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Capacity Of The Multiple-Access  
Arbitrarily Varying Channel**

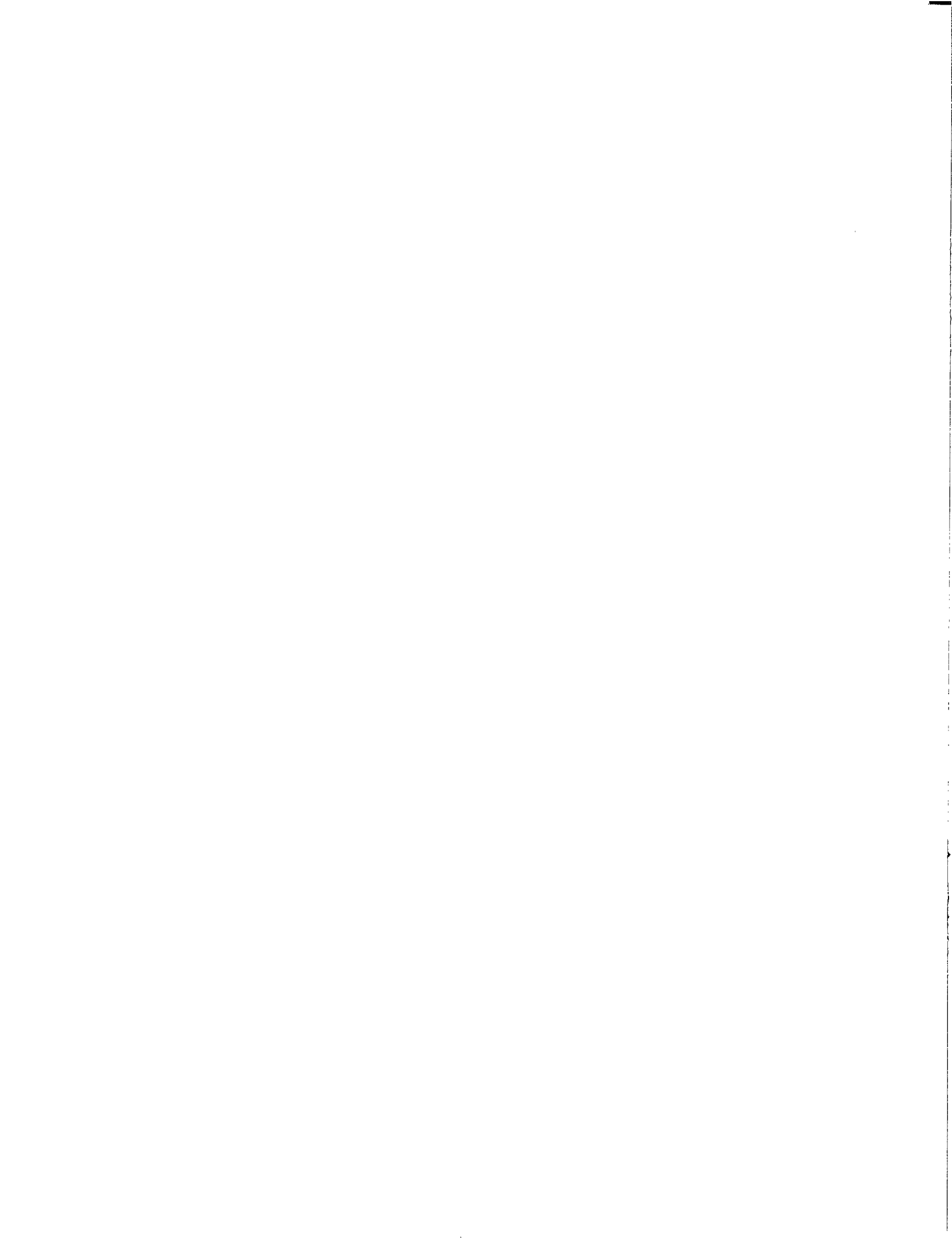
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# On the Deterministic-Code Capacity of the Multiple-Access Arbitrarily Varying Channel

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*Abstract* — The capacity region of the multiple-access arbitrarily varying channel (AVC) was characterized by Jahn, *assuming that the region had a nonempty interior*; however, he did not indicate how one could decide whether or not the capacity region had a nonempty interior. Using the method of *types* and an approach different from Jahn's, we have partially solved this problem. We begin by considering the notion of *symmetrizability* for the two-user AVC as an extension of the same notion for the single-user AVC. We show that if a multiple-access AVC is symmetrizable, then its capacity region has an empty interior. For the two-user AVC, this means that at least one (and perhaps both) users cannot reliably transmit information across the channel. More importantly, we show that if the channel is suitably *nonsymmetrizable*, then the capacity region has a nonempty interior, and both users can reliably transmit information across the channel.

Our proofs rely heavily on a rather complicated decoding rule. This leads us to seek conditions under which simpler multiple-message decoding techniques might suffice. In particular, we give conditions under which the universal *maximum mutual information decoding rule* will be effective.

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## I. INTRODUCTION

A two-user multiple-access arbitrarily varying channel (or AVC for brevity) is a transition probability  $W$  from  $\mathcal{X} \times \mathcal{Y} \times \mathcal{S}$  into  $\mathcal{Z}$ , where  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{S}$ , and  $\mathcal{Z}$  are finite sets, each containing at least two elements. We interpret  $W(z|x, y, s)$  as the conditional probability that the channel output is  $z \in \mathcal{Z}$  given that the channel input symbol from user 1 is  $x \in \mathcal{X}$ , the channel input symbol from user 2 is  $y$ , and that the channel *state* is  $s \in \mathcal{S}$ . The channel operation on  $n$ -tuples  $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$ ,  $\mathbf{y} \in \mathcal{Y}^n$ ,  $\mathbf{s} \in \mathcal{S}^n$ , and  $\mathbf{z} \in \mathcal{Z}^n$  is given by

$$W^n(\mathbf{z}|\mathbf{x}, \mathbf{y}, \mathbf{s}) \triangleq \prod_{k=1}^n W(z_k|x_k, y_k, s_k).$$

Using the average probability of error performance criterion with deterministic codes, Jahn [16] characterized the capacity region,  $C(W)$ , of the multiple-access AVC, *assuming that the region had a nonempty interior*. Jahn did not address the question of how one could decide whether or not  $C(W)$  had a nonempty interior. In this paper we present simple conditions on the channel transition probability  $W$  which determine whether or not the capacity region  $C(W)$  has a nonempty interior. The techniques used to establish these conditions will also be used in a forthcoming paper [15] to study the case in which the channel state sequences are constrained to lie in a certain subset of  $\mathcal{S}^n$ . Our work on the multiple-access AVC was motivated by the recent results of Csiszár and Narayan [8] for the single-user AVC. A preliminary study of the multiple-access AVC quickly reveals the far more complex nature of this channel, and clearly indicates that a more intricate approach is required. By combining extensions of the techniques and results of [8] with new ones of our own, we have uncovered some rather intriguing behavior of the multiple-access AVC. We summarize our results below.

In Section III, we introduce the crucial notion of symmetrizability as an extension of the notion of single-user symmetrizability introduced in [8]. We show that if  $W$  is symmetrizable (in the sense of Definitions 3.1, 3.2, or 3.3), then  $C(W)$  has an empty interior. In Section IV we discuss the decoding rule used to prove our major results. As this rule is rather complicated, in Section IV.B we describe the much simpler and universal *maximum mutual information (MMI) decoding rule*. In Section V we state our major

results. In particular we claim that certain *nonsymmetrizability* conditions are sufficient to imply that  $C(W)$  contains various open rectangles, and thereby possesses a nonempty interior (cf. Theorems 5.1, 5.3, and 5.4). In addition, Theorem 5.5 gives conditions under which the simpler MMI decoding rule can be used in the proof of Theorem 5.1.

## II. THE MULTIPLE-ACCESS AVC

We begin with the definition of a code for a multiple-access AVC.

**Definition 2.1** Let  $N$ ,  $M$ , and  $n$  be positive integers. If  $f$ ,  $g$ , and  $\varphi$  are mappings with

$$f : \{1, \dots, N\} \rightarrow \mathcal{X}^n \quad \text{and} \quad g : \{1, \dots, M\} \rightarrow \mathcal{Y}^n,$$

and

$$\varphi : \mathcal{Z}^n \rightarrow \{1, \dots, N\} \times \{1, \dots, M\},$$

then the triple  $(f, g, \varphi)$  is called a *code*. The mapping  $f$  is called an *encoder for user 1*; the mapping  $g$  is called an *encoder for user 2*, and the mapping  $\varphi$  is called a *decoder*. The *rate pair* of this code is the pair of nonnegative real numbers<sup>1</sup>

$$\left( \frac{\log N}{n}, \frac{\log M}{n} \right). \tag{2.1}$$

Setting  $\mathbf{x}_i \triangleq f(i)$ ,  $i = 1, \dots, N$ , and  $\mathbf{y}_j \triangleq g(j)$ ,  $j = 1, \dots, M$ , we call  $\mathbf{x}_1, \dots, \mathbf{x}_N$  *codewords for user 1*, and we call  $\mathbf{y}_1, \dots, \mathbf{y}_M$  *codewords for user 2*. There is no requirement that the codewords be distinct. Clearly, knowing  $f$  and  $g$  is equivalent to knowing the codewords  $\mathbf{x}_i$  and  $\mathbf{y}_j$ .

**Remark.** In the literature,  $(f, g, \varphi)$  is called a *deterministic code* in order to distinguish it from more general *random codes*. Random codes are discussed in [4, p. 209].

We next introduce the concept of an achievable rate pair.

**Definition 2.2** A pair of nonnegative real numbers,  $(R_1, R_2)$ , is said to be *achievable* for the AVC  $W$  if:

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<sup>1</sup>Throughout this paper,  $\log$  and  $\exp$  are understood as being to the base 2.

For every  $0 < \lambda < 1$ , and every  $\Delta R > 0$ , there exists a positive integer  $n_0$  such that for all  $n \geq n_0$ , there exist positive integers  $N$  and  $M$  such that

$$\frac{\log N}{n} > R_1 - \Delta R \quad \text{and} \quad \frac{\log M}{n} > R_2 - \Delta R,$$

and such that there exists a code  $(f, g, \varphi)$  with the probability of decoding error

$$e(\mathbf{s}) \triangleq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\{\mathbf{z} \in \mathcal{Z}^n : \varphi(\mathbf{z}) \neq (i, j)\} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \lambda, \quad \forall \mathbf{s} \in \mathcal{S}^n. \quad (2.2)$$

**Definition 2.3** The *capacity region* of the AVC  $W$ , denoted  $C(W)$ , is defined by

$$C(W) \triangleq \{(R_1, R_2) : (R_1, R_2) \text{ is achievable}\}.$$

A few comments are in order here. First, it is clear from the definition that  $C(W)$  is a closed set. Second, by the usual time-sharing principle [4, Lemma 2.2, p. 272],  $C(W)$  is also convex. Consequently, if  $E$  is any subset of  $C(W)$ , then the closed convex hull of  $E$  is also a subset of  $C(W)$ .

Before proceeding further, we shall need the following notation. Let  $\mathcal{D}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})$  denote the set of all probability distributions on  $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ . For all  $Q_{XYZ} \in \mathcal{D}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})$ , let

$$\begin{aligned} I_{X \wedge Z}(Q_{XYZ}) &\triangleq I(X \wedge Z), \\ I_{Y \wedge Z}(Q_{XYZ}) &\triangleq I(Y \wedge Z), \\ I_{X \wedge Z | Y}(Q_{XYZ}) &\triangleq I(X \wedge Z | Y), \\ I_{Y \wedge Z | X}(Q_{XYZ}) &\triangleq I(Y \wedge Z | X), \\ I_{XY \wedge Z}(Q_{XYZ}) &\triangleq I(XY \wedge Z), \end{aligned}$$

where the expressions on the right are the usual mutual information quantities computed with the distribution indicated on the left. Next, suppose  $p \in \mathcal{D}(\mathcal{X})$ ,  $q \in \mathcal{D}(\mathcal{Y})$ , and  $r \in \mathcal{D}(\mathcal{S})$ . We can define a probability measure on  $\mathcal{X} \times \mathcal{Y} \times \mathcal{S} \times \mathcal{Z}$  by setting

$$(p \times q \times r \times W)(x, y, s, z) \triangleq p(x)q(y)r(s)W(z|x, y, s).$$

After setting  $(p \times q \times rW)(x, y, z) \triangleq \sum_s (p \times q \times r \times W)(x, y, s, z)$ , we then define

$$\begin{aligned}
I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W) &\triangleq \inf_{r \in \mathcal{D}(S)} I_{\mathcal{X} \wedge \mathcal{Z}}(p \times q \times rW), \\
I_{\mathcal{Y} \wedge \mathcal{Z}}^*(p, q, W) &\triangleq \inf_{r \in \mathcal{D}(S)} I_{\mathcal{Y} \wedge \mathcal{Z}}(p \times q \times rW), \\
I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}^*(p, q, W) &\triangleq \inf_{r \in \mathcal{D}(S)} I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}(p \times q \times rW), \\
I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W) &\triangleq \inf_{r \in \mathcal{D}(S)} I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(p \times q \times rW), \\
I_{\mathcal{X} \mathcal{Y} \wedge \mathcal{Z}}^*(p, q, W) &\triangleq \inf_{r \in \mathcal{D}(S)} I_{\mathcal{X} \mathcal{Y} \wedge \mathcal{Z}}(p \times q \times rW).
\end{aligned} \tag{2.3}$$

**Definition 2.4** Let

$$\begin{aligned}
\mathcal{R}^*(p, q, W) \triangleq \{(R_1, R_2) : & 0 \leq R_1 < I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}^*(p, q, W), \\
& 0 \leq R_2 < I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W), \\
& 0 \leq R_1 + R_2 < I_{\mathcal{X} \mathcal{Y} \wedge \mathcal{Z}}^*(p, q, W)\},
\end{aligned}$$

and denote by  $\mathcal{R}^*(W)$  the closed convex hull of

$$\bigcup_{p \in \mathcal{D}(\mathcal{X}), q \in \mathcal{D}(\mathcal{Y})} \mathcal{R}^*(p, q, W).$$

**Theorem 2.5** (*Jahn* (1981) [16]). *For every AVC  $W$ , we always have*

$$C(W) \subset \mathcal{R}^*(W), \quad (\text{the weak converse}), \tag{2.4}$$

*and, if  $C(W)$  has a nonempty interior, then*

$$\mathcal{R}^*(W) \subset C(W), \quad (\text{the forward part}). \tag{2.5}$$

The weak converse, inclusion (2.4), asserts that all achievable rate pairs must belong to  $\mathcal{R}^*(W)$ . The forward part, inclusion (2.5), asserts that every rate pair in  $\mathcal{R}^*(W)$  is in fact achievable, provided  $C(W)$  has a nonempty interior. Obviously, one would like to know exactly when  $C(W)$  has a nonempty interior. In Section V we will give sufficient conditions under which  $C(W)$  will have a nonempty interior. In fact, using techniques unrelated to Jahn's, we will show that under certain conditions,  $C(W)$  contains certain

open rectangles, proving that  $C(W)$  has a nonempty interior. In Section III we will give sufficient conditions under which  $C(W)$  will have one of the following forms, each with an empty interior,

$$C(W) = \{(0, 0)\},$$

$$C(W) = [0, C_1(W)] \times \{0\}, \quad \text{or} \quad C(W) = \{0\} \times [0, C_2(W)],$$

where

$$C_1(W) \leq \sup_{p \in \mathcal{D}(\mathcal{X}), q \in \mathcal{D}(\mathcal{Y})} I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}^*(p, q, W) \quad \text{and} \quad C_2(W) \leq \sup_{p \in \mathcal{D}(\mathcal{X}), q \in \mathcal{D}(\mathcal{Y})} I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W).$$

### III. SYMMETRIZABILITY

The various notions of *symmetrizability* presented below will play a crucial role in determining whether or not  $C(W)$  has an empty interior. The definitions below generalize the notion of single-user symmetrizability introduced in [8].

