ABSTRACT

Title of Dissertation: PRODUCT DESIGN SELECTION WITH VARIABILITY FOR AN IMPLICIT VALUE FUNCTION

Anil Kumar Maddulapalli, Doctor of Philosophy, 2005

Dissertation directed by: Professor Shapour Azarm
Department of Mechanical Engineering

Often in engineering design selection there is no one design alternative that is better in terms of all attributes, and the preferred design(s) is dependent on the preferences of the Decision Maker (DM). In addition, there is always uncontrollable variability, which is mainly of two types, that has to be accounted for. The first type, preference variability, is caused due to the DM's lack of information on end users' needs. The second type, attribute variability, is caused due to uncontrollable engineering design parameters like manufacturing errors. If variability is not accounted for, the preferred design(s) found might be erroneous. Existing methods presume an explicit form for the DM's "value function" to simplify this selection problem. But, such an assumption is restrictive and valid only in some special cases.

The objective of this research is to develop a decision making framework for product design selection that does not presume any explicit form for the DM's value function and that accounts for both preference and attribute variability.

Our decision making framework has four research components. In the first component, Deterministic Selection, we develop a method for finding the preferred design(s) when the DM gives crisp preference estimates, i.e., best guess of actual preferences. In the second component, Sensitivity Analysis, we develop a method for finding the allowed variation in the preference estimates for which the preferred design(s) do not change. In the third component, Selection with Preference Variability, we develop a method for finding the preferred design(s) when the DM gives a range of preferences instead of crisp estimates. Finally, in our fourth component, Selection with Preference and Attribute Variability, we develop a method in which the DM gives a range of values for attributes of the design alternatives in addition to a range for preferences.

We demonstrate the methods developed in each component with two engineering examples and provide numerical experimental results for verification. Our experiments indicate that the preferred design(s) found in our first, third, and fourth components always include the actual preferred design(s) and that our second component finds the allowed variation in preference estimates efficiently.

PRODUCT DESIGN SELECTION WITH VARIABILITY

FOR AN IMPLICIT VALUE FUNCTION

by

Anil Kumar Maddulapalli

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Advisory Committee:

Professor S. Azarm, Advisor/Chair
Assistant Professor S. A. Gabriel
Associate Professor P. K. Kannan
Professor M. Modarres
Associate Professor L. Schmidt

Dedicated to my Parents

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TABLE OF CONTENTS

NOMENCLATURE

0-60 time	Time taken by an automobile to accelerate from zero to sixty miles per hour
a_i	i^{th} attribute
a_{ij}	i^{th} attribute (variable or fixed) of design D_j
A_{ij}^L	Lower bound of attribute a_i for design D_j when there is attribute variability
A_{ij}^U	Upper bound of attribute a_i for design D_j when there is attribute variability
A_{ij}	Range of attribute a_i for design D_j when there is attribute variability. Also the estimated attribute level of a_i for design D_j
A_{i+}	Range of attribute a_i for design D_+ when there is attribute variability. Also the estimated attribute level of a_i for design D_+
A_{iT}	Range of attribute a_i for a trial design D_T when there is attribute variability. Also the estimated attribute level of a_i for trial design D_T
A_{id}	Range of attribute a_i for a design D_d when there is attribute variability
c	Total number of iterations in a selection method
C_G	Gradient cut at a trial design
D_+	Arbitrary design (other than trial design) that belongs to the original set of designs
D_*	Arbitrary design (other than trial design) in the design attribute space
D_0	Arbitrary design (other than trial design) in the design attribute space
D_i	i^{th} design alternative
D_{CNT}	Set of candidate new trial designs

DM	Decision maker
D_{NT}	New trial design
D_{NTD}	Set of non-eliminated trial designs
D_T	Trial design
D_{Tk}	Trial design for the k^{th} iteration
D_{Tq}	Trial design for the q^{th} iteration
G_{+k}	Objective function in the formulation to verify the robustness index
G_{+*}	Minimum of G_{+k} (k=1,…,c)
h_T	Variable hyper-plane that is perpendicular to the variable gradient ∇v_T and lies between the extremes $[H_T^L, H_T^U]$ at a trial design D_T
h_{Tk}	Variable hyper-plane that is perpendicular to the variable gradient ∇v_{Tk} and lies between the extremes $[H_{Tk}^L, H_{Tk}^U]$ at the k^{th} trial design D_{Tk}
H_T^L	Hyper-plane perpendicular to ∇V_T^L at a trial design D_T
H_{Tk}^L	Hyper-plane perpendicular to ∇V_{Tk}^L at the k^{th} trial design D_{Tk}
H_T^U	Hyper-plane perpendicular to ∇V_T^U at a trial design D_T
H_{Tk}^U	Hyper-plane perpendicular to ∇V_{Tk}^U at the k^{th} trial design D_{Tk}
H_{Tk}^M	Hyper-plane in between $[H_{Tk}^L, H_{Tk}^U]$ at the k^{th} trial design D_{Tk} such that another trial design does not lie in the gradient cut corresponding to the gradients perpendicular to $[H_{Tk}^L, H_{Tk}^M]$
H_T	Hyper-plane perpendicular to ∇V_T at a trial design D_T
H_{Tk}	Hyper-plane perpendicular to ∇V_{Tk} at the k^{th} trial design D_{Tk}
m	Number of attributes

n	Number of designs
MRS	Marginal rate of substitution between attributes
O_T	Region around a trial design D_T where linear approximation of value function is valid
O_{Ti}	Region around the ith trial design D_{Ti} where linear approximation of value function is valid
p^{Tj}_{hjhk}	Perpendicular distance from the jth trial design D_{Tj} to the intersection of h_{Tj} and h_{Tk}
p^{Tk}_{hjhk}	Perpendicular distance from the kth trial design D_{Tk} to the intersection of h_{Tj} and h_{Tk}
p^{Tj}_{HjLHkL}	Perpendicular distance from the jth trial design D_{Tj} to the intersection of H^{L}_{Tj} and H^{L}_{Tk}
p^{Tj}_{HjUHkL}	Perpendicular distance from the jth trial design D_{Tj} to the intersection of H^{U}_{Tj} and H^{L}_{Tk}
p^{Tj}_{Hk}	Perpendicular distance from the jth trial design D_{Tj} to H_{Tk}
p^{Tj}_{HjHk}	Perpendicular distance from the jth trial design D_{Tj} to the intersection of H_{Tj} and H_{Tk}
P_S	Probability of success of a payload in a given scenario
P_{Si}	Probability of success of a payload in the ith scenario
q	Current iteration number in a selection method
r_i	Scale of attribute a_i
R	Radius of O_{Ti}

s_{ijT} Actual (variable) MRS between attributes a_i and a_j at trial design D_T

s_{ijTk} Actual (variable) MRS between attributes a_i and a_j at the k^{th} trial design D_{Tk}

S_{ijT}^{t+} Threshold MRS between attributes a_i and a_j for a design D_+ with respect to a trial design D_T

S_{ijTq}^{tk} Threshold MRS between attributes a_i and a_j for the k^{th} design D_k with respect to the q^{th} trial design D_{Tq}

S_{ijT}^{L} Lower bound of MRS between attributes a_i and a_j at a trial design D_T when there is preference variability

S_{ijTk}^{L} Lower bound of MRS between attributes a_i and a_j at the k^{th} trial design D_{Tk} when there is preference variability

S_{ijT}^{U} Upper bound of MRS between attributes a_i and a_j at a trial design D_T when there is preference variability

S_{ijTk}^{U} Upper bound of MRS between attributes a_i and a_j at the k^{th} trial design D_{Tk} when there is preference variability

S_{ijT} MRS between attributes a_i and a_j at a trial design D_T given by the DM (is crisp when preferences are deterministic and has a range when preferences have variability)

S_{ijTk} MRS between attributes a_i and a_j at the k^{th} trial design D_{Tk} given by the DM (is crisp when preferences are deterministic and has a range when preferences have variability)

V Value function

V_{eff}	Value efficiency
w_{iT}	Coefficient of the variable gradient ∇v_T with respect to attribute a_i at a trial design D_T
w_{iTk}	Coefficient of the variable gradient ∇v_{Tk} with respect to attribute a_i at the k^{th} trial design D_{Tk}
W_{iT}^{t+}	Coefficient of gradient ∇V_T^{t+} with respect to attribute a_i at a trial design D_T
W_{iTq}^{tk}	Coefficient of gradient ∇V_{Tq}^{tk} with respect to attribute a_i at the q^{th} trial design D_{Tq}
W_{iT}	Coefficient of gradient ∇V_T with respect to attribute a_i at a trial design D_T
W_{iTk}	Coefficient of gradient ∇V_{Tk} with respect to attribute a_i at k^{th} trial design D_{Tk}
Z*	Objective function value in the formulations to identify dominated design
α	Parameter of simulant value function
β	Parameter of simulant value function
γ	Parameter of simulant value function
η	A small positive constant
μ	Positive constant
ε	Arbitrarily small positive constant
λ_{ij}	Gradient estimate with respect to attribute a_i at a design D_j for finding a new trial design
λ_{i+}	Gradient estimate with respect to attribute a_i at a design D_+ for finding a new trial design

δ_{+max} Overall elimination robustness of a design D_+[◊]

δ_{+T} Elimination robustness of a design D_+ with respect to a trial design D_T[◊]

δ_{+Tk} Elimination robustness of a design D_+ with respect to the k^{th} trial design D_{Tk}

δ_{jTk} Elimination robustness of the j^{th} design D_j with respect to the k^{th} trial design D_{Tk}

δ Robustness index for D_{NTD}[◊]

ξ_{+T} Attribute elimination robustness of a design D_+ with respect to a trial design D_T

ξ_{+max} Overall attribute elimination robustness of a design D_+

ξ Attribute robustness index for D_{NTD}

∇v_T Variable gradient of value function V at a trial design D_T corresponding to variable MRS s_{ijT}

∇v_{Tk} Variable gradient of value function V at the k^{th} trial design D_{Tk} corresponding to variable MRS s_{ijTk}

∇V_T^1 Gradient at a trial design D_T corresponding to the MRS preferences of the first DM

∇V_T^2 Gradient at a trial design D_T corresponding to the MRS preferences of the second DM

∇V_T^{t+} Threshold gradient of a design D_+ with respect to a trial design D_T

[◊] δ, δ_{+max}, and δ_{+T} represent variation in preferences (e.g., variation in relative importance of attributes, or variation in MRS between attributes)

∇V_{Tq}^{tk}	Threshold gradient of the k^{th} design D_k with respect to the q^{th} trial design D_{Tq}
∇V_{Tk}^{L}	Extreme of gradient range at a trial design D_T when there is preference variability
∇V_{Tk}^{L}	Extreme of gradient range at the k^{th} trial design D_{Tk} when there is preference variability
∇V_{T}^{U}	Other extreme of gradient range at a trial design D_T when there is preference variability
∇V_{Tk}^{U}	Other extreme of gradient range at the k^{th} trial design D_{Tk} when there is preference variability
∇V_T	Gradient of value function V at a trial design D_T corresponding to the crisp MRS S_{ijT}
∇V_{Tk}	Gradient of value function V at the k^{th} trial design D_{Tk} corresponding to the crisp MRS S_{ijTk}

CHAPTER 1

INTRODUCTION

1.1. BACKGROUND, MOTIVATION, AND OVERALL OBJECTIVE

Selecting the "most preferred" design(s) from a set of alternatives that have multiple governing criteria or attributes has been a significant research thrust in engineering design selection and many other decision making processes [Haimes, 1983] [Haimes, 1998] [Hazelrigg, 1998] [Neufville, 1990]. Consider an example wherein a designer is selecting an automobile design. Typical attributes/criteria that the designer has to consider in selecting an automobile design are 0-60 time[1], cost of the automobile, safety of the passengers, and fuel economy to name a few. These criteria are often conflicting (e.g., 0-60 time and fuel economy) and there is no one alternative that is better in every attribute and there are always trade-offs involved. So a designer acting as the Decision Maker (DM) has to be careful in assessing the preferences because the decisions taken during a design selection process are usually irrevocable [Hazelrigg, 1996].

When making a selection, the DM has to satisfy the requirements of the end users. In our automobile design selection example, an automobile user in a normal household would like the automobile to have higher fuel economy, low cost, high passenger safety, and an average 0-60 time. On the other hand, an automobile user with a racer mentality would like the automobile to have less 0-60 time, high passenger safety, average fuel economy, and average cost. So, the designer requires complete information about the end users' needs for making a selection that satisfies the end users. Often, such complete

[1] 0-60 time is the time taken by an automobile to accelerate from zero to sixty miles per hour.

information is not available to the DM (due to lack of resources) which induces "variability" in the DM's preferences [Insua and French, 1991] [Kirkwood and Sarin, 1985]. By variability[2], we mean uncontrollable changes in the parameters (e.g., preferences, attributes) of selection. In the automobile design selection example, when asked to state the relative importance between the attributes: 0-60 time and the fuel economy, the DM might not state the preferences with certainty because he/she does not know what exactly the end users need. We call this preference variability.

In addition to the end user's needs, the DM also has to consider the manufacturing errors, use conditions of the product when making the selection [Hazelrigg, 1998] [Li and Azarm, 2000]. In the automobile design selection example, an automobile that is designed to have an attribute level of six seconds for the 0-60 time might in reality have the 0-60 time between five and eight seconds due to manufacturing errors, modeling errors and so on. Also, the fuel economy of the automobile might vary depending on the use conditions. For example, a higher fuel economy is obtained when the automobile is used on freeways than in congested traffic. Such lack of information on manufacturing errors, use conditions, and so on, causes variability in the attributes, which we call attribute variability.

Hence in a typical engineering design selection process, the DM, like the automobile designer, has to make a selection from a number of design alternatives, with multiple governing attributes/criteria, accounting for preference and attribute variability. This typical design selection situation is depicted in Figure 1.1. If the variability is not accounted for, the selected preferred design(s) might be erroneous [Law, 1996] [Li, 2001]

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[2] Contrary to our definition, some researchers in the literature use the term variability for referring to uncertainty that cannot be quantified and that cannot be reduced by obtaining more information.

[Neufville, 1990]. In such a situation (see Figure 1.1), the DM's engineering judgment alone is not enough to make a selection and a formal, mathematically sound technique is needed to assist the DM in product design selection.

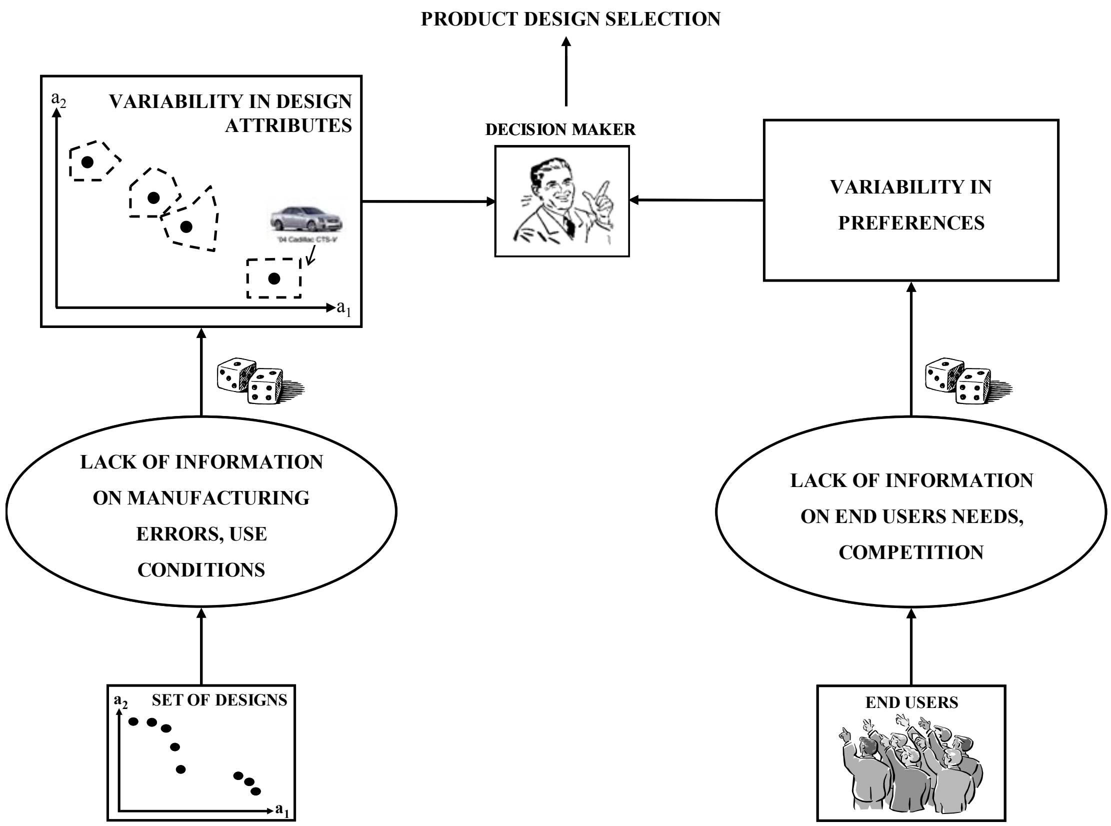

Figure 1.1: Typical design selection problem

Multi-Attribute Decision Making (MADM) is one such popular technique that is used for engineering design selection [Li, 2001] [Neufville, 1990] [Yu, 1985]. MADM methods for product design selection generally assume that the DM has an intuitive value function in mind that he/she maximizes to make the selection [Keeney and Raiffa, 1976] [Olson, 1996]. (Conventionally, the term "value" is used when the attributes are deterministic, and the term "utility" when the attributes are stochastic [Keeney and Raiffa, 1976]. However for simplicity and to avoid confusion, we only use the term value function in this dissertation.) These methods estimate the value function by obtaining from the DM, information about preferences which reflect the value function [Olson, 1996].

Many of the existing MADM methods presume an explicit form for the DM's value function to simplify the selection problem. The most common presumption is that the value function is additive with respect to the attributes [Barzilai, 1997a] [Olson, 1996] [Pomerol and Romeo, 2000] [Saaty, 1980]. When the DM's value function is presumed, methods have been reported in the MADM literature that account for no variability, preference variability alone, attribute variability alone, and both preference and attribute variability (see Chapter 2 for a detailed literature review). However, the assumption that the DM's value function is additive, for instance, is restrictive and valid only in some special cases [Keeney and Raiffa, 1976] [Thurston, 2001]. If that assumption is not valid, then the preferred design(s) found by using existing MADM methods might be erroneous.

Therefore, *the overall objective of this research is to develop a decision making framework for product design selection that does not presume any explicit form for the DM's value function and that accounts for both preference and attribute variability*.

1.2. RESEARCH COMPONENTS

To achieve the overall objective, we developed a step-by-step approach for the research in this dissertation. We developed four research components for different types of variability. These components are: (1) deterministic selection, (2) sensitivity analysis for deterministic selection, (3) selection with preference variability, and (4) selection with preference and attribute variability. A decision making framework integrating the four research components is then developed.

In the next four sections (Section 1.2.1 to Section 1.2.4), an overview and objective of each of the research components is given followed by an overview of the decision making framework in Section 1.2.5.

1.2.1. Research Component 1: Deterministic Selection

Deterministic selection refers to product design selection with no variability. In deterministic selection, we assume that the DM gives crisp (i.e., no variability) preferences and that the attributes of alternatives are an accurate representation of what is expected in reality (i.e., no variability). Many of the existing deterministic selection methods presume some explicit form for the DM's value function (additive being the most popular) [Barzilai, 1997a] [Saaty, 1980]. This is a restrictive assumption and is applicable for some special cases only [Keeney and Raiffa, 1976].

For example, in the selection of an automobile, if the DM is asked for the preferences at a design point that has the attribute levels of 20,000 dollars of cost and nine seconds of 0-60 time, the DM might say: "I would allow an increase in the cost of the automobile by 5000 dollars if the 0-60 time is decreased by two seconds". If the DM is asked the same question at a design point that has the attribute levels of 30,000 dollars of cost and 7 seconds of 0-60 time, the DM might say: "I would allow an increase in the cost of the automobile by 1000 dollars if the 0-60 time is decreased by two seconds". It is generally difficult to represent such a nonlinear preference structure by presuming an explicitly known value function. Also, if the presumed form differs significantly from the DM's implicit value function, the most preferred design might be erroneous.

The objective of the first research component is to develop a deterministic selection method that is applicable when the DM's preferences are implicit and crisp (no variability) and when there is no attribute variability.

1.2.2. Research Component 2: Sensitivity Analysis for Deterministic Selection

Sensitivity Analysis refers to finding the degree of "robustness" of the preferred design(s) to preference variation [Insua and French, 1991]. By robustness, we mean the amount of change (or variation) allowed between the actual preferences and the preference estimates before the preferred design(s) is (are) affected. In general, when the DM gives the preferences, in addition to design requirements (e.g., constraints on the size, price), he/she attempts to satisfy the needs of the end users or customers (e.g., a professional user of a cordless electric drill prefers to have more operations per battery charge, whereas a casual user prefers lower cost) [Urban and Hauser, 1993]. Hence, if the DM does not have complete information about the end users' needs, he/she cannot state the preferences precisely [Insua and French, 1991]. The DM might also have to project into future markets. In cases with such uncertainty, the DM can give only crisp estimates (or a range, see Section 1.2.3 for details) of the actual preferences. Since small variations in preferences could lead to a significant change in the set of preferred design(s) [Korhonen et al., 1992] [White, 1972], it would be useful for the DM to have an idea about the robustness of the preferred design(s) with respect to variation in the preference estimates [Hannan, 1981] [Korhonen et al., 1992].

The objective of the second research component is to develop a sensitivity analysis method to assess the robustness of the preferred design(s) found by the deterministic selection method to variability in DM's preferences (given as crisp estimates in the deterministic selection). In this research component, we assume that there is no attribute variability.

1.2.3. Research Component 3: Selection with Preference Variability

When there is variability in preferences (caused for example due to lack of information on end users' needs, projecting into future markets), the DM would give a range of preferences and would like to know the "potentially optimal designs" (see Chapter 2 for definition) for that range. For example, in the automobile design selection, the DM would say: "I would allow an increase in the cost of the automobile by 4000 dollars to 5000 dollars if the 0-60 time is decreased by two seconds". In the selection of a cordless electric drill, the DM would say: "I would give up between 40 and 50 operations per battery charge to reduce the weight by 0.1 pounds". Each of these potentially optimal designs would be the most preferred for a particular realization of the preferences within the given range.

The objective of the third research component is to develop a selection method that does not assume any explicit form for the DM's value function and finds all the preferred designs when there is variability in DM's preferences. In this research component, we assume that there is no attribute variability.

1.2.4. Research Component 4: Selection with Preference and Attribute Variability

In addition to preference variability, it is quite common in engineering design to have variability in the attributes of the design alternatives. Uncontrollable parameters during the design process (e.g., manufacturing errors, use conditions) are the source for attribute variability. Since it is difficult to identify and quantify the uncontrollable parameters exactly, more often it is only possible to state the ranges of attributes (e.g., 40-45 operations per battery charge, 5-7 seconds of 0-60 time) instead of a number (e.g., 40 operations per battery charge, 6 seconds of 0-60 time) [Eum et al., 2001] [Jimenez et al, 2003].

The objective of the fourth research component is to develop a selection method that does not assume any explicit form for the DM's value function and finds all the preferred designs when there is variability in both the DM's preferences and attributes of design alternatives.

1.2.5. Decision Making Framework: Integrating the Four Research Components

Figure 1.2 shows the schematic of our decision making framework for product design selection with variability for an implicit value function. This decision making framework is iterative and interactive.

In our decision making framework, the DM starts the first iteration by obtaining estimates (due to the inevitable variability in the selection process) of the preferences and the attributes of design alternatives. With these crisp estimates of preferences and attributes, the DM uses our deterministic selection method for finding the set of non-eliminated trial designs, D_{NTD} (this set could be a singleton, see Chapter 2 for definition). Next, the DM finds (and then evaluates) the robustness of D_{NTD} to variations in preference estimates using our sensitivity analysis method. If the DM is satisfied with the robustness of D_{NTD}, he/she stops the iterations and takes the next action, which could be manufacturing one of the non-eliminated trial designs.

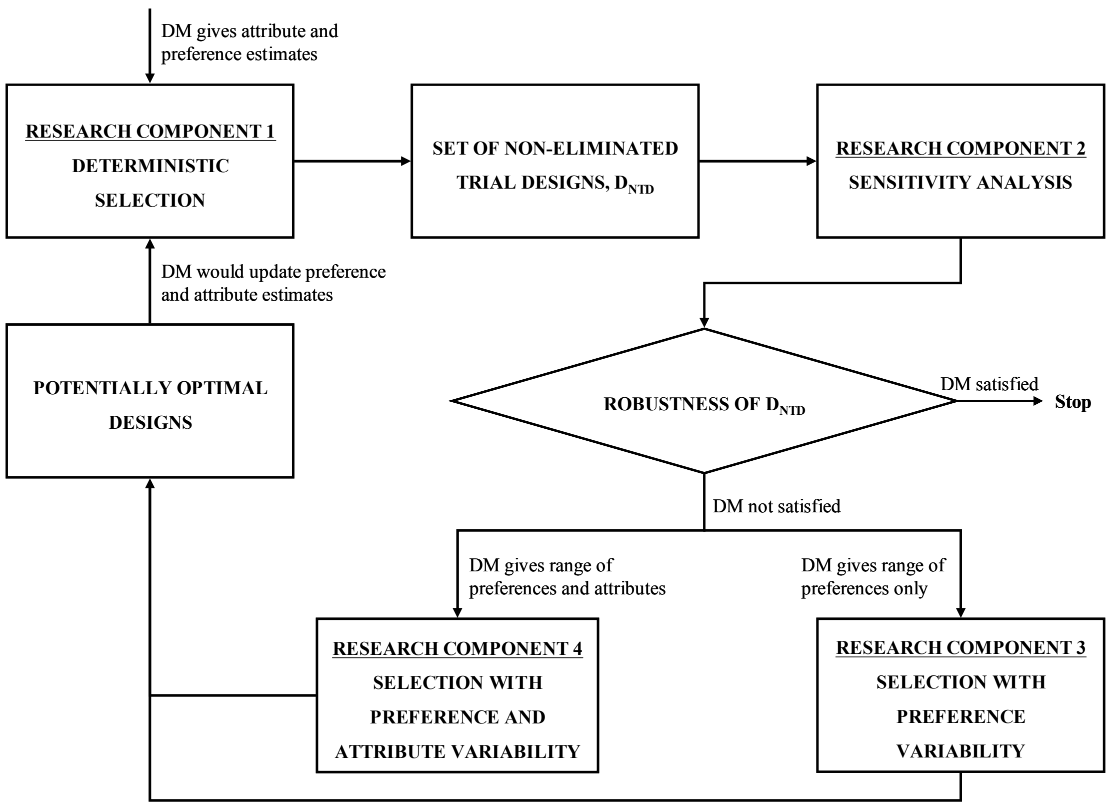

Figure 1.2: Decision making framework integrating four research components

Otherwise, i.e., if the DM is not satisfied with the robustness of D_{NTD}, he/she can give a range either for preferences alone or for preferences and attributes. The DM then uses our method for selection with preference variability or our method for selection with preference and attribute variability (depending on whether DM gives the range for preferences alone or for both preferences and attributes) for finding the set of potentially optimal designs from the set of design alterantives. Since, only one of the potentially optimal designs can be the most preferred (see Definitions in Chapter 2 for details) for the given range of preferences or given range of preferences and attributes, these potentially optimal designs become the set of design alternatives for the next iteration. For the next iteration, the DM would gather more information and improve the estimates for preferences and attributes (from the ranges given in the current iteration) and repeat the above discussed steps starting with deterministic selection.

1.3. ASSUMPTIONS

We make the following assumptions in developing the methods for our four research components.

- There is a single Decision Maker (DM) for making the selection and the DM has enough expertise to state the marginal rate of substitution (see Chapter 2 for definition) between attributes at a design in the attribute space.
- The DM's value function is non-decreasing, differentiable and quasi-concave (see Chapter 2 for definition) with respect to the attributes. The assumption that the value function is non-decreasing with respect to attributes is not required for the applicability of our deterministic selection method (see Chapter 3 for details).
- The design alternatives for selection are discrete and the attributes for selection are specified a priori. Also we assume there is no attribute variability for the methods developed in Chapter 3, Chapter 4 and Chapter 5. We handle attribute variability in Chapter 6.
- The DM can provide ranges for marginal rate of substitution (see Chapter 2 for Definition) and ranges for attributes of the design alternatives when there

is variability. We also assume that the MRS values in the given ranges of preferences are consistent to simplify our approaches in Chapter 4, Chapter 5 and Chapter 6.

- When there is variability in preferences and attributes, we assume that the ranges of marginal rate of substitution (see Chapter 2 for Definition) preferences that the DM gives at a design include the ranges of preferences at any attribute levels in the range of attributes for that design. Also, for simplicity, the DM's risk attitude is not taken into account in the attribute range of a design.
- The value function can be approximated to be linear in a small region around a trial design (see Chapter 2 for definition) for the application of our heuristic approaches in Chapter 3 and Chapter 5.

1.4. ORGANIZATION OF DISSERTATION

The organization of the rest of the dissertation is as follows. In Chapter 2, we give the definitions of concepts and terminologies used throughout the dissertation, as well as a comprehensive review of the related previous work in the literature. In Chapter 3, we present our method for deterministic selection (Research Component 1). Next, in Chapter 4, we present a concept for sensitivity analysis (Research Component 2) and describe the implementation of the concept in our deterministic selection method. In Chapter 5, we develop our method for selection with preference variability (Research Component 3) and extend it to selection with preference and attribute variability (Research Component 4) in Chapter 6. To demonstrate the application and to verify our methods for the four research components, several examples (engineering and numerical) are given in Chapters 3 through 6. Finally, we conclude the dissertation with remarks, drawbacks, contributions, and suggestions for future research directions in Chapter 7.

After reading this chapter and the next, we recommend that the reader continue with Chapter 3 because it contains concepts that are the foundations for Chapters 4, 5 and 6. Chapter 4 and Chapter 5 may be read independently. However, Chapter 6 should be read after Chapter 5.

Figure 1.3 depicts the various chapters, their relationships, and the dissertation's information flow.

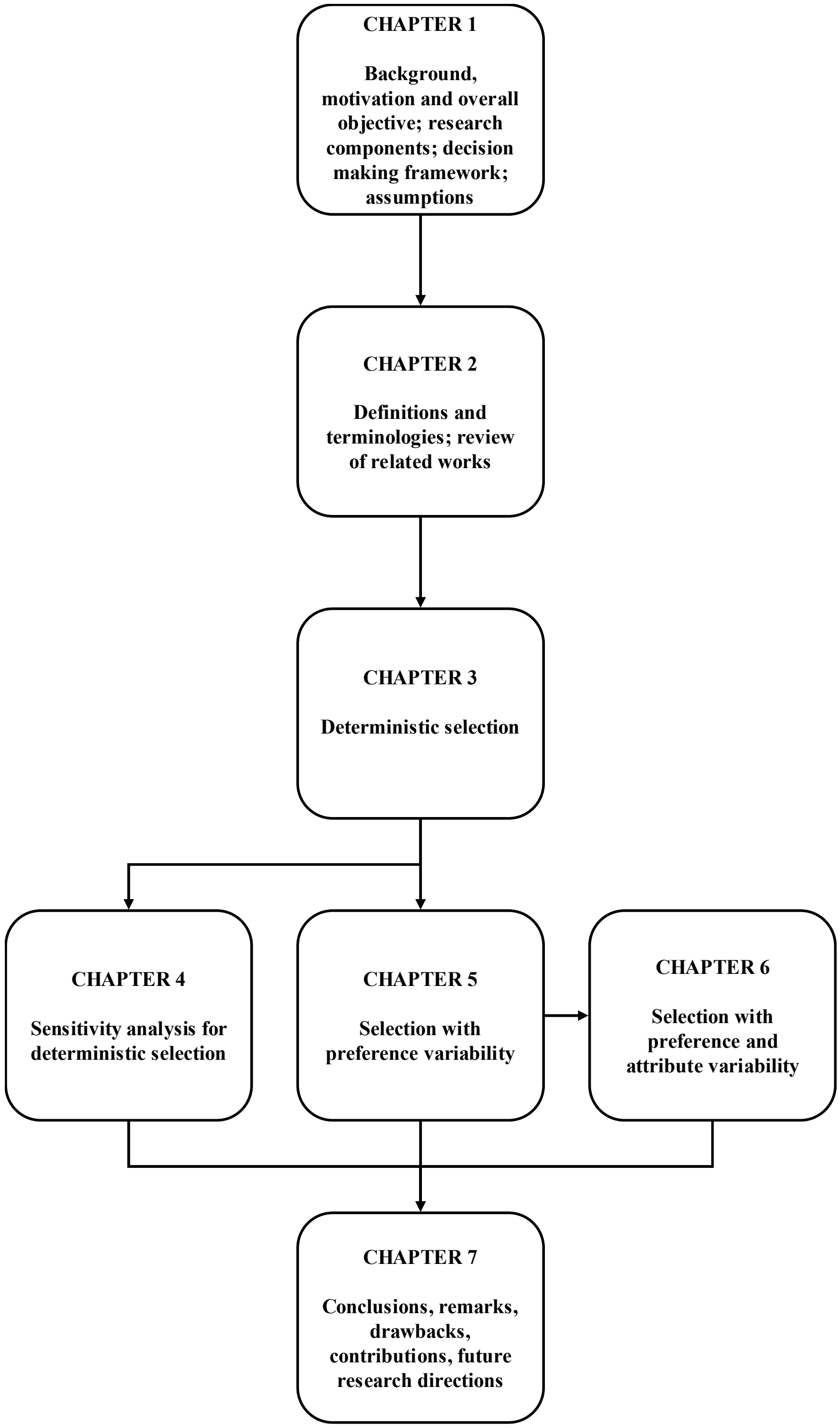

Figure 1.3: Organization of dissertation

CHAPTER 2

DEFINITIONS AND PREVIOUS WORK

2.1. INTRODUCTION

In this chapter, we provide several definitions and terminologies that will be used throughout this dissertation. We also give a comprehensive review of the previous work in the literature related to deterministic selection, sensitivity analysis, selection with preference variability, and selection with preference and attribute variability.

The organization of this chapter is as follows. In Section 2.2, we give related definitions and terminologies. Next in Section 2.3, we provide a literature review for the four research components. Finally we conclude the chapter with a summary in Section 2.4.

2.2. DEFINITIONS AND TERMINOLOGIES

The set of 'n' discrete design alternatives from which the most preferred is to be selected is $\{D_1,\ldots,D_j,\ldots,D_n\}$. Each alternative D_j is represented by the set of attributes $[a_{1j},\ldots,a_{mj}]$ in the m-dimensional design attribute space (i.e., an m-dimensional space in which the coordinates are the attribute values). Let the value function, $V(D_j)$ be a function of attributes $[a_{1j},\ldots,a_{mj}]$ that represents the DM's preferences. V is said to be explicitly known, if we know the form of the equation (e.g., linear, polynomial) of V with some unknown constants or parameters (e.g., weights of attributes). The unknown parameters are determined by capturing the DM's preferences. If the equation of V is not known, we say that V is implicit.

When there is no variability in attributes, a_{ij} values would be exact (i.e., deterministic or fixed or crisp.[3]). However, when there is variability in the attributes, we assume that the ranges of attributes for each design alternative are known. We use the symbol A_{ij}^{L} to represent the lower bound, A_{ij}^{U} to represent the upper bound, and A_{ij} to represent the range $\left[A_{ij}^{L}, A_{ij}^{U}\right]$ of the i^{th} attribute of design D_j. We use the symbol a_{ij} to represent a variable attribute that belongs to the range A_{ij}. (Note that a_{ij} could be fixed or variable depending on whether or not the i^{th} attribute of design D_j is deterministic.)

Next, we provide several more definitions and terminologies used in this dissertation.

2.2.1. Scale of an Attribute, r_i

The scale, r_i, of an attribute a_i is the difference between the maximum and minimum of the attribute over the set of original design alternatives. I.e., $r_i = \left(\max_j a_{ij}\right) - \left(\min_j a_{ij}\right)$. This definition is applicable only when there is no attribute variability.

2.2.2. Quasi-concave Function

A function V defined on a nonempty convex domain is said to be quasi-concave [Bazaraa et al., 1993] [Mangasarian, 1969] if

[3] In this dissertation, we use the terms fixed, deterministic, and crisp interchangeably.

$$V[\theta X_1 + (1-\theta)X_2] \geq \min[V(X_1),V(X_2)] \quad (2.1)$$

for all X_1, X_2 that belong to the domain of V and $\theta \in [0,1]$ (see Figure 2.1 for examples).

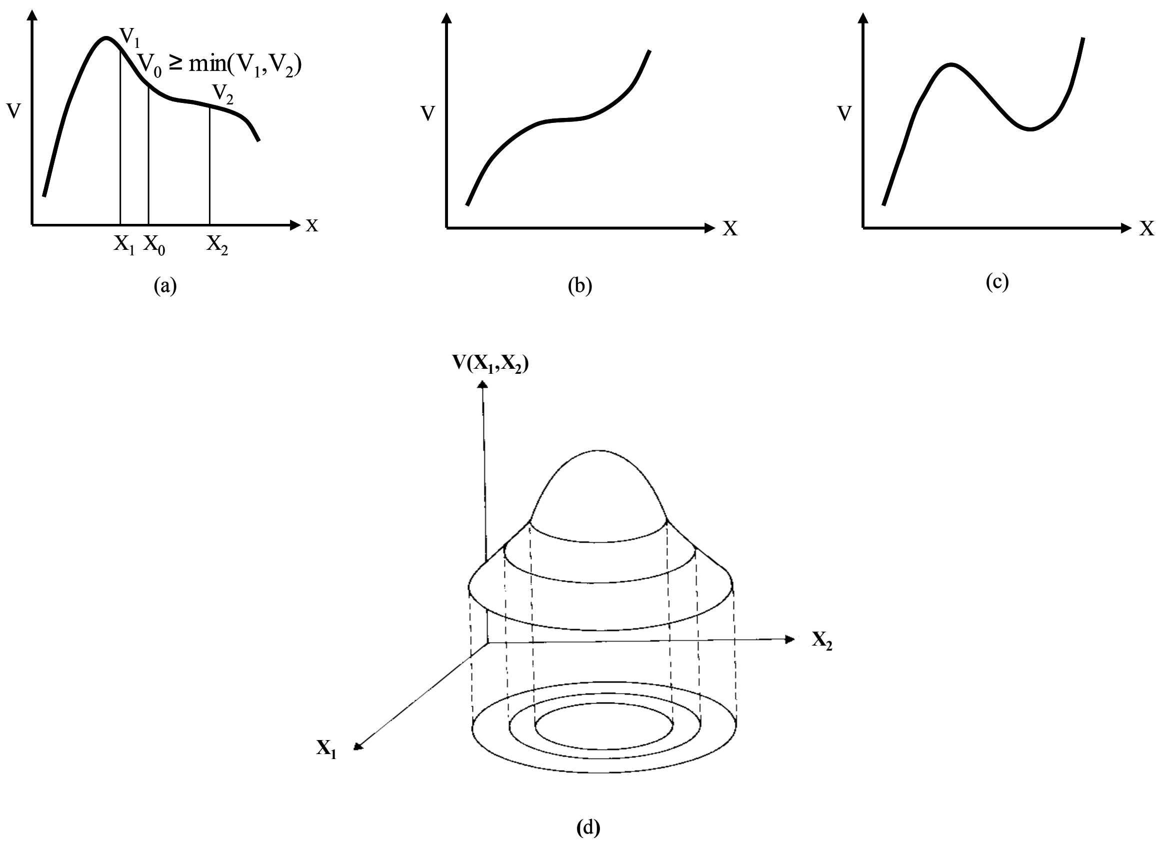

Figure 2.1: Examples of (a) quasi-concave, (b) non-decreasing quasi-concave, and (c) non quasi-concave functions for one variable, and (d) quasi-concave function for two variables

Note that a concave function is always quasi-concave, but the converse might not hold [Takayama, 1993]. For other properties of quasi-concave function refer to the literature e.g., [Avriel et al., 1988] [Crouzeix and Lindberg, 1986] [Greenberg and Pierskalla, 1971] [Schaible and Ziemba, 1981]. In this dissertation we assume that the DM's implicit value function is non-decreasing, differentiable and quasi-concave. However, the assumption that the value function is non-decreasing is not required for the applicability of our deterministic selection method (Research Component 1, see Chapter 3 for details).

2.2.3. Trial Design D_T

The trial design D_T is a particular design under consideration from the original set of design alternatives.

2.2.4. Marginal Rate of Substitution (MRS)

At trial design D_T, let Δa_j be the amount the DM will compromise in attribute a_j in order to gain an amount Δa_i in attribute a_i while maintaining constant value (i.e., the DM remains indifferent [Keeney and Raiffa, 1976] with respect to D_T) according to his/her preferences. The MRS, S_{ijT}, between attributes a_i and a_j at D_T is the ratio $-\Delta a_j / \Delta a_i$. Figure 2.2 illustrates the definition of MRS.

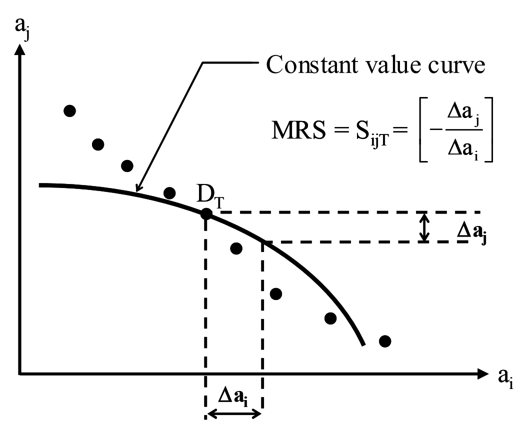

Figure 2.2: Illustration of marginal rate of substitution between attributes

Note that when the attributes are not normalized, S_{ijT} has a dimension that is equal to the ratio of the dimensions of a_j and a_i. For example, in the selection of a cordless electric drill, if a_1 is the attribute "cost" measured in dollars and a_2 is the attribute "weight" measured in pounds, then MRS, S_{12T}, between a_1 and a_2 at D_T has a dimension pound per dollar.

When there is no variability in preferences, both Δa_j and S_{ijT} would be exact (or crisp). However, if there is variability in preferences, the DM would give a range for Δa_j (for a fixed Δa_i) thus leading to a range of MRS. We use the symbol S_{ijT}^{L} to represent the lower bound, the symbol S_{ijT}^{U} to represent the upper bound, and the symbol S_{ijT} to represent the range $\left[S_{ijT}^{L}, S_{ijT}^{U}\right]$ of MRS when there is preference variability. We use the symbol s_{ijT} to represent a variable MRS that belongs to the range S_{ijT}. (In short, S_{ijT} could have a range or be crisp depending on whether or not MRS has variability.)

In Chapter 3 to Chapter 7, in the description, demonstration, and discussion of the proposed method for each research component, when we use the word preference we mean the DM's MRS preferences. However, the word preference might refer to other kind of preferences (e.g., relative importance of attribute) in the introduction and overview of the proposed method for each research component (see Chapter 3 to Chapter 6).

2.2.5. Gradient Cut

The gradient cut [Malakooti, 1988] is the half space C_G bounded by the normal to the gradient of a value function V at a point D_T, ∇V_T, with the gradient pointing in the outward direction from C_G; see Figure 2.3. C_G does not include the boundary line H_T in Figure 2.3.

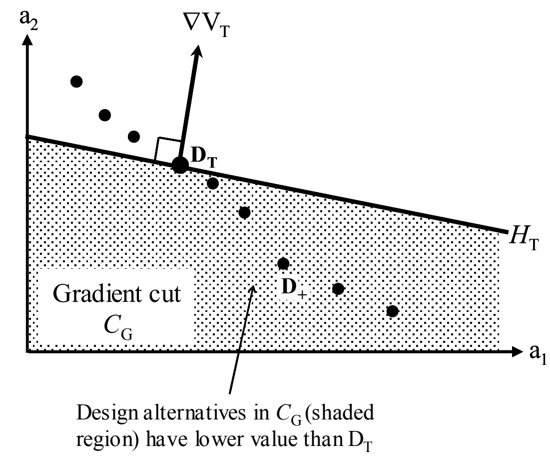

Figure 2.3: Illustration of gradient cut

For a general m-dimensional case, the boundary is a hyper-plane passing through D_T and perpendicular to the gradient at D_T. It can be shown that for a differentiable quasi-concave value function all design alternatives belonging to C_G have a lower value than D_T [Bazaraa et al., 1993] [Sundaram, 1996]. However, design alternatives that are not in C_G might have higher or lower or equal value with respect to D_T [Bazaraa et al., 1993].

2.2.6. Set of Non-eliminated Trial Designs (D_{NTD})

D_{NTD} is a subset of the original designs. Each member of D_{NTD} has been a trial design. No member of D_{NTD} lies in the gradient cut(s) of any other trial design, and so can not be eliminated by any other trial design, including the other members of D_{NTD}. (See Chapter 3 and Chapter 5 for further explanation.)

2.2.7. Dominated Design

When there is no attribute variability, but there is variability in MRS preferences, a design D_+ is said to be dominated by another design D_T, if D_+ has lower value than D_T

(i.e., $V(D_+) < V(D_T)$) for the whole range *of* MRS preferences, S_{ijT}, at D_T. If there is attribute variability also, then D_+ is said to be dominated by D_T if $V(D_+) < V(D_T)$ for the whole range *of* S_{ijT} and the whole range *of* attribute levels A_{i+} and A_{iT} (where i = 1 to m, m is the number of attributes).

2.2.8. Potentially Optimal Design

When there is no attribute variability, but there is variability in MRS preferences, a design D_+ is said to be potentially optimal if D_+ has the highest value among all design alternatives for some subset of S_{ijT}. For example, in Figure 2.4, D_1 has highest value for some part of the MRS range and D_2 has the highest value for some other part of the MRS range. Hence D_1 and D_2 are potentially optimal. On the other hand, D_3 is dominated by D_1 and D_2 because it has lower value than D_1 and D_2 for the whole range of MRS preference.

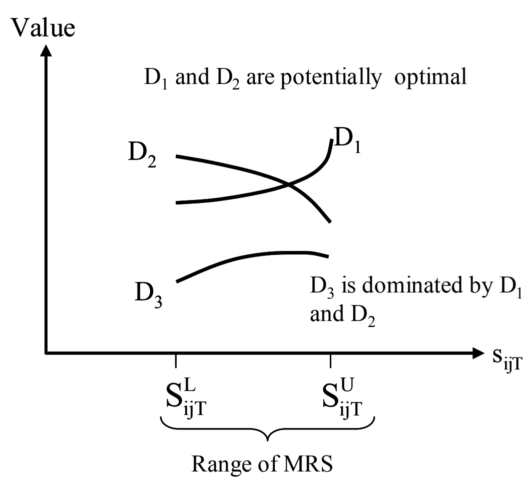

Figure 2.4: Illustration of potentially optimal and dominated designs

If there is attribute variability also, then D_+ is potentially optimal if D_+ is the highest valued design alternative for some subsets of S_{ijT}, A_{i+} and A_{id} (where d = 1 to n, n is the number of design alternatives).

Note that, from the above definition, a design which is not potentially optimal cannot be most preferred for any realization of MRS that belongs to the range of preferences and/or for any realization of attribute that belongs to the range of attributes.

Eum et al., [Eum et al., 2001] gave similar definitions for dominated design and potentially optimal design when the value function is assumed to be additive. In this dissertation, we have extended their definitions for the more general case of an implicit value function.

Note that dominance and potential optimality defined here are different from the component-wise dominance and Pareto optimality [Eum et al., 2001].

2.3. OVERVIEW OF PREVIOUS WORK

Multi-Attribute Decision Making (MADM) is a popular technique that is used for engineering design selection [Li, 2001] [Neufville, 1990]. MADM methods for product design selection in literature can be categorized into five main groups: methods for (i) deterministic selection, (ii) sensitivity analysis, (iii) selection with attribute variability alone, (iv) selection with preference variability alone, and (v) selection with preference and attribute variability. The third group, methods for selection with attribute variability alone, is not the focus of this dissertation and is not reviewed here. However, the interested reader can refer to the literature, e.g., [Bradley and Agogino, 1994] [Jaffray, 1989] [Keeney and Raiffa, 1976] [Li and Azarm, 2002]

[Marston and Mistree, 1998] [Wan and Krishnamurthy, 2001] [Wassenaar and Chen, 2003], for more details. In the next four sections, we provide the literature review of existing methods in the first, second, fourth, and fifth groups.

2.3.1. Literature Review on Deterministic Selection

Existing methods for deterministic selection assume that the DM has an intuitive value function that he/she maximizes to make the selection [Fishburn, 1970] [Keeney and Raiffa, 1976] [Yu, 1985] [Zeleny, 1982]. So, product design selection problem can be viewed as a discrete optimization problem with an implicit (or unexpressed) objective function. Existing deterministic selection methods in the literature try to find the implicit objective function (or the value function) by obtaining from the DM information about quantitative preferences which reflect the value function [Olson, 1996] [Triantaphyllou, 2000]. Various selection methods take the preferences in various forms, e.g., relative importance of attributes [Lootsma, 1999] [Saaty, 1980], comparison of design alternatives [Koksalan et al., 1984] [Malakooti, 1988] [See and Lewis, 2002] [Toubia et al., 2003], or marginal rate of substitution between attributes [Keeney and Raiffa, 1976] [Yu, 1985].

Some MADM methods for deterministic selection estimate the value function completely by presuming its form (e.g., linear, multiplicative) [Barzilai, 1997b] [Saaty, 1980] [Thurston et al., 1994] [Zeleny, 1982]. These methods have two shortcomings. First, presuming a form for the value function is restrictive and is applicable only for some special cases (e.g., preferential independence between attributes) [Keeney and Raiffa, 1976] [Thurston, 2001]. Second, the presumed form can differ significantly from the DM's unexpressed value function, leading to an erroneous selection.

To address those shortcomings, interactive methods have been developed to estimate the value function partially, and to use that information at a series of trial designs (recall Definition in Section 2.2.3) for eliminating lower value design alternatives [Korhonen et al., 1984] [Malakooti, 1988]. Rather than assuming a specific form for the value function, partial estimation methods allow for broad classes of functions (e.g., monotonic, concave). The most generalized value function that has been discussed in the literature is a quasi-concave value function [Koksalan et al., 1984] [Malakooti, 1989a].

There are two components in the partial estimation methods. First is the elimination of lower value design alternatives at a trial design. Second is the search for a better design alternative to use as a new trial design. For the first component, some methods in the literature ask the DM for the pair-wise comparisons of "adjacent" design alternatives [Karwan et al., 1989] [Malakooti, 1989a]. The response to these comparisons is used to construct convex cones and then eliminate lower value designs. These methods become inefficient (i.e., the number of designs eliminated by the convex cones decreases) if the value function is not non-decreasing with respect to the attributes (or cannot be converted to non-decreasing) [Korhonen et al., 1984] [Malakooti, 1988].

Another approach [Malakooti, 1988] for eliminating lower value design alternatives is to find the gradient of the value function at a trial design, and use the gradient cut (recall Definition in Section 2.2.5). The number of designs eliminated by this approach does not depend on the value function being non-decreasing.

Malakooti [Malakooti, 1989a] uses pair-wise comparisons of the adjacent design alternatives to find the gradient of value function. Unfortunately, Malakooti's approach works well only if, for an m-dimensional design attribute space, there are at least m+1 design alternatives in the vicinity of the trial design. In addition, Malakooti's approach of pair-wise comparison of design alternatives to find the gradient has two problems. First, pair-wise comparison of alternatives by the DM might lead to intransitive preferences [Yu, 1985] (which have to be accounted for). Second, comparison of alternatives that are either "far off" or "close by" in the design space is difficult for the DM. Both of these problems are well known in the literature [Yu, 1985]. Also, Malakooti's approach needs "strength of preference" for the comparisons [Malakooti, 1989a] to get a good estimate of the gradient of value function, adding burden to the DM.

