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A Simple Proof of Stability on The Center Manifold for Hopf Bifurcation

by

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ABSTRACT

A simple proof is presented for a well known fact about Hopf bifurcation: if the loss of stability of an equilibrium point results in periodic solutions via Hopf bifurcation, then the stability of these periodic solutions is determined by their stability on an associated center manifold. More precisely, it is shown that the characteristic exponent determining the stability of the periodic solutions is the same whether computed for the original system or the system restricted to the center manifold. Attention is focused on the finite dimensional case of a one parameter family of ordinary differential equations. The proof consists of exhibiting a similarity transformation which uncovers the relationship between the linearized flow of the original system and that of its restriction to the center manifold.

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A standard result in Hopf bifurcation theory is that the stability of the bifurcated periodic solutions agrees with their stability on an associated center manifold. Proofs of this result are available, for instance, in the books by Marsden and McCracken [6], Hassard, Kazarinoff and Wan [4], Chow and Hale [3] as well as in the article [5] by Kazarinoff, the latter emphasizing the infinite dimensional case. The purpose of this note is to give a proof of this basic result which is at once simple and also clarifies the source of this useful fact. The finite dimensional case will be considered here.

Consider the system

$$\dot{z} = f(z,\mu) \tag{1}$$

where $z \in R^n$, μ is a real parameter, $f : R^n \times R \to R^n$ is a C^1 vector field and the dot denotes differentiation with respect to time t. Suppose that $f(0,\mu) \equiv 0$ and that the Jacobian matrix $D_z f(0,0)$ takes the form

$$D_z f(0,0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3}$$

and where the eigenvalues of B lie off the imaginary axis. This can always be achieved by a linear coordinate transformation if the hypotheses for Hopf bifurcation are satisfied. The only interesting case to consider regarding stability is when the eigenvalues of B all have negative real parts, for otherwise the bifurcated periodic orbits are sure to be unstable. For notational convenience, decompose z as $z = (x,y)$ where $x \in R^2$ and $y \in R^{n-2}$, and let $f = (g,h)$ where $g \in R^2$ and $h \in R^{n-2}$. In the sequel, D_i will denote differentiation with respect to the i-th argument of a function.

Denote by $\lambda(\mu)$, $\overline{\lambda(\mu)}$ the continuous extensions of the eigenvalues $\pm i$ for μ small but nonzero. If Hopf's transversality condition

$$\text{Re } \lambda'(0) \neq 0 \tag{4}$$

holds, then the Hopf Bifurcation Theorem asserts that a one parameter family of nonconstant periodic solutions γ_ν, $\nu \in (0, \nu_0]$ emerges from the origin for μ small. Here ν is an auxiliary parameter approximating the amplitude of the periodic solutions and ν_0 is sufficiently small. The periodic solutions γ_ν have precisely two characteristic exponents near zero for ν sufficienty small. One of these is precisely 0 and the other is a smooth function $\beta(\nu)$. If $\beta(\nu)$ is not identically 0, then the sign of $\beta(\nu)$ determines (locally) the stability of the bifurcated periodic orbits. To compute the characteristic exponent $\beta(\nu)$, an often used technique is to restrict the dynamics (1) to a center manifold of a suspended system and compute β instead for the restricted dynamics. This is meaningful since the local attractivity of any center manifold implies that the periodic solutions γ_ν will be confined to the center manifold for small ν. It will now be shown that the characteristic exponent computed in this fashion must indeed agree with that of the periodic orbits in R^n.

Recall that the characteristic multipliers of a periodic solution γ of an autonomous differential system $\dot{x} = f(x)$ may be obtained as the eigenvalues, disregarding unity, of the linearization $D\varphi_t$ of the t-advance map, or flow, of the system, evaluated at any point x of γ with t set to the period of γ [6].

