SRC TR 87-137

Local Feedback Stabilization and Bifurcation Control, II. Stationary Bifurcation

by

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Systems & Control Letters 8 (1987) 467–473
North-Holland

Local feedback stabilization and bifurcation control, II. Stationary bifurcation *

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Received 3 April 1986
Revised 27 August 1986

Abstract: Local feedback stabilization and bifurcation control of nonlinear systems are studied for the case in which the critical linearized system possesses a simple zero eigenvalue. Sufficient conditions are obtained for local stabilizability of the equilibrium point at criticality and for local stabilizability of bifurcated equilibria. These conditions involve assumptions on the controllability of the critical mode for the linearized system. Explicit stabilizing feedback controls are constructed. The Projection Method of analysis of stationary bifurcations is employed. This work complements an earlier study by the same authors (*Systems Control Lett.* 7 (1986) 11–17) of stabilization and bifurcation control in the (Hopf bifurcation) case of two pure imaginary eigenvalues of the linearized system at criticality.

Keywords: Bifurcation, Stability, Stabilization, Control systems, Nonlinear systems, Feedback control.

1. Introduction

Consider a one-parameter family of nonlinear control systems

$$\dot{x} = f_\mu(x, u) \tag{1}$$

where $x \in R^n$, u is a scalar control, μ is a real-valued parameter, and the vector field f_μ is sufficiently smooth. Suppose that for $u \equiv 0$ Eq. (1) has an equilibrium point $x_0(\mu)$ which depends smoothly on μ. In the sequel the system

$$\dot{x} = f_0(x, u). \tag{2}$$

which is simply (1) with $\mu = 0$, will also be of interest.

This paper is concerned with the synthesis of feedback controls $u = u(x)$ achieving certain stability properties for each of the descriptions (1) and (2). The results apply under the following hypothesis, which ensures that new stationary solutions of (1) bifurcate from $x_0(\mu)$ at $\mu = 0$:

(S) Eq. (1) has an equilibrium $x_0(\mu)$ when $u = 0$. Furthermore, the linearization of (1) near x_0, $\mu = 0$ possesses a simple eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = 0$, $\lambda_1'(0) \neq 0$, with the remaining eigenvalues $\lambda_2(0), \ldots, \lambda_n(0)$ in the open left half complex plane.

The assumption that $\lambda_1'(0) \neq 0$ is the familiar strict-crossing (transversality) condition introduced by Hopf [13].

As is well known, hypothesis (S) leads to a *stationary* (or *static*) *bifurcation* for Eq. (1), i.e. a bifurcation involving only equilibrium points. Two stabilization problems are considered in the sequel. One of these pertains to Eq. (1) and the other to Eq. (2). For Eq. (1), the goal is to ensure local asymptotic stability of the bifurcated equilibria. This will be referred to as the *local stationary bifurcation control problem*. For the description (2), it is desired to solve the standard *local feedback stabilization problem* at the equilibrium point $x_0(0)$. Note that under hypothesis (S), (2) is an example of a critical nonlinear system since its linearization possesses an eigenvalue with zero real part. Continuing in the spirit of [1], elementary results from bifurcation theory are used to solve both of these local feedback control problems simultaneously.

The Projection Method (see [14,15]) will be employed to obtain generally applicable formulae for coefficients in the series expansions of the bifurcated equilibrium. These *bifurcation formulae* are then applied to determine sufficient conditions for bifurcation controllability and feedback stabi-

* Research supported in part by the National Science Foundation under Grants ECS-84-04275 and CDR-85-00108, by a grant from the Minta Martin Fund for Aeronautical Research, Univ. of Maryland, and by grants from BG&E, PEPCO, VEPCO, Bechtel Corp. and the General Electric Foundation.

lizability, and to derive explicit stabilizing feedback controls.

The development of the paper is as follows. The problems of local feedback stabilization and local bifurcation control and their connection are discussed in Section 2. Section 3 contains a derivation of the bifurcation formulae needed in the paper. Their application to the stationary bifurcation control problem described above is contained in Section 4. In Section 5, the question of genericity of the results with regard to perturbations in the model is discussed.