To overcome the above shortcomings, in our deterministic selection method we use the DM's Marginal Rate of Substitution (MRS) between attributes (see Chapter 3) to find the gradient of value function at a trial design. MRS captures any nonlinearity, non-monotonicity and coupling (i.e., interdependence between attributes) in the DM's value function [Barzilai, 1998] [Keeney and Raiffa, 1976]. It is generally easier for the DM to provide MRS than to do a pair-wise comparison of alternatives because each MRS involves only trading off between two attributes, rather than comparing two m-attribute designs.

For the second component of the partial estimation methods, finding a design alternative with higher value for the new trial design, Geoffrion et al. [Geoffrion et al., 1972] and Musselman and Talavage [Musselman and Talavage, 1980] have approaches that are applicable only for continuous design alternatives. Koksolan et al. [Koksalan et al., 1984] propose to approximate the value function as linear or quadratic and then to choose the design that has the maximum value as the new trial design. Malakooti [Malakooti, 1988] uses a one-dimensional search approach, which again is dependent on the spread and clustering of the design alternatives. In our deterministic selection method (see Chapter 3) we present a new approach for finding a new trial design that makes efficient use of the gradient information already obtained from the DM at all the previous trial designs. As with the approaches in the literature, our approach requires only that the value function be differentiable and quasi-concave with respect to the attributes. The approach does not depend on the distribution of the design alternatives in the attribute space.

2.3.2. Literature Review on Sensitivity Analysis

Existing literature in sensitivity analysis addresses cases where the DM's value function is presumed to be explicitly known (e.g., known polynomial function of attributes with unknown parameters like weights, utilities [Keeney and Raiffa, 1976]). Sage [Sage, 1981] studied and formalized the allowed errors in the estimation and elicitation of probabilities and utilities before which the preferred design is affected. Barron and Schmidt [Barron and Schmidt, 1987] proposed two procedures: entropy-based and least square (i.e., L_2-metric) to calculate the minimum variation required between the actual weights and the estimates of weights for changing the most preferred design when the value function is linear. Ringuest [Ringuest, 1997] later extended the L_2-metric of Barron and Schmidt [Barron and Schmidt, 1987] to an L_P-metric. Mareschal [Mareschal, 1988] proposed an approach for finding the "weight stability interval", which consists of all possible weights that maintain the rank order obtained using the original estimates of weights.

Insua and French [Insua and French, 1991] proposed some distance based tools to identify the possible competitors to the current most preferred design when the DM's preferences change. Antunes and Climaco [Antunes and Climaco, 1992] proposed a sensitivity analysis approach for their TRIMAP method. However, this approach is applicable only when the number of attributes is three or less, which is a significant limitation [Antunes and Climaco, 1992]. Triantaphyllou and Sanchez [Triantaphyllou and Sanchez, 1997] proposed a sensitivity analysis approach and applied it to popular MADM methods like weighted sum model, weighted product model, and analytical hierarchy process [Saaty, 1980]. Ma et al. [Ma et al., 2001] presented a method for finding the "weight-set" that contains all possible ranges of weights of an additive value function when the rank order of alternatives is given. Triantaphyllou and Shu [Triantaphyllou and Shu, 2001] studied the number of feasible rankings that are possible, assuming an additive value function, for the given set of design alternatives, when the weights of the criteria are allowed to change.

Although the MADM literature describes significant research on sensitivity analysis when the value function is presumed, it is well known that presuming a form for the value function is restrictive and applicable only to special cases [Keeney and Raiffa, 1976] [Thurston, 2001]. In Chapter 4, we present a concept for sensitivity analysis that is applicable for an implicit value function.

2.3.3. Literature Review on Selection with Preference Variability

The literature reports of two ways to account for preference variability in selection. One way is to assume different probability distributions for preferences and then study the affect of these distributions on the most preferred design. Scott [Scott, 2002] studied analytical hierarchy process [Saaty, 1980], assuming uniform distributions for the DM's preferences, and proposed some indices to quantify the changes in the most preferred design. Reeves and Macloed [Reeves and Macloed, 1999] used the Interactive Weighted Tchebycheff [Steuer and Choo, 1983] procedure to study the robustness of the preferred design for various distributions of the preferences. However, the preferred design found by assuming some probability distributions for the preferences might be erroneous if the actual distributions differ from the assumed distributions.

Another way (also popular in the literature) for accounting preference variability in selection is to ask the DM to provide some constraints on the preferences [Claessens et al., 1991] [Insua and French, 1991] [White et al., 1984]. Typical constraints could be some ranges on the preferences, like relative importance of attribute a_1 is between 0.3 and 0.4. The constraints on preferences are then used in finding the non-dominated and potentially optimal designs (see Definition in Section 2.2.8) [Hazen, 1986]. Some people refer to selection with preference variability as selection with partial information [Athanassopoulos and Podinovski, 1997]. Note that it is generally easy for the DM to give some constraints on the preferences than the probability distributions governing the variability in the preferences.

Existing literature in selection with partial information addresses the case when the DM's value function is presumed to be explicitly known (e.g., additive with unknown attribute weights, multiplicative with unknown scaling constants). Hazen [Hazen, 1986] derived a relation between dominance and potential optimality when the value function is explicitly known (additive or multiplicative) with unknown scaling constants or weights. Malakooti [Malakooti, 1989b] proposed the concepts of convex non-dominancy and trade-off non-dominancy and identified their relation to dominance for additive value functions.

Insua and French [Insua and French, 1991] proposed some formal definitions and methods to identify the non-dominated and potentially optimal designs when there is variability in the weights of an additive value function. In their methods, Insua and French [Insua and French, 1991] proposed linear programming problems for checking the dominance and potential optimality of a design. Athanassopoulos and Podinovski [Athanassopoulos and Podinovski, 1997] later developed a dual linear programming method to identify the dominated and potentially optimal designs when there is variability in the weights of an additive value function. Malakooti [Malakooti, 2000] developed a method that can identify a number of dominated designs by solving a single linear programming problem. Carrizosa et al., [Carrizosa et al., 1995] proposed a method for ranking a set of design alternatives with partial information about weights of the additive value function. In their method, Carrizosa et al., [Carrizosa et al., 1995] do not solve any linear programming problem. Instead they use some "quasiorders" for ranking the design alternatives.

Although the MADM literature describes significant research on selection with preference variability when the value function is presumed, it is well known that presuming a form for the value function is restrictive and applicable only to special cases [Keeney and Raiffa, 1976] [Thurston, 2001]. In Chapter 5, we present a method for selection with preference variability for an implicit value function.

2.3.4. Literature Review on Selection with Preference and Attribute Variability

Existing literature in selection with preference and attribute variability addresses the case when the DM's value function is presumed to be explicitly known (e.g., additive with unknown attribute weights, multiplicative with unknown scaling constants). White et al., [White et al., 1984] developed a method for identifying the dominated alternatives when the constraints on the attribute weights, scores of the attributes, and the relative importance between some alternatives are given. They assume that the DM's value function is additive with respect to the attributes. Sage and White [Sage and White, 1984] proposed an interactive decision support system, based on the method proposed by White et al., for selection with preference and attribute variability.

Weber [Weber, 1987] proposed a framework for decision making with preference and attribute variability when the value function is presumed. Weber [Weber, 1987] also surveyed existing methods based on that framework. Moskowitz et al., [Moskowitz et al., 1992] proposed a method called Multi-Criteria Robust Interactive Decision Analysis (MCRID) for eliminating dominated designs when there is preference and attribute variability. In their method, Moskowitz et al., [Moskowitz et al., 1992] expect the DM to give some partial information about the probability distributions governing preference and attribute variability.

Anandalingam and White [Anandalingam and White, 1993] extended the method of White et al., [White et al., 1984] by proposing a penalty function approach for finding the potentially optimal designs. Park and Kim [Park and Kim, 1997] developed a nonlinear programming formulation for finding the dominated and potentially optimal designs when the ranges quantifying the preference and attribute variability are known and when the value function is presumed. Eum et al., [Eum et al., 2001] and Lee et al., [Lee et al., 2001] later proposed linear programming equivalents of the nonlinear programming problems required for checking the dominance and potential optimality of designs. Jimenez et al. [Jimenez et al., 2003] proposed a decision support system that finds the sensitivity of the preferred design to variations in the weights and the attribute for the ranges given by the DM.

Although some research has been reported in the MADM literature on selection with preference and attribute variability when the value function is presumed, it is well known that presuming a form for the value function is restrictive and applicable only to special cases [Keeney and Raiffa, 1976] [Thurston, 2001]. In Chapter 6, we present a method for selection with preference and attribute variability for an implicit value function.

2.4. SUMMARY

In this chapter, we gave important definitions and terminologies that will be used throughout this dissertation. We also provided a detailed literature review for each of our four research components. The shortcomings of the literature related to each of our research components are summarized below.

- In deterministic selection, methods exist in the literature for selection with an implicit value function [Korhonen et al., 1984] [Malakooti, 1988]. These methods ask the DM for the pair-wise comparison of adjacent design alternatives. However, pair-wise comparison of design alternatives can lead to intransitive preferences and it is generally difficult to compare design alternatives that are either far off or close by in the design attribute space. To overcome this shortcoming we ask the DM to provide the marginal rate of substitution between the attributes in our deterministic selection method.
- In sensitivity analysis, existing methods in the literature presume a form for the value function, additive being the most popular [Insua and French, 1991]. However, presuming a form for the value function is restrictive and applicable only in special cases. To overcome this shortcoming, we present a concept for sensitivity analysis that is applicable for an implicit value function.
- In selection with preference variability, some methods [Scott, 2002] assume probability distributions for the preferences and study the affect of the distributions on the most preferred design. However, it is generally difficult to make a good assumption of the actual distributions. Some other methods ask the DM to provide constraints on the preferences [Claessens et al, 1991] [White et al., 1984] and then find the potentially optimal designs for the given constraints. However, existing methods that ask for the constraints on the preferences are applicable only when the DM's value function is presumed

explicitly. We propose a selection method for preference variability that is applicable for an implicit value function to overcome this shortcoming.

- In selection with preference and attribute variability, methods exist in the literature when there is partial information about the probability distributions governing the variability in preferences and attributes [Moskowitz et al., 1992]. Methods for finding the potentially optimal designs, when the ranges quantifying the preference and attribute variability are known, also exist in the literature. However, all of the existing methods presume a form for the DM's value function. To overcome this shortcoming, we propose a selection method for an implicit value function, when the ranges quantifying the preference and attribute variability are known.

In the next chapter, we present the development of the method for our first research component, deterministic selection.

CHAPTER 3

DETERMINISTIC SELECTION

3.1. INTRODUCTION

The amount a DM is willing to give up in one attribute to gain a certain amount in another attribute, for maintaining constant value is, in many instances, dependent on the attribute levels of a design alternative. For example, in the selection of an automobile, if the DM is asked for the preferences at a design point that has the attribute levels of 20,000 dollars of cost and 9 seconds of 0-60 time, the DM might say: “I would allow an increase in the cost of the automobile by 5000 dollars if the 0-60 time is decreased by two seconds”. If the DM is asked the same question at a design point that has the attribute levels of 30,000 dollars of cost and 7 seconds of 0-60 time, the DM might say: “I would allow an increase in the cost of the automobile by 1000 dollars if the 0-60 time is decreased by two seconds”. Similarly, the number of operations per battery charge that a DM would give up to reduce the weight of a cordless electric drill is dependent on the attribute levels of the number of operations per battery charge and the weight of the drill. This kind of nonlinear preference structure is common for a designer acting as the DM in engineering design selection. It is generally difficult to represent such a nonlinear preference structure a priori by presuming an explicitly known value function (e.g., additive, multiplicative, quadratic). Also, if the presumed form differs significantly from the DM’s unexpressed value function, the resulting solution would be erroneous. One might argue that, the DM could be asked for the values of some sample design alternatives and a curve be then fit through the sample values for approximating the DM's value function. There are two problems with such an approach. First, it is extremely difficult for the DM to consistently state the values of some sample design alternatives. Second, one does not know what kind of curve (e.g., polynomial, exponential, multiplicative) to fit through the values of the sample design alternatives.

The purpose of this chapter is to present a deterministic selection method that aids the DM in selecting the preferred design(s) from a set of design alternatives. Our deterministic selection method does not presume any explicit form for the DM's value function, thus allowing the DM's preference structure to be more general.

The organization of this chapter is as follows. We begin this chapter with an overview of our deterministic selection method in Section 3.2. We then present the details of our method in Section 3.3 and present our algorithm for deterministic selection in Section 3.4. Next we give two engineering examples to demonstrate our deterministic selection method in Section 3.5. We present some experimental results to verify our deterministic selection method in Section 3.6 and finally conclude the chapter with a summary in Section 3.7.

3.2. OVERVIEW OF DETERMINISTIC SELECTION METHOD

Figure 3.1 shows the flowchart of our interactive deterministic selection method. This method is iterative and assumes that the DM's preferences reflect an implicit value function that is differentiable and quasi-concave. In this method, we start by picking a trial design, D_T, from the set of design alternatives. If the DM cannot make an informed guess of the highest valued design to use as D_T, we use either the alternative that would have maximum value if the value function were linear with equal importance to the attributes, or a random pick.

As shown in Figure 3.1, in a small region O_T around D_T we approximate the value function to be linear with respect to the attributes. The gradient of V at D_T is $\nabla V_T = [W_{1T},\ldots,W_{mT}]$. The general form for the linear approximation of $V(D_j)$ in O_T would be (considering only the differences between V for design alternatives near D_T):

$$V(D_j) = \sum_{i=1}^{m} W_{iT} \cdot a_{ij} \tag{3.1}$$

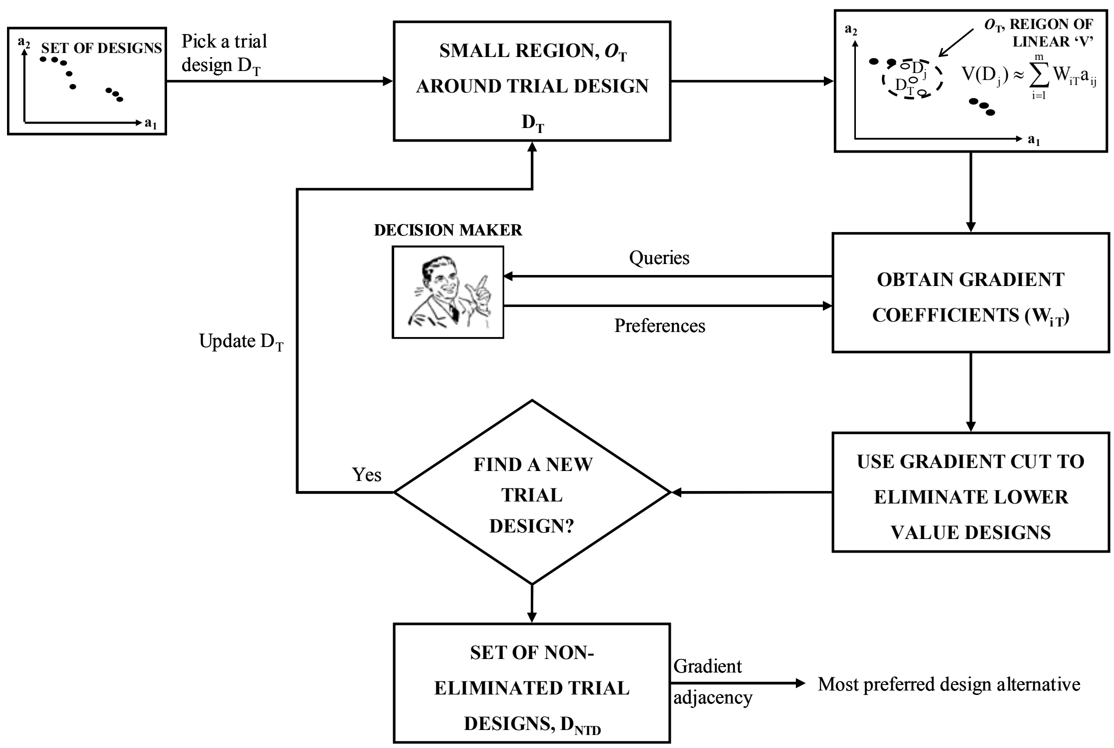

Figure 3.1: Flowchart of our deterministic selection method

Next, we find the gradient coefficients, W_{iT} (i=1,…,m), at D_T by obtaining preference information from the DM interactively (see Section 3.3.1 for details). Then we use the gradient cut for eliminating (to be explained in Section 3.3.2) some of the design alternatives which have a lower value than D_T. Next, we try to find a new trial design from the non-eliminated design alternatives (see Section 3.3.3 for our proposed approach to find a new trial design). If a new trial design is found, we repeat the above steps (recall Figure 3.1), referred to as “an” ‘iteration’ from here on in this chapter. Otherwise (i.e., if a new trial design is not found), we stop the process and collect the non-eliminated trial designs in a set, designated by D_{NTD}. If the set D_{NTD} has a single design then that design alternative would be the most preferred design alternative. If D_{NTD} has more than one design we use a novel approach, called gradient adjacency elimination, (to be explained in Section 3.3.4) for finding the most preferred design alternative from D_{NTD}.

The DM has the option of stopping the process anytime he/she is satisfied that the currently identified new trial design is the most preferred design alternative, even if all the other design alternatives are not eliminated.

Note that our method does not perform a “piecewise linear approximation” of the value function at a series of trial designs. The linear approximation is used to obtain the gradient of the value function at a trial design, and the gradient is used to eliminate lower value designs with respect to the trial design (see Section 3.3.2 for details).

3.3. DESCRIPTION OF DETERMINISTIC SELECTION METHOD

In this section, the individual parts of the deterministic selection method briefly described in Section 3.2 are explained in detail. First, in Section 3.3.1, we present our approach for obtaining the gradient of the value function at a trial design D_T. Then, in Section 3.3.2, we describe how to eliminate lower value design alternatives using the gradient cut. Then we present our approach to find a new trial design in Section 3.3.3. Finally, we discuss gradient adjacency elimination in Section 3.3.4.

3.3.1. Obtaining the Gradient of V at a Trial Design

We ask the DM questions regarding his/her MRS (recall Definition in Section 2.2.4 of Chapter 2) to find the gradient ($\nabla V_T = [W_{1T},\ldots,W_{mT}]$) of the value function. Note that the location of the design alternative in the design attribute space can influence the DM's MRS [Keeney and Raiffa, 1976]. MRS captures any non-linearity, non-monotonicity and coupling in the DM's value function [Barzilai, 1998] [Keeney and Raiffa, 1976]. From the definition of MRS, it can be readily shown that the MRS S_{ijT} between attributes a_i and a_j at D_T is

$$S_{ijT} = \left. \frac{\frac{\partial V}{\partial a_i}}{\frac{\partial V}{\partial a_j}} \right|_{D_T} \tag{3.2}$$

Using Eq. (3.1) as the linear approximation of the value function in O_T

$$\frac{\partial V}{\partial a_i} = W_{iT} \tag{3.3}$$

and

$$S_{ijT} = \frac{W_{iT}}{W_{jT}} \tag{3.4}$$

Accordingly, MRS values when they exist are consistent [Barzilai, 1997b] [Barzilai, 1998]. I.e.,

$$S_{ijT} \cdot S_{jkT} = S_{ikT}. \tag{3.5}$$

Because of this, only MRS values between 'm-1' pairs of attributes are independent when there are 'm' attributes. So, querying the DM for the MRS values gives only 'm-1' independent equations to solve for ‘m’ gradient coefficients, W_{iT}’s, which means that W_{iT}’s, might not be unique.

Further, the solvability of the set of equations depends on the MRS values obtained from the DM being exact and consistent. These conditions are not likely to be met in the responses of a human DM, who is estimating an unexpressed multi-attribute constant-value function. To address these factors, we obtain excess information from the DM by asking for an m-th MRS value. In our method, we ask the DM to provide the MRS between attributes a_i and a_{i+1} (i=1,…,‘m-1’) and the MRS between attribute a_m and attribute a_1 (if m>2), which is the m-th MRS value. (Another source of excess information would be to ask for reciprocal MRS values as well, i.e., S_{jiT} in addition to S_{ijT}.)

When the DM’s value function is differentiable and when there is no information for determining whether or not the DM’s value function is non-decreasing, we use the formulation in Eq. (3.6) to solve for the W_{iT}’s.

$$\text{Minimize}: \sum_{i,j} \left[\left(S_{ijT} - \frac{W_{iT}}{W_{jT}}\right)\cdot\left(\frac{r_i}{r_j}\right)\right]^2 \tag{3.6a}$$

$$\text{subject to}: \sum_{i,j,k} \left[\left(S_{ijT}\cdot S_{jkT} - \frac{W_{iT}}{W_{kT}}\right)\cdot\left(\frac{r_i}{r_k}\right)\right]^2 < \varepsilon, \text{ where } \varepsilon \text{ is arbitrarily small} \tag{3.6b}$$

$$W_{jT} \geq 0;\quad j = 1,\ldots,p\ (p \leq m) \quad \text{and} \quad W_{jT} \leq 0;\quad \text{for the rest} \tag{3.6c}$$

Eq. (3.6a) is for finding the W_{iT}’s that are as close to the given S_{ijT}’s as possible. Eq. (3.6b) is to account for inconsistency in the MRS values. Eq. (3.6c) is to account for the sign of the gradient coefficients, i.e., W_{jT} would be positive if the value function is increasing with respect to attribute a_j at D_T and negative otherwise. It is possible to obtain ambiguous results for the sign of W_{iT} if we assign them directly from MRS values S_{ijT}. So, we ask the DM to provide the sign of any one of the W_{iT}'s at the trial design D_T. The signs of the others follow directly from the MRS values: $W_{jT} = \frac{W_{iT}}{S_{ijT}}$.

The term r_i in Eq. (3.6a) and Eq. (3.6b) is the scale (recall Definition in Section 2.2.1 of Chapter 2) of the i^{th} attribute. Recall from the definition of MRS (Section 2.2.4 of Chapter 2) that S_{ijT} has a dimension which is equal to the ratio of the dimensions of a_j and a_i. Also, from Eq. (3.1) and Eq. (3.3), W_{iT} has a dimension that is the inverse of the dimension of a_i. So, the term $\left(S_{ijT} - \frac{W_{iT}}{W_{jT}}\right)$ in Eq. (3.6a) has a dimension that is equal to the ratio of dimensions of a_j and a_i. Similarly, the term $\left(S_{ijT} \cdot S_{jkT} - \frac{W_{iT}}{W_{kT}}\right)$ in Eq. (3.6b) has a dimension that is equal to the ratio of dimensions of a_k and a_i. So, the terms $\left(S_{ijT} - \frac{W_{iT}}{W_{jT}}\right)$ and $\left(S_{ijT} \cdot S_{jkT} - \frac{W_{iT}}{W_{kT}}\right)$ must be converted to dimensionless quantities before the summation in Eq. (3.6a) and Eq. (3.6b) respectively. Hence, we multiply each term by the ratio of the scales of the attributes to make the term dimensionless.

The solution to the optimization problem in Eq. (3.6) is not unique. Recall we mentioned earlier that the W_{iT}'s are not unique as there are only 'm-1' independent MRS preferences for 'm' attributes (refer Eq. (3.5)). However, as stated in the next lemma, any solution of Eq. (3.6) is a scalar transformation of some other solution.

Lemma: Let ∇V_T be a solution of Eq. (3.6). Any other $\nabla V'_T$ will be a solution of Eq. (3.6) if and only if $\nabla V'_T$ is a positive scalar transformation of ∇V_T, i.e., $\nabla V'_T = \mu \cdot \nabla V_T$, where $\mu>0$.

Proof: It is trivial to see that $\nabla V'_T$ will be a solution of the optimization problem in Eq. (3.6) if it is a scalar transformation of ∇V_T. To prove that if $\nabla V'_T$ is a solution then it is a scalar transformation of ∇V_T, let $\nabla V_T = [W_{1T},\ldots,W_{mT}]$ and $\nabla V'_T = [W'_{1T},\ldots,W'_{mT}]$. Assuming, with out loss of generality, that W_{1T} is not equal to zero we can rewrite ∇V_T and $\nabla V'_T$ as

$$\nabla V_T = W_{1T} \cdot \left[1, \frac{W_{2T}}{W_{1T}}, \ldots, \frac{W_{mT}}{W_{1T}} \right] \tag{3.7a}$$

$$\nabla V'_T = W'_{1T} \cdot \left[1, \frac{W'_{2T}}{W'_{1T}}, \ldots, \frac{W'_{mT}}{W'_{1T}} \right]. \tag{3.7b}$$

But the elements of the ∇V_T and $\nabla V'_T$ in Eq. (3.7) are the MRS values between attribute a_j ($j=2,\ldots,m$) and attribute a_1 and hence are equal, i.e.,

$$\frac{W_{jT}}{W_{1T}} = \frac{W'_{jT}}{W'_{1T}} = S_{j1T}\ ; \quad j = 2,\ldots,m\,. \tag{3.8}$$

So we can rewrite ∇V_T and $\nabla V'_T$ as

$$\nabla V_T = W_{1T} \cdot [1, S_{21T}, \ldots, S_{m1T}] \tag{3.9a}$$

$$\nabla V'_T = W'_{1T} \cdot [1, S_{21T}, \ldots, S_{m1T}]\,. \tag{3.9b}$$

From Eq. (3.9) we can see that ∇V_T and $\nabla V'_T$ are positive scalar transformations of the other because W_{1T} and W'_{1T} have the same sign (depending on whether V is increasing with respect to a_1 or decreasing with respect to a_1). This completes the proof of the lemma. □

It might be suggested that the solution to the optimization problem in Eq. (3.6) can be made unique by adding a normalization constraint (a typical constraint could be $\sum_{i=1}^{m} W_{iT} \cdot r_i = 1$, if W_{iT}'s are non-negative). The reason we do not normalize the gradient coefficients in Eq. (3.6) is that our method needs only the direction of the gradient for eliminating lower value designs (see Section 3.3.2 for details). Since each solution of Eq. (3.6) is a positive scalar transformation of another solution, all the solutions have the same direction. So in our method, adding a normalization constraint might result only in an increase in the complexity of Eq. (3.6).

The gradient, ∇V_T, at a trial design, D_T, gives the increasing direction of the value function at D_T. But, in practice a human DM might have difficulty understanding the significance of the gradient coefficient W_{iT}. However, the DM can usually interpret the relative importance (i.e., the weights) of the attributes [Lootsma, 1999] [Saaty, 1980]. Note that the weights of the attributes are different from the gradient coefficients, W_{iT}'s. Unlike the gradient coefficients, the weights of the attributes are dimensionless and lie between zero and one. In our method, we can easily convert the gradient coefficients, W_{iT}'s, into weights by multiplying each W_{iT} by the corresponding attribute scale, r_i, and then normalizing such that the sum of the weights is one (if W_{iT}'s are non-negative) or the sum of the squares of the weights is one (if W_{iT}'s could be negative).

When the DM's value function is non-decreasing and differentiable (refer Figure 2.1(b)), and when the attributes are normalized between zero and one (with one being the more preferred), our formulation in Eq. (3.6) for finding the gradient coefficients reduces to the formulation in Eq. (3.10).

$$\text{Minimize:} \quad \sum_{i,j} \left[\left(S_{ijT} - \frac{W_{iT}}{W_{jT}} \right) \right]^2 \tag{3.10a}$$

$$\text{subject to:} \quad \sum_{i,j,k} \left[\left(S_{ijT} \cdot S_{jkT} - \frac{W_{iT}}{W_{kT}} \right) \right]^2 < \varepsilon, \text{ where } \varepsilon \text{ is arbitrarily small} \tag{3.10b}$$

$$\sum_{i=1}^{m} W_{iT} = 1, \quad 0 \le W_{iT} \le 1 \tag{3.10c}$$

Eq. (3.10a) and Eq. (3.10b) are similar to Eq. (3.6a) and Eq. (3.6b) respectively, with r_i, r_j, and r_k all equal to one. Eq. (3.10c) is a normalization constraint imposed on gradient coefficients W_{iT}. We use Eq. (3.10c) to normalize W_{iT} in Eq. (3.10) because, W_{iT} is dimensionless when the attributes are normalized and W_{iT} represents the weight of the attributes (which by convention in the literature lies between zero and one). Also W_{iT} is non-negative in Eq. (3.10) because, the formulation in Eq. (3.10) is applicable only when the value function is non-decreasing with respect to the attributes.

The formulations in Eq. (3.6) and Eq. (3.10) can be solved with existing commercial optimization software (e.g., "fmincon" of the MATLAB® optimization toolbox). If one of the MRS values, say S_{jiT}, is zero, then the corresponding W_{jT} would be zero for any non-zero W_{iT}. This would cause a divide-by-zero in attempting to solve Eq. (3.6) and Eq. (3.10). We avoid this difficulty by discarding an attribute if its MRS value is zero, converting to a problem with 'm-1' attributes.

If a feasible solution for Eq. (3.6) or Eq. (3.10) does not exist (for a given ε), it means that the inconsistency in MRS values given by the DM is more than what we allowed for. In such a case, the DM can either change the MRS values or increase the constant ε. However, it should be noted that increasing ε might result in erroneous gradient coefficients. The idea of checking the consistency of the DM's MRS preferences using ε is similar to the idea of consistency index proposed by Saaty for the analytical hierarchy process (AHP) [Saaty, 1980]. However, in AHP, consistency of DM's preferences is checked after finding the weights of the attributes whereas in our approach consistency of DM's MRS preferences is checked while finding the gradient coefficients.

In the next section, we present an approach that efficiently uses the gradient of the value function at a trial design, obtained from the MRS preferences given by the DM, for eliminating lower value designs.

3.3.2. Eliminating Lower Value Designs Using Gradient Cut

If the value function, V, is differentiable and quasi-concave, and if C_G is the gradient cut (recall Definition in Section 2.2.5 of Chapter 2) at D_T (see Figure 3.2), then for all $D \in C_G$, $V(D) < V(D_T)$ [Bazaraa et al., 1993] [Malakooti, 1988]. That is, *any design alternative in C_G has lower value than D_T*, and hence can be eliminated. Applying the property that the gradient of V at D_T is $\nabla V_T = [W_{1T},\ldots,W_{mT}]$, and the attributes at D_T are $[a_{1T},\ldots,a_{mT}]$, then a design D_+ with attributes $[a_{1+},\ldots,a_{m+}]$ is in C_G if [Bazaraa et al., 1993]

$$\sum_{i=1}^{m} W_{iT} \cdot (a_{i+} - a_{iT}) < 0 \ . \qquad (3.11)$$

Figure 3.2 illustrates gradient cut elimination in two dimensions. Note that design alternatives that are not in C_G might have higher or lower or equal value with respect to D_T [Bazaraa et al., 1993] [Sundaram, 1996]. So, gradient cut does not eliminate all designs that have lower value than D_T. From Figure 3.2, we can see that a design D_+ is in C_G if the angle between ∇V_T and the vector joining D_T to D_+ is greater than ninety degrees. Recall Section 3.3.1 (see lemma), wherein we stated that ∇V_T found using Eq. (3.6) is unique up to a positive scalar transformation. Clearly this does not affect the design alternatives eliminated using gradient cut because the angle between the gradient vector and the vector joining D_T to D_+ remains the same even if ∇V_T is changed by a positive transformation. (Note that using Eq. (3.10), when applicable, we get unique gradient coefficients because of the normalization constraint for gradient coefficients.)

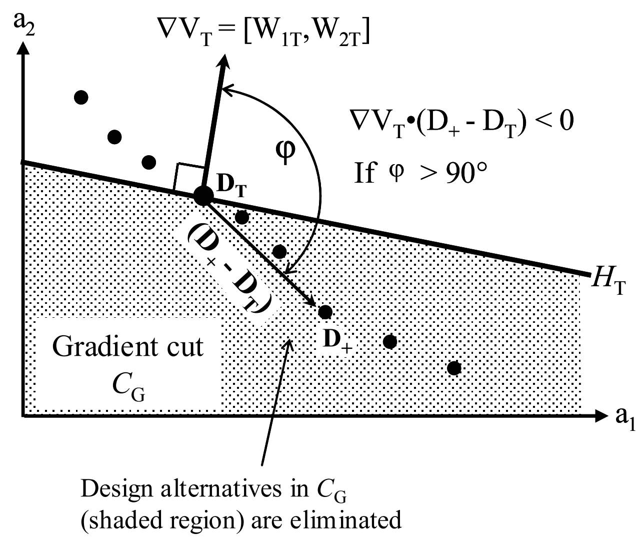

Figure 3.2: Illustration of gradient cut elimination

Our approach for eliminating lower value designs using gradient cut is similar to Malakooti's [Malakooti, 1988]. The difference is that we obtain the gradient of the value function using MRS preferences whereas Malakooti [Malakooti, 1988] uses comparisons of alternatives to obtain the gradient. Also, Malakooti's approach for finding the gradient involves many heuristic components, because of which the gradient cut has to be applied conservatively [Malakooti, 1988].

In the next section, we discuss our approach for finding a new trial design. This approach makes efficient use of the gradient of the value function at all the previous trial designs.

3.3.3. Finding a New Trial Design

An important step in our deterministic selection method (and in its extensions, e.g., see Chapter 5 and Chapter 6) is to find a new trial design for continuing the iterative process shown in Figure 3.1. In order to find the most preferred design in few iterations, a new trial design, D_{NT}, should be chosen, from the set of non-eliminated designs, such that it has higher value than the previous trial designs and it eliminates a large number of design alternatives using gradient cut elimination (recall Section 3.3.2) [Koksalan et al., 1984]. For this, we need to obtain a good estimate of the gradient of the DM's value function at the non-eliminated designs. To reduce the burden on the DM, the estimate of the gradient of the value function should be obtained (in real time) without actually interacting with the DM. In this section, we discuss an approach, which makes good use of the available information about the gradient of the value function at the previous trial designs to estimate the gradient of the value function at a non-eliminated design.

Consider the set of all design alternatives that are not eliminated at the current step in the iterative process described in Figure 3.1. Let the current iteration number be 'q'. Let $D_{T1},\ldots, D_{Tq}$ be the trial designs from the first iteration to the current iteration. Let $\nabla V_{Tj} = [W_{1Tj},\ldots,W_{mTj}]$ be the gradient of the value function at the trial design D_{Tj}: $[a_{1Tj},\ldots,a_{mTj}]$ (j=1,…,q).

Having no information about the behavior of the DM's implicit value function at a non-eliminated design (that has not been a trial design), we presume that the value function is non-decreasing, differentiable and quasi-concave (refer Figure 2.1(b)) with respect to the attributes at a non-eliminated design. *Note that we make this presumption only for the purpose of finding a new trial design. The actual value function at a non-eliminated design could be a general differentiable quasi-concave function.* With this presumption, the resulting formulation for estimating the gradient at a non-eliminated design D_+: $[a_{1+},\ldots,a_{m+}]$, Eq. (3.12), becomes a linear programming problem which can be solved without much computational burden. We use the vector $[\lambda_{1+},\ldots,\lambda_{m+}]$ to represent the estimate of the gradient of the value function at D_+.

For i=1,…, m and j=1,…, q

$$\sum_{i=1}^{m}\lambda_{i+}\cdot(a_{iTj}-a_{i+})<0 \tag{3.12a}$$

$$\left.\begin{array}{l}\lambda_{i+}\le W_{iTj}\text{ if }a_{i+}\ge a_{iTj}\\ \lambda_{i+}\ge W_{iTj}\text{ if }a_{i+}\le a_{iTj}\end{array}\right\} \tag{3.12b}$$

$$\lambda_{i+}\ge 0;\ \ W_{iTj}\ge 0 \tag{3.12c}$$

Eq. (3.12a) is used to check that each D_{Tj} lies within the gradient cut of D_+ for the estimated gradient (recall Eq. (3.11)). Eq. (3.12b) states the constraints imposed on λ_{i+} based on the gradient, ∇V_{Tj}, at each D_{Tj}. Eq. (3.12c) is the constraint on the sign of λ_{i+}. Since we assume that the value function is non-decreasing, the estimate of the gradient at D_+, λ_{i+}, should be less than W_{iTj} if $a_{i+} > a_{iTj}$ and vice versa. Also, if any of the W_{iTj}'s are negative, then we impose only the constraint that the corresponding λ_{i+} is non-negative (Eq. (3.12c)), i.e., we consider only the W_{iTj} that are non-negative in Eq. (3.12b).

Each D_+ for which λ_{i+}'s can be found has then at least one possible value function that eliminates all the previous trial designs D_{Tj} (j=1,…,q) by gradient cut, the gradient being $[\lambda_{1+},...,\lambda_{m+}]$. Each such D_+ then becomes an element of the set of candidate new trial designs, D_{CNT}. (Note that these λ_{i+}'s do not constitute the actual gradient of the value function that the DM has in mind at D_+.) We apply the gradient cut approach at each D_+, belonging to D_{CNT}, using $[\lambda_{1+},...,\lambda_{m+}]$ as the gradient of V, and then choose as the new trial design, D_{NT}, the D_+ which eliminates the greatest number of the original design alternatives. If there is no non-eliminated design, D_+, for which λ_{i+}'s exist, then we relax the constraints in Eq. (3.12), corresponding to the oldest D_{Tj} (i.e., smallest 'j') successively until a D_+ for which λ_{i+}'s exist is found.

If more than one D_{CNT} has the maximum number of alternatives eliminated, we choose as D_{NT} the alternative whose vector from D_T is closest to (i.e., makes the smallest angle with) the gradient, ∇V_{Tq}, at the current trial design, D_{Tq}.

We mentioned earlier that gradient cut can eliminate only some of the designs that have lower value than D_T (recall Section 3.3.2). Because of this property, it is possible that after applying gradient cut elimination at a series of D_T's, each time finding a new D_T, we are left with a set of trial designs that cannot eliminate each other. We call this set of non-eliminated trial designs as D_{NTD} (recall Definition in Section 2.2.6 of Chapter 2). Note that D_{NTD} always contains the most preferred design irrespective of the starting trial design. Otherwise, the most preferred design would have been eliminated by the gradient cut of some trial design, contradicting the property of quasi-concave value function (recall Section 3.3.2). In the next section we discuss a new approach, gradient adjacency elimination, to eliminate between trial designs that belong to D_{NTD} (when it has more than one member).

3.3.4. Gradient Adjacency Elimination

Figure 3.3 illustrates the proposed approach for gradient adjacency elimination for the case of two trial designs D_{T1} and D_{T2}. Lines H_{T1} and H_{T2} pass through D_{T1} and D_{T2}, respectively, and are perpendicular to the gradient of the value function at those points. O_{T1}, O_{T2} are the regions around D_{T1}, D_{T2}, respectively, in which we approximate the value function to be linear (recall Figure 3.1). Note that each D_{Ti} (i = 1, 2) is above the corresponding line H_{Tj} (j = 1, 2) of the other, so neither eliminates the other by gradient cut (recall Section 3.3.2).

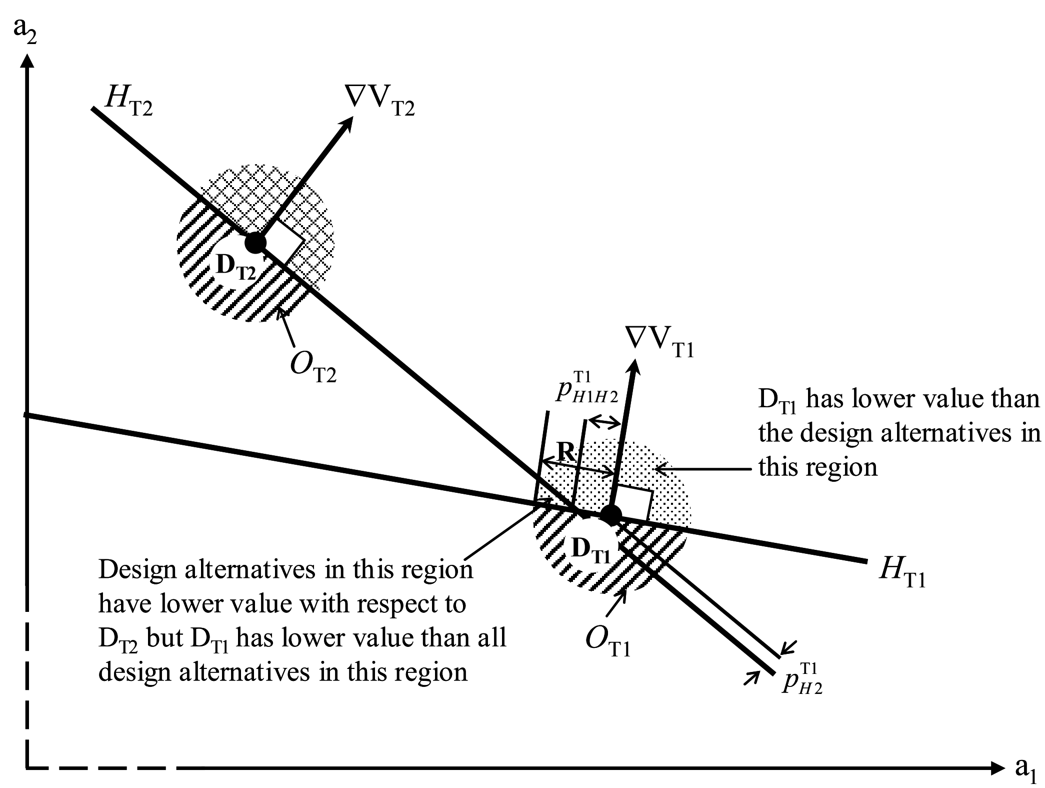

Figure 3.3: Illustration of gradient adjacency elimination

Note that all points in the region O_{T1} above H_{T1} have higher value than D_{T1} because ∇V_{T1} represents the increasing direction of V at D_{T1} and O_{T1} is the region in which the linear approximation of value function is valid. For the case here, H_{T2} passes through that part of O_{T1} which is above H_{T1}. Hence, D_{T2} has higher value than some points in O_{T1} above H_{T1} (recall gradient cut elimination, Section 3.3.2). Therefore, D_{T2} has higher value than D_{T1}. That is, D_{T2} eliminates D_{T1} by transitivity.

For the m-dimensional case H_{T1} and H_{T2} are hyper-planes. We may, for simplicity, take each region O_{Ti} to be a hyper-sphere. We assign O_{Ti} the radius $R = \eta \cdot \min(r_1, r_2, \ldots, r_m)$, where η is a small positive constant and r_i is the scale (recall Definition in Section 2.2.1 of Chapter 2) of the i^{th} attribute. Figure 3.4 illustrates (in two dimensions, for three cases) the relevant geometry and some definitions for determining if D_{T1} has lower value than D_{T2}. The perpendicular distance from D_{T1} to H_{T2} we call p_{H2}^{T1}. It can be seen in Figure 3.4 that H_{T2} passes through the region O_{T1} if

$$p_{H2}^{T1} \leq R. \tag{3.13}$$

The perpendicular distance from D_{T1} to the intersection of H_{T1} and H_{T2} we call p_{H1H2}^{T1}. H_{T2} passes through the region of O_{T1} if

$$p_{H1H2}^{T1} \leq R. \tag{3.14}$$

Thus, D_{T2} will eliminate D_{T1} if Eq. (3.13) and Eq. (3.14) are satisfied. In Figure 3.4(a), Eq. (3.13) is not satisfied; in Figure 3.4(b), Eq. (3.14) is not satisfied. In these two cases it cannot be determined if D_{T1} has lower value than D_{T2}. In Figure 3.4(c), both equations are satisfied, and D_{T2} eliminates D_{T1}.

If there are more than two non-eliminated trial designs in the set D_{NTD}, we apply the tests of Eq. (3.13) and Eq. (3.14) to all ordered pairs of non-eliminated trial designs (‘all ordered pairs’ means testing D_{Ti} against D_{Tj} as well as testing D_{Tj} against D_{Ti}) and eliminate the lower value trial designs. The trial design that remains non-eliminated after testing all the ordered pairs of trial designs would then be the most preferred design.

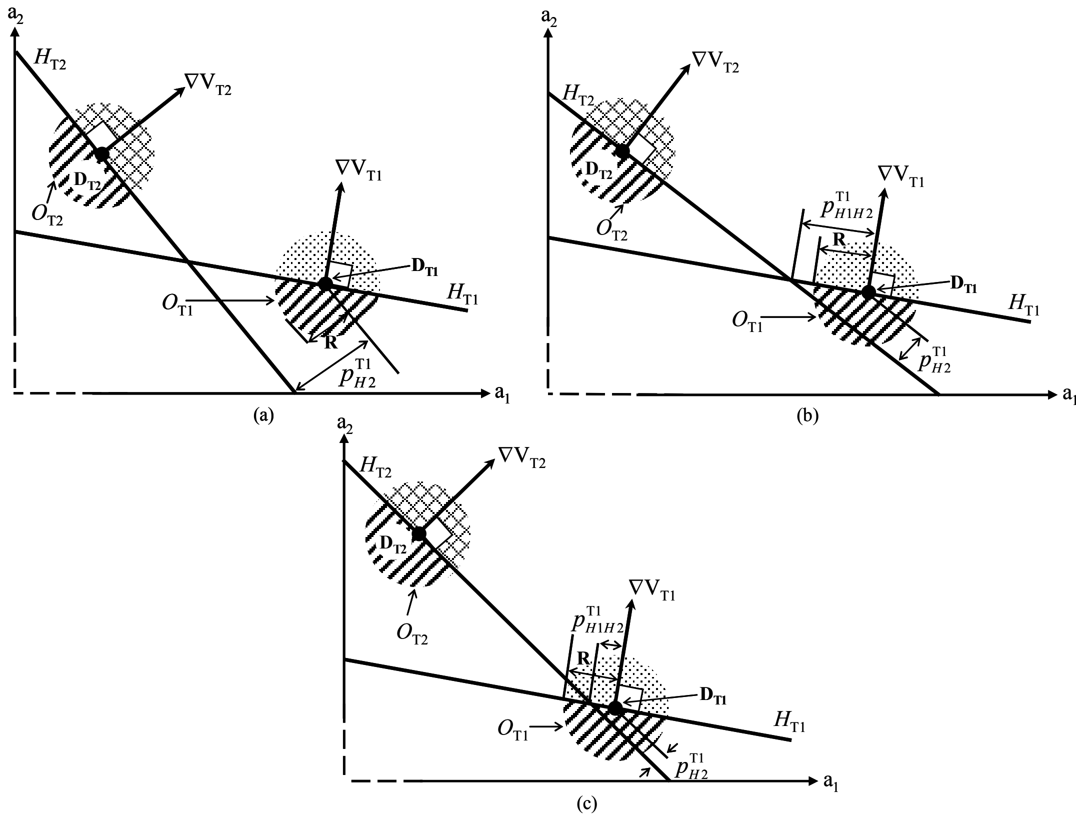

Figure 3.4: Test to find if a trial design is eliminated using gradient adjacency elimination (a) D_{T1} cannot be eliminated with respect to D_{T2} because, p_{H2}^{T1} > R, (b) D_{T1} cannot be eliminated with respect to D_{T2} because, p_{H2}^{T1} < R but $p_{H1\backslash H2}^{T1}$ > R, and (c) D_{T1} has lower value than D_{T2} because, p_{H2}^{T1} < R and $p_{H1\backslash H2}^{T1}$ < R

Gradient adjacency elimination is a heuristic approach and is based on the linear approximation of value function in a small region O_T around D_T. Note that region O_T is not arbitrary; it signifies the region around D_T in which the MRS values at any design point are the same as the MRS values at D_T. This follows from a theorem given by

Barzilai [Barzilai, 1998], which states that MRS values are constant if and only if the value function is linear. So, the DM can choose as the region O_T, the region around D_T where he/she feels that the MRS's are constant. Note that, Eq. (3.13) and Eq. (3.14) involves finding the distances in the attribute space and so this part of our deterministic selection method needs the attributes to be normalized.

If gradient adjacency elimination does not find the most preferred design using the region O_T given by the DM, we increase η, hence the radius of O_T, in small steps (say, 0.02) until a singleton most preferred design is found or the linear approximation of value function is no longer valid. At each step in η we apply the tests of Eq. (3.13) and Eq. (3.14) to all ordered pairs of non-eliminated trial designs.

In the next section, we discuss our algorithm for deterministic selection from a set of discrete design alternatives using the concepts discussed in Section 3.3.

3.4. ALGORITHM FOR DETERMINISTIC SELECTION

Our algorithm for finding the most preferred design alternative for deterministic selection has the following steps.

Step 1: Set the iteration number to one (i.e., q = 1) and pick a starting trial design, D_{T1}, from the set of design alternatives. We choose D_{T1} either as an alternative that would have maximum value if the value function were linear with equal importance to the attributes, or as a random pick.

Step 2: Query the DM for the MRS preferences between attributes at the current trial design D_{Tq}.

Step 3: Find the gradient of the value function at D_{Tq} using the MRS preferences (recall Section 3.3.1).

Step 4: Eliminate lower value designs using the gradient cut at D_{Tq} (recall Section 3.3.2).

Step 5: If all designs except one are eliminated or if the DM is satisfied with the current trial design D_{Tq}, define D_{NTD} to be the singleton set containing D_{Tq}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 7. Otherwise, go to *Step* 6.

Step 6: Find a new trial design from the non-eliminated design alternatives (recall Section 3.3.3). If a new trial design cannot be found, collect all the non-eliminated trial designs in the set D_{NTD}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 7. Otherwise, increase the iteration number by one (i.e., q = q+1), set the new trial design as D_{Tq} and go to *Step* 2.

Step 7: If D_{NTD} is a singleton then that design is the most preferred design alternative. Otherwise, use gradient adjacency elimination (recall Section 3.3.4) for finding the most preferred design alternative from among the D_{NTD}. Increase the radius (R) of the hyper-sphere around the trial designs in steps until all the design alternatives except one are eliminated. Stop.

In the next section, we demonstrate our deterministic selection method by applying the algorithm discussed above to two engineering examples.

3.5. DEMONSTRATION EXAMPLES

As a demonstration, we tested our deterministic selection method by applying our algorithm to two engineering examples. The first example is a two-attribute problem and involves the selection of a payload design for an undersea autonomous vehicle. The second example is a three-attribute problem and involves the selection of a cordless electric drill. The payload design selection example graphically demonstrates the working of our algorithm for deterministic selection. The cordless electric drill selection example demonstrates the applicability of our method to a problem where the attributes are not normalized between zero and one.

3.5.1. Deterministic Selection of Payload Design for Undersea Autonomous Vehicle

Typically, the payload must be effective in several different uses, called "scenarios". Effectiveness in a scenario is measured by a probability of success P_S in that scenario. The design goal is to simultaneously maximize individual P_S's for all scenarios. The payload design is constrained by upper limits on the weight and radiated noise of the payload (see Appendix-I for the description of the payload design optimization problem). For our example, we maximized P_{S1} and P_{S2} for two different scenarios using a Multi-Objective Genetic Algorithm (refer [Gunawan, 2004] for details). Table 3.1 (see Column 2) shows the resulting ten Pareto (see [Gunawan et al., 2003] for definition of Pareto) optimum design alternatives from which we select, with the P_{Si}'s being the attributes.

To verify that our deterministic selection method indeed finds the most preferred design, we use a simulated DM in this example. We constructed the DM's implicit value function to be of the form

$$V = -[(1-P_{S1})^2+(1-P_{S2})^2]. \quad (3.15)$$

We emphasize that the simulant value function given by Eq. (3.15) is not a presumed value function. Rather, it simulates a human DM who is supposedly being queried by our deterministic selection method, providing MRS preferences. The only reason we use this simulant value function is to verify that the most preferred design obtained by our method is indeed accurate. Note in Eq. (3.15), V is non-decreasing, differentiable, and concave.