Consider now the suspended version of (1)

$$\dot{z} = f(z,\mu) \tag{5a}$$

$$\dot{\mu} = 0 \tag{5b}$$

near the equilibrium $(z,\mu) = (0,0)$. The Center Manifold Theorem [6, 4, 3] implies that Eq. (5) has a three dimensional center manifold C near the origin in R^{n+1} which may be represented as the graph of a function $y = u(x,\mu)$:

$$C = \{ (z,\mu) = (x,y,\mu) \in R^{n+1} : y = u(x,\mu) \}. \tag{6}$$

The center manifold need not be unique, but any center manifold is locally invariant and locally attracting. This implies, in particular, that each center manifold of (5) near the origin contains all the local recurrence of (5) near the origin. Since μ is constant along trajectories of (5), it is clear that all small amplitude periodic solutions of (1) for μ small can be obtained by studying periodic solutions of the restriction of (5) to a center manifold. Thus it suffices to use any center manifold and this explains the common usage "the" center manifold.

On the center manifold C, the dynamics (5) reduce to

$$\dot{x} = g(x,u(x,\mu),\mu) \tag{7a}$$

$$\dot{\mu} = 0. \tag{7b}$$

along with the algebraic equation $y = u(x,\mu)$. It is easily verified that Eq. (7a) undergoes a Hopf bifurcation from $x = 0$ at $\mu = 0$.

Denote by $\varphi_t(z,\mu) = (\varphi_t^X(x,y,\mu),\ \varphi_t^Y(x,y,\mu))$ the t-advance map of Eq. (5a), and by $\psi_t(x,\mu)$ the t-advance map of Eq. (7a). Note that the relationships

$$\varphi_t^X(x,u(x,\mu)) = \psi_t(x,\mu) \tag{8a}$$

$$\varphi_t^Y(x,u(x,\mu)) = u(\psi_t(x,\mu),\mu) \tag{8b}$$

hold for $|(x,\mu)|$ sufficiently small along trajectories of Eq. (5a) which remain close to $x = 0$ during the time interval $[0,t]$. Eqs. (8) are thus clearly valid for the bifurcated periodic orbits γ_ν of Eq. (5a), for any

$t \in R$ and for small μ. Differentiation of (8a,b) with respect to x yields, respectively,

$$D_1\varphi_t^X + D_2\varphi_t^X D_1 u = D_1\psi_t \tag{9a}$$

$$D_1\varphi_t^Y + D_2\varphi_t^Y D_1 u = D_1 u D_1\psi_t \tag{9b}$$

where each term is evaluated at $(x, y=u(x,\mu))$. Eqs. (9a,b) immediately yield the identity

$$D_1\varphi_t \begin{bmatrix} I & 0 \\ D_1 u & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ D_1 u & I \end{bmatrix} \begin{bmatrix} D_1\psi_t & D_2\varphi_t^X \\ 0 & D_2\varphi_t^Y - D_1 u D_2\varphi_t^X \end{bmatrix}, \tag{10}$$

where

$$D_1\varphi_t = \begin{bmatrix} D_1\varphi_t^X & D_2\varphi_t^X \\ D_1\varphi_t^Y & D_2\varphi_t^Y \end{bmatrix} \tag{11}$$

and where each term is evaluated at $(x, y=u(x,\mu))$.

Letting $(x, y=u(x,\mu))$ be a point on γ_ν (here $\mu = \mu(\nu)$) and $t = T(\nu)$ be the period of γ_ν, Eq. (10) implies that $n-2$ of the characteristic multipliers of γ_ν are the $n-2$ eigenvalues of $(D_2\varphi_t^Y - D_1 u D_2\varphi_t^X)$, and the remaining 2 characteristic multipliers of γ_ν are unity and the eigenvalue of $D_1\psi_t$ which differs from unity. Thus the characteristic exponents determining stability of the periodic solution γ_ν is the *same* for Eq. (5a) and for the restricted system (7a). This clearly implies that the stability of γ_ν can be determined by considering its stability as a periodic solution of Eq. (7a), which is the original system restricted to the center manifold C with $\mu =$ constant, μ small.

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