**Definition 3.1** The AVC  $W$  is said to be *symmetrizable- $\mathcal{X}\mathcal{Y}$*  if there exists a transition probability  $U$  from  $\mathcal{X} \times \mathcal{Y}$  into  $\mathcal{S}$  such that

$$\sum_s W(z|x, y, s)U(s|x', y') = \sum_s W(z|x', y', s)U(s|x, y), \quad \forall x, x', y, y', z. \quad (3.1)$$

If no such  $U$  exists, we say that  $W$  is *nonsymmetrizable- $\mathcal{X}\mathcal{Y}$* .

**Definition 3.2** The AVC  $W$  is said to be *symmetrizable- $\mathcal{X}$*  if there exists a transition probability  $U$  from  $\mathcal{X}$  into  $\mathcal{S}$  such that

$$\sum_s W(z|x, y, s)U(s|x') = \sum_s W(z|x', y, s)U(s|x), \quad \forall x, x', y, z. \quad (3.2)$$

If no such  $U$  exists, we say that  $W$  is *nonsymmetrizable- $\mathcal{X}$* .



**Definition 3.3** The AVC  $W$  is said to be *symmetrizable- $\mathcal{Y}$*  if there exists a transition probability  $U$  from  $\mathcal{Y}$  into  $\mathcal{S}$  such that

$$\sum_s W(z|x, y, s)U(s|y') = \sum_s W(z|x, y', s)U(s|y), \quad \forall x, y, y', z. \quad (3.3)$$

If no such  $U$  exists, we say that  $W$  is *nonsymmetrizable- $\mathcal{Y}$* .

**Example.** Let  $\mathcal{X} = \mathcal{Y} = \mathcal{S} = \{0, 1\}$ , and let  $\mathcal{Z} = \{0, 1, 2, 3\}$ . Consider the *adder channel* given by  $W(z|x, y, s) = \delta(z - x - y - s)$ , where  $\delta(t) \triangleq 1$  if  $t = 0$ , and  $\delta(t) \triangleq 0$  otherwise. For this channel, it is easy to show that if  $U$  satisfies (3.1), then  $U \equiv 0$ . Since  $U \equiv 0$  is not a transition probability, the adder channel is nonsymmetrizable- $\mathcal{X}\mathcal{Y}$ . Similarly, it is a simple matter to show that if  $U$  satisfies (3.2), then  $U(s|x) = \delta(s - x)$ , and so the adder channel is symmetrizable- $\mathcal{X}$ . Of course, an identical argument shows that the adder channel is symmetrizable- $\mathcal{Y}$ .

**Theorem 3.4** *If the AVC  $W$  is symmetrizable- $\mathcal{X}\mathcal{Y}$ , then*

$$C(W) = \{(0, 0)\}.$$

*Proof.* This result follows almost immediately from Csiszár and Narayan's "Proof of Lemma 1" in [8, p. 187 through equation (3.29)].

**Lemma 3.5** *If the AVC  $W$  is symmetrizable- $\mathcal{X}$ , then*

$$C(W) = \{0\} \times [0, C_2(W)],$$

where  $C_2(W) \leq \sup_{p, q} I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W)$ .

*Proof.* Let  $n$  be any positive integer. Let  $N$  and  $M$  be positive integers. Suppose  $\mathbf{x}_1, \dots, \mathbf{x}_N$ , each in  $\mathcal{X}^n$ , are codewords for user 1, and suppose  $\mathbf{y}_1, \dots, \mathbf{y}_M$ , each in  $\mathcal{Y}^n$ , are codewords for user 2. Let  $\varphi(\mathbf{z}) = (\varphi_1(\mathbf{z}), \varphi_2(\mathbf{z}))$  be any decoder such that  $\varphi_1: \mathcal{Z}^n \rightarrow \{1, \dots, N\}$  and  $\varphi_2: \mathcal{Z}^n \rightarrow \{1, \dots, M\}$ . If  $N \geq 2$ , we will show below, by using a procedure similar to that in [8], that there exists some  $\mathbf{s} \in \mathcal{S}^n$  with

$$\frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\{\mathbf{z} \in \mathcal{Z}^n : \varphi(\mathbf{z}) \neq (i, j)\} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \geq 1/4. \quad (3.4)$$

In other words, if  $N > 1$ , the code can not have an arbitrarily small probability of error for *every*  $\mathbf{s} \in \mathcal{S}^n$ . Since the alternative  $N = 1$  implies  $\frac{\log N}{n} = 0$ , all achievable rate pairs must have the form  $(0, R_2)$ . Clearly,  $C_2(W)$  is the largest value of  $R_2$  such that the pair  $(0, R_2)$  is achievable. By Jahn's weak converse (2.4),  $C_2(W) \leq \sup_{p,q} I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W)$ .

Suppose  $N \geq 2$ . Since  $W$  is symmetrizable- $\mathcal{X}$ , let  $U$  be a symmetrizing transition probability satisfying (3.2). For each  $1 \leq i \leq N$ , let  $\mathbf{S}_i = (S_{i,1}, \dots, S_{i,n})$  be an  $\mathcal{S}^n$ -valued random variable whose components are independent and distributed according to

$$P(S_{i,k} = s) = U(s|x_{i,k}), \quad 1 \leq k \leq n.$$

Observe that for all  $\mathbf{z} \in \mathcal{Z}^n$ , and all  $i, i'$ , and  $j$ ,

$$\begin{aligned} \mathbb{E}[W^n(\mathbf{z}|\mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{S}_i)] &= \prod_{k=1}^n \mathbb{E}[W(z_k|x_{i',k}, y_{j,k}, S_{i,k})] \\ &= \prod_{k=1}^n \sum_s W(z_k|x_{i',k}, y_{j,k}, s)U(s|x_{i,k}) \\ &= \prod_{k=1}^n \sum_s W(z_k|x_{i,k}, y_{j,k}, s)U(s|x_{i',k}) \\ &= \mathbb{E}[W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{S}_{i'})], \end{aligned} \tag{3.5}$$

with the third equality following from symmetrizability- $\mathcal{X}$ . Next, let

$$e(i', i) \triangleq \frac{1}{M} \sum_{j=1}^M \mathbb{E}[W^n(\{\mathbf{z} : \varphi_1(\mathbf{z}) \neq i'\}|\mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{S}_i)].$$

By (3.5),

$$\begin{aligned} e(i', i) &= \frac{1}{M} \sum_{j=1}^M \sum_{\mathbf{z}:\varphi_1(\mathbf{z}) \neq i'} \mathbb{E}[W^n(\mathbf{z}|\mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{S}_i)] \\ &= \frac{1}{M} \sum_{j=1}^M \mathbb{E}\left[ \sum_{\mathbf{z}:\varphi_1(\mathbf{z}) \neq i'} W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{S}_{i'}) \right]. \end{aligned} \tag{3.6}$$

Now, if  $i \neq i'$  and  $\varphi_1(\mathbf{z}) = i$ , then  $\varphi_1(\mathbf{z}) \neq i'$ . With this fact in mind, we can use (3.6) to write, if  $i \neq i'$ ,

$$\begin{aligned} e(i, i') &+ e(i', i) \\ &\geq \frac{1}{M} \sum_{j=1}^M \mathbb{E}\left[ \sum_{\mathbf{z}:\varphi_1(\mathbf{z}) \neq i} W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{S}_{i'}) + \sum_{\mathbf{z}:\varphi_1(\mathbf{z})=i} W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{S}_{i'}) \right] \\ &= 1. \end{aligned} \tag{3.7}$$

Now, recalling the definition of  $e(\mathbf{s})$  in (2.2), we observe that

$$e(\mathbf{s}) \geq e_{\mathcal{X}}(\mathbf{s}) \triangleq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\{\mathbf{z} \in \mathcal{Z}^n : \varphi_1(\mathbf{z}) \neq i\} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}).$$

Then

$$\mathbb{E}[e_{\mathcal{X}}(\mathbf{S}_{i'})] = \frac{1}{N} \sum_{i=1}^N e(i, i').$$

Next, observe that

$$\begin{aligned} \frac{1}{N} \sum_{i'=1}^N \mathbb{E}[e_{\mathcal{X}}(\mathbf{S}_{i'})] &= \frac{1}{N^2} \sum_{i'} \sum_i e(i, i') \\ &\geq \frac{1}{N^2} \frac{N(N-1)}{2}, \quad \text{by (3.7),} \\ &= \frac{N-1}{2N} \\ &\geq 1/4, \quad \text{since } N \geq 2. \end{aligned}$$

From this it follows that for some  $i'$ ,  $\mathbb{E}[e_{\mathcal{X}}(\mathbf{S}_{i'})] \geq 1/4$ , which in turn implies the existence of some  $\mathbf{s} \in \mathcal{S}^n$  with  $e(\mathbf{s}) \geq e_{\mathcal{X}}(\mathbf{s}) \geq 1/4$  so that (3.4) holds.  $\square$

By interchanging the roles of  $\mathcal{X}$  and  $\mathcal{Y}$ , we have the obvious analog of the preceding lemma.

**Lemma 3.6** *If the AVC  $W$  is symmetrizable- $\mathcal{Y}$ , then*

$$C(W) = [0, C_1(W)] \times \{0\},$$

where  $C_1(W) \leq \sup_{p,q} I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}^*(p, q, W)$ .

**Corollary 3.7** *If the AVC  $W$  is symmetrizable- $\mathcal{X}$  and symmetrizable- $\mathcal{Y}$ , then*

$$C(W) = \{(0, 0)\}.$$

Clearly, the three kinds of symmetrizability defined above give simple conditions under which  $C(W)$  will have an empty interior. We conjecture that if  $W$  is nonsymmetrizable- $\mathcal{X}\mathcal{Y}$ , nonsymmetrizable- $\mathcal{X}$ , and nonsymmetrizable- $\mathcal{Y}$ , then every pair  $(R_1, R_2) \in \mathcal{R}^*(W)$  is achievable; we have been unable to prove this. In order to state what we can prove, we need the following two definitions.

**Definition 3.8** For any  $q \in \mathcal{D}(\mathcal{Y})$ , set  $(qW)(z|x, s) \triangleq \sum_y q(y)W(z|x, y, s)$ . We say that  $qW$  is *symmetrizable- $\mathcal{X}$*  if there exists a transition probability  $U$  from  $\mathcal{X}$  into  $\mathcal{S}$  such that

$$\sum_s (qW)(z|x, s)U(s|x') = \sum_s (qW)(z|x', s)U(s|x), \quad \forall x, x', z. \quad (3.8)$$

If no such  $U$  exists, we say that  $qW$  is *nonsymmetrizable- $\mathcal{X}$* .

**Remark.** If  $W$  is *symmetrizable- $\mathcal{X}\mathcal{Y}$*  and  $U$  satisfies (3.1), and if  $q$  is any element of  $\mathcal{D}(\mathcal{Y})$ , then multiplying both sides by  $q(y)q(y')$  and summing over all  $y, y'$  shows that  $qW$  is *symmetrizable- $\mathcal{X}$* . Similarly, if  $W$  is *symmetrizable- $\mathcal{X}$*  and  $U$  satisfies (3.2), multiplying both sides by  $q(y)$  and summing over all  $y$  shows that  $qW$  is *symmetrizable- $\mathcal{X}$*  for every  $q \in \mathcal{D}(\mathcal{Y})$ .

**Definition 3.9** For any  $p \in \mathcal{D}(\mathcal{X})$ , set  $(pW)(z|y, s) \triangleq \sum_x p(x)W(z|x, y, s)$ . We say that  $pW$  is *symmetrizable- $\mathcal{Y}$*  if there exists a transition probability  $U$  from  $\mathcal{Y}$  into  $\mathcal{S}$  such that

$$\sum_s (pW)(z|y, s)U(s|y') = \sum_s (pW)(z|y', s)U(s|y), \quad \forall y, y', z. \quad (3.9)$$

If no such  $U$  exists, we say that  $pW$  is *nonsymmetrizable- $\mathcal{Y}$* .

#### IV. DECODING RULES

Our major results are presented in the form of theorems stated in next section; the purpose of this section is to introduce the decoding rule used in the proof of these results. As this decoding rule is very complicated, we also introduce the much simpler (and universal) *maximum mutual information decoding rule* (MMI decoding rule). Conditions under which an MMI decoder will be effective are given in Theorem 5.5.