Table 3.1: Design alternatives for payload design selection

Design alternative number	Attributes $[P_{S1}, P_{S2}]$ of design alternatives	Values of designs calculated using Eq. (3.15)
1	[0.016, 0.695]	-1.062
2	[0.016, 0.693]	-1.062
3	[0.134, 0.684]	-0.849
4	[0.139, 0.675]	-0.848
5	[0.274, 0.541]	**-0.738**
6	[0.275, 0.114]	-1.310
7	[0.343, 0.093]	-1.254
8	[0.346, 0.091]	-1.254
9	[0.355, 0.090]	-1.244
10	[0.357, 0.075]	-1.267

In the next section, Section 3.5.1.1, we describe the application of our algorithm for deterministic selection (recall Section 3.4) to the payload design selection example, and then discuss the results in Section 3.5.1.2.

3.5.1.1. Application of Algorithm for Deterministic Selection to Payload Design

Following our algorithm in Section 3.4, we set the iteration number to one (i.e., q = 1) and randomly pick D_3 as the starting trial design, i.e., D_{T1}: [P_{S1}, P_{S2}] = [0.134, 0.684] (*Step* 1). Since this is a two attribute problem, we ask the DM to provide only one MRS preference, i.e., MRS preference between P_{S1} (attribute 1) and P_{S2} (attribute 2). Our simulated DM, Eq. (3.15), responds by saying that the MRS preference is, S_{12T1}: 2.74 (*Step* 2).

Using Eq. (3.10), the gradient of the value function at D_{T1} is ∇V_{T1} = [0.73, 0.27] (*Step* 3). We use Eq. (3.10) for finding the gradient because the value function of the simulated DM, Eq. (3.15) is increasing and the attributes P_{Si} are normalized between zero and one. We use an ε value of 0.01 for allowable inconsistency in the MRS values at D_{T1}. Gradient cut at D_{T1} (*Step* 4) eliminates five lower value designs (shown by small rectangles in Figure 3.5(a)).

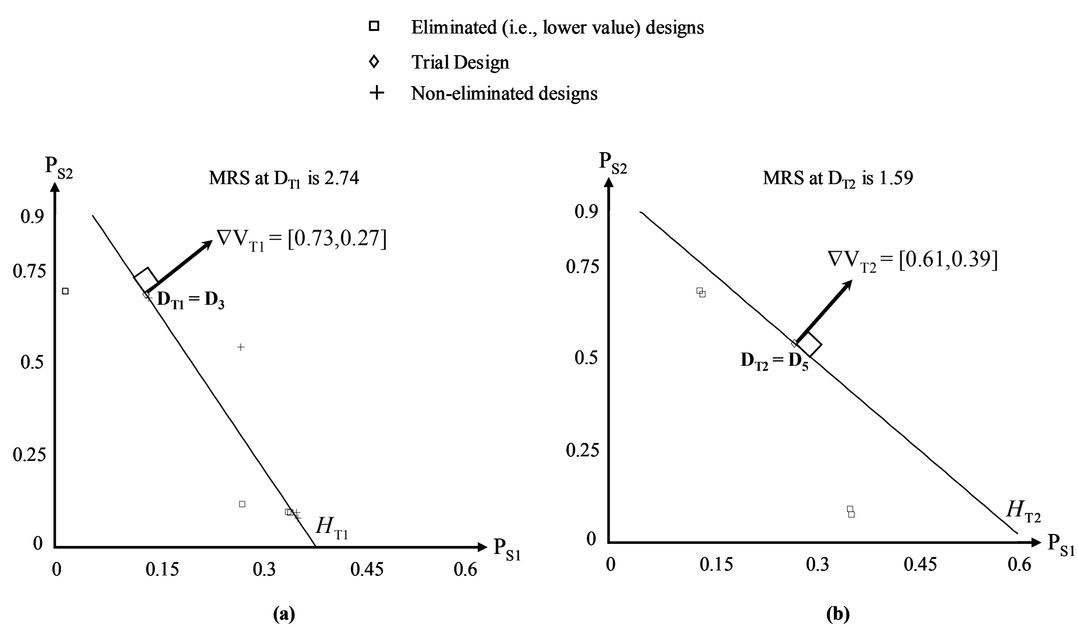

Figure 3.5: Gradient cut at (a) D_{T1}: [0.134, 0.684] and (b) D_{T2}: [0.274, 0.541]

The DM is not satisfied that D_{T1} is the most preferred design, and there are four non-eliminated designs: D_4, D_5, D_9 and D_{10} (shown by '+' in Figure 3.5(a)). So we skip *Step* 5 and find a new trial design (*Step* 6). For D_4, D_5, D_9 and D_{10} we find the gradient estimates λ_{i+}'s in accordance with Eq. (3.12), and apply gradient cut elimination to the other ten members of the original set of alternatives. Table 3.2 lists the number eliminated and λ_{i+}'s for D_4, D_5, D_9 and D_{10}. Since, D_5 eliminates more designs with gradient estimates λ_{i5}'s, it is the new trial design for the second iteration (i.e., $q = 2$), D_{T2}: [0.274, 0.541].

At the second trial design D_{T2} our simulated DM, Eq. (3.15), gives the MRS preference as, S_{12T2}: 1.59 (*Step* 2). Using Eq. (3.10), with ε again 0.01, the gradient of the value function at D_{T2} is $\nabla V_{T2} = [0.61, 0.39]$ (*Step* 3). Gradient cut at D_{T2} (*Step* 4) eliminates all of the non-eliminated designs (shown by small rectangles in Figure 3.5(b)). Since all designs except one are eliminated, D_{NTD} is the singleton set with D_5 as its member (*Step* 5) and D_5: [0.274, 0.541] is the most preferred design alternative (*Step* 7).

Table 3.2: Candidate new trial designs, λ_{i+}'s and number of original designs eliminated for payload design selection

Candidate new trial designs	Gradient estimates λ_{i+}'s	Number of original design alternatives eliminated
D_4: [0.139, 0.675]	[0.715, 0.285]	8
D_5: [0.274, 0.541]	[0.696, 0.304]	9
D_9: [0.355, 0.090]	[0.733, 0.267]	8
D_{10}: [0.357, 0.075]	[0.733, 0.267]	7

3.5.1.2. Discussion

To verify the result obtained by our deterministic selection method we obtained the values of all design alternatives using the simulant value function of Eq. (3.15). Note that the maximum of Eq. (3.15) (which is zero), is obtained when both P_{S1} and P_{S2} are equal to one. Column 3 of Table 3.1 shows the values of each design alternative. From Column 3 of Table 3.1, we can clearly see that D_5 is the most preferred design alternative as found by our deterministic selection method.

3.5.2. Deterministic Selection of Cordless Electric Drill

For cordless electric drill selection, we consider three design attributes: a_1, the number of operations achievable with one charge of a battery pack; a_2, the cost of the drill; and a_3, the weight of the drill. Table 3.3 presents the eighteen design alternatives from which the DM wishes to select the most preferred. The scales (recall Definition in Section 2.2.1 of Chapter 2) of the attributes are 350 to 630 operations; \$70 to \$100; and 5.5 to 7.8 pounds. We emphasize that for the application of our deterministic selection, it does not matter how the design alternatives are obtained.

We made one simplification for this example. The DM's value function would naturally increase with the number of operations, and decrease with cost and weight. We converted the attributes so that the value function is monotonically increasing in all three attributes. For cost and for weight we use

(modified attribute) = (max value in scale of attribute) - (original attribute).

This makes the gradient coefficients, W_{iT}, non-negative. For the convenience of the DM, the MRS questions are asked in terms of the original attributes. (Section 3.6 describes an application of our method to a more general case where in the value function is quasi-concave and non-decreasing.)

Table 3.3: Design alternatives for cordless electric drill selection

Design alternative number	Number of Operations	Cost (in dollars)	Weight (in pounds)
1	350	70	6
2	370	80	5.7
3	380	85	5.5
4	400	72	6.5
5	420	82	6.1
6	430	88	5.8
7	450	74	6.9
8	470	85	6.5
9	480	91	6.1
10	500	79	7.2
11	520	89	6.9
12	530	94	6.4
13	550	84	7.5
14	570	93	7.2
15	580	97	6.7
16	600	90	7.8
17	620	98	7.5
18	630	100	7

We applied our deterministic selection method to three cases of the cordless electric drill example with a different DM in each case. We present in detail the case where the DM is a casual user. We then present in lesser detail the cases for a professional user and for a moderate (i.e., in between a casual and a professional) user. We then discuss the results for all three cases.

3.5.2.1. Application of Algorithm for Deterministic Selection to Cordless Electric Drill Selection by a Casual User

Having no informed guess from the DM for picking the starting trial design, we select randomly the design alternative D_7 as the trial design for the first iteration (i.e., q = 1), D_{T1}: [450 operations, 74 dollars, 6.9 pounds] (*Step* 1). We ask the DM for three MRS preferences at the current trial design: number of operations for cost; cost for weight; and weight for number of operations. We allow the DM to provide the marginal change in both attributes for each MRS (e.g., what change in number of operations for what change in cost). At D_{T1}, the casual user provided the responses in the third column of Table 3.4 (*Step* 2).

Table 3.4: MRS between attributes for DM (a casual user)

MRS	Attributes	Trade-offs for constant value designs at D_{T1}: [450 operations, 74 dollars, 6.9 pounds]	Trade-offs for constant value designs at D_{T2}: [350 operations, 70 dollars, 6 pounds]
S_{12}	Operations	50 operations	50 operations
	Cost	4 dollars	3 dollars
S_{23}	Cost	5 dollars	2 dollars
	Weight	0.5 pounds	0.5 pounds
S_{31}	Weight	0.4 pounds	0.5 pounds
	Operations	50 operations	40 operations

With these data and ε of 0.01 for allowable inconsistency in the MRS values, Eq. (3.6) gives the gradient coefficients (W_{iT1}) at D_{T1}: $W_{1T1} = 0.004$ operation^{-1}; $W_{2T1} = 0.045$ dollar^{-1}; $W_{3T1} = 0.443$ pound^{-1} (*Step* 3). We use Eq. (3.6) for finding the gradient coefficients because the attributes are not normalized in this example. Using the scale of the attributes to convert the gradient coefficients (recall Section 3.3.1), we get the relative importance (i.e., the weights) of the attributes as [0.30, 0.40, and 0.30]. Note that the relative importance of the attributes obtained is consistent with the preferences of a casual user, i.e., the cost of the drill is more important than the number of operations and the weight of the drill.

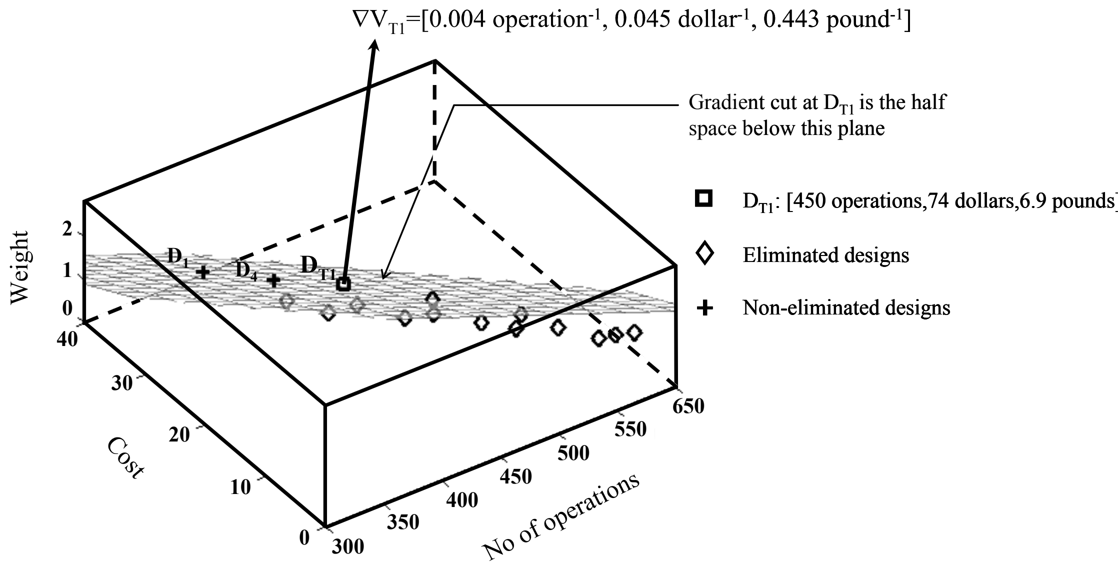

Figure 3.6: Gradient cut at D_{T1}: [450 operations, 74 dollars, 6.9 pounds] in the modified attribute space

Using gradient cut elimination (*Step* 4), fifteen design alternatives are eliminated as shown in Figure 3.6. Figure 3.6 shows the design alternatives in the modified attribute space (recall Section 3.5.2). The DM is not satisfied that D_{T1} is the most preferred design, and there are two non-eliminated designs: D_1 and D_4 (shown by '+' in Figure 3.6). So we skip *Step* 5 and proceed to finding a new trial design (*Step* 6).

For D_1 and D_4 we find the gradient estimates λ_{i1}'s and λ_{i4}'s (i=1, 2 and 3) in accordance with Eq. (3.12), and apply gradient cut elimination to the other seventeen members of the original set of alternatives. Table 3.5 lists the number eliminated and λ_{i1}'s and λ_{i4}'s for D_1 and D_4. Since, D_1 eliminates more design with gradient estimates λ_{i1}'s, it is the new trial design for the second iteration (i.e., q=2), D_{T2}: [350 operations, 70 dollars, 6 pounds].

Table 3.5: Candidate new trial designs, λ_{i+}'s and number of original designs eliminated for cordless electric drill selection

Candidate new trial designs	Gradient estimates λ_{i+}'s	Number of original design alternatives eliminated
D_1: [350 operations, 70 dollars, 6 pounds]	[0.004 operation^{-1}, 0.028 dollar^{-1}, 0.331 pound^{-1}]	17
D_4: [400 operations, 72 dollars, 6.5 pounds]	[0.004 operation^{-1}, 0.029 dollar^{-1}, 0.340 pound^{-1}]	16

Beginning the second iteration, we ask the DM for MRS preferences at D_{T2}, and receive the data in the fourth column of Table 3.4 (*Step* 2). The gradient coefficients (W_{iT2}) at D_{T2} are: W_{1T2} = 0.004 operation^{-1}; W_{2T2} = 0.069 dollar^{-1}; W_{3T2} = 0.248 pound^{-1} (*Step* 3, ε for allowable inconsistency in the MRS values is again 0.01). Using the scale of the attributes to convert the gradient coefficients, we get the relative importance (i.e., the weights) of the attributes as [0.27, 0.57, and 0.16].

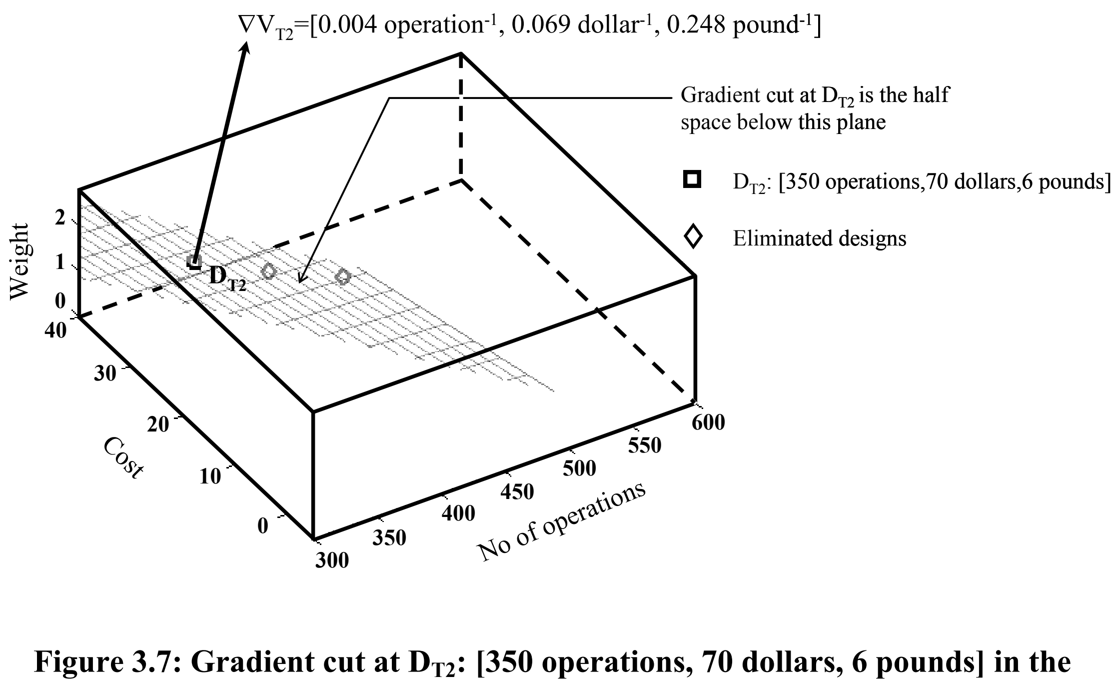

Figure 3.7: Gradient cut at D_{T2}: [350 operations, 70 dollars, 6 pounds] in the modified attribute

We apply gradient cut elimination at D_{T2} (see Figure 3.7) and find that all other design alternatives can be eliminated (*Step* 4). Since all designs except one are eliminated, D_{NTD} is the singleton set with D_1 as its member (*Step* 5) and D_1: [350 operations, 70 dollars, 6 pounds] is the most preferred design alternative (*Step* 7).

3.5.2.2. Application of Algorithm for Deterministic Selection to Cordless Electric Drill Selection by a Professional and Moderate User

In the case where the DM is a professional user, our deterministic selection method found the most preferred design alternative as, D_{18}: [630 operations, 100 dollars, 7 pounds], in one iteration. Table 3.6 shows the MRS preferences given by the professional user. In the case where the DM is a moderate user, the method found the most preferred design alternative as, D_{13}: [550 operations, 84 dollars, 7.5 pounds], in three iterations. Table 3.7 shows the MRS preferences given by the moderate user. In each case, the first trial design was picked randomly.

Table 3.6: MRS between attributes for DM (a professional user)

MRS	Attributes	Trade-offs for constant value designs at D_{T1}: [630 operations, 100 dollars, 7 pounds]
S_{12}	Operations	50 operations
	Cost	10 dollars
S_{23}	Cost	5 dollars
	Weight	0.5 pounds
S_{31}	Weight	0.5 pounds
	Operations	30 operations

Table 3.7: MRS between attributes for DM (a moderate user)

MRS	Attributes	Trade-offs for constant value designs at D_{T1}: [580 operations, 97 dollars, 6.7 pounds]	Trade-offs for constant value designs at D_{T2}: [450 operations, 74 dollars, 6.9 pounds]	Trade-offs for constant value designs at D_{T3}: [550 operations, 84 dollars, 7.5 pounds]
S_{12}	Operations	50 operations	50 operations	50 operations
	Cost	7 dollars	12 dollars	9 dollars
S_{23}	Cost	10 dollars	10 dollars	10 dollars
	Weight	1 pound	1 pound	2 pounds
S_{31}	Weight	1 pound	1 pound	1 pound
	Operations	60 operations	50 operations	40 operations

3.5.2.3. Discussion

We applied the deterministic selection method two times for all three users, each time picking a different starting trial design. We found that for all three users, the most preferred design (i.e. D_1 for casual user, D_{18} for professional user, and D_{13} for moderate user) was not affected by the starting trial design. However, the number of iterations required to reach the most preferred design depended on the starting trial design.

Our method selected the design which might have been selected intuitively by the casual user and the professional user. The casual user's MRS preferences (recall Table 3.4) indicate that cost is most important; number of operations and weight are moderately important. Indeed, our method selected the lowest cost alternative. The professional user's MRS preferences (recall Table 3.6) indicate that number of operations is most important; cost is least important; and weight is moderately important. Our method selected the option having the highest number of operations, highest cost, and relatively high weight. However, for the moderate user the intuitive choice is not clear. The MRS preferences (recall Table 3.7) indicate only that weight is of little concern. Our method selected an alternative having middle values of number of operations and of cost, but relatively high weight.

Next we provide some experimental results that verify our selection method and also support our claim that, within the limit of our experimentation, the most preferred design alternative can be found in just a few iterations.

3.6. VERIFICATION: SOME EXPERIMENTAL RESULTS

In this section, we provide some experimental results to verify our deterministic selection method. We describe the experiments in Section 3.6.1 and discuss the results in Section 3.6.2.

3.6.1. Description of Experiments for Verifying the Deterministic Selection Method

To verify the proposed deterministic selection method, we conducted simulations with fourteen different problem sizes, i.e., (number of attributes) × (number of design alternatives), ranging from two attributes and 50 alternatives to six attributes and 200 alternatives. We generated ten sets of design alternatives for each problem size. For simplicity, the alternatives are uniformly distributed between 0 (worst) and 1 (best) in each attribute.

We used a variety of *simulant value functions* to produce the answers to the MRS questions that our method needs. We tested our method by comparing the most preferred design alternative obtained by our method, with the alternative that has the maximum value according to the simulant value function. *We emphasize here that the role of the simulant value functions is just to represent the preference structure of the human DM* *and to verify the results of our method. In reality (and in our proposed method), the DM does not have any idea about the explicit form of the value function except that it must be differentiable and quasi-concave.* We also recorded some statistical indicators which are based on "value efficiency", V_{eff}. For any simulant value function V, and for each set of design alternatives, we define V_{eff} for a design D_j

$$V_{eff}(D_j) = 100\ \frac{V(D_j) - V_{min}}{V_{max} - V_{min}} \tag{3.16}$$

where V_{min} is the minimum value and V_{max} is the maximum value of V in the set of design alternatives. For each problem size, we found the average number (over the ten sets of alternatives) of iterations and queries needed to find a design alternative that has V_{eff} of at least 95% (i.e., stopping when V_{eff} of a new trial design is greater than 95%), and also the number of iterations and queries needed to find an alternative with 100% V_{eff}. Another statistical indicator is the average V_{eff} of the selected design alternative when the stopping criterion is $V_{eff} \geq 95\%$.

For each of the ten sets of design alternatives in each of the problem sizes we conducted five simulations, each using one of the following simulant value functions to represent the DM's preferences.

$$V_1(D_j) = [-\sum_{i=1}^{m}(a_{ij} - 1)^{\beta}]\,;\ \beta = 2 \tag{3.17}$$

$$V_2(D_j) = -\sum_{i=1}^{m}\gamma_i \cdot e^{(1-a_{ij})}\,;\ \gamma_i = 1/m\,;\ \text{m is the number of attributes} \tag{3.18}$$

$$V_3(D_j) = \prod_{i=1}^{m} a_{ij}^{\alpha_i}\,;\ \alpha_i = 1/m\,;\ \text{m is the number of attributes} \tag{3.19}$$

$$V_4(D_j) = -\sum_{i=1}^{m} \begin{cases} (a_{ij} - 1)^2; & \text{if i is odd} \\ (a_{ij} - 0.5)^2; & \text{if i is even} \end{cases} \quad (3.20)$$

$$V_5(D_j) = \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} a_{ij} a_{kj} \quad (3.21)$$

Note that V_1 is increasing for each attribute. (Malakooti [Malakooti, 1988], used the same value function V_1 for his verification.) V_2 is concave and is exponentially increasing in each attribute. V_3 is the Cobb-Douglas function [Takayama, 1993], which is concave and increasing with respect to the attributes and has inter-dependence between attributes. V_4 is concave and increasing for the odd numbered attributes and uni-modal for the even numbered attributes. Finally, V_5 is the second elementary symmetric function [Greenberg and Pierskalla, 1971], which is quasi-concave with respect to the attributes and has inter-dependence between attributes. We chose these functions to demonstrate that our method works with different forms of the value function as long as it is quasi-concave. However, the highest order polynomial function that we considered in these simulant value functions is two. We use a polynomial of order greater than two (specifically $\beta>2$ in Eq. (3.17)) in the verification of our methods in Chapter 5 and Chapter 6. Also we use a modification of Eq. (3.19) that is quasi-concave but not concave in the verification of our methods in Chapter 5 and Chapter 6.

For the starting trial design in each simulation, we chose from the set of alternatives a design that has less than 40% V_{eff}. In the next section, we present the results of our experiments.

3.6.2. Results of Experiments for the Verification of Deterministic Selection Method

For each problem size and each simulant value function in our experiment, Table 3.8 shows the statistical indicators described in Section 3.6.1. Our experiments show that our method can, indeed, find the DM's most preferred design, the one which has the highest value among the alternatives. From Table 3.8, we observe that when stopping at $V_{eff} \geq 95\%$, the selected designs had V_{eff} ranging from 97.2% to 100%, with an average (over the problem sizes) of 99.1%. It took on average 2.9 iterations to reach 95% and 4 iterations to reach 100% value efficient design alternatives. For the simulant value function V_1, the results shown in Table 3.8 are comparable to the results published by Malakooti [Malakooti, 1988]. However, an exact comparison cannot be made because we do not know the design alternatives used in his verification study. Recall also that our method asks the DM for the comparison of attributes whereas Malakooti's method asks the DM for the comparison of alternatives.

An interesting observation from Table 3.8, is that the number of iterations required for our deterministic selection method depends more on the number of attributes than on the number of designs. For example, we can see that for the simulant value function V_3 given by Eq. (3.19), the average number of iterations required in finding a design alternative with 100% V_{eff} is: '2.5 iterations' when the problem size is '5 attributes' x '50 designs'; '3.1 iterations' when the problem size is '5 attributes' x '100 designs'; and '4.2 iterations' when the problem size is '5 attributes' x '200 designs'. The reason for this is that, gradient cut eliminates all the designs that are in the half space bounded by the gradient at a trial design. So, the number of iterations required by our deterministic selection method will not change if the additional designs lie in the gradient cut of a trial design.

Table 3.8: Results of the verification study for deterministic selection method

Problem size: "# of attributes×# of designs"	Value function	Avg # of queries to get 95% V_{eff}	Avg # of iterations to get 95% V_{eff}	Avg V_{eff} of best design after reaching 95% V_{eff}	Avg # of queries to get 100% V_{eff}	Avg # of iterations to get 100% V_{eff}
2 × 50	V_1	1.9	1.9	99.9	2.3	2.3
	V_2	2.1	2.1	99.8	2.3	2.3
	V_3	2.5	2.5	91.9	3.2	3.2
	V_4	2.9	2.9	98.7	6.2	6.2
	V_5	3	3	99.9	3.1	3.1
2 ×100	V_1	1.9	1.9	99.6	2.9	2.9
	V_2	2.1	2.1	99.9	2.5	2.5
	V_3	3.4	3.4	94.0	3.8	3.8
	V_4	2.7	2.7	98.7	7.1	7.1
	V_5	3	3	99.1	3.7	3.7
3 ×50	V_1	10.2	3.4	99.6	11.1	3.7
	V_2	7.8	2.6	99.6	8.4	2.8
	V_3	6.6	2.2	99.0	10.2	3.4
	V_4	5.4	1.8	99.2	7.5	2.5
	V_5	8.4	2.8	99.4	10.5	3.5
3×100	V_1	9.6	3.2	99.4	11.7	3.9
	V_2	7.2	2.4	99.6	7.8	2.6
	V_3	11.4	3.8	99.1	14.7	4.9
	V_4	8.1	2.7	98.7	13.2	4.4
	V_5	6.3	2.1	99.0	7.8	2.6
3×200	V_1	8.1	2.7	99.3	9.9	3.3
	V_2	6.6	2.2	99.5	8.7	2.9
	V_3	13.8	4.6	99.4	15.3	5.1
	V_4	9.3	3.1	98.5	16.8	5.6
	V_5	5.4	1.8	99.4	7.2	2.4
4X50	V_1	11.2	2.8	99.0	17.6	4.4
	V_2	12	3	98.6	16	4
	V_3	12	3	99.7	14	3.5
	V_4	16.4	4.1	99.3	23.2	5.8
	V_5	7.6	1.9	99.1	8.4	2.1

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Problem size: "# of attributes×# of designs"	Value function	Avg # of queries to get 95% V_{eff}	Avg # of iterations to get 95% V_{eff}	Avg V_{eff} of best design after reaching 95% V_{eff}	Avg # of queries to get 100% V_{eff}	Avg # of iterations to get 100% V_{eff}
4X100	V_1	12	3	98.6	20.4	5.1
	V_2	11.6	2.9	98.8	14	3.5
	V_3	21.2	5.3	99.0	27.6	6.9
	V_4	14.4	3.6	98.2	21.2	5.3
	V_5	9.2	2.3	99.6	10.8	2.7
4X200	V_1	14.8	3.7	98.5	19.6	4.9
	V_2	10.8	2.7	98.7	14	3.5
	V_3	18.8	4.7	99.7	19.6	4.9
	V_4	19.6	4.9	98.7	28.4	7.1
	V_5	15.2	3.8	99.0	17.6	4.4
5×50	V_1	8	1.6	98.9	12.5	2.5
	V_2	11	2.2	99.3	13	2.6
	V_3	12.5	2.5	100.0	12.5	2.5
	V_4	18	3.6	99.3	25.5	5.1
	V_5	11	2.2	99.6	11.5	2.3
5×100	V_1	15.5	3.1	99.2	19	3.8
	V_2	9.5	1.9	98.9	12.5	2.5
	V_3	11.5	2.3	99.2	15.5	3.1
	V_4	13	2.6	98.3	32.5	6.5
	V_5	11	2.2	99.6	11.5	2.3
5×200	V_1	14.5	2.9	98.9	19.5	3.9
	V_2	15	3	99.1	18.5	3.7
	V_3	16.5	3.3	98.9	21	4.2
	V_4	17	3.4	98.3	46	9.2
	V_5	12.5	2.5	99.4	14	2.8
6×50	V_1	14.4	2.4	99.7	15.6	2.6
	V_2	15	2.5	98.7	17.4	2.9
	V_3	14.4	2.4	98.3	21.6	3.6
	V_4	27.6	4.6	98.9	37.2	6.2
	V_5	13.2	2.2	99.5	14.4	2.4
6×100	V_1	15	2.5	99.2	19.8	3.3
	V_2	15	2.5	98.6	18	3
	V_3	14.4	2.4	99.3	16.2	2.7
	V_4	25.2	4.2	98.3	42.6	7.1
	V_5	14.4	2.4	99.5	15	2.5
6×200	V_1	22.2	3.7	98.9	33.6	5.6
	V_2	15.6	2.6	98.3	22.8	3.8
	V_3	24	4	99.4	30	5
	V_4	27.6	4.6	97.2	55.8	9.3
	V_5	20.4	3.4	99.1	24	4

We also conducted an experiment to verify that our approach for finding a new trial design is better than some simplistic approach. In this experiment, we conducted a simulation similar to the simulations discussed in Section 3.6.1. We used the function given by Eq. (3.20) as the simulant value function representing the DM's preference structure. However for finding the new trial design, we used a simplistic approach rather than our approach (recall Section 3.3.3). In this simplistic approach, we use the non-eliminated design alternative which would have maximum value if the value function were linear with equal importance to the attributes as the new trial design. In the simulation, we found the number of iterations required to reach a 95% V_{eff} design alternative.

Table 3.9: Results for the verification of our approach for finding a new trial design

Problem size	2×50	2×100	3×50	3×100	4×50	4×100	5×50	5×100
No of iterations to find 95% V_{eff} design using our approach for finding a new trial design	2.9	2.7	1.8	2.7	4.1	3.6	3.6	2.6
No of iterations to find 95% V_{eff} design using simplistic approach for finding a new trial design	5.4	9.7	3.4	4.3	4.3	4.3	3	3.2

Table 3.9 above shows the number of iterations required to find a design with at least 95% V_{eff} using our approach for finding a new trial design in the first row for different problem sizes (i.e., (number of attributes) × (number of alternatives)). The number of iterations required for finding a design with at least 95% V_{eff} using the simplistic approach is presented in the second row. From Table 3.9, we can see that our approach for finding the new trial design performs much better (for most of the problem sizes) than the simplistic approach. Also our approach for finding the new trial design is a linear programming problem and can be solved quickly.

3.7. SUMMARY

In this chapter, we presented an interactive method for deterministic product design selection with an implicit value function. The method required that the DM state his/her preferences in the form of MRS between attributes at each trial design. We presented an approach for finding the gradient of the DM's value function at a trial design using the DM's response to MRS questions. If the DM's MRS preferences are inconsistent beyond a certain limit (given by ε), our formulation for finding the gradient coefficients becomes infeasible thus alerting the DM about the inconsistency. The deterministic selection method used gradient cut to eliminate lower value designs. We presented an approach that makes good use of the gradient information at all the previous trial designs for finding a better new trial design. We presented a new approach, gradient adjacency elimination, which is useful for eliminating designs that are not eliminated by gradient cut. Finally, we presented an algorithm for deterministic selection using the concepts mentioned above. We demonstrated our deterministic selection with two engineering examples, namely, selection of a payload design for undersea autonomous vehicle and selection of a cordless electric drill. We also presented some experimental results to verify our deterministic selection method.

Our deterministic selection method is applicable when the DM's implicit value function is differentiable and quasi-concave. The main difference between our deterministic selection method and other selection methods for an implicit value function (e.g., [Malakooti, 1988]) is that in our deterministic selection method we query the DM for the marginal rate of substitution (MRS) between the attributes while other methods query the DM for the pair-wise comparison of design alternatives. However, as mentioned in Section 2.3.1 of Chapter 2, pair-wise comparisons of design alternatives are difficult because they involve comparing two m-attribute designs, and, hence, might lead to intransitive preferences.

Our deterministic selection method is iterative and requires the DM to state the MRS preferences at a series of trial designs. Since the method queries the DM for the MRS between attributes, it is presumed that the DM has the requisite level of expertise and consistent judgment to make the trade-offs. Because of its iterative nature, our method might come across as tedious. However, since we have no idea about the DM's implicit value function there is no better way (without explicitly assuming a function) for finding the most preferred design other than eliminating lower value designs with respect to a series of trial designs.

Our deterministic selection method guarantees that the set of non-eliminated trial designs, D_{NTD}, always contains the most preferred design irrespective of the starting trial design. Otherwise, the most preferred design would have been eliminated by the gradient cut of some trial design, contradicting the property of quasi-concave value function (recall Section 3.3.2). However, if D_{NTD}, is not a singleton, the uniqueness of the most preferred design is not guaranteed because the gradient adjacency elimination approach which is used to select from D_{NTD} is a heuristic approach.

For the verification of our deterministic selection method, we used simulant value functions, replacing a human DM, for obtaining the MRS preferences at the trial designs. Although such a numerical approach is mathematically valid, in reality there is no practical way for checking whether the DM gives the MRS preferences consistent with a value function as we move from one trial design to other trial design. But, unfortunately, there are no benchmark problems for validating product design selection methods because of the subjectivity involved with human preferences.

Also in our verification study, we stopped the deterministic selection method after obtaining a design with a value efficiency of 95% and 100%. We could do this in our verification study because we used a simulated DM. In reality, such a stopping criterion cannot be used because of the implicit nature of the DM's value function and the only stopping criterion is that a new trial design cannot be found. However, our experiments showed that, on an average, irrespective of the problem size (for at least up to 'six attributes' × '200 designs') our method finds the most preferred design alternative (i.e., design with value efficiency of 100%) as the new trial design in five to six iterations. So the DM can stop the iterative process after five to six iterations and make a selection from the set of non-eliminated trial designs, D_{NTD}, at that stage

In the next chapter, we present the development of the method for our second research component, sensitivity analysis for deterministic selection. This method is used to find the allowed preference variation for which the set of non-eliminated trial designs, found using the deterministic selection does not change.

CHAPTER 4

SENSITIVITY ANALYSIS FOR DETERMINISTIC SELECTION

4.1. INTRODUCTION

In making a selection from a set of product design alternatives, the DM tries to meet the requirements of the end users of the product. Since, in general, the DM does not have complete information about the end users' needs, he/she may want to know how the preferred design(s) is (are) affected if the preferences vary. For example in automobile design selection, the DM conducts market survey and says that: "I would allow the cost of the automobile to increase *around* 5000 dollars, if the 0-60 time is decreased by two seconds". The DM gives an estimate of his/her actual preference in such a response and he/she cannot state his/her actual preference with certainty. So the DM would like to know how much variation the preferred design(s) can absorb before it is replaced by some other design(s). We call as robustness, the amount of change (or variation) allowed between the actual preferences and the preference estimates before the preferred design(s) is (are) changed. Finding the degree of robustness (or robustness index) of the preferred design(s) to preference variation is generally referred to as sensitivity analysis in the literature [Insua and French, 1991].

The purpose of this chapter is to present a concept for sensitivity analysis for deterministic selection. This concept can be used with any iterative selection scheme that chooses a trial design for each iteration, and uses the DM's estimates of preference parameters at that trial design to eliminate some design options which have lower value than the trial design [Maddulapalli et al., 2002] [Malakooti, 1988]. Such schemes, like our deterministic selection method (recall Chapter 3), are in general applicable to the cases where the DM's value function is implicit.

The organization of this chapter is as follows. In Section 4.2 we present an overview of our concept for sensitivity analysis. Section 4.3 describes an implementation of the concept using our deterministic selection method (recall Chapter 3). Next, in Section 4.4 we discuss our algorithm for sensitivity analysis. In Section 4.5, we demonstrate the application of our sensitivity analysis method with the help of two engineering examples. Then we present some experimental results to verify our sensitivity analysis method in Section 4.6 and finally conclude the chapter with a summary in Section 4.7.

4.2. OVERVIEW OF CONCEPT FOR SENSITIVITY ANALYSIS

Our concept for sensitivity analysis is applicable to iterative selection methods, which choose a trial design D_T at each iteration, and examine every other design D_+ in the original set of design alternatives to eliminate designs having lower value than D_T, e.g., [Malakooti, 1988], our deterministic selection method of Chapter 3. The output of such a method is a set of non-eliminated trial designs D_{NTD}, which could be a singleton. Figure 4.1 shows the flowchart of our concept for calculating three successive metrics, culminating in the "robustness index" of D_{NTD}.

For each D_+ originally eliminated by D_T, and for each preference (e.g., relative importance or MRS between attributes) estimate, there is a certain variation (i.e., a difference) between the estimate and the actual preference for which D_+ becomes non-eliminated. Our first metric δ_{+T}, which we call elimination robustness of design D_+ with respect to trial design D_T, is defined as the *smallest* of those variations, where they are considered in magnitude, expressed as fractions of their estimates. Thus, so long as the variation in *every* preference is less than δ_{+T}, D_+ will always be eliminated by D_T. If D_+ is originally non-eliminated, δ_{+T} can conveniently be taken as zero (i.e., no preference variation is needed to make it non-eliminated). For each D_T, δ_{+T} is calculated for each design D_+ in the original set of design alternatives.

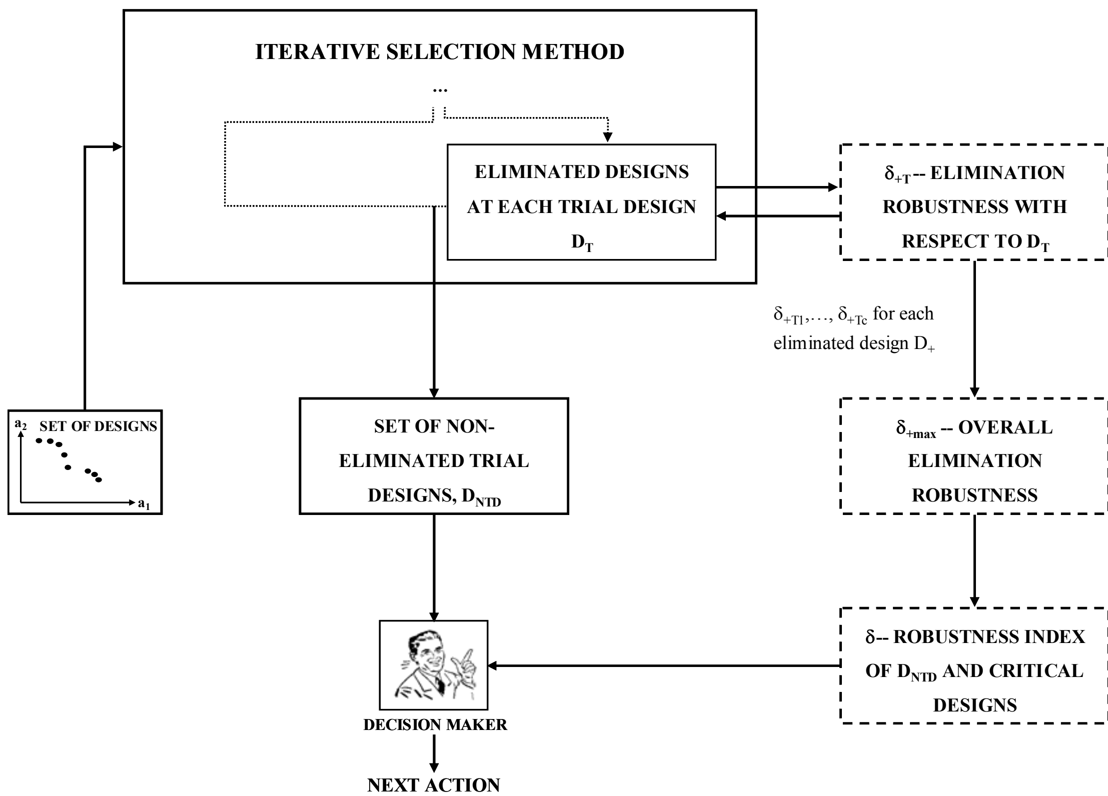

Figure 4.1: Flowchart of the concept for sensitivity analysis

The second metric, δ_{+max}, is the overall elimination robustness of a design D_+. δ_{+max} is the *largest* of the δ_{+T}'s for D_+ over all D_T's. Thus, so long as the variation in *every* preference is less than δ_{+max} at *all* trial designs, D_+ will be eliminated by at least one trial design.

The final metric is robustness index, δ, which is the *minimum* of all the δ_{+max}'s. All designs not in the set of non-eliminated trial designs, D_{NTD}, remain eliminated so long as the variation in *every* preference is less than δ.

Table 4.1: Overall elimination robustness of design alternatives

Design alternative number	3	4	1	2	9	10	8	7	6
Overall elimination robustness δ_{+max}	**0.35**	0.38	0.97	0.97	2.52	2.52	2.95	3.08	223.14

To explain the usefulness of our robustness index, we present the results of one of our examples in Table 4.1 (see Section 4.5.1.1 for details). In this example, we selected from ten designs. The set D_{NTD} consists of a single element, D_5. Table 4.1 shows the overall elimination robustness, the δ_{+max}'s, of the other nine designs in an ascending order. The minimum δ_{+max} occurs for D_3, and this value becomes the robustness index of D_{NTD}: $\delta = 0.35$. Thus, as long as all actual preferences differ from their estimates by less than 35%, D_5 will be the most preferred design. Any design for which $\delta_{+max} = \delta$ we call a "critical design"; it becomes a member of D_{NTD} if the preference variation is δ or more. In the example, D_3 is a singleton critical design. The DM can consider the robustness index δ and the identified critical designs to choose what action to take next. If the DM feels that the robustness index is acceptable or that the critical designs are not important, then he/she can make a selection from the set D_{NTD}. Otherwise, he/she can give ranges for the preferences and then find the potentially optimal designs for those ranges (see Chapter 5 for our method to find the potentially optimal designs for a range of MRS preferences).

Note that δ_{+max} for each design is the preference variation at the trial designs that would cause that design to become a member of D_{NTD}. Arranging the designs in the ascending order of δ_{+max} (as in Table 4.1) lets the DM see which designs (other than the critical designs) are next nearest to becoming members of D_{NTD}, and what amount of preference variation would cause that to happen. In the above example, D_4 is next nearest to becoming a member of D_{NTD} and that will happen if the preference variation is 38%. Also, the DM can give ranges of preferences symmetric about the preference estimates (i.e., preference estimate is the mid point of the range) at the trial designs to account for the preference variability and then find the set of non-eliminated designs for those ranges directly using δ_{+max}. However, if the ranges of preferences are not symmetric about the preference estimates then the DM should use selection with preference variability for finding the set of non-eliminated trial designs (see Chapter 5 for our method for selection with preference variability).

The robustness index also gives the bounds or intervals within which the actual preferences at all trial designs must lie in order to not affect D_{NTD}: {estimated preference value}$\cdot\{1\pm\delta\}$. The bounds on the preferences are similar to the weight stability intervals proposed by Mareschal [Mareschal, 1988]. However, Mareschal's approach is applicable only for an additive value function (with unknown weights). In contrast, our concept for sensitivity analysis is applicable to selection with an implicit value function.

4.3. SENSITIVITY ANALYSIS IMPLEMENTATION

In this section, we describe the implementation of the concept for sensitivity analysis (recall Figure 4.1) in our deterministic selection method (recall Chapter 3). In this implementation, we assume that the DM's implicit value function is differentiable, quasi-concave and non-decreasing with respect to the attributes. (Note that the assumption that the DM's value function is non-decreasing with respect to the attributes is not necessary for the application of the method developed in Chapter 3.) Because the DM's value function is assumed non-decreasing with respect to attributes, for selection, it is enough to consider only those designs that are Pareto optimal from the original set of design alternatives [Malakooti, 1988]. Figure 4.2 shows the flowchart of the implementation.

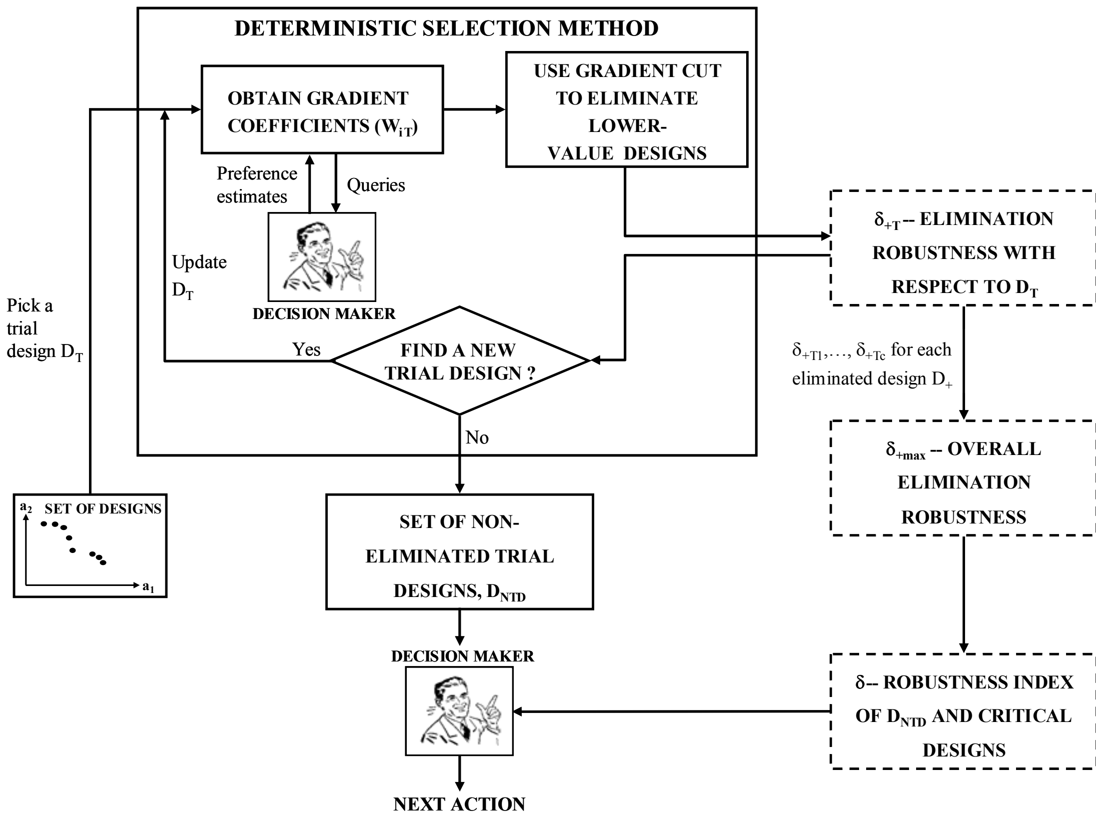

Figure 4.2: Flowchart of the concept for sensitivity analysis applied to our deterministic selection method

The flowchart in Figure 4.2 is similar to that of Figure 4.1 except that the box titled "Iterative Selection Method" in Figure 4.1 is replaced by the flowchart of our deterministic selection method (recall Figure 3.1) in Figure 4.2. Recall from Chapter 3 that, in our deterministic selection method, we start by picking a trial design D_T from the set of design alternatives. Next, we capture the DM's preferences by querying about the MRS between attributes. We use the DM's response to these MRS queries as the preference (or MRS) estimates in the implementation of our sensitivity analysis concept.

In our deterministic selection method, after eliminating lower value designs using the gradient coefficients at a series of trial designs, we collect the non-eliminated trial designs in the set, designated by D_{NTD}. We then use the gradient adjacency elimination approach for eliminating more designs from D_{NTD}. However, in the implementation of the concept of sensitivity analysis in our deterministic selection method, the robustness index that we find is the robustness of the set D_{NTD} and not of the most preferred design.

In the next two sections we describe the individual components of the implementation shown in Figure 4.2. In Section 4.3.1, we explain our approach for finding δ_{+T}, followed in Section 4.3.2 by our approach for finding δ. Refer to Chapter 3, for our approaches for: finding the gradient coefficients, eliminating lower value designs using gradient cut and finding a new trial design. Note, from here on in this chapter, δ_{+T}, δ_{+max}, and δ represent the preference variation between actual MRS preferences and their estimates.

4.3.1. Finding Elimination Robustness of a Design with respect to a Trial Design

Let S_{ijT} be the MRS estimate between attributes a_i and a_j given by the DM at the current trial design D_T in our deterministic selection method (recall Figure 4.2), and let ∇V_T be the corresponding gradient of the value function. Also, let D_+ be an arbitrary design that belongs to the original set of design alternatives and that lies in the gradient cut C_G corresponding to ∇V_T at D_T (therefore D_T eliminates D_+).