2. Stability of critical systems and bifurcation analysis

Under the stationary bifurcation hypothesis (S), it is well known [8], [12], [14], [15], [18] that Eq. (1) exhibits a stationary (or static) bifurcation from x_0 at $\mu = 0$. That is, new stationary solutions (i.e. equilibrium points) bifurcate from x_0 at $\mu = 0$. The stability characteristics of the new solutions are intimately related to those of $x_0(\mu)$ at criticality, i.e. at $\mu = 0$. It is this intrinsic relationship that allows the joint consideration of local stabilization for Eq. (2) and bifurcation control for Eq. (1).

To establish this relationship and motivate the derivations to follow, consider a general one-parameter family of nonlinear ordinary differential equations

$$\dot{x} = F_\mu(x) \tag{3}$$

having an equilibrium point $x_0(\mu)$ at which hypothesis (S) holds. Then near $x_0(0)$, $\mu = 0$ in (x, μ) space there exists a locally unique curve of points $(x(\varepsilon), \mu(\varepsilon))$, distinct from the μ axis and passing through $(0, 0)$, such that for all sufficiently small $|\varepsilon|$, $x(\varepsilon)$ is an equilibrium point of (3) when $\mu = \mu(\varepsilon)$. Moreover, the parameter ε may be chosen so that $x(\varepsilon)$ and $\mu(\varepsilon)$ are smooth.

Denote the series expansions of $\mu(\varepsilon)$, $x(\varepsilon)$ by

$$\mu(\varepsilon) = \mu_1\varepsilon + \mu_2\varepsilon^2 + \cdots, \tag{4}$$

$$x(\varepsilon) = x_1\varepsilon + x_2\varepsilon^2 + \cdots, \tag{5}$$

respectively. Generically, $\mu_1 \neq 0$, and there is a second equilibrium point besides $x_0(\mu)$ for all small $|\mu|$. However, if $\mu_1 = 0$ and $\mu_2 > 0$ (resp. $\mu_2 < 0$), then there are two new equilibrium points, one for positive and negative values of ε. These occur only for sufficiently small positive (resp. negative) values of μ. The new equilibrium points also have an eigenvalue β which vanishes at $\mu = 0$, with a series expansion

$$\beta(\varepsilon) = \beta_1\varepsilon + \beta_2\varepsilon^2 + \cdots. \tag{6}$$

Moreover, the exchange of stability formula [13,14]

$$\beta_1 = -\mu_1\lambda_1'(0) \tag{7}$$

holds. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the appropriate exchange of stability formula is [13]

$$\beta_2 = -2\mu_2\lambda_1'(0). \tag{8}$$

(Note: Eqs. (7) and (8) may be derived using the Factorization Theorem in Iooss and Joseph [15, pp. 90–91].) Suppose $\lambda_1'(0) > 0$. Then these facts imply that supercritical solution branches are stable while subcritical branches are unstable.

The following result follows from an application of the Center Manifold Theorem [6,10,12,18] to a suspended version of Eq. (3) at $x_0(0)$, $\mu = 0$.

Theorem 1. *If $\mu_1 \neq 0$, then the equilibrium point $x_0(0)$ is unstable for Eq. (3). If $\mu_1 = 0$ and $\mu_2 \neq 0$, then $x_0(0)$ is asymptotically stable if $\beta_2 < 0$ but is unstable if $\beta_2 > 0$.*

Thus, the equilibrium point $x_0(0)$ will be assured asymptotically stable if one can arrange that $\beta_1 = 0$ and $\beta_2 < 0$. If explicit formulae can be derived for β_1 and β_2, this provides a starting point for the construction of locally stabilizing feedback controls for the critical system (2). In fact, by the exchange of stability formulae, it is clear that this also ensures the stability of the bifurcated stationary solution, by ensuring that the bifurcation is a supercritical *pitchfork bifurcation*. This is a desirable outcome, as compared to the *transcritical bifurcation* which would occur if $\beta_1 \neq 0$, in which the bifurcated equilibrium point in stable on one side of $\mu = 0$ and unstable on the other. Indeed, under hypothesis (S), a supercritical pitchfork bifurcation ensures that, even though the nominal equilibrium solution $x_0(\mu)$ loses stability as μ varies through 0, the new equilibrium solution attracts a neighborhood of initial conditions about $x_0(0)$. This can also be shown through an application of the Center Manifold Theorem and the theory of normal forms (see [10] for details).

