ABSTRACT

Title of Dissertation:	OPTIMAL FEEDBACK CONTROL FOR HYBRID SYSTEMS, WITH APPLICATION TO VEHICLE DYNAMICS.
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Dissertation Directed By:	Dr. William Levine, Professor, Department of Electrical Engineering and Dr. Bala Balachandran, Professor, Department of Mechanical Engineering

Hybrid dynamical systems are common throughout the physical and computer world, and they consist of dynamical systems that contain both continuous time and discrete time dynamics. Examples of this type of system include thermostat controlled systems, multi-geared transmission based systems, and embedded computer systems. Sometimes, complicated non-linear continuous time systems can be simplified by breaking them up into a set of less complicated continuous systems connected through discrete interactions (referred to as system hybridization). One example is modeling of vehicle dynamics with complicated tire-to-ground interaction by using a tire slipping or no slip model. When the hybrid system is to be a controlled dynamical system, a limited number of tools exist in the literature to synthesize feedback control solutions in an optimal way. The purpose of this dissertation is to develop necessary and sufficient conditions for finding optimal feedback control solutions for a class of hybrid problems that applies to a variety of engineering problems. The necessary and sufficient conditions are developed by decomposing the hybrid problem into a series of non-hybrid optimal feedback control problems that are coupled together with the appropriate boundary conditions. The

conditions are developed by using a method similar to Bellman's Dynamic Programming Principle. The solution for the non-hybrid optimal control problem that contains the final state is found and then propagated backwards in time until the solution is generated for every node of the hybrid problem. In order to demonstrate the application of the necessary and sufficient conditions, two hybrid optimal control problems are analyzed. The first problem is a theoretical problem that demonstrates the complexity associated with hybrid systems and the application of the hybrid analysis tools. Through the control problem, a solution is found for the feedback control that minimizes the time to the origin problem for a hybrid system that is a combination of two standard optimal control problems found in the literature; the double integrator system and a harmonic oscillator. Through the second problem, optimal feedback control is found for the drag racing and hot-rodding control problems for any initial reachable state of the system and a hybrid model of a vehicle system with tire-to-ground interaction.

OPTIMAL FEEDBACK CONTROL FOR HYBRID SYSTEMS, WITH APPLICATION TO VEHICLE DYNAMICS.

By

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Dissertation submitted to the Faculty of the Graduate School of the
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Dedication

I would like to dedicate this dissertation to my wife Beth, daughter Leah, and my new little one who is on the way. I love you all! I would also like to dedicate this dissertation to my mother, who had faith in me and knew someday I would finish this work.

Acknowledgements

I would like to acknowledge my faculty co-advisor Professor William Levine for his patience and guidance as I walked along my path and finished this work. I would also like to thank my faculty co-advisor Professor Balakumar Balachandran for his help in making sure that I had the opportunity to finish this work. Finally I would like to thank Dr. Gregory Schultz, Dr. Norman Wereley, and Dr. Nikhil Chopra for serving on my faculty committee. Finally, I would like to thank the rest of my family and friends for their support and encouragement along the way.

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Chapter 1: Introduction

The purpose of this dissertation is to extend necessary and sufficient conditions found in the literature to find feedback controls for hybrid optimal control problems. Loosely speaking, hybrid control problems are mixed dynamical systems that contain both continuous time and discrete time dynamics. Necessary and sufficient conditions exist for both smooth and non-smooth continuous time systems and necessary conditions exist for hybrid dynamical systems. The aim of this dissertation is to augment the hybrid necessary conditions with a sufficient condition so that optimal feedback controls can be calculated for a class of hybrid systems. The class of hybrid systems considered in this dissertation is general enough so that it embodies a large number of engineering problems, but is specific enough that the necessary and sufficient conditions are not theoretically overwhelming.

The development of necessary conditions for hybrid optimal control problems is not a new idea as many hybrid maximum principles can be found in the literature. In fact, the necessary conditions developed using non-smooth analysis can be classified as hybrid maximum principles because they have the ability to analyze optimal control problems with discontinuous dynamics. The problem with the non-smooth analysis results and the hybrid maximum principles is that the theoretical development of the material is performed by mathematicians, so the results are as general as possible. Generality is good, but in this case the theoretical development is very complex making the material difficult to understand and hard to apply. More restrictive assumptions can make it easier to understand and apply the theory.

Furthermore, necessary conditions only provide an open-loop solution to the hybrid optimal control problem, whereas in practical application of control system design, a feedback solution is desired.

The problem presented above is important because many physical systems exhibit hybrid behavior. Simple examples are the household thermostat and multi-gear power transmission systems. Analytical tools are required to study the behavior of these systems and to design controllers for implementation. If not treated properly, the non-linearity associated with the coupling of the discrete and continuous dynamics can cause undesirable system behavior.

1.1 Dissertation Contribution

This dissertation makes two contributions to the state-of-art. The first contribution develops a hybrid model that is applicable to many engineering problems and presents a method for finding optimal feedback controls for this class of problems. The method solves the hybrid optimal control problem in the spirit of Bellman's Principle of Dynamic Programming, by decomposing the control problem into a series of non-hybrid optimal control problems and applying the non-smooth necessary and sufficient conditions found in Chapter 4. The hybrid non-smooth necessary and sufficient condition only requires the decomposed non-hybrid control problem's dynamic constraints to be Lipschitz continuous and can be applied to a large class of engineering problems. However, the assumptions for the hybrid problem simplify the analysis enough so that the theoretical implementation is not overwhelmingly complex and can be performed by control engineers. An important feature of this method is that a feedback control solution for the hybrid optimal control problem is produced, which can provide proof that an optimal control strategy is applicable to the entire state-space of the system. Because a feedback solution is required, the solution is harder to compute, but a feedback solution will be a global solution and can be implemented on physical systems.

The second contribution is that two example problems are solved using the method. The first example problem is purely theoretical and demonstrates the complexity of a "simple" hybrid control problem. The second example is practical and provides a precise proof of the well known solution to the traction control problem.

1.2 Dissertation Organization

The dissertation is organized in the following way. In Chapter 2 an introduction to hybrid systems and includes a brief literature survey is provided. In Chapter 3, the hybrid optimal control problem is presented. In this chapter, the general hybrid control problem is defined, and the hybrid optimal control problem is constrained appropriately so that the analysis tools can be derived. Furthermore, the basic mathematical tools that are required in the derivation of the various necessary and sufficient conditions are given. In Chapter

4, a brief review of optimal control principles is given. The Pontryagin Maximum Principle is introduced as an optimization tool to compute the open-loop solution to a non-hybrid non-linear optimal control problem, and a sufficient condition is given for non-hybrid problems that are sufficiently smooth. Further the maximum principles of Bardi and Clarke-Vinter and the sufficient condition of Bardi are introduced to extend the results for problems that are non-smooth. In Chapter 5, the hybrid maximum principles of Sussmann, Riedinger, and Caines are introduced. In Chapter 6, the non-smooth necessary and sufficient condition of Chapter 5 is collected into a theorem for hybrid systems that satisfy the assumptions included in the hybrid optimal control problem of Chapter 3. In Chapter 7, the theory of Chapter 6 is used to solve two example problems. For Chapter 8, a summary is given and recommendations for future research work in this field are given.

The first contribution of this dissertation can be found in Chapters 3 and 6. In Chapter 3 the hybrid optimal control problem, which defines the class of problems that the theory applies to is provided. In Chapter 6 the theoretical material required for computing the feedback optimal control solution is provided. The second contribution of this dissertation can be found in Chapter 7. Both hybrid optimal control problems are formulated and the solutions are computed by using the theory given in Chapter 6.

In the introductory portion of this dissertation, the term non-smooth will be used without precise definition and will mean a system that has some discontinuity and/or lack of differentiability (for example discontinuous dynamic equations of motion or state trajectory). When the theoretical portion of the dissertation is presented, this term will be precisely defined and the ambiguity is removed. Also, the references in this dissertation are organized alphabetically by author, not in order of appearance. As such, the reference numbers do not start at one and they increase as the reader proceeds through the dissertation.

Chapter 2: Introduction to Hybrid Systems

2.1 Overview of Hybrid System

The purpose of this dissertation is to develop extensions of current optimization techniques for a subclass of generalized hybrid systems in order to design state feedback controllers that can be applied to mechanical systems. Current optimization techniques can deal with finding open-loop controls for a variety of hybrid systems, but cannot be used to find optimal feedback controls. Before optimization of hybrid systems can be discussed, hybrid systems must be defined and a brief discussion of the state-of-the-art of the theory of hybrid systems presented.

A general definition of a hybrid system is: a hybrid system is a dynamical system whose evolution depends on a coupling between variables that take values in a continuum and variables that take values in a finite or countable set [71]. Two well known examples of hybrid systems are computer-controlled systems and a typical household thermostat.

A common way to represent hybrid systems is with a hybrid automaton [3]. A hybrid automaton is similar to a finite state machine, but allows each node to contain continuous dynamics with switching constraints associated with the system variables. The specific structure of the hybrid automaton varies with different modeling formalisms, which will be presented later, but the general idea of a hybrid automaton is described in Figure 2.1

and is based on the work in [1][2][3][4][5]. Figure 2.1 contains two nodes/locations named i and j. Each node has a set of local dynamics that define the continuous behavior of the system. Also, each node has a discrete part that defines the switching from one node to the next node.

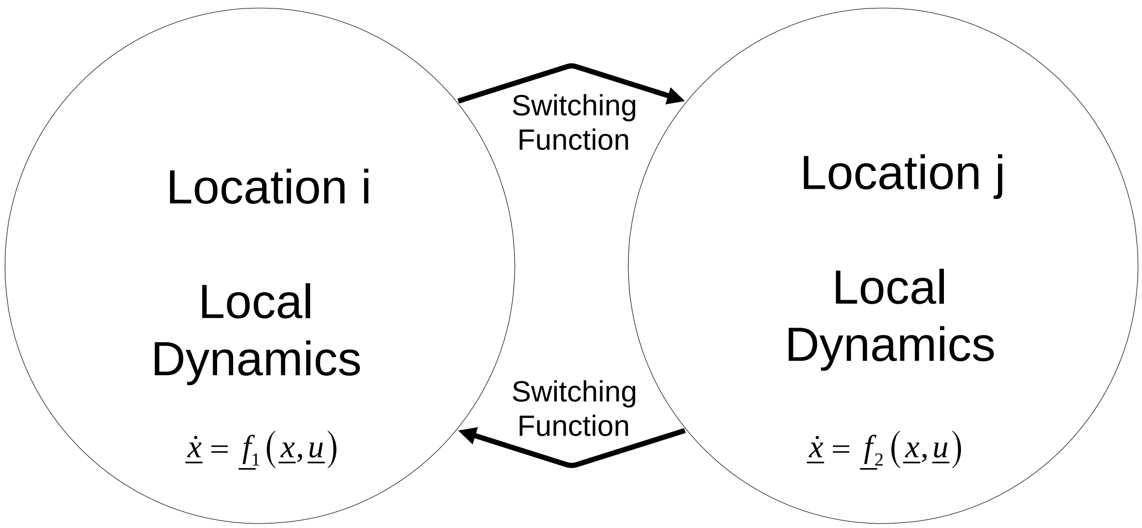

Figure 2.1: General hybrid automaton with two nodes.

A hybrid automaton consists of a series of locations or nodes that define the discrete states of the system. For each location, the local dynamics are the continuous processes that define the evolution of the state variables within that location. Switching functions are used to define the discrete behavior of the system and are functions of both the location (discrete) variables and the state variables used to describe the local dynamics. This representation is general in that the continuous processes can be non-linear and there are no restrictions on the switching functions. A precise definition of a hybrid system will not be presented here, but will be given later.

A typical household thermostat is a good example of a simple hybrid automaton. The thermostat is a good example [1][2][3][4][71] because most people have them in their homes and are familiar with their operation. Figure 2.2 depicts a hybrid automaton model of the thermostat [3].

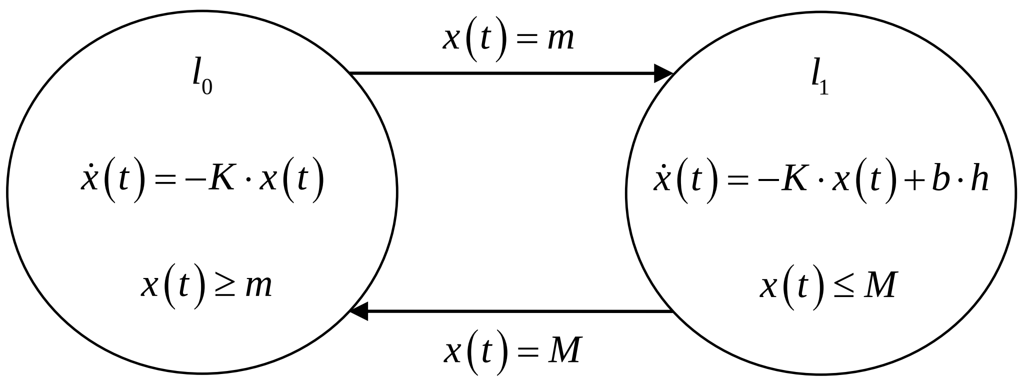

Figure 2.2: Hybrid Automaton for the thermostat example problem.

The thermostat has two nodes (or discrete states) and one continuous state. The continuous state is the temperature, $x(t)$, and the discrete states are the operational nodes, heater on and heater off, l_1 and l_0 respectively. At node l_0, the heater is off and the room temperature decreases exponentially with coefficient K. When the temperature equals the minimum value m, a discrete event occurs and the system switches from node l_0 to node l_1. In node l_1, the heater is on and heats the room with respect to the heater input $b \cdot h$. The heater stays on until the maximum desired room temperature, M, is reached. Another discrete event occurs and the system switches back to node, l_0. This cycle continues for the life of the thermostat.

2.2 Importance of Hybrid Systems

The study of hybrid systems is important for two reasons. First, many examples of hybrid systems exist so it is important to understand how they work. Second, as will be demonstrated, if not treated properly, the non-linearities of the hybrid system can destabilize the system under automatic control.

Two techniques are used to analyze and develop controllers for hybrid systems by un-hybridizing the problem. The first technique is often applied to embedded systems and involves discretizing the physics of the mechanical system (i.e. discretizing the whole problem) and applying discrete analysis and control design tools. The second technique involves ignoring the discrete part and analyzing the specific dynamics at each location. The control design involves developing controllers for each set of continuous dynamics and assigning them to their respective nodes.

Both of these techniques work in some instances, but the coupling of the discrete and the continuous dynamics can introduce non-linearities in the problem that result in unpredictable and/or strange behavior. Assume, for example, that the hybrid system contains two nodes with stable continuous dynamics and a switch that determines which set of continuous dynamics is currently driving the state variables. For certain systems with this type of structure, it can be shown that switching sequences exist that will destabilize the resulting hybrid system, which demonstrates the importance of including the interaction of the discrete dynamics with the continuous dynamics in the analysis of the system [46].

2.3 Examples of Hybrid Systems

Any mechanical or physical system that has an embedded computer used for control is a hybrid system. With the miniaturization of computers, their integration into mechanical systems is becoming more common. Applications ranging from robotics to vehicles to home appliances use computers to control their behavior.

Familiar examples of hybrid systems exist in everyday life. These examples include the automotive transmission, the household thermostat, and automotive anti-lock braking systems (ABS). Also, complicated non-linear continuous systems can take advantage of hybrid system tools, which may provide an easier way to analyze complicated behaviors or motions, such as human locomotion. Further examples include gain-scheduled control, pulse-width modulated control, and supervisory control.

To clarify the definition of hybrid systems and set the stage for explanation of hybrid modeling and analysis techniques, the manual transmission, jumping baton, and vehicle traction control problems will be described in detail. The typical household thermostat will also be used, but was presented earlier in the introductory section.

2.3.1 Manual Transmission

The second example of a hybrid system that will be discussed is an automotive powertrain with a manual transmission [42].

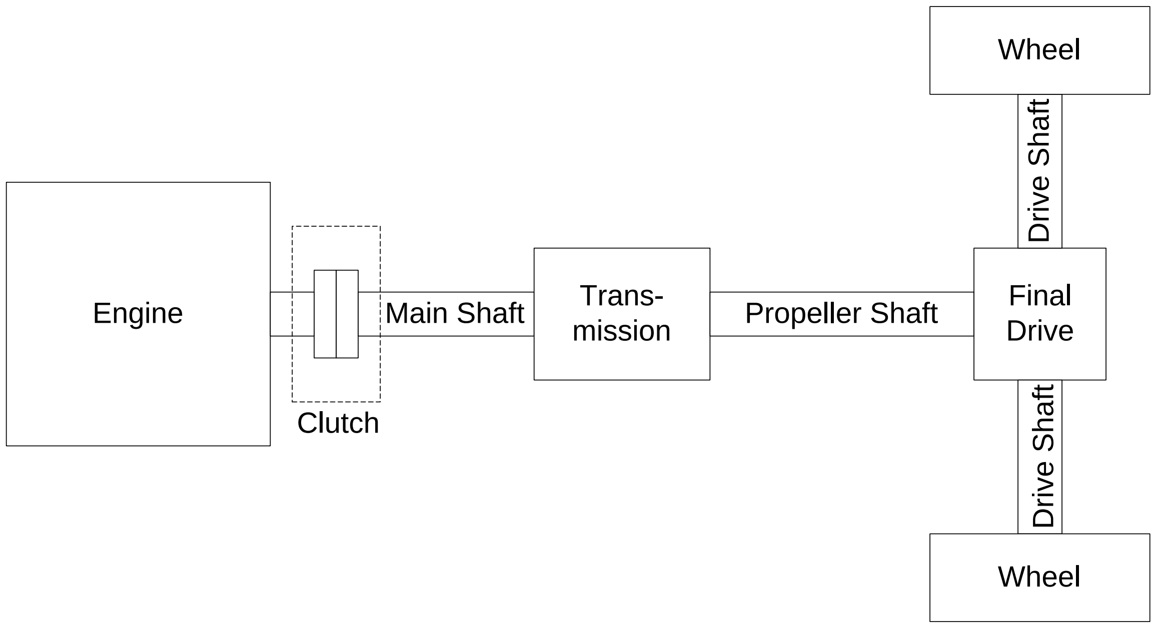

Figure 2.3: Depiction of an automotive powertrain.

Figure 2.3 depicts the major components in the automotive powertrain. The powertrain can be classified as a hybrid system because of the clutch and transmission. The automotive clutch is a non-linear component that transmits torque from the engine to the main shaft of the powertrain. The clutch can be considered a hybrid component because its friction disks have two discrete states. In the first node, the friction disks slide with respect to one another, and in the second node, the friction disks are locked together. The transmission transmits torque from the main shaft to the propeller shaft. The transmission typically has three to six forward gears and one reverse gear. Each gear location provides a different gear ratio which changes the relation between propeller torque and shaft speed. So each gear behaves like a node in a hybrid system. This system has four inputs, two continuous and two discrete, and two outputs. The inputs are the torque supplied by the internal combustion engine, the deceleration torque supplied by the brakes, the gear, and the normal load pressing the friction disks together in the clutch. The outputs of the system are the vehicle velocity and the engine speed. Typically, the driver will use the accelerator to control the engine torque output and maintain a desired vehicle velocity with respect to small perturbations. For large perturbations, the driver will use a combination of the accelerator and brake, and clutch load and gear position to control the vehicle velocity. Note that the continuous dynamics of the system are non-linear. The torque generation process by the engine is a non-linear function of multiple variables, non-linear effects exist in the gear trains, and tire-to-ground interaction is a non-linear phenomenon.

In a more detailed model, several other hybrid features could be included. The firing of the cylinder in the engine as well as the transition from normal tire contact with the road to skidding can be described by hybrid systems [19][24].

2.3.2 Automotive Traction Control

Electronics are being embedded into automotive systems to improve safety and performance. One example deals with the automotive traction control system. The traction control system consists of an ABS system [24][42] and a system to control engine torque. The ABS system is used to minimize the stopping distance for the automobile while maintaining vehicle stability and the engine torque controller manages the torque delivered to the tires to maximize stability and acceleration performance.

Traction control maximizes acceleration and minimizes vehicle stopping distance by controlling the torque to the wheels in such a way that the vehicle's tires operate close to their coefficient of maximum adhesion. Under acceleration or braking torque the tire deforms (due to static friction) at the tire-to-road interaction point and produces a reaction force that acts to accelerate or brake the vehicle. Under hard braking or acceleration, or on slippery roadway surfaces, the tire may transition to a state of pure sliding between the tire and ground. As such the coefficient of maximum adhesion occurs right at the transition point between the tire not sliding and sliding.

High level modeling of traction control systems can be accomplished with a hybrid model with two nodes. The first node contains the continuous dynamics of the rolling tire under acceleration or braking torque up to the coefficient of maximum adhesion. Once the coefficient of maximum adhesion is passed (i.e. the tires lock up), a discrete event occurs and the system switches to the second node, which contains the continuous dynamics describing the pure sliding motion of the locked tire. The control objective is then to operate the system at the switching point between the two nodes, maximizing the coefficient of adhesion for the tires.

This problem will be analyzed in much greater detail later in this dissertation.

2.3.4 Jumping Baton

The jumping baton [45] is a useful model to help study the mechanics of human jumping. The jumping baton model consists of a rod with one end free and the other end fixed to the ground in such a way that the fixed end cannot translate until the vertical reaction force exceeds some threshold value. As soon as the vertical reaction force exceeds the threshold value, the baton leaves the ground and is free to rotate until it strikes the ground. Figure 2.4 depicts the jumping baton problem [45].

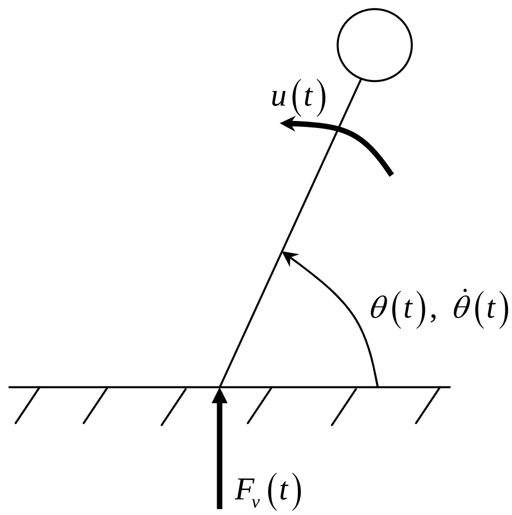

Figure 2.4: Diagram describing the jumping baton problem.

When fixed to the ground, one set of continuous dynamics governs the motion of the baton while another set of dynamics govern the rod's motion after it leaves the ground, yielding two discrete states. An interesting thing to note about this model is that the dimension of the state space changes when the continuous dynamics switch from node l_0 to l_1. Because of the problem constraints, when the rod is fixed to the ground the dimension of the state space is two and the state dimension jumps to twelve after the rod leaves the ground (although the eight states associated to the out of plane motion of the rod can be ignored, resulting in a four state model) .

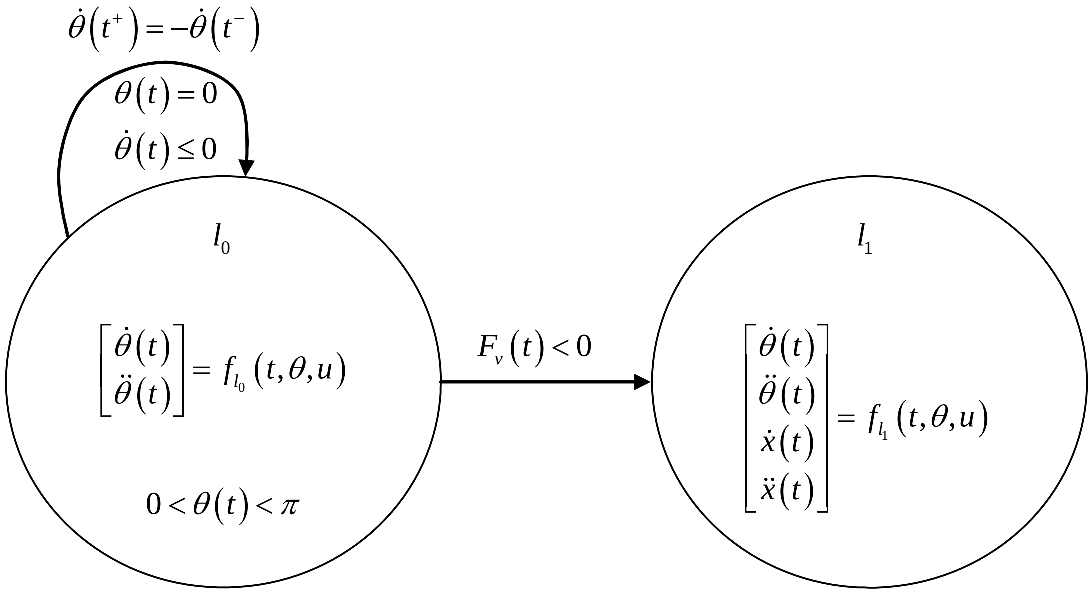

Figure 2.5: Hybrid automaton for the jumping baton

Figure 2.5 depicts the model of the jumping baton. Notice that when the baton is still pinned to the ground it can transition from node l_0 to l_0. This behavior models the elastic impact of the baton with the ground, which causes a jump in the state (the velocity of the rod instantaneously switches sign under a perfect elastic collision assumption).