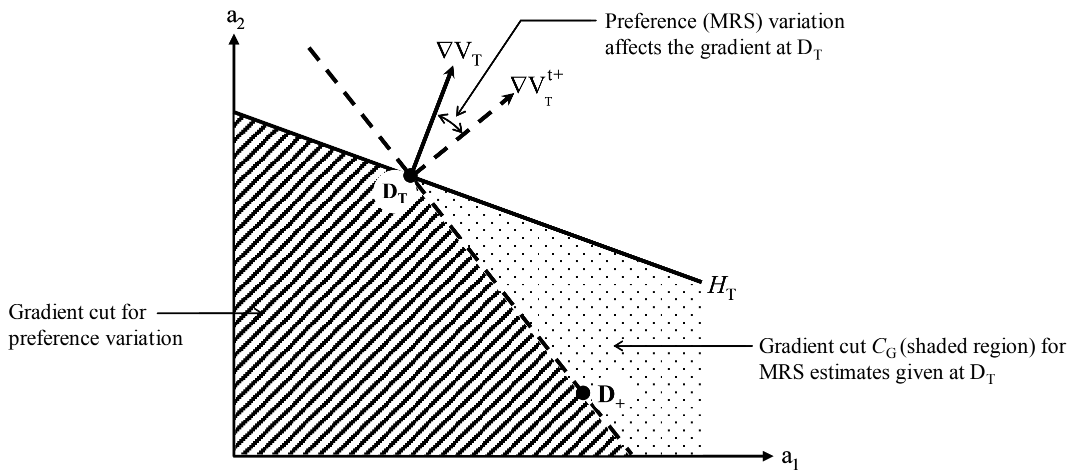

Figure 4.3: Illustration of threshold gradient of D_+ with respect to D_T

As illustrated in Figure 4.3, if the actual preferences (stated when there is complete information about end users' needs) at D_T are such that the gradient is $\nabla V_T^{t+} : [W_{1T}^{t+}, ..., W_{mT}^{t+}]$, then D_+ would not be eliminated. We call ∇V_T^{t+} the threshold gradient of D_+ with respect to D_T. δ_{+T}, the elimination robustness of D_+ with respect to D_T, is the smallest of the 'm-1' preference variations at D_T for which the gradient reaches the threshold gradient. Note from Eq. (3.11) (recall Section 3.3.2 of Chapter 3) that Eq. (4.1) holds at ∇V_T^{t+} (where the terms a_{iT} and a_{i+} are the attributes of D_T and D_+, respectively).

$$\sum_{i=1}^{m} W_{iT}^{t+} \cdot (a_{i+} - a_{iT}) \geq 0 . \tag{4.1}$$

We use the symbol s_{ijT} to represent the actual MRS values between attributes a_i and a_j at D_T and $\nabla v_T = [w_{1T}, \ldots, w_{mT}]$ to represent the gradient corresponding to s_{ijT}. As long as all s_{ijT}'s lie in the range given by Eq. (4.2), their corresponding gradient will not reach the threshold gradient ∇V_T^{t+}.

$$(1 - \delta_{+T}) \cdot S_{ijT} < s_{ijT} < (1 + \delta_{+T}) \cdot S_{ijT} \tag{4.2}$$

In the next section, we present our formulation for finding δ_{+T} when D_+ lies in the gradient cut of ∇V_T. Recall from Section 4.2 that for convenience we set δ_{+T} of a design D_+ not in the gradient cut of ∇V_T to zero.

4.3.1.1. Formulation for Finding the Elimination Robustness of a Design with respect to a Trial Design

We use the formulation in Eq. (4.3) for finding δ_{+T} of a design D_+: $[a_{1+},\ldots,a_{m+}]$ with respect to a trial design D_T: $[a_{1T},\ldots,a_{mT}]$ treating δ_{+T} and the w_{iT} as the variables that are to be found. In Eq. (4.3), a_{iT} and a_{i+} are fixed (or deterministic).

$$\text{Minimize} \quad \delta_{+T} \tag{4.3a}$$

$$\text{subject to:} \sum_{i=1}^{m} w_{iT} \cdot (a_{i+} - a_{iT}) \geq 0 \tag{4.3b}$$

$$\sum_{i=1}^{m} w_{iT} = 1; \quad w_{iT} \geq 0 \tag{4.3c}$$

$$(1-\delta_{+T}) \cdot S_{ijT} \leq \frac{w_{iT}}{w_{jT}} \leq (1+\delta_{+T}) \cdot S_{ijT}; \quad \text{'m-1' such constraints} \tag{4.3d}$$

$$\delta_{+T} \geq 0 \tag{4.3e}$$

Eq. (4.3b) is used to check that D_+ is not in the gradient cut corresponding to the gradient coefficients, w_{iT}, at D_T (recall Eq. (4.1)). Note that, in the formulation of Eq. (4.3), we are looking for preference variations that would make D_+ not eliminated.

Eq. (4.3c) is a normalization constraint on the gradient coefficients, w_{iT}. We impose the constraint that the gradient coefficients, w_{iT}, are non-negative because we assume that the value function is non-decreasing with respect to the attributes. We use Eq. (4.3c) to normalize w_{iT} because, we assume that the attributes are normalized in

Eq. (4.3). When the attributes are normalized, w_{iT} is dimensionless and represents the relative importance or weight of the attribute (which by convention in the literature lies between zero and one). However, if the attributes are not normalized we neglect Eq. (4.3c) in the above formulation. One could also modify Eq. (4.3c) as $\sum_{i=1}^{m} w_{iT} \cdot r_i = 1$, where r_i is the scale of the i^{th} attribute (recall Definition in Section 2.2.1 of Chapter 2).

Eq. (4.3d) is to check that s_{ijT} (recall from Eq. (3.4) that $s_{ijT} = \frac{w_{iT}}{w_{jT}}$) are within the bounds, given by δ_{+T}, of the MRS estimates, S_{ijT}. Also, if the lower bound in Eq. (4.3d) becomes negative, we set it equal to zero because s_{ijT} cannot be negative. Eq. (4.3e) is a constraint imposed on δ_{+T}.

In Eq. (4.3d), we assume that the actual MRS values can lie in either direction of (i.e., greater or lesser than) the MRS estimate S_{ijT}. I.e., we assume the preference variation to be symmetric about the MRS estimates. We make this assumption because we do not have any information about where the DM's actual MRS preference is. However, we can readily modify Eq. (4.3d) if the DM says that the actual preference is in a particular direction of S_{ijT}.

Note that it is important to obtain the global optimum of δ_{+T} when using the formulation in Eq. (4.3). A local optimum could differ significantly from the global optimum giving misleading conclusions about the allowed preference variation at D_T for which D_+ is always eliminated.

Note that in Eq. (4.3) we assume the s_{ijT}'s are exact and consistent. By exact and consistent we mean that Eq. (4.4) is satisfied (recall Eq. (3.4) and Eq. (3.5)).

$$s_{ijT} = \frac{w_{iT}}{w_{jT}}, \text{ and } s_{ijT} \cdot s_{jkT} = \frac{w_{iT}}{w_{kT}} \tag{4.4}$$

Since only 'm-1' MRS values are independent when they are consistent, we use only 'm-1' constraints for the bounds on s_{ijT} (recall Eq. (4.3d)), even though we obtain 'm' MRS estimates from the DM. However, if one feels that the exactness and consistency assumption is not appropriate then, the formulation in Eq. (4.3) can be easily modified by adding two more constraints as given by Eq. (4.5) or Eq. (4.6) depending on whether or not the attributes are normalized. In Eq. (4.6), r_i is the scale of the i^{th} attribute.

$$\sum_{i,j}\left[s_{ijT} - \frac{w_{iT}}{w_{jT}}\right]^2 \leq \varepsilon, \text{where } \varepsilon \text{ is arbitrarily small} \tag{4.5a}$$

$$\sum_{i,j,k}\left[s_{ijT} \cdot s_{jkT} - \frac{w_{iT}}{w_{kT}}\right]^2 \leq \varepsilon \tag{4.5b}$$

$$\sum_{i,j}\left[\left(s_{ijT} - \frac{w_{iT}}{w_{jT}}\right) \cdot \left(\frac{r_i}{r_j}\right)\right]^2 \leq \varepsilon \tag{4.6a}$$

$$\sum_{i,j,k}\left[\left(s_{ijT} \cdot s_{jkT} - \frac{w_{iT}}{w_{kT}}\right) \cdot \left(\frac{r_i}{r_k}\right)\right]^2 \leq \varepsilon \tag{4.6b}$$

Eq. (4.5a) or Eq. (4.6a) would be used to check how close the s_{ijT}'s are to the w_{iT}'s (recall Eq. (3.6a) and Eq. (3.10a)) and Eq. (4.5b) or Eq. (4.6b) would be used to check that s_{ijT} are consistent (recall Eq. (3.6b) and Eq. (3.10b)). However, note that adding the constraints in Eq. (4.5) or Eq. (4.6) (which are nonlinear and non-convex) to the formulation in Eq. (4.3) would increase the computational burden for finding δ_{+T}.

The Eq. (4.3) formulation (with or without additional constraints of Eq. (4.5) or Eq. (4.6)) can be solved with existing commercial optimization software (e.g., "fmincon" from the MATLAB® optimization toolbox). One might argue that the bounds on s_{ijT}, hence δ_{+T}, could be obtained more easily by finding the threshold gradient ∇V_T^{t+} (see Figure 4.3) analytically. $\nabla V_T^{t+} : [W_{1T}^{t+},\ldots,W_{mT}^{t+}]$ could be found by minimizing the angle between ∇V_T and ∇V_T^{t+} subject to the constraint of Eq. (4.1), where $\nabla V_T = [W_{1T},\ldots,W_{mT}]$ is the gradient of the value function at D_T obtained from the MRS estimates S_{ijT}. Once ∇V_T^{t+} is found, the corresponding threshold MRS, S_{ijT}^{t+}, can be found using Eq. (3.4). The threshold MRS can then be used in finding the bounds on MRS values, s_{ijT}, as given by Eq. (4.7a) or Eq. (4.7b) as the case may be, and the bounds can then be used in finding δ_{+T}.

$$\frac{W_{iT}}{W_{jT}} < s_{ijT} < \frac{W_{iT}^{t+}}{W_{jT}^{t+}} \tag{4.7a}$$

$$\frac{W_{iT}^{t+}}{W_{jT}^{t+}} < s_{ijT} < \frac{W_{iT}}{W_{jT}} \tag{4.7b}$$

Even though the above discussed approach looks tempting, it is not applicable for problems with more than two attributes, as is proven in the lemma in Appendix-II.

In the next section, we present our approach for finding the robustness index δ of D_{NTD}.

4.3.2. Finding Robustness Index of D_{NTD}

Let D_+ be any arbitrary design alternative that does not belong to the set of non-eliminated trial designs D_{NTD}. Let $\delta_{+T1},\ldots, \delta_{+Tc}$ be the elimination robustness of D_+ with respect to trial designs $D_{T1},\ldots, D_{Tc}$, respectively (c is the total number of iterations).

The overall elimination robustness of D_+, δ_{+max} (recall Section 4.2), is then the maximum of all δ_{+Tj}'s (j = 1, ... , c). And the robustness index δ of D_{NTD} is then the minimum of all δ_{+max}'s.

Our definition of robustness index δ is conservative because it restricts the variation between the actual MRS and its estimate to be the same for all pairs of attributes at all the trial designs. However, if we find the allowed preference variation for each pair of attributes at each trial design, the resulting amount of information is likely to overwhelm the DM. Note that we can readily modify our approach if the DM is interested in the robustness of D_{NTD} with respect to a particular pair of attributes and/or a particular trial design.

We mentioned earlier that our robustness index can be used for finding the critical designs – those that become non-eliminated if the variation between the actual MRS preferences and the estimates at the trial designs is δ or more. If the intermediate data (the δ_{+T}'s and the δ_{+max}'s) are retained, they can be traced back as follows to identify a "critical pair" -- the two attributes whose MRS variation has the largest influence in determining the critical designs. First, find the D_+ whose δ_{+max} equals δ (the minimum of all δ_{+max}'s). For that D_+, find the trial design D_T whose δ_{+T} equals its δ_{+max} (maximum of the δ_{+T}'s for that D_+). Next, for that D_T and D_+ find which constraint(s) out of the 'm-1' constraints on the bounds of MRS (recall Eq. (4.3d)) are active. The attributes corresponding to that constraint(s) are the critical pair(s). Improving the MRS estimate for the critical pair would give the largest increase in the robustness index, so knowing the critical pair can help the DM. The DM can also assign various ranges for the MRS preference between the critical pair and see how the potentially optimal designs change using our method for selection with preference variability (see Chapter 5 for details).

In the next section, we present the algorithm we developed for finding the robustness index of D_{NTD} using the concepts developed in the earlier sections.

4.4. ALGORITHM FOR SENSITIVITY ANALYSIS

Our algorithm for finding the robustness index has the following steps. In this algorithm, *Step* 1 to *Step* 6 are similar to the algorithm we presented for our deterministic selection method in Section 3.4 of Chapter 3.

Step 1: Set the iteration number to one (i.e., q = 1) and pick a starting trial design, D_{T1}, from the set of design alternatives. We choose D_{T1} either as an alternative that would have maximum value if the value function were linear with equal importance to the attributes, or as a random pick.

Step 2: Query the DM for the preference (MRS) estimates at the current trial design D_{Tq}.

Step 3: Find the gradient of the value function at D_{Tq} using the preference (MRS) estimates (recall Section 3.3.1 of Chapter 3).

Step 4: Eliminate lower value designs using the gradient cut at D_{Tq} (recall Section 3.3.2 of Chapter 3). Store the designs that are eliminated by D_{Tq}.

Step 5: If all designs except one are eliminated, define D_{NTD} to be the singleton set containing the non-eliminated design, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 7. Otherwise, go to *Step* 6.

Step 6: Find a new trial design from the non-eliminated design alternatives (recall Section 3.3.3 of Chapter 3). If a new trial design cannot be found, collect all the non-eliminated trial designs in the set D_{NTD}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 7. Otherwise, increase the iteration number by one (i.e., q = q+1), set the new trial design as D_{Tq} and go to *Step* 2.

Step 7: For each D_+ that does not belong to D_{NTD}, find δ_{+q} (q=1,…,c). If D_+ is eliminated by D_{Tq}, use Eq. (4.3) (recall Section 4.3.1.1) for finding δ_{+Tq} otherwise, set δ_{+Tq} to zero.

Step 8: For each D_+ that does not belong to D_{NTD}, find δ_{+max}, the overall elimination robustness of D_+, by finding the maximum of the δ_{+Tq}'s (q=1,…,c).

Step 9: Find δ, the robustness index of D_{NTD}, by finding the minimum of all δ_{+max}'s and present this δ and the corresponding critical design(s) to the DM. Stop.

In the next section, we demonstrate our sensitivity analysis method by applying the algorithm discussed above to two engineering examples.

4.5. DEMONSTRATION EXAMPLES

As a demonstration, we tested our sensitivity analysis concept by applying our algorithm to two engineering examples. These examples are same as the examples in Section 3.5 of Chapter 3. The first example is a two-attribute problem and involves the selection of a payload design for an undersea autonomous vehicle. The second example is a three-attribute problem and involves the selection of a cordless electric drill. The payload design selection example graphically demonstrates the working of our algorithm for sensitivity analysis. The cordless electric drill selection example demonstrates the applicability of our method to a problem where the attributes are not normalized between zero and one.

4.5.1. Sensitivity Analysis for Deterministic Selection of Payload Design for Undersea Autonomous Vehicle

For the payload design selection example, we set the ten Pareto optimum design alternatives, shown in Table 3.1 (reproduced in Column 2 of Table 4.2), as the design alternatives from which we select, with the P_{Si}'s being the attributes. We again use the simulant value function given by Eq. (3.15) as the simulated DM for this example (recall Section 3.5.1 of Chapter 3).

In the next section, Section 4.5.1.1, we describe the application of our algorithm for sensitivity analysis (recall Section 4.4) to the payload design selection example, and then discuss the results in Section 4.5.1.2. Some of the steps we describe in Section 4.5.1.1 are similar to the steps in Section 3.5.1.1 of Chapter 3.

4.5.1.1. Application of Algorithm for Sensitivity Analysis to Payload Design Selection

Following our algorithm in Section 4.4., we set the iteration number to one (i.e., $q = 1$) and randomly pick D_3 as the starting trial design, i.e., D_{T1}: $[P_{S1}, P_{S2}] = [0.134, 0.684]$ (*Step* 1). The simulated DM of Eq. (3.15) responds with the MRS estimate as, S_{12T1}: 2.74 (*Step* 2). The gradient of the value function at D_{T1} is then $\nabla V_{T1} = [0.73, 0.27]$ (*Step* 3). Gradient cut at D_{T1} (*Step* 4) eliminates five lower value designs; i.e., D_1, D_2, D_6, D_7, and D_8 (shown by small rectangles in Figure 4.4(a)). Since more than one design is non-eliminated we skip *Step* 5 and find a new trial design

($Step$ 6). Using our approach for finding a new trial design (recall Section 3.3.3), we find D_5 as the new trial design. We increase the iteration number by one (i.e., q = 2), set D_5 as D_{T2}: [0.274, 0.541] and go to *Step* 2.

The simulated DM of Eq. (3.15) gives the MRS estimate at D_{T2} as, S_{12T2}: 1.59 (*Step* 2). The gradient of the value function at D_{T2} is then $\nabla V_{T2} = [0.61, 0.39]$ (*Step* 3). Gradient cut at D_{T2} (*Step* 4) eliminates all other designs (shown by small rectangles in Figure 4.5(a)). Since all designs except one are eliminated, D_{NTD} is the singleton set with D_5 as its member (*Step* 5) and we set the total number of iterations to two, i.e., c = 2 and go to *Step* 7.

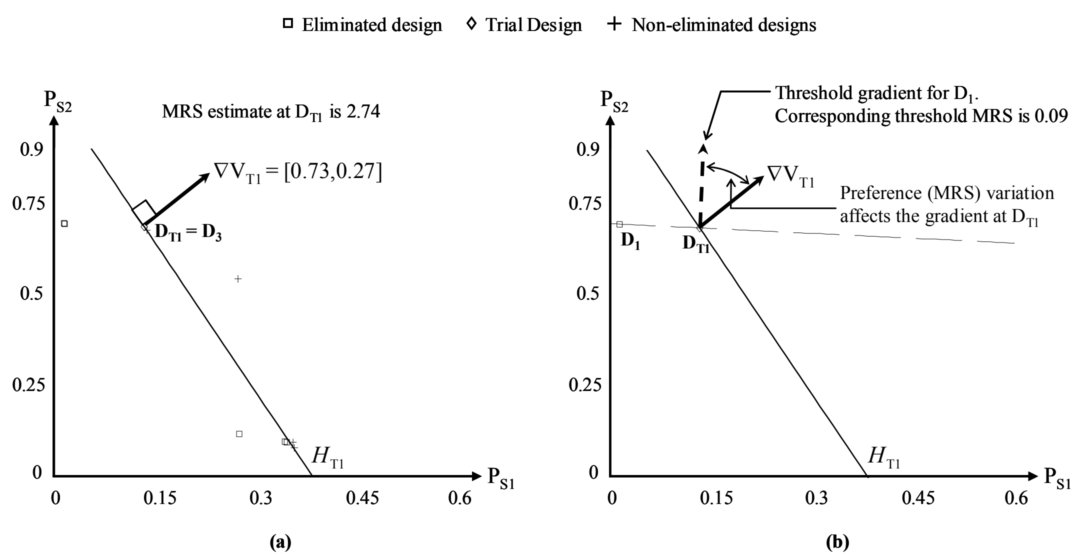

Figure 4.4: Payload design selection (a) gradient cut at D_{T1} and (b) threshold gradient of D_1 with respect to D_{T1}

We then find the elimination robustness of each eliminated design D_+ with respect to $D_{T1,}$ i.e., δ_{+T1}'s and D_{T2}, i.e., δ_{+T2}'s (*Step* 7). Column 3 of Table 4.2 shows the δ_{+T1}'s of all designs. For example, using Eq. (4.3), the elimination robustness of D_1 with respect to

D_{T1} is 0.97. From Eq. (4.2), we can then say that D_1 will be eliminated by D_{T1} as long as the actual MRS value at D_{T1}, i.e.,s_{12T1}, is in the range

$$0.09 < s_{12T1} < 5.40 . \tag{4.8}$$

When solving Eq. (4.3) for δ_{1T1}, the inequality $s_{12T1} > 0.09$ is active, so the threshold MRS, S^{t1}_{12T1}, is 0.09 (see Figure 4.4(b)). The elimination robustness of D_3, D_4, D_9, and D_{10} is zero because these designs are not eliminated by D_{T1}. The elimination robustness of D_5 is not listed in Table 4.2 because D_5 is the most preferred design and belongs to D_{NTD}.

Table 4.2: Elimination robustness of eliminated payload design alternatives

Design alternative number	Attributes $[P_{S1}, P_{S2}]$ of design alternatives	δ_{+T1}'s elimination robustness of D_+ with respect to D_{T1}	δ_{+T2}'s elimination robustness of D_+ with respect to D_{T2}	δ_{+max}, overall elimination robustness of D_+
3	[0.134, 0.684]	0.00	0.35	**0.35**
4	[0.139, 0.675]	0.00	0.38	0.38
1	[0.016, 0.695]	0.97	0.62	0.97
2	[0.016, 0.693]	0.97	0.63	0.97
9	[0.355, 0.090]	0.00	2.52	2.52
10	[0.357, 0.075]	0.00	2.52	2.52
8	[0.346, 0.091]	0.02	2.95	2.95
7	[0.343, 0.093]	0.03	3.08	3.08
6	[0.275, 0.114]	0.47	223.14	223.14

Column 4 of Table 4.2 shows the δ_{+T2}'s of all designs. Figure 4.5(b) illustrates the threshold gradients of D_1 and D_3 (recall D_3 was trial design for first iteration) with respect to D_{T2}. Also, from Eq. (4.2), D_1 and D_3 will be eliminated by D_{T2} so long as the actual MRS value at D_{T2}, s_{12T2}, is in the range given by Eq. (4.9a) and Eq. (4.9b), respectively.

$$0.59 < s_{12T2} < 2.57 \tag{4.9a}$$

$$1.02 < s_{12T2} < 2.13 \tag{4.9b}$$

When solving Eq. (4.3) for δ_{1T2}, the inequality $s_{12T2} > 0.59$ is active, so the threshold MRS of D_1 at D_{T2}, S^{t1}_{12T2}, is 0.59 (see Figure 4.5(b)). Also the inequality $s_{12T2} > 1.02$ is active when solving Eq. (4.3) for δ_{3T2}, so the threshold MRS for D_3 at D_{T2}, S^{t3}_{12T2}, is 1.02.

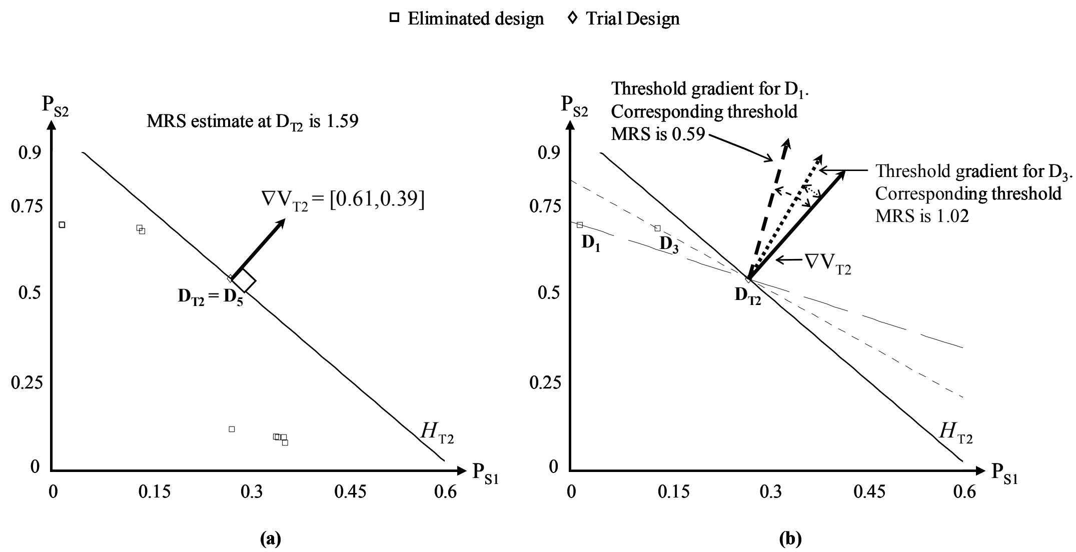

Figure 4.5: Payload design selection (a) gradient cut at D_{T2} and (b) threshold gradient of D_1 and D_3 with respect to D_{T2}

Column 5 of Table 4.2 shows the overall elimination robustness, δ_{+max}'s, for all eliminated designs (*Step* 8). Finally we find the robustness index of D_5, δ, by finding the minimum of all δ_{+max}'s (*Step* 9).

From Table 4.2, it can be seen that δ is 0.35, which implies that D_5 will be the most preferred design as long as the difference between the actual MRS value and its estimate is less than 35%. From Table 4.2, we also see that D_3 is the singleton critical design alternative, i.e., D_3 will not be eliminated if the actual MRS value differs by 35% from the MRS estimate. Also, from Table 4.2, we observe that the actual MRS value at the trial designs can change by 38% from the MRS estimate before a design alternative other than D_3 becomes non-eliminated.

In the next section, we discuss the verification of the results for payload design selection.

4.5.1.2. Discussion

Since, the payload design selection problem has only two attributes, it is possible to find analytically, the threshold MRS at which an eliminated design D_+ is not eliminated by a trial design D_T. Recall that Eq. (4.1) gives the necessary condition for the threshold gradient of D_+ with respect to D_T. Assuming that the MRS values are consistent, Eq. (4.1) can be modified to

$$S_{12T}^{t+} \cdot (P_{S1+} - P_{S1T}) + (P_{S2+} - P_{S2T}) \geq 0, \qquad (4.10)$$

where, $[P_{S1+}, P_{S2+}]$ and $[P_{S1T}, P_{S2T}]$ are the attributes of the designs D_+ and D_T respectively and S_{12T}^{t+} is the threshold MRS. Solving Eq. (4.10) then gives the threshold MRS at which D_+ is no longer eliminated by D_T.

Table 4.3: Verification of threshold MRS for payload selection

Design alternative number	Threshold MRS with respect to D_{T1} found using Eq. (4.10)	Threshold MRS with respect to D_{T1} found using our approach, Eq.(4.3)	Threshold MRS with respect to D_{T2} found using Eq. (4.10)	Threshold MRS with respect to D_{T2} found using our approach, Eq.(4.3)
1	0.09	0.09	0.60	0.60
2	0.07	0.07	0.59	0.59
3	-	-	1.03	1.03
4	-	-	0.99	0.99
6	4.04	4.04	354.31	354.34
7	2.82	2.82	6.45	6.45
8	2.80	2.80	6.25	6.25
9	-	-	5.56	5.56
10	-	-	5.56	5.56

To verify the results in Section 4.5.1.1, we found the threshold MRS of each design with respect to both trial designs (i.e., D_3 and D_5) using Eq. (4.10) and then compared them with the threshold MRS found by our approach, i.e., Eq. (4.3). Table 4.3

shows the results. The second and fourth columns of Table 4.3 show the threshold MRS for each D_+ with respect to D_{T1} and D_{T2}, respectively, found using Eq. (4.10). The third and fifth columns in Table 4.3 show the threshold MRS obtained using our approach (i.e., Eq. (4.3)). It can be seen that these results match closely, thus verifying our sensitivity analysis method.

4.5.2. Sensitivity Analysis for Deterministic Selection of Cordless Electric Drill

In this section, we present the cordless electric drill selection example to demonstrate our algorithm for a problem where the attributes are not normalized. This example is similar to the example in Section 3.5.2 of Chapter 3. We use the eighteen design alternatives shown in Table 3.3 (reproduced in Column 2 of Table 4.4), as the design alternatives for selection. We consider three design attributes: a_1, the number of operations achievable with one charge of a battery pack; a_2, the cost of the drill; and a_3, the weight of the drill. We present, in Section 4.5.2.1, the application of our algorithm for sensitivity analysis to cordless electric drill selection by a casual user. Some of the steps we describe in Section 4.5.2.1 are similar to the steps in Section 3.5.2.1 of Chapter 3 and so are discussed briefly.

4.5.2.1. Application of Algorithm for Sensitivity Analysis to Cordless Electric Drill Selection by a Casual User

Having no informed guess from the DM for picking the starting trial design, we select randomly the design alternative D_7 as the trial design for the first iteration (i.e., q = 1) D_{T1}: [450 operations, 74 dollars, 6.9 pounds] (*Step* 1). The DM, a casual user, provides the MRS estimates as shown in the third column of Table 3.4 (*Step* 2). The gradient coefficients (W_{iT1}) at D_{T1} are then: $W_{1T1} = 0.004$ operation^{-1}; $W_{2T1} = 0.045$ dollar^{-1}; $W_{3T1} = 0.443$ pound^{-1} (*Step* 3). Gradient cut at D_{T1} (*Step* 4) eliminates fifteen lower value designs (only D_1 and D_4 are not eliminated). Since more than one design is non-eliminated we skip *Step* 5 and find a new trial design (*Step* 6). Using our approach for finding a new trial design (recall Section 3.3.3), we find D_1 as the new trial design. We increase the iteration number by one (i.e., q = 2), set D_1 as D_{T2}: [350 operations, 70 dollars, 6 pounds] and go to *Step* 2.

Table 4.4: Elimination robustness of cordless electric drill designs

Design alternative number	Attributes of design alternatives [Number of operations, Cost, Weight]	δ_{+T1}'s elimination robustness of D_+ with respect to D_{T1}	δ_{+T2}'s elimination robustness of D_+ with respect to D_{T2}
1	[350 operation, 70 dollars, 6.0 pounds]	-	-
2	[370 operation, 80 dollars, 5.7 pounds]	0.02	0.85
3	[380 operation, 80 dollars, 5.5 pounds]	0.12	0.83
4	[400 operation, 72 dollars, 6.5 pounds]	0.00	0.22
5	[420 operation, 82 dollars, 6.1 pounds]	0.19	1.89
6	[430 operation, 88 dollars, 5.8 pounds]	0.27	0.91
7	[450 operation, 74 dollars, 6.9 pounds]	0.00	**0.18**
8	[470 operation, 85 dollars, 6.5 pounds]	0.53	1.21
9	[480 operation, 91 dollars, 6.1 pounds]	0.41	1.71
10	[500 operation, 79 dollars, 7.2 pounds]	0.69	0.39
11	[520 operation, 89 dollars, 6.9 pounds]	1.68	1.04
12	[530 operation, 94 dollars, 6.4 pounds]	0.51	1.29
13	[550 operation, 84 dollars, 7.5 pounds]	0.69	0.50
14	[570 operation, 93 dollars, 7.2 pounds]	1.13	0.93
15	[580 operation, 97 dollars, 6.7 pounds]	0.65	1.06
16	[600 operation, 90 dollars, 7.8 pounds]	0.76	0.63
17	[620 operation, 98 dollars, 7.5 pounds]	0.99	0.92
18	[630 operation, 100 dollars, 7.0 pounds]	0.84	0.91

The casual user of the cordless electric drill gives the MRS estimates as shown in the fourth column of Table 3.4 (*Step* 2). The gradient coefficients (W_{iT2}) at D_{T2} are then: W_{1T2} = 0.004 operation^{-1}; W_{2T2} = 0.069 dollar^{-1}; W_{3T2} = 0.248 pound^{-1} (*Step* 3). Gradient cut at D_{T2} (*Step* 4) eliminates all of the non-eliminated designs. Since all designs except one are eliminated, D_{NTD} is the singleton set with D_1 as its member (*Step* 5) and we set the total number of iterations to two, i.e., c = 2 and go to *Step* 7.

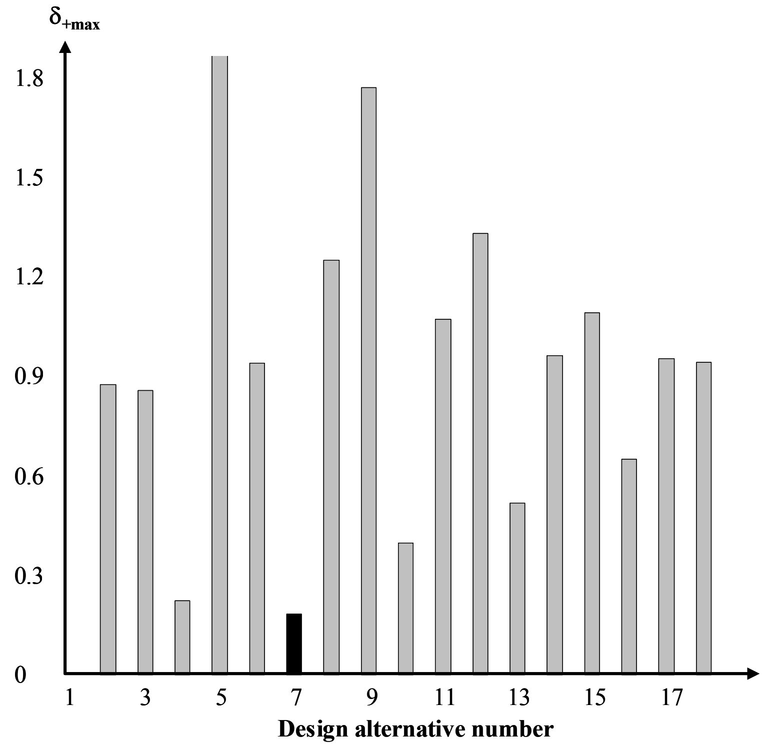

Figure 4.6: Overall elimination robustness of cordless electric drill designs

We then find the elimination robustness of each eliminated design D_+ with respect to D_{T1}, i.e., δ_{+T1}'s and D_{T2}, i.e., δ_{+T2}'s (*Step* 7). Column 3 of Table 4.4 shows the δ_{+T1}'s and Column 4 of Table 4.4 shows the δ_{+T2}'s of all designs. δ_{7T1} is zero because, D_7 is the trial design for the first iteration. δ_{4T1} is zero because, D_4 is not eliminated by the first trial design. Also δ_{1T1} and δ_{1T2} are empty because D_1 is the most preferred design and belongs to the set D_{NTD}. While calculating the δ_{+Ti}'s of eliminated designs, we neglect Eq. (4.3c) because the attributes are not normalized for the cordless electric drill selection.

Figure 4.6 shows the overall elimination robustness, δ_{+max}'s, for all designs (*Step* 8). Finally we find the robustness index of D_1, δ, by finding the minimum of all δ_{+max}'s (*Step* 9).

From Figure 4.6, it can be seen that δ is 0.18 (shown by a black bar), meaning that D_1 will be the most preferred design as long as the difference between the actual MRS values and their estimates is less than 18%. From Figure 4.6, we also see that D_7 is the singleton critical design alternative, i.e., D_7 will not be eliminated if the actual MRS values differ by 18% from the MRS estimates. Also, from Figure 4.6, we observe that the actual MRS values at the trial designs can change by 22% from the MRS estimates before a design alternative other than D_7 becomes non-eliminated. As described in Section 4.3.2, we traced back through the data about the δ_{+T}'s and the δ_{+max}'s and found that the critical pair of attributes for this example is, cost of the drill and weight of the drill.

Next we provide some experimental results that verify our sensitivity analysis method.

4.6. VERIFICATION: SOME EXPERIMENTAL RESULTS

To verify the proposed sensitivity analysis method, we conducted simulations with four different problem sizes i.e., (number of attributes) × (number of design alternatives), ranging from three attributes and fifty alternatives to six attributes and fifty alternatives. For each problem size, we used MATLAB® to generate the fifty random

Pareto design points. For simplicity, the alternatives are uniformly distributed between 0 (worst) and 1 (best) in each attribute. We chose the four different problem sizes to demonstrate the applicability of our method to problems with high number of attributes. Appendix-III shows the design alternatives that we used for each problem size.

For each problem size, we conducted three simulations, each using a different simulant value function to produce the MRS preference estimates that our method needs. The simulant value functions we used are given by Eq. (3.17), Eq. (3.18), and Eq. (3.19) (recall Section 3.6 of Chapter 3). Note that these simulant value functions are non-decreasing, differentiable, and quasi-concave.

In each simulation for a problem size, we applied our algorithm for sensitivity analysis and found the overall elimination robustness δ_{+max} of the designs. We then found the robustness index δ of the set of non-eliminated trial designs D_{NTD} by finding the minimum of the overall elimination robustness of all eliminated designs. Figure 4.7 shows, as an example, the overall elimination robustness δ_{+max} of the designs for the simulation with problem size 'three attributes' × 'fifty designs' and Eq. (3.17) as the simulant value function. From Figure 4.7, we can see that the set of non-eliminated trial designs D_{NTD} for this simulation consists of designs D_{32}, D_{38}, and D_{49} and that D_{45} is the critical design. The robustness index of D_{NTD} for this simulation is 2.8% (shown by a black bar in Figure 4.7). The δ_{+max} values of D_{32}, D_{38} and D_{49} are zero in Figure 4.7 because they are members of D_{NTD}.

Since each simulation has more than two attributes, we cannot do an analytic verification as we did with the payload selection example (recall Section 4.5.1.2).

Instead, we used a numerical approach to verify the results obtained from each simulation. This numerical approach is explained below.

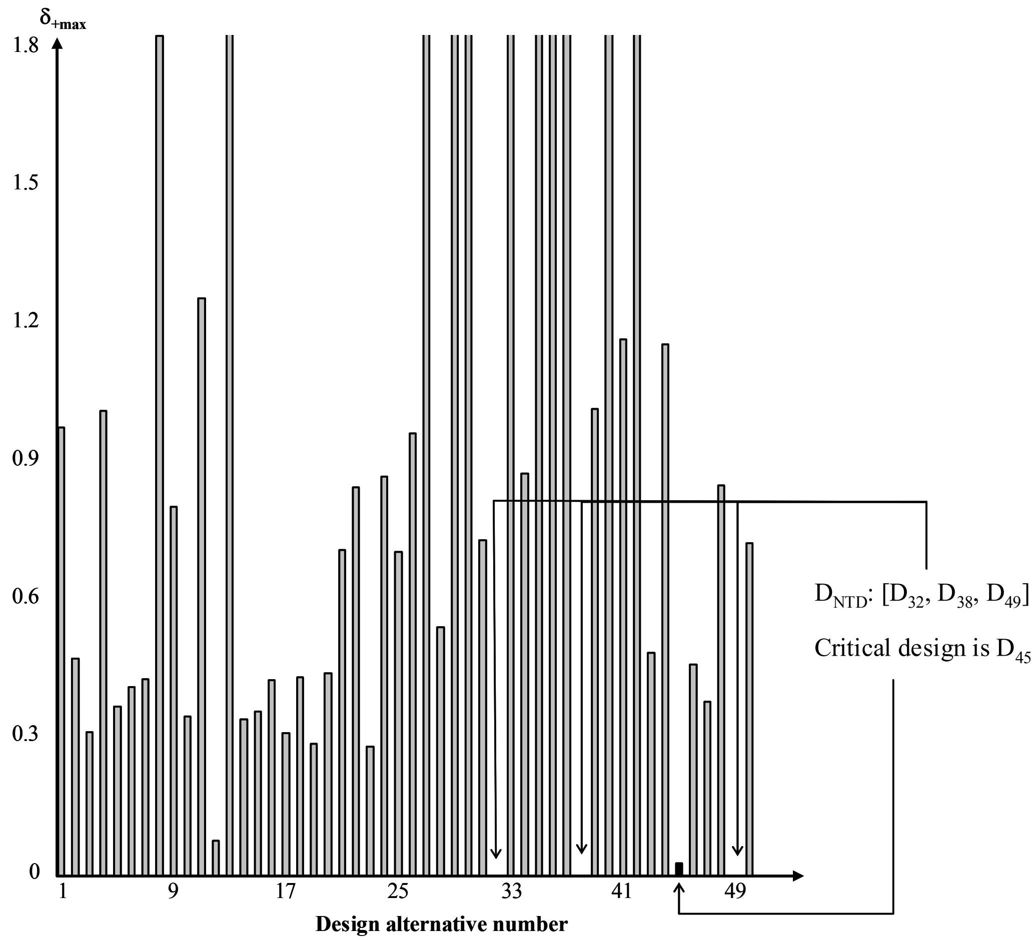

Figure 4.7: δ_{+max} of design alternatives for the experiment with 'three attributes' × 'fifty designs' with Eq. (3.17) as the simulant value function

Once the robustness index δ of D_{NTD} is found, we can define bounds on the variation in the MRS between attributes a_i and a_j at a trial design D_{Tk} (k=1,…c; c is the number of iterations), s_{ijTk}, as shown in Eq. (4.11a) if δ is less than one and as shown in Eq. (4.11b) if δ is greater than or equal to one. In Eq. (4.11) S_{ijTk} is the MRS estimate between a_i and a_j at D_{Tk}.

$$(1-\delta)\cdot S_{ijTk} \; < \; s_{ijTk} \; < \; (1+\delta)\cdot S_{ijTk} \tag{4.11a}$$

$$0 \; < \; s_{ijTk} \; < \; (1+\delta)\cdot S_{ijTk} \tag{4.11b}$$

We set the lower bound on s_{ijTk} to zero if δ is greater than or equal to one because, MRS values cannot be negative if the value function is non-decreasing with respect to the attributes. Since we assume that the MRS values are consistent, it is enough to state the bounds on 'm-1' MRS values. Also we can define the bounds on the gradient coefficients, w_{iTk}, corresponding to s_{ijTk} as shown in Eq. (4.12a) if δ is less than one and as shown in Eq. (4.12b) if δ is greater than or equal to one.

$$(1-\delta)\cdot S_{ijTk} \; < \; \frac{w_{iTk}}{w_{jTk}} \; < \; (1+\delta)\cdot S_{ijTk} \tag{4.12a}$$

$$0 \; < \; \frac{w_{iTk}}{w_{jTk}} \; < \; (1+\delta)\cdot S_{ijTk} \tag{4.12b}$$

If the δ of D_{NTD} found by our approach is accurate, then every design D_+ that does not belong to D_{NTD} will be eliminated by at least one trial design D_{Tk} for a gradient ∇v_{Tk} whose coefficients, w_{iTk}, satisfy either Eq. (4.12a) or Eq. (4.12b) (as appropriate). To check this, we first solve the optimization formulation shown in Eq. (4.13).

For k=1,…, c (c is the total number if iterations)

$$\text{Maximize} \quad G_{+k} = \sum_{i=1}^{m} w_{iTk}\cdot(a_{i+} - a_{iTk}) \tag{4.13a}$$

$$\text{subject to}: \sum_{i=1}^{m} w_{iTk} = 1; \quad w_{iTk} \geq 0 \tag{4.13b}$$

$$\begin{aligned} (1\text{-}\delta)\cdot S_{ijTk} \leq \frac{w_{iTk}}{w_{jTk}} \leq (1+\delta)\cdot S_{ijTk};\ \text{'m-1' such constraints; if } \delta < 0 \\ 0 \leq \frac{w_{iTk}}{w_{jTk}} \leq (1+\delta)\cdot S_{ijTk};\ \text{'m-1' such constraints; if } \delta \geq 0 \end{aligned} \tag{4.13c}$$

Next, we find the minimum of G_{+k}, which we call G_{+*}, over all trial designs D_{Tk} (k=1,…,c). Eq. (4.13a) is used to check whether or not D_+ is in the gradient cut corresponding to w_{iTk} at D_{Tk} (recall Eq. (3.11)). If G_{+*} is negative, then it means that D_+ lies in the gradient cut of all the gradients that satisfy either Eq. (4.12a) or Eq. (4.12b) for at least one D_{Tk}. I.e., D_+ will remain eliminated by at least one D_{Tk}.

Eq. (4.13b) is the normalization constraint for the gradient coefficients. Eq. (4.13c) is the constraints imposed on w_{iTk} using Eq. (4.12). The formulation in Eq. (4.13) is similar to the formulation we use for eliminating dominated designs when the DM gives a range of MRS preferences in selection with preference variability (see Chapter 5 for more details).

To verify the robustness index found in each simulation, we solved the optimization problem in Eq. (4.13) for all eliminated design alternatives for each trial design in each simulation. In each simulation, we then found that the G_{+*} values are negative for all eliminated designs. This means that in each simulation, D_{NTD} is not affected for all possible gradients whose coefficients satisfy Eq. (4.12). This verifies our sensitivity analysis approach.

Figure 4.8 shows, as an example, the G_{+*} values of the designs for the simulation with problem size 'three attributes' × 'fifty designs' and Eq. (3.17) as the simulant value function. From Figure 4.8, we can see that G_{+*} values of all designs except D_{32}, D_{38}, D_{45}, and D_{49} are negative. The G_{+*} values of D_{32}, D_{38}, and D_{49} are zero because they are members of D_{NTD} (recall Figure 4.7), and so their G_{+*} values are not calculated. G_{+*} value of D_{45} is zero because it is the critical design. Recall, from Section 4.2, that robustness index corresponds to the δ_{+max} of the critical design and δ_{+max} corresponds to the threshold gradient of the critical design. G_{+*} of the critical design D_{45} is zero because it lies on the plane perpendicular to the threshold gradient (recall Figure 4.3). For all the simulations we conducted, we found that the G_{+*} value of the critical design is zero.

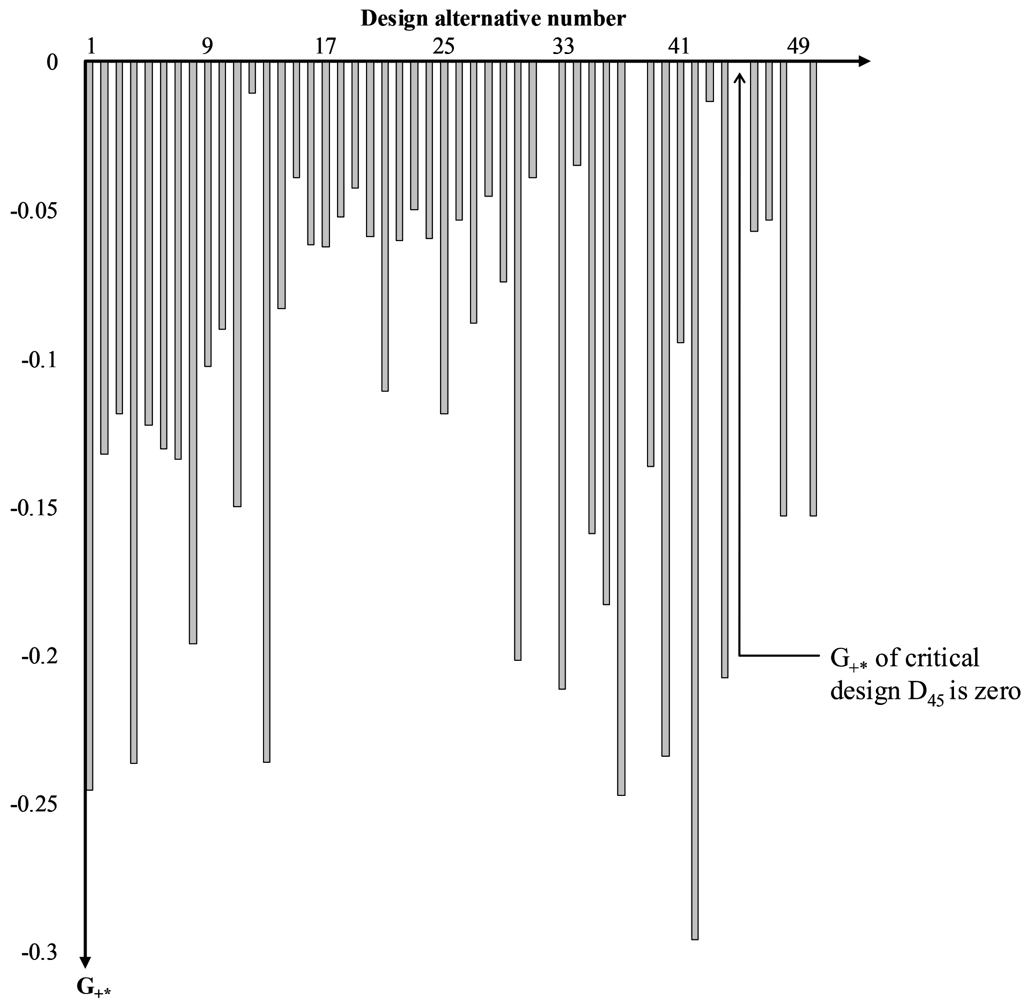

Figure 4.8: G_{+*} values of design alternatives for the experiment with 'three attributes' × 'fifty designs' with Eq. (3.17) as the simulant value function

4.7. SUMMARY

In this chapter, we presented a concept for sensitivity analysis in product design selection when the DM gives only estimates of the actual preferences. Our concept is applicable to the class of iterative selection methods which eliminate some design options at the trial design chosen for the current iteration. Such methods are generally used when the DM's value function is implicit rather than known.

In our concept, we calculate three successive metrics, culminating in the robustness index for the set of non-eliminated trial designs D_{NTD}, and we identify the critical design(s). The robustness index defines the bounds (or an interval) on the actual preferences at all trial designs for which D_{NTD} does not change. The DM can use the robustness index and the critical design(s) as guidance for further actions (e.g., select from the present D_{NTD}, assign ranges for preferences and find the potentially optimal designs). Also using the overall elimination robustness δ_{+max} of designs, the DM can find the designs that will be members of D_{NTD} for ranges of preferences symmetric about the preference estimates given at the trial designs. However, if the ranges of preferences are not symmetric about the preference estimates, the DM has to use selection with preference variability (see Chapter 5) for finding D_{NTD}.

We showed an implementation of our concept, using our deterministic selection method. In this implementation, we presented an approach for finding δ_{+T}, elimination robustness of a design D_+ with respect to a trial design D_T. Our formulation finds δ_{+T} in real time (i.e., not much computational burden) when the MRS values are assumed to be consistent. Also we introduced the concept of critical pair, i.e., the two attributes whose MRS variation has the largest influence in determining the critical design(s). Critical pair tells the DM the MRS estimate that needs to be improved, if necessary. Also the DM can analyze how the potentially optimal designs (see Chapter 5 for details) are affected by assigning various ranges to the MRS preference between the critical pair.

We presented an algorithm for sensitivity analysis for our deterministic selection and demonstrated the algorithm with two engineering examples: payload design selection and cordless electric drill selection. We also provided some experimental results that numerically verified our sensitivity analysis method. Our results show that the set of non-eliminated trial designs D_{NTD} does not change if the DM's actual MRS preferences lie within the bounds given by our robustness index.

In sensitivity analysis for deterministic selection, we find the robustness of the set of non-eliminated designs D_{NTD}. However the DM might be interested in the robustness of the most preferred design found from D_{NTD} (i.e., when D_{NTD} is not a singleton), which cannot be handled by our sensitivity analysis concept. Our approach for finding the robustness index is a worst case approach and restricts the variation in the MRS between all pairs of attributes at all trial designs to be the same and symmetric about the preference estimates. However, we can readily modify our approach if the DM is interested in the robustness of the set of non-eliminated trial designs with respect to a particular pair of attributes and/or a particular trial design and/or in a particular direction of (i.e., greater or less than) the preference estimates.

Note that for finding δ_{+T}, elimination robustness of a design D_+ with respect to a trial design D_T, using Eq. (4.3), it is important to obtain the global optimum. However, in our simulations and examples, we used "fmincon" from the MATLAB® optimization toolbox, which might converge to a local optimum, as the optimizer. We used MATLAB® to maintain uniformity with the methods developed in the other chapters. But our experimental results indicate that the set of non-eliminated trial designs D_{NTD} remains unaffected as long as the DM's actual MRS preferences lie within the bounds given by our robustness index. This could be due to the conservative nature of our approach for finding the robustness index (recall Section 4.3.2). However to be sure that the robustness index is accurate, one should use a global optimizer (e.g., genetic algorithm) or use different starting points to converge to the global optimum using a local optimizer (e.g., "fmincon" from the MATLAB® optimization toolbox) in solving the optimization problem of Eq. (4.3). A better approach (and an area for future research) would be to modify the formulation in Eq. (4.3) so that it becomes convex optimization problem.

In the next chapter, we present the development of the method for our third research component, selection with preference variability. This method is used for finding the potentially optimal designs when a range of preferences (instead of preference estimates), due to preference variability, are given by the DM.

CHAPTER 5

SELECTION WITH PREFERENCE VARIABILITY

5.1. INTRODUCTION

We mentioned earlier that when the DM does not have enough information about the end users' needs, he/she may provide ranges of the actual preferences. In this chapter, we present a method for selection when the DM gives ranges for the actual preferences. We call such a selection process: selection with preference variability.

When the DM gives ranges of the preferences, often, it is likely that for a particular subset of ranges some design is preferred (i.e., has highest value) and for another subset of ranges some other design is preferred. For example, in the automobile design selection, consider that the DM says: "I would allow the cost of the automobile to increase *between* 4000 dollars and 5000 dollars, if the 0-60 time is decreased by two seconds". If the DM's actual preference is to allow an increase of 4000 dollars for the desired reduction in the 0-60 time, then one design alternative might be preferred and if the DM's actual preference is to allow an increase of 5000 dollars, then some other design alternative might be preferred. Since the DM cannot say with certainty what his/her actual preference is, both design alternatives have a chance to be the most preferred for the given range of preference. Such designs are referred to as "potentially optimal designs" (recall Definition in Section 2.2.8 of Chapter 2) in the literature [Eum et al., 2001]. In selection with preference variability, the task is to find the set of designs that are potentially optimal from the original set of designs. Note that some people refer to selection with preference variability as selection with partial information [Eum et al., 2001].