3. Bifurcation formulae

In this section bifurcation formulae for Eq. (3) are derived. The results will be the main tool in the construction of stabilizing feedbacks for Eqs. (1) and (2) in Section 4. The Projection Method, as elaborated in [15], will be employed in the derivation.

By assumption, the Jacobian matrix $D_x F_0(x_0(0))$ of (3) at criticality possesses a simple zero eigenvalue $\lambda_1(0)$. Denote by r (resp. l) the right column (resp. left row) eigenvector of the critical Jacobian matrix corresponding to this eigenvalue. Using the fact that 0 is a simple eigenvalue, it is not difficult to see that the vectors l and r may be chosen to have only real elements. To be more specific, set the first component of r to 1 and then choose l so that $lr = 1$.

Without loss of generality, assume that for small $|\mu|$ the known equilibrium point $x_0(\mu)$ of (3) is the origin, i.e. $x_0(\mu) \equiv 0$ for small $|\mu|$. This can always be achieved by a smooth change of variables $x \to x + x_0(\mu)$. Rewrite (3) in the series form

$$\begin{aligned}\dot{x} &= L(\mu)x + Q_\mu(x, x) + C_\mu(x, x, x) + \cdots \\ &= L_0 x + \mu L_1 x + \mu^2 L_2 x + \cdots \\ &\quad + Q_0(x, x) + \mu Q_1(x, x) + \cdots \\ &\quad + C_0(x, x, x) + \cdots .\end{aligned} \tag{9}$$

Here, $L(\mu)$, L_1, L_2 are $n \times n$ matrices, $Q_\mu(x, x)$, $Q_0(x, x)$, $Q_1(x, x)$ are vector valued quadratic forms generated by symmetric bilinear forms $Q_\mu(x, y)$, $Q_0(x, y)$, $Q_1(x, y)$, respectively, and $C_0(x, x, x)$ is a vector valued cubic form generated by a symmetric trilinear form $C(x, y, z)$. The terms not explicitly written in (9) are of higher order in x and μ than those which are.

A convenient outcome of this representation is the formula

$$\lambda_1'(0) = lL_1 r. \tag{10}$$

See [14] or [15] for a proof.

If x is any real (unknown) solution of $F_\mu(x) = 0$, define the parameter ε by

$$\varepsilon := lx, \tag{11}$$

and attempt a series expansion of the form

$$\begin{pmatrix} x(\varepsilon) \\ \mu(\varepsilon) \end{pmatrix} = \sum_{k=1}^{\infty} \varepsilon^k \begin{pmatrix} x_k \\ \mu_k \end{pmatrix}. \tag{12}$$

Substituting the expansion (12) in the equation obtained by equating the right side of (9) to 0, and equating coefficients of like powers of ε yields the following relationships:

$$0 = L_0 x_1, \tag{13}$$

$$0 = L_0 x_2 + \mu_1 L_1 x_1 + Q_0(x_1, x_1), \tag{14}$$

$$\begin{aligned}0 &= L_0 x_3 + \mu_1 L_1 x_2 + \mu_2 L_1 x_1 + \mu_1^2 L_2 x_1 \\ &\quad + 2Q_0(x_1, x_2) + \mu_1 Q_1(x_1, x_1) \\ &\quad + C_0(x_1, x_1, x_1).\end{aligned} \tag{15}$$

By Eqs. (11) and (12),

$$\begin{aligned}\varepsilon &= lx(\varepsilon) \\ &= \varepsilon l x_1 + \varepsilon^2 l x_2 + \varepsilon^3 l x_3 + \cdots .\end{aligned} \tag{16}$$

Hence,

$$lx_1 = 1 \quad \text{and} \quad lx_k = 0 \text{ for } k \geq 2. \tag{17}$$

Eqs. (13) and (17), and the assumption that 0 is a simple eigenvalue of L_0, now imply

$$x_1 = r. \tag{18}$$

Substituting this in Eq. (14) gives the following equation, which should evidently be solved for *both* x_2 and μ_1:

$$L_0 x_2 = -\mu_1 L_1 r - Q_0(r, r). \tag{19}$$

Recall that L_0 is singular. From elementary linear algebra (or the Fredholm Alternative), this equation has a solution x_2 if and only if the right side of (19) is orthogonal to all left eigenvectors of L_0 corresponding to the zero eigenvalue. Since zero is a simple eigenvalue of L_0, one need only require that

$$\mu_1 l L_1 r + l Q_0(r, r) = 0, \tag{20}$$

so that μ_1 is determined as

$$\mu_1 = -\frac{1}{\lambda_1'(0)} l Q_0(r, r) \tag{21}$$

where Eq. (10) has been employed.