##### A. The Primary Decoding Rule

The decoding rule we use is an extension of the single-user rule developed in [8]. To define our decoding rule, we proceed as follows. Recall that the *type*<sup>2</sup> of an  $n$ -tuple  $\mathbf{x} \in \mathcal{X}^n$

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<sup>2</sup>The reader unfamiliar with the notion of types may consult Csiszár and Körner [4, pp. 29-33].

is defined to be the probability distribution  $P_{\mathbf{x}}$  given by  $P_{\mathbf{x}}(a) \triangleq N(a|\mathbf{x})/n$  for  $a \in \mathcal{X}$ , where  $N(a|\mathbf{x})$  denotes the number of occurrences of  $a$  in the  $n$ -tuple  $\mathbf{x}$ . In an analogous way, the *joint type* of a pair of  $n$ -tuples,  $\mathbf{x}$  and  $\mathbf{y}$ , is defined by  $P_{\mathbf{x},\mathbf{y}}(a,b) \triangleq N(a,b|\mathbf{x},\mathbf{y})/n$  for  $a \in \mathcal{X}$  and  $b \in \mathcal{Y}$ , where  $N(a,b|\mathbf{x},\mathbf{y})$  denotes the number of occurrences of  $(a,b)$  in the  $n$ -tuple  $((x_1,y_1),\dots,(x_n,y_n))$ . Let  $\mathcal{D}_n(\mathcal{X})$  denote the set of types generated by  $\mathcal{X}^n$ ; more precisely,  $\mathcal{D}_n(\mathcal{X})$  is the set of  $P \in \mathcal{D}(\mathcal{X})$  such that  $P = P_{\mathbf{x}}$  for some  $\mathbf{x} \in \mathcal{X}^n$ . Let  $P \in \mathcal{D}_n(\mathcal{X})$  and  $Q \in \mathcal{D}_n(\mathcal{Y})$  be types chosen as in the proof of Theorem 5.1 in Appendix A. Also, let  $\eta$  be as in Appendix A. We shall use the following subsets of  $\mathcal{Z}^n$ . For  $\mathbf{s} \in \mathcal{S}^n$ , and  $i = 1, \dots, N$ , let

$$J_i^0(\mathbf{s}) \triangleq \{\mathbf{z} \in \mathcal{Z}^n : D(P_{\mathbf{x}_i, \mathbf{s}, \mathbf{z}} \| P \times P_{\mathbf{s}} \times QW) \leq \eta\}, \quad (4.1)$$

where  $(P \times P_{\mathbf{s}} \times QW)(x, s, z) \triangleq \sum_y P(x)Q(y)P_{\mathbf{s}}(s)W(z|x, y, s)$ , and where  $D(\cdot\|\cdot)$  denotes the *Kullback-Leibler informational divergence* [4, p. 20]. Next, let

$$J_i^0 \triangleq \bigcup_{\mathbf{s}'' \in \mathcal{S}^n} J_i^0(\mathbf{s}'').$$

If  $\mathbf{z} \in J_i^0(\mathbf{s})$ , then we say that  $(\mathbf{x}_i, \mathbf{s}, \mathbf{z})$  is *jointly typical*. Thus, if  $\mathbf{z} \in J_i^0$ , there must be some  $\mathbf{s}'' \in \mathcal{S}^n$  with  $(\mathbf{x}_i, \mathbf{s}'', \mathbf{z})$  jointly typical. What we would like to do is use a decoder which decides message  $i$  was sent whenever  $\mathbf{z} \in J_i^0$  and  $\mathbf{z} \notin J_{i'}^0$  for all  $i' \neq i$ . In other words, if there is a unique  $i$  such that  $\mathbf{z} \in J_i^0$ , then we would decide message  $i$  was sent. Unfortunately, this approach, sometimes called *typicality decoding*, will not suffice for a general AVC [8]. We need a stronger decoding rule. To help us decide between  $i$  and  $i'$  when  $\mathbf{z}$  belongs to both  $J_i^0$  and  $J_{i'}^0$ , we will use the set

$$J_i^1(\mathbf{s}) \triangleq \{\mathbf{z} \in \mathcal{Z}^n : \forall i' \neq i, \mathbf{z} \in J_{i'}^0 \implies I(\mathbf{x}_i; \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) \leq \eta\}, \quad (4.2)$$

where  $I(\mathbf{x}_i; \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s})$  denotes  $I(XZ \wedge X' | S)$  computed using  $P_{X X' S Z} = P_{\mathbf{x}_i, \mathbf{x}_{i'}, \mathbf{s}, \mathbf{z}}$ . Let

$$F_i \triangleq \bigcup_{\mathbf{s}' \in \mathcal{S}^n} [J_i^0(\mathbf{s}') \cap J_i^1(\mathbf{s}')]. \quad (4.3)$$

We note that this definition implies that for any fixed  $\mathbf{s} \in \mathcal{S}^n$ ,

$$F_i^c = \bigcap_{\mathbf{s}' \in \mathcal{S}^n} [J_i^0(\mathbf{s}')^c \cup J_i^1(\mathbf{s}')^c] \subset [J_i^0(\mathbf{s})^c \cup J_i^1(\mathbf{s})^c]. \quad (4.4)$$

We claim that  $F_1, \dots, F_N$  are pairwise disjoint. This is a consequence of the assumption in the statement of Theorem 5.1 that  $q|W$  is nonsymmetrizable- $\mathcal{X}$ .<sup>3</sup> Let  $\varphi_1$  be any mapping defined on  $\mathcal{Z}^n$  such that for each  $i$ ,

$$\mathbf{z} \in F_i \implies \varphi_1(\mathbf{z}) = i, \quad (4.5)$$

i.e.,  $F_i \subset \mathcal{Z}^n$  is the *decoding set* for message  $i$ . Note that in general,  $\bigcup_{i=1}^N F_i$  is a proper subset of  $\mathcal{Z}^n$ ; however, it will turn out that any  $\varphi_1$  satisfying (4.5) will suffice. To summarize, the mapping  $\varphi_1$  will assign message  $i$  to the output  $\mathbf{z}$  if for some  $\mathbf{s}'$ ,  $(\mathbf{x}_i, \mathbf{s}', \mathbf{z})$  is jointly typical *and*, whenever  $i' \neq i$  is such that  $(\mathbf{x}_{i'}, \mathbf{s}'', \mathbf{z})$  is jointly typical for some  $\mathbf{s}''$ , then  $I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}') \leq \eta$ . It remains to define the decoding rule for the messages of user 2. To this end, for each  $i = 1, \dots, N$  and each  $j = 1, \dots, M$ , let

$$\begin{aligned} K_{ij}^0(\mathbf{s}) &\triangleq \{\mathbf{z} \in \mathcal{Z}^n : D(P_{\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}, \mathbf{z}} \| P \times Q \times P_{\mathbf{s}} \times W) \leq \eta\}, \\ K_{ij}^0 &\triangleq \bigcup_{\mathbf{s}'' \in \mathcal{S}^n} K_{ij}^0(\mathbf{s}''), \\ K_{ij}^1(\mathbf{s}) &\triangleq \{\mathbf{z} \in \mathcal{Z}^n : \forall j' \neq j, \mathbf{z} \in K_{ij'}^0 \implies I(\mathbf{x}_i \mathbf{y}_{j'} \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) \leq \eta\}. \end{aligned} \quad (4.6)$$

Now let

$$G_{ij} \triangleq \bigcup_{\mathbf{s}' \in \mathcal{S}^n} [K_{ij}^0(\mathbf{s}') \cap K_{ij}^1(\mathbf{s}')]. \quad (4.7)$$

For future reference, note that for any fixed  $\mathbf{s} \in \mathcal{S}^n$ ,

$$G_{ij}^c \subset [K_{ij}^0(\mathbf{s})^c \cup K_{ij}^1(\mathbf{s})^c]. \quad (4.8)$$

We claim that for each  $i$ ,  $G_{i1}, \dots, G_{iM}$  are pairwise disjoint. This is a consequence of the assumption in Theorem 5.1 that  $W$  is nonsymmetrizable- $\mathcal{Y}$ ; we establish this claim in Appendix B. Let  $\varphi_2$  be any mapping defined on  $\mathcal{Z}^n$  such that for all  $i, j$ ,

$$\mathbf{z} \in F_i \cap G_{ij} \implies \varphi_2(\mathbf{z}) = j.$$

Let

$$\varphi(\mathbf{z}) \triangleq (\varphi_1(\mathbf{z}), \varphi_2(\mathbf{z})).$$

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<sup>3</sup>As a consequence of Lemma A.2, if  $q|W$  is nonsymmetrizable- $\mathcal{X}$ , then a *type*  $Q$  can be chosen so that  $Q|W$  is also nonsymmetrizable- $\mathcal{X}$ . See also Appendix B.

In other words, we first try to decode message  $i$  from user 1, and only then do we try to decode message  $j$  from user 2. (The idea of first decoding message  $i$  and then decoding message  $j$  also appears in the context of source coding; see Slepian and Wolf [20].)

As demonstrated in Appendix A, this decoding rule enables us to prove Theorem 5.1. Since this decoding rule is so complicated, we now describe the much simpler and universal MMI decoding rule.

### B. The Maximum Mutual Information Decoding Rule

The decoder  $\varphi$  above was described in terms of the sets  $\{F_i\}$  and  $\{G_{ij}\}$  defined by (4.3) and (4.7). For a given  $\mathbf{z} \in \mathcal{Z}^n$ , determining which  $F_i$  and  $G_{ij}$  that  $\mathbf{z}$  belongs to would be a complex task. Consider the following maximum mutual information (MMI) decoder. Let

$$\hat{F}_i \triangleq \{\mathbf{z} \in \mathcal{Z}^n : I(\mathbf{x}_i \wedge \mathbf{z}) > I(\mathbf{x}_{i'} \wedge \mathbf{z}), \forall i' \neq i\} \quad (4.9)$$

and

$$\hat{G}_{ij} \triangleq \{\mathbf{z} \in \mathcal{Z}^n : I(\mathbf{y}_j \wedge \mathbf{z} | \mathbf{x}_i) > I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i), \forall j' \neq j\}. \quad (4.10)$$

Obviously, the  $\{\hat{F}_i\}_{i=1}^N$  are disjoint, as are the  $\{\hat{G}_{ij}\}_{j=1}^M$  for each  $i$ . If  $\hat{\varphi}(\mathbf{z}) = (\hat{\varphi}_1(\mathbf{z}), \hat{\varphi}_2(\mathbf{z}))$  has the property that

$$\mathbf{z} \in \hat{F}_i \implies \hat{\varphi}_1(\mathbf{z}) = i \quad (4.11)$$

and

$$\mathbf{z} \in \hat{F}_i \cap \hat{G}_{ij} \implies \hat{\varphi}_2(\mathbf{z}) = j, \quad (4.12)$$

then we say  $\hat{\varphi}$  is an MMI decoder. Clearly, the decoder  $\hat{\varphi}$  is much simpler than the decoder  $\varphi$  described above. More importantly,  $\hat{\varphi}$  is *universal* in the sense that the definition of the sets  $\{\hat{F}_i\}$  and  $\{\hat{G}_{ij}\}$  does not depend in any way on  $\mathcal{W}$ . In the next section we will discuss conditions under which an MMI decoder will be useful.

## V. MAJOR RESULTS

In this section we present nonsymmetrizability conditions under which the capacity region  $C(W)$  contains certain nonempty open rectangles of achievable rate pairs. We also give conditions under which an MMI decoder can be used to achieve rate pairs in these same rectangles.

**Theorem 5.1** *Suppose  $W$  is nonsymmetrizable- $\mathcal{Y}$ . Fix any  $p \in \mathcal{D}(\mathcal{X})$  and  $q \in \mathcal{D}(\mathcal{Y})$ . Further, suppose  $qW$  is nonsymmetrizable- $\mathcal{X}$ . If*

$$0 < R_1 < I_{\mathcal{X}\wedge\mathcal{Z}}^*(p, q, W) \tag{5.1}$$

and

$$0 < R_2 < I_{\mathcal{Y}\wedge\mathcal{Z}|\mathcal{X}}^*(p, q, W), \tag{5.2}$$

then  $(R_1, R_2)$  is achievable in the sense of Definition 2.2.

*Proof.* See Appendix A.

**Remark 5.2** Suppose  $p \in \mathcal{D}(\mathcal{X})$  and  $q \in \mathcal{D}(\mathcal{Y})$  are strictly positive. If  $qW$  is nonsymmetrizable- $\mathcal{X}$ , and if  $W$  is nonsymmetrizable- $\mathcal{Y}$ , then  $I_{\mathcal{X}\wedge\mathcal{Z}}^*(p, q, W)$  and  $I_{\mathcal{Y}\wedge\mathcal{Z}|\mathcal{X}}^*(p, q, W)$  are both strictly positive. To see this, suppose  $I_{\mathcal{X}\wedge\mathcal{Z}}^*(p, q, W) = 0$ . Then there is some  $r \in \mathcal{D}(\mathcal{S})$  with  $I_{\mathcal{X}\wedge\mathcal{Z}}(p \times q \times rW) = 0$ . This implies  $\sum_s (qW)(z|x, s)r(s)$  is not a function of  $x$ . But then taking  $U(s|x) = r(s)$  will symmetrize  $qW$ . Similarly, if  $I_{\mathcal{Y}\wedge\mathcal{Z}|\mathcal{X}}^*(p, q, W) = 0$ , there is some  $r \in \mathcal{D}(\mathcal{S})$  with  $I_{\mathcal{Y}\wedge\mathcal{Z}|\mathcal{X}}(p \times q \times rW) = 0$ . This implies  $\sum_s W(z|x, y, s)r(s)$  is not a function of  $y$ . Taking  $U(s|y) = r(s)$  shows that  $W$  is symmetrizable- $\mathcal{Y}$ . An analogous observation for single-user AVC's was made in [8].

Upon proving Theorem 5.1 we must also have the following analog obtained by interchanging the roles of  $\mathcal{X}$  and  $\mathcal{Y}$ .

**Theorem 5.3** *Suppose  $W$  is nonsymmetrizable- $\mathcal{X}$ . Fix any  $p \in \mathcal{D}(\mathcal{X})$  and  $q \in \mathcal{D}(\mathcal{Y})$ . Further, suppose  $pW$  is nonsymmetrizable- $\mathcal{Y}$ . If*

$$0 < R_1 < I_{\mathcal{X}\wedge\mathcal{Z}|\mathcal{Y}}^*(p, q, W) \quad \text{and} \quad 0 < R_2 < I_{\mathcal{Y}\wedge\mathcal{Z}}^*(p, q, W),$$

then  $(R_1, R_2)$  is achievable in the sense of Definition 2.2.



We now state our major result.

**Theorem 5.4** *If  $W$  is nonsymmetrizable- $\mathcal{Y}$  and there exists a  $q \in \mathcal{D}(\mathcal{Y})$  such that  $qW$  is nonsymmetrizable- $\mathcal{X}$ , or if  $W$  is nonsymmetrizable- $\mathcal{X}$  and there exists a  $p \in \mathcal{D}(\mathcal{X})$  such that  $pW$  is nonsymmetrizable- $\mathcal{Y}$ , then  $C(W) = \mathcal{R}^*(W)$ .*

*Proof.* See Appendix A.

As the proof in Appendix A shows, Theorem 5.4 relies on Jahn's forward result, inclusion (2.5), together with our forward results, Theorems 5.1 and 5.3. We will say more about this in the next section.

Our final result gives conditions under which an MMI decoder can be used in the proof of Theorem 5.1.

**Theorem 5.5** *If  $p \in \mathcal{D}(\mathcal{X})$  and  $q \in \mathcal{D}(\mathcal{Y})$  are such that*

$$I_{\mathcal{X} \wedge \mathcal{Z}}(p \times q \times r \times W) > I_{\mathcal{S} \wedge \mathcal{Z}}(p \times q \times r \times W), \quad \forall r \in \mathcal{D}(\mathcal{S}), \quad (5.3)$$

and

$$I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(p \times q \times r \times W) > I_{\mathcal{S} \wedge \mathcal{Z} | \mathcal{X}}(p \times q \times r \times W), \quad \forall r \in \mathcal{D}(\mathcal{S}), \quad (5.4)$$

then for

$$0 < R_1 < I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W) \quad \text{and} \quad 0 < R_2 < I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W),$$

there exists an  $\varepsilon > 0$  such that for all sufficiently large  $n$ , if  $N \triangleq \lfloor \exp(nR_1) \rfloor$  and  $M \triangleq \lfloor \exp(nR_2) \rfloor$ , then there exist codewords for user 1,  $\mathbf{x}_1, \dots, \mathbf{x}_N$ , each in  $\mathcal{X}^n$ , and there exist codewords for user 2,  $\mathbf{y}_1, \dots, \mathbf{y}_M$ , each in  $\mathcal{Y}^n$ , and there exists an MMI decoder  $\hat{\varphi}$  with

$$\frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\{\mathbf{z} \in \mathcal{Z}^n : \hat{\varphi}(\mathbf{z}) \neq (i, j)\} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp(-n\varepsilon)$$

holding uniformly for every  $\mathbf{s} \in \mathcal{S}^n$ .