The purpose of this chapter is to present a method for selection with preference variability. Our method for selection with preference variability is applicable when the DM's value function is implicit. In our method, we expect the DM to give ranges for the MRS preferences at a series of trial designs.

The organization of the rest of this chapter is as follows. We give an overview of our method for selection with preference variability in Section 5.2. We then present the details of our method in Section 5.3 and present an algorithm for selection with preference variability in Section 5.4. Next in Section 5.5, we give two engineering examples to demonstrate our method for selection with preference variability. Then we present some experimental results to verify our method for selection with preference variability in Section 5.6, and finally we conclude the chapter with a summary in Section 5.7.

5.2. OVERVIEW OF METHOD FOR SELECTION WITH PREFERENCE VARIABILITY

Figure 5.1 shows the flowchart of our method for selection with preference variability. This method is iterative and assumes that the DM's value function is differentiable, non-decreasing and quasi-concave with respect to the attributes. Since we assume the DM's value function to be non-decreasing with respect to the attributes, for selection it is enough to consider only those designs that are Pareto optimal from the original set of design alternatives [Malakooti, 1988].

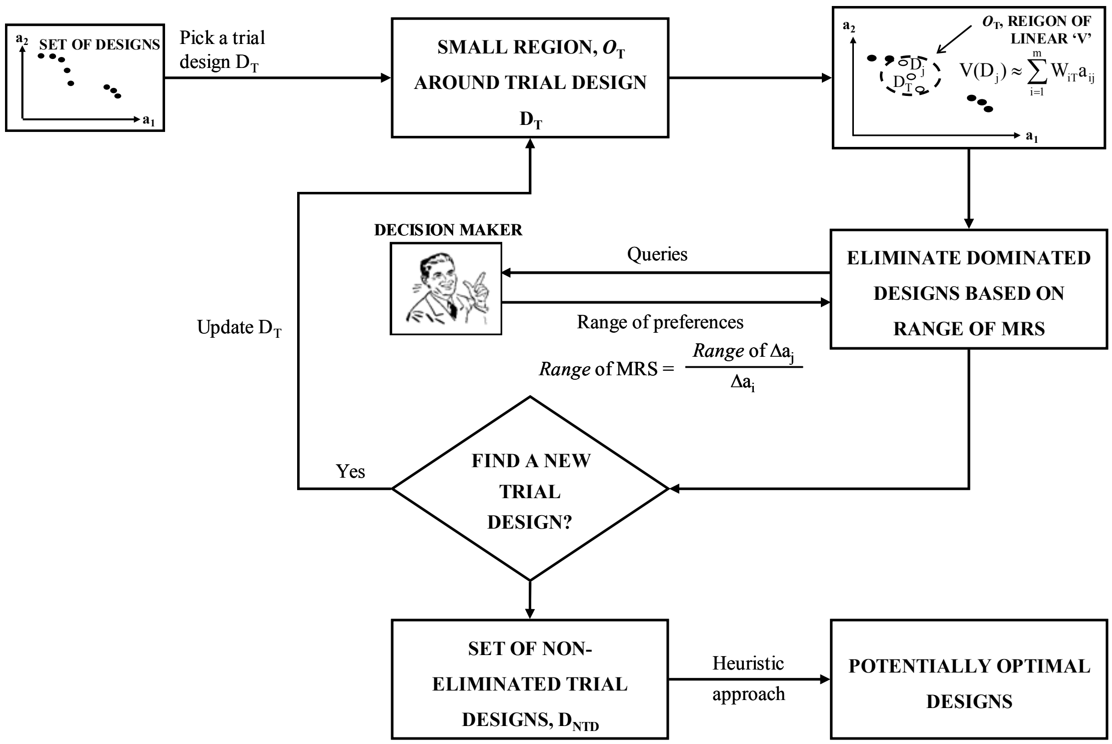

Figure 5.1: Flowchart of our method for selection with preference variability

In this method (see Figure 5.1), similar to our deterministic selection method (recall Figure 3.1), we start by picking an initial trial design, D_T, from the set of design alternatives. In a small region O_T around D_T we then approximate the value function to be linear with respect to the attributes. Next, we query the DM for the MRS preferences at D_T. Due to variability, the DM gives a range of MRS preferences. For example, in the selection of a cordless electric drill, the DM might say: "I would give up between 40 and 50 operations per battery charge to reduce the weight by 0.1 pounds".

When the DM gives a range for MRS preferences, the gradient coefficients, which are a function of MRS preferences (recall Eq. (3.4)), also have a range. Because of this, the gradient cut approach we used for deterministic selection (recall Figure 3.2) is not applicable for eliminating dominated designs. So, we use a modified version of the gradient cut for eliminating dominated designs based on the range of MRS preferences (see Section 5.3.1 for details).

Next, we try to find a new trial design (see Section 5.3.2 for details) from the non-eliminated design alternatives. If a new trial design is found, we repeat the above steps (recall Figure 5.1), referred to as “an” ‘iteration’ from here on in this chapter. Otherwise, we stop the process and collect the non-eliminated trial designs in a set, designated by D_{NTD}. All the designs that are not in D_{NTD} are dominated (recall Definition in Section 2.2.7 of Chapter 2) by at least one design in the original set of designs.

However, it is possible that the elements of the set D_{NTD} are not all potentially optimal (i.e., they might be dominated by some designs belonging to D_{NTD}, see Section 5.3.1 for a detailed explanation). So, we present a heuristic approach to test whether or not the elements of D_{NTD} are potentially optimal. This heuristic approach (see Section 5.3.3 for details) is based on the gradient adjacency elimination approach of our deterministic selection method (recall Section 3.3.4 of Chapter 3).

5.3. DESCRIPTION OF METHOD FOR SELECTION WITH PREFERENCE VARIABILITY

In this section, we discuss in detail the individual parts of our method for selection with preference variability. In Section 5.3.1, we describe our approach for eliminating dominated designs based on the range of MRS preferences. Next, we present our approach for finding a new trial design in Section 5.3.2. Finally we discuss the heuristic approach for finding potentially optimal designs from the set of non-eliminated trial designs in Section 5.3.3.

5.3.1. Eliminating Dominated Designs based on the Range of MRS Preferences

As mentioned earlier, when the DM gives a range of MRS preferences at a trial design D_T, the corresponding gradient coefficients at D_T also have a range. Due to this, some designs might lie only in the gradient cuts of some part of the range and not lie in the gradient cuts of some other part of the range, e.g., D_* in Figure 5.2. I.e., D_* is guaranteed to have a lower value than D_T only for some part of the range of MRS preferences. So, we adopt a conservative approach and eliminate, as dominated designs, those designs that lie in all possible gradient cuts for the entire range of MRS preferences (e.g., D_0 in Figure 5.2).

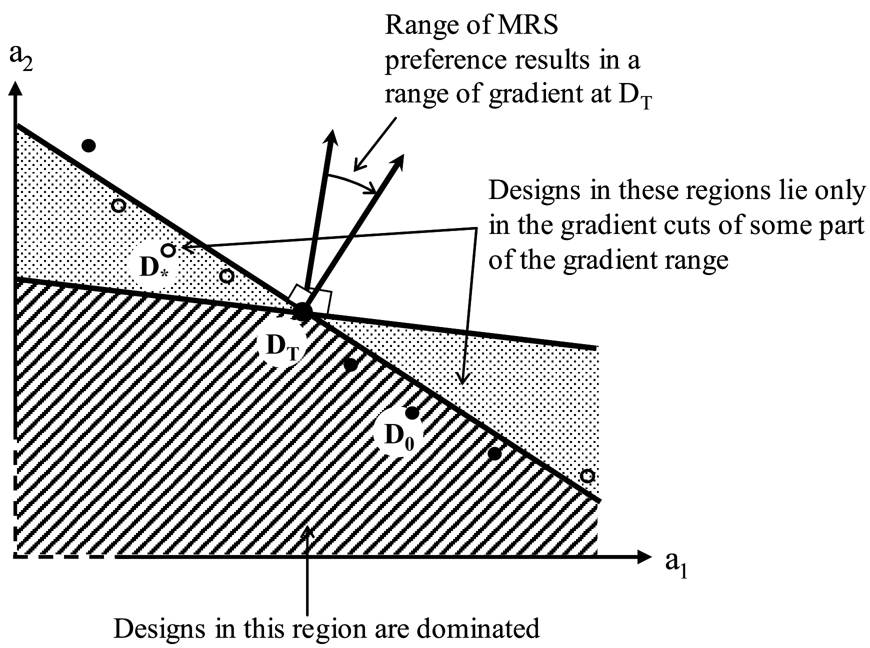

Figure 5.2: Illustration of our approach for finding dominated designs based on the range of MRS preferences

Based on Figure 5.2, a simple way to check whether or not a design is dominated by D_T is to find the extremes of the range of gradient and then use Eq. (3.11) (recall Section 3.2 of Chapter 3) to check if that design lies in the gradient cut for the extremes of the range of gradient. A design is dominated by D_T if Eq. (3.11) is satisfied for both extremes of the range of the gradient, otherwise that design is not dominated. However, there is no easy (and general) way to find the extremes of the range of gradients from the range of MRS preferences. We observed that, the extremes of the range of gradients correspond to the upper and lower bounds (i.e., S_{ijT}^{U} and S_{ijT}^{L}, respectively, recall Definition in Section 2.2.4 of Chapter 2) of MRS preferences at D_T only when the number of attributes is two. Unfortunately, this might not hold when the number of attributes exceeds two.

Below, we present a formulation that uses the range of MRS preferences, S_{ijT}, directly (i.e., without mapping them to a range of gradient coefficients) for checking whether or not a design D_+: $[a_{1+},\ldots,a_{m+}]$ is dominated by D_T: $[a_{1T},\ldots,a_{mT}]$. This linear programming (LP) problem is simple to solve by any LP solver (e.g., "linprog" from the MATLAB® optimization toolbox). In this formulation, w_{iT} (i=1,…,m) are the variables and $[a_{1+},\ldots,a_{m+}]$, $[a_{1T},\ldots,a_{mT}]$ are fixed (or deterministic).

$$\text{Maximize} \quad Z^* = \sum_{i=1}^{m} w_{iT} \cdot (a_{i+} - a_{iT}) \tag{5.1a}$$

$$\text{subject to}: \sum_{i=1}^{m} w_{iT} = 1; \quad w_{iT} \geq 0 \tag{5.1b}$$

$$S_{ijT}^{L} \leq \frac{w_{iT}}{w_{jT}} \leq S_{ijT}^{U}; \text{ 'm-1' such constraints} \tag{5.1c}$$

The objective function Z^* in the above formulation, Eq. (5.1a), is used for checking whether or not D_+ is dominated by D_T (recall Figure 5.2). If there exists a vector ∇v_T: $[w_{1T},\ldots,w_{mT}]$ from the possible range of gradient at D_T for which D_+ does not lie in the corresponding gradient cut, then the value of Z^* in Eq. (5.1a) will be non-negative (recall Eq. (3.11)) otherwise Z^* will be negative. So, if the maximum value of Z^* is negative then we can conclude that D_+ lies in the gradient cuts of all the gradients for the given range of MRS preferences at D_T. Hence D_+ is dominated by D_T.

Eq. (5.1b) is a normalization constraint on the gradient coefficients, w_{iT}. We impose the constraint that the gradient coefficients, w_{iT}, are non-negative because we assume that the value function is non-decreasing with respect to the attributes. We use Eq. (5.1b) to normalize w_{iT} because, we assume that the attributes are normalized in Eq. (5.1). When the attributes are normalized, w_{iT} is dimensionless and represents the relative importance or weight of the attribute (which by convention in the literature lies between zero and one). However, if the attributes are not normalized we neglect Eq. (5.1b) in the above formulation, i.e., w_{iT} (i=1,..,m) are not normalized. One could also modify Eq. (5.1b) as $\sum_{i=1}^{m} w_{iT} \cdot r_i = 1$, where r_i is the scale of the i^{th} attribute (recall Definition in Section 2.2.1 of Chapter 2).

Eq. (5.1c) imposes the constraint that the variable MRS values $s_{ijT} = \frac{w_{iT}}{w_{jT}}$ should belong to the range of MRS S_{ijT}: $\left[S_{ijT}^L, S_{ijT}^U\right]$ given by the DM at D_T. Note that the condition $s_{ijT} = \frac{w_{iT}}{w_{jT}}$ holds when the MRS values are assumed to be exact and consistent (recall Eq. (4.4)). Since only 'm-1' MRS values are independent when they are consistent, we use 'm-1' constraints for the bounds on $s_{ijT} = \frac{w_{iT}}{w_{jT}}$ (recall Eq. (5.1c)). However, if one feels that the exactness and consistency assumption is not appropriate then Eq. (5.1) can be easily modified by adding two more constraints as given by

Eq. (4.5) if the attributes are normalized and by Eq. (4.6) if the attributes are not normalized. However, note that adding the constraints in Eq. (4.5) or Eq. (4.6) (which are nonlinear and non-convex) to the formulation in Eq. (5.1) would increase the computational burden in eliminating the dominated designs.

Note that Eq. (5.1) should be applied to each design D_+ (that belongs to the original set of design alternatives and is not already eliminated) to check whether or not that design is dominated by D_T. Based on the definition of dominated design (recall Definition in Section 2.2.7 of Chapter 2), for a design D_+, if Z^* in Eq. (5.1) is negative then it is guaranteed that D_+ is dominated by the trial design D_T. However, it is possible that D_+ might be dominated by D_T even if Z^* is positive. Recall that for a differentiable quasi-concave value function, design alternatives not in the gradient cut C_G, i.e., above the hyper-plane, H_T, (recall Figure 3.2) might have higher or lower or equal value with respect to D_T. I.e., gradient cut does not necessarily eliminate all designs that have lower value than D_T. Added to that, for eliminating dominated designs when the MRS preferences have a range, we use a worst case (i.e., conservative) approach and eliminate only those designs that are in all possible gradient cuts (recall Figure 5.2).

Because Eq. (5.1) cannot guarantee that all dominated designs with respect to a trial design are eliminated, it is possible that some designs in the set of non-eliminated designs D_{NTD} are dominated. We present, in Section 5.3.3, a heuristic approach to identify dominated designs from D_{NTD}.

In the next section, we present our approach for finding a new trial design.

5.3.2. Finding a New Trial Design

An important step in our method for selection with preference variability is to find a new trial design for continuing the iterative process shown in Figure 5.1. Ideally the new trial design should be such that it eliminates a large number of dominated designs from the original set of designs. To check this, we need to approximate the range of MRS preferences at the candidate new trial designs and then choose as a new trial design the candidate new trial design that eliminates the maximum number of dominated designs from the original set of designs with the estimated range of MRS. But there is no easy way for approximating the range of MRS preferences at a candidate new trial design without interacting with the DM. So we use the same approach we presented for finding a new trial design in deterministic selection (recall Section 3.3.3 of Chapter 3) even when there is preference variability.

In order to find a new trial design using the approach discussed in Section 3.3.3 of Chapter 3, we need the deterministic gradient of the value function at the previous trial designs. Since there is variability in the MRS preferences, for simplicity, we take the gradient corresponding to the mid-point of the range of MRS preferences at a previous trial design as the nominal (or deterministic) gradient for that trial design.

In the next section, we present our heuristic approach for identifying dominated designs from D_{NTD}, and hence find the set of potentially optimal designs.

5.3.3. Heuristic Approach for Finding Potentially Optimal Designs

In deterministic selection, we use the gradient adjacency elimination approach for finding the most preferred design alternative(s) from the set of non-eliminated trial designs (recall Section 3.3.4 of Chapter 3). In this section, we extend the gradient adjacency elimination approach to the case when MRS preferences at a trial design D_T have a range.

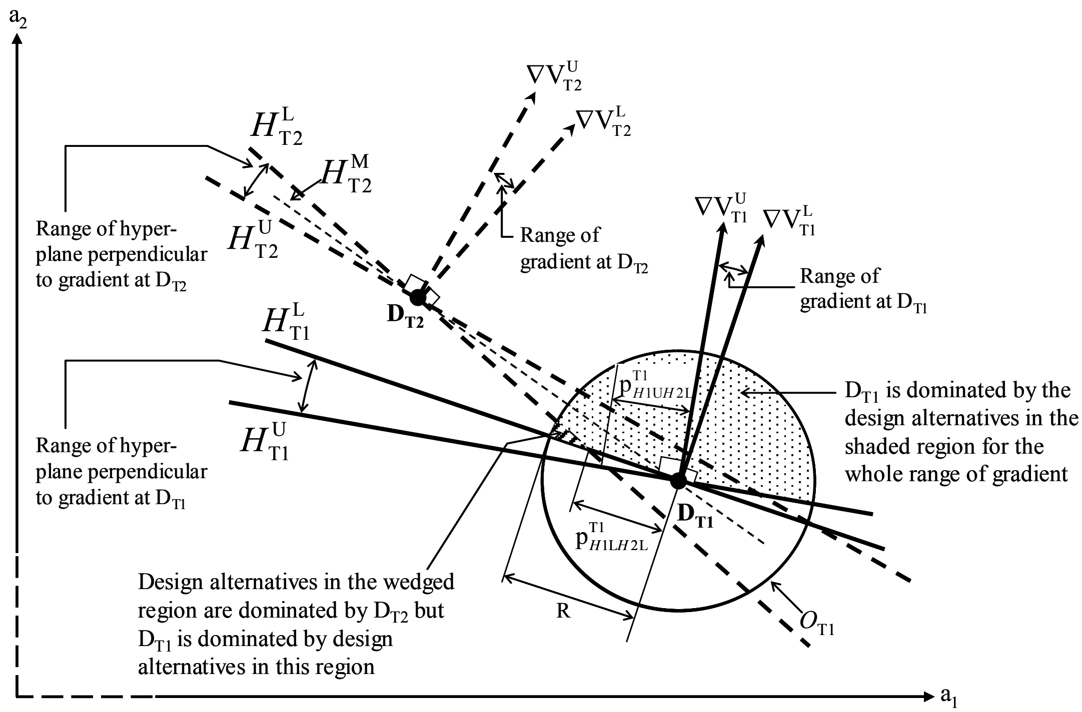

Figure 5.3: Illustration of our heuristic approach for eliminating dominated designs

Let D_{T1} and D_{T2} be two non-eliminated trial designs (i.e., they belong to D_{NTD}). Let S_{ijT1}: $\left[S^L_{ijT1}, S^U_{ijT1}\right]$ and S_{ijT2}: $\left[S^L_{ijT2}, S^U_{ijT2}\right]$ be the range of MRS preference between attributes a_i and a_j at D_{T1} and D_{T2}, respectively. Recall that since MRS preferences at D_{T1} and D_{T2} have a range, the corresponding gradients at D_{T1} and D_{T2} also have a range. Let ∇V^L_{T1} and ∇V^U_{T1} be the extremes of the range of gradient at D_{T1} (note that the extremes of the range of gradient might not necessarily correspond to S^L_{ijT1} and S^U_{ijT1}) and ∇V^L_{T2} and ∇V^U_{T2} be the extremes of the range of gradient at D_{T2}. Figure 5.3 illustrates our heuristic approach for checking if D_{T1} is dominated by D_{T2} in a two attribute space.

Lines $H_{\mathrm{T1}}^{\mathrm{L}}$ and $H_{\mathrm{T1}}^{\mathrm{U}}$ pass through $\mathrm{D_{T1}}$ and are perpendicular to the extremes of the range of gradient, $\nabla\mathrm{V}_{\mathrm{T1}}^{\mathrm{L}}$ and $\nabla\mathrm{V}_{\mathrm{T1}}^{\mathrm{U}}$, respectively. Lines $H_{\mathrm{T2}}^{\mathrm{L}}$ and $H_{\mathrm{T2}}^{\mathrm{U}}$ pass through $\mathrm{D_{T2}}$ and are perpendicular to the extremes of the range of gradient, $\nabla\mathrm{V}_{\mathrm{T2}}^{\mathrm{L}}$ and $\nabla\mathrm{V}_{\mathrm{T2}}^{\mathrm{U}}$, respectively. O_{T1} is the region around $\mathrm{D_{T1}}$ in which we approximate the value function to be linear (recall Figure 5.1). I.e., at every point inside O_{T1}, the range of MRS preferences is the same as the range of MRS preferences given by the DM at $\mathrm{D_{T1}}$ [Barzilai, 1998]. Note that neither of the two trial designs ($\mathrm{D_{T1}}$ or $\mathrm{D_{T2}}$) is dominated by the other (recall Figure 5.2) for the ranges of gradient.

As shown in Figure 5.3, all points in the shaded region of O_{T1} have a higher value than $\mathrm{D_{T1}}$ for the entire range of gradient at $\mathrm{D_{T1}}$ (i.e., those points dominate $\mathrm{D_{T1}}$). For the case shown in Figure 5.3, all the lines that lie between the extremes $H_{\mathrm{T2}}^{\mathrm{L}}$ and $H_{\mathrm{T2}}^{\mathrm{U}}$ at $\mathrm{D_{T2}}$ pass through the shaded region of O_{T1}. Hence $\mathrm{D_{T2}}$ dominates some points (recall Figure 5.2) in the shaded region of O_{T1} that have higher value than $\mathrm{D_{T1}}$. Hence $\mathrm{D_{T2}}$ dominates $\mathrm{D_{T1}}$ by transitivity.

For an m-dimensional case, $H_{\mathrm{T1}}^{\mathrm{L}}$, $H_{\mathrm{T1}}^{\mathrm{U}}$, $H_{\mathrm{T2}}^{\mathrm{L}}$, and $H_{\mathrm{T2}}^{\mathrm{U}}$ are hyper-planes. Note that in Figure 5.3, $\mathrm{D_{T1}}$ lies in the gradient cuts of the gradients perpendicular to the hyper-planes in the range $H_{\mathrm{T2}}^{\mathrm{M}}$ and $H_{\mathrm{T2}}^{\mathrm{U}}$. So it is enough to check that $\mathrm{D_{T2}}$ dominates $\mathrm{D_{T1}}$ for the other part of the range, i.e., $H_{\mathrm{T2}}^{\mathrm{L}}$ and $H_{\mathrm{T2}}^{\mathrm{M}}$. Let h_{T1} be a hyper-plane that lies between $H_{\mathrm{T1}}^{\mathrm{L}}$ and $H_{\mathrm{T1}}^{\mathrm{U}}$ at $\mathrm{D_{T1}}$. Let h_{T2} be a hyper-plane that lies between $H_{\mathrm{T2}}^{\mathrm{L}}$ and $H_{\mathrm{T2}}^{\mathrm{M}}$ at $\mathrm{D_{T2}}$. Also, let $\mathrm{p}_{h1h2}^{\mathrm{T1}}$ be the perpendicular distance from $\mathrm{D_{T1}}$ to the intersection of h_{T1} and h_{T2}. Assigning the radius R (typical value of R is 0.1) to O_{T1}, we can use Eq. (5.2) to geometrically check that $\mathrm{D_{T2}}$ dominates $\mathrm{D_{T1}}$.

$$\left\{ \underset{h_{T1}\in\left[H_{T1}^{L},H_{T1}^{U}\right],h_{T2}\in\left[H_{T2}^{L},H_{T2}^{M}\right]}{\text{maximum}} p_{h1h2}^{T1} \right\} \le R \tag{5.2}$$

It can be seen that for the case shown in Figure 5.3, the maximum distance from D_{T1} to the intersection of h_{T1} and h_{T2} (i.e., maximum p_{h1h2}^{T1}) corresponds to p_{H1LH2L}^{T1} (i.e., intersection of the hyper-planes H_{T1}^{L} and H_{T2}^{L}). Also p_{H1LH2L}^{T1} is less than R, the radius of O_{T1}. So we can say that D_{T2} dominates D_{T1} for the case shown in Figure 5.3.

The case shown in Figure 5.3 is simple in that any hyper-plane h_{T1} that lies between H_{T1}^{L} and H_{T1}^{U} at D_{T1} is not parallel to any hyper-plane h_{T2} that lies between H_{T2}^{L} and H_{T2}^{U} at D_{T2}. But, this might not hold for some cases in the given range of MRS preferences S_{ijT1}: $\left[S_{ijT1}^{L},S_{ijT1}^{U}\right]$ and S_{ijT2}: $\left[S_{ijT2}^{L},S_{ijT2}^{U}\right]$ at D_{T1} and D_{T2}, respectively, resulting in the maximum p_{h1h2}^{T1} to be infinity.

However, for the case where h_{T1} is parallel to h_{T2}, it is implied that ∇v_{T1} is equal to ∇v_{T2}; where ∇v_{T1} and ∇v_{T2} are the gradients perpendicular to h_{T1} and h_{T2} at D_{T1} and D_{T2}, respectively. When ∇v_{T1} is equal to ∇v_{T2}, we can find the value of the designs directly by using Eq. (3.1) based on a linear approximation of value function. In such a case, Eq. (5.3) can be used to check that D_{T2}: $[a_{1T2},\ldots,a_{mT2}]$ dominates D_{T1}: $[a_{1T1},\ldots,a_{mT1}]$ (here $[a_{1T2},\ldots,a_{mT2}]$, $[a_{1T1},\ldots,a_{mT1}]$ are fixed or deterministic).

$$\left\{ \underset{\nabla v_{T2}=\nabla v_{T1}:[w_{1T1},\ldots,w_{mT1}]}{\text{maximum}} \sum_{i=1}^{m} w_{iT1}\cdot(a_{iT1}-a_{iT2}) \right\} \le 0 \tag{5.3}$$

In our heuristic approach, to mathematically check that a trial design D_{T2} dominates another trial design D_{T1}, we need to conduct two tests. First test, Eq. (5.2), is for the case in which any hyper-plane h_{T1} at D_{T1} is not parallel to any hyper-plane h_{T2} at

D_{T2} (see Section 5.3.3.1 for the formulation). Second test, Eq. (5.3), is for the case in which some of the hyper-planes at D_{T1}, i.e., h_{T1}'s, are parallel to some of the hyper-planes at D_{T2}, i.e., h_{T2}'s (see Section 5.3.2.2 for the formulation).

In the next section we discuss the formulation for the first test, Eq. (5.2).

5.3.3.1. First test to Check whether D_{T2} Dominates D_{T1}

Let ∇v_{T1} be any gradient that lies between the extremes of the range of gradient, i.e., ∇V_{T1}^{L} and ∇V_{T1}^{U} at D_{T1}. Let ∇v_{T2} be any gradient that lies between the extremes of the range of gradient ∇V_{T2}^{L} and ∇V_{T2}^{U} at D_{T2}. Let w_{iT1} and w_{iT2} (i=1 to m) be the gradient coefficients corresponding to ∇v_{T1} and ∇v_{T2}, respectively. Also, let h_{T1} and h_{T2} be the hyper-planes perpendicular to ∇v_{T1} and ∇v_{T2} at D_{T1} and D_{T2}, respectively. The perpendicular distance from D_{T1} to the intersection of h_{T1} and h_{T2} we call p_{h1h2}^{T1}.

The maximum p_{h1h2}^{T1} that is required to conduct the test of Eq. (5.2) can be calculated from Eq. (5.4). In this formulation, w_{iT} (i=1,…,m) are the variables and $[a_{1+},\ldots,a_{m+}]$, $[a_{1T},\ldots,a_{mT}]$ are fixed (or deterministic).

$$\text{Maximize} \quad p_{h1h2}^{T1} \tag{5.4a}$$

$$\text{subject to:} \quad \frac{\left[\sum_{i=1}^{m} w_{iT1} \cdot w_{iT2}\right]^2}{\left(\sum_{i=1}^{m} w_{iT1}^2\right)\left(\sum_{i=1}^{m} w_{iT2}^2\right)} < 1 \tag{5.4b}$$

$$\sum_{i=1}^{m} w_{iT1} = 1; \quad w_{iT1} \geq 0 \tag{5.4c}$$

$$S_{ijT1}^{L} \leq \frac{w_{iT1}}{w_{jT1}} \leq S_{ijT1}^{U}; \ \text{'m-1' such constraints} \tag{5.4d}$$

$$\sum_{i=1}^{m} w_{iT2} = 1; \quad w_{iT2} \geq 0 \tag{5.4e}$$

$$S_{ijT2}^{L} \leq \frac{w_{iT2}}{w_{jT2}} \leq S_{ijT2}^{U}; \text{ 'm - 1' such constraints} \tag{5.4f}$$

$$\sum_{i=1}^{m} w_{iT2} \cdot (a_{iT1} - a_{iT2}) \geq 0 \tag{5.4g}$$

Eq. (5.4b) is a constraint for checking that the angle between ∇v_{T1} and ∇v_{T2} is greater than zero (hence h_{T1} is not parallel to h_{T2}). Note that the angle between the vectors ∇v_{T1} and ∇v_{T2} is zero only when the cosine of the angle (given by the square root of LHS of Eq. (5.4b)) is one. Eq. (5.4c) to Eq. (5.4f) imposes the normalization constraint on w_{iT1} and w_{iT2} and the constraints that MRS preferences $s_{ijT1} = \frac{w_{iT1}}{w_{jT1}}$ and $s_{ijT2} = \frac{w_{iT2}}{w_{jT2}}$ should belong to the range of MRS preferences given by the DM at D_{T1} and D_{T2}, respectively. Eq. (5.4g) is a constraint for checking that ∇v_{T2} belongs to the range of the gradients that are perpendicular to the hyper-planes belonging to the range H_{T2}^{L} and H_{T2}^{M} (recall Section 5.3.3).

In the next section we discuss the formulation for the second test, Eq. (5.3).

5.3.3.2. Second test to Check whether D_{T2} Dominates D_{T1}

The formulation required for conducting the test of Eq. (5.3) is given by Eq. (5.5). As mentioned earlier, when ∇v_{T1} is equal to ∇v_{T2} (i.e., h_{T1} is parallel to h_{T2}), we can find the values of the designs directly by using Eq. (3.1) (recall Section 3.2 of Chapter 3) based on a linear approximation of the value function. So, if the maximum of the difference between the values of D_{T1} and D_{T2} (i.e., objective function of Eq. (5.5)) is negative, then we can conclude that D_{T2} dominates D_{T1} for the case where h_{T1} is parallel to h_{T2}. Eq. (5.5b) is a constraint for checking that ∇v_{T1} is equal to ∇v_{T2} (hence h_{T1} is parallel to h_{T2}). Eq. (5.5c) to Eq. (5.5f) are similar to Eq. (5.4c) to Eq. (5.5f). In the formulation of Eq. (5.5), w_{iT} (i=1,…,m) are the variables and $[a_{1+},\ldots,a_{m+}]$, $[a_{1T},\ldots,a_{mT}]$ are fixed (or deterministic).

$$\text{Maximize} \quad \sum_{i=1}^{m} w_{iT1} \cdot (a_{iT1} - a_{iT2}) \tag{5.5a}$$

$$\text{subject to:} \quad \frac{\left[\sum_{i=1}^{m} w_{iT1} \cdot w_{iT2}\right]^2}{\left(\sum_{i=1}^{m} w_{iT1}^2\right)\left(\sum_{i=1}^{m} w_{iT2}^2\right)} = 1 \tag{5.5b}$$

$$\sum_{i=1}^{m} w_{iT1} = 1; \quad w_{iT1} \geq 0 \tag{5.5c}$$

$$S_{ijT1}^{L} \leq \frac{w_{iT1}}{w_{jT1}} \leq S_{ijT1}^{U}; \ \text{'m - 1' such constraints} \tag{5.5d}$$

$$\sum_{i=1}^{m} w_{iT2} = 1; \quad w_{iT2} \geq 0 \tag{5.5e}$$

$$S_{ijT2}^{L} \leq \frac{w_{iT2}}{w_{jT2}} \leq S_{ijT2}^{U}; \ \text{'m - 1' such constraints} \tag{5.5f}$$

Using our heuristic approach, we can say that a trial design D_{T2} dominates D_{T1} only when the tests of Eq. (5.2) (i.e., objective function of Eq. (5.4) is less than or equal to R, the radius of O_{T1}) and Eq. (5.3) (i.e., objective function of Eq. (5.5) is non-positive) are both satisfied. However, it is possible that Eq. (5.4) or Eq. (5.5) is infeasible. If Eq. (5.4) is infeasible, the test of Eq. (5.3) alone is enough to conclude that D_{T2} dominates D_{T1}. Similarly, the test of Eq. (5.2) alone is enough to conclude that D_{T2} dominates D_{T1} if Eq. (5.5) is infeasible. Note that Eq. (5.5) becomes infeasible only when no h_{T1} is parallel to any h_{T2} and Eq. (5.4) becomes infeasible only when any h_{T1} is parallel to some h_{T2} (hence Eq. (5.4) and Eq. (5.5) cannot be infeasible simultaneously).

If there are more than two non-eliminated trial designs in the set D_{NTD}, we apply the heuristic approach to all ordered pairs of non-eliminated trial designs (recall Section 3.3.4 of Chapter 3). Also note that the maximum of p^{T1}_{h1h2} (i.e., perpendicular distance from D_{T1} to the intersection of h_{T1} and h_{T2}) might not be the same as the maximum p^{T2}_{h1h2} (i.e., perpendicular distance from D_{T2} to the intersection of h_{T1} and h_{T2}). If it so happens that the maximum values of both p^{T1}_{h1h2} and p^{T2}_{h1h2} are less than the given radius R of O_{Ti} (i=1,2), it means that R is too large for the linear approximation to be valid. The designs that are not eliminated after the application of heuristic approach will be denoted as the potentially optimal designs. Note however, it is possible that some dominated designs are not eliminated even after applying our heuristic approach. Also, Eq. (5.4) and Eq. (5.5) involve finding the distances in the attribute space, so the attributes should be normalized before the application of our heuristic approach.

In the next section, we discuss our algorithm for selection with preference variability using the concepts discussed in Section 5.3.

5.4. ALGORITHM FOR SELECTION WITH PREFERENCE VARIABILITY

Our algorithm for selection with preference variability has the following steps.

Step 1: Set the iteration number to one (i.e., q = 1) and pick a starting trial design, D_{T1}, from the set of design alternatives. We choose D_{T1} either as an alternative that would have maximum value if the value function were linear with equal importance to the attributes, or as a random pick.

Step 2: Query the DM for the MRS preferences between attributes at the current trial design D_{Tq}. Due to variability, DM responds with a range of preferences.

Step 3: Eliminate dominated designs based on the range of MRS preferences at D_{Tq} (recall Section 5.3.1).

Step 4: If all designs except one are eliminated, define D_{NTD} to be the singleton set containing D_{Tq}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 6. Otherwise, go to *Step* 5.

Step 5: Find a new trial design from the non-eliminated design alternatives (recall Section 5.3.2). If a new trial design cannot be found, collect all the non-eliminated trial designs in the set D_{NTD}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 6. Otherwise, increase the iteration number by one (i.e., q = q+1), set the new trial design as D_{Tq} and go to *Step* 2.

Step 6: If D_{NTD} is a singleton then that design is the most preferred design alternative. Otherwise, use our heuristic approach (recall Section 5.3.3) for finding the potentially optimal design alternatives from among the D_{NTD}. Stop.

5.5. DEMONSTRATION EXAMPLES

As a demonstration, we tested our method for selection with preference variability by applying our algorithm to two engineering examples. These examples are the same as the examples in Section 3.5 of Chapter 3. The first example, selection of a payload design for undersea autonomous vehicle, graphically demonstrates the working of our algorithm for selection with preference variability. The second example, selection of a cordless electric drill, demonstrates the applicability of our method to a problem where the attributes are not normalized between zero and one.

5.5.1. Selection of Payload Design for Undersea Autonomous Vehicle with Preference Variability

For the payload design selection example, we set the ten Pareto optimum design alternatives, shown in Table 3.1 (reproduced in Column 2 of Table 5.1), as the design alternatives from which we select, with the P_{Si}'s being the attributes. Since it is difficult for a human DM to verify that the potentially optimal designs found by our method for selection with preference variability are indeed accurate (i.e., the designs are indeed most preferred for some subset of the original range of preferences), we again use a simulated DM in this example. We constructed the DM's implicit value function to be of the form

$$V = -[(1-P_{S1})^{\beta}+(1-P_{S2})^{2}]. \quad (5.6)$$

Eq. (5.6) is similar to Eq. (3.15) except that we have a parameter β in Eq. (5.6) for creating variability in MRS preferences between the attributes. We assign a range to β (note that in Eq. (5.6), V is non-decreasing, differentiable, and quasi-concave for any β greater than or equal to one). As β varies in its specified range, the MRS preference between attributes also varies. As the range of β increases, the variability in the MRS preference also increases. We again emphasize that the variability construct of Eq. (5.6) is not a presumed value function. Rather, it simulates a human DM who is supposedly being queried by our selection method, providing a range of MRS preference. The only reason we use this variability construct is to verify that the potentially optimal designs obtained by our method are indeed accurate.

We applied our method to three cases with different ranges for β in each case. We discuss in detail the case where variability in β is large (thus resulting in large variability in MRS preference) in Section 5.5.1.1. Next, in Section 5.5.1.2 we present briefly the results for the other two cases, where in the variability in β is moderate. Finally, we discuss the results of all three cases in Section 5.5.1.3.

5.5.1.1. Payload Design Selection with Large Variability in MRS Preference

For this case, we fix the range of β to be "11 to 18". We choose this range because according to Eq. (5.6) different designs will have the highest value for different values of β in this range (see Section 5.5.1.3 for details). The range of MRS preference at a trial design corresponding to a range of β can be found from Eq. (5.6) by solving a simple optimization problem (see Appendix-IV for details).

Table 5.1: Z^* values of payload design alternatives for selection with preference variability

Design alternative number	Attributes $[P_{S1}, P_{S2}]$ of design alternatives	Z^* values at D_{T1}, objective function in Eq. (5.1), of designs	Z^* values at D_{T2}, objective function in Eq. (5.1), of designs	Z^* values at D_{T3}, objective function in Eq. (5.1), of designs
1	[0.016, 0.695]	-0.0812		
2	[0.016, 0.693]	-0.0814		
3	[0.134, 0.684]	0	0.1215	-0.0001
4	[0.139, 0.675]	0.0018	0.1127	0
5	[0.274, 0.541]	0.0847	0	0.0792
6	[0.275, 0.114]	0.0024	-0.2866	
7	[0.343, 0.093]	0.0532	-0.2786	
8	[0.346, 0.091]	0.0549	-0.2792	
9	[0.355, 0.090]	0.062	-0.2768	
10	[0.357, 0.075]	0.0612	-0.2858	

Following our algorithm in Section 5.4., we set the iteration number to one (i.e., $q = 1$) and randomly pick D_3 as the starting trial design, i.e., D_{T1}: $[P_{S1}, P_{S2}] = [0.134, 0.684]$ (*Step* 1). Since this is a two attribute problem, we ask the

DM to provide the range of only one MRS preference, i.e., MRS preference between P_{S1} (attribute 1) and P_{S2} (attribute 2). Our simulated DM, Eq. (5.6), responds by saying that the range of MRS preference is, S_{12T1}: [2.46, 4.13] (*Step* 2).

We then use Eq. (5.1) with the given MRS range to eliminate some dominated designs (*Step* 3). Table 5.1 (Column 3) shows the Z^* values (objective function in Eq. (5.1)) at D_{T1} for the payload design alternatives. We can see that Z^* is negative for D_1, D_2 (hence dominated by D_{T1}) and non-negative for the rest of the design alternatives except D_3. Z^* of D_3 is zero because it is the trial design for this iteration.

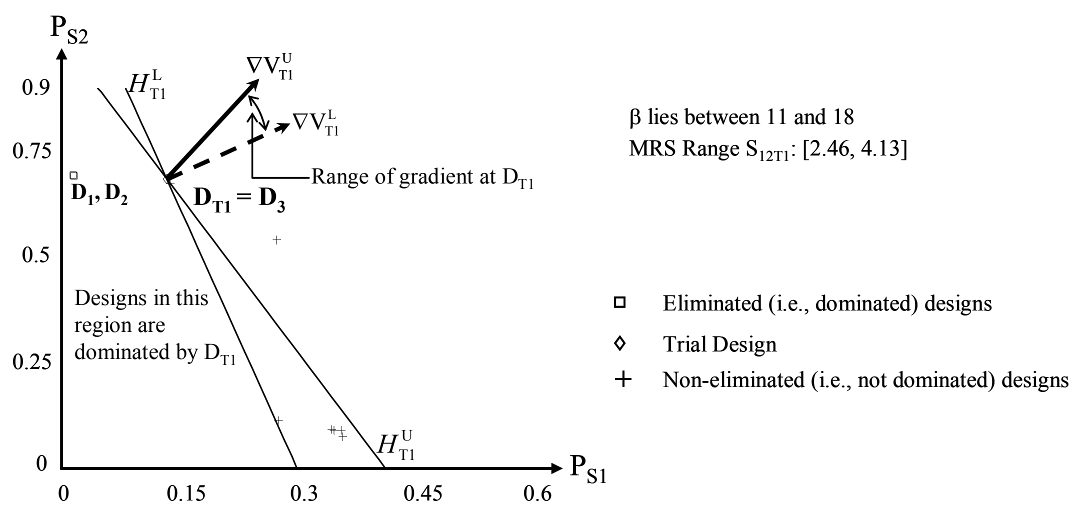

Figure 5.4: Dominated designs at D_{T1} when β lies between 11 and 18 for payload design selection

Since this is a two attribute example, the upper bound, S^U_{12T1}, and the lower bound, S^L_{12T1}, of the range of MRS preference correspond to the extremes, ∇V^U_{T1} and ∇V^L_{T1}, of the range of gradient at D_{T1}. So we can visualize the attribute space with the range of gradients as shown in Figure 5.4. From Figure 5.4, we can see that only D_1 and D_2 lie in all the possible gradient cuts that belong to the range of gradient at D_{T1}. Hence, only D_1 and D_2 are dominated by D_{T1} and can be eliminated.

Since more than one design is not eliminated, we skip *Step* 4 and find a new trial design (*Step* 5). Using our approach for finding a new trial design, we find D_5 as the new trial design. So we increase the iteration number by one (i.e., q = 2), set D_5 as D_{T2}: [0.274, 0.541] and go to *Step* 2.

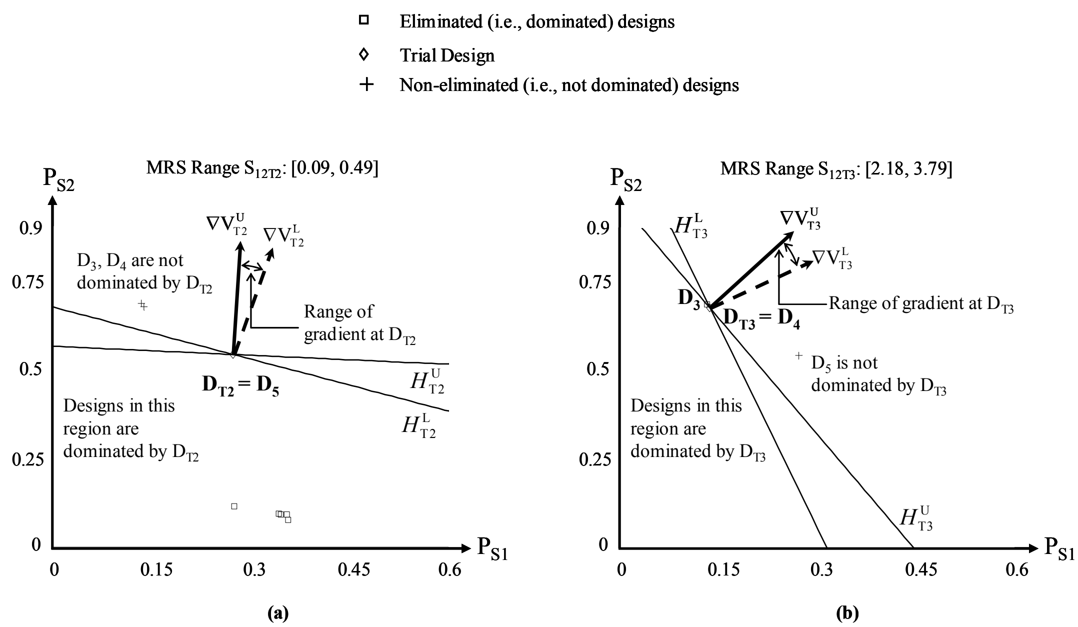

Figure 5.5: Dominated designs at (a) D_{T2} and (b) D_{T3} when β lies between 11 and 18 for payload design selection

Our simulated DM, Eq. (5.6), gives the range of MRS preference at D_{T2} as, S_{12T2}: [0.09, 0.49] (*Step* 2). We then use Eq. (5.1) for eliminating dominated designs based on the given range of MRS, S_{12T2} (*Step* 3). Table 5.1 (Column 4) shows the Z^* values at D_{T2} for the payload design alternatives. We can see that Z^* is negative for D_6, D_7, D_8, D_9, and D_{10} (hence dominated by D_{T2}) and positive for D_3 and D_4. Z^* of D_5 is zero because it is the trial design for this iteration. Z^* is empty for D_1 and D_2 because they are already eliminated by D_{T1}.

We can see from Figure 5.5(a) also that D_{T2} does not dominate D_3 and D_4 because D_3 and D_4 do not lie in any of the gradient cuts that belong to the range of gradient at D_{T2}. Since more than one design is not eliminated (recall D_3, D_4 and D_5 are not eliminated), we skip *Step* 4 and find a new trial design. Perforce, D_4 is the new trial design because it is the only non-eliminated design which has not been a trial design (*Step* 5). So we increase the iteration number by one (i.e., q = 3), set D_4 as D_{T3}: [0.139, 0.675] and go to *Step* 2.

Our simulated DM, Eq. (5.6), gives the range of MRS preference at D_{T3} as, S_{12T3}: [2.18, 3.79] (*Step* 2). We then use Eq. (5.1) for eliminating dominated designs based on the given range of MRS, S_{12T3} (*Step* 3). Table 5.1 (Column 5) shows the Z^* values at D_{T3} for the payload design alternatives. We can see that Z^* is negative for D_3 (hence dominated by D_{T3}) and positive for D_5. We can see from Figure 5.5(b) also that D_{T3} does not dominate D_5 and D_3 is dominated by D_{T3}. D_4 and D_5 are the only non-eliminated designs at this stage. Since both of them have already been trial designs we stop the iterative process and collect the two designs in the set D_{NTD} (*Step* 4) and go to *Step* 6.

We then apply our heuristic approach to see if any of the two trial designs can be eliminated (*Step* 6). We fix the radius of the region, O_{Ti} (i=2, 3), around D_{Ti} (i=2, 3) where the linear approximation of value function is estimated to be valid as: R = 0.12 (the R value is chosen arbitrarily). Using the formulation in Eq. (5.4), we then find the maximum p^{T2}_{h2h3} for D_{T2} as 0.13 and the maximum p^{T3}_{h2h3} for D_{T3} as 0.14. Since the maximum values of both p_{h2h3}^{T2} and p_{h2h3}^{T3} are greater than R neither design dominates the other. So we conclude that D_4 and D_5 are potentially optimal for the case when β lies between 11 and 18.

In the next section, we present briefly our results for the selection of payload design when the variability in β (hence the variability in MRS preference) is moderate.

5.5.1.2. Payload Design Selection with Moderate Variability in MRS Preference

We applied our method for selection with preference variability for different β ranges. In the case where β lies between 11 and 14.4, the method found D_5 and D_4 as the members of the set of non-eliminated trial designs D_{NTD} in three iterations starting with D_3 as the initial trial design (D_5 was the second trial design, D_{T2}, and D_4 was the third trial design, D_{T3}). We then applied our heuristic approach to see if any of the two trial designs can be eliminated (*Step* 6). We fixed the radius of the region, O_{Ti} (i=2, 3), around D_{Ti} (i=2, 3) where the linear approximation of value function is estimated to be valid as: R = 0.12. Using the formulation in Eq. (5.4), we then found the maximum p_{h2h3}^{T2} for D_{T2} as 0.13 and the maximum p_{h2h3}^{T3} for D_{T3} as 0.11. Since the maximum value p_{h2h3}^{T3} is less than R, the first test (recall Eq. (5.2)) for checking whether D_5 dominates D_4 is satisfied. So we conducted the second test by solving the formulation in Eq. (5.5). We found that the formulation in Eq. (5.5) is infeasible meaning that there is no hyper-plane h_{T2} that lies between H_{T2}^{L} and H_{T2}^{U} at D_{T2} that is parallel to any hyper-plane h_{T3} that lies between H_{T3}^{L} and H_{T3}^{U} at D_{T3}. Since the test of Eq. (5.2) alone is enough when Eq. (5.5) is infeasible (recall Section 5.3.3.2), we conclude that for the case when β lies between 11 and 14.4, D_5 is the singleton potentially optimal (hence the most preferred) design.

In the case where β lies between 14.6 and 18, the method again found D_5 and D_4 as the members of the set D_{NTD} in three iterations starting with D_3 as the initial trial design (again D_5 was the second trial design, D_{T2}, and D_4 was the third trial design, D_{T3}). We then applied our heuristic approach to see if any of the two trial designs can be eliminated (*Step* 6). We again fixed the radius of the region, O_{Ti} (i=1, 2) as: R = 0.12. Using the formulation in Eq. (5.4), we then found the maximum p_{h2h3}^{T2} for D_{T2} as 0.10 and the maximum p_{h2h3}^{T3} for D_{T3} as 0.14. Since the maximum value p_{h2h3}^{T2} is less than R, the first test (recall Eq. (5.2)) for checking whether D_4 dominates D_5 is satisfied. So we conducted the second test by solving the formulation in Eq. (5.5). We found that the formulation in Eq. (5.5) is infeasible meaning that there is no hyper-plane h_{T2} that lies between H_{T2}^{L} and H_{T2}^{U} at D_{T2} that is parallel to any hyper-plane h_{T3} that lies between H_{T3}^{L} and H_{T3}^{U} at D_{T3}. Since the test of Eq. (5.2) alone is enough when Eq. (5.5) is infeasible, we conclude that for the case when β lies between 14.6 and 18, D_4 is the singleton potentially optimal (hence the most preferred) design.

In the next section, we discuss the verification of the results for payload design selection with preference variability.

5.5.1.3. Discussion

To verify the results obtained by our method we use the variability construct shown in Eq. (5.6). Substituting different values for β in Eq. (5.6), we can obtain the values of the design alternatives for that β (see Figure 5.6). Note that the maximum of Eq. (5.6) (which is zero), for each β, is obtained when both P_{S1} and P_{S2} are equal to one.

From Figure 5.6 we can see that when β lies between 11 and 14.4, D_5 has the highest value. When β is equal to 14.5, both D_5 and D_4 have the highest value. When β lies between 14.6 and 18, D_4 alone has the highest value. Even though we showed in Figure 5.6 the values of the design alternatives for only some discrete β in the range 11 to 18, it can be verified that D_5 has highest value when β lies between 11 and 14.5 and D_4 has highest value when β lies between 14.5 and 18 (see Section 5.6 for an approach for finding the potentially optimal design alternatives according to a simulant value function).