2.4 Brief Survey of Mathematical Modeling of Hybrid Systems

In the literature, a variety of modeling approaches have been developed to describe hybrid systems. The modeling approaches differ in the amount of structure associated with the hybrid system. The three most common modeling approaches will be presented; the first being a control engineering approach, the second a computer science approach, and a third approach that generalizes the previous models. The Brockett modeling approach [17][18] will be presented first and uses non-linear differential equations to model hybrid systems. Second, the Alur approach [1][2][3][4] is introduced. The Alur modeling approach extends the classic idea of a finite state machine to include continuous dynamics and extensively uses this model to study reachability and verification. And finally, the Branicky approach [15] is described. The Branicky model extends an Alur hybrid model with a general structure that encompasses both the Alur and Brockett modeling techniques.

The three modeling approaches described are fundamental to hybrid systems and will be developed in further detail. Other models exist in the literature but are derivatives of the three presented in this section and will not be summarized here. The model used for feedback control synthesis in this dissertation is a version of the Branicky model and will be developed in detail in the next chapter.

2.4.1 Brockett Model

The purpose of Brockett's modeling approach [17][18] was to develop a simple motion description language that can be used to guide a robot along a specific trajectory while allowing for compliance in the robot's actions. Specifically, he tried to design controllers to not only control the position of the various parts of the robot, but also control the compliance associated with that position, where the compliance is specified by an incremental force-displacement relationship along the nominal path.

Brockett assumes that systems of this type have two kinds of inputs and outputs, symbolic (discrete event) inputs and continuous time inputs. Brockett extends standard sampling theory techniques to incorporate reading symbols or changing controller gains based on the evolution of the state vector. He does this by incorporating a real-valued monotonically increasing trigger signal that could be used to switch controller gains or read symbols every time the value of the signal passed an integer value. Furthermore, Brockett couples this triggering signal with a rate equation which allows the triggering signal to be a function of time and state.

Brockett introduces four different modeling techniques to describe hybrid dynamical systems in varying detail; Type A, Type B, Type C, and Type D. The Type A and Type B models and Type C and Type D models are identical in structure except the state variables for the Type A/C models lie in a discrete state space and the state variables for the Type B/D system lie in a continuous state space. The Type C and Type D models extend the Type A and Type B models by associating a rate equation to the discrete event state variables. Only the most general model, Type D, will be described here.

The Type D model contains continuous state variables that are a function of time and discrete event variables that are driven by a triggering function. The structure of the Type D model is:

$$
\begin{aligned}
\dot{x}(t) &= a\left(u(t), x(t), z\lceil p(t)\rceil\right) \\
\dot{p}(t) &= r\left(u(t), y(t), z\lceil p(t)\rceil\right) \\
y(t) &= c\left(x(t), z\lceil p(t)\rceil\right) \\
w\lfloor p(t)\rfloor &= h\left(y(t_p), z\lceil p(t)\rceil\right) \\
z\lceil p(t)\rceil &= f\left(u(t), v\lfloor p(t)\rfloor, y(t_p), z\lfloor p(t)\rfloor\right)
\end{aligned}
\tag{2.1.1}
$$

Here, the variables have the following meaning:

1. x is the continuous time state vector
2. a is a function that describes how the state vector changes with the continuous time input u, and the current value of the state variable x, and the discrete event state variable z.
3. p is the triggering signal variable
4. r is a real-valued function that describes the rate of change of the triggering signal as a function of the continuous time input u, the continuous time output y, and the discrete event state variable z. Note that $r > 0$, $\forall u, y, z$, to ensure that p is monotonic increasing.
5. y is the continuous time output
6. c is a function that describes the continuous time output as a function of the state vector x, and the discrete event state variable z
7. w is the symbolic output for the system
8. h is a function that describes the symbolic output as a function of the continuous output y at the last triggering signal event and the discrete event state variable z
9. f is a function that defines the evolution of the discrete event state variable z and is a function of the continuous time input u, the symbolic input v, the continuous time output y at the last triggering time, and the discrete event state variable z

Note that $\lfloor \bullet \rfloor$ represents the floor operator and indicates the integer value of $\bullet$ computed by rounding down the current value of $\bullet$ while the $\lceil \bullet \rceil$ operator is the ceiling operator and is the next larger integer found by rounding up the value of $\bullet$. Thus the discrete event state variable z changes each time p passes through an integer value.

When using this model to analyze a hybrid system, constraints needed to be placed on the function r to avoid skipping symbolic input values. For Type A and Type C systems where the state variable x is a discrete time variable, the function r needed to be bounded with the following constraint: $0 < r < 1$. This constraint doesn't allow p to skip any integer values as it increases monotonically. For Types B and D systems where the state vector is continuous time, only a lower bound for the function r is required and is: $r > 0$.

Given initial conditions for x and p, a unique solution over a given time interval will exist for Brockett's model, as long as, on any finite interval of time, p, passes through only a finite number of integers. Because p is constrained to pass through a finite number of integers, the symbolic input v will only produce a finite number of discontinuities to the derivatives of x and p. From theory based on the study of uniqueness and existence of solutions of ordinary differential equations with weak continuity hypotheses, there exists a unique x and p, with x and p continuous and differentiable almost everywhere, satisfying the equations.

Furthermore, due to the structure of Brockett's model, a series of these models (Types A and C and Types B and D) can be interconnected, allowing for the abstraction of a complex model as a series of interconnected simpler hybrid systems. The connection of two continuous time ports is trivial because time is universal, but since the symbolic ports are event driven their connection requires special attention. If two symbolic ports are interconnected, the alphabets of the two ports must agree. Multiple triggering signals may be required to force the alphabets of the two symbolic ports to agree. Buffering can be introduced to handle this lack of synchronization and buffering can be easily modeled with a Type D model.

A simple two-speed transmission can be modeled using Brockett's Type D form. This example is not found in the Brockett literature; it was created to clarify the Type D model. Assume that it is desired to model the longitudinal motion of a vehicle with a simple two-speed transmission with high and low gears. For simplicity, it will also be assumed that the system cannot provide braking torque, only acceleration torque. This assumption does restrict the applicability of the model, but it can be easily relaxed to incorporate braking torque. The continuous input to the model is the desired engine torque and the continuous outputs are the engine speed and velocity of the vehicle at every time t during the run. The discrete input to the model is the desired gear position and the discrete output is the actual gear position. The continuous state variable is the longitudinal vehicle velocity and the discrete state variable is a real number that represents the desired gear. Next, the following definitions are made:

1. u is the accelerator position or desired engine torque
2. x is the longitudinal velocity of the vehicle

3. y is the engine speed and longitudinal velocity of the vehicle
4. v is the desired gear position
5. z is 0 or 1 depending on the actual gear position

Now that the model variables have been defined, the functions that define the behavior of the system can be defined. Assume that

$$\dot{x}(t) = \left(1 - z\lceil p(t)\rceil\right) \cdot g_{Low}\left(x(t), u(t), y(t), z\lceil p(t)\rceil\right) + z\lceil p(t)\rceil \cdot g_{High}\left(x(t), u(t), y(t), z\lceil p(t)\rceil\right) \tag{2.1.2}$$

where g_{Low} and g_{High} define the continuous dynamics of the vehicle for the low and high gears respectively. When $z\lceil p(t)\rceil = 0$ equation (2.1.2) defines the dynamics associated with low gear and when $z\lceil p(t)\rceil = 1$, the equation defines the dynamics associated with high gear. The function r will be assumed constant and represents the amount of time required to execute a gear change. Since the discrete output is equal to the discrete state, h will evaluate to $z\lceil p(t)\rceil$. Finally, the function f will have the following form

$$f = \begin{cases} 1, \text{ for } v\lfloor p(t)\rfloor = Gear_{High} \\ 0, \text{ for } v\lfloor p(t)\rfloor = Gear_{Low} \end{cases} \tag{2.1.3}$$

A run of the model will begin with the vehicle in the initial gear (assume low gear) moving with its initial velocity and initial value of the trigger function. By model definition, the value of the triggering signal p will pass through an integer value at constant intervals of time defined by the switching delay required to change gears. When the driver (or controller) wants a gear change at time $t_{g,requested}$, they will request the gear change through the symbolic input v and wait until p passes through the next integer value at time $t_{g,initiated}$. So the system has the following properties

$$\begin{aligned} z\left\lceil p\left(t_{g,initiated}\right)\right\rceil = v\left\lfloor p\left(t_{g,requested}\right)\right\rfloor = 0 \\ \dot{x}(t) = g_{Low}\left(x(t), u(t), y(t), z\left\lceil p(t)\right\rceil\right) \end{aligned} \tag{2.1.4}$$

When p passes through the next integer value at time $t_{g,finished}$, the system will have the properties

$$\begin{aligned} z\left\lceil p\left(t_{g,finished}\right)\right\rceil = v\left\lfloor p\left(t_{g,initiated}\right)\right\rfloor = 1 \\ \dot{x}(t) = g_{High}\left(x(t), u(t), y(t), z\left\lceil p(t)\right\rceil\right) \end{aligned} \tag{2.1.5}$$

and the continuous dynamic equations are "switched" to the dynamics representing the desired gear. This model of the transmission is not perfect because it must wait for p to pass through an integer value before the gear change can be initiated, but it does adequately allow for the "hybrid" nature of the control problem. Note that a simple function for r was used for this example. In fact the function r only needs to be a real-valued function greater than zero, so it could be redefined to reduce the delay for gear change initiation, improving the behavior of the model.

2.4.2 Alur Model

The next modeling technique that will be discussed was developed by Alur, et al. [1][2][3][4]. The structure of this model was developed specifically to analyze reachability and verification problems for hybrid systems. Alur first developed the timed automaton, where the dynamics were simply clocks, and then extended the timed automaton model with enough structure to model hybrid systems that have simple continuous dynamics.

Alur used formal language theory and finite state machine definitions to develop timed automata theory. Timed automata hybrid systems are finite state machines that use clocks to track time. These clocks can be reset on a transition, allowing the model to track not only time, but delays produced by not switching from node to node. The advantage of this model is that it used a dense set of the real line, to represent time, not a discretized set or a fictitious clock to track time. A dense set in $\mathbb{R}$ is a set of real numbers P, such that every interval (a,b), with $a<b$, contains a member of P (i.e. time has a continuous representation).

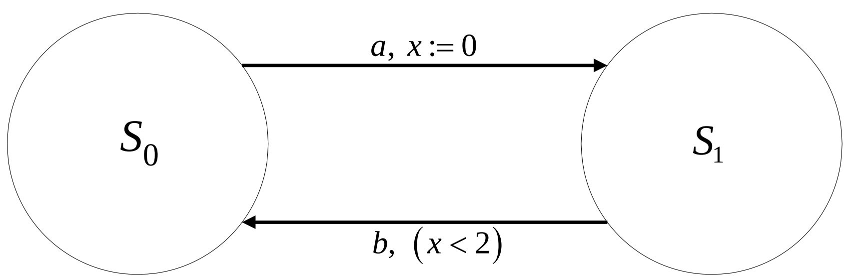

Figure 2.6: Timed automaton described by Alur.

Figure 2.6 depicts a timed automaton model that can be found in [4] and is defined by the tuple:

$$G=\langle S,\mu,s_{init},E,C,\pi,\tau\rangle \tag{2.1.6}$$

Where:

1. S is a finite set of nodes.
2. $\mu : S \to 2^{AP}$ is a function that assigns to each node the set of atomic propositions true in that node, where an atomic proposition is of the form $\alpha = \beta$ or $\alpha > \beta$, with α and β being algebraic terms.
3. $s_{init} \in S$ is the initial node.
4. $E \subseteq S \times S$ is a set of edges between nodes.
5. C is a finite set of clocks. A clock is a variable that strictly increases uniformly with respect to the system time variable that drives the system. Each clock can be reset, but all clocks increase at the same rate.
6. $\pi : E \to 2^{C}$ is a function that indicates which clocks should be reset with each edge (C is the set of clocks).
7. τ is a function that labels each edge with an enabling condition, constructed of Boolean connectives of the form $x \le c$ and $x \in C$ and $c \in \text{Natural Numbers}$ (all positive integers), that indicates when a transition can, but doesn't have to occur.

In the example of Figure 2.6, $S = \{S_0, S_1\}$ contains two nodes, s_{init} is not defined, μ assigns the symbolic input to a when the transition along edge E_1 occurs and to b when the transition along edge E_2 occurs, E contains two edges $E_1 = \{S_0, S_1\}$ and $E_2 = \{S_1, S_0\}$, one clock $C = \{x\}$, one reset function $\pi_1(E_1)$ which resets the clock x to zero when the transition along edge E_1 occurs, and two enabling conditions such that $\tau(E_1)$ evaluates to true whenever the system is in node S_0 and $\tau(E_2)$ evaluates to true when $x < 2$ and the system is in node S_1.

In Figure 2.6 if $s_{init} = S_0$ and the system is started, the transition to node S_1 is always enabled. When the transition finally occurs, the symbolic input a is read by the system and the clock x is reset to zero. Next, the transition back to node S_0 is enabled as long as $x < 2$. When the transition occurs the symbolic input b is read by the system and the cycle can repeat. Note that the enabling condition allows for modeling time delays in a system and in this case limits the time in location S_1 to less than 2 seconds if a switch back to S_0 is required.

In general when the model is initialized, all of the clocks are set to zero and the starting node is s_{init}. The model is started and the clocks increase uniformly with time. If the system clocks meet an edge's enabling condition, the system can transition along that edge to the connecting node. When the edge is enabled, the transition is not forced and does not have to occur. If a transition e occurs all of the clocks in $\pi(e)$ are reset to zero and start counting again. The current node and the clock values define the state of the system at that instant in time. Further, the clock values can be described by a function $\Gamma(G)$ that maps all of the clocks to the positive real numbers. Note that $\Gamma(G)$ can map the clock values into different parts of the positive real numbers, so each clock may have different magnitudes but all clocks will increase at the same rate. A state of the system is described by,

$$\langle s,\nu\rangle,\ s\in S,\ \nu\in\Gamma(G) \tag{2.1.7}$$

Note that the $\langle\bullet,\bullet\rangle$ notation indicates a run of the timed transition system and not the typical inner product. This notation will continue throughout this section.

Alur defined a run of the model as a series of states of G along with the times at which the transitions occur. A run is defined as,

$$\left(\langle s_0, \nu_0, t_0 \rangle, \langle s_1, \nu_1, t_1 \rangle, \langle s_2, \nu_2, t_2 \rangle, \ldots\right),\ s_i \in S,\ \nu_i \in \Gamma(G),\ t_i \in \mathbb{R} \tag{2.1.8}$$

The states in the run satisfy the following constraints:

1. Initialization: The run starts in a state $\langle s_0, \nu_0 \rangle$ at time t_0 equal to zero.
2. Consecution: For every $i \geq 0$:
   a. The time of the (*i* + 1)th transition is strictly greater than that of the *i*th transition.
   b. $e_i = \langle s_i, s_{i+1} \rangle$ is an edge contained in E.
   c. The clock assignment ν_{i+1} at time t_{i+1} equals $\left[\pi(e_i) \to 0\right]\left(\nu_i + t_{i+1} - t_i\right)$. The $\left[\pi(e_i) \to 0\right]$ term refers to all of the clocks that are reset due to the edge transition and the $\left(\nu_i + t_{i+1} - t_i\right)$ term refers to the value of the ν_{i+1} assignment if it is not reset to a specific value given by the first term.
   d. The clock assignment $\left[\pi(e_i) \to 0\right]\left(\nu_i + t_{i+1} - t_i\right)$ satisfies the enabling condition $\tau(e_i)$.
3. Progress of time: Every time value is eventually reached, that is, for any $t \in \mathbb{R}$, there exists some j such that $t_j \geq t$.

The constraints allow for progression of the system without anomalies, such as Zeno behavior, which occurs when an infinite number of transitions occur during a bounded interval of time.

Now that the timed automaton has been defined, the Alur hybrid automaton can be introduced. Alur started with the timed automaton and instead of a clock at each node (i.e., $\dot{x}(t) = 1$), he associated a set of continuous dynamics to each node. Alur completely redefines the timed transition model to form the hybrid automaton.

Alur, [3], defined a hybrid system as a system consisting of six components:

$$H = \left(Loc, Var, Lab, Edg, Act, Inv\right) \tag{2.1.9}$$

Here,

1. Loc is a finite set of vertices called locations.
2. Var is a finite set of real-valued variables. A valuation v for the variables is a function that assigns a real-value $v(x) \in \mathbb{R}$ to each variable $x \in Var$. Denote the set of valuations as V. A state for an Alur hybrid system is defined as a 2-tuple, (l, v), consisting of a location $l \in Loc$ and a valuation $v \in V$.
3. Lab is a finite set of synchronization labels. These labels allow for state resetting or jumping.
4. Edg is a finite set of edges, called transitions. Each transition $e \in Edg$, $e = (l, a, \mu, l')$, consists of a source node $l \in Loc$, a target node $l' \in Loc$, a synchronization label $a \in Lab$, and a transition relation $\mu \subseteq V^2$. A transition is enabled if for some valuation, $v \in V$, in the source node and some valuation, $v' \in V$, in the target node, $(v, v') \in \mu$.
5. Act is a labeling function that assigns to each location a set of activities. The activities are time-invariant functions from the nonnegative reals to the set of valuations, where a valuation is a function that assigns a real number to each variable. The activities are the continuous dynamics of the system.
6. Inv is a labeling function that assigns to each node an invariant such that $Inv(l) \subseteq V$. If at some time the invariant is not met, the system must transition to another node.

Note that the hybrid automaton appears to resemble the time transition automaton, but has a different structure. The Loc variable is similar to the S variable in equation (2.1.6) in that they both define a set of finite nodes that represent the discrete states of the system. The Lab function is similar to the π function of equation (2.1.6), in that they both label the locations and define the resetting of the state (i.e. synchronization of the states) after the jump. The Edg set incorporates the E and τ sets and represent the finite set of edges representing the possible discrete location changes and enabling conditions. Act incorporates the set of clocks C from equation (2.1.6) while also including dynamics that are not constant. The Inv and μ functions are similar as they both give conditions that must be true while the system is in the particular node.

Figure 2.7 depicts the Alur hybrid automaton with its corresponding structure [3]. The state of a hybrid system can change in only two ways. First, a discrete transition instantaneously changes both the location and the value of the state according to the control forcing the transition. The second type of transition is a time delay that changes only the values of the variable according to the activities of the current location (i.e. time evolution of the continuous dynamics). The only way the system can stay at one location is if the invariant for that location is true. The system transitions to another location the instant the invariant becomes false.

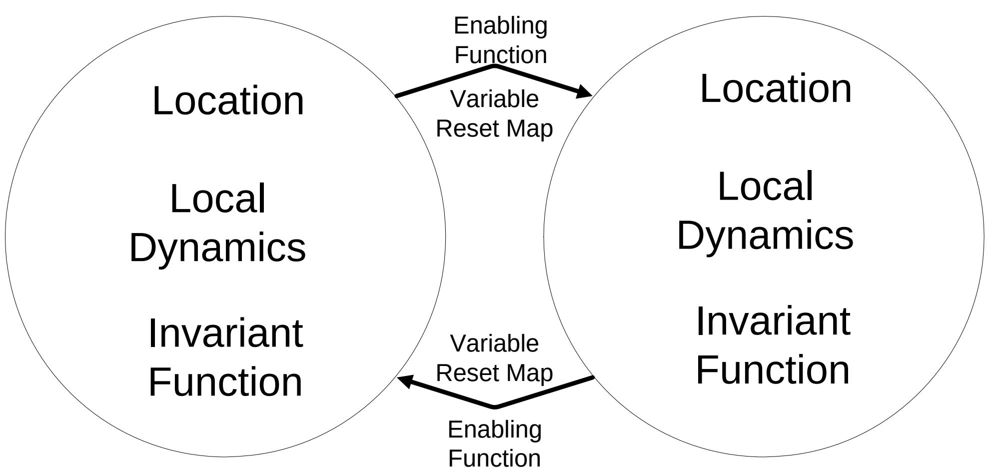

Figure 2.7: Hybrid Automaton as defined by Alur

The best way to clarify the description of an Alur hybrid automaton is through a simple example. The thermostat example given in Figure 2.2 is in the Alur hybrid automaton form. The thermostat has two locations and the dynamics at each location are different. At node l_0, the room is cooled at an exponential rate. When the system temperature drops to m, the system transitions to node l_1. At node l_1 the heater is turned on, and the room is heated accordingly. When the temperature exceeds M , the system transitions back to node l_0 and the cycle repeats.

The Alur model for the simple thermostat is given by $H = (Loc, Var, Lab, Edg, Act, Inv)$ where

1. $Loc = \{l_0, l_1\}$ are the two nodes of the system representing it's discrete states.
2. $Var = \{x\}$ is the continuous state variable for the system, room temperature.
3. $Lab = \{\ \}$ is empty because the state is not "reset" during the transition.
4. $Edg = \{(l_0, [], x(t) = m, l_1), (l_1, [], x(t) = M, l_0)\}$ defines all of the possible system transitions. So when the thermostat is in node l_0 the system can only transition to node l_1 and when the system is in node l_0 it can only transition to node l_0. Furthermore when the system is in location l_0 the transition is enabled when $x(t) = m$ and when the system is in location l_1, the transition is enabled when $x(t) = M$.
5. $Act = \{-K \cdot x(t), -K \cdot x(t) + b \cdot h\}$ are the continuous dynamics associated with the thermostat in nodes l_0 and l_1 respectively.
6. $Inv = \{x(t) \geq m, x(t) \leq M\}$ are the invariants that must be true while the system is in each node. Note that when the system is in location l_0 and the state is $x(t) = m$, the transition is enabled and the invariant is still true. At an infinitely small time later, t_s, the state will be $x(t_s) < m$ and the invariant fails to be true, forcing a transition to l_1 according to the edge condition.

2.4.3 Branicky Model

The Branicky model [15] unified the modeling approaches presented above and yielded a model structure that allowed for the analysis of general hybrid systems. The Branicky model incorporates the structure of both the Brockett model and Alur model allowing it to be less conservative and more general than each of the simpler models.

Branicky requires that four types of hybrid phenomena be captured in order for the model to be general. They are

1. Autonomous Switching – The continuous dynamics (vector field) abruptly changes (e.g. switches) when the state trajectory intersects a certain boundary.
2. Autonomous Impulse – The state changes impulsively when it hits a prescribed region of the state space (i.e. the dynamics don't change but the state jumps instantaneously to another value). An example of this type of system is object collisions.
3. Controlled Switching – The vector field changes in response to a control command with an associated cost.
4. Controlled Impulses – The state changes impulsively in response to a control input with an associated cost.

Note that the Brockett model only includes autonomous switching, since the triggering signal p is a bounded continuous function of the discrete and continuous state variables. Furthermore, the Alur model allows for autonomous switching and autonomous impulse, but does not include the ability to provide controlled switching or impulse. In fact the Alur model is only designed for hybrid systems where the continuous dynamics are independent of a control input.

The mathematical model that Branicky presents is an indexed collection of dynamical systems along with a map that defines the jumping and resetting of the states and vector fields and a map that defines the jump conditions. Formally, Branicky [15] defines a generalized model of a hybrid system as a seven-tuple H_c where

$$H_c = \left[Q, \Sigma, A, G, V, C, F\right] \tag{2.1.10}$$

and

1. Q is a countable set of indexed states representing the discrete nodes.

2. $\Sigma = \{\Sigma_q\}_{q \in Q}$ is the collection of controlled dynamical systems. $\Sigma_q = [X_q, \Gamma_q, \phi_q, U_q]$, where $X_q \in \mathbb{R}^{n_q}$ is the continuous state space, $\Gamma_q \in \mathbb{R}$ is a transition semi-group with identity (which is time for continuous systems and represents transitions for discrete systems), $\phi_q : X_q \times \Gamma_q \times U_q \to X_q$ are the continuous dynamics, and U_q is the set containing all possible controls. Note that the state space is an element of $\mathbb{R}^{n_q}$ which indicates that the number of states can change during a discrete event.
3. $A = \{A_q\}_{q \in Q}, A_q \subset X_q$ for each $q \in Q$. A is the set of all autonomous jump sets indexed by q.
4. $G = \{G_q\}_{q \in Q}, G_q : A_q \times V_q \to S$ is the autonomous jump transition map which describes to which node the system will transition after an autonomous jump. V_q represents the transition control set that defines which nodes the system can transition to once the autonomous jump set A_q is encountered. G_q defines the discrete dynamics of the system. Furthermore, $S = \cup_{q \in Q} X_q \times \{q\}$ is the hybrid state space of the system.
5. $V = \{V_q\}_{q \in Q}$ is the set of all possible transition controls.
6. $C = \{C_q\}_{q \in Q}, C_q \subset X_q$ is the collection of controlled jump sets. The controlled jump sets are subsets of the continuous state space where if the state is a member of the subset, a jump can occur. The jump doesn't have to occur, which is why it is defined as being a controlled jump.
7. $F = \{F_q\}_{q \in Q}, F_q : C_q \to 2^S$ is the collection of controlled jump destination maps.