*Proof.* See Appendix D.

Observe that Theorem 5.5 makes no explicit assumptions concerning nonsymmetrizability. We emphasize that the proof of Theorem 5.1 uses the nonsymmetrizability assumptions only to show that the decoding sets  $\{F_i\}$  and the  $\{G_{ij}\}$  are disjoint. For the MMI decoder, the  $\{\hat{F}_i\}$  and the  $\{\hat{G}_{ij}\}$  are obviously disjoint. We also point out that while no knowledge of  $W$  is required to define  $\hat{\varphi}$ , the conditions (5.3) and (5.4) *do* depend on  $W$ .

## VI. CONCLUSIONS

In 1981, Jahn [16] characterized the capacity region  $C(W)$  of the multiple-access AVC, *assuming that  $C(W)$  had a nonempty interior*. Jahn did not address the question of how one could decide *a priori* whether or not  $C(W)$  had a nonempty interior. In Section III, we showed that if  $W$  is symmetrizable in the sense of Definitions 3.1, 3.2, or 3.3, then  $C(W)$  has an empty interior. We then gave sufficient *nonsymmetrizability* conditions under which  $C(W)$  contains various open rectangles, and thereby possesses a nonempty interior (cf. Theorems 5.1 and 5.3). However, we still have the following open problem. If  $W$  is nonsymmetrizable- $\mathcal{X}$ , nonsymmetrizable- $\mathcal{Y}$ , and nonsymmetrizable- $\mathcal{X}\mathcal{Y}$ , does it follow that  $C(W)$  has a nonempty interior? We conjecture that this is the case.

To prove Theorem 5.4, in Appendix A we appeal to Jahn's forward result, inclusion (2.5), to show that if  $W$  is nonsymmetrizable- $\mathcal{Y}$  and  $qW$  is nonsymmetrizable- $\mathcal{X}$ , then  $\mathcal{R}^*(W) \subset C(W)$ . To see why we took this approach, suppose that  $W$  is nonsymmetrizable- $\mathcal{X}$  and nonsymmetrizable- $\mathcal{Y}$ , and suppose that for some  $p \in \mathcal{D}(\mathcal{X})$  and  $q \in \mathcal{D}(\mathcal{Y})$ ,  $pW$  is nonsymmetrizable- $\mathcal{Y}$  and  $qW$  is nonsymmetrizable- $\mathcal{X}$ . Then Theorem 5.1 and Theorem 5.3 do not in general combine even to show that  $C(W)$  contains the region

$$\{(R_1, R_2) : 0 \leq R_1 \leq I_{\mathcal{X}\wedge\mathcal{Z}|\mathcal{Y}}^*(p, q, W), \quad 0 \leq R_2 \leq I_{\mathcal{Y}\wedge\mathcal{Z}|\mathcal{X}}^*(p, q, W), \\ 0 \leq R_1 + R_2 \leq I_{\mathcal{X}\mathcal{Y}\wedge\mathcal{Z}}^*(p, q, W)\}. \quad (6.1)$$

This can be seen by considering the inequalities

$$I_{\mathcal{X}\mathcal{Y}\wedge\mathcal{Z}}^*(p, q, W) \triangleq \inf_{r \in \mathcal{D}(\mathcal{S})} I_{\mathcal{X}\mathcal{Y}\wedge\mathcal{Z}}(p \times q \times rW) \\ = \inf_{r \in \mathcal{D}(\mathcal{S})} \left[ I_{\mathcal{X}\wedge\mathcal{Z}}(p \times q \times rW) + I_{\mathcal{Y}\wedge\mathcal{Z}|\mathcal{X}}(p \times q \times rW) \right]$$

$$\begin{aligned}
&\geq \inf_{r \in \mathcal{D}(\mathcal{S})} I_{\mathcal{X} \wedge \mathcal{Z}}(p \times q \times rW) + \inf_{r \in \mathcal{D}(\mathcal{S})} I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(p \times q \times rW) \\
&= I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W) + I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W)
\end{aligned} \tag{6.2}$$

and

$$I_{\mathcal{X} \mathcal{Y} \wedge \mathcal{Z}}^*(p, q, W) \geq I_{\mathcal{Y} \wedge \mathcal{Z}}^*(p, q, W) + I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}^*(p, q, W). \tag{6.3}$$

If either inequality is strict, then the closed convex hull of the union of the open rectangles

$$\{(R_1, R_2) : 0 < R_1 < I_{\mathcal{X} \wedge \mathcal{Z} | \mathcal{Y}}^*(p, q, W), 0 < R_2 < I_{\mathcal{Y} \wedge \mathcal{Z}}^*(p, q, W)\}$$

and

$$\{(R_1, R_2) : 0 < R_1 < I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W), 0 < R_2 < I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W)\}$$

will be a proper subset of the region in (6.1). It follows that in general, our approach cannot give a direct proof that  $\mathcal{R}^*(W) \subset C(W)$ . As a possible topic of future research, we suggest that a more complicated decoding rule might overcome this difficulty (cf. the Remark at the end of Appendix A). Of course, in the special case that for every  $p \in \mathcal{D}(\mathcal{X})$  and every  $q \in \mathcal{D}(\mathcal{Y})$ , one can show that each of the five different infima in (6.3) and (6.2) is achieved by the same  $\hat{r} \in \mathcal{D}(\mathcal{S})$  ( $\hat{r}$  depending on  $p$  and  $q$ ), Theorems 5.1 and 5.3 can be combined with a time-sharing argument to give a proof that  $\mathcal{R}^*(W) \subset C(W)$  without appealing to Jahn's result.

A very important part of our proof of Theorem 5.1 was the decoding rule defined in terms of the decoding sets  $F_i$  and  $G_{ij}$  (cf. (4.3) and (4.7)). As we pointed out, this decoding rule is significantly more powerful than the so-called typicality decoding rule. However, as seen from the definition of the sets  $F_i$  and  $G_{ij}$ , our decoding rule is quite complicated. For this reason, we included Theorem 5.5 to give conditions under which the simpler (and universal) maximum mutual information decoding rule could be used in the proof of Theorem 5.1.

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APPENDIX A

PROOF OF THEOREMS 5.1 AND 5.4

Recall that the *variational distance* between any two distributions,  $P, Q \in \mathcal{D}(\mathcal{X})$ , is given by

$$d(P, Q) \triangleq \sum_{x \in \mathcal{X}} |P(x) - Q(x)|.$$

**Lemma A.1** (*Projection*). For every  $P_{XY}$  and  $Q_{XY}$  in  $\mathcal{D}(\mathcal{X} \times \mathcal{Y})$ ,

$$d(P_X, Q_X) \leq d(P_{XY}, Q_{XY}), \quad \text{and} \quad D(P_X \| Q_X) \leq D(P_{XY} \| Q_{XY}).$$

**Notation.** For joint types, if  $P_{XY} \in \mathcal{D}_n(\mathcal{X} \times \mathcal{Y})$ , we say that  $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}_{XY}$  if and only if  $P_{\mathbf{x}, \mathbf{y}} = P_{XY}$ . We denote by  $\mathcal{D}_n(\mathcal{Y} | \mathbf{x})$  the set of all  $P_{XY} \in \mathcal{D}_n(\mathcal{X} \times \mathcal{Y})$  such that  $P_X = P_{\mathbf{x}}$ .

Before proceeding with the proofs of Theorems 5.1 and 5.4, we need to introduce two auxiliary functions and an associated lemma. Observe that  $W$  is *symmetrizable- $\mathcal{Y}$*  if and only if for some transition probability  $U$  from  $\mathcal{Y}$  into  $\mathcal{S}$  we have

$$F_{\mathcal{Y}}^W(U) = 0,$$

where

$$F_{\mathcal{Y}}^W(U) \triangleq \max_{x, y, y', z} \left| \sum_s W(z|x, y, s) U(s|y') - \sum_s W(z|x, y', s) U(s|y) \right|. \quad (\text{A.1})$$

Now, let

$$\xi_{\mathcal{Y}}(W) \triangleq \inf_U F_{\mathcal{Y}}^W(U). \quad (\text{A.2})$$

Since  $F_{\mathcal{Y}}^W$  is a continuous function on the compact set of transition probabilities from  $\mathcal{Y}$  into  $\mathcal{S}$ , the infimum in (A.2) is always achieved. It follows that  $W$  is *symmetrizable- $\mathcal{Y}$*  if and only if  $\xi_{\mathcal{Y}}(W) = 0$ . Similarly, if  $q \in \mathcal{D}(\mathcal{Y})$ , and if  $U$  is any transition probability from  $\mathcal{X}$  into  $\mathcal{S}$ , we let

$$F_{\mathcal{X}}^W(q, U) \triangleq \max_{x, x', z} \left| \sum_s (qW)(z|x, s) U(s|x') - \sum_s (qW)(z|x', s) U(s|x) \right|, \quad (\text{A.3})$$

and we set

$$\xi_{\mathcal{X}}(q, W) \triangleq \inf_U F_{\mathcal{X}}^W(q, U). \quad (\text{A.4})$$

The following lemma says that  $\xi_{\mathcal{X}}(q, W)$  is a uniformly continuous function of  $q \in \mathcal{D}(\mathcal{Y})$ .

**Lemma A.2** For any  $q, \hat{q} \in \mathcal{D}(\mathcal{Y})$ ,

$$|\xi_{\mathcal{X}}(q, W) - \xi_{\mathcal{X}}(\hat{q}, W)| \leq d(q, \hat{q}).$$

*Proof.* See [14, Lemma 3.18, p. 39].

*Proof of Theorem 5.4.* By Jahn's weak converse, inclusion (2.4),  $C(W) \subset \mathcal{R}^*(W)$ . Now, suppose that  $W$  is nonsymmetrizable- $\mathcal{Y}$  and that for some  $q$ ,  $qW$  is nonsymmetrizable- $\mathcal{X}$ . By combining the preceding lemma with the fact that every  $q \in \mathcal{D}(\mathcal{Y})$  can be approximated by a strictly positive distribution, we may assume that  $q$  is strictly positive. Choose any positive  $p \in \mathcal{D}(\mathcal{X})$ . By Remark 5.2,  $I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W)$  and  $I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W)$  are both positive. By Theorem 5.1,  $C(W)$  has a nonempty interior, and by Jahn's forward result, inclusion (2.5),  $\mathcal{R}^*(W) \subset C(W)$ .  $\square$

*Proof of Theorem 5.1.* Let us first state explicitly what we shall prove.

Provided that the hypotheses of the theorem hold, we shall prove that there exists an  $\varepsilon > 0$  such that for all sufficiently large  $n$ , if we take  $N = \lfloor \exp(nR_1) \rfloor$  and  $M = \lfloor \exp(nR_2) \rfloor$ , then there exist codewords  $\mathbf{x}_1, \dots, \mathbf{x}_N$  for user 1, each in  $\mathcal{X}^n$ , and there exist codewords  $\mathbf{y}_1, \dots, \mathbf{y}_M$  for user 2, each in  $\mathcal{Y}^n$ , and there exists a decoder  $\varphi$  with

$$\frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\{\mathbf{z} \in \mathcal{Z}^n : \varphi(\mathbf{z}) \neq (i, j)\} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp(-n\varepsilon/8), \quad \forall \mathbf{s} \in \mathcal{S}^n. \quad (\text{A.5})$$

Now, suppose that  $R_1$  satisfies (5.1) and  $R_2$  satisfies (5.2). Then we can choose  $\delta > 0$  so small that (cf. (A.2) and (A.4))

$$\begin{aligned} 0 < 2\delta &< \min\{\xi_{\mathcal{X}}(q, W), \xi_{\mathcal{Y}}(W)\}, \\ 0 < R_1 &< I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W) - 2\delta, \\ 0 < R_2 &< I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W) - 2\delta. \end{aligned} \quad (\text{A.6})$$

Next, observe that we can always find  $\hat{p} \in \mathcal{D}(\mathcal{X})$  and  $\hat{q} \in \mathcal{D}(\mathcal{Y})$  such that for all  $x \in \mathcal{X}$ ,  $\hat{p}(x) > 0$ , and for all  $y \in \mathcal{Y}$ ,  $\hat{q}(y) > 0$ , and such that  $d(p, \hat{p})$  and  $d(q, \hat{q})$  are both sufficiently

small so that

$$\begin{aligned}\xi_{\mathcal{X}}(q, W) &\leq \xi_{\mathcal{X}}(\hat{q}, W) + \delta/2, \\ I_{\mathcal{X} \wedge \mathcal{Z}}^*(p, q, W) &\leq I_{\mathcal{X} \wedge \mathcal{Z}}^*(\hat{p}, \hat{q}, W) + \delta/2, \\ I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(p, q, W) &\leq I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(\hat{p}, \hat{q}, W) + \delta/2.\end{aligned}$$

Let  $\beta \triangleq \frac{1}{2} \min\{\min_x \hat{p}(x), \min_y \hat{q}(y)\} > 0$ . Choose  $\eta$  with

$$0 < \eta < \min\left\{\frac{\delta}{2}, \frac{\beta^4 \delta^2}{16 \ln 2}, \frac{\beta^6 \delta^2}{16 \ln 2}\right\}, \quad (\text{A.7})$$

and so small that if  $P^{(1)}$  and  $P^{(2)}$  are any two distributions on  $\mathcal{X} \times \mathcal{Z}$  or on  $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$  with  $D(P^{(1)} \| P^{(2)}) \leq \eta$ , then

$$|I_{\mathcal{X} \wedge \mathcal{Z}}(P^{(1)}) - I_{\mathcal{X} \wedge \mathcal{Z}}(P^{(2)})| < \delta/2 \quad (\text{A.8})$$

and

$$|I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(P^{(1)}) - I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(P^{(2)})| < \delta/2. \quad (\text{A.9})$$

Choose  $0 < \varepsilon < \min\{R_1, R_2, \eta/5\}$ . We must now specify how large  $n$  must be for the theorem to hold. Observe that for all sufficiently large  $n$ , we can find *types*  $P \in \mathcal{D}_n(\mathcal{X})$  and  $Q \in \mathcal{D}_n(\mathcal{Y})$  with  $d(\hat{p}, P)$  and  $d(\hat{q}, Q)$  both so small that not only do we have  $P(x) \geq \beta$  for all  $x \in \mathcal{X}$  and  $Q(y) \geq \beta$  for all  $y \in \mathcal{Y}$ , but also

$$\begin{aligned}\xi_{\mathcal{X}}(\hat{q}, W) &\leq \xi_{\mathcal{X}}(Q, W) + \delta/2, \\ I_{\mathcal{X} \wedge \mathcal{Z}}^*(\hat{p}, \hat{q}, W) &\leq I_{\mathcal{X} \wedge \mathcal{Z}}^*(P, Q, W) + \delta/2, \\ I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(\hat{p}, \hat{q}, W) &\leq I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(P, Q, W) + \delta/2,\end{aligned}$$

which implies

$$\delta \leq \xi_{\mathcal{X}}(Q, W), \quad (\text{A.10})$$

$$R_1 + \delta \leq I_{\mathcal{X} \wedge \mathcal{Z}}^*(P, Q, W), \quad (\text{A.11})$$

$$R_2 + \delta \leq I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(P, Q, W). \quad (\text{A.12})$$