Since the Fredholm Alternative conditions are now satisfied, Eqs. (14) and (17) for x_2 have a solution. This solution is easily verified to be unique. Equations (14), (17) are conveniently expressed as the single equation

$$\begin{pmatrix} L_0 \\ l \end{pmatrix} x_2 = \begin{pmatrix} -\mu_1 L_1 r - Q_0(r, r) \\ 0 \end{pmatrix}. \tag{22}$$

Since (22) has a unique solution, the coefficient matrix

$$R := \begin{pmatrix} L_0 \\ l \end{pmatrix} \tag{23}$$

is full rank. Hence, $R^{\mathrm{T}}R$ is a nonsingular square matrix and x_2 is given by

$$x_2 = (R^{\mathrm{T}}R)^{-1}R^{\mathrm{T}}\begin{pmatrix} -\mu_1 L_1 r - Q_0(r, r) \\ 0 \end{pmatrix}. \tag{24}$$

With x_2 now available, one applies the Fredholm Alternative to Eq. (15) to solve for the coefficient μ_2. Multiplying both sides of (15) by l and solving for μ_2, one obtains

$$\mu_2 = -\frac{1}{\lambda_1'(0)}\{\mu_1 l L_1 x_2 + \mu_1^2 l L_2 r + 2lQ_0(r, x_2) + \mu_1 l Q_1(r, r) + lC_0(r, r, r)\}. \tag{25}$$

Using the exchange of stability formula (7) and Eq. (21) for μ_1, the coefficient β_1 is found to be

$$\beta_1 = lQ_0(r, r). \tag{26}$$

If $\mu_1 = 0$ (implying also $\beta_1 = 0$), then the exchange of stability formula (8) is valid. In that case, one finds that β_2 is given by

$$\beta_2 = 2l(2Q_0(r, x_2) + C_0(r, r, r)) \quad (\text{if } \mu_1 = 0 \text{ only}). \tag{27}$$

The formulae (21), (24)–(27) will be employed in the next section to obtain sufficient conditions for bifurcation controllability and local stabilizability for Eqs. (1) and (2), respectively.

4. Stationary bifurcation control

Motivated by the bifurcation formulae derived above, and by the results of [1], one expands the vector field of Eq. (1) as

$$\begin{aligned} \dot{x} &= f_\mu(x, u) \\ &= L_0 x + \mu L_1 x + u\tilde{L}_1 x + u\gamma + Q_0(x, x) \\ &\quad + \mu^2 L_2 x + \mu Q_1(x, x) + u\tilde{Q}_1(x, x) \\ &\quad + C_0(x, x, x) + \cdots . \end{aligned} \tag{28}$$

The notation here is similar to that in Eq. (9). As in [1], a feedback control consisting of quadratic and cubic terms is assumed. That is, $u = u(x)$ is taken as

$$u(x) = x^{\mathrm{T}}Q_u x + C_u(x, x, x), \tag{29}$$

where Q_u is a real symmetric $n \times n$ matrix and $C_u(x, x, x)$ is a cubic form generated by a scalar valued symmetric trilinear form. Note that $u(x)$ contains no terms linear in x. This ensures that the left and right eigenvectors corresponding to the zero eigenvalue, and the value of μ at criticality, will be unaffected by the feedback control. (Further discussion on this choice follows Theorem 2 below.) The closed loop dynamics with a feedback of the form (29) become (starred quantities below denote values after feedback)

$$\begin{aligned} \dot{x} &= L_0^* x + Q_0^*(x, x) + C_0^*(x, x, x) \\ &\quad + \mu L_1^* x + \mu^2 L_2^* x + \mu Q_1^*(x, x) + \cdots \end{aligned} \tag{30}$$

where the matrices L_i^*, $i = 0, 1, 2$, the quadratic forms $Q_0^*(x, x)$, $Q_1^*(x, x)$ and the cubic form $C_0^*(x, x, x)$ are

$$L_i^* = L_i, \quad i = 0, 1, 2, \tag{31a}$$

$$Q_0^*(x, x) = (x^{\mathrm{T}}Q_u x)\gamma + Q_0(x, x), \tag{31b}$$

$$Q_1^*(x, x) = Q_1(x, x), \tag{31c}$$

and

$$C_0^*(x, x, x) = C_u(x, x, x)\gamma + C_0(x, x, x) + (x^{\mathrm{T}}Q_u x)\tilde{L}_1 x. \tag{31d}$$