A run of the model consists of the following steps. First, the system will start in some initial state that is not located in A or C. The system will evolve according to $\phi_{q,0}$. If the state enters $A_{q,0}$, it must transition using $G_{q,0}$ to a different location in the state space. Alternately, if the state enters $C_{q,0}$ it may transition through $F_{q,0}$ to another location in the state space. Note that when the system transitions, the state trajectory can jump according to maps $G_{q,0}$ or $F_{q,0}$. The structure of this model allows for both autonomous and controlled transitions, which gives rise to two types of system control. First, the model allows for control of the continuous dynamics in each node. Second, the model allows for control of the discrete events. Control of the continuous dynamics can cause an autonomous jump if the control forces the state to enter A_q. Furthermore, if the control moves the state into the set defined by C_q, a discrete event control can be applied to the system forcing a transition to another node.

The model can be augmented to make it more multi-purpose:

1. Outputs can be added to the model. $O = \{O_q\}_{q \in Q}$, $\eta = \{\eta_q\}_{q \in Q}$ where $\eta_q : A_q \to O_q$ produces an output at each jump time.
2. $\Delta : A_q \to \mathbb{R}_+$, is a jump delay map that can be used to accommodate transitions that are not instantaneous and take a specified amount of time.
3. $\tau : X \times \Gamma \to \mathbb{R}$ is a transition time map which provides a mechanism for reconciling different time scales incorporated in the continuous dynamics.

Note that both the Alur and Brockett models can be described by a Branicky type model, so it unifies the modeling approach for hybrid system analysis.

The simple two-speed manual transmission model presented earlier can be used to demonstrate the Branicky model structure. First, two nodes exist for this problem, so $Q = \{q_1, q_2\}$ where the nodes represent q_1 for low gear and q_2 for high gear. Next, the controlled dynamic systems, Σ can be defined as

$$\begin{aligned} \Sigma_1 &= \left[x \in \mathbb{R}, t \in \mathbb{R}, g_{Low}, U\right] \\ \Sigma_2 &= \left[x \in \mathbb{R}, t \in \mathbb{R}, g_{High}, U\right] \end{aligned} \tag{2.1.11}$$

where g_{Low} and g_{High} are the continuous dynamics associated with the low and high gears, respectively and $U \subset \mathbb{R}$ is a closed set that contains the set of desired torques that can be produced by the engine. A will be empty for this problem because no autonomous jumps will occur. If it were desired to force a gear change as a function of vehicle velocity (i.e. engine speed for this problem) it would be defined in A. Since A is empty and no autonomous jumps are defined, then G and V will be empty as well. $C = \{\mathbb{R}, \mathbb{R}\}$ are the regions of the state space where the controlled transition is enabled. It will be assumed for this problem that the transition can occur for any value of the state space at any time. Note that for a more accurate model there would be a subset of the state space in each location that enables the transition. If the longitudinal velocity (i.e. engine speed) is too high then a restriction would be placed on shifting from high gear to low gear because the engine would be over revved. Finally, $F = \{2,1\}$ are the controlled jump destination maps. If the system is in node q_1 and a controlled jump is requested (i.e. a gear change is desired), then F requires the system to jump to node q_2. Conversely if the system is in node q_2 then F requires the system to jump to node q_1.

Branicky also created a restricted version of his hybrid dynamical system model [15]. He did this so that control synthesis can be performed for the system. For simplicity, Branicky added a set D, to represent the set of destinations for every possible transition. In turn, he removed the set F of set-valued maps that describe the transitions. Branicky also added time delay sets Δ_a and Δ_c to account for autonomous and controlled non-instantaneous jump times. Branicky also restricted the general model with the following assumptions:

1. Restricted Model Assumptions - For every $i \in Z_+$, X_i is the closure of a connected open subset of Euclidean space $\mathbb{R}^{d_i}$ and $d_i \in Z_+$, with Lipschitz

boundary ∂X_i. $A_i, C_i, D_i \subset X_i$ are closed. Further, ∂A_i is Lipschitz and contains ∂X_i.

2. Jump Set Separation - $d(A_i, C_i) > 0$ and $\inf_{i \in Z_+} d(A_i, D_i) > 0$ where $d(\bullet,\bullet)$ is the appropriate Euclidean distance.
3. Transversality of A - For each *i*, ∂A_i is an oriented C^1-manifold without boundary and at each point x on ∂A_i, $f_i(x,z,u)$ is transversal to ∂A_i for all choices of z and u.
4. Transversality of C - For each *i*, ∂C_i is an oriented C^1-manifold without boundary and at each point x on ∂C_i, $f_i(x,z,u)$ is transversal to ∂C_i for all choices of z and u.

These assumptions provide a well-defined dynamical system in that they assure the existence and uniqueness of the state in each constituent system, where switching times are well defined, and that autonomous switching times do not accumulate (i.e., no Zeno problems).

2.5 Brief Survey of Analysis Results for Hybrid Systems

The study of hybrid systems is a relatively new field of research; so many perspectives on how to analyze these systems exist. The two main approaches were born out of computer science and non-linear system theory. The first approach, computer science, abstracts the continuous dynamics away from the problem and uses timed automaton theory to analyze the behavior of the system, [1][2][3][4][5][13][39]. Bounds on the continuous dynamics are used as clock constraints, which control the discrete behavior of the problem. Once the problem is in this form, finite automaton analysis techniques can be applied to study reachability from the initial state and safety verification problems. The study of these two problems gives insight into the required control, but doesn't provide theory for finding a controller that satisfies a set of system specifications. Algorithms have been developed that partition the state space into regions according to the discrete events and the set of states that can be reached by the continuous dynamics. Further, some of these algorithms incorporate a labeling scheme to their partitioning algorithm that labels each region of the state space as either satisfying or not satisfying a set of constraints, solving the verification problem. To improve the computational efficiency of the partitioning algorithms, variants of these algorithms exist in the literature that try to reduce computational time, minimize the number of regions required in the partitioning of the state space, while still solving reachability and verification problems.

The second approach abstracts the discrete behavior away from the hybrid problem and treats it as a non-smooth system, [15][17][18][43]. Different types of analysis using this approach have been reported in the literature. Differential inclusion theory can be applied to a sub-class of hybrid systems that meet the differential inclusion assumptions. Calculus that can be used to study the dynamic behavior has been defined for differential inclusion problems. Lyapunov stability theory has been applied to hybrid systems to determine their stability properties [9][14][24][40][46][54][73]. Specifically, the Lyapunov stability technique has been applied to switched systems to help determine if any/all switching sequences will produce stable dynamic behavior. Finally, non-smooth optimization techniques have been applied to hybrid systems to find open-loop controls that satisfy necessary conditions of optimality [56][61][63][64][65][67].

The results reported in this dissertation will be based on the non-smooth non-linear system method and extend the results of non-smooth optimization to a subclass of hybrid systems, and provide enough structure to solve some practical engineering control problems. The theory developed will be applied to two problems. The first problem is a simple hybrid problem consisting of a harmonic oscillator and double integrator with a defined switching rule. The second problem analyzes the traction control problem and analytically proves that the theory developed does indeed provide at least a suboptimal solution.

Three areas of hybrid systems provide a set of hybrid system analysis tools. Alur et al. [1][2][3][4][5] provided theory to analyze the reachability and verification problem for hybrid systems that have the structure dictated by his modeling formalism. Alur’s work in this area of hybrid systems has spawned a large set of tools that analyze the reachability and verification problem for a wide range of hybrid systems [39][66]. Second, an array of tools has been developed for a sub-class of hybrid systems called switched systems. Finally, a suite of tools has been developed to find optimal controllers for various subclasses of hybrid systems.

Switched systems are a sub-class of hybrid systems where the control algorithm causes the system to switch between a set of autonomous continuous time systems. Analysis tools exist for switched systems that examine system stability and controller synthesis [14][24][41][46][54][73].

A good part of the literature on switched systems [14][24][41][46][54][73] examined stability of the switched system state trajectory because certain switching sequences for the switched system can cause strange behavior. Theoretical examples of strange behavior can be easily generated. A switched system with two nodes each of which contains stable dynamics, can have a switching sequence that produces an unstable response. On the other hand, a switched system with two nodes each of which contains unstable dynamics, can have a switching sequence that produces a stable response. Lyapunov stability ideas from non-linear control system theory provided the theory to analyze the stability of switched systems. The general idea was to show that the switching sequence monotonically decreases the overall “energy” of the switched system. When the switched system contained only continuous linear dynamics, the Lyapunov theory provided a general analytical method to determine if and what switching sequences produce an unstable or stable response.

The ideas introduced to study the stability of the switched system can be applied to controller synthesis for switched systems. Controller synthesis tools are used to design controllers that stabilize switched systems [9][46][53][73]. If the continuous systems use state feedback control, then the control synthesis tools will design the individual controller gains as well as the required switching sequence to stabilize the system while meeting design constraints. As before when the continuous dynamics are linear, exact computational methods are presented in the literature to perform the design and produce locally and globally stable responses.

The focus of this research will not be switched systems or stability of switched systems, so the author refers the reader to the references for more information.

Chapter 3: Optimal Control Problem Definition

The purpose of this chapter is to introduce the general hybrid optimal control problem that will be analyzed in this dissertation. The problem will be given in its most general form. The standard analysis tools found in the literature will be developed using a version of the general optimal control problem that is constrained by assumptions. These analysis tools form the basis of the hybrid Maximum Principle and hybrid sufficient condition found in this dissertation.

This chapter is organized in the following way. First, a brief overview of optimal control will be provided. And then the general hybrid control problem will be given and the optimal control problem defined.

3.1 Overview of Optimal Control

Optimal control of non-linear systems has been a popular area of research for many years and many papers and textbooks have been written on the subject. [6][7][55][72] are four examples of texts that introduce the concepts of optimal control of non-linear systems. The purpose of optimal control research is to develop a set of necessary and sufficient conditions the optimal control must satisfy, given a specific class of control problems. Necessary conditions are a set of conditions every optimal control candidate must satisfy and sufficient conditions are a set of conditions that only the optimal control will satisfy.

Necessary conditions have been the main focus of the research because they are relatively easy to apply and narrow down the set of all controls to a set of candidate optimal controls. The usual necessary conditions, however, only provide an open-loop solution to the problem because they are associated with the optimal trajectory from a known initial state to the final state. The sufficient conditions are more powerful than the necessary conditions because only the optimal control solutions (there may be more than one) will satisfy the sufficient conditions, and the known methods for determining sufficient conditions produce a feedback control solution. Application of the sufficient conditions without the necessary conditions is generally difficult because it usually requires solving a partial differential equation along every trajectory that can reach the final state of the system. Typically in a control synthesis process, the necessary conditions are used to generate a set of candidate optimal controls and then the sufficient conditions are used to determine which candidate controls are optimal (if an optimal control exists). In special cases (as in the case of linear systems), the necessary conditions can be shown to be sufficient, so every candidate control solution identified by the necessary condition is optimal.

A general definition of optimization is given in [8] and is, “Optimization is the process of maximizing or minimizing a desired objective function while satisfying the prevailing constraints.” As such, all optimization problems contain a cost function (or an objective function) and a set of constraints, where the objective function and the constraints are related through a common set of variables. The optimal control problem has a form that is identical to the optimization problem given in the previous definition. Optimal control problems consist of a cost (or objective) function and a set of constraints that are related to the variables in the cost function. The constraints consist of the dynamics of the system to be controlled and any other constraints the problem may have (for instance the state may be constrained to a subspace of the state space). The necessary conditions are typically derived by assuming that an optimal trajectory exists (i.e. an initial condition and control are defined) with an associated cost, computing the effect on the cost function of temporal, spatial, and control variations of the reference trajectory, and finally applying the definition of optimality. The sufficient conditions are usually derived by computing the optimal cost-to-go function (or value function) to the final state from every initial state in the state space (assuming that a control exists such that the initial state can reach the final state), computing the variation in the value function by temporal and spatial variations, and finally applying the definition of optimality.

Two main methods are used to derive necessary conditions for optimal control. The first method uses the Calculus of Variations to compute the variation in cost associated with smooth variations in the temporal, spatial, and control variables [6]. The second method, the Maximum Principle [6][55][72], derives the necessary conditions in a more general form by computing the variation in cost associated with non-smooth (i.e. jumps) variations in the spatial and control variables along the reference trajectory. The sufficient conditions [6][7][72] are derived by using an infinitesimal version of the Principle of Dynamic programming to compute the variation in the optimal cost-to-go function to the final state of the system.

The optimal control of hybrid systems is a field of research that is starting to grow. The main focus in analytical computation of optimal controls has been in the development of hybrid maximum principles. [56][57][58][59][61][63][64][65][67][68][69] extend non-smooth optimization principles to classes of hybrid systems and present necessary conditions for a maximum principle that extends the classic maximum principle developed by Pontryagin. Development of sufficient conditions has had little attention, compared to the hybrid maximum principles, in the analytical computation of hybrid optimal controls. However, [47][62] develop sufficient conditions for optimal controls for restricted classes of hybrid systems. The restricted class of hybrid systems is smaller than the class studied in this dissertation.

A variety of tools have been developed to solve optimal control problems for different subclasses of hybrid systems. For example, optimal control tools have been developed for hybrid systems where the continuous part is sampled into discrete form and the discrete event part is described by inequality constraints [10]. In this form, [10] demonstrated that these problems can be recast as mixed integer optimization problems, where commercial solvers can be used to solve the problem. Furthermore, [74] applied dynamic programming principles in conjunction with a quadratic optimal control principle to find the optimal controller gains when the continuous dynamics are linear and the switching sequence between discrete modes is given. In [16] the authors presented three different optimization techniques that involved discretizing the state and control spaces and solved the resulting discrete boundary-value problem. The first technique extended an algorithm that was developed to solve optimal control problems with impulsive controls. The second technique involved value and policy iteration for systems that are piecewise continuous and encompassed a generalized Bellman function. The third technique was a linear programming technique that encapsulated the impulsive control technique previously developed. [9] developed a technique to synthesize state feedback optimal controllers for switched systems where the continuous dynamics are linear and autonomous. This technique constructs switching tables that identify regions of the state space where an optimal switch should occur when the state enters that region. Further, [21] studied a type of hybrid optimal control problem that resembled a first in first out buffer, which is common in manufacturing engineering. This model assumed that there exists a queue that sends jobs to a central processor in the order that they were received. The idea is that each job represents a discrete event and while it is processing, the continuous state evolves according to a set of continuous dynamics defined by the job. The optimization problem utilized a model of this type and minimized the total processing time, while ensuring that the individual jobs met specific quality requirements. Cassandras developed first order optimality conditions to solve this optimization problem. Finally, [75] used a genetic algorithm approach to find test inputs that can be applied to the physical system to assess its functionality. The purpose of this research was to automatically develop test inputs that will ensure functionality of the system and/or find potential operational faults. This problem can be cast as an optimal control problem by defining a cost function that is dependent on whether the design criteria are met. The genetic algorithm approach is a derivative free numerical optimization technique that evolves through the state space by a process of competition and controlled variation. Since the solver utilized a derivative free approach, the system did not need to be represented analytically so this technique could be applied directly to system simulations.

Numerical algorithms have also been developed to solve the optimal control problem for hybrid systems. Studies [37] and [38] used dynamic programming to numerically solve the optimal control problem for hybrid systems where the cost function is convex. The authors used a linear programming technique to find the control law that provides an automatic discrete event to select the optimal mode for the system. The authors demonstrated their technique on a simple gear shifting model for a truck with a flexible transmission. Study [19] also presented a dynamic programming process to solve the hybrid optimal control problem, but their method applied to hybrid systems that can be represented by a bisimulation. The authors used the equivalence relations of the bisimulation to find the cost-to-go value of any region of the bisimulation based on all of the strictly smaller cost-to-go regions leading to that region. Essentially, since the bisimulation partitions the state space according to regions that transition to other regions, the algorithm is able to compute the minimum cost-to-go function based on the path of the smallest cost-to-go regions that eventually reach the region being computed. This process continues until the optimal region path is computed for every region in the partition.

Since the focus of this dissertation is on the analytical analysis of hybrid systems, the hybrid maximum principles of Sussmann [67][68][69], Riedinger [57][58][59], and Caines [61][63][64][65] will be discussed in more detail in Chapter 5.

3.2 Hybrid Control System

The hybrid control system will utilize the hybrid model form developed by Sussmann, given in [67], that is similar to the Branicky model formulation [15] given in the introduction of this dissertation, but does not allow for controlled switching.

Let the hybrid control system, Σ, be given by the seven-tuple

$$\Sigma = \left(Q, M, U, f, u, I, S\right) \tag{3.1.1}$$

Where:

1. Q is a finite set that is used to define the locations of the hybrid model.
2. $M = \left\{\mathbb{R}^{n_q}\right\}_{q\in Q}$ is a family of real spaces of order n_q indexed by $q \in Q$ which represent the family of state spaces where the respective continuous dynamics are defined.
3. $U = \left\{U_q\right\}_{q\in Q}$ is a family of control spaces for Σ indexed by $q \in Q$.
4. $f = \left\{f_q\right\}_{q\in Q}$ is a family of functions such that $f_q : \mathbb{R}^{n_q} \times U_q \times \mathbb{R} \to \mathbb{R}^{n_q}$, where f_q define the continuous dynamics for each location $q \in Q$.
5. $u = \left\{u_q\right\}_{q\in Q}$ is the set of all admissible controls for each location q. For every interval of time over which the dynamics are defined, $u_q \subseteq U_q$.
6. $I = \left\{I_q\right\}_{q\in Q}$ is a family of subintervals of $\mathbb{R}$, that are allowed to be empty and represent the ability to bound the switching event to a specific interval of time.
7. S is a subset of $\hat{M}^2\left(\Sigma\right)$ and defines all of the switching criteria between the locations of the hybrid system. $\hat{M}^2\left(\Sigma\right)$ is defined as:

$$\hat{M}^2\left(\Sigma\right) \equiv \left\{\left(q, x, q', x'\right) : q, q' \in Q, x \in \mathbb{R}^{n_q}, x' \in \mathbb{R}^{n_{q'}}\right\}$$

   $\hat{M}^2\left(\Sigma\right)$ is the set of all of the potential pre- and post-switch values of x, for all possible pre- and post-switch values for the discrete state q. The actual switching sets for the hybrid system that define the switching dynamics from location q to location q' are given by:

$$S_{q,q'} \equiv \left\{ (x, x') \in \mathbb{R}^{n_q} \times \mathbb{R}^{n_{q'}} : (q, x, q', x') \in S \right\}$$

This equation says that the switching set that defines the switching dynamics from location q to location q' is the pre and post switch values of x that are associated with the location q and location q' defined in S, respectively. The switching set is used to define surfaces in the state space where an autonomous jump will occur.

Note that this model can be represented in Branicky form. Recall that the Branicky model is given by equation (2.1.10) and is

$$H = [Q, \Sigma, A, G, V, C, F] \tag{3.1.2}$$

where the elements of the model are defined in the introduction section of this dissertation. Q in the Branicky model is identical to Q in equation (3.1.1). Both represent the number of finite states in the model. The Σ term in the Branicky model represents the controlled dynamical system and encapsulates the (M, U, f, u) terms of the Sussmann model. The I and S terms of the Sussmann model incorporate the ability to bound the switching time and to define state based switching constraints. If another state variable is introduced to the Sussmann system to represent the system clock, then the I and S terms can be represented by the autonomous jump set and jump transition map given in the Branicky model as the A, G and V terms. The final two tuples in

(3.1.2) are not captured in the Sussmann model so in the Branicky formulation are empty. As such the Sussmann model does not capture control ordered jumps. The Sussmann model given in equation (3.1.1) is less general than the Branicky model given in equation (3.1.2), but is a useful model for representing physical systems and development of analysis tools.

The fact that the Sussmann model doesn't include the controlled switching phenomenon limits the applicability of this work. For example in the drag racing problem, the tire is modeled as either being in a sliding mode or non-sliding mode. When the system is non-sliding, a "large enough" braking or accelerating torque can cause the system to instantaneously transition to the sliding case. However, by assuming that once the system is in the non-sliding state, it stays in the non-sliding state (which will be verified as valid), this model will apply. Since sufficient conditions are developed along with necessary conditions, the inclusion of controlled dynamic switching to this work is straightforward because the optimal solution to the final set of states is always known.

The Sussmann model provides a formalism for representing physical hybrid control problems, but must include more structure before the optimal control analysis tools can be developed. In order to satisfy the assumptions required for the non-smooth necessary and sufficient conditions in Chapter 4 [7], it will be assumed that the hybrid optimal control problem will satisfy the following set of assumptions:

1. For every control space, U_q, $U_q = \mathbb{R}^{m_q}$, where $m_q \leq n_q$ is the dimension of the control space for all discrete locations indexed by q.

2. For all $q = [1, \ldots, n]$, f_q is Lipschitz continuous in the state variable, x, uniformly in the control variable, $\alpha \in U_q$, and the time $t \in J_q$. Equivalently, for all $x, y \in \mathbb{R}^{n_q}$, $u \in U_q$, $t \in I_q$, there exists a constant $l > 0$ such that
$$\left\| f_q(x, \alpha, t) - f_q(y, \alpha, t) \right\| \le l \cdot \|x - y\| \tag{3.1.3}$$
3. For all $q = [1, \ldots, n]$, f_q is bounded on a ball centered at $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ with radius $R > 0$ for all admissible controls $\alpha \in U_q$.
4. f_q is differentiable with respect to x and $\frac{\partial f_q}{\partial x}$ is continuous for all $x \in \mathbb{R}^n$ and $u \in \bar{U}_q$, where $\bar{U}_q$ is the closure of the control set U_q.
5. For every $q \in Q$, $I_q = \mathbb{R}$. This assumption insures that the switching times for the discrete dynamics are always free and not bounded.
6. For all $x \in S_{q,q'}$ and $q \in Q$, there exists a scalar function $w_q : \mathbb{R}^{n_q} \to \mathbb{R}$ such that $w_q(x) = 0$ (i.e. w_q is a hypersurface in $\mathbb{R}^{n_q}$). Furthermore, assume that there exists a function $h_q : \mathbb{R}^{n_q} \to \mathbb{R}^{n_{q+1}}$ such that $(x') = h_q(x)$ and x' does not satisfy $w_{q'}(x') = 0$. This assumption restricts the discrete dynamics to only occur when the state of the system is an element of a pre-defined surface in the state space. Furthermore, this assumption requires that the "jump" in the system trajectory is only dependent on the pre-jump state, and when the jump occurs the trajectory does not jump to a point where it can instantaneously jump again. Note that the post-jump state is only indirectly dependent on the control. The pre-jump controls determine the pre-jump state, which determines the post-jump state.

Note that these assumptions are actually more restrictive that what is required for necessary conditions of Sussmann [67], but allow for the addition of the sufficient condition to the theory. Furthermore, the assumptions on the dynamical constraints are similar to the assumptions required by the Branicky model [15]. The rest of the assumptions are technically required for the validity of the hybrid necessary and sufficient conditions developed later and are not included in the Branicky model.

A trajectory of a hybrid system can be defined for the time interval $t_0 \le \tau \le t$ once the initial condition for the system is given and a control function for the system are defined over the interval $t_0 \le \tau \le t$. Assume that the initial condition for the trajectory is the 3-tuple, (q_1, x_0, t_0), where $q_1 \in Q$, $x_0 \in \mathbb{R}^{n_{q_0}}$, $t_0 \in \mathbb{R}$ $t_0 \ge 0$, and $w_{q_1}(x_0) \ne 0$. Furthermore, let the set $v = \{v_1, \ldots, v_k\}$ be a set of controls where each element of v, v_i $i = 1 \ldots k$, is a control function, $v_i(\tau_i) \in u_q$, defined over an interval of time $\tau_i^+ < \tau_i \le \tau_i^-$, the times τ_i^- satisfy $t_0 < \tau_1^- < \tau_2^- < \ldots < \tau_{k-1}^- < t$, $\tau_1^+ = t_0$, $\tau_{i+1}^+ = \tau_i^-$ $i = 1, \ldots, k-1$, $\tau_k^- = t$, and for all $i = 1 \ldots k-1$, $w_i\left(x\left(\tau_i^-\right)\right) = 0$. Then a trajectory of the system, Ξ, is the set of 3-tuples, $\left(q_j, x, t\right)$, $j = 2 \ldots k$, defined for every τ, $t_0 \le \tau \le t$, where $q_j \in Q$, $t \in \mathbb{R}$ and $t > t_0$, $x \in \mathbb{R}^{n_i}$, and x is the solution of the differential equation $\dot{x}(t) = f_i(x, \alpha, t)$, $\alpha(t) \in u_i$, with initial condition $x\left(\tau_i^+\right)$ and τ_i^+ over the interval of time $\tau_i^+ \le \tau_i \le \tau_i^-$, for all $i = 1 \ldots k$.