We further assume that  $n$  is large enough that if

$$N \triangleq \lfloor \exp(nR_1) \rfloor \quad \text{and} \quad M \triangleq \lfloor \exp(nR_2) \rfloor, \quad (\text{A.13})$$

and if

$$\underline{R}_1 \triangleq \frac{\log N}{n} \quad \text{and} \quad \underline{R}_2 \triangleq \frac{\log M}{n},$$

then  $\varepsilon < \underline{R}_1 \leq R_1$  and  $\varepsilon < \underline{R}_2 \leq R_2$ . (We need  $\underline{R}_1 > \varepsilon$  and  $\underline{R}_2 > \varepsilon$  in order to apply Theorem C.1 below.) Hence, from (A.11) and (A.12) we get

$$\underline{R}_1 + \delta \leq I_{\mathcal{X} \wedge \mathcal{Z}}^*(P, Q, W), \quad (\text{A.14})$$

and

$$\underline{R}_2 + \delta \leq I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}^*(P, Q, W). \quad (\text{A.15})$$

In the remainder of this proof we drop the underscore from  $\underline{R}_1$  and  $\underline{R}_2$ ; hence, from here until the end of the proof, references to  $R_1$ ,  $R_2$ , (A.11), and (A.12) are actually references to  $\underline{R}_1$ ,  $\underline{R}_2$ , (A.14), and (A.15). Also note that this convention means that instead of (A.13), we can write

$$N = \exp(nR_1) \quad \text{and} \quad M = \exp(nR_2).$$

Regard  $n$  as fixed so large that we have found  $P \in \mathcal{D}_n(\mathcal{X})$  and  $Q \in \mathcal{D}_n(\mathcal{Y})$  satisfying (A.10), (A.11), and (A.12). Now, assuming  $n$  is large enough,<sup>4</sup> we select codewords for user 1,  $\mathbf{x}_1, \dots, \mathbf{x}_N$ , each of type  $P \in \mathcal{D}_n(\mathcal{X})$ , and we select codewords for user 2,  $\mathbf{y}_1, \dots, \mathbf{y}_M$ , each of type  $Q \in \mathcal{D}_n(\mathcal{Y})$ , such that the codeword properties we use below in the proof will hold. The fact that we can do this is the subject of Theorem C.1 in Appendix C. Since the properties that we need seem quite strange at first, we will not introduce them until they appear naturally in the course of the proof. The reader is referred to Theorem C.1 in Appendix C for a complete description of these properties.

The next step in the proof is to define the decoding rule. This was done in Section IV.A. After rereading Section IV.A, it is now easy to see that

$$\varphi(\mathbf{z}) \neq (i, j) \implies \varphi_1(\mathbf{z}) \neq i \quad \text{or} \quad \varphi_2(\mathbf{z}) \neq j,$$

or, in terms of the decoding sets,

$$\varphi(\mathbf{z}) \neq (i, j) \implies \mathbf{z} \in F_i^c \cup [F_i^c \cup G_{ij}^c] = F_i^c \cup G_{ij}^c.$$

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<sup>4</sup>How large depends only on  $\varepsilon$  and on the cardinalities of the sets  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{S}$ .

Now, for fixed  $\mathbf{s} \in \mathcal{S}^n$ , let  $e(\mathbf{s})$  be as in (2.2). Clearly, applying the union bound followed by (4.4) and (4.8) yields,

$$\begin{aligned}
e(\mathbf{s}) &\leq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(F_i^c \cup G_{ij}^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\
&\leq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(F_i^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\
&\quad + \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(G_{ij}^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\
&\leq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(J_i^0(\mathbf{s})^c \cup J_i^1(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\
&\quad + \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(K_{ij}^0(\mathbf{s})^c \cup K_{ij}^1(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}).
\end{aligned}$$

Next, observe that

$$K_{ij}^0(\mathbf{s}) \subset J_i^0(\mathbf{s}) \implies J_i^0(\mathbf{s})^c \subset K_{ij}^0(\mathbf{s})^c,$$

and apply the union bound again to obtain

$$\begin{aligned}
e(\mathbf{s}) &\leq \frac{2}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(K_{ij}^0(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\
&\quad + \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(J_i^1(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\
&\quad + \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(K_{ij}^1(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}).
\end{aligned} \tag{A.16}$$

We now turn to the task of bounding  $e(\mathbf{s})$  uniformly for  $\mathbf{s} \in \mathcal{S}^n$ . Each of the three preceding sums will be treated separately. To begin, let

$$e_0(\mathbf{s}) \triangleq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(K_{ij}^0(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}). \tag{A.17}$$

Set

$$A(\mathbf{s}) \triangleq \{i : I(\mathbf{x}_i \wedge \mathbf{s}) > \varepsilon\},$$

and

$$B(\mathbf{s}, \mathbf{x}_i) \triangleq \{j : I(\mathbf{y}_j \wedge \mathbf{x}_i \mathbf{s}) > \varepsilon\}.$$

Then

$$e_0(\mathbf{s}) \leq \frac{1}{N} |A(\mathbf{s})| + \frac{1}{N} \sum_{i \notin A(\mathbf{s})} \frac{1}{M} \sum_{j=1}^M W^n(K_{ij}^0(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}).$$



Continuing,

$$\begin{aligned}
e_0(\mathbf{s}) &\leq \frac{1}{N}|A(\mathbf{s})| + \frac{1}{N} \sum_{i \notin A(\mathbf{s})} \left( \frac{1}{M}|B(\mathbf{s}, \mathbf{x}_i)| + \frac{1}{M} \sum_{j \notin B(\mathbf{s}, \mathbf{x}_i)} W^n(K_{ij}^0(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \right) \\
&\leq \frac{1}{N}|A(\mathbf{s})| + \frac{1}{M} \max_i |B(\mathbf{s}, \mathbf{x}_i)| \\
&\quad + \frac{1}{NM} \sum_{i \notin A(\mathbf{s})} \left( \sum_{j \notin B(\mathbf{s}, \mathbf{x}_i)} W^n(K_{ij}^0(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \right). \tag{A.18}
\end{aligned}$$

Fix  $i \notin A(\mathbf{s})$  and  $j \notin B(\mathbf{s}, \mathbf{x}_i)$  and observe that if  $P_{XYSZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s})$ , then  $I(X \wedge S) \leq \varepsilon$  and  $I(Y \wedge XS) \leq \varepsilon$ . Now, write (cf. (4.6))

$$K_{ij}^0(\mathbf{s})^c = \bigcup_{P_{XYSZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) : D(P_{XYSZ} \| P \times Q \times P_S \times W) > \eta} T_{Z|XYS}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}).$$

Then using [4, inequality (2.8), p. 32], the union bound, and the Type Counting Lemma [4, Lemma 2.2, p. 29] (see Notes following (A.19) below),

$$\begin{aligned}
W^n(K_{ij}^0(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) &\leq \sum \exp(-nD(P_{XYSZ} \| P_{XYS} \times W)) \\
&= \sum \exp[-n(D(P_{XYSZ} \| P \times Q \times P_S \times W) \\
&\quad - I(X \wedge S) - I(Y \wedge XS))] \\
&\leq \sum \exp[-n(\eta - 2\varepsilon)] \\
&\leq (n+1)^{|\mathcal{X}||\mathcal{Y}||\mathcal{S}||\mathcal{Z}|} \exp[-n(\eta - 2\varepsilon)] \\
&\leq \exp[-n(\eta - 3\varepsilon)] \\
&\leq \exp(-2n\varepsilon), \quad \text{since } \eta \geq 5\varepsilon. \tag{A.19}
\end{aligned}$$

**Notes.** (i) The summations are understood to be over all

$$P_{XYSZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \text{ such that } D(P_{XYSZ} \| P \times Q \times P_S \times W) > \eta.$$

(ii) We assume  $n$  is so large that  $(n+1)^{|\mathcal{X}||\mathcal{Y}||\mathcal{S}||\mathcal{Z}|} \leq \exp(n\varepsilon)$ . We caution the reader that we will make similar assumptions as needed below without comment.

Now, it is a property of our codewords (Theorem C.1, inequalities (C.1) and (C.2)) that for all  $\mathbf{s} \in \mathcal{S}^n$ , and all  $\mathbf{x} \in \mathcal{X}^n$ ,

$$\frac{1}{N}|A(\mathbf{s})| \leq \exp(-n\varepsilon/2) \quad \text{and} \quad \frac{1}{M}|B(\mathbf{s}, \mathbf{x})| \leq \exp(-n\varepsilon/2).$$

Putting these inequalities along with (A.19) into (A.18) yields

$$e_0(\mathbf{s}) \leq 3 \exp(-n\varepsilon/4). \quad (\text{A.20})$$

We now bound the third sum in (A.16). The second sum is treated similarly. Let

$$e_1(\mathbf{s}) \triangleq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(K_{ij}^1(\mathbf{s})^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}). \quad (\text{A.21})$$

For each  $i, j$ , and  $\mathbf{s}$ , write

$$K_{ij}^1(\mathbf{s})^c = \bigcup_{j' \neq j} [K_{ij'}^0 \cap \{\mathbf{z} : I(\mathbf{x}_i, \mathbf{y}_j, \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\}].$$

We claim that

$$K_{ij'}^0 \subset \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta\}.$$

This is easily seen as follows. Suppose that  $\mathbf{z} \in K_{ij'}^0$ . Then there is some  $\mathbf{s}'' \in \mathcal{S}^n$  such that  $\mathbf{z} \in K_{ij'}^0(\mathbf{s}'')$ . This means that

$$D(P_{\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{s}'', \mathbf{z}} \| P \times Q \times P_{\mathbf{s}''} \times W) \leq \eta.$$

Applying the Projection Lemma A.1,

$$D(P_{\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{z}} \| P \times Q \times P_{\mathbf{s}''} W) \leq \eta. \quad (\text{A.22})$$

By (A.22) and the definition of  $\eta$  in regard to (A.9), we can write

$$\begin{aligned} I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) &= I_{\mathbf{y} \wedge \mathbf{z} | \mathcal{X}}(P_{\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{z}}) > I_{\mathbf{y} \wedge \mathbf{z} | \mathcal{X}}(P \times Q \times P_{\mathbf{s}''} W) - \delta/2 \\ &\geq I_{\mathbf{y} \wedge \mathbf{z} | \mathcal{X}}^*(P, Q, W) - \delta/2. \end{aligned}$$

Using (A.12), followed by the fact that we chose  $\eta < \delta/2$ ,

$$\begin{aligned} I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) &> (R_2 + \delta) - \delta/2 \\ &> R_2 + \eta. \end{aligned}$$

Having established our claim, we then see that

$$K_{ij}^1(\mathbf{s})^c \subset \bigcup_{j' \neq j} \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta \text{ and } I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\}. \quad (\text{A.23})$$

The next step is to write

$$\begin{aligned} \{j' \neq j\} &= \{j' \neq j\} \cap \bigcup_{P_{XY'Y'S}} \{j' : (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}\} \\ &= \bigcup_{P_{XY'Y'S}} \{j' \neq j : (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}\}, \end{aligned}$$

where the union is over *all* joint types  $P_{XY'Y'S} \in \mathcal{D}_n(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{S})$ . So, we can write

$$K_{ij}^1(\mathbf{s})^c \subset \bigcup_{P_{XY'Y'S}} \left( \bigcup_{j' \neq j : (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}} \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta \text{ and } I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\} \right).$$

We use this inclusion as follows. By setting

$$\theta_{ij}(\mathbf{s}) \triangleq \bigcup_{j' \neq j : (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}} \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta \text{ and } I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\}, \quad (\text{A.24})$$

it is clear that we can write

$$e_1(\mathbf{s}) \leq \sum_{P_{XY'Y'S}} \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}). \quad (\text{A.25})$$

Fix a type  $P_{XY'Y'S}$  and consider

$$\frac{1}{N} \sum_{i=1}^N \left( \frac{1}{M} \sum_{j=1}^M W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \right). \quad (\text{A.26})$$

Observe that if  $i$  has the property that there is no  $j'$  with  $(\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'S}$ , then  $\theta_{ij}(\mathbf{s}) = \phi$ , and  $W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) = 0$  for all  $j$ . Now, it is a property of our codewords (Theorem C.1, inequality (C.7)) that if

$$I(X \wedge Y'S) > |R_2 - I(Y' \wedge S)|^+ + \varepsilon, \quad (\text{A.27})$$

then

$$\frac{1}{N} |\{i : \exists j' \text{ with } (\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'S}\}| \leq \exp(-n\varepsilon/2).$$

Similarly, for fixed  $i$ , if  $j$  has the property that there is no  $j'$  with  $(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}$ , then  $\theta_{ij}(\mathbf{s}) = \phi$ , and  $W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) = 0$ . Again, it is a property of our codewords (Theorem C.1, inequality (C.8)) that if

$$I(Y \wedge XY'S) > |R_2 - I(Y' \wedge XS)|^+ + \varepsilon, \quad (\text{A.28})$$

then

$$\frac{1}{M} |\{j : \exists j' \neq j \text{ with } (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}\}| \leq \exp(-n\varepsilon/2).$$

From this we see that if (A.27) or (A.28) holds, then the quantity in (A.26) is bounded above by  $\exp(-n\varepsilon/2)$ .