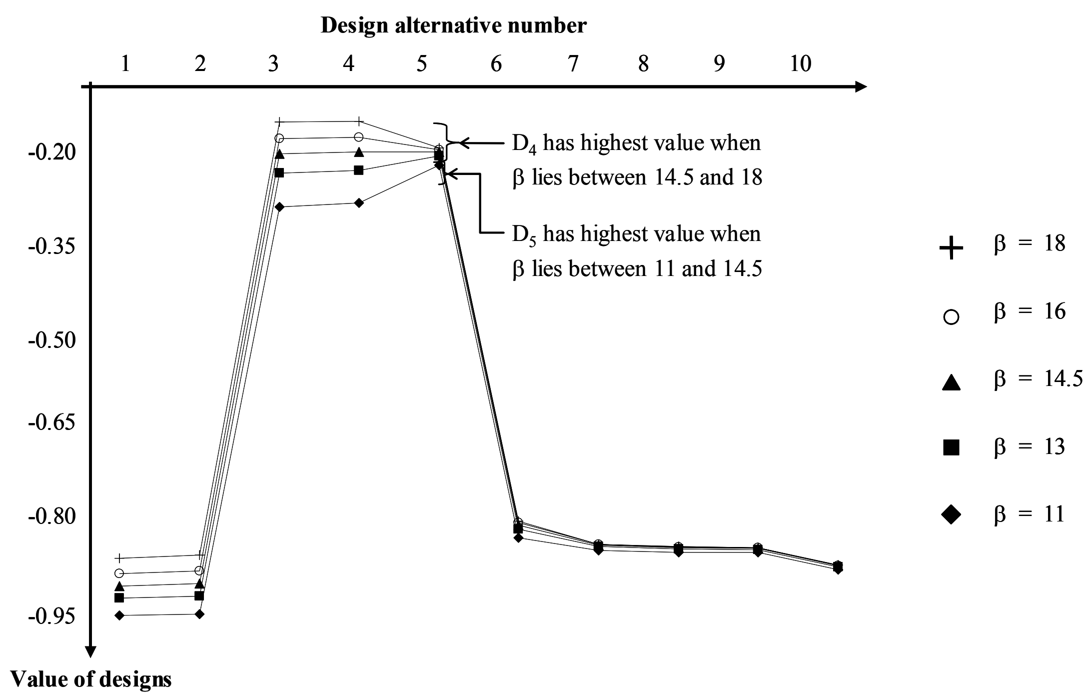

Figure 5.6: Value of payload design alternatives for different β's

Recall that using our method we obtained D_4 and D_5 as the potentially optimal designs when β lies between 11 and 18. From Figure 5.6 this is expected because D_4 has the highest value for some part of the β range and D_5 has the highest value for some other part of the β range. When β lies between 11 and 14.4, our method obtained a single most preferred design D_5 and when β lies between 14.6 and 18, our method again obtained a single most preferred design D_4, as expected from Figure 5.6. This verifies the results of our method for selection with preference variability for payload design selection.

5.5.2. Selection of Cordless Electric Drill with Preference Variability

In this section, we present the cordless electric drill selection example to demonstrate our algorithm for selection with preference variability to a problem where the attributes are not normalized. This example is similar to the example in Section 3.5.2 of Chapter 3. We use the eighteen design alternatives shown in Table 3.3 (reproduced in Column 2 of Table 5.4), as the design alternatives for selection. We consider three design attributes: a_1, the number of operations achievable with one charge of a battery pack; a_2, the cost of the drill; and a_3, the weight of the drill. We present, in Section 5.5.2.1, the application of our algorithm for selection with preference variability to cordless electric drill selection by a casual user. Next, in Section 5.5.2.2, we discuss our results.

5.5.2.1. Cordless Electric Drill Selection with Preference Variability by a Casual User

Having no informed guess from the DM for picking the starting trial design, we select randomly the design alternative D_7 as the first trial design D_{T1}: [450 operations, 74 dollars, 6.9 pounds] (*Step* 1). The DM, a casual user, provides the trade-offs as shown in the third column of Table 5.2 (*Step* 2) and says that the range of MRS preferences is ±25% around these trade-offs. For example, from the third column of Table 5.2, the trade-off between attributes a_1 and a_2 at D_{T1} is 4 dollars per 50 operations. The range of MRS between attributes a_1 and a_2 at D_{T1}, S_{12T1}, for a ±25% variability is then: [0.06 dollar operation^{-1}, 0.1 dollar operation^{-1}] (recall the MRS has a dimension equal to the ratio of attributes). Table 5.3 (Column 3) shows the range of MRS preferences at D_{T1}.

Table 5.2: Casual user's trade-offs between attributes of cordless electric drill

MRS	Attributes	Trade-offs for constant value designs at D_{T1}: [450 operations, 74 dollars, 6.9 pounds]	Trade-offs for constant value designs at D_{T2}: [350 operations, 70 dollars, 6 pounds]	Trade-offs for constant value designs at D_{T3}: [400 operations, 72 dollars, 6.5 pounds]
S_{12}	Operations	50 operations	50 operations	50 operations
	Cost	4 dollars	3 dollars	3 dollars
S_{23}	Cost	5 dollars	2 dollars	2 dollars
	Weight	0.5 pounds	0.5 pounds	0.5 pounds
S_{31}	Weight	0.4 pounds	0.5 pounds	0.5 pounds
	Operations	50 operations	40 operations	45 operations

Table 5.3: Range of MRS preferences with ±25% variability around the trade-offs of Table 5.2 for a casual user

MRS	Attributes	Range of MRS at D_{T1}: [450 operations, 74 dollars, 6.9 pounds]	Range of MRS at D_{T2}: [350 operations, 70 dollars, 6 pounds]	Range of MRS at D_{T3}: [400 operations, 72 dollars, 6.5 pounds]
S_{12}	Operations Cost	[0.06 dollar operation^{-1}, 0.1 dollar operation^{-1}]	[0.045 dollar operation^{-1}, 0.075 dollar operation^{-1}]	[0.045 dollar operation^{-1}, 0.075 dollar operation^{-1}]
S_{23}	Cost Weight	[0.075 pound dollar^{-1}, 0.125 pound dollar^{-1}]	[0.188 pound dollar^{-1}, 0.313 pound dollar^{-1}]	[0.188 pound dollar^{-1}, 0.313 pound dollar^{-1}]
S_{31}	Weight Operations	[93.75 operation pound^{-1}, 156.25 operation pound^{-1}]	[60.0 operation pound^{-1}, 100.0 operation pound^{-1}]	[67.5 operation pound^{-1}, 112.5 operation pound^{-1}]

Using Eq. (5.1) with the MRS range shown in third column of Table 5.3, we eliminate some dominated designs (*Step* 3). Table 5.4 (Column 3) shows the Z^* values (objective function in Eq. (5.1)) at D_{T1} for the cordless electric drill design alternatives.

We can see that Z^* is non-negative for D_1, D_2, D_3, D_4 and D_5 and negative (hence dominated by D_{T1}) for the rest of the design alternatives except D_7. Z^* of D_7 is zero because it is the trial design for this iteration. Since the attributes of cordless electric drill are not normalized, we neglect the constraint of Eq. (5.1b) in eliminating dominated designs for this example.

Table 5.4: Z^* values of cordless electric drill design alternatives for selection with preference variability

Design alternative number	Attributes of design alternatives [Number of operations, Cost, Weight]	Z^* values at D_{T1}, objective function of Eq. (5.1), of designs	Z^* values at D_{T2}, objective function of Eq. (5.1), of designs	Z^* values at D_{T3}, objective function of Eq. (5.1), of designs
1	[350 operation, 70 dollars, 6.0 pounds]	135549975.38	0.00	107085597.49
2	[370 operation, 80 dollars, 5.7 pounds]	45029242.11	-0.09	
3	[380 operation, 80 dollars, 5.5 pounds]	12234502.72	-0.13	
4	[400 operation, 72 dollars, 6.5 pounds]	27953942.30	1204143.97	0.00
5	[420 operation, 82 dollars, 6.1 pounds]	1239183.05	-0.09	
6	[430 operation, 88 dollars, 5.8 pounds]	-0.01		
7	[450 operation, 74 dollars, 6.9 pounds]	0.00	17568139.55	7088283.62
8	[470 operation, 85 dollars, 6.5 pounds]	-0.04		
9	[480 operation, 91 dollars, 6.1 pounds]	-0.03		
10	[500 operation, 79 dollars, 7.2 pounds]	-0.02		
11	[520 operation, 89 dollars, 6.9 pounds]	-0.08		
12	[530 operation, 94 dollars, 6.4 pounds]	-0.05		
13	[550 operation, 84 dollars, 7.5 pounds]	-0.05		
14	[570 operation, 93 dollars, 7.2 pounds]	-0.09		
15	[580 operation, 97 dollars, 6.7 pounds]	-0.07		
16	[600 operation, 90 dollars, 7.8 pounds]	-0.08		
17	[620 operation, 98 dollars, 7.5 pounds]	-0.12		
18	[630 operation, 100 dollars, 7.0 pounds]	-0.09		

Since more than one design is not eliminated, we skip *Step* 4 and find a new trial design (*Step* 5). Using our approach for finding a new trial design, we find D_1 as the new trial design. So we increase the iteration number by one (i.e., $q = 2$), set D_1 as D_{T2}: [350 operations, 70 dollars, 6 pounds] and go to *Step* 2.

The DM, a casual user, gives the range of MRS between attributes as shown in the fourth column of Table 5.3 (*Step* 2). Again, the range of MRS is obtained with ±25% variability around the trade-offs shown in the fourth column of Table 5.2. Using Eq. (5.1), D_2, D_3, and D_5 are then eliminated as dominated designs by D_{T2} (*Step* 3). Table 5.4 (Column 4) shows the Z^* values at D_{T2}. Since more than one design is not eliminated (recall D_1, D_4 and D_7 are not eliminated), we skip *Step* 4 and find a new trial design. Perforce, D_4 is the new trial design because it is the only non-eliminated design which has not been a trial design (*Step* 5). So we increase the iteration number by one (i.e., $q = 3$), set D_4 as D_{T3}: [400 operations, 72 dollars, 6.5 pounds] and go to *Step* 2.

Table 5.3 (Column 5) shows the range of MRS between attributes given by the casual user (*Step* 2). Once again the range of MRS is obtained with ±25% variability around the trade-offs shown in the fifth column of Table 5.2. Using Eq. (5.1), the Z^* values (*Step* 3) of both D_1 and D_7 are non-negative (see fifth column of Table 5.4). D_1, D_4 and D_7 are the only non-eliminated designs at this stage. Since all of them have already been trial designs we stop the iterative process and collect the three designs in the set D_{NTD} (*Step* 4) and go to *Step* 6.

We then apply our heuristic approach to see if any of the three trial designs can be eliminated (*Step* 6). For the application of our heuristic approach, we normalize the attributes using the scale of the attributes (recall Section 5.3.3). We fix the radius of the region, O_{Ti} (i=1, 2, 3), around D_{Ti} (i=1, 2, 3) where the linear approximation of value function is estimated to be valid as: $R = 0.12$ (the R value is chosen arbitrarily). Our approach eliminates D_7 using the test of Eq. (5.2) and Eq. (5.3). So, we conclude that D_1 and D_4 are the potentially optimal designs for the ranges of MRS preferences given by the casual user.

In the next section, we discuss the results of cordless electric drill selection with preference variability.

5.5.2.2. Discussion

The trade-offs at D_{T1} and D_{T2} shown in Table 5.2 are the same as the trade-offs (or MRS estimates) given by the casual user for deterministic selection (recall Table 3.4) and for sensitivity analysis for deterministic selection (recall Section 4.5.2.1 of Chapter 4). Recall that in deterministic selection we found the most preferred design as D_1: [350 operations, 70 dollars, 6 pounds] in two iterations, so there was no need for a third iteration. However, the trade-offs at D_{T3} shown in Column 5 of Table 5.2 are consistent with the casual user's preferences at other trial designs.

Recall from Figure 4.6, that the overall elimination robustness is 0.18 for D_7, 0.22 for D_1 and greater than 0.25 for the rest of the design alternatives. This shows that if the difference between the actual MRS values and their estimates is greater than or equal to 18%, D_7 will not be eliminated using gradient cut. Also, D_4 will not be eliminated using gradient cut if the difference between the actual MRS values and their estimates is greater than or equal to 22%. Stated otherwise, D_7 will be eliminated if the range of MRS preferences at D_1 is within ±18% around the trade-offs of Table 5.2 and D_4 will be eliminated if the range of MRS preferences at D_1 is within ±22% around the trade-offs of Table 5.2.

When the DM gives the range of MRS preferences as ±25% around the trade-offs of Table 5.2, our method for selection with preference variability, as expected, found D_1, D_4 and D_7 as the non-eliminated trial designs. But D_7 is eliminated as dominated design using the heuristic approach.

We applied our method for selection with preference variability to two more cases with different MRS ranges. In the first case, the range of MRS preferences was ±20% around the trade-offs of Table 5.2 and in the second case, the range of MRS preferences was ±15% around the trade-offs of Table 5.2. As expected, in the first case, our method for selection with preference variability found D_1, D_7 as the non-eliminated trial designs (D_7 was again eliminated using the heuristic approach). In the second case, our method found D_1 as the singleton non-eliminated trial design.

Next we provide some experimental results that verify our method for selection with preference variability.

5.6. VERIFICATION: SOME EXPERIMENTAL RESULTS

To verify the proposed method for selection with preference variability, we conducted simulations with four different problem sizes i.e., (number of attributes) × (number of design alternatives), ranging from three attributes and fifty alternatives to six attributes and fifty alternatives. For each problem size, we used MATLAB® to generate the fifty random Pareto design points. For simplicity, the alternatives are uniformly distributed between 0 (worst) and 1 (best) in each attribute. We chose the four different problem sizes to demonstrate the applicability of our method to problems with high number of attributes. Appendix-III shows the design alternatives that we used for each problem size.

We used three *simulant value functions* given by Eq. (5.7), Eq. (5.8), and Eq. (5.9) to produce the range of MRS preferences that our method needs.

$$V_1(D_j) = \begin{cases} \left[-\sum_{i=1}^{m} (1 - a_{ij})^{\beta} \right] \\ 2 \leq \beta \leq 2.5 \end{cases} \tag{5.7}$$

$$V_2(D_j) = \begin{cases} -\sum_{i=1}^{m} \gamma_i \cdot e^{(1-a_{ij})} \\ \sum_{i=1}^{m} \gamma_i = 1 \\ 0.9\left(1/m\right) \leq \gamma_i \leq 1.1\left(1/m\right) \\ \text{m is the number of attributes} \end{cases} \tag{5.8}$$

$$V_3(D_j) = \begin{cases} \prod_{i=1}^{m} a_{ij}^{\alpha_i} \\ \sum_{i=1}^{m} \alpha_i = 2 \\ 1.8\left(1/m\right) \leq \alpha_i \leq 2.2\left(1/m\right) \\ \text{m is the number of attributes} \end{cases} \tag{5.9}$$

These simulant value functions are similar to the simulant value functions of Eq. (3.17), Eq. (3.18), and Eq. (3.19), respectively, that we used for the verification of our deterministic selection method (recall Section 3.6 of Chapter 3). The only difference is that the parameters, β in Eq. (5.7), γ_i in Eq. (5.8), and α_i in Eq. (5.9), have an assigned range and thus create variability in the MRS preferences. Note that the simulant value function of Eq. (5.9) is quasi-concave but not concave [Avriel et al., 1988] whereas the simulant value functions of Eq. (5.7) and Eq. (5.8) are concave. All three simulant value functions are non-decreasing and differentiable with respect to the attributes. The range of MRS preferences at a trial design corresponding to the range of parameters can be found from Eq. (5.7), Eq. (5.8), and Eq. (5.9) by solving a simple optimization problem (see Appendix-IV for details).

We tested our method for selection with preference variability by comparing the potentially optimal design alternatives obtained by our method with the potentially optimal design alternatives according to the simulant value function. A design D_j would be potentially optimal according to a simulant value function if D_j has the highest value for some value of the parameter in the assigned parameter range. I.e., Eq. (5.10) is satisfied for some parameter (i.e., β in V_1 of Eq. (5.7), γ_i in V_2 of Eq. (5.8), and α_i in V_3 of Eq. (5.9)) value, in the assigned parameter range, for all designs D_k ($k = 1,\ldots,n$; $k \neq j$; n is the number of other designs) other than D_j.

$$V_i(D_k) - V_i(D_j) < 0;\ \text{'n - 1' such constraints} \tag{5.10}$$

For each problem size we conducted three simulations, each using a different simulant value function to represent the DM's preferences. For each problem size and each simulant value function, Table 5.5 shows the non-eliminated trial designs D_{NTD} (recall Figure 5.1) in the third column, potentially optimal designs after applying our heuristic approach (with R, radius of the region where the linear approximation of value function is estimated to be valid, equal to 0.05) in the fourth column, and the potentially optimal designs according to the simulant value function in the fifth column.

From Table 5.5, we can see that the set of non-eliminated designs D_{NTD}, always includes the potentially optimal designs according to the simulant value function (Column 5 of Table 5.5). However, D_{NTD} contains a number of design alternatives that are not potentially optimal according to the simulant value function. This is expected because we use a conservative approach for eliminating dominated designs (recall Section 5.3.1).

Table 5.5: Results of verification study for selection with preference variability

Problem size: "# of attributes×# of designs"	Value function	Non-eliminated trial designs, members of D_{NTD}	Potentially optimal designs after applying heuristic with R=0.05	Potentially optimal designs according to the simulant value functions
3×50	V_1	D_{12}, D_{32}, D_{38}, D_{45}	D_{12}, D_{32}, D_{38}, D_{45}	D_{32}
	V_2	D_{12}, D_{32}	D_{32}	D_{32}
	V_3	D_{12}, D_{32}	D_{12}, D_{32}	D_{32}
4×50	V_1	D_{18}, D_{19}, D_{21}, D_{24}, D_{28}, D_{29}, D_{35}, D_{37}, D_{38}	-	D_{37}
	V_2	D_7, D_{19}, D_{35}, D_{37}	D_{35}, D_{37}	D_{35}, D_{37}
	V_3	D_{18}, D_{19}, D_{21}, D_{24}, D_{29}, D_{35}, D_{37}	D_{19}, D_{35}, D_{37}	D_{35}, D_{37}
5×50	V_1	D_4, D_{12}, D_{14}, D_{15}, D_{16}, D_{20}, D_{21}, D_{22}, D_{26}, D_{29}, D_{31}, D_{33}, D_{35}, D_{36}, D_{43}, D_{49}	D_{12}	D_{12}
	V_2	D_{12}, D_{16}, D_{21}, D_{33}, D_{35}, D_{43}, D_{45}, D_{49}	D_{12}, D_{35}	D_{12}
	V_3	D_{12}, D_{16}, D_{20}, D_{21}, D_{22}, D_{26}, D_{33}, D_{35}, D_{43}, D_{45}, D_{49}	D_{12}, D_{16}, D_{26}, D_{43}, D_{49}	D_{12}
6×50	V_1	D_{20}, D_{21}, D_{32}, D_{36}, D_{39}, D_{40}, D_{41}, D_{44}, D_{46}, D_{49}	D_{39}, D_{41}, D_{46}, D_{49}	D_{39}
	V_2	D_{20}, D_{32}, D_{39}, D_{41}, D_{44}, D_{46}, D_{49}	D_{20}, D_{39}, D_{44}, D_{46}	D_{39}
	V_3	D_{20}, D_{32}, D_{39}, D_{40}, D_{41}, D_{44}, D_{46}, D_{49}	D_{20}, D_{32}, D_{39}, D_{40}, D_{44}, D_{46}, D_{49}	D_{20}, D_{39}

For example, consider the experiment with 'three attributes' × 'fifty designs' with V_3, Eq. (5.9), as the simulant value function. Our method for selection with preference variability found D_{12} and D_{32} as the members of the set of non-eliminated trial designs and only D_{32} was the potentially optimal design according to the simulant value function. Figure 5.7, shows the value range (i.e., minimum and the maximum value) for each design alternative found using Eq. (5.9). From Figure 5.7, we can see that value ranges of no design overlaps with the value range of D_{32}. So we can clearly see that D_{32} dominates other designs. However, note that, if there is overlap between value ranges of two designs, it does not necessarily mean that those designs are potentially optimal.

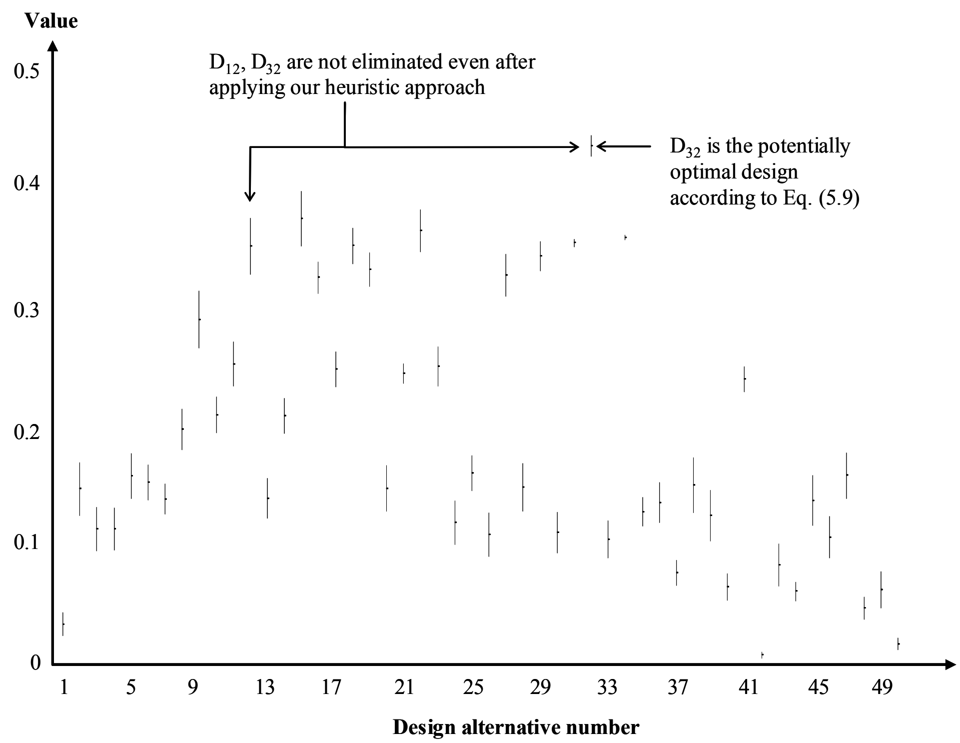

Figure 5.7: Value ranges of design alternatives for the experiment with 'three attributes' × 'fifty designs' with Eq. (5.9) as the simulant value function

Our heuristic approach is successful in most of the experiments in reducing the size of D_{NTD}. However, the potentially optimal designs that remain after applying the heuristic approach (Column 4 of Table 5.5) still contain design alternatives that are not potentially optimal according to the simulant value function (Column 5 of Table 5.5). This supports our earlier statement that some designs that are actually dominated might not be eliminated even after applying our heuristic approach. In Table 5.5, Column 4 data for 'four attributes' × 'fifty designs' is empty because our heuristic approach returned an error saying that the radius R of the region where the linear approximation of value function is estimated to be valid is too large for that experiment (recall Section 5.3.3.2).

5.7. SUMMARY

In this chapter, we presented a method for product design selection with preference variability for an implicit value function. Our method assumed that the DM's implicit value function is differentiable, quasi-concave and non-decreasing with respect to the attributes. This assumption is more general and less restrictive than other popular assumptions as reported in the literature (e.g., additive value function) [Athanassopoulos and Podinovski, 1997] [Insua and French, 1991].

Our method for selection with preference variability is iterative and requires that the DM give a range for MRS preference between attributes at a series of trial designs. We presented an approach for eliminating dominated designs using the range of MRS preferences directly. The mathematical formulation of this approach under certain conditions becomes a linear programming problem and can be solved quickly to obtain the set of non-eliminated trial designs. We also presented a heuristic for identifying the dominated designs from the set of non-eliminated trial designs. Finally, we presented an algorithm for selection with preference variability and demonstrated the algorithm with two engineering examples: payload design selection and cordless electric drill selection. We also provided some experimental results that numerically verified our method for selection with preference variability. Our experiments showed that the potentially optimal designs found using our method always include the actual potentially optimal designs according to the simulant value functions.

Our approach for eliminating dominated designs is conservative and does not always eliminate all the dominated designs. But on the bright side, our approach does not eliminate a design that is actually potentially optimal. So the set of non-eliminated trial designs always includes the actual potentially optimal designs. Also our method does not need presumed probability distributions governing the variability in MRS preferences since our approach for eliminating dominated designs is a worst case approach. Even though we assumed that in our method for selection with preference variability the DM gives a range of MRS, our formulation for eliminating dominated designs can accommodate other constraints on the MRS preferences (see Section 7.4.2 of Chapter 7 for details).

Our heuristic approach (recall Section 5.3.3) does not necessarily eliminate all dominated designs from the set of non-eliminated trial designs D_{NTD}. The formulation for our heuristic approach is non-convex and is computationally expensive. Also, as the variability in preferences becomes large, the number of iterations required for finding D_{NTD} might increase. This might be tedious for the DM. One way to reduce the number of iterations is to improve the approach for finding the new trial design. Recall, from Section 5.3.2, that the approach we use for finding a new trial design does not account for the range of preferences at the trial designs.

In the next chapter, we present the development of the method for our fourth research component, selection with preference and attribute variability. This method is used for finding the set of non-eliminated trial designs when the DM gives a range of preferences and a range of attributes for design alternatives because of preference and attribute variability.

CHAPTER 6

SELECTION WITH PREFERENCE AND ATTRIBUTE VARIABILITY

6.1. INTRODUCTION

In Chapter 5 we presented a method for selection with preference variability. In that method and also in the methods of Chapter 3 and Chapter 4, we assumed that there is no attribute variability. However, it is quite common in engineering design to have variability in the attributes of the design alternatives as well. Uncontrollable parameter variations during the design process (e.g., manufacturing errors, use conditions) are the source for attribute variability. For example, in the automobile design selection, an automobile that is designed to have an attribute level of six seconds for the 0-60 time might in reality have the 0-60 time between five and eight seconds due to manufacturing errors, use conditions, modeling errors and so on.

The purpose of this chapter is to present a method for selection with both preference and attribute variability. Specifically, we extend our method for selection with preference variability (described in Chapter 5) to account for attribute variability also. In this chapter, we assume that the attribute variability can be quantified with a known range for each attribute of a design alternative.

The organization of the rest of this chapter is as follows. We give an overview of our method for selection with preference and attribute variability in Section 6.2. We then present the details of our method in Section 6.3, and present an algorithm for selection with preference and attribute variability in Section 6.4. Next in Section 6.5, we give two engineering examples to demonstrate our method for selection with preference and attribute variability. Then we present some experimental results to verify our method for selection with preference and attribute variability in Section 6.6, and finally we conclude the chapter with a summary in Section 6.7.

6.2. OVERVIEW OF METHOD FOR SELECTION WITH PREFERENCE AND ATTRIBUTE VARIABILITY

Figure 6.1 shows the flowchart of our method for selection with preference and attribute variability. This method is iterative and assumes that the DM's value function is differentiable, non-decreasing, and quasi-concave with respect to the attributes. Because of this assumption, for selection it is enough to consider only those designs that are Pareto optimal from the original set of design alternatives [Malakooti, 1988]. The individual components of the method shown in Figure 6.1 are similar to the method for selection with preference variability (recall Figure 5.1) except for the dashed boxes.

In our method for selection with preference and attribute variability, we assume that the ranges of the attributes (shown by dotted rectangles in Figure 6.1) quantifying the variability in the attributes of the design alternatives are known. The black dot in the middle of small dashed rectangles represents the nominal attribute levels of the design alternatives. By nominal attribute levels we mean the attribute levels that would occur if there were no variability.

With the range of MRS preferences (obtained by querying the DM at a trial design) and the range of the attributes of design alternatives, we use a modified version of gradient cut (recall Section 3.3.2 of Chapter 3) for eliminating some of the dominated designs with respect to a trial design (see Section 6.3.1 for details). We assume that the DM gives the range of MRS preferences at a trial design keeping in mind the range of attributes at that trial design. In other words, the given range of MRS preferences should include the range of MRS preferences at any attribute levels belonging to the range of attributes at a trial design.

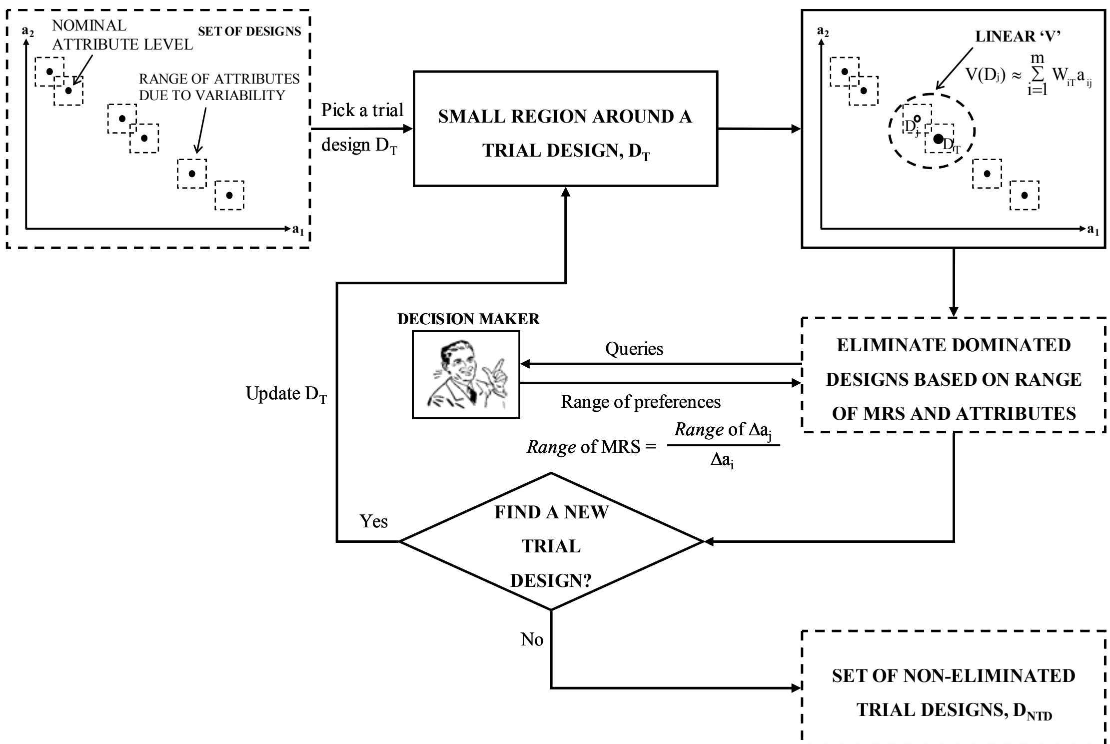

Figure 6.1: Flowchart of our method for selection with preference and attribute variability

Next, we try to find a new trial design (see Section 6.3.2 for details) from the non-eliminated design alternatives. If a new trial design is found, we repeat the above steps (see Figure 6.1), referred to as “an” ‘iteration’ from here on in this chapter. Otherwise, we stop the process and collect the non-eliminated trial designs in a set, designated by D_{NTD}. Ideally none of the designs in the set D_{NTD} should be dominated. But due to the properties of quasi-concave function (to be explained in Section 6.3.1), it is possible that some dominated designs belong to D_{NTD}.

6.3. DESCRIPTION OF METHOD FOR SELECTION WITH PREFERENCE AND ATTRIBUTE VARIABILITY

In this section, we discuss in detail the individual parts of our method for selection with preference and attribute variability. In Section 6.3.1, we describe our approach for eliminating dominated designs based on the range of MRS preferences and the range of attributes of design alternatives. Next, we present our approach for finding a new trial design in Section 6.3.2.

6.3.1. Eliminating Dominated Designs based on the Range of MRS Preferences and the Range of Attributes

Figure 6.2 illustrates, in two attribute space, our approach for eliminating dominated designs based on the range of MRS preferences and the range of attributes. Let D_T be the current trial design with the solid rectangle as the known range of attributes and the black dot in the middle as the nominal design with the given attribute levels. Because the DM gives a range of MRS (due to variability) at D_T, the corresponding gradient coefficients at D_T also have a range as shown in Figure 6.2.

Because of the variability in the MRS preferences and the attributes, a number of gradient cuts are possible at D_T, the union of which is shown by the dotted region in Figure 6.2. The shaded area in Figure 6.2 is the intersection of all the possible gradient cuts at D_T. We eliminate as dominated designs, those designs (e.g., D_0 in Figure 6.2) whose range of attributes lie completely inside the shaded area of Figure 6.2. Due to this reason, in our approach, if there is overlap between the ranges of attributes for two designs (i.e., the rectangles intersect), then those two designs will not dominate one another irrespective of the ranges of MRS preferences.

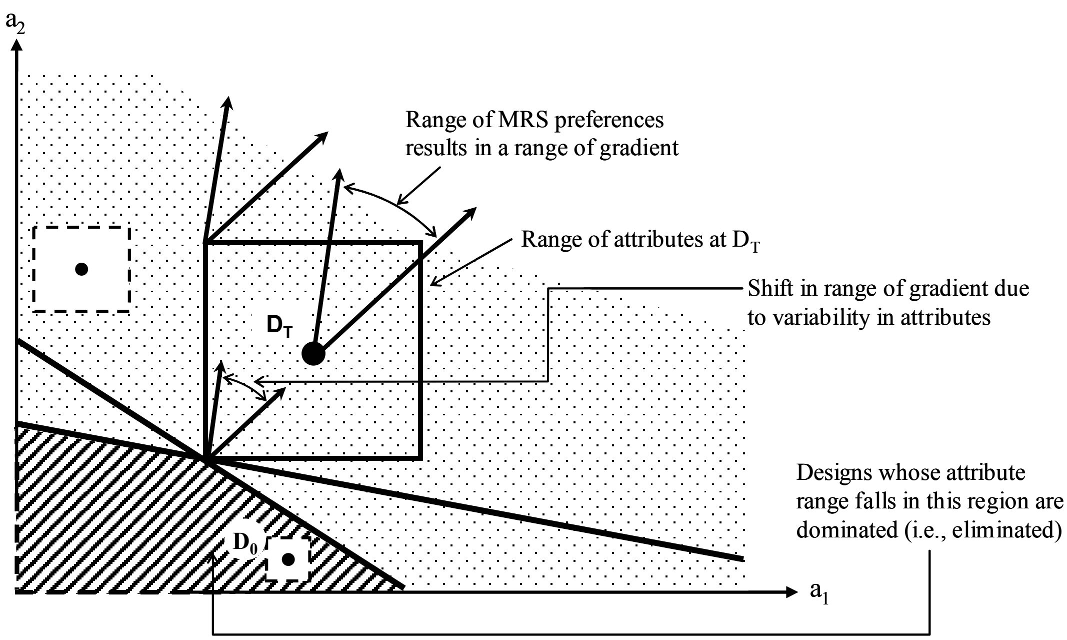

Figure 6.2: Illustration of our approach for eliminating dominated designs based on the range of MRS preferences and the range of attributes of design alternatives

However, visualizing the range of the gradient corresponding to the range of MRS preferences and the range of attributes as shown in Figure 6.2 is easy in two dimensions but is difficult for higher dimensions. So, we present a mathematical formulation in Eq. (6.1) for checking whether or not a design D_+ is dominated by a trial design D_T. In this formulation, w_{iT}, a_{i+} and a_{iT} (i=1,…,m) are the variables.

$$\text{Maximize} \quad Z^* = \sum_{i=1}^{m} w_{iT} \cdot (a_{i+} - a_{iT}) \tag{6.1a}$$

$$\text{subject to:} \quad \sum_{i=1}^{m} w_{iT} = 1; \quad w_{iT} \geq 0 \tag{6.1b}$$

$$S^L_{ijT} \leq \frac{w_{iT}}{w_{jT}} \leq S^U_{ijT}; \text{ 'm - 1' such constraints} \quad (6.1c)$$

$$A^L_{i+} \leq a_{i+} \leq A^U_{i+}; \text{ 'm' such constraints} \quad (6.1d)$$

$$A^L_{iT} \leq a_{iT} \leq A^U_{iT}; \text{ 'm' such constraints} \quad (6.1e)$$

The formulation in Eq. (6.1) is similar to the formulation in Eq. (5.1) except that two new sets of constraints are added to account for the variability in attributes. Eq. (6.1d) is to check that the variable attributes of D_+, a_{i+}, belong to the range of attributes at D_+. Eq. (6.1e) imposes a similar constraint on the variable attributes of D_T, a_{iT}.

If there exists a vector ∇v_T: $[w_{1T},\ldots,w_{mT}]$ in the range of gradient at D_T, and vectors $[a_{1+},\ldots a_{m+}]$ and $[a_{1T},\ldots,a_{mT}]$ in the ranges of attributes at D_+ and D_T respectively, for which D_+ does not lie in the corresponding gradient cut, then the value of Z^* in Eq. (6.1a) will be non-negative (recall Eq. (3.11)) otherwise Z^* will be negative. So, if the maximum value of Z^* is negative then we can conclude that D_+ lies in the gradient cuts of all the gradients at D_T. Hence D_+ is dominated by D_T.

The formulation is Eq. (6.1) has a nonlinear objective function with linear constraints and can be solved by existing commercial software (e.g., "fmincon" of the MATLAB® optimization toolbox). Note that in Eq. (6.1) we impose a normalization constraint on gradient coefficients w_{iT}. However, if the attributes are not normalized then we neglect the normalization constraint of Eq. (6.1b). One could also modify Eq. (6.1b) as $\sum_{i=1}^{m} w_{iT} \cdot r_i = 1$, where r_i is the scale of the i^{th} attribute (recall Definition in Section 2.2.1 of Chapter 2). Also, in Eq. (6.1), we assume that the MRS preferences s_{ijT} are exact and consistent (recall Eq. (4.4)). However, if one feels that the exactness and consistency assumption is not appropriate then Eq. (6.1) can be easily modified by adding two more constraints, as given by Eq. (4.5) if the attributes are normalized and by Eq. (4.6) if the attributes are not normalized.

Note that Eq. (6.1) should be applied to each design D_+ (that belongs to the original set of design alternatives and is not already eliminated) for checking whether or not that design is dominated by D_T. Based on the definition of dominated design (recall Section 2.2.7 of Chapter 2), for a design D_+, if Z^* in Eq. (6.1) is negative then it is guaranteed that D_+ is dominated by the trial design D_T. However, it is possible that D_+ might be dominated by D_T even if Z^* is positive. This is because, gradient cut does not eliminate all lower value designs with respect to D_T, and we use a conservative approach and eliminate only those designs whose attribute ranges lie completely in all possible gradient cuts (recall Figure 6.2).

In the next section, we present our approach for finding a new trial design.

6.3.2. Finding a New Trial Design

For finding a new trial design in our method for selection with preference and attribute variability, we again use the same approach we presented for finding a new trial design in deterministic selection (recall Section 3.3.3 of Chapter 3). In order to find a new trial design using the approach discussed in Section 3.3.3 of Chapter 3, we need the deterministic gradient of the value function at previous trial designs and the deterministic attributes for the design alternatives. Since there is preference and attribute variability, for simplicity, we take the gradient corresponding to the mid-point of the range of MRS preferences at the a previous trial design as the nominal (or deterministic) gradient for that trial design and the nominal attribute levels of the design alternatives as the deterministic attributes.

In the next section, we discuss our algorithm for selection with preference and attribute variability using the concepts discussed in Section 6.3.

6.4. ALGORITHM FOR SELECTION WITH PREFERENCE AND ATTRIBUTE VARIABILITY

Our algorithm for selection with preference and attribute variability has the following steps.

Step 1: Obtain the ranges of attributes and the nominal attribute levels for the design alternatives.

Step 2: Set the iteration number to one (i.e., q = 1) and pick a starting trial design, D_{T1}, from the set of design alternatives. We choose D_{T1} either as an alternative (with the nominal attribute levels) that would have the maximum value if the value function were linear with equal importance to the attributes, or as a random pick.

Step 3: Query the DM for the MRS preferences between attributes at the current trial design D_{Tq}. Due to variability, DM responds with a range of preferences.

Step 4: Eliminate dominated designs based on the range of MRS preferences at D_{Tq} and the range of attributes for design alternatives (recall Section 6.3.1).

Step 5: If all designs except one are eliminated, define D_{NTD} to be the singleton set containing D_{Tq}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 7. Otherwise, go to *Step* 6.

Step 6: Find a new trial design from the non-eliminated design alternatives (recall Section 6.3.2). If a new trial design cannot be found, collect all the non-eliminated trial designs in the set D_{NTD}, set total number of iterations to current iteration number (i.e., c = q), and go to *Step* 7. Otherwise, increase the iteration number by one (i.e., q = q+1), set the new trial design as D_{Tq} and go to *Step* 3.

Step 7: Stop.

6.5. DEMONSTRATION EXAMPLES

As a demonstration, we tested our method for selection with preference and attribute variability by applying our algorithm to two engineering examples. These examples are the same as the examples in Section 3.5 of Chapter 3. The first example involves the selection of a payload design for undersea autonomous vehicle and the second example involves the selection of a cordless electric drill.

6.5.1. Selection of Payload Design for Undersea Autonomous Vehicle with Preference and Attribute Variability

For the payload design selection example, we once again set the ten Pareto optimum design alternatives, shown in Table 3.1 (reproduced in Column 2 of Table 6.1), as the design alternatives from which we select, with the P_{Si}'s being the attributes. The attribute levels in the second column of Table 6.1 are the nominal attribute levels of the payload design alternatives. Once again we use the simulated DM given by Eq. (5.6) for verifying the results obtained by our method. However for this example, in Eq. (5.6), in addition to the parameter β, the attributes P_{S1} and P_{S2} also have variability quantified by a known range.

In the next section, Section 6.5.1.1, we describe the application of our algorithm for selection with preference and attribute variability (recall Section 6.4) to the payload design selection example, and then discuss the results in Section 6.5.1.2.

6.5.1.1. Application of Algorithm for Selection with Preference and Attribute Variability to Payload Design Selection

We fix the range of β in Eq. (5.6) to be "11 to 18" (recall Section 5.5.1.1 of Chapter 5). Also, we fix the range of attribute levels P_{S1} and P_{S2} to be ±5% around the nominal attribute levels (*Step* 1). I.e., if the nominal attribute level of a design alternative, say D_1, for the attribute, say P_{S1}, is 0.016, then the variability in the attribute P_{S1} for D_1 is quantified by the range [0.015, 0.017]. Also, for all the designs, we ensure that the lower bound on the range of an attribute does not become less than zero and that the upper bound on the range of an attribute does not become greater than one. The range of MRS preferences at a trial design for the given ranges of β, P_{S1} and P_{S2} can be found from Eq. (5.6) by solving a simple optimization problem (see Appendix-IV for details).

Following our algorithm in Section 6.4., we set the iteration number to one (i.e., $q = 1$) and randomly pick D_3 as the starting trial design, i.e., D_{T1} (*Step* 2). Since this is a two attribute problem, we ask the DM to provide the range of only one MRS preference, i.e., MRS preference between P_{S1} (attribute 1) and P_{S2} (attribute 2). Our simulated DM, Eq. (5.6), responds by saying that the range of MRS preferences is, S_{12T1}: [1.95, 5.00] (*Step* 3).

Table 6.1: Z^* values of payload design alternatives for selection with preference and attribute variability

Design alternative number	Nominal attribute levels [P_{S1}, P_{S2}] of design alternatives	Z^* values at D_{T1}, objective function in Eq. (6.1), of designs	Z^* values at D_{T2}, objective function in Eq. (6.1), of designs	Z^* values at D_{T3}, objective function in Eq. (6.1), of designs
1	[0.016, 0.695]	-0.0463		
2	[0.016, 0.693]	-0.0467		
3	[0.134, 0.684]	0	0.1871	0.0343
4	[0.139, 0.675]	0.0318	0.1776	0
5	[0.274, 0.541]	0.12	0	0.115
6	[0.275, 0.114]	0.0464	-0.2316	
7	[0.343, 0.093]	0.1024	-0.2179	
8	[0.346, 0.091]	0.1043	-0.2182	
9	[0.355, 0.090]	0.112	-0.2151	
10	[0.357, 0.075]	0.1117	-0.2236	

We then use Eq. (6.1) with the given MRS range and the ranges of attributes for eliminating some dominated designs (*Step* 4). Table 6.1 (Column 3) shows the Z^* values (objective function in Eq. (6.1)) at D_{T1} for the payload design alternatives. We can see that Z^* is negative for D_1, D_2 (hence dominated by D_{T1}) and non-negative for the rest of the design alternatives except D_3. Z^* of D_3 is zero because it is the trial design for this iteration.

Since more than one design is not eliminated, we skip *Step* 5 and find a new trial design (*Step* 6). Using our approach for finding a new trial design, we find D_5 as the new trial design. So we increase the iteration number by one (i.e., q = 2), set D_5 as D_{T2} and go to *Step* 3.

Our simulated DM, Eq. (5.6), gives the range of MRS preference at D_{T2} as, S_{12T2}: [0.06, 0.62] (*Step* 3). We then use Eq. (6.1) for eliminating dominated designs based on the given range of MRS preference, S_{12T2}, and the ranges of attributes (*Step* 4). Table 6.1 (Column 4) shows the Z^* values at D_{T2} for the payload design alternatives. We can see that Z^* is negative for D_6, D_7, D_8, D_9, and D_{10} (hence dominated by D_{T2}) and positive for D_3 and D_4. Z^* of D_5 is zero because it is the trial design for this iteration. Z^* is empty for D_1 and D_2 because they are already eliminated by D_{T1}.

Since more than one design is not eliminated (recall D_3, D_4 and D_5 are not eliminated), we skip *Step* 5 and find a new trial design. Perforce, D_4 is the new trial design because it is the only non-eliminated design which has not been a trial design (*Step* 6). So we increase the iteration number by one (i.e., q = 3), set D_4 as D_{T3} and go to *Step* 3.

Our simulated DM, Eq. (5.6), gives the range of MRS preference at D_{T3} as, S_{12T3}: [1.72, 4.58] (*Step* 3). We then use Eq. (6.1) for eliminating dominated designs based on the given range of MRS, S_{12T3}, and the ranges of attributes (*Step* 4). Table 6.1 (Column 5) shows the Z^* values at D_{T3} for the payload design alternatives. We can see that Z^* is positive for D_3 and D_5. D_3, D_4 and D_5 are the only non-eliminated designs at this stage. Since all of them have already been trial designs we stop the iterative process and collect the three designs in the set D_{NTD} (*Step* 6) and stop the selection process (*Step* 7).

In the next section, we discuss the verification of the results for payload design selection with preference and attribute variability.

6.5.1.2. Discussion

Figure 6.3 shows the value range (i.e., minimum and the maximum value) for each design alternative found using Eq. (5.6), and the assigned ranges for β, P_{S1}, and P_{S2}. We can see from Figure 6.3 that designs D_1, D_2, D_6, D_7, D_8, D_9, and D_{10} are clearly dominated by D_3, D_4 and D_5 and the value ranges of D_3, D_4, and D_5 have some overlap.

Using Eq. (5.10) (recall Section 5.6 of Chapter 5), we found that only D_3, D_4 and D_5 are potentially optimal for the given ranges of β, P_{S1}, and P_{S2}. (Note that, if there is an overlap between value ranges of two designs, it does not necessarily mean that those designs are potentially optimal, recall Section 5.6 of Chapter 5.)

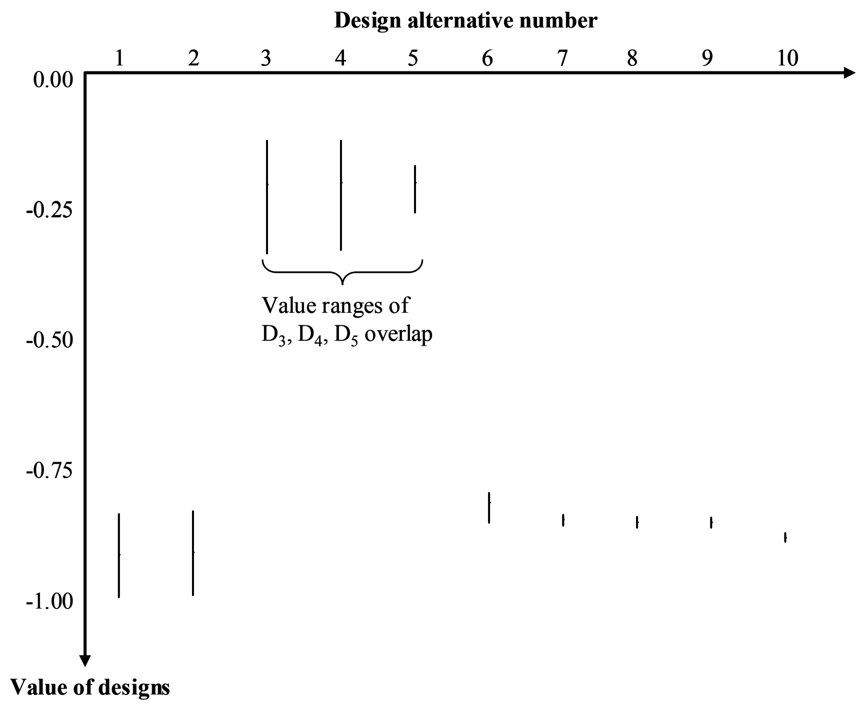

Figure 6.3: Value ranges of payload design alternatives for the simulant value function of Eq. (5.6) with $11 \leq \beta \leq 18$ and ±5% variability in P_{S1} and P_{S2}

We applied our method for selection with preference and attribute variability to another case of payload design selection problem with different ranges for the attributes P_{S1} and P_{S2}. In this case, we fixed the range of attributes P_{S1} and P_{S2} to be ±15% around the nominal attribute levels. Also, for all the designs, we ensure that the lower bound on the range of an attribute does not become less than zero and that the upper bound on the range of an attribute does not become greater than one. Starting with an initial trial design of D_3, our method found designs D_1, D_2, D_3, D_4 and D_5 to be the elements of D_{NTD}.

However, using Eq. (5.10), we found that only D_3, D_4 and D_5 are the potentially optimal designs for this case also.

These results support our earlier statement that, in our method for selection with preference and attribute variability some designs that are actually dominated might be included in the set of non-eliminated trial designs D_{NTD}.

6.5.2. Selection of Cordless Electric Drill with Preference and Attribute Variability

In this section, we present the cordless electric drill selection example to demonstrate our algorithm for selection with preference and attribute variability to a problem where the attributes are not normalized. This example is similar to the example in Section 3.5.2 of Chapter 3. We use the eighteen design alternatives shown in Table 3.3 (reproduced in Column 2 of Table 6.3), as the design alternatives for selection. The attribute levels in the second column of Table 6.3 are the nominal attribute levels of the cordless electric drill design alternatives. We consider three design attributes: a_1, the number of operations achievable with one charge of a battery pack; a_2, the cost of the drill; and a_3, the weight of the drill. We present, in Section 6.5.2.1, the application of our algorithm for selection with preference and attribute variability to cordless electric drill selection by a casual user.

6.5.2.1. Cordless Electric Drill Selection with Preference and Attribute Variability by a Casual User

We fix the range of attributes to be ±1% around the nominal attribute levels given in the second column of Table 6.3 (*Step* 1). I.e., for a design alternative, say D_1, if the nominal attribute level for the attribute, say number of operations, is 350, then the variability in the attribute, number of operations, for D_1 is quantified by the range [346.5, 353.5]. Following our algorithm in Section 6.4., we set the iteration number to one (i.e., q = 1) and randomly pick D_7 as the starting trial design, i.e., D_{T1} (*Step* 2). The DM, a casual user, provides the range of MRS as shown in the third column of Table 6.2 (*Step* 3). Note that the ranges of MRS shown in Table 6.2 are obtained with ±20% variability around the trade-offs shown in Table 5.2 of Section 5.5.2.1 in Chapter 5.