Note the inherent relationship between the control and the discrete events. The control for the hybrid system model steers the trajectory to the hyperspace defined by w_i and thereby forces the system to perform a discrete event. Further, the definition of the control set forces the number of discrete locations (or events), k, to the set $k \in [1, \ldots, \infty)$.

A simple example of the relationship between the control and the discrete events is the thermostat problem given in Chapter 2. When the temperature drops below a specified level, the thermostat turns the heater on. As the heater heats the air the control (the heater) causes the state (the room temperature) to rise until the switching surface (the threshold temperature) is reached. The heater then is switched off and the state begins to fall and the cycle repeats. Since the control drives the state to the switching surface, the discrete event is dependent on the control.

An example where the discrete events and the control are not related is the manual transmission problem in which the driver input is exogenous and the control u is the engine torque. Since the driver can select any gear at any time they desire, the choice of gear ratio and time at which the gear ratio is changed is not a function of the state or time. Hence, the driver forces the discrete event to occur at some time that can be independent of the state of the system and the environment under which the system is operating. As such, the discrete event (the gear change) is independent of the system control.

3.3 Hybrid Optimal Control Problem

Now that the hybrid control system has been defined, the optimal control problem that is going to be studied throughout this dissertation can be defined.

First, pick $k \in [1,\infty)$ discrete locations and order them in the sequence $\{q_1, q_2, \ldots . q_k\}$. Next, pick an initial condition (q_1, x_0, t_0), time $t > t_0$, and control set $v = \{v_1, \ldots, v_k\}$, where $x_0 \in \mathbb{R}^{n_{q_1}}$, $w_{q_0}(x_0) \neq 0$, $t_0 \geq 0$, such that at time τ_i^-, $x(\tau_i^-) \in \mathbb{R}^{n_{q_1}}$ and $w_{q_i}(x(\tau_i^-)) = 0$ for all $i = 1 \ldots k$. And let the resulting control/trajectory pair for the hybrid system be denoted Ξ.

Furthermore, let the function, $J : \Xi \to \mathbb{R}$, be a real valued function of the control/trajectory pair that defines the performance of the control/trajectory pair, called the cost (or objective) function, which satisfies the equation

$$J(\Xi) = \sum_{i=1}^{k} \int_{\tau_i^+}^{\tau_i^-} L_i(x, \alpha, t) \cdot dt + \sum_{i=1}^{k-1} \Phi_i\left(x\left(\tau_i^-\right), \tau_i^-\right) + \Phi_0\left(x\left(t_0\right), t_0\right) + \Phi_k\left(x\left(\tau_k^-\right), \tau_k^-\right) \tag{3.2.1}$$

where for all i, $L_i : \mathbb{R}^{n_{q_i}} \times \mathbb{R}^{m_{q_i}} \times \mathbb{R} \to \mathbb{R}$ is a real valued function called the Lagrangian, $\Phi_i : \mathbb{R}^{n_{q_i}} \times \mathbb{R} \to \mathbb{R}$ is a cost associated with the discrete event characterized by $\left(x\left(\tau_i^-\right), \tau_i^-\right)$, $x\left(\tau_i^-\right) \in \mathbb{R}^{n_{q_i}}$, $\Phi_0 : \mathbb{R}^{n_{q_1}} \times \mathbb{R} \to \mathbb{R}$ is a cost associated with the initial condition of the system, and $\Phi_k : \mathbb{R}^{n_{q_k}} \times \mathbb{R} \to \mathbb{R}$ is a cost associated with the final condition of the system at time $\tau_k^- = t_f$.

Assume the cost function in equation (3.2.1) satisfies the following assumptions

1. For all $i = 1, \ldots, k$, L_i is Lipschitz continuous in the state variable, x, uniformly in the control variable, $\alpha \in u_{q_i}$, and the time $t \in \left(\tau_i^+, \tau_i^-\right]$. Equivalently, for all $x, y \in M_{q_i}$, $\alpha \in u_{q_i}$, $t \in \left(\tau_i^+, \tau_i^-\right]$ there exists a constant $L > 0$ such that
$$\left\| L_i(x, \alpha, t) - L_i(y, \alpha, t) \right\| \le L \cdot \|x - y\| \tag{3.2.2}$$
2. For all $i = 1, \ldots, k$, L_i is bounded on a ball centered at $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ with radius $R > 0$ for all admissible controls $\alpha \in u_{q_i}$.
3. For all $i = 1, \ldots, k-1$, Φ_i is Lipschitz continuous in the state variable, x, uniformly in the time t. Equivalently, for all $x, y \in M_{q_i}$, there exists a constant $G > 0$ such that
$$\left\| \Phi_i\left(x, \tau_i^-\right) - \Phi_i\left(y, \tau_i^-\right) \right\| \le G \cdot \|x - y\| \tag{3.2.3}$$
4. Φ_0 is Lipschitz continuous in the state variable, x, uniformly in the time t_0, i.e. for all $x, y \in M_{q_k}$, there exists a constant $G_0 > 0$ such that

$$\left\|\Phi_0\left(x,\tau_1^+\right)-\Phi_0\left(y,\tau_1^+\right)\right\| \le G_0 \cdot \|x-y\| \tag{3.2.4}$$

5. Φ_k is Lipschitz continuous in the state variable, x, uniformly in the time t_k, i.e. for all $x, y \in M_{q_1}$, there exists a constant $G_k > 0$ such that

$$\left\|\Phi_k\left(x,\tau_k\right)-\Phi_k\left(y,\tau_k\right)\right\| \le G_k \cdot \|x-y\| \tag{3.2.5}$$

As before these assumptions are required to apply the theory of Chapter 4 [7] to the hybrid optimal control problem.

Then the optimal control problem is to find the set of controls $v^* = \left\{v_1^*, v_2^*, \ldots v_k^*\right\}$ that minimizes the cost given in equation (3.2.1) for all possible initial conditions, while satisfying the constraints imposed by the hybrid control system defined by equation (3.1.2) and its associated assumptions.

Note that the Lipschitz continuity assumptions given above allow for the non-autonomous hybrid control problem to be written as an autonomous hybrid control problem by adding an extra state variable, $x_{n+1}(t)$, that represents time. Let

$$\dot{x}_{n+1}(t) = 1 \tag{3.2.6}$$

with initial condition $x_{n+1}(t_0) = t_0$, then the state variable $x_{n+1}(t)$ represents the time of the system and can replace all explicit references to time in the non-autonomous system, transforming it to an autonomous system.

Chapter 4: Optimal Control

The purpose of this chapter is to present the necessary and sufficient conditions for the optimal control of non-hybrid systems. The theory presented in this chapter provides the fundamental tools for understanding the hybrid maximum principles in Chapter 5 and the development of the necessary and sufficient condition in Chapter 6.

The Maximum Principles of Pontryagin (PMP) [6][55], Bardi [7], and Clarke/Vinter (CMP) [23][24][25][26][72] will be discussed. Each Maximum Principle will be presented without proof, but the proof will be discussed to facilitate understanding of the material. Furthermore, a “smooth” sufficient condition [6] and the non-smooth necessary and sufficient conditions of Bardi [7] will also be presented.

The maximum principles assume that a reference trajectory exists for a controlled system and analyzes the system’s properties when that control function is optimal. Unfortunately, since the conditions are only necessary, some or all of the control functions that satisfy the necessary conditions may **not** be optimal, and more analysis tools are required to identify the optimal control function(s) if they exist. Sufficient conditions provide the analysis tools that prove the control function(s) identified by the necessary conditions are optimal. Loosely speaking, the known sufficient conditions are difficult to use without the necessary conditions, because the sufficient conditions find all optimal control functions that produce trajectories to the final point from every point in the state space. Since the sufficient conditions compute all optimal solutions, they will produce a feedback optimal control. Feedback controls that satisfy the sufficient conditions are more useful than the open-loop control given by the necessary conditions. The main problem with using the sufficient conditions is that since a feedback control is calculated, much more computational power may be required. Combining the sufficient conditions with the necessary conditions allows for using the necessary conditions to narrow down all possible optimal controls and verification of the optimal control is given by the sufficient conditions.

The optimal control material will be presented in the following order. First the PMP will be given, then the "smooth" sufficient condition will be developed, next the non-smooth necessary and sufficient conditions of Bardi will be presented, and finally the non-smooth CMP will be discussed and compared to the work of Bardi.

4.1 Pontryagin's Maximum Principle (PMP)

Pontryagin's Maximum Principle (PMP) provides necessary conditions for a control to be the solution to a class of optimal control problems. Assuming an optimal control exists and is unique, the PMP necessary conditions narrow down the set of admissible controls to a set of controls that contains the optimal control.

The complete presentation and derivation of the PMP as given by Pontryagin can be found in [55]. The main result found in [55] is given to provide insight into the theoretical contribution of this dissertation. Most optimal control textbooks also provide a version of the PMP necessary conditions, for example see [6]. The PMP will be presented in the following order. First, the optimal control problem will be stated. Then the necessary conditions given by the PMP will be given for the proposed optimal control problem. Finally, the proof will be outlined for completeness.

4.1.1 Problem Formulation

The purpose of this section is to define the optimal control problem [55]. A simplified version of the optimal control problem given in Chapter 3 will be used as the basis for the derivation of the PMP. Assume that for the hybrid control problem, $q = 1$, so that the initial discrete location is the only discrete location that contains the system trajectory. Furthermore assume that:

1. Q has only one element.
2. $M = \mathbb{R}^n$, is the state space
3. $U \subset \mathbb{R}^m$, is the control space and is a subset of $\mathbb{R}^m$.
4. $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$, is a function that represents the dynamics of the system.
5. $u \in U$ is the set of admissible controls for the problem.
6. $I = \mathbb{R}$, is the bound on the switching time.
7. $S = \varnothing$, is empty.
8. The control problem satisfies the assumptions given in Chapter 3, pg. 46-47.

Given an admissible control function, $u(t) \in U$, defined for every $t_0 \le t < t_f$, the control problem produces a trajectory $x(t) \in M$, defined for every $t_0 \le t \le t_f$, that is the solution to the differential equation

$$\dot{x}(t) = f\left(x(t), u(t), t\right) \tag{4.1.1}$$

with initial condition $x(t_0) = x_0 \in M$.

Assume that the cost function has the form

$$J\left(x(t_0), u(t), t_f - t_0\right) = \int_{t_0}^{t_f} L(x,u,t) \cdot dt \tag{4.1.2}$$

where J satisfies the assumptions in Chapter 3, pg. 50. Note that in Chapter 3, the cost function was a function of the complete hybrid trajectory. For this control problem, the trajectory is completely defined by the initial condition $x(t_0)$, the control function $u(t)$ defined for every $t_0 \le t < t_f$, and the initial and final times t_0 and t_f, respectively.

Then the optimal control problem is to find the control function $u(t)$, defined over $t_0 \le t < t_f$, that minimizes the cost function in equation (4.1.2) while satisfying the constraints of the control problem, i.e. equation (4.1.1) and $x(t_0) = x_0$.

4.1.2 Necessary Conditions

The PMP provides a set of necessary conditions that the control must satisfy in order to be optimal. Since the PMP only gives necessary conditions, a control that is not optimal may satisfy the necessary conditions, but every optimal control must satisfy the necessary condition. So the PMP necessary conditions provide a set of candidate optimal control functions which must contain the optimal control (assuming that it exists). The necessary conditions can be summarized in the following theorem.

Theorem 4.1.1 [55]

Let $\hat{u}(t) \in U$, for $t_0 \le t \le t_f$, be an admissible control and $\hat{x}(t)$, for $t_0 \le t \le t_f$, be the solution to the control problem with initial condition $x(t_0)$. Furthermore, let equation (4.1.2) be the cost associated with the trajectory $\hat{x}(t)$, $t_0 \le t \le t_f$.

If $\hat{u}(t)$ and $\hat{x}(t)$, $t_0 \le t \le t_f$, are the optimal control function and corresponding (state) trajectory, then there exists a nonzero absolutely continuous vector function $\lambda(t)$ which is the solution of the differential equation,

$$\frac{d\lambda(t)}{dt} = -\frac{\partial H\left(\hat{x}(t),\hat{u}(t),\lambda(t),t\right)}{\partial x} \tag{4.1.3}$$

with final condition $\lambda(t_f)$, where $H\left(\hat{x}(t),\hat{u}(t),\lambda(t),t\right)$ is a real function, $H:\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, defined as

$$H\left(\hat{x}(t),\hat{u}(t),\lambda(t),t\right) = \left\langle \lambda(t), f\left(\hat{x}(t),\hat{u}(t)\right)\right\rangle - \lambda_0(t)\cdot L(x,u,t) \tag{4.1.4}$$

where $\langle \bullet,\bullet \rangle$ denotes the standard inner product, viz. $\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i$ for any n-vectors x and y, such that the following conditions are true:

1. $H\left(\hat{x}(t),\hat{u}(t),\lambda(t),t\right) = \sup_{v\in U} H\left(\hat{x}(t),v,\lambda(t),t\right)$.
2. At the terminal time t_f, $\lambda_0(t_f) \le 0$, $H\left(\hat{x}(t_f),\hat{u}(t_f),\lambda(t_f),t_f\right) = 0$, and if condition 1 is satisfied, then $\lambda_0(t)$ and $H\left(\hat{x}(t_f),\hat{u}(t_f),\lambda(t_f),t_f\right)$ are constant at almost every time $t_0 \le t \le t_f$.

Further, the concepts of Theorem 3.1.1 can be extended to allow for variations in the endpoints (both temporal and spatial) as well as allowing the final and initial points to lie in a defined subset of the state space.

These extensions require more necessary conditions that come in the form of transversality conditions. The transversality conditions allow the MP to be applied to problems with variable endpoints. Let S_0 be an r_0 dimensional smooth manifold with $r_0 < n$ and S_1 be an r_1 dimensional smooth manifold with $r_1 < n$ and impose the constraints that $x(t_0) \in S_0$ and $x(t_f) \in S_1$. Then the transversality conditions provide the additional necessary conditions given in Theorem 4.1.2.

Theorem 4.1.2 [55]

Assume the optimal control problem given above with the additional constraints $x(t_0) \in S_0$ and $x(t_f) \in S_f$.

If $\hat{u}(t)$ and $\hat{x}(t)$, $t_0 \le t \le t_f$, are the optimal control function and corresponding (state) trajectory, then there exists a nonzero absolutely continuous vector function $\lambda(t)$ which satisfies equation (4.1.3), the necessary conditions given in Theorem 4.1.1 and the transversality conditions at both endpoints of the trajectory $x(t)$, where the transversality conditions require

$$\begin{aligned} \langle \lambda(t_0), p_0 \rangle &= 0 \\ \langle \lambda(t_f), p_1 \rangle &= 0 \end{aligned} \tag{4.1.5}$$

where $p_0 \in \mathbb{R}^n$ is any vector that belongs to or is parallel to the tangent hypersurface, T_0, of the set S_0 at the point $x(t_0)$, and $p_1 \in \mathbb{R}^n$ is any vector that belongs to or is parallel to the tangent hypersurface, T_1, of the set S_1 at the point $x(t_f)$.

4.1.3 PMP Proof Outline

The proof of the PMP can be found in **[55]** and will **not** be given here. The purpose of this section is to provide insight into the proof, to help understand its derivation.

The PMP is proved in four steps. The first step is to assume that a reference control function exists that produces a resultant trajectory that satisfies the problem assumptions. The next step is to perform variations in the temporal, spatial and control variables to this reference trajectory, in order to calculate the variation in the cost functional. Third, all possible variations in the cost functional are calculated and collected in the cone of attainability. Finally, since the reference trajectory is assumed to be optimal the cone of attainability will not contain the vector of improved cost, and the necessary conditions are developed. See **[55]** for the complete development of the proof and [6], Chapter 5, for a less rigorous heuristic proof of the PMP.

Variations in Trajectory

The purpose of this section is to develop the variation in reference trajectory associated with variations in temporal, spatial and control variables. The information is presented to give the reader an idea about the proof of the PMP and not provide a proof of the PMP.

The first step in developing the variation is calculating the variation in trajectory associated with a variation in initial condition. Given the control problem in the previous section, let $u(t)$, $t_0 \le t \le t_f$, be some arbitrary admissible control function and let $x(t)$, $t_0 \le t \le t_f$, represent the corresponding solution to equation (4.1.1) with the initial condition $x(t_0)$.

Further let $y(t)$, $t_0 \le t \le t_f$, represent another solution to equation (4.1.1) using the same control function $u(t)$, $t_0 \le t \le t_f$, only starting at the initial condition defined by

$$y(t_0) = x(t_0) + \varepsilon \cdot \xi(t_0) + o(\varepsilon) \tag{4.1.6}$$

Define $\delta x(t)$ to be a vector not dependent on ε that is the solution of the differential equation

$$\frac{d(\delta x(t))}{dt} = \frac{\partial f(x(t), u(t), t)}{\partial x} \cdot \delta x(t) \tag{4.1.7}$$

with the initial condition:

$$\delta x(t_0) = \xi(t_0) \tag{4.1.8}$$

Now, let $\delta x_{\xi_0}(t)$ be the solution of equation (4.1.7) with initial condition ξ_0, and let $\delta x_{\xi_0}(t)$ be bounded for all $\delta x_{\xi_0}(t)$, $t_0 \le t \le t_f$. Next let $\Phi(t, t_0)$ be the state transition matrix for the differential equation (4.1.7). Thus,

$$\delta x_{\xi_0}(t) = \Phi(t, t_0) \cdot \delta x_{\xi_0}(t_0) \tag{4.1.9}$$

Using the work in Appendix A, the variation in trajectory associated with a variation in initial condition can be calculated as

$$y(t) - x(t) = \Phi(t, t_0) \cdot \left(y(t_0) - x(t_0) \right) + o(\varepsilon) \tag{4.1.10}$$

Now that the variation in initial condition has been developed, a needle variation in control will be calculated and the equation describing the variation in trajectory will be given.

A needle variation in control is a control function that is defined with respect to a reference control, where over a finite number of small time intervals, the value of the control jumps from the reference value to some other value in the control space. Since the control instantaneously changes value, a discontinuity in the control is produced. This variation in control makes the PMP's necessary conditions more general than the conditions derived from the classical Calculus of Variations arguments, since the Calculus of Variations necessary conditions only analyze smooth variations in control [6].

Let $u(t)$ be an admissible control function defined on the interval $t_0 \le t \le t_f$. Now pick specific instants of time $t_1, t_2, \ldots, t_s,\ \tau \in \left(t_0, t_f\right)$, where s is finite, which are regular points for $u(t)$, and satisfy the inequality:

$$t_0 < t_1 \le t_2 \le \ldots \le t_s \le \tau < t_f \tag{4.1.11}$$

the time τ is associated with a special variation, so is spelled out specifically in equation (4.1.11) for future use.

Now, pick an arbitrary $\delta t \in \mathbb{R}$, an arbitrary set of non-negative real numbers $[\delta t_1, \delta t_2, \ldots, \delta t_s] \in \mathbb{R}$, and an arbitrary set of admissible control values $v_i \in U$, $i = 1\ldots s$ and define for i, $i = 1\ldots s$, lengths, l_i, as

$$l_i = \begin{cases} \delta t - \left(\delta t_i + \ldots + \delta t_s\right),\ t_i = \tau \\ -\left(\delta t_i + \ldots + \delta t_s\right),\ t_i = t_s < \tau \\ -\left(\delta t_i + \ldots + \delta t_j\right),\ t_i = t_{i+1} = \ldots = t_j < t_{j+1}\left(j < s\right) \end{cases} \tag{4.1.12}$$

Next, define the s open half intervals, $I_1,\ I_2, \ldots I_s$ as

$$I_i = \left\{t : t_i + \varepsilon l_i < t \le t_i + \varepsilon\left(l_i + \delta t_i\right)\right\} \tag{4.1.13}$$

Equation (4.1.13) is going to be used to define a finite set of time intervals over which a variation in control is applied.

In order to understand the definition of the length l_i and interval I_i, three examples will be given. For all three assume that $s = 3$. First pick the following distinct times

$$t_0 < t_1 < t_2 < t_3 < \tau < t_f \tag{4.1.14}$$

Since $s = 3$ and the times t_1, t_2, and t_3 are distinct, the lengths l_i and time intervals are as follows:

1. For $i = 1$, the third equation in equation (4.1.12) is used, $i = j = 1$ and

$$l_1 = -\delta t_1,\ I_1 = \left\{t : t_1 - \varepsilon \cdot \delta t_1 < t \le t_1\right\} \tag{4.1.15}$$

2. For $i = 2$, the third equation in equation (4.1.12) is used, $i = j = 2$ and

$$l_2 = -\delta t_2,\ I_2 = \left\{ t : t_2 - \varepsilon \cdot \delta t_2 < t \le t_2 \right\} \tag{4.1.16}$$

3. For $i = 3$, the second equation in equation (4.1.12) is used, and

$$l_3 = -\delta t_3,\ I_3 = \left\{ t : t_3 - \varepsilon \cdot \delta t_3 < t \le t_3 \right\} \tag{4.1.17}$$

Now pick the following non-distinct times to define the intervals I_i,

$$t_0 < t_1 = t_2 < t_3 < \tau < t_f \tag{4.1.18}$$

Since $s = 3$, the times t_1 and t_2 are non-distinct and t_3 and τ are distinct, the lengths l_i and time intervals are as follows:

1. For $i = 1$, the third equation in equation (4.1.12) is used, $i = 1$, $j = 2$ and

$$l_1 = -\delta t_1 - \delta t_2,\ I_1 = \left\{ t : t_1 - \varepsilon \cdot \left(\delta t_1 + \delta t_2 \right) < t \le t_1 - \varepsilon \cdot \delta t_2 \right\} \tag{4.1.19}$$

2. For $i = 2$, the third equation in equation (4.1.12) is used, $i = j = 2$ and

$$l_2 = -\delta t_2,\ I_2 = \left\{ t : t_2 - \varepsilon \cdot \delta t_2 < t \le t_2 \right\} = \left\{ t : t_1 - \varepsilon \cdot \delta t_2 < t \le t_1 \right\} \tag{4.1.20}$$

3. For $i = 3$, the second equation in equation (4.1.12) is used, and

$$l_3 = -\delta t_3,\ I_3 = \left\{ t : t_3 - \varepsilon \cdot \delta t_3 < t \le t_3 \right\} \tag{4.1.21}$$

Finally, pick the following non-distinct times to define the intervals I_i,

$$t_0 < t_1 = t_2 < t_3 = \tau < t_f \tag{4.1.22}$$

Since $s = 3$, the times t_1 and t_2 are non-distinct and the times t_3 and τ are non-distinct, the lengths l_i and time intervals are as follows:

1. For $i = 1$, the third equation in equation (4.1.12) is used, $i = 1$, $j = 2$ and

$$l_1 = -\delta t_1 - \delta t_2,\ I_1 = \left\{ t : t_1 - \varepsilon \cdot \left(\delta t_1 + \delta t_2 \right) < t \le t_1 - \varepsilon \cdot \delta t_2 \right\} \tag{4.1.23}$$

2. For $i = 2$, the third equation in equation (4.1.12) is used, $i = j = 2$ and

$$l_2 = -\delta t_2,\ I_2 = \left\{ t : t_2 - \varepsilon \cdot \delta t_2 < t \le t_2 \right\} = \left\{ t : t_1 - \varepsilon \cdot \delta t_2 < t \le t_1 \right\} \tag{4.1.24}$$

3. For $i = 3$, the first equation in equation (4.1.12) is used, and

$$\begin{aligned} l_3 &= \delta t - \delta t_3, \\ I_3 &= \left\{ t : t_3 + \varepsilon \cdot \left(\delta t - \delta t_3 \right) < t \le t_3 + \varepsilon \cdot \delta t \right\} \\ &= \left\{ t : \tau + \varepsilon \cdot \left(\delta t - \delta t_3 \right) < t \le \tau + \varepsilon \cdot \delta t \right\} \end{aligned} \tag{4.1.25}$$

Note that if all of the t_i are distinct, then each t_i represents the right hand endpoint of the interval I_i and if ε is small enough, then all of the intervals I_i are mutually disjoint. If $t_i = t_{i+1} = \ldots = t_j < t_{j+1}$, then the intervals $I_i,\ I_{i+1},\ \ldots,\ I_j$ all border one another and the right endpoint of $\bigcup_{k=i}^{j} I_k$ is t_i.