Symmetric bilinear and trilinear forms $Q_0^*(x, y)$, $C_0^*(x, y, z)$ generating the quadratic and cubic forms $Q_0^*(x, x)$ and $C_0^*(x, x, x)$, respectively, are now chosen:

$$Q_0^*(x, y) = (x^{\mathrm{T}}Q_u y)\gamma + Q_0(x, y), \tag{32}$$

$$\begin{aligned} C_0^*(x, y, z) &= C_u(x, y, z)\gamma + C_0(x, y, z) \\ &\quad + \tfrac{1}{3}\{(y^{\mathrm{T}}Q_u z)\tilde{L}_1 x + (x^{\mathrm{T}}Q_u y)\tilde{L}_1 z \\ &\quad + (z^{\mathrm{T}}Q_u x)\tilde{L}_1 y\}. \end{aligned} \tag{33}$$

After feedback, the coefficient β_1 becomes, using Eq. (26),

$$\begin{aligned} \beta_1^* &= lQ_0^*(r, r) \\ &= l\{Q_0(r, r) + (r^{\mathrm{T}}Q_u r)\gamma\} \\ &= \beta_1 + (r^{\mathrm{T}}Q_u r)l\gamma, \end{aligned} \tag{34}$$

where β_1 denotes the value of β_1 with no feedstabilize a bifurcation and not merely to stabilize an equilibrium point for a fixed parameter value. Second, it should not be surprising that in some situations a linear feedback which locally stabilizes an equilibrium may result in globally unbounded behavior, whereas nonlinear feedbacks exist which stabilize the equilibrium both locally and globally. For an example, see Moon and Rand [20]. Hence, even if stabilization, rather than bifurcation control, is the issue being studied, nonlinear feedback controls can be superior. These comments apply also to Theorem 1 of [1].

Now consider the case $l\gamma = 0$, i.e. let the critical (zero) eigenvalue be *uncontrollable* for the linearized system. In the setting of [1], under the analogous assumption it was found that *generically* local feedback stabilization of the nonlinear system is achievable. However, Eq. (34) reveals that in the present setting feedback has *no effect* on the value of β_1 in case $l\gamma = 0$. The discussion in Section 2 therefore implies that the local feedback stabilization problem for Eq. (2) will then be *unsolvable*, unless perhaps it happens that $\beta_1 = 0$ in the absence of a control effort (a nongeneric assumption). Similarly, the local stationary bifurcation control problem is also generically unsolvable in case $l\gamma = 0$.

Theorem 3. *Let hypothesis* (S) *hold and assume* $l\gamma = 0$, *that is, the critical zero eigenvalue is uncontrollable for the linearized version of* (2). *Then if* $\beta_1 \neq 0$ *for Eq.* (1) *with* $u(x) \equiv 0$, *both the local stationary bifurcation control problem for Eq.* (1) *and the local feedback stabilization problem for Eq.* (2) *are not solvable by a smooth feedback control with vanishing linear part.*

It is natural, given the negative conclusion of this theorem, to consider the possibility of constructing 'nearly stabilizing' feedback controls for the case $l\gamma = 0$. For the local stabilization problem for Eq. (2), this would correspond to rendering a *neighborhood* of the equilibrium point attracting [4]. For the local stationary bifurcation control problem, it is of interest to consider the possibility of controlling the bifurcation in Eq. (1) so that it *approximates* a supercritical (i.e. stable) pitchfork bifurcation to any desired degree of accuracy. This avenue of investigation is currently being pursued by the authors.