We now consider (A.26) when (A.27) and (A.28) do not hold. Rewrite (A.24) as

$$\theta_{ij}(\mathbf{s}) = \bigcup_{j' \neq j: (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}} \zeta_{j'},$$

where

$$\zeta_{j'} \triangleq \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta \text{ and } I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\}.$$

We would like to apply the union bound to  $W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s})$ . Before doing so, we bound the quantity  $W^n(\zeta_{j'} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s})$  uniformly for  $j'$  such that  $(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'Y'S}$ . To do this, write

$$\begin{aligned} \zeta_{j'} &= \zeta_{j'} \cap \mathcal{Z}^n \\ &= \zeta_{j'} \cap \bigcup_{P_{XY'Y'SZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})} \mathcal{T}_{Z|XY'Y'S}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \\ &= \bigcup_{P_{XY'Y'SZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})} [\zeta_{j'} \cap \mathcal{T}_{Z|XY'Y'S}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})]. \end{aligned} \quad (\text{A.29})$$

Now, consider a set of the form

$$\gamma = \zeta_{j'} \cap \mathcal{T}_{Z|XY'Y'S}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})$$

for some joint type  $P_{XY'Y'SZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})$ . The first step is to bound  $W^n(\gamma | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s})$  independently of the particular type  $P_{XY'Y'SZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})$ . In other words, we need a bound that depends only on  $P_{XY'Y'S} = P_{\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}}$ . Now, if  $\mathbf{z} \in \gamma$ , then  $P_{\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}, \mathbf{z}} = P_{XY'Y'SZ}$  and

$$I(Y' \wedge Z | X) > R_2 + \eta, \quad (\text{A.30})$$

and

$$I(XYZ \wedge Y' | S) > \eta. \quad (\text{A.31})$$

In other words, either  $\gamma = \emptyset$  or (A.30) and (A.31) both hold. Now, observe that by [4, Lemma 2.5, p. 31],

$$|\gamma| \leq |\mathcal{T}_{Z|XY Y'S}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})| \leq \exp[nH(Z|XY Y'S)].$$

Then note that since

$$\gamma \subset \mathcal{T}_{Z|XY Y'S}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \subset \mathcal{T}_{Z|XY S}(\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}),$$

we have, by [4, equation (2.7), p. 32],

$$\mathbf{z} \in \gamma \implies W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-nH(Z|XY S)].$$

We can now write

$$\begin{aligned} W^n(\gamma|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) &= \sum_{\mathbf{z} \in \gamma} W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\ &\leq |\gamma| \exp[-nH(Z|XY S)] \\ &\leq \exp[nH(Z|XY Y'S)] \cdot \exp[-nH(Z|XY S)] \\ &= \exp[-nI(Y' \wedge Z|XY S)], \end{aligned} \tag{A.32}$$

where the distribution  $P_{XY Y'SZ}$  satisfies both (A.30) and (A.31). We must still lower bound  $I(Y' \wedge Z|XY S)$  independently of  $Z$ . There are four cases to consider:

1.  $R_2 \geq I(Y' \wedge XY S)$
2.  $I(Y' \wedge XY S) \geq R_2 \geq I(Y' \wedge XS)$
3.  $I(Y' \wedge XS) \geq R_2 \geq I(Y' \wedge S)$
4.  $I(Y' \wedge S) \geq R_2$ .

In the first three cases, we will use the inequality

$$\begin{aligned} I(Y' \wedge Z|XY S) &= I(Y' \wedge YSZ|X) - I(Y' \wedge YS|X) \\ &= I(Y' \wedge YSZ|X) - I(Y' \wedge XY S) + I(Y' \wedge X) \\ &\geq I(Y' \wedge YSZ|X) - I(Y' \wedge XY S) \\ &\geq I(Y' \wedge Z|X) - I(Y' \wedge XY S). \end{aligned} \tag{A.33}$$

By (A.30),

$$I(Y' \wedge Z | XYS) \geq R_2 + \eta - I(Y' \wedge XYS).$$

Substituting this into (A.32) yields

$$W^n(\gamma | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-n(\eta + R_2 - I(Y' \wedge XYS))],$$

independently of  $P_{XY'YSZ} \in \mathcal{D}_n(\mathcal{Z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s})$ . Applying the Type Counting Lemma to (A.29), we get

$$W^n(\zeta_{j'} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-n(\eta - \varepsilon + R_2 - I(Y' \wedge XYS))].$$

By another property of our codewords (Theorem C.1, inequality (C.5)),

$$|\{j' : (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'YS}\}| \leq \exp[n(|R_2 - I(Y' \wedge XYS)|^+ + \varepsilon)]. \quad (\text{A.34})$$

Thus

$$W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) = W^n\left(\bigcup_{j' \neq j: (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'YS}} \zeta_{j'} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}\right)$$

is bounded above by

$$\exp[-n(\eta - 2\varepsilon + R_2 - I(Y' \wedge XYS) - |R_2 - I(Y' \wedge XYS)|^+)].$$

In case 1 we get

$$W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-n(\eta - 2\varepsilon)].$$

In case 2 we use the fact that the inequality in (A.28) fails. This leads to

$$I(Y \wedge XY'S) \leq R_2 - I(Y' \wedge XS) + \varepsilon.$$

Rewriting this as

$$I(Y \wedge XY'S) + I(Y' \wedge XS) \leq R_2 + \varepsilon,$$

or equivalently as

$$I(Y' \wedge XYS) + I(Y \wedge XS) \leq R_2 + \varepsilon,$$

we obtain  $I(Y' \wedge XYS) \leq R_2 + \varepsilon$ . Thus in case 2,

$$W^n(\theta_{ij}(\mathbf{s}) | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-n(\eta - 3\varepsilon)].$$

In case 3 we use the fact that both inequalities, (A.27) and (A.28), fail. So,

$$I(Y \wedge XY'S) \leq \varepsilon,$$

and

$$I(X \wedge Y'S) + I(Y' \wedge S) \leq R_2 + \varepsilon.$$

Write

$$\begin{aligned} I(Y' \wedge XY'S) &= I(Y \wedge XY'S) + I(X \wedge Y'S) + I(Y' \wedge S) \\ &\quad - [H(X) + H(Y) + H(S) - H(XYS)] \\ &\leq R_2 + 2\varepsilon. \end{aligned}$$

So, in case 3,

$$W^n(\theta_{ij}(\mathbf{s})|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-n(\eta - 4\varepsilon)].$$

Since the bounds for the first two cases imply the third, in the first three cases, we may use the preceding inequality. Now, in case 4 use (A.31) to write

$$\begin{aligned} I(Y' \wedge Z|XY'S) &= I(XYZ \wedge Y'|S) - I(XY \wedge Y'|S) \\ &\geq \eta - I(XY \wedge Y'|S). \end{aligned}$$

We claim that  $I(XY \wedge Y'|S) \leq 2\varepsilon$ . Since the inequalities in (A.27) and (A.28) fail,

$$I(X \wedge Y'S) \leq \varepsilon \quad \text{and} \quad I(Y \wedge XY'S) \leq \varepsilon.$$

Writing

$$\begin{aligned} I(XY \wedge Y'|S) &= I(Y \wedge XY'S) + I(X \wedge Y'S) \\ &\quad - [H(X) + H(Y) + H(S) - H(XYS)] \\ &\leq 2\varepsilon, \end{aligned}$$

we have  $I(Y' \wedge Z|XY'S) \geq \eta - 2\varepsilon$ . Combining this with (A.32), and applying the Type Counting Lemma to (A.29) yields

$$W^n(\zeta_{j'}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq \exp[-n(\eta - 3\varepsilon)].$$

Since in case 4, the upper bound in (A.34) reduces to  $\exp(n\varepsilon)$ , we have

$$\begin{aligned} W^n(\theta_{ij}(\mathbf{s})|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) &= W^n\left(\bigcup_{j' \neq j: (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XYYS}} \zeta_{j'}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}\right) \\ &\leq \exp[-n(\eta - 4\varepsilon)]. \end{aligned}$$

We then have, in all four cases, when (A.27) and (A.28) both fail,

$$\begin{aligned} W^n(\theta_{ij}(\mathbf{s})|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) &\leq \exp[-n(\eta - 4\varepsilon)] \\ &\leq \exp(-n\varepsilon), \quad \text{since } \eta > 5\varepsilon, \\ &\leq \exp(-n\varepsilon/2). \end{aligned}$$

To summarize, regardless of (A.27) and (A.28), we always have the quantity in (A.26) bounded above by  $\exp(-n\varepsilon/2)$ . By (A.25) we have

$$\begin{aligned} e_1(\mathbf{s}) &\leq \sum_{P_{XYYS}} \exp(-n\varepsilon/2) \\ &\leq (n+1)^{|\mathcal{X}||\mathcal{Y}|^2|S|} \exp(-n\varepsilon/2) \\ &\leq \exp(n\varepsilon/4) \exp(-n\varepsilon/2) \\ &= \exp(-n\varepsilon/4). \end{aligned} \tag{A.35}$$

Finally, by using a similar procedure, whose main difference is that instead of (A.33) we use the fact that

$$\begin{aligned} I(X' \wedge Z|XY S) &= I(X' \wedge XYSZ) - I(X' \wedge XYS) \\ &\geq I(X' \wedge Z) - I(X' \wedge XYS), \end{aligned}$$

and also the fact that

$$\begin{aligned} I(X' \wedge Z|XY S) &= I(XYZ \wedge X'|S) - I(XY \wedge X'|S) \\ &\geq I(XZ \wedge X'|S) - I(XY \wedge X'|S), \end{aligned}$$

we can bound the middle term in (A.16) by  $\exp(-n\varepsilon/4)$ . Combining this with (A.20) and (A.35), we have, for every  $\mathbf{s} \in \mathcal{S}^n$ ,

$$e(\mathbf{s}) \leq 8 \exp(-n\varepsilon/4).$$



Since for all sufficiently large  $n$ ,  $8 \leq \exp(n\varepsilon/8)$ , we see that (A.5) holds.  $\square$

**Remark.** Before arriving at the decoding rule described in Section IV.A, we tried the following. Let

$$K_{ij}^2(\mathbf{s}) \triangleq \{\mathbf{z} \in \mathcal{Z}^n : \forall i' \neq i, \mathbf{z} \in K_{i'j}^0 \implies I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) \leq \eta\},$$

$$K_{ij}^3(\mathbf{s}) \triangleq \{\mathbf{z} \in \mathcal{Z}^n : \forall i' \neq i, \forall j' \neq j, \mathbf{z} \in K_{i'j'}^0 \implies I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{x}_{i'} \mathbf{y}_{j'} | \mathbf{s}) \leq \eta\},$$

and set

$$E_{ij} \triangleq \bigcup_{\mathbf{s}' \in \mathcal{S}^n} [K_{ij}^0(\mathbf{s}') \cap K_{ij}^1(\mathbf{s}') \cap K_{ij}^2(\mathbf{s}') \cap K_{ij}^3(\mathbf{s}')].$$

Then with only a little extra care, one can show that the  $\{E_{ij}\}$  are pairwise disjoint, provided that  $W$  is nonsymmetrizable- $\mathcal{X}\mathcal{Y}$ , nonsymmetrizable- $\mathcal{X}$ , and nonsymmetrizable- $\mathcal{Y}$ . One would then like to use any decoder  $\varphi$  with the property that

$$\mathbf{z} \in E_{ij} \implies \varphi(\mathbf{z}) = (i, j).$$

Our problem with this approach is that we have been unable to find a *useful* bound on (compare (A.34))

$$|\{(i', j') : (\mathbf{x}_i, \mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XX'YY'S}\}|.$$

## APPENDIX B

### THE DECODING SETS

In this appendix we prove our claim that for each  $i$ , the decoding sets  $\{G_{ij}\}_{j=1}^M$  defined in equation (4.7) are pairwise disjoint. Based on this proof, it can easily be shown that the sets  $\{F_i\}$  defined in equation (4.3) are also pairwise disjoint.

Suppose that for some pair  $j \neq j'$ ,  $\mathbf{z} \in G_{ij} \cap G_{ij'}$ . Since  $\mathbf{z} \in G_{ij}$ , there must be some  $\mathbf{s} \in \mathcal{S}^n$  with

$$\mathbf{z} \in K_{ij}^0(\mathbf{s}) \cap K_{ij}^1(\mathbf{s}).$$

Similarly, since  $\mathbf{z} \in G_{ij'}$ , there must be some  $\mathbf{s}' \in \mathcal{S}^n$  with

$$\mathbf{z} \in K_{ij'}^0(\mathbf{s}') \cap K_{ij'}^1(\mathbf{s}').$$

Now, since  $\mathbf{z} \in K_{ij'}^0(\mathbf{s}')$ ,  $\mathbf{z} \in K_{ij'}^0$ . Since we also have  $\mathbf{z} \in K_{ij}^1(\mathbf{s})$ , we conclude that

$$I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) \leq \eta. \quad (\text{B.1})$$

Arguing similarly, since  $\mathbf{z} \in K_{ij}^0(\mathbf{s})$ ,  $\mathbf{z} \in K_{ij}^0$ . Since we also have

$$\mathbf{z} \in K_{ij'}^1(\mathbf{s}') = \{\mathbf{z} \in \mathcal{Z}^n : \forall j \neq j', \mathbf{z} \in K_{ij}^0 \implies I(\mathbf{x}_i \mathbf{y}_{j'} \mathbf{z} \wedge \mathbf{y}_j | \mathbf{s}') \leq \eta\},$$

we conclude that

$$I(\mathbf{x}_i \mathbf{y}_{j'} \mathbf{z} \wedge \mathbf{y}_j | \mathbf{s}') \leq \eta. \quad (\text{B.2})$$

We also obviously have

$$D(P_{\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}, \mathbf{z}} \| P \times Q \times P_{\mathbf{s}} \times W) \leq \eta \quad \text{and} \quad D(P_{\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{s}', \mathbf{z}} \| P \times Q \times P_{\mathbf{s}'} \times W) \leq \eta. \quad (\text{B.3})$$

Let  $P_{XY Y' S S' Z} = P_{\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}, \mathbf{s}', \mathbf{z}}$ . Note that  $P_Y = P_{Y'} = Q$  and  $P_X = P$ . Thus,

$$D(P_{XY S Z} \| P \times Q \times P_S \times W) \leq \eta \quad \text{and} \quad I(X Y Z \wedge Y' | S) \leq \eta \quad (\text{B.4})$$

and

$$D(P_{XY' S' Z} \| P \times Q \times P_{S'} \times W) \leq \eta \quad \text{and} \quad I(X Y' Z \wedge Y | S') \leq \eta. \quad (\text{B.5})$$

We can rewrite the two inequalities in (B.5) as

$$\sum_{x, y, y', s, z} P_{XY Y' S' Z}(x, y, y', s, z) \log \frac{P_{XY' S' Z}(x, y', s, z)}{P(x)Q(y')P_{S'}(s)W(z|x, y', s)} \leq \eta$$

and

$$\sum_{x, y, y', s, z} P_{XY Y' S' Z}(x, y, y', s, z) \log \frac{P_{XY Y' S' Z}(x, y, y', s, z)P_{S'}(s)}{P_{XY' S' Z}(x, y', s, z)P_Y(y, s)} \leq \eta.$$

Adding these two inequalities yields

$$\sum_{x, y, y', s, z} P_{XY Y' S' Z}(x, y, y', s, z) \log \frac{P_{XY Y' S' Z}(x, y, y', s, z)}{P(x)Q(y)Q(y')W(z|x, y', s)P_{S'|Y}(s|y)} \leq 2\eta.$$

We recognize the preceding expression as an informational divergence. If we let

$$V'(z|x, y, y') \triangleq \sum_s W(z|x, y', s)P_{S'|Y}(s|y),$$

then applying the Projection Lemma A.1 yields

$$D(P_{XY Y' Z} \| P \times Q \times Q \times V') \leq 2\eta.$$

By Pinsker's Inequality [4, Problem 17, p. 58],

$$\begin{aligned} d(P_{XY Y' Z}, P \times Q \times Q \times V') &\leq \sqrt{(2 \ln 2) D(P_{XY Y' Z} \| P \times Q \times Q \times V')} \\ &\leq 2\sqrt{\eta \ln 2}. \end{aligned}$$

Next, starting with (B.4) and proceeding as above, we arrive at

$$d(P_{XY Y' Z}, P \times Q \times Q \times V) \leq 2\sqrt{\eta \ln 2},$$

where

$$V(z|x, y, y') \triangleq \sum_s W(z|x, y, s) P_{S|Y'}(s|y').$$

Since  $d$  is a metric, we can use the triangle inequality to get

$$\sum_{x, y, y', z} P(x) Q(y) Q(y') |V(z|x, y, y') - V'(z|x, y, y')| \leq 4\sqrt{\eta \ln 2}.$$

Recalling that  $P(x) \geq \beta > 0$  and  $Q(y) \geq \beta > 0$ ,

$$\max_{x, y, y', z} |V(z|x, y, y') - V'(z|x, y, y')| \leq \frac{4\sqrt{\eta \ln 2}}{\beta^3} < \delta,$$

since we chose  $\eta < \beta^6 \delta^2 / (16 \ln 2)$  in (A.7). Now, observe that the preceding maximum does not change if we interchange  $y$  and  $y'$  and then interchange  $V$  and  $V'$ . Hence, we also have

$$\max_{x, y, y', z} |V'(z|x, y', y) - V(z|x, y', y)| < \delta.$$

It is then easy to show that

$$\max_{x, y, y', z} \left| \frac{1}{2} [V(z|x, y, y') + V'(z|x, y', y)] - \frac{1}{2} [V'(z|x, y, y') + V(z|x, y', y)] \right| < \delta.$$

If we set  $U(s|y) \triangleq \frac{1}{2} [P_{S|Y}(s|y) + P_{S|Y'}(s|y)]$ , this becomes

$$\max_{x, y, y', z} \left| \sum_s W(z|x, y, s) U(s|y') - \sum_s W(z|x, y', s) U(s|y) \right| < \delta. \quad (\text{B.6})$$

In other words (cf. (A.1))  $F_{\mathcal{Y}}^W(U) < \delta$ , and so we must have (cf. (A.2))

$$\xi_{\mathcal{Y}}(W) < \delta,$$

contradicting (A.6).