Now the needle variation in control can be defined.

Definition 4.1.3 Needle Variation of Control [55]

Pick a set of s times that satisfy equation (4.1.11) and let $v = \{v_1, v_2, \ldots, v_s\}$ be a set of values v_i such that $v_i \in U$, for all $i = 1, 2, \ldots, s$ and define a control function, $u(t)$ for $t_0 \le t < t_f$, as a reference control. Then a needle variation in control is a control function, $u^*(t)$, defined over the time interval $t_0 \le t \le \tau + \varepsilon\delta t$ which has the following form

$$u^*(t) = \begin{cases} u(t),\ t \notin I_i \\ v_i,\ t \in I_i \end{cases} \tag{4.1.26}$$

for all $i = 1 \ldots s$.

Note for small enough ε, the control defined by equation (4.1.26) is admissible, and hence is a permissible control for the system.

Now using Definition 4.1.3 and the definitions in Appendix A, the variation in trajectory can be calculated. Let $x(t_0)$ be the initial condition of the reference trajectory, $x(t)$, defined by the admissible control, $u(t)$, and dynamic equation

$$\dot{x}(t) = f(x,u,t) \tag{4.1.27}$$

defined over the time interval $t_0 \le t \le t_f$.

Furthermore, let $x^*(t_0)$ be the initial condition, parameterized by ξ_0, for the perturbed trajectory $x^*(t)$ associated with the new control, $u^*(t)$, which is a needle variation of the reference control, and the dynamics given in equation (4.1.27), defined over the time interval $t_0 \le t \le \tau + \varepsilon \cdot \delta t$.

The trajectories $x(t)$ and $x^*(t)$ are solutions to equation (4.1.27) under their respective initial conditions and controls and can be written as

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(x,u,t) \cdot dt \\ x^*(t) &= x^*(t_0) + \int_{t_0}^{t} f(x^*,u^*,t) \cdot dt \end{aligned} \tag{4.1.28}$$

The work in Appendix A can be used to calculate the value of the trajectory under the various types of variation and will be summarized here. Three possible situations exist and are

1. When $u^*(t) = u(t)$ for the interval of time $t_1 \le t \le t_2$, which is not an interval of size $\varepsilon \cdot \delta t$, $x^*(t)$ can be represented as $x^*(t) = x(t) + \varepsilon \cdot \Phi(t,t_1) \cdot \xi(t_1) + o(\varepsilon)$,

where $\xi(t_1)$ is a vector representing a variation in initial condition and $\Phi(t,t_1)$ is the state transition matrix describing $\delta x(t)$.

2. When $u^*(t)=u(t)$ for the interval of time $\tau \le t \le \tau+\varepsilon\delta t$, then:

$$\begin{aligned} x^*(\tau+\varepsilon\delta t) &= x^*(\tau)+\varepsilon\cdot f\left(x(\tau),u(\tau),\tau\right)\cdot\delta t+o(\varepsilon) \\ x(\tau+\varepsilon\delta t) &= x(\tau)+\varepsilon\cdot f\left(x(\tau),u(\tau),\tau\right)\cdot\delta t+o(\varepsilon) \end{aligned} \tag{4.1.29}$$

3. When $u^*(t)=v_i$ for the interval of time $t_i+\varepsilon\cdot l_i \le t \le t_i+\varepsilon\cdot(l_i+\delta t_i)$, then:

$$x^*\left(t_i+\varepsilon\cdot(l_i+\delta t_i)\right)=x^*(t_i+\varepsilon\cdot l_i)+\varepsilon\cdot\delta t_i\cdot\left(f\left(x(t_i),v_i,t_i\right)\right)+o(\varepsilon) \tag{4.1.30}$$

Note that the three cases given above can be pieced together to derive a general equation for $x^*(\tau+\varepsilon\delta t)$, associated with temporal variations of the trajectory, spatial variations in the initial condition of the trajectory, and needle variations of the reference control. Let $\xi(t_0)=\xi_0$ and assume that $u^*(t)$ is a needle variation of the reference control, then

$$x^*(\tau+\varepsilon\delta t)=x(\tau)+\varepsilon\cdot\Phi(\tau,t_0)\cdot\xi(t_0)+\varepsilon\cdot\vartriangle x(\tau)+o(\varepsilon) \tag{4.1.31}$$

where:

$$\begin{aligned} \vartriangle x(\tau) = {} & f\left(x(\tau),u(\tau),\tau\right)\cdot\delta t \\ & +\sum_{i=1}^{s}\Phi(\tau,t_i)\cdot\left[f\left(x(t_i),v_i,t_i\right)-f\left(x(t_i),u(t_i),t_i\right)\right]\cdot\delta t_i \end{aligned} \tag{4.1.32}$$

Equation (4.1.31) is a very important equation and provides the backbone for the proof of the maximum principle because as long as every needle variation, spatial variation and temporal variation provides an increase in the cost function, then $x(t)$ must be the optimal (state) trajectory and $u(t)$ the optimal control.

[55] (pg. 89) uses an induction argument to prove that equations (4.1.31) and (4.1.32) provide the first order approximation to $x^*(\tau+\varepsilon\delta t)$ for any finite number of needle variations of control, and the interested reader is referred there for the complete proof.

Cone of Attainability and Necessary Conditions

Now that a formula has been derived for the variation in the reference trajectory associated with variations in control, space, and time, it can be used to show that the resultant state trajectory and cost lie in a convex cone whose apex lies on the reference trajectory, called the cone of attainability.

Now, the $\triangle x$ term in equation (4.1.32) can be thought of as a bounded vector that originates from the point $x(\tau)$ on the reference trajectory. For a fixed τ, all possible variation vectors $\triangle x$ will fill out a set K_τ in X_τ, where X_τ is a subspace of $\mathbb{R}^n$ and K_τ is a convex cone with origin at $x(\tau)$, called the cone of attainability.

It is the assumption that the reference trajectory is optimal and the properties of the cone of attainability that provide the necessary conditions of the PMP. [55] proves that for any regular point, $x(t)$, along the reference trajectory, with a curve that starts at $x(t)$ and a tangent vector that is completely contained in the cone of attainability, there exists an admissible needle variation in the reference control whose trajectory, with initial condition $x(t_0)$, intersects that curve. Figure 4.1 depicts a curve Λ emanating from the regular point $x(\tau)$ that has a tangent vector L completely contained in the cone of attainability K_τ.

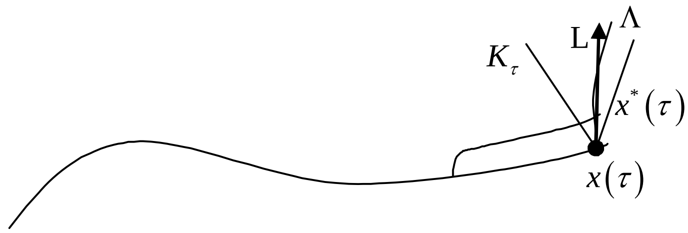

Figure 4.1: Geometric depiction of the Cone of Attainability.

Finally, [55] uses the geometric concepts in Appendix B to prove that since the vector of improved cost is not contained in the cone of attainability, a plane exists that passes through $x(\tau)$ and separates the vector of improved cost from the cone of attainability. It is the properties of this plane and the system adjoint to the variation in trajectory which form the basis for the necessary conditions.

4.2 Smooth Sufficient Condition for Optimality

The PMP only provides necessary conditions for finding the optimal solution of a control problem. A complete theory of optimal control needs methods to distinguish those controls that are truly optimal from those that satisfy the necessary conditions but are not optimal. Sufficient conditions are one way to do this.

The problem of finding sufficient conditions satisfied by an optimal solution is classical. Well known sufficient conditions are associated with the names Hamilton, Jacobi, Caratheodory, and Bellman. Bellman called the technique he developed Dynamic Programming.

In this section, the sufficient conditions for a control to be optimal for a very restrictive class of problems will be developed.

The sufficient conditions are based on the Hamilton-Jacobi-Caratheodory-Bellman (HJCB) partial differential equation (PDE). The HJCB PDE is an equation that provides conditions on the optimal control that are dependent upon the differentiability with respect to the state of the optimal cost-to-go from the current state to the final state. What makes this theory restrictive is that the optimal cost-to-go function must be differentiable along the reference trajectory, which is often not the case.

4.2.1 Sufficient Conditions for Differentiable Value Functions

The purpose of this section is to develop the HJCB PDE and a sufficient condition for a candidate control to be optimal. This development comes from [6] (Section 5-18 to Section 5-20) and provides a fundamental formulation of the material.

First the control problem will be given and the cost-to-go function will be defined. Then the HJCB PDE will be developed and finally the sufficient condition given.

4.2.2 Control Problem

Let the control problem be the one presented in the PMP section on pages 54-55 with the dynamic constraints given in equation (4.1.1), cost function given in equation (4.1.2), and the following additional assumptions

1. f is continuous in x and measurable, in the sense of Lebesgue, in u and t

2. Let the final state $\left(x\left(t_f\right),t_f\right)\in S$, where S is a smooth manifold in $\mathbb{R}^n\times\mathbb{R}$

3. L and $\dfrac{\partial L}{\partial x}$ are given and continuous on the direct product $\mathbb{R}^n\times\bar{U}$, where $\bar{U}$ is the closure of U.

In order to develop the sufficient condition, this control problem will be embedded into a larger control problem. Find the feedback control $u\left(x(t),t\right)$, $t_0\le t\le t_f$, for all $x(t)$ such that $\left(x\left(t_f\right),t_f\right)\in S$ the cost function

$$J\left(x(t),u(\tau),t_f-t\right)=\int_t^{t_f}L\left(x,u,\tau\right)\cdot d\tau \tag{4.2.1}$$

is minimized for all $t_0\le t\le t_f$, where $x(t)$ is the initial state, $u(\tau)$, $t\le\tau\le t_f$, is the control that transfers the state from $x(t)$ to $\left(x\left(t_f\right),t_f\right)\in S$, and t_f-t is the time to go. Note that the larger problem defined for derivation of the sufficient conditions is an embedding of all possible optimal control problems that satisfy the problem constraints, so the general solution will result in an optimal closed-loop feedback control function for the system.

4.2.3 Sufficient Condition

The purpose of this section is to derive the sufficient condition using the Hamilton-Jacobi-Caratheodory-Bellman partial differential equation. Define $u\left(x(t),t\right)$, for all t

such that $t_0 \le t \le t_f$, as the feedback control that transfers the state from initial condition $\left(x(t),t\right) \in X$ to $\left(x\left(t_f\right),t_f\right) \in S$. For every time t_1 such that $t \le t_1 \le t_f$, denote the value of the control at time t_1 by $u\left(t_1\right)$ and the trajectory by $x\left(t_1\right)$.

Definition 4.2.1 [6]

Let $u(\tau)$, $t_0 \le \tau \le t_f$, be an admissible control that transfers the state from $x\left(t_0\right)$ to the final set S along the trajectory $x(\tau)$, $t_0 \le \tau \le t_f$. Then the cost-to-go function will be defined as the function

$$J_c\left(x(t),t_f - t\right) = \int_t^{t_f} L\left(x(\tau),u(\tau),\tau\right)\cdot d\tau \qquad (4.2.2)$$

where $J_c\left(x(t),t_f - t\right)$ is a differentiable function defined on a region $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ such that $\left(x(t),t\right) \in \Sigma$.

Note that the cost-to-go function can be written in the shorthand notation (the dependency of J_c on u has been dropped) because the control has now been defined in terms of the state variable.

Because of the differentiability assumption, equation (4.2.2) can be differentiated resulting in

$$H\left(x,-\frac{\partial J_c\left(x(t),t_f - t\right)}{\partial x},u,1,t\right) - \frac{\partial J_c\left(x(t),t_f - t\right)}{\partial t} = 0 \qquad (4.2.3)$$

where H is the Hamiltonian given by

$$H\left(x,-\frac{\partial J_c\left(x(t),t_f-t\right)}{\partial x},u,1,t\right) = \left\langle -\frac{\partial J_c\left(x(t),t_f-t\right)}{\partial x}, f\left(x,u,t\right)\right\rangle - L\left(x,u,t\right) \tag{4.2.4}$$

and $\lambda_0 = 1$ without loss of generality.

Now the following Caratheodory lemma gives properties of the control that produces the minimum cost-to-go to the target set, S, for all trajectories that lie in the subset X.

Lemma 4.2.2, Caratheodory [6]

Suppose that for each point (x,t) in $X \subseteq \mathbb{R}^n \times \mathbb{R}$, a function $G(x,\omega,t)$ has, as a function of ω, zero as its unique absolute minimum with respect to all ω in U at $\omega = u^o(x,t)$, hence that

$$0 = G\left(x,u^o(x,t),t\right) < G\left(x,\omega,t\right) \tag{4.2.5}$$

for all $\omega \in U$, such that $\omega \neq u^0(x,t)$. Furthermore, let $\hat{u}$ be an admissible control such that:

1. $\hat{u}\left(x(t_0),t_0\right)$ transfers (x_0,t_0) to $\left(x(t_f),t_f\right) \in S$
2. if $\hat{x}(t)$ is the trajectory corresponding to $\hat{u}(\tau)$, then for all $t \in \left[t_0,t_f\right]$, $\left(\hat{x}(t),t\right) \in X$.
3. for all $\tau \in \left[t_0,t_f\right)$, $\hat{u}(\tau)$ satisfies the relation $\hat{u}(\tau) = u^o(\hat{x},\tau)$.

Then $\hat{u}(\tau)$ is an optimal control relative to the set of controls u that generate trajectories lying entirely in X, and the cost $J\left(\hat{x}(t),\hat{u}(\tau),t_f - t\right)$ is zero for $t \in [t_0, t_1)$.

The Caratheodory Lemma says that for all admissible controls that transfer the state from (x_0, t_0) to S over the interval $t_0 \le t \le t_f$, such that $\left(x(t),t\right) \in X$, the control $\hat{u}(\tau)$ is optimal. Note that $\hat{u}(\tau)$ may not be the absolute minimum control, because a control may exist that produces a trajectory that leaves X, returns to X, hits the target set S, and still produces a cost that is lower than $J\left(\hat{x}(t),\hat{u}(\tau),t_f - t\right)$. As such the theorem only produces a local result.

Now the Hamiltonian will be used to define the H-Maximal control.

Definition 4.2.3 H-Maximal Control [6]

Let the Hamiltonian be defined in its usual sense

$$H(x,\lambda,u,t) = \left\langle \lambda, f(x,u,t) \right\rangle - \lambda_0 \cdot L(x,u,t) \tag{4.2.6}$$

If for each point $(x,t) \in X$ the function $H(x,\lambda,\omega,t)$ has, as a function of ω, a unique absolute maximum with respect to all $\omega \in U$ at $\omega = \tilde{u}(x,\lambda,t)$, then H is normal relative to X and $\tilde{u}(x,\lambda,t)$ is the H-maximal control.

Note that this definition differs slightly from that given in [6]. [6] defines the Hamiltonian as

$$H\left(x,\lambda,u,t\right)=\left\langle \lambda, f\left(x,u,t\right)\right\rangle+\lambda_0\cdot L\left(x,u,t\right) \tag{4.2.7}$$

and as such minimizes the Hamiltonian instead of maximizing it. Since it is desirable to use similar notation as that in the development of the Maximum Principle, equation (4.2.6) will be used to define the Hamiltonian and the sufficient condition.

Now that the H-maximal control has been defined, the HJCB PDE can be given.

Definition 4.2.4 HJCB [6]

If H is normal relative to X and $\hat{u}\left(t\right)=\tilde{u}\left(x,\lambda,t\right)$ is the H-maximal control relative to X, then the HJCB is

$$H\left(\hat{x},-\frac{\partial\hat{J}_c\left(\hat{x}\left(t\right),t_f-t\right)}{\partial x},\hat{u},1,t\right)-\frac{\partial\hat{J}_c\left(\hat{x}\left(t\right),t_f-t\right)}{\partial t}=0 \tag{4.2.8}$$

where

$$\hat{J}_c\left(\hat{x}\left(t\right),t_f-t\right)=0 \tag{4.2.9}$$

for all $\left(\hat{x},t\right)\in S$.

Now a theorem similar to the previous Lemma can be given which provides the smooth sufficient condition.

Theorem 4.2.5, Local Sufficient [6]

Let H be normal relative to X and $\tilde{u}(x,\lambda,t)$ be the corresponding H-maximal control relative to X, see Definition 4.2.3. Further let $\hat{u}$ be an admissible control such that:

1. $\hat{u}$ transfers (x_0,t_0) to $(x(t_f),t_f)\in S$
2. if $\hat{x}(t)$ is the trajectory corresponding to $\hat{u}(\tau)$, then for all $t\in[t_0,t_f]$, $(\hat{x}(t),t)\in X$.
3. there is a solution $\hat{J}_c(x(t),t_f-t)$ of the HJCB such that $\hat{J}_c(x(t),t_f-t)=0$ for all $(\hat{x},t)\in S$ and for all $t\in[t_0,t_f)$, $\hat{u}(t)=\tilde{u}\left(\hat{x}(t),-\frac{\partial\hat{J}_c(\hat{x}(t),t_f-t)}{\partial x},t\right)$.

Then $\hat{u}(\tau)$ is an optimal control relative to the set of controls $u\in U$ that generate trajectories lying entirely in X, and for all $t\in[t_0,t_f)$

$$J(\hat{x}(t),\hat{u}(\tau),t_f-t)=\hat{J}_c(\hat{x}(t),t_f-t) \tag{4.2.10}$$

The proof of the theorem follows from definition of the Hamiltonian and Lemma 4.2.2 by letting

$$G(x,\omega,t)=H\left(x,-\frac{\partial J_c(x(t),t_f-t)}{\partial x},\omega,1,t\right)-\frac{\partial J_c(x(t),t_f-t)}{\partial t} \tag{4.2.11}$$

and when $u^o(x,t)=\hat{u}(t)$

$$G(x,u^o(x,t),t)=H\left(\hat{x},-\frac{\partial\hat{J}_c(\hat{x}(t),t_f-t)}{\partial x},\hat{u},1,t\right)-\frac{\partial\hat{J}_c(\hat{x}(t),t_f-t)}{\partial t} \tag{4.2.12}$$

The result of this theorem is that if a candidate control exists that satisfies conditions 1-3 of the Theorem 4.2.5, then the control is optimal with respect to controls that produce trajectories contained in X and the cost-to-go function associated with the candidate control equals the cost-to-go function associated with the optimal control. Since this method only compares trajectories that lie within a specific region of the state-space, it only provides a local optimal control solution. If the region X can be expanded to the entire state-space, then it will provide a global optimal control solution. Note that if $X_1 \subset X$ and the trajectory $\hat{x} \in X_1$ is the optimal trajectory defined by the optimal control $\hat{u}$ relative to X, then $\hat{u}$ is the optimal control relative to X_1 as well. Obviously, the converse statement violates the conditions of the Caratheodory Lemma, so the optimal control may not be optimal with respect to a larger set X.

4.3 Bardi Non-Smooth Necessary and Sufficient Conditions

The purpose of this section is to generalize the concepts found in the previous section to a much larger and more useful class of optimal control problems. It removes many of the restrictive assumptions and uses abstract mathematical methods to develop sufficient conditions for optimality of the candidate control. Furthermore, the theory is general enough that necessary conditions are given as well, providing a complete set of necessary and sufficient conditions for optimality of control for a large class of optimal control problems.

As in the previous section, the sufficient conditions are developed from the HJCB PDE. The main difference between the previous work and the work in this section is that the assumption on the differentiability of the optimal cost-to-go function is relaxed. The previous work assumed the optimal cost-to-go function was differentiable in a neighborhood around and along the reference trajectory while the work in this section allows the optimal cost-to-go function to have a finite number of points where it is not differentiable.

4.3.1 Non-Smooth HJCB

The sufficient conditions in the last section can be extended to problems where the first and second partial derivatives of the optimal cost-to-go function sometimes fail to exist. Under assumptions of continuity (which also can be relaxed), problems with non-smooth optimal cost-to-go functions do satisfy the HJCB equation and have the same equivalence to the adjoint. The theory required to prove these results is based on the theory of viscosity solutions to partial differential equations [7] which will be briefly introduced in this section.

This section is outlined as follows: first a simple example will be given that demonstrates the non-smoothness of the optimal cost-go-function, then the viscosity solution theory will be presented and then the HJCB and the necessary and sufficient conditions will be developed for systems with dynamics and cost functions that do not explicitly depend on time. The assumption that the system doesn't depend explicitly on time is not that restrictive because it will be shown that systems that satisfy the assumptions in Chapter 3 and depend explicitly on time can be transformed into a system that is independent of time.

4.3.2 Example

A very simple example which demonstrates the non-smoothness of the cost-to-go function is the minimum time to the origin problem for the double integrator, presented in [6]. The optimal control problem is a free time, fixed endpoint problem that minimizes the time, t_f, to the origin from any initial state $x(t_0) \in \mathbb{R}^2$, with the dynamic constraints

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{4.3.1}$$

and the admissible control set $u(t) \in [-1,1]$.

The cost function for this problem is

$$\begin{aligned} J(u) &= \int_{t_0}^{t_f} 1 \cdot dt \\ J(u) &= (t_f - t_0) \end{aligned} \tag{4.3.2}$$

This problem is a standard optimal control problem that is used to demonstrate the usefulness of the PMP.

The optimal control solution is bang-bang; the optimal control satisfies the following feedback control law

$$u(t) = \begin{cases} 1, \ x_1(t) < -\dfrac{1}{2} x_2(t) \left| x_2(t) \right| \wedge \left[x_1(t) = \dfrac{1}{2} x_2{}^2(t), x_2(t) \le 0 \right] \\ -1, \ x_1(t) > \dfrac{1}{2} x_2(t) \left| x_2(t) \right| \wedge \left[x_1(t) = -\dfrac{1}{2} x_2{}^2(t), x_2(t) \ge 0 \right] \end{cases} \tag{4.3.3}$$

where $\wedge$ is the “and” operator, see [6] for derivation.

The cost-to-go to the origin can be computed for this problem given the feedback control law, equation (4.3.3), and the initial condition $x(t_0)$. Let $x(t_0) \in \mathbb{R}^2$ be the initial state for the problem and the control given by equation (4.3.3), the time to go to the origin from the initial state is

$$t_f\left(x(t_0)\right) = \begin{cases} x_2(t_0) + \sqrt{4x_1(t_0) + 2x_2^{\,2}(t_0)},\ x_1(t_0) > -\frac{1}{2}x_2(t_0)\left|x_2(t_0)\right| \\ -x_2(t_0) + \sqrt{-4x_1(t_0) + 2x_2^{\,2}(t_0)},\ x_1(t_0) < -\frac{1}{2}x_2(t_0)\left|x_2(t_0)\right| \\ \left|x_2(t_0)\right|,\ x_1(t_0) = -\frac{1}{2}x_2(t_0)\left|x_2(t_0)\right| \end{cases} \qquad (4.3.4)$$

Note that equation (4.3.4) is the value of optimal cost-to-go function, $\hat{J}_c\left(\hat{x}(t_0), t_0\right)$, in the field of extremal trajectories defined by the control law given in equation (4.3.3).

Now from equation (4.3.4), the cost function $\hat{J}_c\left(\hat{x}(t_0), t_0\right)$ is continuous, but is not differentiable for x_1 and x_2 lying on the trajectory $x_1(t) = -\frac{1}{2}x_2(t)\left|x_2(t)\right|$ and hence the HJCB results given in the previous section do not apply because $\frac{\partial \hat{J}_c\left(\hat{x}(t_0), t_0\right)}{\partial x}$ doesn't exist along the entire optimal trajectory.

4.3.3 Viscosity Solutions

The purpose of this section is to introduce continuous viscosity solutions of partial differential equations, specifically Hamilton-Jacobi equations, and their associated properties. The definitions and properties of viscosity solutions of partial differential equations will be presented here without proof. These results with proof can be found in [7][32][49].

Let F be a Hamilton-Jacobi equation that is a real-valued continuous Hamiltonian function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ of the form

$$F\left(x, u(x), Du(x)\right) = \frac{\partial u(x)}{\partial t} + H\left(x, Du(x)\right) \tag{4.3.5}$$

satisfying

$$F\left(x, u(x), Du(x)\right) = 0 \tag{4.3.6}$$

where $x \in \Omega$, D is the gradient function, Ω is an open domain of $\mathbb{R}$ and H is a Hamiltonian function.

The Hamiltonian function used in optimal control theory is a more general version of the function developed by Hamilton in classical mechanics. The Hamiltonian function provides a convenient form to embody the optimal control problem while allowing for the necessary conditions to be written in a more compact form. The interested reader is referred to [69] for a historical perspective of optimal control and the relationship between the control Hamiltonian and the Hamiltonian used in classical mechanics.

Now a viscosity solution will be defined.