Table 6.2: Ranges of MRS given by a casual user for cordless electric drill selection with preference and attribute variability

MRS	Attributes	Ranges of MRS at D_{T1} with nominal attributes: [450 operations, 74 dollars, 6.9 pounds]	Ranges of MRS at D_{T2} with nominal attributes: [350 operations, 70 dollars, 6 pounds]	Ranges of MRS at D_{T3} with nominal attributes: [400 operations, 72 dollars, 6.5 pounds]
S_{12}	Operations Cost	[0.06 dollar operation^{-1}, 0.1 dollar operation^{-1}]	[0.05 dollar operation^{-1}, 0.07 dollar operation^{-1}]	[0.05 dollar operation^{-1}, 0.07 dollar operation^{-1}]
S_{23}	Cost Weight	[0.08 pound dollar^{-1}, 0.12 pound dollar^{-1}]	[0.20 pound dollar^{-1}, 0.30 pound dollar^{-1}]	[0.20 pound dollar^{-1}, 0.30 pound dollar^{-1}]
S_{31}	Weight Operations	[100.0 operation pound^{-1}, 150.0 operation pound^{-1}]	[64.0 operation pound^{-1}, 96.0 operation pound^{-1}]	[72.0 operation pound^{-1}, 108.0 operation pound^{-1}]

Using Eq. (6.1) with the MRS ranges shown in third column of Table 6.2, we eliminate some dominated designs (*Step* 4). Table 6.3 (Column 3) shows the Z^* values (objective function in Eq. (6.1)) at D_{T1} for the cordless electric drill design alternatives. We can see that Z^* is non-negative for D_1, D_2, D_3, D_4 and D_5 and negative (hence dominated by D_{T1}) for the rest of the design alternatives except D_7. Z^* of D_7 is zero because it is the trial design for this iteration. Since the attributes of cordless electric drill are not normalized, we neglect the constraint of Eq. (6.1b) in eliminating dominated designs for this example.

Since more than one design is not eliminated, we skip *Step* 5 and find a new trial design (*Step* 6). Using our approach for finding a new trial design, we find D_1 as the new trial design. So we increase the iteration number by one (i.e., q = 2), set D_1 as D_{T2} and go to *Step* 3.

Table 6.3: Z^* values of cordless electric drill design alternatives for selection with preference and attribute variability

Design alternative number	Nominal attribute levels of design alternatives [Number of operations, Cost, Weight]	Z^* values at D_{T1}, objective function of Eq. (6.1), of designs	Z^* values at D_{T2}, objective function of Eq. (6.1), of designs	Z^* values at D_{T3}, objective function of Eq. (6.1), of designs
1	[350 operation, 70 dollars, 6.0 pounds]	26494.080	0.000	17272.60
2	[370 operation, 80 dollars, 5.7 pounds]	6913.779	-0.081	
3	[380 operation, 80 dollars, 5.5 pounds]	2844.758	-0.127	
4	[400 operation, 72 dollars, 6.5 pounds]	6703.409	4337.930	0.00
5	[420 operation, 82 dollars, 6.1 pounds]	494.789	-0.085	
6	[430 operation, 88 dollars, 5.8 pounds]	-0.004		
7	[450 operation, 74 dollars, 6.9 pounds]	0.000	6589.772	7218.79
8	[470 operation, 85 dollars, 6.5 pounds]	-0.026		
9	[480 operation, 91 dollars, 6.1 pounds]	-0.027		
10	[500 operation, 79 dollars, 7.2 pounds]	-0.012		
11	[520 operation, 89 dollars, 6.9 pounds]	-0.070		
12	[530 operation, 94 dollars, 6.4 pounds]	-0.047		
13	[550 operation, 84 dollars, 7.5 pounds]	-0.041		
14	[570 operation, 93 dollars, 7.2 pounds]	-0.089		
15	[580 operation, 97 dollars, 6.7 pounds]	-0.068		
16	[600 operation, 90 dollars, 7.8 pounds]	-0.080		
17	[620 operation, 98 dollars, 7.5 pounds]	-0.117		
18	[630 operation, 100 dollars, 7.0 pounds]	-0.085		

The DM, a casual user, gives the ranges of MRS between attributes as shown in fourth column of Table 6.2 (*Step* 3). Using Eq. (6.1), D_2, D_3, and D_5 are then eliminated as dominated designs by D_{T2} (*Step* 4). Table 6.3 (Column 4) shows the Z^* values at D_{T2}.

Since more than one design is not eliminated (recall D_1, D_4 and D_7 are not eliminated), we skip *Step* 5 and find a new trial design. Perforce, D_4 is the new trial design because it is the only non-eliminated design which has not been a trial design (*Step* 6). So we increase the iteration number by one (i.e., q = 3), set D_4 as D_{T3} and go to *Step* 3.

Table 6.2 (Column 5) shows the range of MRS between attributes given by the casual user (*Step* 3). Using Eq. (6.1), the Z^* values (*Step* 4) of both D_1 and D_7 are non-negative (see fifth column of Table 6.3). D_1, D_4 and D_7 are the only non-eliminated designs at this stage. Since all of them have already been trial designs we stop the iterative process and collect the three designs in the set D_{NTD} (*Step* 6) and stop the selection process (*Step* 7).

We applied our method for selection with preference variability to another case of cordless electric drill selection problem with different ranges for the attributes. In this case, we fixed the range of attributes to be ±5% around the nominal attribute levels. However, we used the same ranges of MRS preferences given in Table 6.2 for this case also. Starting with an initial trial design of D_7, our method found designs D_1, D_4, D_7, D_{10} and D_{14} to be the elements of D_{NTD}. This shows that, as expected, increasing the variability in attributes increases the number of designs in the set D_{NTD}.

Next we provide some experimental results that verify our method for selection with preference and attribute variability.

6.6. VERIFICATION: SOME EXPERIMENTAL RESULTS

To verify the proposed method for selection with preference and attribute variability, we conducted simulations with four different problem sizes i.e., (number of attributes) × (number of design alternatives), ranging from three attributes and fifty alternatives to six attributes and fifty alternatives. For each problem size, we used MATLAB® to generate the fifty random Pareto design points. For simplicity, the alternatives are uniformly distributed between 0 (worst) and 1 (best) in each attribute. We chose the four different problem sizes to demonstrate the applicability of our method to higher attribute problems. Appendix-III shows the nominal attribute levels of the design alternatives that we used for each problem size.

For each problem size we conducted three simulations, each using a different simulant value function for producing the range of MRS preferences that our method needs. The simulant value functions we used are given by Eq. (5.7), Eq. (5.8), and Eq. (5.9) (recall Section 5.6 of Chapter 5). In addition, for each simulation, we fix the range of attributes of the random design alternatives to be ±5% around the nominal attribute levels given in Appendix-III. Also, for all the designs, we ensure that the lower bound on the range of an attribute does not become less than zero and that the upper bound on the range of an attribute does not become greater than one. The range of MRS preferences at a trial design corresponding to the range of parameters and the range of attributes can be found from Eq. (5.7), Eq. (5.8), and Eq. (5.9) by solving a simple optimization problem (see Appendix-IV for details). Note that these three simulant value functions are non-decreasing, differentiable and quasi-concave even with variability in the parameters and the attributes.

To test our method for selection with preference and attribute variability, we found the designs that are potentially optimal according to the simulant value function. A design D_j would be potentially optimal according to a simulant value function if, D_j has the highest value for some value of the parameter in the assigned parameter range and for some attribute levels of design alternatives in the assigned ranges of attributes. I.e., Eq. (5.10) is satisfied for some parameter (i.e., β in V_1 of Eq. (5.7), γ_i in V_2 of Eq. (5.8), and α_i in V_3 of Eq. (5.9)) value in the assigned parameter range and for some attribute levels of design alternatives in the assigned ranges of attributes, for all designs D_k ($k = 1,\ldots,n$; $k \neq j$; n is the number of other designs) other than D_j.

If the results of our method are accurate then the set of non-eliminated designs, D_{NTD}, should be a super set of the set of potentially optimal designs according to the simulant value function. Table 6.4 shows the non-eliminated trial designs D_{NTD} in the third column and the potentially optimal designs according to the simulant value function in the fourth column. From Table 6.4, we can see that the set of non-eliminated designs D_{NTD}, always includes the potentially optimal designs according to the simulant value function, thus verifying our method for selection with preference and attribute variability.

From Table 6.4, we can see that the set D_{NTD} contains a number of design alternatives that are not potentially optimal according to the simulant value function. This shows that our method for selection with preference and attribute variability is very conservative in eliminating dominated designs as mentioned before (recall Section 6.3.1).

Table 6.4: Results of verification study for selection with preference and attribute variability

Problem size: "# of attributes × # of designs"	Value function	Non-eliminated trial designs, members of D_{NTD} for ±0.05% attribute variability	Potentially optimal designs according to the simulant value functions for ±0.05% attribute variability
3×50	V_1	D_{12}, D_{15}, D_{16}, D_{18}, D_{19}, D_{22}, D_{23}, D_{31}, D_{32}, D_{34}, D_{38}, D_{39}, D_{43}, D_{45}	D_{31}, D_{32}, D_{34}
	V_2	D_9, D_{12}, D_{15}, D_{16}, D_{18}, D_{19}, D_{22}, D_{23}, D_{27}, D_{29}, D_{31}, D_{32}, D_{34}, D_{38}, D_{43}, D_{45}, D_{49}	D_{12}, D_{15}, D_{18}, D_{22}, D_{31}, D_{32}, D_{34}
	V_3	D_{12}, D_{15}, D_{16}, D_{17}, D_{18}, D_{19}, D_{22}, D_{23}, D_{27}, D_{28}, D_{29}, D_{31}, D_{32}, D_{34}, D_{38}, D_{39}, D_{45}, D_{47}	D_{12}, D_{15}, D_{18}, D_{22}, D_{31}, D_{32}, D_{34}
4×50	V_1	D_4, D_6, D_7, D_{18}, D_{19}, D_{21}, D_{23}, D_{24}, D_{28}, D_{29}, D_{31}, D_{35}, D_{36}, D_{37}, D_{38}, D_{42}, D_{43}, D_{44}, D_{46}	D_{18}, D_{21}, D_{24}, D_{28}, D_{35}, D_{37}
	V_2	D_4, D_5, D_7, D_{14}, D_{15}, D_{18}, D_{19}, D_{21}, D_{24}, D_{28}, D_{29}, D_{31}, D_{35}, D_{36}, D_{37}, D_{38}, D_{41}, D_{42}, D_{44}, D_{46}	D_{18}, D_{21}, D_{24}, D_{35}, D_{36}, D_{37}, D_{38}, D_{41}, D_{44}
	V_3	D_4, D_7, D_{14}, D_{15}, D_{16}, D_{18}, D_{19}, D_{21}, D_{24}, D_{28}, D_{29}, D_{31}, D_{35}, D_{36}, D_{37}, D_{38}, D_{42}, D_{44}, D_{46}	D_{18}, D_{21}, D_{24}, D_{29}, D_{35}, D_{37}
5×50	V_1	D_4, D_7, D_9, D_{10}, D_{11}, D_{12}, D_{14}, D_{15}, D_{16}, D_{20}, D_{21}, D_{22}, D_{26}, D_{27}, D_{29}, D_{31}, D_{33}, D_{35}, D_{36}, D_{37}, D_{39}, D_{41}, D_{42}, D_{43}, D_{44}, D_{45}, D_{49}	D_{12}, D_{14}, D_{16}, D_{22}, D_{26}
	V_2	D_4, D_6, D_7, D_{11}, D_{12}, D_{14}, D_{15}, D_{16}, D_{19}, D_{20}, D_{21}, D_{22}, D_{26}, D_{31}, D_{33}, D_{35}, D_{36}, D_{38}, D_{39}, D_{42}, D_{43}, D_{44}, D_{45}, D_{49}	D_{12}, D_{14}, D_{16}, D_{21}, D_{22}, D_{26}
	V_3	D_4, D_7, D_{11}, D_{12}, D_{14}, D_{15}, D_{16}, D_{19}, D_{20}, D_{21}, D_{22}, D_{26}, D_{29}, D_{31}, D_{33}, D_{35}, D_{36}, D_{38}, D_{39}, D_{41}, D_{42}, D_{43}, D_{44}, D_{45}, D_{49}	D_{12}, D_{14}, D_{16}, D_{22}, D_{26}
6×50	V_1	D_{15}, D_{17}, D_{19}, D_{20}, D_{21}, D_{22}, D_{26}, D_{27}, D_{29}, D_{30}, D_{31}, D_{32}, D_{33}, D_{34}, D_{36}, D_{38}, D_{39}, D_{40}, D_{41}, D_{43}, D_{44}, D_{45}, D_{46}, D_{49}	D_{20}, D_{39}
	V_2	D_{19}, D_{20}, D_{21}, D_{32}, D_{39}, D_{40}, D_{41}, D_{44}, D_{46}, D_{49}	D_{20}, D_{39}, D_{44}, D_{46}
	V_3	D_{19}, D_{20}, D_{21}, D_{32}, D_{33}, D_{36}, D_{39}, D_{40}, D_{41}, D_{44}, D_{45}, D_{46}, D_{49}	D_{20}, D_{39}

We conducted the simulations (with Eq. (5.7), Eq. (5.8), and Eq. (5.9) as simulant value functions) for each problem size one more time for another case. In this case, for each simulation, we fix the range of attributes of the random design alternatives to be ±1% around the nominal attribute levels given in Appendix-III. We found that the size of D_{NTD} (i.e., number of designs in the set) for this case is on average about half the size of D_{NTD} (see Column 3 of Table 6.4) for the case when there is ±5% variability in attributes. This indicates that in our method for selection with preference and attribute variability, the number of designs eliminated as dominated designs decrease significantly with increases in the attribute variability.

6.7. SUMMARY

In this chapter, we presented a method for product design selection with preference and attribute variability for an implicit value function. Our method assumed that the DM's implicit value function is differentiable, quasi-concave and non-decreasing with respect to the attributes. This assumption is more general and less restrictive than other popular assumptions as reported in the literature (e.g., additive value function) [Eum et al., 2001] [Lee et al., 2001].

Our method for selection with preference and attribute variability requires that the range of attributes of design alternatives be known in addition to the range of MRS preferences. We presented a mathematical formulation for eliminating dominated designs using the ranges of attributes and MRS preferences. When the MRS values are assumed consistent, this formulation can be solved without much computational burden. We presented an algorithm for selection with preference and attribute variability and demonstrated the algorithm with two engineering examples: payload design selection and cordless electric drill selection. We also provided some experimental results that numerically verified that the set of non-eliminated trial designs found by our method always includes the set of potentially optimal designs.

Our method for selection with preference and attribute variability is conservative and does not always eliminate all the dominated designs. But on the bright side, our approach does not eliminate a design that is actually potentially optimal. Also our method does not need presumed probability distributions governing the variability in MRS preferences and attributes of design alternatives since our approach for eliminating dominated designs is a worst case approach.

Note that for eliminating dominated designs using Eq. (6.1), it is important to obtain the global optimum. A local optimum for Eq. (6.1) could be negative while the global optimum is positive leading to erroneous conclusions. However, in our simulations and examples, we used “fmincon” from the MATLAB® optimization toolbox, which might converge to a local optimum, as the optimizer. We used MATLAB® to maintain uniformity with the methods developed in the previous chapters. But our experimental results indicate (recall Table 6.4) that our method never eliminated as dominated design a design that is potentially optimal according to a simulant value function. This could be due to the conservative nature of our approach for eliminating dominated designs (recall Section 6.3.1). However to be sure that only the actual dominated designs are eliminated using Eq. (6.1) one should use a global optimizer (e.g., genetic algorithm) or use different starting points to converge to the global optimum using a local optimizer (e.g., “fmincon” from the MATLAB® optimization toolbox). A better approach (and an area for future research) would be to modify the formulation in Eq. (6.1) so that it becomes a convex optimization problem.

In the next chapter we provide the conclusions for this dissertation.

CHAPTER 7

CONCLUSIONS

7.1. INTRODUCTION

This dissertation has four research components in the context of engineering product design selection with an implicit value function. In our first research component, Deterministic Selection (Chapter 3), we developed a new method that uses the DM's marginal rate of substitution (MRS) between the attributes for finding the preferred design alternative(s). In the second research component, Sensitivity Analysis for Deterministic Selection (Chapter 4), we developed a concept for finding the robustness of a set of non-eliminated trial designs to variations in DM's preference estimates. Our third research component, Selection with Preference Variability (Chapter 5), helped us produce a new method for identifying dominated designs and potentially optimal designs for the given ranges of MRS preferences. Finally, in our fourth research component, Selection with Preference and Attribute Variability (Chapter 6), we extended our method for selection with preference variability to account for variability in the attributes of design alternatives.

We presented the objectives of our research components in Chapter 1 and reviewed the previous works in Chapter 2. In Chapters 3-6, we demonstrated the application of the proposed method for each research component to a couple of engineering examples. Also in Chapter 3-6, we provided numerical experimental results to verify our proposed method for each research component.

The purpose of this chapter is to conclude this dissertation. In Section 7.2 we give concluding remarks for each research component. Next, in Section 7.3 we highlight the contributions of this research. Finally, in Section 7.4 we provide specific ideas and extensions concerning future research directions.

7.2. CONCLUDING REMARKS

In Section 7.2.1 to Section 7.2.4, we provide the concluding remarks for each of the four research components. Next, in Section 7.2.5 we give a common advantage and the common disadvantages for all of our research components. Finally, in Section 7.2.6 we give some remarks about the computational cost for the methods of each research component.

7.2.1. Research Component 1: Deterministic Selection

Our deterministic selection is iterative and requires the DM to give the marginal rate of substitution (MRS) between the attributes at a series of trial designs. The MRS preferences are used in finding the gradient of the value function at the trial designs. The gradient is then used for eliminating some lower value designs with respect to the trial designs. Our proposed deterministic selection method has the following advantages and disadvantages.

7.2.1.1. Advantages

Our deterministic selection method has the following advantages.

- Captures the DM's preferences in the form of MRS between attributes at a series of trial designs. Capturing the MRS preferences accounts for any non-monotonicity and coupling (i.e., interdependence between attributes) in the DM's value function. Such a nonlinear preference structure is common for a designer acting as a DM in engineering design selection (recall Section 3.1 of Chapter 3). Our formulation for finding the gradient coefficients allows for some inconsistency in the DM's MRS preferences and alerts the DM if the inconsistency is more than a threshold that is allowed for (recall Section 3.3.1 of Chapter 3). Also our formulation for finding the gradient coefficients does not require normalization of the attributes (see Lemma in Section 3.3.1 of Chapter 3).
- Eliminates only those designs that have lower value than the trial designs thus ensuring that the set of non-eliminated trial designs, D_{NTD}, (which is usually small) always includes the most preferred design irrespective of the starting trial design. So it will be much easier for the DM to identify (e.g., using our gradient adjacency elimination, recall Section 3.3.4 of Chapter 3, or using his/her judgment/expertise) the most preferred design from the usually small set D_{NTD} than identifying the most preferred design from the original set of design alternatives.
- Uses gradient information at all the previous trial designs in finding a new trial design. Such an effective usage of information reduces the number of iterations required in finding the most preferred design alternative as the new trial design (recall Section 3.6.2 of Chapter 3).

7.2.1.2. Disadvantages

Our deterministic selection method has the following disadvantages.

- Because of its iterative nature, our deterministic selection method might come across as tedious to a DM. Also our deterministic selection method presumes that the DM has the requisite level of expertise and consistent judgment to state the MRS preferences between attributes. Because of this presumption, our deterministic selection method, in its current state, cannot be applied to a common man's selection problem (e.g., a consumer who wants to buy a laptop but does not know much about laptops).
- The most preferred design found using our heuristic gradient adjacency elimination (when the set of non-eliminated trial designs D_{NTD} is not a singleton) might be sub-optimal (recall Section 3.3.4 of Chapter 3). I.e., our gradient adjacency elimination might eliminate as lower value design, a design which is actually the most preferred.
- In our deterministic selection method, we cannot check if the DM is giving the MRS preferences consistent with a quasi-concave value function as we move from one trial design to the other. Note, however, that one need to obtain additional information, form the DM, about the actual values of the design alternatives in order to check whether or not the DM's preferences are consistent with a quasi-concave value function.

In the next section, we give the concluding remarks of our second research component.

7.2.2. Research Component 2: Sensitivity Analysis for Deterministic Selection

Our concept for sensitivity analysis is applicable to the class of iterative selection methods that eliminate some design options at the trial design chosen for the current iteration. Such methods are generally used when the DM's value function is implicit rather than known. In our sensitivity analysis concept, we calculate three successive metrics, culminating in the robustness index for the set of non-eliminated trial designs D_{NTD}, and we identify the critical design(s). Our proposed concept for sensitivity analysis as applied to our deterministic selection method has the following advantages and disadvantages.

7.2.2.1. Advantages

Our method for sensitivity analysis for deterministic selection has the following advantages.

- Identifies critical design(s) and critical pair of attributes (recall Section 4.3.2 of Chapter 4). If the DM thinks that the critical design(s) is (are) not important, he/she can decide to make a selection from the set of non-eliminated trial designs. Otherwise, the DM can find the potentially optimal designs by assigning a range for MRS preferences. In particular, the DM can analyze how the potentially optimal designs change by assigning different ranges to the MRS preference between the critical pair of attributes.

- Identifies, using the overall elimination robustness metric, the amount of preference variation (for the given preference estimates) that would cause each eliminated design to become a member of the set of non-eliminated trial designs. Using this information, the DM can directly find the set of non-eliminated designs (i.e., without using our method for selection with preference variability) for ranges of MRS preferences that are symmetric about the preference estimates given at the trial designs in deterministic selection.
- Finds elimination robustness of a design with respect to a trial design without much computational burden when the MRS values are consistent (recall Eq. (4.4)). Thus the DM can make judgments about the robustness of the set of non-eliminated trial designs and decide what action to take in real time.

7.2.2.2. Disadvantages

Our method for sensitivity analysis for deterministic selection has the following disadvantages.

- The robustness index found by our sensitivity analysis method is the allowed preference variation for which the set of non-eliminated trial designs is not affected. However, the DM might actually want to know the robustness index of the most preferred design alternative. This is a drawback of our sensitivity analysis method and requires future research.

- Our approach for finding the robustness index is a worst case approach and restricts the variation in the MRS between all pairs of attributes at all trial designs to be the same. Also, the bounds given by the robustness index are always symmetric about the preference estimates given at the trial designs. But in reality, the ranges of preferences that the DM has in mind might not be symmetric about the preference estimates.

In the next section, we give the concluding remarks of our third research component.

7.2.3. Research Component 3: Selection with Preference Variability

Our method for selection with preference variability is iterative and requires that the DM give some constraints (e.g., ranges) on the marginal rate of substitution (MRS) between the attributes at a series of trial designs. The constraints on the MRS preferences at the trial designs are then used in eliminating some dominated designs. Our proposed method for selection with preference variability has the following advantages and disadvantages.

7.2.3.1. Advantages

Our method for selection with preference variability has the following advantages.

- Queries the DM for constraints (e.g., ranges) on the preferences which are easier to state than giving the probability distributions governing the preference variability.
- Finds the dominated designs without much computational burden. Our formulation for finding the dominated designs is a linear programming problem when, the MRS values are consistent (recall Eq. (4.4)) and the DM gives linear constraints on the MRS. Hence the DM can be presented with the set of non-eliminated trial designs in real time.
- Eliminates only those designs that are dominated with respect to the trial designs thus ensuring that the set of non-eliminated trial designs, D_{NTD}, always includes the potentially optimal designs. Hence, the DM can be sure that the most preferred design for a subset of the given ranges of preferences is always in D_{NTD}. So, the DM can make a selection directly from D_{NTD} (instead of the original set of designs) once he/she improves the preference estimates by obtaining more information about the end users' needs (recall Section 1.2.5 of Chapter 1).

7.2.3.2. Disadvantages

Our method for selection with preference variability has the following disadvantages.

- Our approach for eliminating dominated designs is conservative. Because of which, the set of non-eliminated trial designs might contain some designs that are actually dominated.

- Our heuristic approach (recall Section 5.3.3 of Chapter 5) does not eliminate all dominated designs from the set of non-eliminated trial designs. Also, the formulation for our heuristic approach is non-convex and is computationally expensive.
- As the variability in DM's preferences increase, the number of designs dominated by a trial design might decrease thus resulting in an increase in the number of iterations to find the set of non-eliminated trial designs. This might become tedious for the DM.

In the next section, we give the concluding remarks of our fourth research component.

7.2.4. Research Component 4: Selection with Preference and Attribute Variability

Our method for selection with preference and attribute variability is iterative and requires that the range of attributes of design alternatives be known in addition to the range of MRS preferences between the attributes at a series of trial designs. The range of MRS preferences and the range of attributes are then used in eliminating some dominated designs. Our proposed method for selection with preference and attribute variability has the following advantages and disadvantages.

7.2.4.1. Advantages

Our method for selection with preference and attribute variability has the following advantages.

- Queries the DM for ranges of preferences and ranges of attributes which are easier to state than giving the probability distributions governing the preference variability and attribute variability.
- Finds the dominated designs without much computational burden. Our formulation for finding the dominated designs has a nonlinear objective function with linear constrains when the MRS values are consistent (recall Eq. (4.4)). Hence the DM can be presented with the set of non-eliminated trial designs in real time.
- Eliminates only those designs that are dominated with respect to the trial designs thus ensuring that the set of non-eliminated trial designs, D_{NTD}, always includes the potentially optimal designs. Hence, the DM can be sure that the most preferred design for a subset of the given ranges of preferences and the given ranges of attributes is always in D_{NTD}. So, the DM can make a selection directly from D_{NTD} (instead of the original set of designs) once he/she improves the preference estimates and the attribute estimates by obtaining more information (recall Section 1.2.5 of Chapter 1).

7.2.4.2. Disadvantages

Our method for selection with preference and attribute variability has the following disadvantages.

- Our approach for eliminating dominated designs is conservative. Because of which, the set of non-eliminated trial designs might contain some designs that are actually dominated.

- As the variability in DM's preferences or variability in attributes increase, the number of designs dominated by a trial design might decrease thus resulting in an increase in the number of iterations to find the set of non-eliminated trial designs. This might become tedious for the DM.

In the next section, we give a common advantage and the common disadvantages of all our research components.

7.2.5. Common Advantage and Disadvantages of All Research Components

Our research components have the following common advantage.

- Our methods in each research component account for an implicit value function that is non-decreasing, differentiable, and quasi-concave with respect to the attributes. An implicit quasi-concave value function is more general [Malakooti, 1988] and less restrictive than other popular assumptions for the DM's value function as reported in the literature (e.g., additive value function). Our first research component, Deterministic Selection, is applicable even when the DM's value function is non-decreasing.

Our research components have the following common disadvantages.

- Because we assume that the DM's implicit value function is differentiable, our methods in each research component cannot be applied when the attributes are discrete or when then the DM's value function is not differentiable. However, it is not uncommon in engineering design selection to have discrete attributes, e.g., color of an automobile.

- Even though assuming that the DM's value function is non-decreasing, differentiable, and quasi-concave, is more general than other popular assumptions, there is no evidence to suggest that, in practice, the DM's value function is always quasi-concave. If the DM's value function is not quasi-concave, then the preferred design(s) found by our methods in each research component might be erroneous.

In the next section, we give some remarks about the computational cost of the methods in each research component.

7.2.6. Remarks on the Computational Cost of the Research Components

In this section, we provide some remarks about how the computational cost of the methods in each research component depends on the number of design alternatives 'n'.

- In deterministic selection, at each iteration, we need to solve the optimization problem in Eq. (3.6) or Eq. (3.10) for finding the gradient coefficients at the trial design for that iteration. Once the gradient coefficients are found, gradient cut elimination and gradient adjacency elimination can be applied with out much computational burden irrespective of the number of design alternatives. Computational time taken for solving Eq. (3.6) or Eq. (3.10) depends on the number of attributes and the consistency of the DM's MRS preferences.

- In sensitivity analysis for deterministic selection, at each iteration, we need to solve the optimization problem in Eq. (4.3) for each eliminated design to find the elimination robustness of that design. I.e., if ‘c’ is total number of iterations and ‘t’ is the average number of designs eliminated at each iteration ($t < n$), we need to solve Eq. (4.3) ‘$c \cdot t$’ times. So the computational burden increases linearly with the number of design alternatives in sensitivity analysis.
- In selection with preference variability, at each iteration, we need to solve the optimization problem in Eq. (5.1) for checking whether or not a design alternative is dominated by the trial design for that iteration. I.e., for the first iteration, Eq. (5.1) has to be solved ‘n-1’. For the subsequent iterations, Eq. (5.1) has to be solved for less than or equal to ‘n-1’ times because some designs might be eliminated in the previous iterations. So, at most, Eq. (5.1) has to be solved ‘$c \cdot (n-1)$’ times, where ‘c’ is total number of iterations. Hence the computational cost for finding the set of non-eliminated trial designs, D_{NTD}, increases linearly with the number of design alternatives in selection with preference variability. However, our heuristic approach (recall Eq. (5.4) and Eq. (5.5)) is computationally expensive because it involves solving non-convex optimization problems. Also in our heuristic approach we apply the tests of Eq. (5.2) and Eq. (5.3) to all ordered pairs (recall Section 5.3.3) of non-eliminated trial designs. Because of this the computational burden in applying the heuristic approach increases quadratically with the number of designs in D_{NTD}.

- In selection with preference and attribute variability, at each iteration, we need to solve the optimization problem in Eq. (6.1) for checking whether or not a design alternative is dominated by the trial design for that iteration. I.e., for the first iteration, Eq. (6.1) has to be solved 'n-1' times. For the subsequent iterations, Eq. (6.1) has to be solved for less than or equal to 'n-1' times because some designs might be eliminated in the previous iterations. So, at most, Eq. (6.1) has to be solved 'c·(n-1)' times, where 'c' is total number of iterations. Hence the computational cost for finding the set of non-eliminated trial designs, D_{NTD}, increases linearly with the number of design alternatives in selection with preference and attribute variability.

In the next section, we discuss the contributions of this dissertation.

7.3. CONTRIBUTIONS

The contributions of the research presented in this dissertation are summarized below.

- Developed a first of its kind formal decision making framework for product design selection in that the DM's value function is not presumed explicitly and both preference and attribute variability are accounted for. This decision making framework is applicable when the DM's value function is non-decreasing, differentiable and quasi-concave with respect to the attributes.
- Developed a new mathematical formulation that does not need normalization of attributes for finding the gradient of the DM's implicit value function using

marginal rate of substitution (MRS) between attributes. We showed that normalization of attributes, hence the normalization of gradient coefficients, is not necessary for eliminating lower value designs using the gradient cut.

- Developed novel heuristic approaches in Chapter 3 and Chapter 5 that make use of already existing gradient information at the trial designs for eliminating more designs from the set of non-eliminated trial designs.
- Introduced the concept of a robustness index for measuring the allowed variation in the preference estimates for which the set of non-eliminated trial designs is not affected when the DM's value function is implicit. Such a concept for robustness index exists in the literature when the DM's value function is presumed explicitly. Our concept for robustness index is the first such concept when the DM's value function is implicit.
- Developed a novel approach, based on the gradient cut notion, for eliminating dominated designs when the DM's value function is implicit and when there is preference variability or both preference and attribute variability. We presented mathematical formulation for identifying the dominated designs without finding the actual gradient range for the given range of preferences (and attributes when attribute variability is also present).

In the next section, we give suggestions for future research.

7.4. FUTURE RESEARCH DIRECTIONS

The research presented in this dissertation addresses a variety of situations for product design selection with an implicit value function. However, there are many important research issues left unresolved. In this section, we briefly discuss some of these issues and provide some general research directions to address them. Some of the discussions presented here are based on currently known shortcomings of our proposed methods (summarized in Section 7.2).

7.4.1. Robustness Index of the Set of Non-eliminated Trial Designs to Variations in the Attribute Levels of Design Alternatives

In Chapter 4, we proposed a concept for sensitivity analysis for deterministic selection. In that concept, we find the robustness index of the set of non-eliminated trial designs, D_{NTD}, to variations in the preference estimates given by the DM. However, as mentioned in Chapter 6, in addition to preference variability, it is quite common in engineering design selection to have attribute variability. In this section, we suggest a method for finding the robustness index of D_{NTD} to variation in the attribute levels of design alternatives.

For finding the robustness index of D_{NTD} to variations in the attribute levels of design alternatives, we propose to calculate three successive metrics culminating in the robustness index.

Let D_+ be an arbitrary design belonging to the original set of designs. Let D_T be the current trial design in our deterministic selection method (recall Figure 3.1), and let ∇V_T be the gradient of the value function at D_T (see Figure 7.1(a)). In Figure 7.1(a), for the estimated attribute levels of D_+: $[A_{1+}, A_{2+}]$ and D_T: $[A_{1T}, A_{2T}]$, D_+ lies in the gradient cut of D_T. However, if the actual attribute levels of D_+ become $\left[A^t_{1+},\ldots,A^t_{m+}\right]$ (assume for simplicity that the attributes of D_T do not vary) then D_+ no longer lies in the gradient cut at D_T (see Figure 7.1(a)). Also, if the actual attribute levels of D_T become $\left[A^t_{1T},\ldots,A^t_{mT}\right]$ then D_+ (even with the estimated attribute levels) will no longer lie in the gradient cut at D_T (see Figure 7.1(b)).

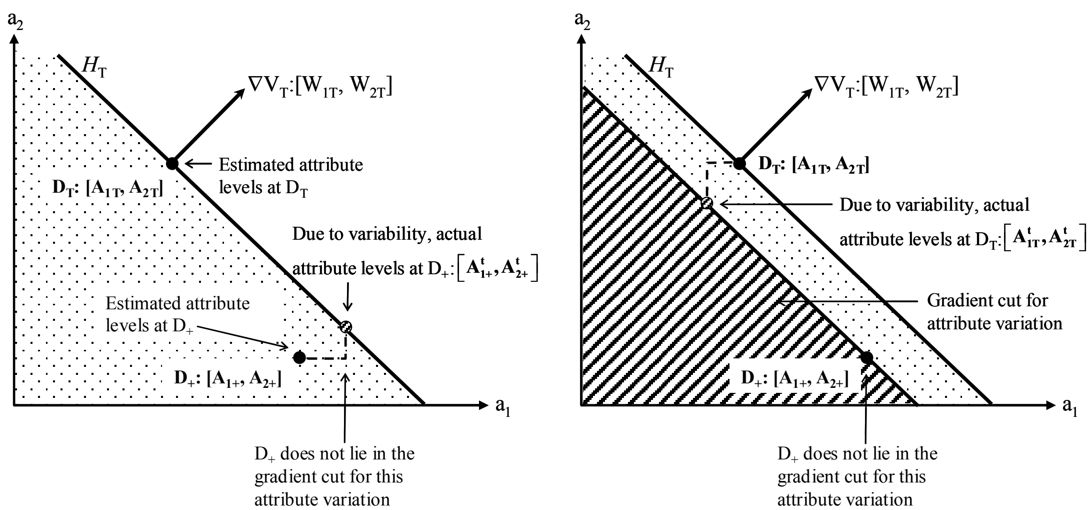

Figure 7.1: Illustration of attribute elimination robustness of D_+ with respect to D_T

Our first metric ξ_{+T}, which we call attribute elimination robustness of design D_+ with respect to trial design D_T, is defined as the *smallest* variation in the attribute levels of D_+ and D_T for which D_T does not eliminate D_+. By variation we mean difference between the estimated attribute level and the actual attribute level. Here, ξ_{+T} is dimensionless and is expressed as a fraction of the estimated attribute levels. We propose to use the formulation in Eq. (7.1) for finding ξ_{+T}. In Eq. (7.1), a_{i+} and a_{iT} are the variables, W_{iT} (i=1,...,m) are the gradient coefficients corresponding to ∇V_T, and A_{i+} and A_{iT} are the estimated attribute values at D_+ and D_T respectively.

$$\text{Minimize} \quad \xi_{+T} \tag{7.1a}$$

$$\text{subject to:} \sum_{i=1}^{m} W_{iT} \cdot (a_{i+} - a_{iT}) \geq 0 \tag{7.1b}$$

$$(1-\xi_{+T}) \cdot A_{iT} \leq a_{iT} \leq (1+\xi_{+T}) \cdot A_{iT}; \text{'m' such constraints} \tag{7.1c}$$

$$(1-\xi_{+T}) \cdot A_{i+} \leq a_{i+} \leq (1+\xi_{+T}) \cdot A_{i+}; \text{'m' such constraints} \tag{7.1d}$$

$$0 \leq \xi_{+T} \leq \left(\frac{R}{\sqrt{2}}\right) \tag{7.1e}$$

Eq. (7.1b) is used to check that D_+ is not in the gradient cut of D_T corresponding to the attribute levels, a_{i+} and a_{iT}, of D_+ and D_T respectively. Eq. (7.1c) is to check that a_{iT} (i=1,…,m) are within the bounds, given by ξ_{+T}, of the estimated attribute levels at D_T. Eq. (7.1d) imposes a similar constraint on a_{i+}. Also, if the lower bound in Eq. (7.1c) and in Eq. (7.1d) becomes negative, we set it equal to zero. Eq. (7.1e) is a constraint imposed on ξ_{+T}. In Eq. (7.1e), R is the radius of region O_T around D_T in which the value function is approximated to be linear. Recall (see Section 3.3.4 of Chapter 3) that the MRS preferences given at D_T are valid only in the region O_T. The upper bound on ξ_{+T} in Eq. (7.1) ensures that the attribute variations at D_T are within the region O_T. If a feasible solution for Eq. (7.1) does not exist then we propose to set ξ_{+T} as $\left(R/\sqrt{2}\right)$. Note that for each D_T (i.e., at each iteration), ξ_{+T} is calculated for each design D_+ in the input set.

The second metric ξ_{+max} we call the overall attribute elimination robustness of a design D_+. ξ_{+max} is the *largest* of the ξ_{+T}'s for D_+ over all D_T's. Thus, so long as the variation in *every* attribute level is less than ξ_{+max} at *all* trial designs and D_+, D_+ will be eliminated by at least one trial design.

The final metric is attribute robustness index, ξ, which is the *minimum* of all the ξ_{+max}'s. All designs not in D_{NTD} remain eliminated so long as the variation in *every* attribute level is less than ξ at all design alternatives.

Using the attribute robustness index, the DM can then decide whether to make a selection from D_{NTD} (e.g., using gradient adjacency elimination, recall Section 3.3.4 of Chapter 3) or take some other step (e.g., finding potentially optimal designs with a range of attributes for design alternatives).

In the next section, we discuss our next future research direction.

7.4.2. Extensions to Our Method for Selection with Preference Variability

In Section 5.7 of Chapter 5, we mentioned that our formulation for eliminating dominated designs in selection with preference variability can accommodate constraints other than the ranges on the MRS preferences. In this section, we suggest an extension to the formulation in Eq. (5.1) for accommodating other type of constraints on the MRS preferences.

It is not necessary that the DM can always state a range of preferences to account for preference variability. Sometimes (due to lack of information) the DM might give some other type of constraints on the MRS preferences. For example, in payload design selection, the DM might say: "I would give up more in the probability of success of scenario 2, P_{S2}, than in the probability of success of scenario 3, P_{S3}, to gain an increase of 0.1 in the probability of success of scenario 1, P_{S1}". Such a response means that the DM's MRS preference between P_{S1} and P_{S2} (i.e., S_{12}) is greater than his/her MRS preference between P_{S1} and P_{S3} (i.e., S_{13}). Also, it is possible that the DM can provide a crisp (or deterministic) MRS between some attributes and give some constraints for the MRS between other attributes. Next, we provide an example for modifying the formulation in Eq. (5.1) for different types of constraints on the MRS.

Example: Consider the payload design selection problem with the probability of success, P_S, in five scenarios as the attributes, i.e., P_{Si} (i= 1 to 5) are the attributes. Due to lack of information on futuristic scenarios the DM can only give the MRS between P_{S5} and P_{S1}, i.e., S_{51}, with certainty as 0.2. For P_{S1} and P_{S2}, the DM says that the MRS S_{12} is between 0.12 and 0.18. For P_{S2} and P_{S3}, the DM says that MRS S_{23} is between 0.08 and 0.10. The DM also says the MRS between P_{S3} and P_{S4}, i.e., S_{34}, is always greater than the MRS between P_{S4} and P_{S5}, i.e., S_{45}. With these preferences, modify the formulation in Eq. (5.1) for eliminating dominated designs.

Solution: Let w_{iT} (i=1,…,5) be the coefficients of the variable gradient at a trial design D_T: [P_{S1T},…,P_{S5T}]. Let D_+: [P_{S1+},…,P_{S5+}] be an arbitrary design that belongs to the original set of designs. Assuming that the MRS values, s_{ijT}, are consistent (recall Eq. (4.4)), if the maximum value of Z^* in Eq. (7.2) is negative then we can conclude that design D_+ is dominated by D_T. In Eq. (7.2), w_{iT}, (i=1,…,5) are the variables.

$$\text{Maximize} \quad Z^* = \sum_{i=1}^{5} w_{iT} \cdot (P_{Si+} - P_{SiT}) \tag{7.2a}$$

$$\text{subject to}: \sum_{i=1}^{m} w_{iT} = 1; \quad w_{iT} \geq 0 \tag{7.2b}$$

$$0.12 \leq \frac{w_{1T}}{w_{2T}} \leq 0.18 \tag{7.2c}$$

$$0.08 \le \frac{w_{2T}}{w_{3T}} \le 0.10 \tag{7.2d}$$

$$\frac{w_{3T}}{w_{4T}} \ge \frac{w_{4T}}{w_{5T}} \tag{7.2e}$$

$$\frac{w_{4T}}{w_{5T}} = 0.2 \tag{7.2f}$$

Eq. (7.2) is a simple extension of Eq. (5.1). However, for the given constraints on MRS preferences, the formulation in Eq. (7.2) is no longer linear (see Eq. (7.2e)). Note that, similar extensions can be done to the formulation in Eq. (6.1) for eliminating dominated designs in selection with preference and attribute variability.

We mentioned earlier (recall Section 5.7) that our method for selection with preference variability does not need probability distributions governing the MRS preferences. However, in addition to the range of MRS preferences, if the DM can provide the probability distributions (with in the given range) of the MRS preferences, our method can be extended as follows for finding the preferred design(s). Since the designs not in the set of non-eliminated trial designs D_{NTD} are dominated irrespective of the probability distributions for the given ranges of MRS preferences, D_{NTD} can be used as the set of designs from which the selection has to be made. Pick a design D_+ from D_{NTD} and then conduct Monte Carlo runs. At the beginning of the Monte Carlo runs, assign a number called likelihood of elimination to each design belonging to D_{NTD} and set it to zero. In each Monte Carlo run, sample the MRS preferences at D_+ from the given probability distributions and then find the gradient corresponding to the sampled MRS preferences. If a design belonging to D_{NTD} lies in the gradient cut corresponding to the sampled MRS preferences at D_+, then increase the likelihood of elimination of that design by one. After completing the stipulated Monte Carlo runs, compare the likelihood of elimination of a design belonging to D_{NTD} with the threshold likelihood (a percentage of the number of simulation runs) specified by the DM and eliminate that design if the likelihood is greater than the threshold. The design(s) that are not eliminated after conducting the Monte Carlo runs at all members of D_{NTD} would then be the preferred design(s).

In the next section, we discuss our next future research direction.

7.4.3. Bayesian Statistics for Predicting the Actual Values of Design Alternatives

We mentioned earlier (recall Section 7.3.1.2) that our heuristic gradient adjacency elimination approach, which is used for selecting from the set of non-eliminated trial designs, D_{NTD}, might sometimes result in a sub-optimal most preferred design. An interesting research issue is to consider using Bayesian statistics for predicting the actual values of the designs in D_{NTD} and then pick the design with the highest predicted value as the most preferred design. Using gradient cut elimination, at each iteration of our deterministic selection method (recall Section 3.3.2 of Chapter 3), we obtain some information about the relative ranking of the designs. If this information can be used to predict the actual value using Bayesian statistics then heuristics like gradient adjacency elimination can be avoided.

In the next section, we discuss our next future research direction.

7.4.4. Heuristic for Selection with Preference and Attribute Variability

The output of our method for selection with preference and attribute variability contains a number of designs that are actually dominated. The possibility of developing a heuristic to reduce the set of non-eliminated trial designs to the set of potentially optimal designs should be investigated.

In the next section, we discuss our next future research direction.

7.4.5. Selection with Multiple Decision Makers

An important issue that has been not addressed in this dissertation is that product design selection often involves multiple decision makers (DMs) instead of a single DM. In selection with multiple DMs, researchers acknowledge that it is difficult to find a design that satisfies the preferences of all the DMs. So the task in selection with multiple DMs is to find a compromise solution. Our deterministic selection method can be readily extended for eliminating those designs that have lower value according to all the DMs (see Figure 7.2). The challenge, however, is to reduce the set of non-eliminated designs to the compromise solution and this challenge requires future investigation.

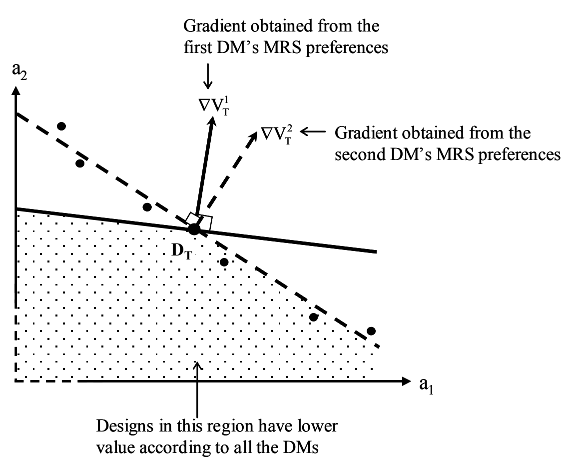

Figure 7.2: Illustration of eliminating lower value designs according to all DMs

In the next section, we discuss our next future research direction.

7.4.6. Multi-Attribute Multi-Disciplinary Selection

In our deterministic selection method, we assume that the DM can state the MRS between any two attributes. However it might be difficult for the DM to state the preferences if the two attributes belong to separate disciplines. For example, in automobile design selection, it would be extremely difficult for the DM to compare between the gear ratio of the transmission and the passenger leg room inside the automobile. Stating the MRS preferences would be considerably simpler for the DM if the attributes with some similarities were grouped into disciplines. For instance, in automobile design selection, attributes like passenger leg room and dashboard display can be grouped into an interior of the automobile discipline. Grouping similar attributes into disciplines will result in a two-level selection (disciplines in the upper-level and attributes of a discipline in the lower-level) rather than a single-level selection. We call such a two-level selection problem: Multi-Attribute Multi-Disciplinary Selection. Some methods exist in the literature for two-level selection when the DM's value function is explicitly known (e.g., analytical hierarchy process [Saaty, 1980]). However, to the best of our knowledge, no method exists in the literature for two-level selection with an implicit value function.

APPENDIX-I

DESCRIPTION OF PAYLOAD DESIGN OPTIMIZATION PROBLEM AND DEMONSTRATION OF SOFTWARE FOR DETERMINISTIC SELECTION

In this appendix, we first provide the description of the optimization problem for payload design of an undersea autonomous vehicle in Section A.I-1 and then discuss the software we developed for payload design optimization and deterministic selection in Section A.I-2.

A.I-1. DESCRIPTION OF THE OPTIMIZATION PROBLEM FOR PAYLOAD DESIGN OF AN UNDERSEA AUTONOMOUS VEHICLE (UAV)

The original formulation for the payload design optimization problem can be found in [Gunawan, 2004]. Typically, the payload of a UAV must be effective in several different uses, called "scenarios". Effectiveness in a scenario is measured by the probability of success, P_S, of payload delivery in that scenario. The design goal is to simultaneously maximize the individual P_S's for all scenarios. The payload design is constrained by upper limits on the weight of the payload and on the radiated noise generated by the payload.

There are six design variables: the payload length (PL), the hull diameter (DH), the material of the hull (HM), the payload type (PT), the first inner material type (I1), and the second inner material type (I2). Four of the variables are discrete: HM, PT, I1, and I2. The choices for HM, PT and I1 are [6061AL, 7075AL], [BULK, MULTI_MISS], and

[TYPE_1A, TYPE_1B], respectively. For the discrete variable I2, the options available are [TYPE_2A, TYPE_2B, TYPE_1B], but I2 can be TYPE_1B only if the variable I1 is TYPE_1B also. The other two variables are continuous and they are bounded as: $6.0 \le DH \le 12.75$ and $1.0(DH) \le PL \le 5.0(DH)$. In addition to the six design variables, there is a fixed continuous design parameter, the maximum depth (= 3000 ft), at which the payload operates. There are no closed-form relationships to map the design variables to the constraints and to the P_S's. Rather, we are provided with a design analyzer (a computer program) that maps the design variables to the payload weight, the radiated noise, and the P_S's for the scenarios.

For the example in Section 3.5.1 of Chapter 3, we address a two objective payload design optimization with two constraints. The two objectives are to maximize P_{S1} and P_{S2} for two different scenarios (typical names of the scenarios are ASW Small, ASW Medium, ASW Large, ATT Small, ATT Medium, and ATT Large). The two constraints are an 85 lb upper bound on the payload weight and a 0.16 Watt/m^2 upper bound on the radiated noise generated. The problem is then mathematically formulated as follows.

$$\text{Maximize} \quad P_{S1}\left(\text{PL, DH, HM, PT, I1, I2}\right) \tag{A.I-1a}$$

$$\text{Maximize} \quad P_{S2}\left(\text{PL, DH, HM, PT, I1, I2}\right) \tag{A.I-1b}$$

$$\text{subject to: Weight}\left(\text{PL, DH, HM, PT, I1, I2}\right) - 85 \le 0 \tag{A.I-1c}$$

$$\text{Noise}\left(\text{PL, DH, HM, PT, I1, I2}\right) - 0.16 \le 0 \tag{A.I-1d}$$

The Pareto optima obtained by solving the formulation in Eq. (A.I-1) using a Multi Objective Genetic Algorithm (refer [Gunawan et al., 2003] for details) is then used as the set of design alternatives, with P_{Si} as the attributes, for selection in Section 3.5.1of Chapter 3.

In the next section, we discuss the software we developed for payload design optimization and deterministic selection.

A.I-2. SOFTWARE FOR PAYLOAD DESIGN OPTIMIZATION AND DETERMINISTIC SELECTION

Figure A.I-1 shows the flowchart of the software we developed in MATLAB® for payload design optimization and selection. The software has interfaces for various stages of the optimization and selection process. In our software, the DM first chooses the scenarios for optimization. Next, the DM chooses the optimizer that he/she wants to use for the optimization. Then the DM sets the objectives, constraints, and other parameters for the optimizer. The optimizer then generates the Pareto optima which are used as the design alternatives for selection, with the objectives as attributes. The DM then starts the selection process by invoking our deterministic selection method (recall Chapter 3).

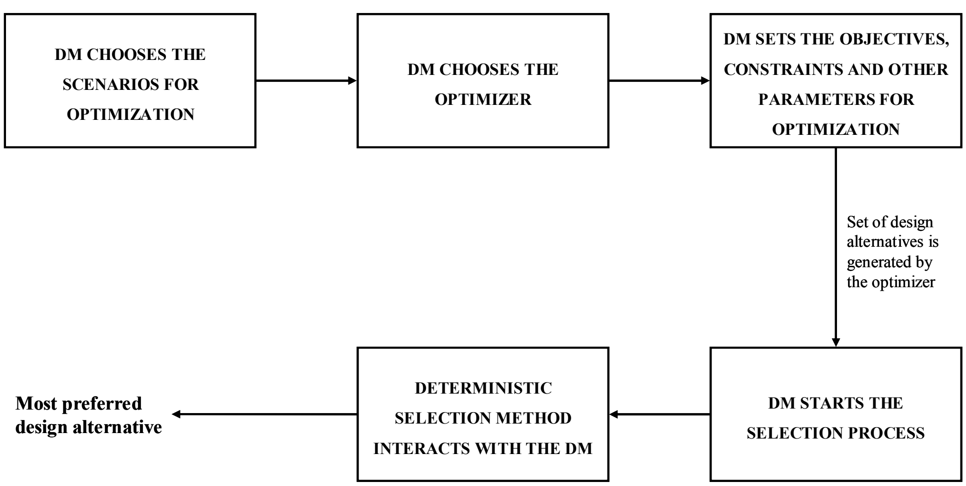

Figure A.I-1: Flowchart of our software for payload design optimization and deterministic selection

At present, our software has interfaces only for the deterministic selection method. The interface development for our other research components, sensitivity analysis method (recall Chapter 4), selection with preference variability (recall Chapter 5), and selection with preference and attribute variability (recall Chapter 6) are left out for future work. Dr. Gunawan [Gunawan, 200] has developed the background codes (i.e., not the interfaces) for the optimization part of the payload design.