Having established that for each  $i$ ,  $G_{i1}, \dots, G_{iM}$  are pairwise disjoint, it can be similarly established that  $F_1, \dots, F_N$  are pairwise disjoint; simply contradict (A.10) instead of (A.6).

## APPENDIX C

### THE CODEWORD PROPERTIES

In this appendix we present a theorem which establishes that for all sufficiently large  $n$ , we can always find a set of codewords for each user such that the properties used to prove Theorem 5.1 in Appendix A will hold. (See [8] for the analogous single-user properties.)

**Theorem C.1** (*Codeword Properties*). *Given  $\varepsilon > 0$ , there exists an  $n_0$  depending only on  $\varepsilon$ ,  $|\mathcal{X}|$ ,  $|\mathcal{Y}|$ , and  $|\mathcal{S}|$ , such that for every  $n \geq n_0$ , if  $P \in \mathcal{D}_n(\mathcal{X})$  and  $Q \in \mathcal{D}_n(\mathcal{Y})$ , and if  $N$  and  $M$  are positive integers with*

$$\varepsilon \leq R_1 = \frac{\log N}{n} \quad \text{and} \quad \varepsilon \leq R_2 = \frac{\log M}{n},$$

*then there exist codewords,  $\mathbf{x}_1, \dots, \mathbf{x}_N$ , each of type  $P$ , and there exist codewords,  $\mathbf{y}_1, \dots, \mathbf{y}_M$ , each of type  $Q$  such that (C.1) – (C.10) all hold simultaneously:*

$$\frac{1}{N} |\{i : I(\mathbf{x}_i \wedge \mathbf{s}) > \varepsilon\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{s} \in \mathcal{S}^n, \quad (\text{C.1})$$

$$\frac{1}{M} |\{j : I(\mathbf{y}_j \wedge \mathbf{x}\mathbf{s}) > \varepsilon\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{x} \in \mathcal{X}^n, \mathbf{s} \in \mathcal{S}^n, \quad (\text{C.2})$$

$$\frac{1}{M} |\{j : I(\mathbf{y}_j \wedge \mathbf{s}) > \varepsilon\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{s} \in \mathcal{S}^n, \quad (\text{C.3})$$

$$\frac{1}{N} |\{i : I(\mathbf{x}_i \wedge \mathbf{y}\mathbf{s}) > \varepsilon\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{y} \in \mathcal{Y}^n, \mathbf{s} \in \mathcal{S}^n. \quad (\text{C.4})$$

*For every type  $P_{X'Y'Y'S} \in \mathcal{D}_n(\mathcal{X}' \times \mathcal{X}' \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{S})$ ,*

$$|\{j' : (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{X'Y'Y'S}\}| \leq \exp[n(|R_2 - I(Y' \wedge XY'S)|^+ + \varepsilon)], \quad \forall \mathbf{s} \in \mathcal{S}^n, \quad (\text{C.5})$$

and

$$|\{i' : (\mathbf{x}_i, \mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{s}) \in \mathcal{T}_{X X' Y S}\}| \leq \exp[n(|R_1 - I(X' \wedge X Y S)|^+ + \varepsilon)], \quad \forall \mathbf{s} \in \mathcal{S}^n; \quad (\text{C.6})$$

if  $I(X \wedge Y' S) > |R_2 - I(Y' \wedge S)|^+ + \varepsilon$ ,

$$\frac{1}{N} |\{i : \exists j' \text{ with } (\mathbf{x}_i, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{X Y' S}\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{s} \in \mathcal{S}^n; \quad (\text{C.7})$$

if  $I(Y \wedge X Y' S) > |R_2 - I(Y' \wedge X S)|^+ + \varepsilon$ ,

$$\frac{1}{M} |\{j : \exists j' \neq j \text{ with } (\mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_{j'}, \mathbf{s}) \in \mathcal{T}_{X Y Y' S}\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{s} \in \mathcal{S}^n; \quad (\text{C.8})$$

if  $I(Y \wedge X' S) > |R_1 - I(X' \wedge S)|^+ + \varepsilon$ ,

$$\frac{1}{M} |\{j : \exists i' \text{ with } (\mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{s}) \in \mathcal{T}_{X' Y S}\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{s} \in \mathcal{S}^n; \quad (\text{C.9})$$

if  $I(X \wedge X' Y S) > |R_1 - I(X' \wedge Y S)|^+ + \varepsilon$ ,

$$\frac{1}{N} |\{i : \exists i' \neq i \text{ with } (\mathbf{x}_i, \mathbf{x}_{i'}, \mathbf{y}_j, \mathbf{s}) \in \mathcal{T}_{X X' Y S}\}| \leq \exp(-n\varepsilon/2), \quad \forall \mathbf{s} \in \mathcal{S}^n. \quad (\text{C.10})$$

*Proof.* Observe that if  $P_{\mathbf{x}_i} = P$  and  $P_{\mathbf{y}_j} = Q$  for all  $i$  and  $j$  respectively, then in order that all of the bounds (C.5) – (C.10) be nonvacuous, it is necessary that  $P_X = P_{X'} = P$  and  $P_Y = P_{Y'} = Q$ . Most of the properties follow easily from their single-user counterparts proved in the appendix of [8]. The exceptions are (C.7) and (C.9); for these, a proof is required. It will suffice to prove (C.7).

Let  $\{\mathbf{X}_1, \dots, \mathbf{X}_N\}$  and  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_M\}$  be two independent families of independently identically distributed random variables such that each member of the family  $\{\mathbf{X}_1, \dots, \mathbf{X}_N\}$  is uniformly distributed on  $\mathcal{T}_P$ , and each member of the family  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_M\}$  is uniformly distributed on  $\mathcal{T}_Q$ .

Let  $t \triangleq \exp(-n\varepsilon/2)$ , and define the events

$$A(\mathbf{s}, \mathcal{T}_{X Y' S}) \triangleq \left\{ \frac{1}{N} |\{i : \exists j' \text{ with } (\mathbf{X}_i, \mathbf{Y}_{j'}, \mathbf{s}) \in \mathcal{T}_{X Y' S}\}| \leq t \right\},$$

and

$$A \triangleq \bigcap A(\mathbf{s}, \mathcal{T}_{X Y' S}),$$

where the intersection is over all  $\mathbf{s} \in \mathcal{S}^n$  and all types  $P_{XY'S}$  such that

$$I(X \wedge Y'S) > |R_2 - I(Y' \wedge S)|^+ + \varepsilon.$$

The assertion in (C.7) will be proved if  $P(A) > 0$ , or equivalently,  $P(A^c) < 1$ . Now, we will show below that, *uniformly* for every  $\mathbf{s} \in \mathcal{S}^n$  and every set  $\mathcal{T}_{XY'S}$ ,

$$P(A(\mathbf{s}, \mathcal{T}_{XY'S})^c) \leq 2 \exp[-\frac{1}{2} \exp(n\varepsilon/4)]. \quad (\text{C.11})$$

It will then follow that

$$\begin{aligned} P(A^c) &\leq |\mathcal{S}|^n (n+1)^{|\mathcal{X}||\mathcal{Y}||\mathcal{S}|} \cdot 2 \exp[-\frac{1}{2} \exp(n\varepsilon/4)] \\ &= 2 \exp[n \log |\mathcal{S}| + |\mathcal{X}||\mathcal{Y}||\mathcal{S}| \log(n+1) - \frac{1}{2} \exp(n\varepsilon/4)]. \end{aligned}$$

Clearly, if  $n$  is sufficiently large (obviously, how large depends only on  $\varepsilon$ ,  $|\mathcal{X}|$ ,  $|\mathcal{Y}|$ , and  $|\mathcal{S}|$ ),

$$P(A^c) < 1.$$

In order to prove (C.11), we will prove and employ the following: *uniformly* for every  $\mathbf{s} \in \mathcal{S}^n$  and every set  $\mathcal{T}_{Y'S}$ , we have

$$P(G(\mathbf{s}, \mathcal{T}_{Y'S})^c) \leq \exp[-\frac{1}{2} \exp(n\varepsilon/4)], \quad (\text{C.12})$$

where

$$G(\mathbf{s}, \mathcal{T}_{Y'S}) \triangleq \left\{ |\{j' : (\mathbf{Y}_{j'}, \mathbf{s}) \in \mathcal{T}_{Y'S}\}| \leq t' \right\},$$

and  $t' \triangleq \exp[n(|R_2 - I(Y' \wedge S)|^+ + \varepsilon/4)]$ . To prove (C.12), let

$$g_{j'} \triangleq \begin{cases} 1, & \text{if } \mathbf{Y}_{j'} \in \mathcal{T}_{Y'S}(\mathbf{s}) \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$|\{j' : (\mathbf{Y}_{j'}, \mathbf{s}) \in \mathcal{T}_{Y'S}\}| = \sum_{j'=1}^M g_{j'}.$$

So, using Markov's inequality, followed by the independence of the random variables  $\{g_{j'}\}$ ,

$$\begin{aligned}
P(G(\mathbf{s}, \mathcal{T}_{Y'S})^c) &= P\left(\sum_{j'=1}^M g_{j'} > t'\right) \\
&= P\left(\exp\left(\sum_{j'=1}^M g_{j'}\right) > \exp(t')\right) \\
&\leq \exp(-t') \cdot E\left[\exp\left(\sum_{j'=1}^M g_{j'}\right)\right] \\
&= \exp(-t') \cdot \prod_{j'=1}^M E[\exp(g_{j'})].
\end{aligned}$$

Now, since exp's are to the base 2, and since

$$2^x \leq 1 + x, \quad \text{when } x \in [0, 1],$$

we can write

$$\begin{aligned}
E[\exp(g_{j'})] &\leq E[1 + g_{j'}] \\
&= 1 + E[g_{j'}] \\
&\leq e^{E[g_{j'}]} \\
&= \exp(E[g_{j'}] \log e).
\end{aligned}$$

To upper bound  $E[g_{j'}]$ , we appeal to [4, Lemmas 2.3 and 2.5, pp. 30-31], and then assume that  $n$  is so large that  $(n+1)^{|Y|} \leq \exp(n\varepsilon/4)/(2 \log e)$ . This yields

$$\begin{aligned}
E[g_{j'}] &= P(\mathbf{Y}_{j'} \in \mathcal{T}_{Y'S}(\mathbf{s})) \\
&= \frac{1}{|\mathcal{T}_Q|} |\mathcal{T}_{Y'S}(\mathbf{s})| \\
&\leq \exp(nH(Y'|S))/[\exp(nH(Y'))(n+1)^{-|Y|}] \\
&\leq \exp[-n(I(Y' \wedge S) - \varepsilon/4)]/2 \log e.
\end{aligned}$$

With this upper bound,

$$\begin{aligned}
P(G(\mathbf{s}, \mathcal{T}_{Y'S})^c) &\leq \exp[-t' + ME[g_1] \log e] \\
&\leq \exp[-(t' - \frac{1}{2} \exp[n((R_2 - I(Y' \wedge S)) + \varepsilon/4)])] \\
&\leq \exp[-\frac{1}{2} \exp(n\varepsilon/4)].
\end{aligned}$$

Having established (C.12), we proceed to verify (C.11). Write

$$\begin{aligned} \mathbb{P}(A(\mathbf{s}, \mathcal{T}_{XY'S})^c) &= \mathbb{P}(A(\mathbf{s}, \mathcal{T}_{XY'S})^c \cap G(\mathbf{s}, \mathcal{T}_{Y'S})) + \mathbb{P}(A(\mathbf{s}, \mathcal{T}_{XY'S})^c \cap G(\mathbf{s}, \mathcal{T}_{Y'S})^c) \\ &\leq \mathbb{P}(A(\mathbf{s}, \mathcal{T}_{XY'S})^c \cap G(\mathbf{s}, \mathcal{T}_{Y'S})) + \exp[-\frac{1}{2} \exp(n\varepsilon/4)]. \end{aligned} \quad (\text{C.13})$$

Keeping in mind that

$$\mathbf{X}_i \in \mathcal{T}_{X|Y'S}(\mathbf{Y}_{j'}, \mathbf{s}) \iff \mathbf{X}_i \in \mathcal{T}_{X|Y'S}(\mathbf{Y}_{j'}, \mathbf{s}) \text{ and } \mathbf{Y}_{j'} \in \mathcal{T}_{Y'S}(\mathbf{s}),$$

let

$$f_i \triangleq \begin{cases} 1, & \text{if } \mathbf{X}_i \in \bigcup_{j': \mathbf{Y}_{j'} \in \mathcal{T}_{Y'S}(\mathbf{s})} \mathcal{T}_{X|Y'S}(\mathbf{Y}_{j'}, \mathbf{s}) \\ 0, & \text{otherwise.} \end{cases}$$

Let  $g$  denote the indicator function of the event  $G(\mathbf{s}, \mathcal{T}_{Y'S})$ . Then

$$\begin{aligned} \mathbb{P}(A(\mathbf{s}, \mathcal{T}_{XY'S})^c \cap G(\mathbf{s}, \mathcal{T}_{Y'S})) &= \mathbb{P}\left(\left\{\frac{1}{N} \sum_{i=1}^N f_i > t\right\} \cap G(\mathbf{s}, \mathcal{T}_{Y'S})\right) \\ &\leq \exp[-Nt] \cdot \mathbb{E}[g \cdot \exp\left(\sum_{i=1}^N f_i\right)]. \end{aligned} \quad (\text{C.14})$$