Definition 4.3.1 [7]

1. A function $u \in C(\Omega)$ is a viscosity sub-solution of equation (4.3.6), if for any $\varphi \in C^1(\Omega)$, $F(x_0, u(x_0), D\varphi(x_0)) \leq 0$, at any local maximum point $x_0 \in \Omega$ of $u - \varphi$.
2. A function $u \in C(\Omega)$ is a viscosity super-solution of equation (4.3.6), if for any $\varphi \in C^1(\Omega)$, $F(x_1, u(x_1), D\varphi(x_1)) \geq 0$, at any local minimum point $x_1 \in \Omega$ of $u - \varphi$.
3. u is a viscosity solution of equation (4.3.6) if it is a viscosity sub-solution and super-solution.

For example, see [7], the function $u(x) = |x|$ is a viscosity solution to the equation

$$-\left|\frac{du(x)}{dx}\right| + 1 = 0 \tag{4.3.7}$$

for $x \in \left]-1, 1\right[$.

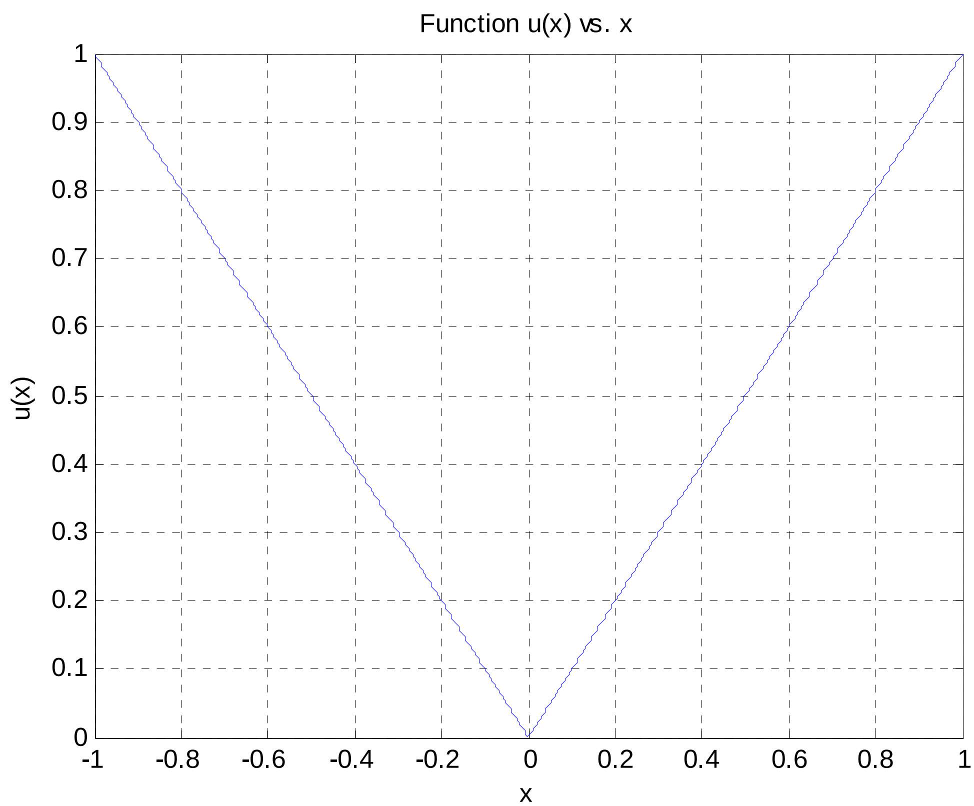

Figure 4.2: Plot of the function u(x) for the viscosity solution example.

Figure 4.2 is graphical representation of the function $u(x)$. Note that for any $x \neq 0$, where $x \in]-1,1[$, $u(x)$ is differentiable with respect to x, so $\frac{d\varphi(x)}{dx} = \frac{du(x)}{dx}$ is true and it is easy to see that the definition of viscosity super-solution and sub-solution are satisfied, so $u(x)$ is a viscosity solution of equation (4.3.7). When $x = 0$, $u(x)$ is not differentiable, and the definitions of the super-solution and sub-solution are required.

In order for $u(x) = |x|$ to be a viscosity super-solution of (4.3.7) it must be true that

$$-\left|\frac{d\varphi(0)}{dx}\right|+1\geq 0 \tag{4.3.8}$$

or equivalently

$$-1\leq\frac{d\varphi(0)}{dx}\leq 1 \tag{4.3.9}$$

for any $\varphi(x)$ such that $u(x)-\varphi(x)$ has a local minimum at $x=0$. There are many $\varphi(x)$ satisfying equation (4.3.9), for example

$$\varphi(x)=\frac{x^2}{2} \tag{4.3.10}$$

Furthermore, in order for $u(x)=|x|$ to be a viscosity sub-solution of (4.3.7) it must be true that

$$-\left|\frac{d\varphi(0)}{dx}\right|+1\leq 0 \tag{4.3.11}$$

or equivalently

$$-1\geq\frac{d\varphi(0)}{dx}\geq 1 \tag{4.3.12}$$

for any $\varphi(x)$ such that $u(x)-\varphi(x)$ has a local maximum at $x=0$. Obviously, a $\varphi(x)$ does not exist that satisfies equation (4.3.11) and condition (1) of Definition 4.2.1 is satisfied and $u(x)$ is a viscosity sub-solution and viscosity solution of equation (4.3.7).

Viscosity solutions can also be defined in terms of a sub-differential and super-differential. Let the following sets be associated with a function $u \in C(\Omega)$ and $x \in \Omega$

1. The set $D^{+}u(x) := \left\{ p \in \mathbb{R}^n : \limsup_{y \to x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \leq 0 \right\}$ is the super-differential of u at x
2. The set $D^{-}u(x) := \left\{ p \in \mathbb{R}^n : \liminf_{y \to x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \geq 0 \right\}$ is the sub-differential of u at x

The relationship between the super-differential/sub-differential and the viscosity super-solution/sub-solution can now be presented.

Lemma 4.3.2 [7]

Let $u \in C(\Omega)$, then

1. $p \in D^{+}u(x)$ if and only if there exists a function $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at x
2. $p \in D^{-}u(x)$ if and only if there exists a function $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local minimum at x

The following example is given to provide a geometric interpretation of Lemma 4.3.2. Figure 4.3 depicts a piecewise linear continuous function $u(x)$, that is not differentiable at the point x_0. Let $\varphi(x)$ be any function that is differentiable at x_0 and is a local minimum of $u(x_0) - \varphi(x_0)$. Then Lemma 4.3.2 says that all vectors $D\varphi(x_0) = p$ are elements of the sub-differential of the function $u(x_0)$. Figure 4.3 depicts four functions

$\varphi(x)$ that satisfy the conditions of the Lemma. Using these four functions it is easy to see that the vector p is constrained to the following set

$$\left.\frac{\partial u(x)}{\partial x}\right|_{x_0}^{-} \le p \le \left.\frac{\partial u(x)}{\partial x}\right|_{x_0}^{+} \tag{4.3.13}$$

where the $+$ and $-$ notation refer to the right side and left side derivates respectively. Further from Figure 4.3 one can see that a function $\varphi(x)$ does not exist such $u(x_0)-\varphi(x_0)$ has a local maximum at x_0, so the super-differential is empty at x_0.

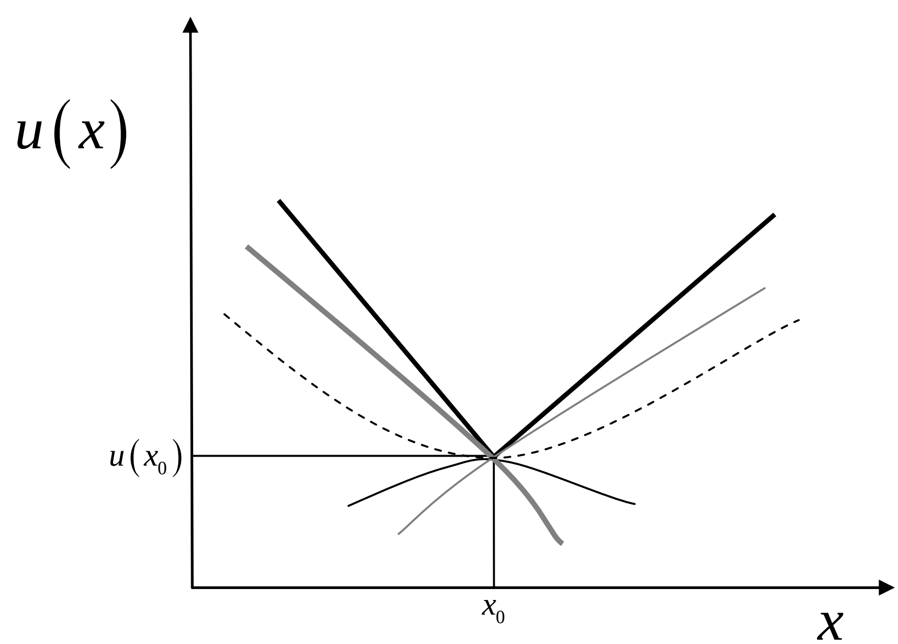

Figure 4.3: Graphical representation of the sub-differential in Lemma 4.3.2

Note that if the function $u(x)$ is concave instead of convex, the sub-differential will be empty and the vector p is an element of the super-differential and is constrained by

$$\left.\frac{\partial u(x)}{\partial x}\right|_{x_0}^{+} \leq p \leq \left.\frac{\partial u(x)}{\partial x}\right|_{x_0}^{-} \tag{4.3.14}$$

Lemma 4.3.2 can now be used to develop a new definition of viscosity solution that is equivalent to Definition 4.3.1.

Definition 4.3.3 [7]

1. A function $u \in C(\Omega)$ is a viscosity sub-solution of equation (4.3.6), if for all $x \in \Omega$ and $p \in D^{+}u(x)$, $F\left(x_0, u(x_0), p\right) \leq 0$.
2. A function $u \in C(\Omega)$ is a viscosity super-solution of equation (4.3.6), if for all $x \in \Omega$ and $p \in D^{-}u(x)$, $F\left(x_0, u(x_0), p\right) \geq 0$.
3. u is a viscosity solution of equation (4.3.6) if it is a viscosity sub-solution and super-solution.

The next lemma gives properties of the sub- and super-differentials.

Lemma 4.3.4 [7]

Let $u \in C(\Omega)$ and $x \in \Omega$, then

1. $D^{+}u(x)$ and $D^{-}u(x)$ are closed convex (possible empty) subsets of $\mathbb{R}^n$
2. if u is differentiable at x, then $Du(x) = D^{+}u(x) = D^{-}u(x)$
3. if for some x both $D^{+}u(x)$ and $D^{-}u(x)$ are nonempty, then $D^{+}u(x) = D^{-}u(x) = Du(x)$
4. the sets $A^{+} = \left\{x \in \Omega : D^{+}u(x) \neq \varnothing\right\}$ and $A^{-} = \left\{x \in \Omega : D^{-}u(x) \neq \varnothing\right\}$ are dense

Finally, the previous lemmas and properties can be used to develop a result concerning the differentiability of viscosity solutions.

Proposition 4.3.5 [7]

1. If $u \in C(\Omega)$ is a viscosity solution of equation (4.3.6), then $F(x, u(x), Dx(u)) = 0$ at any point $x \in \Omega$ where u is differentiable
2. if u is locally Lipschitz continuous and is a viscosity solution of equation (4.3.6), then $F(x, u(x), Dx(u)) = 0$ almost everywhere in Ω.

4.3.4 Control Problem and Associated Assumptions

The purpose of this section is to present the optimal control problem and it's associated assumptions.

As before, let $f : \mathbb{R}^n \times U \times \mathbb{R} \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, be a function that describes the dynamics for the control system and satisfies the assumptions given in Chapter 3, pg. 46-47

$$\dot{x}(t) = f(x(t), u(t), t) \tag{4.3.15}$$

which is defined over the interval of time, $t_0 \le t \le t_f$, where $x(t) \in \mathbb{R}^n$ is the state trajectory and $u(t) \in U$ is an admissible control.

For the interval of time $t \le \tau \le t_f$, let there exist a real-valued function J that describes the cost associated with transferring the state from the initial value $x(t)$ to the final value $x(t_f)$, using the admissible control $u(\tau)$, defined as

$$J(x(t), u, t_f - t) = \int_t^{t_f} L(\overline{x}(\tau), \overline{u}(\tau), \tau)\, d\tau + g(x(t_f), t_f) \tag{4.3.16}$$

where $x(t)$ is the initial state, u describes an admissible control function $\bar{u}(\tau)\in U$, $t\le\tau\le t_f$, $\bar{x}(\tau)$, $t\le\tau\le t_f$, is the trajectory that represents the solution of equation (4.3.15), and $t_f - t$ is the interval of time over which the trajectory transitions from its initial state to final state, and L satisfies the assumptions given in Chapter 3, pg. 50. Furthermore, let the terminal cost g also satisfy the assumptions given in Chapter 3, pg. 50.

Note that the cost function and dynamic constraints are written in non-autonomous form [7]. The continuity assumptions of Chapter 3, pg 46-47, allow the non-autonomous equations to be written in autonomous form by augmenting the system of dynamic constraints with

$$\begin{aligned}\dot{x}_{n+1}(t) &= 1\\ x_{n+1}(t_0) &= t_0\end{aligned} \tag{4.3.17}$$

so without loss of generality the autonomous case will be studied here and the explicit dependence on time will be dropped from the resultant equations.

Now, a shorthand notation will be introduced that will simplify writing the equations in the following theorems and proofs. Let

$$\bullet(x,u) = \bullet\big(x(t),u(t)\big) \tag{4.3.18}$$

where $\bullet$ is a placeholder for f, L, J and any other function that has the same type of dependency on the state, control and time.

Before the sufficient conditions can be derived, some definitions and equivalence relationships are required for the proofs and theorems given in the following sections.

First, fix the control function $u(\tau),\ t \le \tau \le t_f$, then from the initial state $x(t)$ and control function $u(\tau)$, the cost-to-go function J_c can be defined [7] as

$$J_c\left(x(t),u,t_f - t\right) = \int_t^{t_f} L\left(x(\tau),u(\tau)\right)d\tau + g\left(x\left(t_f\right)\right) \qquad (4.3.19)$$

where $x(t)$ is the initial state at time t and $t_f - t$ is the interval of time over which the trajectory transitions from its initial state to final state. Note that given the problem assumptions the cost-to-go function is defined for any initial state, time, and control as long as the function g is defined at $x\left(t_f\right)$.

Next, let $\hat{u}(t),\ t \le \tau \le t_f$, be the control function that minimizes the cost-to-go to the final condition, for the initial condition $x(t)$ and define [7] the optimal cost-to-go function as

$$\hat{J}_c\left(x(t),t_f - t\right) = \inf_{u(t)\in U} J_c\left(x(t),u,t_f - t\right) \qquad (4.3.20)$$

Note that the left hand side of equations (4.3.20) and (4.3.19) are identical except for the inclusion of the variable u in equation (4.3.19). u is dropped from equation (4.3.20) to indicate that the cost-to-go function is defined in terms of the optimal control.

Now recall that the Hamiltonian for PMP is defined as

$$H\left(x,\lambda,u,\lambda_0\right)=\left\langle \lambda(t),\, f\left(x,u\right)\right\rangle-\lambda_0\cdot L\left(x,u\right) \tag{4.3.21}$$

where $\lambda(t)$ is the adjoint, f defines the dynamics of the system, L is the Lagrangian, and λ_0 is a constant greater than zero. When the control is optimal for the reference trajectory, the PMP requires that the Hamiltonian is maximized, and satisfies

$$H\left(x,\lambda,u,\lambda_0\right)=\sup_{u(t)\in U}\left\{\left\langle \lambda(t),\, f\left(x,u\right)\right\rangle-\lambda_0\cdot L\left(x,u\right)\right\} \tag{4.3.22}$$

By using equation (4.3.22), and letting $\lambda_0=1$, the Hamiltonian for the HJCB PDE along the optimal trajectory was previously derived as

$$\begin{aligned} &H\left(x,-\frac{\partial \hat{J}_c\left(x(t),t_f-t\right)}{\partial x},u,1,t\right) \\ &\qquad=\sup_{u(t)\in U}\left\{\left\langle -\frac{\partial \hat{J}_c\left(x(t),t_f-t\right)}{\partial x},\, f\left(x,u\right)\right\rangle-L\left(x,u\right)\right\} \end{aligned} \tag{4.3.23}$$

The following propositions and theorems show that for the problem statement, the cost-to-go function associated with the optimal control, $\hat{J}_c\left(x(t),t_f-t\right)$ is the unique viscosity solution to the HJCB equation and which in turn can be used to derive the PMP. All of the results can be found in [7][49].

Note that the value function (the optimal cost-to-go function) is not in the same form as the viscosity solution definitions given in the previous section. The sub-differential and super-differential have the following form using the definition of the viscosity solution that has a $x\in\Omega$ and $t\in\mathbb{R}$ component.

From [7], let the following sets be associated with a function $u \in C(\Omega \times \mathbb{R})$ with $x \in \Omega$ and $t \in \mathbb{R}$

1. The set $$D^{+}u(x,t) := \left\{ p \in \mathbb{R}^n : \lim_{\substack{s\to t\\ y\to x}} \sup_{y\in\Omega} \frac{u(y,s) - u(x,t) - \langle p, y-x \rangle - p_0 \cdot (s-t)}{|x-y| + |s-t|} \le 0 \right\}$$ is the super-differential of u at (x,t)
2. The set $$D^{-}u(x,t) := \left\{ p \in \mathbb{R}^n : \lim_{\substack{s\to t\\ y\to x}} \sup_{y\in\Omega} \frac{u(y,s) - u(x,t) - \langle p, y-x \rangle - p_0 \cdot (s-t)}{|x-y| + |s-t|} \ge 0 \right\}$$ is the sub-differential of u at (x,t)

In terms of the value function $\hat{J}_c(x(t), t_f - t)$ the definition of the sub-differential and super-differential become

1. The set $$D^{+}u(x,t_f - t) := \left\{ \begin{array}{l} (p_{0,}p) \in \mathbb{R} \times \mathbb{R}^n : \\ \lim\limits_{\substack{s\to t\\ y\to x}} \sup\limits_{y\in\Omega} \dfrac{u(y,t_f - s) - u(x,t_f - t) - \langle p, y-x \rangle + p_0 \cdot (s-t)}{|x-y| + |s-t|} \le 0 \end{array} \right\}$$ is the super-differential of u at $(x, t_f - t)$
2. The set $$D^{-}u(x,t_f - t) := \left\{ \begin{array}{l} (p_{0,}p) \in \mathbb{R} \times \mathbb{R}^n : \\ \lim\limits_{\substack{s\to t\\ y\to x}} \sup\limits_{y\in\Omega} \dfrac{u(y,t_f - s) - u(x,t_f - t) - \langle p, y-x \rangle + p_0 \cdot (s-t)}{|x-y| + |s-t|} \ge 0 \end{array} \right\}$$ is the sub-differential of u at $(x, t_f - t)$

The first proposition that will be presented is the Principle of Dynamic Programming. It will be used throughout the rest of this section in the presentation of the various theorems.

Proposition 4.3.6, Dynamic Programming Principle (DPP) ([7], pg. 149)

Assume that control problem and cost-to-go function presented in the previous section is given, with initial condition $x(t_0) \in \mathbb{R}^n$ and an admissible control function $u(\tau)$ defined over the interval $t_0 \le \tau < t_f$. Also for a fixed initial condition $x(t_1)$, define the value function, as

$$\hat{J}_c\left(x(t_1), t_f - t_1\right) = \inf_{u \in U}\left\{\int_{t_1}^{t_f} L(x,u)\,dt + g\left(x(t_f)\right)\right\} \tag{4.3.24}$$

for $t_0 \le t_1 < t_f$ and for $t_1 = t_f$

$$\hat{J}_c\left(x(t_f), t_f - t_f\right) = g\left(x(t_f)\right) \tag{4.3.25}$$

Then for all $x \in \mathbb{R}^n$ and $t_1 \le \tau < t_f$

$$\hat{J}_c\left(x(t_1), t_f - t_1\right) = \inf_{u \in U}\left\{\int_{t_1}^{\tau} L\left(x(s), u(s)\right) ds + \hat{J}_c\left(x(\tau), t_f - \tau\right)\right\} \tag{4.3.26}$$

The proof of the dynamic programming principle first shows that equation (4.3.26) is true for the case where the '=' is replaced by '$\le$' and then is shown true for the case where the '=' is replaced by '$\ge$'. The detailed proof can be found in **[7]**.

The next proposition shows that the value function is a viscosity solution of the HJCB PDE.

Proposition 4.3.7 ([7], pg. 150):

Given the previous problem statement and assumptions, then the value function $\hat{J}_c\left(x(t_0), t_f - t_0\right)$ is a viscosity solution of the HJCB PDE

$$-\frac{\partial \hat{J}_c\left(x\left(t_0\right), t_f - t_0\right)}{\partial t} + \sup_{u \in U} H\left(x, -\frac{\partial \hat{J}_c\left(x\left(t_0\right), t_f - t_0\right)}{\partial x}, u, 1\right) = 0 \qquad (4.3.27)$$

in $\mathbb{R}^n \times \left]0, +\infty\right[$.

The proof of the proposition requires showing that $\hat{J}_c\left(x\left(t_0\right), t_f - t_0\right)$ is both a viscosity sub-solution at any local maximal point of $\hat{J}_c\left(x\left(t\right), t_f - t\right) - \phi\left(x\left(t\right), t_f - t\right)$ and viscosity super-solution at any local minimum point of $\hat{J}_c\left(x\left(t\right), t_f - t\right) - \phi\left(x\left(t\right), t_f - t\right)$, for $t_0 \le t \le t_f$. Again, the proof can be found in [7].

The next theorem to be presented is a comparison principle which is used to show uniqueness of the viscosity solution to the HJCB PDE for the given optimal control problem.

Theorem 4.3.8 Comparison Principle ([7], pg. 152):

Assume $H : \mathbb{R}^{2n} \to \mathbb{R}$ is continuous and satisfies the specific regularity conditions, $T \in \left]0, +\infty\right[$. If $\hat{J}_{c,1}, \hat{J}_{c,2} \in BC\left(\mathbb{R}^n \times \left[0, T\right]\right)$ are viscosity sub- and super-solutions, respectively, of

$$-\frac{\partial \hat{J}_{c,i}\left(x\left(t\right), t_f - t\right)}{\partial t} + H\left(x, -\frac{\partial \hat{J}_{c,i}\left(x\left(t\right), t_f - t\right)}{\partial x}, u, 1\right) = 0 \qquad (4.3.28)$$

in $\mathbb{R}^n \times \left]0, +\infty\right[$, then

$$\sup_{\mathbb{R}^n \times [0,T]} \left(\hat{J}_{c,1} - \hat{J}_{c,2} \right) \leq \sup_{\mathbb{R}^n \times \{0\}} \left(\hat{J}_{c,1} - \hat{J}_{c,2} \right) \tag{4.3.29}$$

and if $H(x,-\lambda,u,1) = \sup_{u \in U} \left\{ \left\langle -\lambda, f(x,u) \right\rangle - L(x,u) \right\}$, then the value function $\hat{J}_c\left(x(t_0), t_f - t_0\right)$ is the unique solution of

$$\begin{gathered} -\frac{\partial \hat{J}_c\left(x(t_0), t_f - t_0\right)}{\partial t} + H\left(x, -\frac{\partial \hat{J}_c\left(x(t_0), t_f - t_0\right)}{\partial x}, u, 1 \right) = 0 \\ \hat{J}_c\left(x(t_f), t_f - t_f\right) = g\left(x(t_f)\right) \end{gathered} \tag{4.3.30}$$

The proof of the comparison principle [7] consists of finding two viscosity solutions to the problem and then showing that they have to be equal.

The next lemma and two theorems are the main result and provide necessary and sufficient conditions for the optimal control problem with the given assumptions. The necessary conditions are the same as the PMP and the sufficient condition is given by the HJCB and the non-smooth value function.

First, a lemma will be given that relates the adjoint variable to the value function. Second, the necessary and sufficient conditions will be given and the proof of the conditions will be sketched out.

Note that the Lagrangian form of the cost-to-go function is given in equation (4.3.24), but without loss of generality can transformed into Mayer form [7] which only tracks an end cost that is a function of the trajectory. To ease the proofs, the following theorems will use the Mayer form of the cost function. As long as $L(x,u)$ is Lipschitz continuous with respect to the state variables uniformly in the control variables, then the cost function

$$J\left(x(t_0),u,t_f-t\right)=\int_{t_0}^{t_f} L(x,u)\,dt+g^*\left(x(t_f)\right) \tag{4.3.31}$$

can be converted into

$$J\left(x(t_0),u,t_f-t\right)=g\left(x(t_f)\right) \tag{4.3.32}$$

by augmenting the dynamic system with the new state variable

$$\begin{aligned}\dot{x}_{n+1}&=L(x,u)\\ x_{n+1}&=0\end{aligned} \tag{4.3.33}$$

The new cost function can be written as

$$\psi\left(x(t_f),t_f\right)=x_{n+1}(t_f)+g^*\left(x(t_f)\right) \tag{4.3.34}$$

Equation (4.3.34) can be an unbounded function, so by assuming that g is a bounded function equation (4.3.34) can be written as

$$g\left(\psi\left(x(t_f)\right)\right) \tag{4.3.35}$$

or $g\left(x(t_f)\right)$.