5. Remarks on genericity

The goal pursued in this paper of using feedback to transform a given transcritical bifurcation into a supercritical pitchfork bifurcation deserves some scrutiny. Neither transcritical bifurcations nor pitchfork bifurcations are robust to perturbations in the vector field. Under small perturbations, it is well known [3,11,15] that these bifurcations tend to be destroyed and replaced by either two *saddle–node bifurcations* or a single saddle–node bifurcation and a nonbifurcating equilibrium path. Recall [8,10] that a saddle–node bifurcation is typified by the bifurcation diagram of $F_\mu(x) = x^2 - \mu$. Here, there are no equilibria for $\mu < 0$, one for $\mu = 0$ and two equilibrium points (a saddle and a node of the associated differential system) for $\mu > 0$. Arnold [3] discusses this situation in detail, and shows that the saddle–node bifurcation is the only generic bifurcation for one-dimensional equations. Since a simple zero eigenvalue has been assumed in this paper, a reduction of the dynamics to a center manifold shows that this bifurcation problem is intrinsically one-dimensional. (For more details on this line of reasoning, the reader is referred to the discussions of the Shoshitaishvili Reduction Theorem in Arnold [3, pp. 265–267] and Kubicek and Marek [17, p. 216].)

These considerations lead to the conclusion that the assumed nominal equilibrium point $x_0(\mu)$ will no longer depend smoothly on μ near a bifurcation point for small perturbations in the system model. This equilibrium path will, instead, exhibit a jump for some value of the parameter μ near 0. The conclusions of this paper should be considered as a first approximation in the design of bifurcation control laws for systems whose models may be subject to some error.

Acknowledgment

The authors wish to thank the reviewer for his comments.

References

[1] E.H. Abed and J.-H. Fu, Local feedback stabilization and bifurcation control, I. Hopf bifurcation, *Systems Control Lett.* 7 (1986) 11–17.

[2] D. Aeyels, Stabilization of a class of nonlinear systems by a smooth feedback control, *Systems Control Lett.* **5** (1985) 289–294.

[3] V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer, New York, 1983).

[4] S. Behtash and S.S. Sastry, Stabilization of nonlinear systems with degenerate linearization, in: *Proc. 1986 IEEE Int. Symp. Circ. Syst.*, San Jose, CA (May 1986) pp. 912–915.

[5] R.W. Brockett, Asymptotic stability and feedback stabilization, in: R.W. Brockett, R.S. Millman and H.J. Sussmann, Eds., *Differential Geometric Control Theory* (Birkhauser, Boston, MA, 1983) pp. 181–191.

[6] J. Carr, *Applications of Centre Manifold Theory* (Springer, New York, 1981).

[7] J. Casti, *Connectivity, Complexity, and Catastrophe in Large-Scale Systems* (Wiley–Interscience, Chichester, 1979).

[8] S.-N. Chow and J.K. Hale, *Methods of Bifurcation Theory* (Springer, New York, 1982).

[9] H.I. Freedman, The nonbifurcation of periodic solutions when the variational matrix has a zero eigenvalue, *J. Math. Anal. Appl.* **51** (1975) 429–439.

[10] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).

[11] J. Guckenheimer, Persistent properties of bifurcations, *Physica D* **7** (1983) 105–110.

[12] B.D. Hassard, N.D. Kazarinoff and Y.-H. Wan, *Theory and Applications of Hopf Bifurcation* (Cambridge Univ. Press, London, 1981).

[13] E. Hopf, Bifurcation of a periodic solution from a stationary solution of a system of differential equations, *Ber. Math.-Phys. Kl. Sachs. Akad. Wiss. Leipzig* **94** (1942) 1–22 (Translation to English with commentary in [18]).

[14] L.N. Howard, Nonlinear oscillations, in: F.C. Hoppensteadt, Ed., *Nonlinear Oscillations in Biology* (Amer. Math. Society, Providence, RI, 1979) pp 1–68.

[15] G. Iooss and D.D. Joseph, *Elementary Stability and Bifurcation Theory* (Springer, New York, 1980).

[16] T. Kailath, *Linear Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1980).

[17] M. Kubicek and M. Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures* (Springer, New York, 1983).

[18] J.E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications* (Springer, New York, 1976).

[19] R.K. Mehra, Catastrophe theory, nonlinear system identification and bifurcation control, in: *Proc of the Joint Automatic Control Conference* (1977) pp. 823–831.

[20] F.C. Moon and R.H. Rand, Parametric stiffness control of flexible structures, in: *NASA Conf. Proc.: Workshop on Identification and Control of Flexible Space Structures*, San Diego, CA (May 1984) pp. 329–338.