In the next section, we demonstrate our software with a simple example.

A.I-2.1. Demonstration of Software with an Example

In this section, we demonstrate our software with an example. Specifically, we provide some snapshots of the interfaces for payload design optimization and selection using the software we developed.

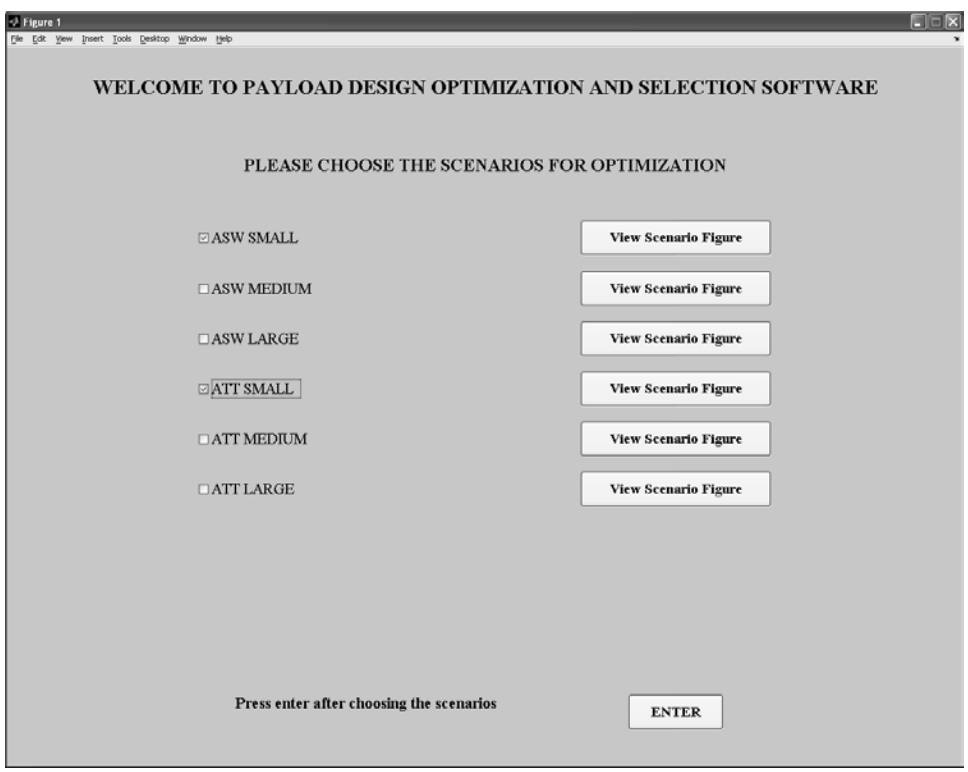

Figure A.I-2: Interface for choosing the scenarios

As mentioned earlier, in our software, the DM first chooses the scenarios for optimization. Figure A.I-2, shows the interface with list of scenarios available for the DM. The button, “View Scenario Figure” corresponding to each scenario name, when clicked on shows the DM the figure of that scenario. In our example, the DM chooses scenarios ASW Small and ATT Small as the scenarios to optimize for and presses the button “Enter”.

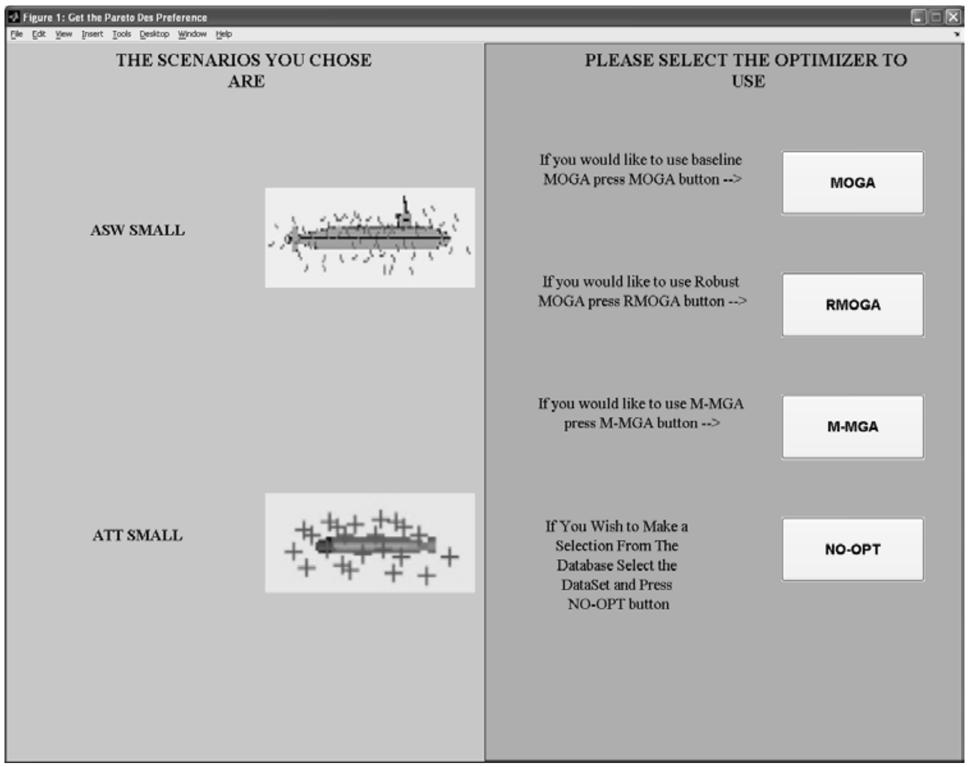

Figure A.I-3: Interface for optimizer selection

Our software then generates the interface with the chosen scenarios on the left hand side and the list of available optimizers on the right hand side (see Figure A.I-3). In our example, the DM chooses ASW Small and ATT Small as the scenarios. The left hand side of Figure A.I-3 shows the scenario names and the corresponding figures. The DM has different choices of optimizers to choose from. The choices are: a baseline Multi-Objective Genetic Algorithm (MOGA) with no uncertainty handling capability; a

Robust Multi-Objective Genetic Algorithm (RMOGA) with capability to handle uncertainty; a Multi-Objective Multi-Disciplinary Genetic Algorithm (M-MGA) for optimization with multiple disciplines; and selection from existing set of design alternatives with no optimization (NO-OPT). Currently, only the option of baseline MOGA is in working condition. So perforce, for our example, the DM chooses MOGA as the optimizer.

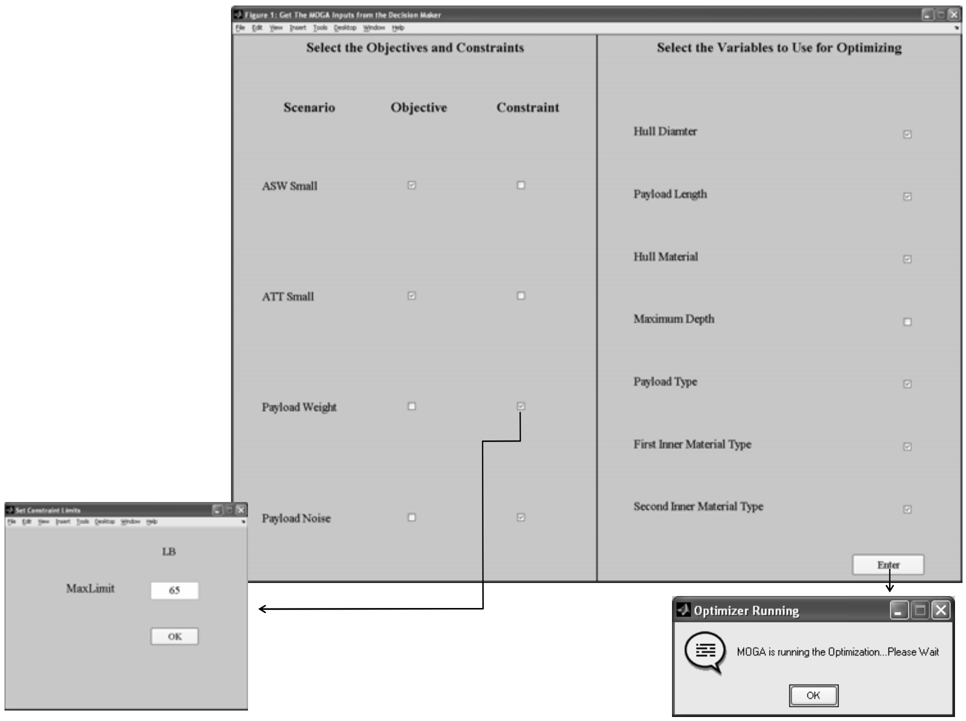

Figure A.I-4: Interface for choosing the parameters of optimization

Next our software generates the interface for obtaining the inputs to the optimizer, MOGA (see Figure A.I-4). Using this interface, the DM can select the objectives, the constraints, and the design variables for optimization. In this interface, the left hand side provides the DM with the options for the objectives and constraints. In payload design selection, the probability of success (P_S) in a scenario is usually an objective. However, the DM has an option to set the P_S in a scenario as a constraint also. The DM can also choose whether, the weight of the payload and the noise generated by the payload, are objectives or constraints or both. For our example, the DM chooses the P_S of ASW Small and the P_S of ATT Small as the objectives by checking the boxes (with a ✓ mark, see Figure A.I-4). The DM then chooses the weight of the payload as a constraint. Our software then generates an interface (see Figure A.I-4) for the DM to enter the maximum limit on the weight of the payload. The DM enters 65 lbs in our example. In our example, the DM also chooses the noise generated by the payload as a constraint and enters the maximum allowable noise as 0.16 Watt/m^2.

The right hand side of Figure A.I-4 provides the DM with the options for the variables of optimization. As mentioned in Section A.I-1, in payload design selection, the hull diameter (DH), the payload length (PL), the material of the hull (HM), the payload type (PT), the first inner material type (I1), and the second inner material type (I2) are typically the variables. So our software checks the boxes (with a ✓ mark) corresponding to DH, PL, HM, PT, I1, and I2 by default, thus considering them as variables. The DM can uncheck any of the boxes if he/she does not want to consider the corresponding property as a variable. Also, as mentioned in Section A.I-1, the property maximum depth at which the payload operates is a parameter and usually set to 3000 ft. However, the DM can choose the maximum depth as a variable by checking the box corresponding to it. In our example, the DM opts to leave maximum depth as a parameter thus choosing DH, PL, HM, PT, I1, and I2 as variables.

The DM then presses the button “Enter”. Our software invokes the optimizer MOGA for generating the Pareto optimum designs for the inputs given by the DM. Since MOGA takes a few minutes for generating the Pareto optima, our software generates a message box saying that: “MOGA is running the Optimization… Please Wait” (see Figure A.I-4).

After MOGA finishes generating the Pareto optimum designs, our software displays the design alternatives in the attribute space (recall that probability of success in the scenarios, P_{Si}, are the attributes in payload design selection). If the attributes are more than two, our software uses bar charts to display the designs. However, in our example, the DM chooses only two scenarios. Hence the number of attributes is two and our software shows the Pareto optimum design alternatives in the two attribute space as shown in Figure A.I-5. In our example, MOGA generate 31 designs as the Pareto optimum designs.

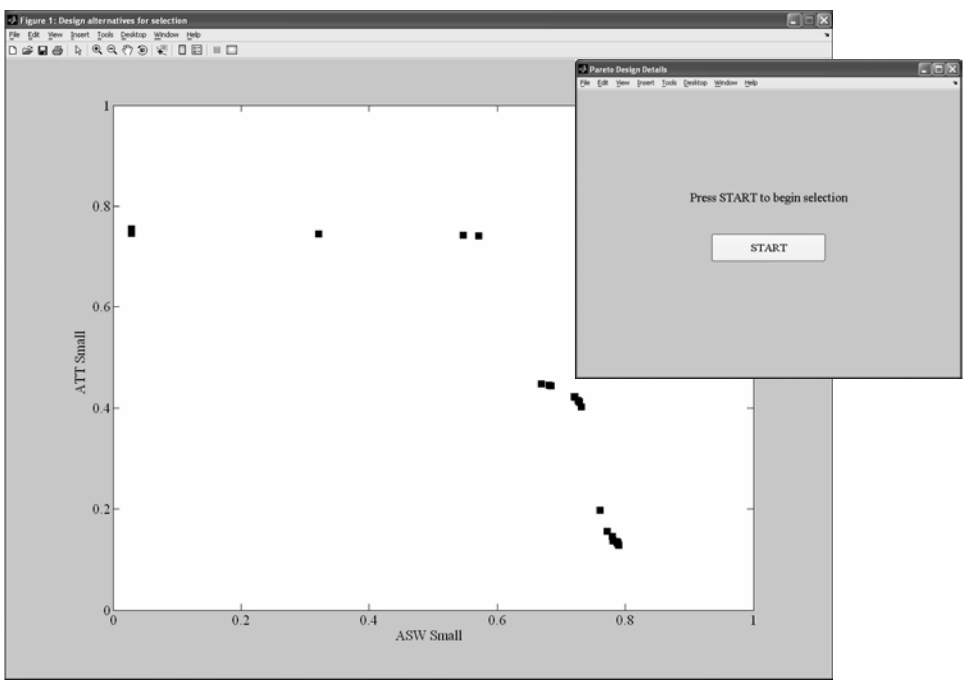

Figure A.I-5: Interface to display Pareto optimum designs in the attribute space

Our software then asks the DM to start the process of selecting from the Pareto optimum design alternatives by pressing the button “Start” (see Figure A.I-5). Once the

DM presses the "Start" button, our software invokes our deterministic selection method (recall Chapter 3).

Recall that *Step* 1 of our algorithm for deterministic selection is to choose a trial design for the first iteration from the set of design alternatives (recall Section 3.4 of Chapter 3). Our software, displays the current trial design on the left hand side of the next interface (see Figure A.I-6). The trial design is displayed in the attribute space using a bar chart. However, the DM can see the values for the variables of the trial design by pressing the button "ViewDes" just below the bar chart (see Figure A.I-6). Note the scenarios are numbered in the order displayed in the list provided in Figure A.I-2. For our example, ASW Small is Scenario 1, and ATT Small is Scenario 2.

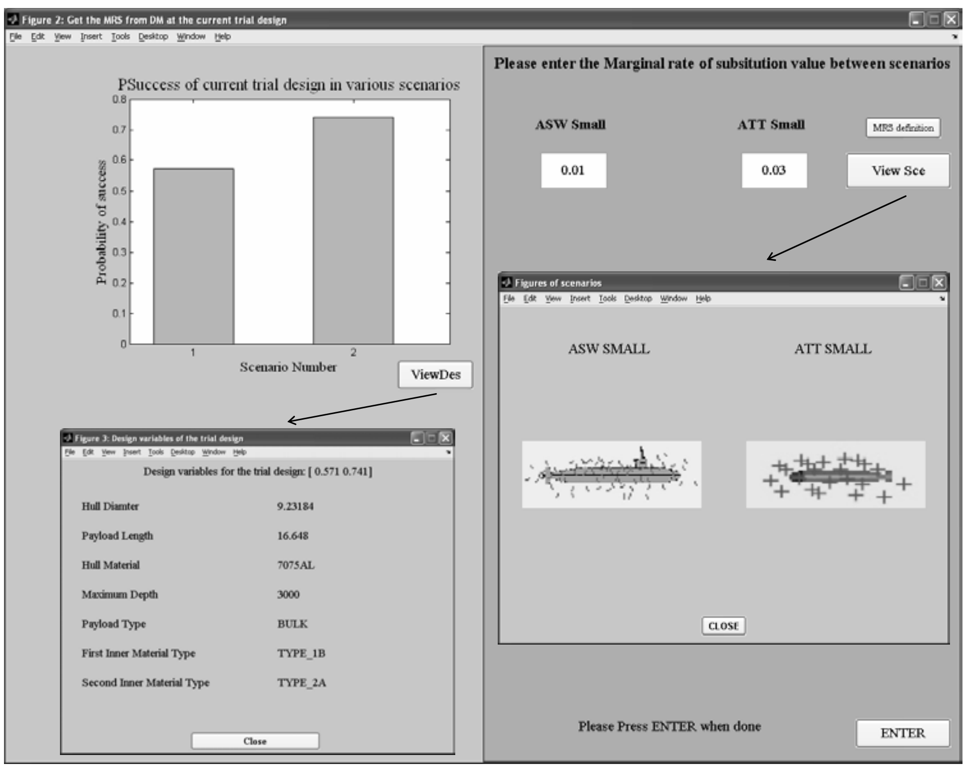

Figure A.I-6: Interface for obtaining MRS preferences at the first trial design

The right hand side of the interface in Figure A.I-6 asks the DM to enter the marginal rate of substitution (MRS) between the attributes (i.e., P_S in a scenario). Recall that in our deterministic selection method (Chapter 3), we ask the DM for 'm' MRS questions when there are 'm>2' number of attributes. However, when there are only two attributes, as in our example, we ask for only one MRS. Using the interface in Figure A.I-6, the DM can provide the trade-offs in both the attributes while stating the MRS. When the DM presses the button "View Sce", our software generates an interface with the pictures of the attributes (i.e., the scenarios for payload design) the DM is comparing. The DM can see the definition of MRS by pressing the button "MRS definition" (above the button "View Sce" in Figure A.I-6). For our example, the DM says: "I would give up 0.03 in the P_S of ATT Small to gain 0.01 in the P_S of ASW Small" (see Figure A.I-6). I.e., the MRS between the attributes at the first trial design D_{T1} is, $S_{12T1} = 3$ (*Step* 3 of our algorithm, recall Section 3.4 of Chapter 3).

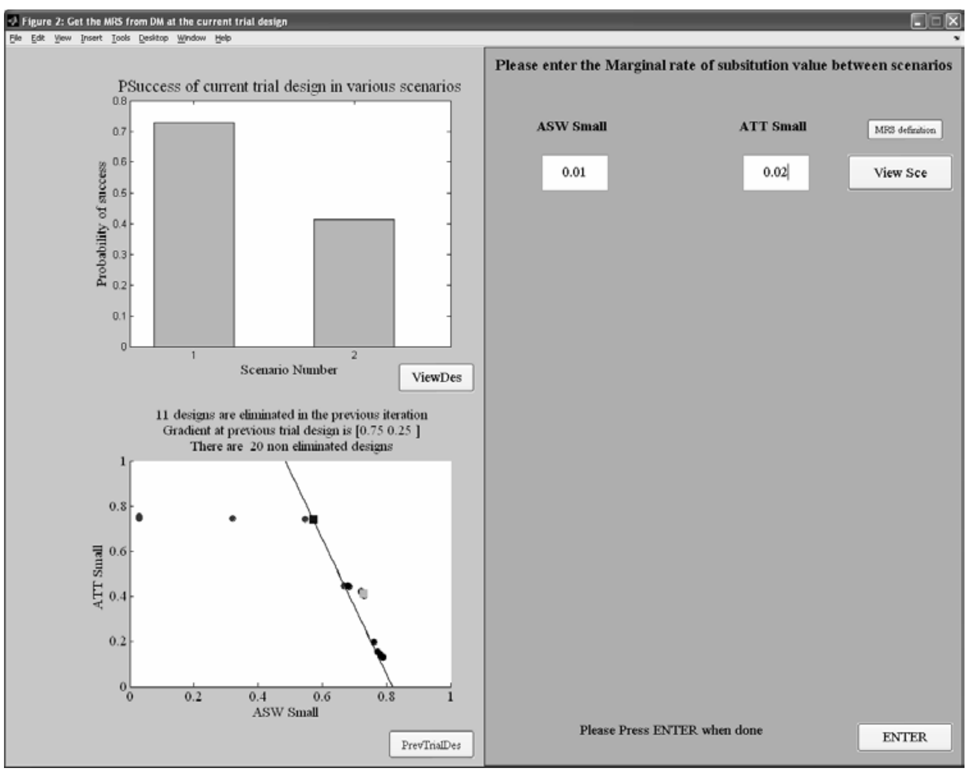

Figure A.I-7: Interface for obtaining the MRS preferences at the current trial design

Our algorithm for deterministic selection then finds the gradient at the first trial design (i.e., D_{T1}) and eliminates lower value designs using gradient cut (recall Section 3.4 of Chapter 3). If all the designs except one are eliminated then the non-eliminated trial design would be the most preferred design. Otherwise, our algorithm for deterministic selection finds a new trial design for the next iteration. In our example, gradient at D_{T1} is ∇V_{T1}: [0.75, 0.25] and eleven design alternatives are eliminated using the gradient cut. Our software shows these data and also the new trial design D_{T2} on the left hand side of the next interface (see Figure A.I-7).

In Figure A.I-7, the bar chart on the top is the new trial design in the attribute space and the chart on the bottom is the illustration of the gradient cut at the previous trial design (i.e., D_{T1}). Note that gradient cut at a trial design is displayed only when the number of attributes is two. If the number of attributes is more than two it is difficult to visualize the gradient cut. The right hand side of Figure A.I-7 queries the DM for the MRS at the current trial design (i.e., D_{T2}). For our example, the DM says that at D_{T2}: "I would give up 0.02 in the P_S of ATT Small to gain 0.01 in the P_S of ASW Small" (see Figure A.I-7).

Our algorithm for deterministic selection then finds the gradient at the current trial design (i.e., D_{T2}) and eliminates lower value designs using gradient cut. In our example, gradient at D_{T2} is ∇V_{T2}: [0.67, 0.33] and eighteen design alternatives are eliminated using the gradient cut. Only two design alternatives are non-eliminated at this stage and both of them have been trial designs already. Our gradient adjacency elimination approach (recall Section 3.3.4 of Chapter 3) finds the most preferred design as the design shown on the top of the left hand side of Figure A.I-8.

Our software generates the interface shown in Figure A.I-8 for displaying the most preferred design alternative using our deterministic selection. The top half of the left hand side shows the most preferred design in the attribute space using a bar chart. The bottom half of the left hand side shows the gradient cut (when the number of attributes is two) at the previous trial design. The right hand side of Figure A.I-8 shows the values of the variables for the most preferred design alternative.

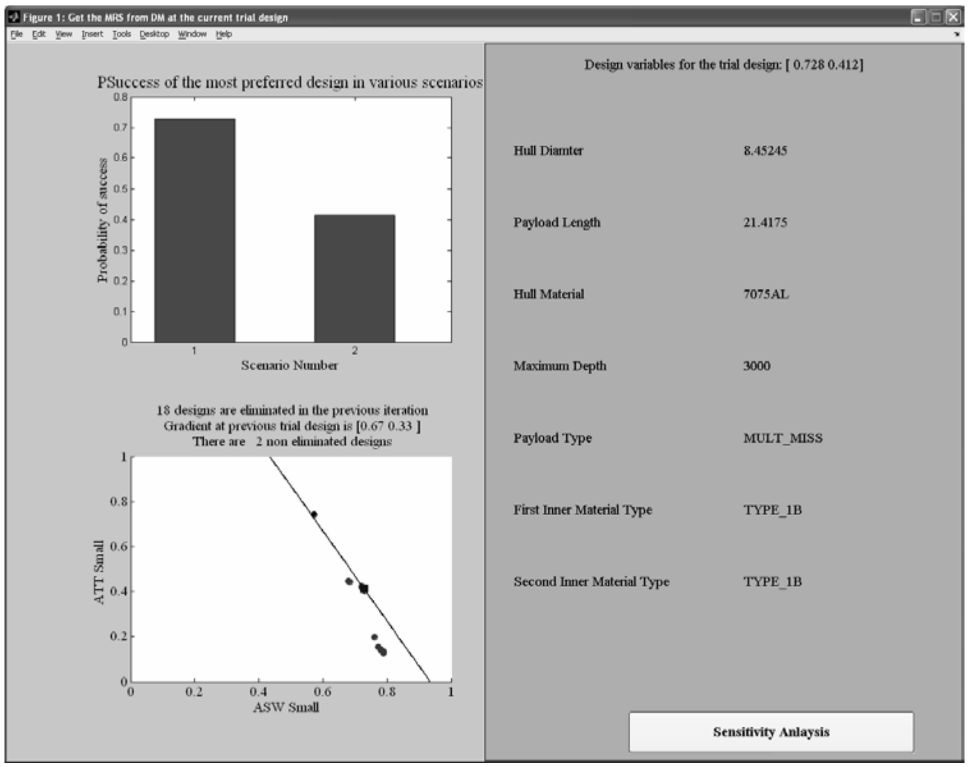

Figure A.I-8: Interface for displaying the most preferred design alternative

The button "Sensitivity Analysis" when pressed should invoke our method for sensitivity analysis for deterministic selection. However, as we mentioned earlier, the interfaces for our research components other than deterministic selection are not yet available.

APPENDIX-II

PROOF OF LEMMA IN SECTION 4.3.1.1 OF CHAPTER 4

Lemma: Let ∇V_T:$[W_{1T},...,W_{mT}]$ be the gradient of the value function corresponding to the MRS estimate, S_{ijT}, at D_T. Let D_+ be a design that lies in the gradient cut C_G at D_T corresponding to ∇V_T. Let $\nabla V_T^{t+}:[W_{1T}^{t+},...,W_{mT}^{t+}]$ be the threshold gradient at which D_+ no longer lies in the gradient cut. Let s_{ijT} be an MRS satisfying either $\frac{W_{iT}}{W_{jT}} < s_{ijT} \leq \frac{W_{iT}^{t+}}{W_{jT}^{t+}}$ or $\frac{W_{iT}^{t+}}{W_{jT}^{t+}} < s_{ijT} < \frac{W_{iT}}{W_{jT}}$. The design D_+, does not necessarily lie in the gradient cut for the gradient ∇v_T: $[w_{1T},\ldots,w_{mT}]$ corresponding to s_{ijT} at D_T.

(We prove the lemma for 'm=3' attributes. The lemma can be proved in a similar way if the number of attributes is greater than three.)

Proof: Without loss of generality, assume that the MRS values s_{ijT} are consistent (i.e., $s_{ijT} \cdot s_{jkT} = s_{ikT}$), and let: s_{12T} be the MRS between attributes a_1 and a_2; s_{23T} be the MRS between attributes a_2 and a_3; and s_{31T} be the MRS between attributes a_3 and a_1. Let ∇v_T: $[w_{1T}, w_{2T}, w_{3T}]$ be the gradient corresponding to s_{ijT}. Assuming that w_{iT} (i=1, 2, and 3) are normalized according to Eq. (4.3c), we can find w_{iT} from s_{ijT} using

$$w_{1T} = \frac{s_{12T} \cdot s_{23T}}{1 + s_{23T} + s_{12T} \cdot s_{23T}} \quad \text{(A.II-1a)}$$

$$w_{2T} = \frac{s_{23T}}{1 + s_{23T} + s_{12T} \cdot s_{23T}} \quad \text{(A.II-1b)}$$

$$w_{3T} = \frac{1}{1 + s_{23T} + s_{12T} \cdot s_{23T}} \tag{A.II-1c}$$

Since D_+ lies in the gradient cut corresponding to ∇V_T, Eq. (A.II-2) is satisfied (recall Eq. (3.11)).

$$\sum_{i=1}^{m} W_{iT} \cdot (a_{i+} - a_{iT}) < 0 \tag{A.II-2}$$

Since, W_{iT} are assumed to be non-negative, Eq. (A.II-2) is satisfied if and only if at least one of $(a_{i+} - a_{iT})$ is negative. Without loss of generality, assume that $(a_{1+} - a_{1T})$ is negative. Now, s_{12T}, s_{23T}, and s_{31T} satisfy either Eq. (A.II-3a) or Eq. (A.II-3b) (from hypothesis of lemma).

$$\frac{W_{iT}}{W_{jT}} < s_{ijT} < \frac{W_{iT}^{t+}}{W_{jT}^{t+}} \tag{A.II-3a}$$

$$\frac{W_{iT}^{t+}}{W_{jT}^{t+}} < s_{ijT} < \frac{W_{iT}}{W_{jT}} \tag{A.II-3b}$$

It is easy to see that both Eq. (A.II-3a) and Eq. (A.II-3b) are satisfied by at least one s_{ijT}. The third s_{ijT} can satisfy either of Eq. (A.II-3a) or Eq. (A.II-3b) depending on ∇V_T and ∇V_T^t. Let us consider the case when s_{12T} and s_{23T} satisfy Eq. (A.II-3a), i.e.,

$$\frac{W_{1T}}{W_{2T}} < s_{12T} < \frac{W_{1T}^{t+}}{W_{2T}^{t+}} \tag{A.II-4a}$$

$$\frac{W_{2T}}{W_{3T}} < s_{23T} < \frac{W_{2T}^{t+}}{W_{3T}^{t+}} \tag{A.II-4b}$$

Using Eq. (A.II-1) and Eq. (A.II-4) and some simple algebra we obtain the following inequalities (see Eq. (A.II-5)) for ∇v_T: $[w_{1T}, w_{2T}, w_{3T}]$.

$$\frac{W_{1T} \cdot W_{3T}^{t+}}{W_{3T}} < w_{1T} < \frac{W_{1T}^{t+} \cdot W_{3T}}{W_{3T}^{t+}} \tag{A.II-5a}$$

$$\frac{W_{2T} \cdot W_{3T}^{t+}}{W_{3T}} < w_{2T} < \frac{W_{2T}^{t+} \cdot W_{3T}}{W_{3T}^{t+}} \quad \text{(A.II-5b)}$$

$$W_{3T}^{t+} < w_{3T} < W_{3T} \quad \text{(A.II-5c)}$$

From Eq. (A.II-5c) and Eq. (A.II-5a), we can see that, for w_{1T}, Eq. (A.II-6) holds.

$$\frac{W_{1T} \cdot W_{3T}^{t+}}{W_{3T}} < W_{1T} < w_{1T} < W_{1T}^{t+} < \frac{W_{1T}^{t+} \cdot W_{3T}}{W_{3T}^{t+}} \quad \text{(A.II-6)}$$

Now, if D_+ lies in the gradient corresponding to ∇v_T then Eq. (A.II-7) should be satisfied.

$$\sum_{i=1}^{m} w_{iT} \cdot (a_{i+} - a_{iT}) < 0 \quad \text{(A.II-7)}$$

Since $(a_{1+} - a_{1T})$ is negative and w_{1T} can be less thanW_{1T} (thus resulting in w_{2T} or w_{3T} to be more than W_{2T} or W_{3T} respectively), there is no guarantee that Eq. (A.II-7) is always satisfied. So it is possible that D_+ might not lie in the gradient cut corresponding to a gradient that satisfy the bounds on MRS given by Eq. (A.II-3). This proves the lemma for three attributes. □

APPENDIX-III

RANDOMLY GENERATED DESIGN ALTERNATIVES FOR VERIFICATION STUDY IN CHAPTERS 4, 5 AND 6

Table A.III-1 to Table A.III-4 shows the design alternatives for various problem sizes that are randomly generated using MATLAB® for the verification study of Chapters 4, 5 and 6.

Table A.III-1: Design alternatives for problem size 'three attributes' × 'fifty designs'

Design alternative number	Attribute: a_1	Attribute: a_2	Attribute: a_3
1	0.393	0.640	0.725
2	0.635	0.150	0.474
3	0.094	0.996	0.398
4	0.502	0.914	0.400
5	0.213	0.305	0.902
6	0.639	0.695	0.111
7	0.666	0.370	0.085
8	0.279	0.967	0.566
9	0.367	0.927	0.660
10	0.301	0.351	0.893
11	0.425	0.408	0.686
12	0.989	0.013	0.169
13	0.564	0.571	0.644
14	0.719	0.369	0.436
15	0.585	0.793	0.609
16	0.413	0.581	0.783
17	0.509	0.120	0.917
18	0.725	0.150	0.007
19	0.242	0.234	0.902
20	0.321	0.719	0.886
21	0.586	0.725	0.078
22	0.785	0.034	0.864
23	0.153	0.440	0.913
24	0.468	0.091	0.936
25	0.319	0.969	0.402
Continued at right			
26	0.433	0.902	0.553
27	0.595	0.570	0.622
28	0.013	0.973	0.469
29	0.425	0.136	0.958
30	0.866	0.074	0.531
31	0.499	0.076	0.929
32	0.963	0.023	0.664
33	0.882	0.042	0.273
34	0.882	0.144	0.486
35	0.088	0.425	0.996
36	0.563	0.821	0.078
37	0.087	0.854	0.743
38	0.443	0.320	0.868
39	0.670	0.092	0.918
40	0.395	0.354	0.867
41	0.835	0.148	0.121
42	0.331	0.980	0.157
43	0.267	0.984	0.328
44	0.539	0.798	0.459
45	0.853	0.084	0.690
46	0.489	0.181	0.717
47	0.682	0.913	0.069
48	0.395	0.756	0.687
49	0.691	0.442	0.053
50	0.337	0.311	0.895

Table A.III-2: Design alternatives for problem size ‘four attributes’ × ‘fifty designs’

Design alternative number	Attribute: a_1	Attribute: a_2	Attribute: a_3	Attribute: a_4
1	0.393	0.640	0.725	0.695
2	0.906	0.943	0.635	0.150
3	0.996	0.398	0.093	0.060
4	0.647	0.502	0.914	0.400
5	0.806	0.364	0.706	0.956
6	0.684	0.957	0.213	0.305
7	0.902	0.639	0.695	0.111
8	0.666	0.370	0.085	0.524
9	0.307	0.550	0.985	0.416
10	0.279	0.967	0.566	0.065
11	0.475	0.494	0.367	0.927
12	0.786	0.140	0.923	0.999
13	0.987	0.169	0.805	0.271
14	0.402	0.987	0.418	0.628
15	0.303	0.895	0.338	0.638
16	0.895	0.576	0.349	0.472
17	0.989	0.013	0.169	0.126
18	0.569	0.564	0.571	0.644
19	0.416	0.719	0.369	0.436
20	0.246	0.173	0.788	0.622
21	0.128	0.254	0.925	0.999
22	0.485	0.585	0.793	0.609
23	0.783	0.874	0.728	0.339
24	0.887	0.074	0.980	0.773
25	0.264	0.321	0.719	0.886
26	0.197	0.366	0.797	0.586
27	0.819	0.310	0.864	0.546
28	0.982	0.194	0.119	0.345
29	0.468	0.091	0.936	0.319
30	0.057	0.827	0.213	0.653
31	0.147	0.816	0.693	0.721
32	0.077	0.901	0.124	0.770
33	0.342	0.959	0.534	0.027
34	0.310	0.564	0.842	0.546
35	0.106	0.937	0.525	0.617
36	0.493	0.388	0.968	0.275
37	0.428	0.663	0.584	0.769
38	0.322	0.607	0.440	0.866
39	0.499	0.076	0.929	0.115
40	0.947	0.130	0.025	0.474
41	0.531	0.375	0.279	0.656
42	0.775	0.372	0.963	0.023
43	0.949	0.866	0.490	0.135
44	0.845	0.225	0.468	0.319
45	0.277	0.982	0.925	0.169
46	0.767	0.388	0.049	0.500
47	0.678	0.445	0.811	0.355
48	0.025	0.724	0.442	0.997
49	0.832	0.652	0.654	0.008
50	0.088	0.425	0.996	0.933

Table A.III-3a: Design alternatives for problem size ‘five attributes’ × ‘fifty designs’

Design alternative number	Attribute: a_1	Attribute: a_2	Attribute: a_3	Attribute: a_4	Attribute: a_5
1	0.393	0.640	0.725	0.695	0.906
2	0.943	0.635	0.150	0.474	0.966
3	0.778	0.696	0.025	0.190	0.700
4	0.094	0.996	0.398	0.093	0.060
5	0.647	0.502	0.914	0.400	0.806
6	0.364	0.706	0.956	0.684	0.957
7	0.550	0.985	0.416	0.279	0.967
8	0.566	0.065	0.475	0.494	0.367
9	0.927	0.660	0.301	0.351	0.893
10	0.140	0.923	0.999	0.987	0.169
11	0.796	0.402	0.987	0.418	0.628
12	0.303	0.895	0.338	0.638	0.895
13	0.576	0.349	0.472	0.492	0.035
14	0.686	0.989	0.013	0.169	0.126
15	0.564	0.571	0.644	0.546	0.987
16	0.914	0.393	0.219	0.502	0.513
17	0.254	0.925	0.999	0.485	0.585
18	0.793	0.609	0.215	0.402	0.218
19	0.783	0.874	0.728	0.339	0.305
20	0.515	0.958	0.141	0.379	0.177

Continued in the next page

Table A.III-3b: Design alternatives for problem size 'five attributes' × 'fifty designs'

Continued from the last page

Design alternative number	Attribute: a_1	Attribute: a_2	Attribute: a_3	Attribute: a_4	Attribute: a_5
21	0.452	0.774	0.495	0.996	0.625
22	0.673	0.509	0.120	0.917	0.536
23	0.667	0.788	0.004	0.071	0.741
24	0.902	0.395	0.887	0.074	0.980
25	0.773	0.264	0.321	0.719	0.886
26	0.078	0.833	0.533	0.819	0.310
27	0.864	0.546	0.785	0.034	0.864
28	0.452	0.767	0.733	0.477	0.624
29	0.936	0.319	0.969	0.402	0.433
30	0.902	0.553	0.330	0.111	0.219
31	0.921	0.230	0.817	0.709	0.088
32	0.173	0.838	0.994	0.470	0.897
33	0.978	0.479	0.476	0.066	0.223
34	0.147	0.816	0.693	0.721	0.077
35	0.870	0.534	0.312	0.371	0.342
36	0.959	0.534	0.027	0.290	0.347
37	0.562	0.958	0.981	0.106	0.937
38	0.525	0.617	0.493	0.388	0.968
39	0.136	0.958	0.149	0.291	0.540
40	0.964	0.238	0.668	0.793	0.326
41	0.609	0.186	0.363	0.597	0.536
42	0.472	0.135	0.300	0.803	0.531
43	0.375	0.279	0.656	0.775	0.372
44	0.963	0.023	0.664	0.882	0.042
45	0.273	0.949	0.866	0.490	0.135
46	0.319	0.277	0.982	0.925	0.169
47	0.889	0.119	0.262	0.719	0.998
48	0.662	0.487	0.152	0.674	0.243
49	0.355	0.667	0.510	0.843	0.841
50	0.956	0.289	0.423	0.577	0.025

Table A.III-4a: Design alternatives for problem size 'six attributes' × 'fifty designs'

Design alternative number	Attribute: a_1	Attribute: a_2	Attribute: a_3	Attribute: a_4	Attribute: a_5	Attribute: a_6
1	0.393	0.640	0.725	0.695	0.906	0.943
2	0.635	0.150	0.474	0.966	0.778	0.696
3	0.025	0.190	0.700	0.094	0.996	0.398
4	0.093	0.060	0.647	0.502	0.914	0.400
5	0.806	0.364	0.706	0.956	0.684	0.957
6	0.213	0.305	0.902	0.639	0.695	0.111
7	0.666	0.370	0.085	0.524	0.624	0.030
8	0.087	0.395	0.307	0.550	0.985	0.416
9	0.279	0.967	0.566	0.065	0.475	0.494
10	0.367	0.927	0.660	0.301	0.351	0.893
11	0.881	0.707	0.982	0.458	0.786	0.140
12	0.923	0.999	0.987	0.169	0.805	0.271
13	0.113	0.344	0.114	0.796	0.402	0.987
14	0.418	0.628	0.303	0.895	0.338	0.638
15	0.895	0.576	0.349	0.472	0.492	0.035
16	0.989	0.013	0.169	0.126	0.099	0.736
17	0.546	0.987	0.914	0.393	0.219	0.502
18	0.246	0.173	0.788	0.622	0.128	0.254
19	0.413	0.581	0.783	0.874	0.728	0.339
20	0.509	0.120	0.917	0.536	0.667	0.788

Continued in the next page

Table A.III-4b: Design alternatives for problem size 'six attributes' × 'fifty designs'

Continued from the last page

Design alternative number	Attribute: a_1	Attribute: a_2	Attribute: a_3	Attribute: a_4	Attribute: a_5	Attribute: a_6
21	0.242	0.234	0.902	0.395	0.887	0.074
22	0.980	0.773	0.264	0.321	0.719	0.886
23	0.197	0.366	0.797	0.586	0.725	0.078
24	0.833	0.533	0.819	0.310	0.864	0.546
25	0.785	0.034	0.864	0.452	0.767	0.733
26	0.477	0.624	0.444	0.540	0.711	0.280
27	0.755	0.658	0.244	0.983	0.316	0.466
28	0.449	0.318	0.982	0.194	0.119	0.345
29	0.468	0.091	0.936	0.319	0.969	0.402
30	0.921	0.230	0.817	0.709	0.088	0.173
31	0.213	0.653	0.147	0.816	0.693	0.721
32	0.077	0.901	0.124	0.770	0.877	0.844
33	0.312	0.371	0.342	0.959	0.534	0.027
34	0.290	0.347	0.090	0.013	0.973	0.469
35	0.310	0.564	0.842	0.546	0.781	0.562
36	0.958	0.981	0.106	0.937	0.525	0.617
37	0.493	0.388	0.968	0.275	0.428	0.663
38	0.584	0.769	0.553	0.165	0.112	0.863
39	0.964	0.238	0.668	0.793	0.326	0.540
40	0.322	0.607	0.440	0.866	0.074	0.531
41	0.116	0.277	0.885	0.283	0.952	0.003
42	0.499	0.076	0.929	0.115	0.440	0.698
43	0.363	0.597	0.536	0.015	0.947	0.130
44	0.531	0.375	0.279	0.656	0.775	0.372
45	0.963	0.023	0.664	0.882	0.042	0.273
46	0.949	0.866	0.490	0.135	0.077	0.547
47	0.518	0.404	0.460	0.377	0.845	0.225
48	0.468	0.319	0.277	0.982	0.925	0.169
49	0.767	0.388	0.049	0.500	0.118	0.889
50	0.119	0.262	0.719	0.998	0.662	0.487

APPENDIX-IV

FORMULATIONS FOR FINDING THE RANGE OF MRS PREFERENCES FOR THE SIMULANT VALUE FUNCTIONS

A.IV-1. SIMULANT VALUE FUNCTION V_1

The simulant value function V_1 at a trial design D_T is given by Eq. (A.IV-1) (recall Eq. (5.7)).

$$V_1(D_T) = \begin{cases} \left[-\sum_{i=1}^{m} (1 - a_{iT})^{\beta} \right] \\ 2 \leq \beta \leq 2.5 \end{cases} \tag{A.IV-1}$$

From Eq. (A.IV-1), the partial derivative of V_1 with respect to attributes a_i is

$$\frac{\partial V_1(D_T)}{\partial a_i} = \beta \cdot \left(1 - a_{iT}\right)^{\beta-1}. \tag{A.IV-2}$$

From Eq. (3.2), the MRS between attributes a_i and a_j, i.e., S_{ijT}, for V_1 is then

$$S_{ijT} = \frac{\left(1 - a_{iT}\right)^{\beta-1}}{\left(1 - a_{jT}\right)^{\beta-1}}. \tag{A.IV-3}$$

We use the formulation in Eq. (A.IV-4) for finding the range of MRS, S_{ijT}, between attributes a_i and a_j. We minimize S_{ijT} for the lower bound, S_{ijT}^{L}, and maximize S_{ijT}, S_{ijT}^{U}, for the upper bound.

$$\text{Minimize/Maximize}: \; S_{ijT} = \frac{\left(1 - a_{iT}\right)^{\beta-1}}{\left(1 - a_{jT}\right)^{\beta-1}} \tag{A.IV-4a}$$

$$\text{subject to}: 2 \leq \beta \leq 2.5 \tag{A.IV-4b}$$

If there is attribute variability also, we add the constraints of Eq. (A.IV-4c) and Eq. (A.IV-4d) to the formulation for finding the range of MRS preference.

$$A_{iT}^{L} \le a_{iT} \le A_{iT}^{U} \tag{A.IV-4c}$$

$$A_{jT}^{L} \le a_{jT} \le A_{jT}^{U} \tag{A.IV-4d}$$

A.IV-2. SIMULANT VALUE FUNCTION V_2

The simulant value function V_2 at a trial design D_T is given by Eq. (A.IV-5) (recall Eq. (5.8)).

$$V_2(D_T) = \begin{cases} -\sum_{i=1}^{m} \gamma_i \cdot e^{(1\text{-}a_{iT})} \\ \sum_{i=1}^{m} \gamma_i = 1 \\ 0.9\left(1/m\right) \le \gamma_i \le 1.1\left(1/m\right) \\ \text{m is the number of attributes} \end{cases} \tag{A.IV-5}$$

From Eq. (A.IV-5), the partial derivative of V_2 with respect to attributes a_i is

$$\frac{\partial V_2(D_T)}{\partial a_i} = \gamma_i \cdot e^{(1\text{-}a_{iT})}. \tag{A.IV-6}$$

From Eq. (3.2), the MRS between attributes a_i and a_j, i.e., S_{ijT}, for V_2 is then

$$S_{ijT} = \frac{\gamma_i \cdot e^{(1\text{-}a_{iT})}}{\gamma_j \cdot e^{(1\text{-}a_{jT})}}. \tag{A.IV-7}$$

We use the formulation in Eq. (A.IV-8) for finding the range of MRS, S_{ijT}, between attributes a_i and a_j. We minimize S_{ijT} for the lower bound, S_{ijT}^{L}, and maximize S_{ijT}, S_{ijT}^{U}, for the upper bound.

$$\text{Minimize/Maximize}: \; S_{ijT} = \frac{\gamma_i \cdot e^{(1\text{-}a_{iT})}}{\gamma_j \cdot e^{(1\text{-}a_{jT})}} \tag{A.IV-8a}$$

$$\text{subject to}: 0.9\left(\frac{1}{m}\right) \le \gamma_i \le 1.1\left(\frac{1}{m}\right) \tag{A.IV-8b}$$

$$0.9\left(\frac{1}{m}\right) \le \gamma_j \le 1.1\left(\frac{1}{m}\right) \tag{A.IV-8c}$$

If there is attribute variability also, we add the constraints of Eq. (A.IV-8d) and Eq. (A.IV-8e) to the formulation for finding the range of MRS preference.

$$A_{iT}^{L} \le a_{iT} \le A_{iT}^{U} \tag{A.IV-8d}$$

$$A_{jT}^{L} \le a_{jT} \le A_{jT}^{U} \tag{A.IV-8e}$$

A.IV-3. SIMULANT VALUE FUNCTION V_3

The simulant value function V_3 at a trial design D_T is given by Eq. (A.IV-9) (recall Eq. (5.9)).

$$V_3(D_T) = \begin{cases} \prod_{i=1}^{m} a_{iT}^{\alpha_i} \\ \sum_{i=1}^{m} \alpha_i = 2 \\ 1.8\left(\frac{1}{m}\right) \le \alpha_i \le 2.2\left(\frac{1}{m}\right) \\ \text{m is the number of attributes} \end{cases} \tag{A.IV-9}$$

From Eq. (A.IV-9), the partial derivative of V_3 with respect to attributes a_i is

$$\frac{\partial V_3(D_T)}{\partial a_i} = \left(\frac{\alpha_i}{a_{iT}}\right) \cdot \prod_{i=1}^{m} a_{iT}^{\alpha_i}\ . \tag{A.IV-10}$$

From Eq. (3.2), the MRS between attributes a_i and a_j, i.e., S_{ijT}, for V_2 is then

$$S_{ijT} = \left(\frac{\alpha_i}{a_{iT}}\right) \cdot \left(\frac{a_{jT}}{\alpha_j}\right). \tag{A.IV-11}$$

We use the formulation in Eq. (A.IV-12) for finding the range of MRS, S_{ijT}, between attributes a_i and a_j. We minimize S_{ijT} for the lower bound, S_{ijT}^{L}, and maximize S_{ijT}, S_{ijT}^{U}, for the upper bound.

$$\text{Minimize/Maximize}: \ S_{ijT} = \left(\frac{\alpha_i}{a_{iT}}\right)\cdot\left(\frac{a_{jT}}{\alpha_j}\right) \tag{A.IV-12a}$$

$$\text{subject to}: 1.8\left(\tfrac{1}{m}\right) \le \alpha_i \le 2.2\left(\tfrac{1}{m}\right) \tag{A.IV-12b}$$

$$1.8\left(\tfrac{1}{m}\right) \le \alpha_j \le 2.2\left(\tfrac{1}{m}\right) \tag{A.IV-12c}$$

If there is attribute variability also, we add the constraints of Eq. (A.IV-12d) and Eq. (A.IV-12e) to the formulation for finding the range of MRS preference.

$$A_{iT}^{L} \le a_{iT} \le A_{iT}^{U} \tag{A.IV-12d}$$

$$A_{jT}^{L} \le a_{jT} \le A_{jT}^{U} \tag{A.IV-12e}$$

A.IV-4. SIMULANT VALUE FUNCTION V FOR PAYLOAD SELECTION

The simulant value function V at a trial design D_T for payload design selection is given by Eq. (A.IV-13) (recall Eq. (5.6)).

$$V = -[(1-P_{S1T})^{\beta}+(1-P_{S2T})^{2}] \tag{A.IV-13}$$

From Eq. (A.IV-13), the partial derivative of V with respect to attributes P_{S1} is

$$\frac{\partial V(D_T)}{\partial P_{S1}} = \beta\cdot\left(1-P_{S1T}\right)^{\beta\text{-}1}. \tag{A.IV-14}$$

From Eq. (A.IV-13), the partial derivative of V with respect to attributes P_{S2} is

$$\frac{\partial V(D_T)}{\partial P_{S2}} = 2\cdot\left(1-P_{S2T}\right). \tag{A.IV-15}$$

From Eq. (3.2), the MRS between attributes P_{S1} and P_{S2}, i.e., S_{12T}, for V is then

$$S_{12T} = \frac{\beta \cdot \left(1 - P_{S1T}\right)^{\beta\text{-}1}}{2 \cdot \left(1 - P_{S2T}\right)}. \tag{A.IV-16}$$

We use the formulation in Eq. (A.IV-17) for finding the range of MRS, S_{12T}, between P_{S1} and P_{S2}. We minimize S_{12T} for the lower bound, S_{12T}^{L}, and maximize S_{12T}, S_{12T}^{U}, for the upper bound. In Eq. (A.IV-17), β^{L} and β^{U} are the lower bound and upper bound, respectively, on β.

$$\text{Minimize/Maximize}: \; S_{12T} = \frac{\beta \cdot \left(1 - P_{S1T}\right)^{\beta\text{-}1}}{2 \cdot \left(1 - P_{S2T}\right)} \tag{A.IV-17a}$$

$$\text{subject to}: \; \beta^{L} \le \beta \le \beta^{U} \tag{A.IV-17b}$$

If there is attribute variability also, we add the constraints of Eq. (A.IV-17c) and Eq. (A.IV-17d) to the formulation for finding the range of MRS preference.

$$P_{S1T}^{L} \le P_{S1T} \le P_{S1T}^{U} \tag{A.IV-17c}$$

$$P_{S2T}^{L} \le P_{S2T} \le P_{S2T}^{U} \tag{A.IV-17d}$$

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