The first lemma that will be presented is Lemma 3.43 from ([7], pg. 175). Lemma 3.43 provides the equivalence between the adjoint and the variation of the value function with respect to x. Note that for a given optimal control, the problem assumptions require that the cost function (not the optimal cost-to-go function) is differentiable with respect to x.

Lemma 4.3.9 ([7], pg. 175)

Under the assumptions and notations of the control problem, the cost functional

$$J\left(x(t),u,\tau-t\right)=g\left(x^{*}(\tau)\right) \tag{4.3.36}$$

where $x^{*}(\tau)$, $t_0 \le t \le \tau$, is the solution to

$$x^{*}(\tau)=x(t)+\int_{t}^{\tau} f\left(x(s),u(s)\right)ds \tag{4.3.37}$$

with any admissible control function $u(s)\in U$, is differentiable with respect to x and for all $t_0 \le t \le \tau$ and when the control function $u(s)\in U$ is optimal,

$$\lambda(t)=\frac{\partial J\left(x(t),u,\tau-t\right)}{\partial x} \tag{4.3.38}$$

The proof of Lemma 4.3.9 first develops the relationship between the adjoint and the variation in initial condition and then the cost function is differentiated with respect to x and the equivalence to the solution to the differential equation describing the evolution of adjoint system with final condition

$$Dg\left(x^{*}(\tau)\right)\cdot\Phi(\tau,t) \tag{4.3.39}$$

is provided. See [7] for the complete proof of the lemma.

Now that Lemma 4.3.9 has been given, the theorem describing the necessary and sufficient conditions can be given. Theorem 3.42 ([7], pg. 175) provides a version of the Maximum Principle that is necessary and sufficient.

Theorem 4.3.9, Maximum Principle ([7], pg. 175)

Assume that the control problem is the one previously presented with the following additional assumptions; $U \subseteq \mathbb{R}^m$ is compact, $g \in C^1(\mathbb{R}^n)$, f differentiable with respect to x and $\frac{\partial f}{\partial x}$ is continuous for all $x \in \mathbb{R}^n$. Let $\hat{u}(\tau)$, $t \le \tau \le t_f$, be a control that moves the state from a given point $x(t) \in \mathbb{R}^n$ to the final state $x(t_f)$ along the optimal trajectory $\hat{x}(\tau)$, $t \le \tau \le t_f$. Further, define the adjoint, $\lambda(t)$, to be the solution to the following system of equations

$$\begin{aligned} \dot{\lambda}(t) &= -\left\langle \lambda(t), \frac{\partial f\left(\hat{x}(t), \hat{u}(t)\right)}{\partial x} \right\rangle \\ \lambda(t_f) &= \frac{\partial g\left(\hat{x}(t_f)\right)}{\partial x} \end{aligned} \tag{4.3.40}$$

Then $\hat{u}(\tau)$ is optimal for the initial state $x(t)$ and final state $x(t_f)$ **if and only if** for almost all $\tau \in \left]t, t_f\right[$

1. $\left\langle -\lambda(\tau), f\left(\hat{x}(\tau), \hat{u}(\tau)\right)\right\rangle = \max_{v \in U}\left[\left\langle -\lambda(\tau), f\left(\hat{x}(\tau), v\right)\right\rangle\right] = H\left(\hat{x}, -\lambda, \hat{u}, 1\right)$
2. The $n+1$-tuple $\left(\lambda(\tau), -H\left(\hat{x}, -\lambda, \hat{u}, 1\right)\right) \in D^+ \hat{J}_c\left(\hat{x}(\tau), t_f - \tau\right)$

Where $D^+ \hat{J}_c\left(\hat{x}(\tau), t_f - \tau\right)$ is the viscosity super-solution of the HJCB PDE.

The proof of necessity found in [7] uses the definitions of the value function, viscosity super-differential (the value function is the maximal sub-solution), and properties of dynamic equations to prove that if the control function $u(\tau)$ is optimal then condition 2 is true. Condition 1 is a direct consequence of the fact that the optimal control satisfies the non-smooth form of the HJCB PDE. See [7] for the details of the proof.

The sufficient portion of the Maximum Principle will now be given in Theorem. The sufficient condition is given in Theorem 3.38 ([7], pg. 173).

Theorem 4.3.10 Sufficient Condition of Optimality ([7], pg. 173)

Assume the optimal control problem is the one presented in Theorem 4.3.9. Suppose that there exists a verification function u, where u is a viscosity solution of the HJCB, which is locally Lipschitz in a neighborhood of $x(t)$, for all $t \in \left[t_0, t_f\right]$and an admissible control $\alpha \in U$ defined over the interval $t \le \tau \le t_f$ such that at the final point, $u\left(x\left(t_f\right), t_f - t_f\right) = g\left(x\left(t_f\right)\right)$. Then $\alpha(\tau)$ is optimal over the interval $t \le \tau \le t_f$, if for all most every $t \in \left[t_0, t_f\right]$ the following condition holds:

$$\exists\left(p_0, p\right) \in D^{\pm}u\left(x(t), t_f - t\right): p_0 - \left\langle p(t), f(x, \alpha)\right\rangle \ge 0 \tag{4.3.41}$$

The proof of the sufficient condition given in [7] uses the definition of Dini derivatives to prove that $u\left(x(t), t_f - t\right)$ is a non-increasing function as $t \to t_f$. As shown in the proof, if this is the case, then the verification function has to equal the value function. The proof will not be given in it entirety, but will be outlined below for clarity. Instead of using Dini derivates, the definitions of viscosity sub- and super-solutions can be used to prove that $u\left(x(t), t_f - t\right)$ is a non-increasing function as $t \to t_f$.

If it can be shown that $u\left(x(t), t_f - t\right)$ is a non-increasing function as $t \to t_f$, then

$$g\left(x\left(t_f\right)\right) = u\left(x\left(t_f\right), t_f - t_f\right) \le u\left(x\left(t_0\right), t_f - t_0\right) \tag{4.3.42}$$

Since $u\left(x\left(t_0\right), t_f - t_0\right)$ and $\hat{J}_c\left(x\left(t_0\right), t_f - t_0\right)$ are viscosity solutions to the HJCB PDE, the comparison principle (Theorem 4.3.8) for viscosity solutions and theorem assumptions imply

$$\begin{aligned} &u\left(x\left(t_0\right), t_f - t_0\right) - \hat{J}_c\left(x\left(t_0\right), t_f - t_0\right) \\ &\qquad \le u\left(x\left(t_f\right), t_f - t_f\right) - \hat{J}_c\left(x\left(t_f\right), t_f - t_f\right) = 0 \end{aligned} \tag{4.3.43}$$

or

$$u\left(x\left(t_0\right), t_f - t_0\right) \le \hat{J}_c\left(x\left(t_0\right), t_f - t_0\right) \tag{4.3.44}$$

but since $\hat{J}_c\left(x\left(t_0\right), t_f - t_0\right) = g\left(x\left(t_f\right)\right)$, equations (4.3.44) and (4.3.42) imply

$$\hat{J}_c\left(x\left(t_0\right), t_f - t_0\right) = u\left(x\left(t_0\right), t_f - t_0\right) \tag{4.3.45}$$

and finishes the proof of sufficient.

In order to prove that $u\left(x(t), t_f - t\right)$ is a non-increasing function as $t \to t_f$, the fact that $u\left(x(t), t_f - t\right)$ is a viscosity solution and the definition of the super-differential is applied.

Next the differentiability of g in Theorem 4.3.9 can be relaxed to form an extended maximum principle.

Theorem 4.3.11 Extended Maximum Principle ([7], pg. 179)

Assume that the control problem is the one given before with the following assumptions; U is a compact set, $\dfrac{\partial f}{\partial x}$ exists and is continuous, $g\left(x\left(t_f\right)\right) \in C\left(\mathbb{R}^n\right)$, and that $D^+ g\left(x\left(t_f\right)\right) \neq \varnothing$. Furthermore, let the adjoint vector satisfy the following equations

$$\begin{aligned} \dot{\lambda}(t) &= -\lambda(t) \cdot \frac{\partial f(x,u,t)}{\partial x} \\ \lambda\left(t_f\right) &= \bar{\lambda} \end{aligned} \tag{4.3.46}$$

where $\bar{\lambda} \in D^+ g\left(x\left(t_f\right)\right)$. α is an optimal control defined over the interval $t_0 \leq t \leq t_f$, if and only if for almost all $t \in \left]t_0, t_f\right[$ the Hamiltonian is maximized and

$$\left(\lambda(t), -H(x,-\lambda,\alpha,1)\right) \in D^+ \hat{J}_c\left(x(t), t_f - t\right) \tag{4.3.47}$$

where

$$\hat{J}_c\left(x(t), t_f - t\right) = \inf_{v \in U}\left\{J\left(x(t), v, t_f - t\right)\right\} \tag{4.3.48}$$

The proof of Theorem 4.3.11 can be found in [7] and will not be provided here.

Finally, two corollaries will be given that describes the behavior of the Hamiltonian and adjoint at the endpoints of the problem.

Corollary 4.3.12 ([7] pg. 180):

Assume that the control problem is the one given before with the following assumptions;

U is a compact set, $g\left(x\left(t_f\right)\right) \in C\left(\mathbb{R}^n\right)$, $D^+ g\left(x\left(t_f\right)\right) \neq \varnothing$, and that $\frac{\partial f}{\partial x}$ exists and is continuous. Let the adjoint vector satisfy the following equations

$$\begin{aligned} \dot{\lambda}(t) &= -\lambda(t) \cdot \frac{\partial f(x,u)}{\partial x} \\ \lambda\left(t_f\right) &\in D^+ g\left(x\left(t_f\right)\right) \end{aligned} \tag{4.3.49}$$

If $\hat{u}$ is optimal for the initial state $\left(x\left(t_0\right), t_0\right)$ and final time t_f, then for all $t \in \left[t_0, t_f\right]$

$$D^- \hat{J}_c\left(x(t), t_f - t\right) \subseteq \left\{\lambda(t), H\left(\hat{x}, -\lambda, \hat{u}, 1\right)\right\} \subseteq D^+ \hat{J}_c\left(x(t), t_f - t\right) \tag{4.3.50}$$

And if either of the inclusions is an equality, then $\hat{J}_c$ is differentiable at $\left(x(t), t\right)$ and both inclusions are equality.

The proof of Corollary 4.3.12 comes directly from the definitions and properties of the sub-differential, super-differential, viscosity solutions, and PMP.

In order to prove that equation (4.3.50) holds at the endpoints, all that need to be shown is that for t_0 the one-sided differential exists for any $\tau > t_0$ as $\tau \to t_0$ and for t_f the one-sided differential exists for any $\tau < t_f$ as $\tau \to t_f$. The definition of viscosity solution implies that at any points of non-differentiability, the one sided differentials still exists, which finishes the proof of the corollary.

Corollary 4.3.13 ([7] pg. 180):

Assume the hypotheses of Theorem 4.3.11. If $\hat{u}$ is optimal for the initial state $\left(x\left(t_0\right),t_0\right)$ and final time t_f, then for all $t \in \left[t_0,t_f\right]$ $H\left(\hat{x},-\lambda,\hat{u},1\right)$ is constant.

Corollary 4.3.13 is proved by showing that the differential of H is zero at every $t \in \left[t_0,t_f\right]$ that the differential exists (which is almost everywhere because H is locally Lipschitz). The proof can be found in [7] (pg. 180) and will be excluded here.

Endpoint Constraints for Non-smooth Value Functions

The previous work was based on the assumption that the final time was fixed. The purpose of this section is to expand this theory to include other endpoint constraints. The first endpoint constraint that will be studied is when the endpoints of the state variable are constrained to a surface in the state space and the final time is free. The second endpoint constraint that will be analyzed is the case where the endpoint of the state variable is constrained to a surface in the state space and the end time is fixed.

The extension of the theory is not straightforward for these endpoint constraints because discontinuities in the value function can occur. For example if the final state is constrained to a surface in the state space and the final time is fixed, then any spatial variation that does not intersect the surface at the final time produces an infinite value function, and hence a discontinuity in the value function with respect to state. For the free end time problem, when the set of states that can reach the surface of final states is not the entire state space, the value function is infinite for the set of non-reachable states and hence is discontinuous along the boundary of the reachable set.

Viscosity Solutions on Boundaries

Before the results of the endpoint constraints can be given, further properties of viscosity solutions need to be given. The behavior of viscosity solutions along smooth boundaries needs to be defined. This behavior will be used to develop the "transversality" conditions for the necessary and sufficient conditions given in Chapter 6.

Proposition 4.3.14 ([7], pg. 40)

Assume that there exists a subset, Ω, of $\mathbb{R}^n$ such that

$$\Omega = \Omega^1 \cup \Omega^2 \cup \Gamma \tag{4.3.51}$$

where Ω^i is an open subset of Ω and Γ is a smooth surface in $\mathbb{R}^n$ that is a boundary (possibly incomplete) between Ω^1 and Ω^2. Define $n(x)$ as the unit vector normal to Γ at x, pointing into Ω^1 and $T(x)$ the tangent space to Γ at x. Also denote P_N as the orthogonal projection of $\mathbb{R}^n$ onto the space spanned by $n(x)$ and P_T as the orthogonal projection of $\mathbb{R}^n$ onto $T(x)$.

Let $u \in C(\Omega)$ and assume that its restrictions u^i to $\Omega^i \cup \Gamma$ belong to $C^1(\Omega^i \cup \Gamma)$, $i = 1, 2$. Then u is a viscosity solution of the HJCB in Ω if and only if the following conditions hold

a. u^i is a classical solution of the HJCB in Ω^i, $i = 1, 2$

b. $F\left(x, u(x), P_T Du^i(x) + \xi^+ \cdot n(x)\right) \le 0$, for all $\xi^+ \in \left[Du^1(x) \cdot n(x), Du^2(x) \cdot n(x)\right]$ and all $x \in \Gamma$

c. $F\left(x, u(x), P_T Du^i(x) + \xi^- \cdot n(x)\right) \ge 0$, for all $\xi^- \in \left[Du^2(x) \cdot n(x), Du^1(x) \cdot n(x)\right]$ and all $x \in \Gamma$

The proof of this proposition comes directly from the definitions of the viscosity solution and the sub- and super-solutions. Proposition 4.3.14 provides conditions upon which u is a viscosity solution when a boundary in the state-space is reached.

Now that the behavior of viscosity solutions on smooth boundaries has been defined, the conditions under which the viscosity solution exists on boundaries need to be derived. The first case that will be analyzed is the case where the final time is free and the endpoint is constrained to lie in a smooth surface embedded in the state space of the system.

Free End Time/Fixed Surface of Final Conditions

When the optimal control problem requires that the final state is an element of a surface in the state space, the resulting value function can be discontinuous. Clearly if the surface of final states is not reachable from some initial condition, the value function is infinite at that initial state and a solution to the problem does not exist. So when the set of states that can reach the surface of final conditions is not the entire state space, a discontinuity occurs along the boundary of the reachable set.

Before the development of the free end time/fixed surface of final conditions necessary and sufficient conditions can be developed, the reachable set needs to be defined and the definition given in [7] will be used. Define $S \subset \mathbb{R}^n$ as a target set that is closed with a compact boundary ∂S. Let $\Re(t)$ be the set of points reachable from the target set, S, in time less than t by the backwards system $\frac{dx(t)}{dt} = -f(x,\alpha)$, $\alpha \in U$, then

$$\Re(t) = \left\{ x \in \mathbb{R}^n : T(x) < t \right\} \tag{4.3.52}$$

where $t > 0$, and $T(x)$ is the minimum time to the target set from initial condition x, over all admissible control functions. The entire reachable set, $\Re$, for the control problem can now be defined as

$$\Re = \bigcup_{t>0} \Re(t) = \left\{ x \in \mathbb{R}^n : T(x) < \infty \right\} \tag{4.3.53}$$

Now that the reachable set has been defined, small-time controllability on the surface S (STC S) can be defined.

Definition 4.3.16 ([7], pg. 228)

The controlled system $\dot{x} = f(x,a)$, $a \in U$, is STC S if $S \subseteq \operatorname{int} \Re(t)$, for all $t > 0$.

The next proposition gives the conditions under which an optimal control problem is STC S.

Proposition 4.3.17 ([7], pg. 229)

Under the assumptions of Chapter 3 of this dissertation, pg. 46-47, the system is STC S .

The proof of Proposition 4.3.17 will not be presented here, but can be found in [7]. The next proposition lists the properties of a system being STC S .

Proposition 4.3.18 ([7], pg. 230)

Assume that a control system satisfies the assumptions of Proposition 4.3.17 and is STC S , then the following properties are true.

1. $\Re$ is an open set
2. T is continuous in $\Re$
3. $\lim_{x \to x_0} T(x) = \infty$ for all $x_0 \in \partial\Re$

where T is the minimum time to the target set from initial condition x , and $\partial\Re$ is the boundary of the reachable set $\Re$.

The reader is referred to **[7]** for the proof of Proposition 4.3.18. Note that Proposition 4.3.18 demonstrates the discontinuity of the value function for the minimum time problem along the boundary of the reachable set, but also proves the continuity of the value function while inside the reachable set.

Now that the properties of the reachable set have been given, the next proposition will generalize these concepts to systems where a nonzero terminal cost exits.

Proposition 4.3.19 ([7], pg. 249)

Assume the control problem satisfies the assumptions in Chapter 3, pg. 46-47, pg. 50. Assume that the purpose of the control problem is to find a control that minimizes the cost function

$$J(x,\alpha) = \int_0^{t_f} L(x,\alpha)\cdot ds + g\left(x\left(t_f\right)\right) \tag{4.3.54}$$

where t_f is the first time $x\left(t_f\right) \in S$. Further assume $g \in C(S)$ and $g(x) \geq 0$ for all $x \in S$.

If the value function, $\hat{J}_c\left(x\left(t_f\right), t_f - t_f\right)$, is continuous for all $x\left(t_f\right) \in \partial S$, then

$$\Re = \left\{x : \hat{J}_c\left(x(t), t_f - t\right) < \infty\right\} \tag{4.3.55}$$

and

$$\lim_{x \to x_0} \hat{J}_c\left(x(t), t_f - t\right) = \infty \tag{4.3.56}$$

for any $x_0 \in \partial\Re$ and $\hat{J}_c\left(x(t), t_f - t\right)$ is an element of the union of the space of functions $u : \kappa \to \mathbb{R}$, with $\|u\|_\infty < \infty$, and the space of uniformly continuous functions $u : \kappa \to \mathbb{R}$, for any closed set κ where the minimum time function T is bounded.

The proof of Proposition 4.3.19 can be found in [7]. Proposition 4.3.19 redefines the reachable set as the set of states that can reach the final surface and has a cost that is not infinite.

A notion of compatibility of the terminal cost is required to prove that the value function $\hat{J}_c\left(x(t), t_f - t\right)$ is the unique viscosity solution of the HJCB PDE. Definition 3.5 [7] defines the compatibility condition, Theorem 3.6 [7] equates the continuity of the value function with the compatibility of the terminal cost, and finally Proposition 3.13 [7] provides the conditions under which the value function is the unique viscosity solution of the HJCB PDE. The definition, theorem, and proposition will be given here for completeness.

Definition 4.3.20 ([7], pg. 251)

The terminal cost g is compatible with the continuity of $\hat{J}_c$, if g has an extension to a neighborhood $B(S,\delta)$ of S, which is lower semicontinuous at points of ∂S and such that $g(x) \leq J(x,\alpha)$ for all $\alpha \in U$ and $x \in B(S,\delta)$.

Definition 4.3.20 can now be used with the definition of STC S to relate the compatibility condition with the continuity of the value function.

Theorem 4.3.21 ([7], pg. 251)

Assume the control problem satisfies the assumptions of Proposition 4.3.19 and is STC S. Then the compatibility of g is equivalent to $\hat{J}_c\left(x(t), t_f - t\right) \in C(\Re)$.

Theorem 4.3.21 says that the terminal cost g will be compatible with the value function $\hat{J}_c$ if the value function is continuous in the set of reachable states. This point becomes important in next proposition which proves that $\hat{J}_c$ is a viscosity solution of the HJCB PDE.

For this problem, the Hamiltonian in the HJCB PDE is defined as

$$H(x,u,-\lambda,\lambda_0) = \sup_{u \in U}\left\{\left\langle -\lambda, f(x,u)\right\rangle - \lambda_0 \cdot L(x,u)\right\} \tag{4.3.57}$$

and all $x, \lambda \in \mathbb{R}^n$.

Proposition 4.3.22 ([7], pg. 256)

Assume the control problem satisfies the assumptions in Theorem 4.3.21 and g satisfies the compatibility condition. Then $\hat{J}_c$ is the unique solution to the HJCB PDE, $H\left(x,u,D\hat{J}_c,\lambda_0\right) = 0$ continuous in $\Re \setminus \operatorname{int} S$, bounded below, and satisfying the boundary condition $\hat{J}_c\left(x(t_f), t_f - t_f\right) = g\left(x(t_f)\right)$ on ∂S and $\hat{J}_c\left(x(t), t - t_f\right) \to \infty$ as $x(t) \to x_0 \in \partial\Re$.

The proof to Proposition 4.3.22 can be found in [7]. This proposition is very important because it proves that as long as $x(t) \in \Re$, $\hat{J}_c\left(x(t), t_f - t\right)$ is the viscosity solution to the HJCB PDE for the optimal control problem. The necessary and sufficient conditions for optimality of the control developed earlier in this chapter can now be applied to the optimal control problem.

Fixed End Time/Fixed Surface of Final Conditions

Assume that the control problem is the standard problem given at the beginning of this Chapter, except that the final time t_f is fixed and the final value of the state $x(t_f)$ is required to be an element of the surface S that is contained in $\mathbb{R}^n$, defined by the equation $h\left(x(t_f)\right) = 0$. The result of imposing the two extra constraints on the optimal control problem is that the value function maybe discontinuous along the surface of final states because for all $x(t_f) \in S$, the value function is defined as

$$\hat{J}_c\left(x(t), t_f - t\right) = \inf_{u \in U}\left\{\int_t^{t_f} L(x,u)\cdot ds + g\left(x(t_f)\right)\right\} \tag{4.3.58}$$

but when $x(t_f) \notin S$

$$\hat{J}_c\left(x(t), t_f - t\right) = \infty \tag{4.3.59}$$

A theoretical development of the behavior of the viscosity solution for this case can be found in ([7], Chapter 5) and will be excluded here. When the optimal control problem can be represented in Mayer form with only a terminal cost, g, the optimal control problem can be transformed into an equivalent form where the new terminal cost function, $\tilde{g}$, is continuous in x. Penalization methods for approximate control problems are used to perform this transformation. Once the control problem is converted into this form, the necessary and sufficient conditions for the optimal control developed earlier apply. The important point here is that the necessary and sufficient conditions developed earlier still apply to this optimal control problem and the interested reader is referred to ([7], Chapter 5) for the technical development.

An approximate terminal cost function that eliminates the discontinuity can be found in ([7], Section 5.2), and is

$$\tilde{g}_n\left(x\left(t_f\right)\right) = \min\begin{cases} g\left(x\left(t_f\right)\right) + n \cdot \left(\inf_{y \in S}\left|x\left(t_f\right) - y\right|\right)^2 \\ \sup_{x \in S}\left|g\left(x\right)\right| + 2 \cdot T \cdot \sup_{x \in \Omega, \alpha \in U}\left|L\left(x, \alpha\right)\right| + 1 \end{cases} \tag{4.3.60}$$

where $g\left(x\left(t_f\right)\right)$ is the terminal cost for the original control problem, L is the Lagrangian for the original cost function, and T, $t_f \le T < \infty$, is the upper bound for all finite horizons of interest.

The first term in equation (4.3.60) is a continuous function of x that may be unbounded. The second term in equation (4.3.60) is constant and provides an upper bound to the terminal cost.

Since the terminal cost for the approximate problem is continuous and as $n \to \infty$, $g_n \to g$, the theorems developed earlier apply, and as $n \to \infty$, the solution of the approximate problem goes to the solution of the original problem, so the theorems developed previously apply to the original problem.

The details in this section are provided without proof and only serve to raise the point that the necessary and sufficient conditions developed in Theorems 4.3.10 and 4.3.11 still apply to optimal control problems where the final time is fixed and the state is required to lie on a surface of final conditions.