To upper bound the preceding expectation, we need the following  $\sigma$ -fields. Let

$$\mathcal{F}_0 \triangleq \sigma(\mathbf{Y}_1, \dots, \mathbf{Y}_M),$$

and for  $i = 1, \dots, N$ , let

$$\mathcal{F}_i \triangleq \sigma(\mathbf{Y}_1, \dots, \mathbf{Y}_M, \mathbf{X}_1, \dots, \mathbf{X}_i).$$

Write

$$\mathbb{E}[g \cdot \exp\left(\sum_{i=1}^N f_i\right)] = \mathbb{E}[\mathbb{E}[\exp(f_N) | \mathcal{F}_{N-1}] g \exp\left(\sum_{i=1}^{N-1} f_i\right)]. \quad (\text{C.15})$$

First, observe that by independence,

$$\mathbb{E}[f_N | \mathcal{F}_{N-1}] \cdot g = g \cdot \frac{1}{|\mathcal{T}_P|} \left| \bigcup_{j': \mathbf{Y}_{j'} \in \mathcal{T}_{Y'S}(\mathbf{s})} \mathcal{T}_{X|Y'S}(\mathbf{Y}_{j'}, \mathbf{s}) \right|.$$

If  $g = 0$  in the preceding equation, the left-hand side is 0, and any nonnegative number will be an upper bound. If  $g = 1$ , we have

$$|\{j' : \mathbf{Y}_{j'} \in \mathcal{T}_{Y'S}(\mathbf{s})\}| \leq t'.$$



If  $n$  is sufficiently large,

$$\begin{aligned} \mathbb{E}[f_N | \mathcal{F}_{N-1}] \cdot g &\leq g \cdot t' \cdot \exp[-n(I(X \wedge Y'S) - \varepsilon/4)]/2 \log e \\ &= g \cdot \exp[n(|R_2 - I(Y' \wedge S)|^+ - I(X \wedge Y'S) + \varepsilon/2)]/2 \log e. \end{aligned}$$

So, if  $I(X \wedge Y'S) > |R_2 - I(Y' \wedge S)|^+ + \varepsilon$ ,

$$\mathbb{E}[f_N | \mathcal{F}_{N-1}] \cdot g \leq g \cdot \exp(-n\varepsilon/2)/2 \log e.$$

Invoking the inequality  $2^x \leq 1 + x$ , we then get

$$\begin{aligned} \mathbb{E}[\exp(f_N) | \mathcal{F}_{N-1}] \cdot g &\leq (1 + \mathbb{E}[f_N | \mathcal{F}_{N-1}]) \cdot g \\ &\leq (1 + \exp(-n\varepsilon/2)/2 \log e) \cdot g \\ &\leq g \cdot \exp[\tfrac{1}{2} \exp(-n\varepsilon/2)]. \end{aligned}$$

Applying the preceding analysis inductively to (C.15),

$$\mathbb{E}[g \cdot \exp\left(\sum_{i=1}^N f_i\right)] \leq \mathbb{E}[g] \cdot \exp[\tfrac{1}{2}N \exp(-n\varepsilon/2)].$$

Since  $\mathbb{E}[g] = \mathbb{P}(g = 1) \leq 1$ ,

$$\mathbb{E}[g \cdot \exp\left(\sum_{i=1}^N f_i\right)] \leq \exp[\tfrac{1}{2}N \exp(-n\varepsilon/2)].$$

Combining this with (C.14),

$$\begin{aligned} \mathbb{P}(A(\mathbf{s}, \mathcal{T}_{XY'S})^c \cap G(\mathbf{s}, \mathcal{T}_{Y'S})) &\leq \exp[-N(t - \tfrac{1}{2} \exp(-n\varepsilon/2))] \\ &= \exp[-N(\tfrac{1}{2} \exp(-n\varepsilon/2))] \\ &= \exp[-\tfrac{1}{2} \exp[n(R_1 - \varepsilon/2)]] \\ &\leq \exp[-\tfrac{1}{2} \exp(n\varepsilon/2)], \end{aligned} \tag{C.16}$$

where the last step follows because  $R_1 \geq \varepsilon$ . Combining (C.13) with (C.16) yields (C.11).  $\square$

**Remark.** We point out that the procedure that established (C.12) also shows that with positive probability we can find  $\mathbf{Y}_1, \dots, \mathbf{Y}_M$ , each of type  $Q$ , such that for all  $P_{XY'S}$ ,

$$|\{j' : (\mathbf{x}, \mathbf{y}, \mathbf{Y}_{j'}, \mathbf{s}) \in \mathcal{T}_{XY'S}\}| \leq \exp[n(|R_2 - I(Y' \wedge XY'S)|^+ + \varepsilon)], \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{s}.$$

Thus we have also proved (C.5). Since (C.8) and (C.10) are more intricate, we refer the reader to the appendix in [8].

## APPENDIX D

### PROOF OF THEOREM 5.5

In this appendix we prove Theorem 5.5. We first need the following general lemma.

**Lemma D.1** *Let  $\{\hat{F}_i\}$  and  $\{\hat{G}_{ij}\}$  be arbitrary subsets of  $\mathcal{Z}^n$ , not necessarily given by (4.9) and (4.10). Suppose that the  $\{\hat{F}_i\}_{i=1}^N$  are disjoint and that the  $\{\hat{G}_{ij}\}_{j=1}^M$  are disjoint for each  $i$ . Let  $\hat{\varphi}$  satisfy (4.11) and (4.12). Let  $J_i^0(\mathbf{s})$  and  $K_{ij}^0(\mathbf{s})$  be given by (4.1) and (4.6) respectively. If*

$$\hat{F}_i^c \cap J_i^0(\mathbf{s}) \subset \bigcup_{i' \neq i} \{\mathbf{z} : I(\mathbf{x}_{i'} \wedge \mathbf{z}) > R_1 + \eta \text{ and } I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) > \eta\}, \quad \forall \mathbf{s} \in \mathcal{S}^n, \quad (\text{D.1})$$

and if

$$\hat{G}_{ij}^c \cap K_{ij}^0(\mathbf{s}) \subset \bigcup_{j' \neq j} \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta \text{ and } I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\}, \quad \forall \mathbf{s} \in \mathcal{S}^n, \quad (\text{D.2})$$

then we can use  $\hat{\varphi}$  instead of  $\varphi$  in the proof of Theorem 5.1.

*Proof.* Observe that

$$\hat{\varphi}(\mathbf{z}) \neq (i, j) \implies \mathbf{z} \in \hat{F}_i^c \cup \hat{G}_{ij}^c.$$

Hence

$$\begin{aligned} & \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\{\mathbf{z} \in \mathcal{Z}^n : \hat{\varphi}(\mathbf{z}) \neq (i, j)\} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \\ & \leq \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\hat{F}_i^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) + \frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\hat{G}_{ij}^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}). \end{aligned}$$

We consider only the second term. The first is treated similarly. Write

$$\begin{aligned} \hat{G}_{ij}^c &= [\hat{G}_{ij}^c \cap K_{ij}^0(\mathbf{s})^c] \cup [\hat{G}_{ij}^c \cap K_{ij}^0(\mathbf{s})] \\ &\subset K_{ij}^0(\mathbf{s})^c \cup [\hat{G}_{ij}^c \cap K_{ij}^0(\mathbf{s})]. \end{aligned}$$

By (D.2),

$$\hat{G}_{ij}^c \subset K_{ij}^0(\mathbf{s})^c \cup \bigcup_{j' \neq j} \{\mathbf{z} : I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) > R_2 + \eta \text{ and } I(\mathbf{x}_i; \mathbf{y}_j; \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) > \eta\}.$$

A review of the proof of Theorem 5.1 giving special attention to equations (A.17) and (A.20) as well as (A.21), (A.23), and (A.35) shows that

$$\frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M W^n(\hat{G}_{ij}^c | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \leq 4 \exp(-n\varepsilon/4).$$

□

*Proof of Theorem 5.5.* Assume  $\delta > 0$  has been chosen small enough that

$$I_{\mathcal{X} \wedge \mathcal{Z}}(p \times q \times r \times W) - I_{\mathcal{S} \wedge \mathcal{Z}}(p \times q \times r \times W) > 2\delta, \quad \forall r \in \mathcal{D}(\mathcal{S}),$$

and

$$I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(p \times q \times r \times W) - I_{\mathcal{S} \wedge \mathcal{Z} | \mathcal{X}}(p \times q \times r \times W) > 2\delta, \quad \forall r \in \mathcal{D}(\mathcal{S}).$$

Then as in the proof of Theorem 5.1, we can assume that  $\hat{p}$  and  $\hat{q}$  and  $P$  and  $Q$  have been chosen so that

$$I_{\mathcal{X} \wedge \mathcal{Z}}(P \times Q \times r \times W) - I_{\mathcal{S} \wedge \mathcal{Z}}(P \times Q \times r \times W) > \delta, \quad \forall r \in \mathcal{D}(\mathcal{S}), \quad (\text{D.3})$$

and

$$I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(P \times Q \times r \times W) - I_{\mathcal{S} \wedge \mathcal{Z} | \mathcal{X}}(P \times Q \times r \times W) > \delta, \quad \forall r \in \mathcal{D}(\mathcal{S}). \quad (\text{D.4})$$

Let  $\mathbf{x}_1, \dots, \mathbf{x}_N$  be the codewords for user 1, all of type  $P$ , and let  $\mathbf{y}_1, \dots, \mathbf{y}_M$  be the codewords for user 2, all of type  $Q$  and having the properties listed in Theorem C.1. With  $\hat{F}_i$  and  $\hat{G}_{ij}$  given by (4.9) and (4.10) respectively, we have  $\mathbf{z} \in \hat{F}_i^c \cap J_i^0(\mathbf{s})$  if and only if

$$\mathbf{z} \in \bigcup_{i' \neq i} \{\mathbf{z} : I(\mathbf{x}_{i'} \wedge \mathbf{z}) \geq I(\mathbf{x}_i \wedge \mathbf{z}) \text{ and } D(P_{\mathbf{x}_i, \mathbf{s}, \mathbf{z}} \| P \times P_{\mathbf{s}} \times QW) \leq \eta\}.$$

By the Projection Lemma A.1,  $D(P_{\mathbf{x}_i, \mathbf{s}, \mathbf{z}} \| P \times P_{\mathbf{s}} \times QW) \leq \eta$  implies  $D(P_{\mathbf{x}_i, \mathbf{z}} \| P \times QP_{\mathbf{s}}W) \leq \eta$ , and so by (A.8), (A.11), and the fact that  $\eta < \delta/2$ , if  $\mathbf{z} \in \hat{F}_i^c \cap J_i^0(\mathbf{s})$ , then there is some

$i' \neq i$  with

$$\begin{aligned}
I(\mathbf{x}_{i'} \wedge \mathbf{z}) &\geq I(\mathbf{x}_i \wedge \mathbf{z}) \geq I_{\mathcal{X} \wedge \mathcal{Z}}(P \times Q \times P_s W) - \delta/2 \\
&\geq I_{\mathcal{X} \wedge \mathcal{Z}}^*(P, Q, W) - \delta/2 \\
&> R_1 + \eta.
\end{aligned}$$

We claim that  $I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) > \eta$  as well. Observe that

$$\begin{aligned}
I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) &\geq I(\mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) \\
&= I(\mathbf{x}_{i'} \mathbf{s} \wedge \mathbf{z}) - I(\mathbf{s} \wedge \mathbf{z}) \\
&\geq I(\mathbf{x}_{i'} \wedge \mathbf{z}) - I(\mathbf{s} \wedge \mathbf{z}) \\
&\geq I(\mathbf{x}_i \wedge \mathbf{z}) - I(\mathbf{s} \wedge \mathbf{z}) \\
&\geq [I_{\mathcal{X} \wedge \mathcal{Z}}(P \times P_s \times QW) - I_{\mathcal{S} \wedge \mathcal{Z}}(P \times P_s \times QW)] - \delta/2, \\
&\quad \text{since } \mathbf{z} \in J_i^0(\mathbf{s}) \text{ (see Note below),} \\
&\geq \delta/2 > \eta, \quad \text{by (D.3).}
\end{aligned}$$

**Note.** We assume  $\eta$  was chosen so small that not only do we have (A.8) and (A.11), but also if  $P^{(1)}$  and  $P^{(2)}$  are any two distributions on  $\mathcal{X} \times \mathcal{S} \times \mathcal{Z}$  or on  $\mathcal{X} \times \mathcal{Y} \times \mathcal{S} \times \mathcal{Z}$  with  $D(P^{(1)} \| P^{(2)}) \leq \eta$ , then

$$\left| [I_{\mathcal{X} \wedge \mathcal{Z}}(P^{(1)}) - I_{\mathcal{S} \wedge \mathcal{Z}}(P^{(1)})] - [I_{\mathcal{X} \wedge \mathcal{Z}}(P^{(2)}) - I_{\mathcal{S} \wedge \mathcal{Z}}(P^{(2)})] \right| < \delta/2$$

and

$$\left| [I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(P^{(1)}) - I_{\mathcal{S} \wedge \mathcal{Z} | \mathcal{X}}(P^{(1)})] - [I_{\mathcal{Y} \wedge \mathcal{Z} | \mathcal{X}}(P^{(2)}) - I_{\mathcal{S} \wedge \mathcal{Z} | \mathcal{X}}(P^{(2)})] \right| < \delta/2.$$

Thus  $\hat{F}_i^c \cap J_i^0(\mathbf{s}) \subset \bigcup_{i' \neq i} \{\mathbf{z} : I(\mathbf{x}_{i'} \wedge \mathbf{z}) > R_1 + \eta \text{ and } I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{x}_{i'} | \mathbf{s}) > \eta\}$ .

The proof of (D.2) follows similarly if one observes that

$$\begin{aligned}
I(\mathbf{x}_i \mathbf{y}_j \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) &\geq I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) \\
&= I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s}) - I(\mathbf{x}_i \mathbf{z} \wedge \mathbf{s}) \\
&= [I(\mathbf{x}_i \wedge \mathbf{y}_{j'} | \mathbf{s}) + I(\mathbf{z} \wedge \mathbf{y}_{j'} | \mathbf{s} | \mathbf{x}_i)] - [I(\mathbf{x}_i \wedge \mathbf{s}) + I(\mathbf{z} \wedge \mathbf{s} | \mathbf{x}_i)] \\
&= [I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) - I(\mathbf{s} \wedge \mathbf{z} | \mathbf{x}_i)] + [I(\mathbf{x}_i \wedge \mathbf{y}_{j'} | \mathbf{s}) - I(\mathbf{x}_i \wedge \mathbf{s})] \\
&\geq I(\mathbf{y}_{j'} \wedge \mathbf{z} | \mathbf{x}_i) - I(\mathbf{s} \wedge \mathbf{z} | \mathbf{x}_i).
\end{aligned}$$

□



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