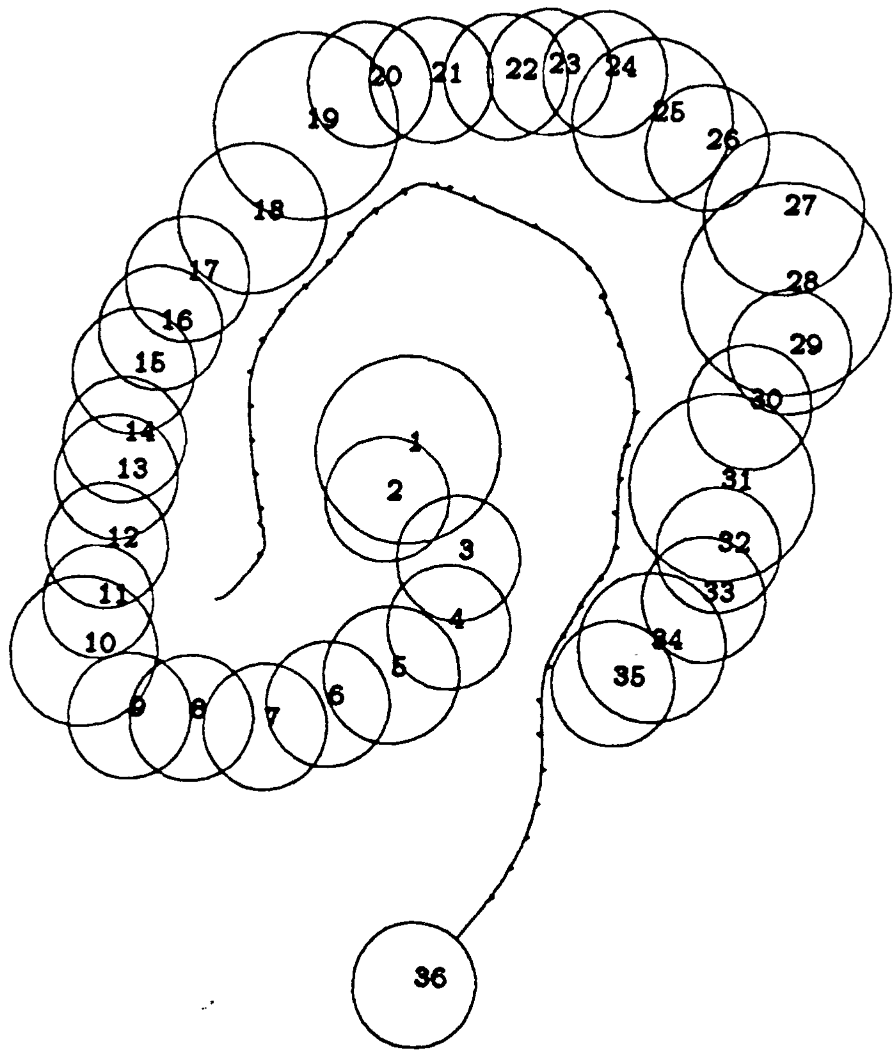

Figure 5.2: Intersected disks step two illustration.

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