4.4 Clarke-Vinter Non-Smooth Necessary Conditions

For completeness, the Clarke-Vinter non-smooth maximum principle will be discussed. Clarke [23][24][25][26][72] provides a complete development of a non-smooth maximum principle similar the Bardi necessary conditions, which uses a generalized gradient instead of the viscosity solution sub- and super- solutions. Clarke's non-smooth maximum principle (CMP) provides the ability to apply the maximum principle to problems where either the cost functional or dynamic constraints have points that are not differentiable. However, the CMP will not be presented here because under the assumptions of the control problem in Chapter 3, the generalized gradient will be equal to the sub- and super- differentials presented in the previous section [7][72], hence the CMP and the Bardi necessary conditions will be identical. The interested reader is referred to the references given in the beginning of this section for the complete development of the CMP.

Chapter 5: Hybrid Maximum Principle

Generalizations of Pontryagin's Maximum Principle that handle hybrid optimal control problems can be found in the literature. These generalizations will be called Hybrid Maximum Principles (HMP). Three of those generalizations will be presented here. The first set of necessary conditions presented is the conditions developed by Sussmann [67]. Next the work of Riedenger [58][59] will be given and finally the work of Caines [61][63][64][65] will be presented.

The Sussmann Hybrid Maximum Principle (SHMP) is a hybrid maximum principle that provides necessary conditions for optimal control in the most general setting [67][68]. Because of its applicability to a wide range of hybrid optimal control problems its theory will be presented in the most detail. The two other HMP are presented here in less detail to provide justification for using a less general HMP to solve engineering problems, as is done later in this dissertation. The work given by Riedinger [57][58][59] is a less general version of the SHMP, but can be applied to basic engineering problems. This HMP also allows for systems that have asynchronous discrete switching. In [61][63][64][65] Caines and his collaborators use the Riedinger HMP to develop the Caines' HMP and associated algorithms for numerical implementation. The Caines' HMP is much less general than the SHMP, but the simplification provides a framework for development of numerical algorithms that can be used to analyze hybrid control problems. The Riedinger and Caines' work is less general than is required for this dissertation, but provides justification for deriving a HMP that may not be as general as the SHMP, but can be implemented by engineers to solve practical engineering problems.

5.1 Sussmann's HMP

Sussmann's hybrid maximum principle (SHMP) [67][68] is an extension of the non-smooth maximum principle allowing for the analysis of optimal control solutions for hybrid systems. It provides necessary conditions for the optimal trajectories for the hybrid system, given a function that represents the cost associated with moving the state from the initial condition to the final condition with the optimal control. As with the PMP, the SHMP does not provide a rigorous means to study the uniqueness and existence of the optimal solution. The following section will summarize Sussmann's results.

This section is organized into the following sections. First, the definitions required to state the SHMP will be presented along with the general necessary conditions. Then the assumptions given in Chapter 3 will be applied to the problem and the proof will be outlined.

5.1.1 Definitions

Before the SHMP can be introduced, some definitions and nomenclature need to be introduced that will be used in the presentation of the SHMP. First the general hybrid model will be presented and then some associated assumptions and definitions will be provided.

Let

$$\Sigma = (Q, M, U, f, u, I, S) \tag{5.1.1}$$

be a general Sussmann hybrid system (see pg. 44-45).

A control for the hybrid system Σ is defined as a triple $\zeta = (q, t, \eta)$ consisting of the following variables.

1. q is a finite sequence of locations that describe the progression of the discrete part of the hybrid system.
2. t is a finite sequence of real numbers that describe the switching times for the evolution of the discrete part.
3. η is a finite sequence of maps such that η_j belongs to U_{q_i} and $P_{\eta_j} = P_j$ for $j = 1 \ldots \nu$. By definition the map η_j ensures that the control string u_j is admissible while in location q_j.

The definition of a control can be used to define the trajectory for a hybrid system. Let ζ be a control for Σ and let $\nu = \nu(\zeta)$ be used to index the final location visited by the hybrid system. Let the control ζ have the following properties; $q(\zeta) = (q_1, \cdots, q_\nu)$, $t(\zeta) = (t_1, \cdots, t_\nu)$, and $\eta(\zeta) = (\eta_1, \cdots, \eta_\nu)$. A trajectory for ζ is a ν-tuple $\xi = (\xi_1, \cdots, \xi_\nu)$ such that for each $j \in \{1, \cdots, \nu\}$:

1. ξ_j is an absolutely continuous map from $[t_{j-1}, t_j]$ to M_{q_j}, such that $\dot{\xi}_j = f_{q_j}(\xi_j(t), \eta_j(t))$ for almost all $t \in [t_{j-1}, t_j]$.
2. And the switching condition $(\xi_j(t_j), \xi_{j+1}(t_{j+1})) \in S_{q_j, q_{j+1}}$ holds if $j < \nu$.

A trajectory control pair for the hybrid system Σ is a pair (ξ,ζ) such that ξ is a trajectory for Σ and ζ is a control for Σ. All of the trajectory control pairs for the hybrid system Σ are denoted as $TCP(\Sigma)$.

Let Σ be a hybrid system as defined above and let $\Xi=(\xi,\zeta)\in TCP(\Sigma)$, then the endpoint condition $\partial\Xi$ of Ξ is the 4-tuple $(q_1,\xi_1(a_\zeta),q_\nu,\xi_\nu(b_\zeta))$ where:

1. $\xi=(\xi_1,\cdots,\xi_\nu)$
2. $q_\zeta=(q_1,\cdots,q_\nu)$
3. a_ζ is the initial time
4. b_ζ is the final time

Now the endpoint constraints for the system can be defined. As long as $\Xi\in TCP(\Sigma)$ and $\partial\Xi\in\hat{M}^2(\Sigma)$ the endpoint constraint for Σ, E, will be the subset of $\hat{M}^2(\Sigma)$ that corresponds to all of the $\partial\Xi$ associated with every Ξ. Given any initial and final location (q,q'), the endpoint constraint can be written as $E_{q,q'}=\{(x,x'):(q,x,q',x')\in E\}$.

The Lagrangian for Σ can be defined as a family of functions $L=\{L_q\}$ such that

1. each L_q is a real-valued function on $M_q\times U_q$
2. whenever $q\in Q$, $\eta\in u_q$ has domain $[\alpha,\beta]$, and $\xi:[\alpha,\beta]\to M_q$ is an absolutely continuous solution of $\dot{\xi}(t)=f_q(\xi(t),\eta(t))$ a.e. then the function $[\alpha,\beta]\ni t\to L(\xi(t),\eta(t))$ is integrable

The corresponding Lagrangian cost function $C_L : TCP(\Sigma) \to \mathbb{R} \cup \{+\infty\}$, is

$C_L(\xi,\zeta) = \sum_{j=1}^{\nu} \int_{t_{j-1}}^{t_j} L_{q_j}\left(\xi_j(t), \eta_j(t)\right) dt$, where all of the variables have been defined above.

Similar cost functions can be defined for the switching terms and the endpoints for Σ. Let Φ be a switching cost function and let φ be an endpoint cost function, then the cost functional, $\hat{C}_{\Phi,\varphi}(\xi,\zeta) = \varphi(\partial\Xi) + \sum_{j=1}^{\nu-1} \Phi\left(q_j, \xi_j(t_j), q_{j+1}, \xi_{j+1}(t_j)\right)$ incorporates both of these costs.

Now, the hybrid Bolza cost functional for Σ is defined as the sum of the Lagrangian cost function and the switching and endpoint function and can be written as, $J = C_L + \hat{C}_{\Phi,\varphi}$, and given the previous definition $J : TCP(\Sigma) \to \mathbb{R} \cup \{+\infty\}$.

Next define the free time and fixed time problem. Given a hybrid control system $\sum$, a Bolza cost function J, and an endpoint constraint $E \subseteq \hat{M}^2(\sum)$, the free time problem control problem $P(\sum, J, E)$ will minimize J for $TCP(\sum, E)$ and the fixed time problem $P(\sum, J, E, a, b)$ will minimize J for $TCP(\sum, E, a, b)$ for each compact subinterval $[a,b] \in \mathbb{R}$.

Now that the problem definition has been given, the assumptions on the control problem can be given. Assume:

1. $\Sigma = (Q, M, U, f, u, I, S)$ is a hybrid control system
2. $J = C_L + \hat{C}_{\Phi,\varphi}$ is a hybrid Bolza cost function
3. $E \subseteq \hat{M}^2(\Sigma)$ is an endpoint constraint
4. $\Xi^{\#} = (\xi^{\#}, \zeta^{\#}) \in TCP(\Sigma)$ is an optimal trajectory for the system

The SHMP will give a necessary condition for the control $\Xi^{\#}$ to be optimal. The necessary condition only compares trajectories that have a control with the same switching sequence that are "close" to $\Xi^{\#}$. As such the result is only a local solution, not a global solution.

Finally, define the local solution of P as a trajectory-control pair $\Xi^{\#} = (\xi_1^{\#}, \ldots, \xi_{v^{\#}}^{\#}, \zeta^{\#})$ such that there exists neighborhoods $N_1, \ldots, N_{v^{\#}}$ of the graphs of $\xi_1^{\#}, \ldots, \xi_{v^{\#}}^{\#}$ with the property that $\Xi^{\#}$ minimizes the cost J in the class of all the trajectory-control pairs $\Xi = (\xi_1, \ldots, \xi_v, \zeta)$ such that $\partial\Xi \in E$, $q(\zeta) = q(\zeta^{\#})$, and the graph of ξ_j is contain in N_j for $j = 1, \ldots, v^{\#}$.

5.1.2 Hybrid Maximum Principle

Now that the general problem definition has been given, the SHMP can be given.

Theorem 5.1.1 [67][68]

Assume that the hybrid optimal control problem is the one given earlier in this section with its associated assumptions. Then there exists an adjoint pair (λ, λ_0) along $\Xi^{\#}$ that satisfies:

1. the Hamiltonian maximization condition,
2. nontriviality condition,
3. transversality condition,
4. and Hamiltonian value conditions.

Sussmann is purposely vague when he presents the SHMP this way because he wants the reader to understand the generality of his solution. The four necessary conditions presented in the SHMP apply regardless of the problem structure and assumptions. Specifically, Sussmann proves the SHMP for systems that contain classical dynamics and for systems that use generalized differential (such as the generalized gradient given by Clarke [25]. In order to simplify the necessary conditions the SHMP will be defined here for systems that use classic differentials.

Assume the control problem satisfies the assumptions in Chapter 3, pg. 46-47, pg. 50

Further, assume:

1. each set u_q is invariant under time translations, restrictions and concatenations – that is, if a control $\eta:[a,b]\to U_q$ belongs to u_q, $a \le c \le d \le b$, and $\tau \in \mathbb{R}$, then the maps $[c,d] \ni t \to \eta(t) \in U_q$ and $[a-\tau, b-\tau] \ni t \to \eta(t+\tau) \in U_q$ also belong to u_q and, we define $\eta''(t) = \eta(t)$ for $a < t \le b$, $\eta''(t) = \eta'(t)$ for $b < t \le b'$, then $\eta'' \in u_q$ - and contains all the constant U_q-valued maps defined on compact intervals.

and define the j th jump of the discrete part of the hybrid system as $\gamma_j\left(\Xi^{\#}\right)$, where $\gamma_j\left(\Xi^{\#}\right)=\left(\hat{x}_j\left(t_j\right),\hat{x}_{j+1}\left(t_j\right)\right)$. Using similar notation $\gamma_e\left(\Xi^{\#}\right)$ defines the pair $\gamma_e\left(\Xi^{\#}\right)=\left(\hat{x}_1\left(a\right),\hat{x}_{v^{\#}}\left(b\right)\right)$.

Also, let the real valued function H_q be defined as

$$H_q\left(x,\lambda,u,\lambda_0,t\right)=\left\langle\lambda, f_q\left(x,u,t\right)\right\rangle-\lambda_0\cdot L_q\left(x,u,t\right) \tag{5.1.2}$$

where $x\in M_q$, $\lambda\in M_q^*$, $u\in U_q$, $\lambda_0\in\mathbb{R}$, and $t\in\mathbb{R}$ and M_q^* is the dual space to M_q.

Before the statements of the SHMP are presented, the adjoint pair and the transversality conditions require a special notion of a tangent cone, so the tangent cone will be introduced first.

A Boltyanskii approximating cone is used to define the required notion of a tangent cone.

Definition 5.1.2 Boltyanskii Approximating Cone [67][68]

Let S be a subset of a smooth manifold X and let $\overline{x}\in S$. A Boltyanskii approximating cone to S at $\overline{x}$ is a closed convex cone K in the tangent space $T_{\overline{x}}X$ to X at $\overline{x}$ such that there exists a neighborhood V of 0 in $T_{\overline{x}}X$ and a continuous map $\mu:V\cap K\to X$ with the property that $\mu\left(V\cap K\right)\subseteq S$, $\mu\left(0\right)=\overline{x}$, and $\mu\left(v\right)=\overline{x}+v+o\left(\left\|v\right\|\right)$ as $v\to 0$ via values in $V\cap K$.

Figure 5.1 and Figure 5.2 depict the Boltyanskii approximating cone at points in two different subsets of $\mathbb{R}^2$. Figure 5.1 depicts an intersection of two curves that meet at a point. The Boltyanskii approximating cone at this point will be a pie shaped slice because a small enough V around the zero point in the tangent space can be selected such that the intersection of the approximating cone, K, and V maps back into the set S.

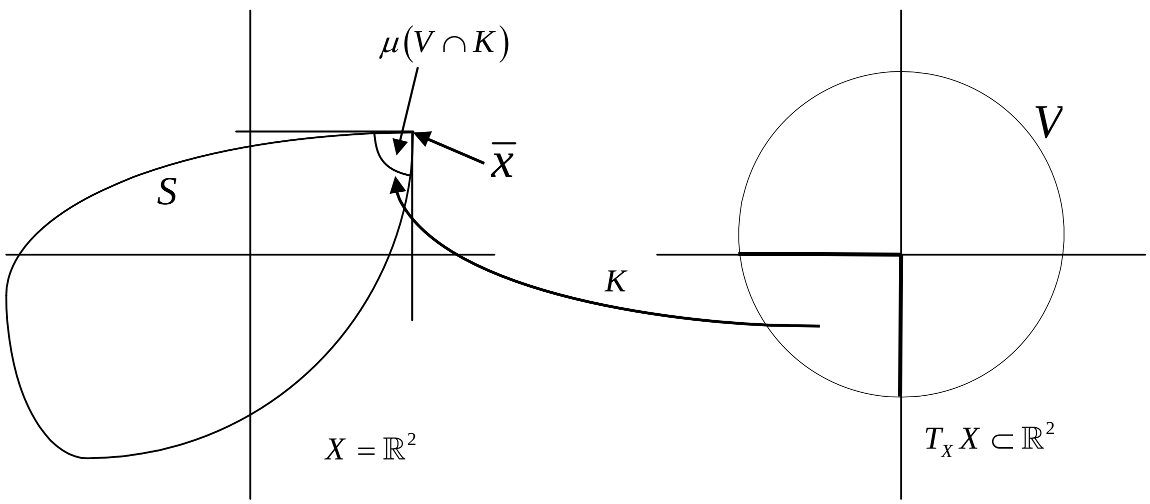

Figure 5.1: Example demonstrating a Boltyanskii approximating cone.

Figure 5.2 depicts the Boltyanskii approximating cone to a set S at a cusp in the set. The Boltyanskii approximating cone at this point is a line. The intersection of V and K for this example has to be a line because a line is the only convex cone that will map back into S according to μ, as zero is approached in the tangent space.

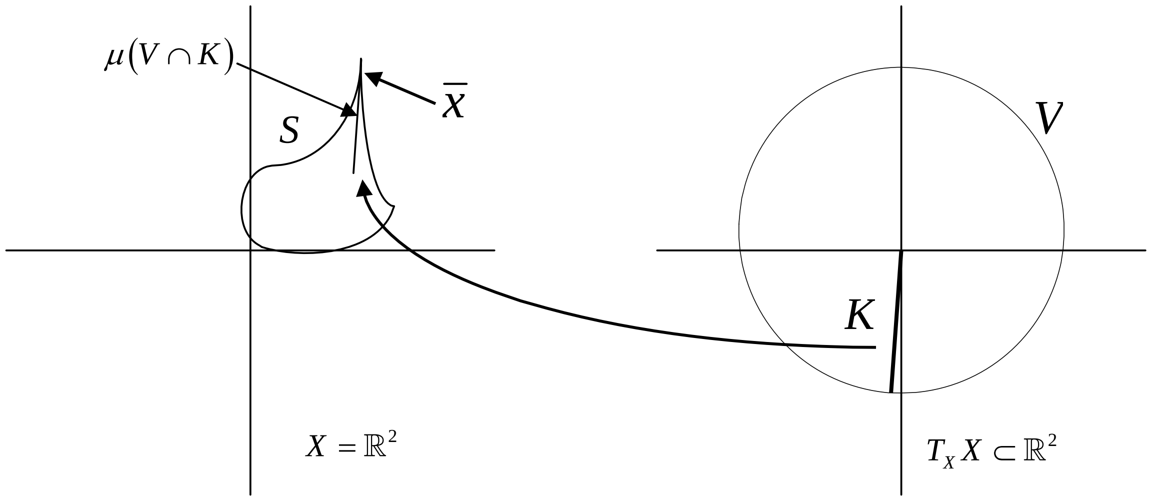

Figure 5.2: Example demonstrating a Boltyanskii approximating cone for a cusp.

The idea of the Boltyanskii approximating cone is to use a closed convex cone in the tangent space to approximate the subset S for some neighborhood of $\bar{x}$. The function μ is the map between the tangent space and the real space that describes this approximating cone. The cone is such that at the point of interest $\bar{x}$, the function evaluates to zero and if V is a neighborhood around zero in the tangent space, the intersection of the cone and V maps back into the original subset S as V approaches zero.

Three additional examples will be presented to clarify this notion of the approximating cone; for all three examples assume $X \equiv \mathbb{R}^2$. For the first example, let $S = \{(x, y) \in \mathbb{R}^2\}$ and let $\bar{x} = (x_f, y_f)$, find the Boltyanskii approximating cone, K, to set S at $\bar{x}$. The tangent space to all of $\mathbb{R}^2$ is also $\mathbb{R}^2$, so for any neighborhood of 0 in the tangent space, the function μ will map the whole neighborhood back into the set S and $K = \mathbb{R}^2$.

For the second example, let $S=\left\{(x,y)\in\mathbb{R}^2:(x,y)=(x_1,y_1)\right\}$ where x_1 and y_1 are constant. Again the tangent space to X is $\mathbb{R}^2$, but if $\bar{x}=(x_1,y_1)$ then any neighborhood of $\bar{x}$ contained in S consists of $\bar{x}$ itself, so the Boltyanskii tangent cone will consist of one point and is $K=\left\{(u,v)=(0,0)\right\}$. If $\bar{x}$ is any other point in $\mathbb{R}^2$, then a tangent cone will not exist because $\bar{x}$ is not in the subset S.

Lastly, let $S=\left\{(x,y)\in\mathbb{R}^2:x\le y\right\}$. If $\bar{x}=\left\{(x,y):x>y\right\}$, then a tangent cone will not exist, since μ will not map any point back into the set S. If $\bar{x}=\left\{(x,y):x<y\right\}$, then a neighborhood of size δ will be mapped back from the tangent space into S, where δ is a function of $\bar{x}$, resulting in $K=\mathbb{R}^2$. Finally let $\bar{x}=\left\{(x,y):x=y\right\}$, then for a neighborhood V of 0 in the tangent space, μ will map $V\cap K$ into S if $K=\left\{(v,w)\in\mathbb{R}^2:v\le w\right\}$.

The necessary conditions require the notion of a polar of the Boltyanskii approximating cone. Let K be a cone that is a subset of a finite dimensional linear space, V. The polar of the cone K is $K^{\perp}=\left\{w\in V^*:w\cdot v\le 0,\forall v\in K\right\}$, where V^* is the dual space of V. For example, if

$$K=\left\{(x,y)\in\mathbb{R}^2:y\ge|x|\right\} \tag{5.1.3}$$

then

$$\begin{aligned}K^{\perp}&=\left\{(d,f)\in\mathbb{R}_2:dx+fy\le 0,\forall y\ge|x|\right\}\\&=\left\{(d,f)\in\mathbb{R}_2:(|d|+f)|x|\le 0,\ \forall x\in K\right\}\end{aligned} \tag{5.1.4}$$

where $\mathbb{R}_2$ represents the row vector space. Consequently, the values of d and f that satisfy equation (5.1.4) are all values such that $f \le -|d|$.

Consider the Boltyanskii approximating cone

$$K = \{(x, y) \in \mathbb{R}^2 : x = y\} \tag{5.1.5}$$

then

$$\begin{aligned} K^{\perp} &= \{(d, f) \in \mathbb{R}_2 : dx + fy \le 0, \forall x = y\} \\ &= \{(d, f) \in \mathbb{R}_2 : (d + f) \cdot x \le 0, \forall x \in K\} \end{aligned} \tag{5.1.6}$$

Since $x \in \mathbb{R}$, equation (5.1.6) reduces to

$$K^{\perp} = \{(d, f) \in \mathbb{R}_2 : d + f = 0\} \tag{5.1.7}$$

or that

$$K^{\perp} = \{(x, y) \in \mathbb{R} : \alpha \cdot \nabla w(x, y), \forall \alpha \in \mathbb{R}\} \tag{5.1.8}$$

where

$$w(x, y) = x - y = 0 \tag{5.1.9}$$

and ∇ is the standard gradient operator.

The precise definitions of the necessary conditions for the SHMP can now be presented.

Proposition 5.1.3 Adjoint Equation [67][68]

The adjoint equation, or adjoint pair, is a pair, (λ, λ_0) along $\Xi^{\#}$ with the following properties:

1. λ is a $v^{\#}$-tuple $(\lambda_1, \ldots, \lambda_{v^{\#}})$ such that each λ_j is a field of covectors along $\xi_j^{\#}$, where covector is defined by; if V is a vector space contained in $\mathbb{R}^n$, then a covector is a linear map $\alpha : V \to \mathbb{R}^n$. The set of all covectors is a vector space V^* that is the dual of V. Note that for this case the local trajectory $\xi_j^{\#}$ is composed of the state $\hat{x}_j$ and control $\hat{u}_j$.
2. each λ_i is absolutely continuous function of t for the entire interval contained in I_{q_i}.
3. $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \geq 0$
4. each λ_j satisfies the adjoint equation $\dot{\lambda}_j = -\dfrac{\partial H_{q_j^{\#}}}{\partial x}\left(\hat{x}_j(t), \lambda_j(t), \hat{u}_j(t), \lambda_0\right)$
5. and for each $j \in \{1, \ldots, v^{\#} - 1\}$, the switching condition $\begin{bmatrix} -\lambda_j\left(t_j^{\#}\right) \\ \lambda_{j+1}\left(t_j^{\#}\right) \\ h_j^{+} \\ h_j^{-} \end{bmatrix} - \lambda_0 \cdot \nabla \Phi_{q_j^{\#}, q_{j+1}^{\#}}\left(\gamma_j\left(\Xi^{\#}\right)\right) \in K_j^{\perp}$ is true, where ∇ is the conventional gradient of a function, $K_j^{\perp}$ is the polar of the Boltyanskii approximating cone to the set $S_{q_j^{\#}, q_{j+1}^{\#}}$ at $\gamma_j\left(\Xi^{\#}\right)$, and

$$h_j^{+} = \begin{cases} \lim\limits_{s \downarrow 0} \dfrac{1}{s} \cdot \displaystyle\int_{\tau_j - s}^{\tau_j} H_{q_i}\left(x_j(t), \lambda_j(t), u_j(t), \lambda_0, t\right) \cdot dt, & \text{if the limit exists} \\ 0, & \text{if the limit does not exist} \end{cases}$$
$$h_j^{-} = \begin{cases} \lim\limits_{s \downarrow 0} \dfrac{1}{s} \cdot \displaystyle\int_{\tau_j}^{\tau_j + s} H_{q_i}\left(x_j(t), \lambda_j(t), u_j(t), \lambda_0, t\right) \cdot dt, & \text{if the limit exists} \\ 0, & \text{if the limit does not exist} \end{cases} \tag{5.1.10}$$

   Further, $K_j^{\perp}$ is a subset of $\mathbb{R}^{2n}$ (n is the dimension of adjoint vector), so the left hand side of the equation is formed by concatenating the adjoint vectors and gradient of the switch cost function forming a vector of length $2n$.

Proposition 5.1.4 Hamiltonian Maximization Condition [67][68]

If (λ, λ_0) is an adjoint pair along $\Xi^{\#}$, then the adjoint pair will satisfy the Hamiltonian maximization condition if there exist real numbers $h_1 \ldots h_{v^{\#}}$ such that for each $j \in \{1, \ldots, v^{\#}\}$, $h_j = H_{q_j^{\#}}\left(\hat{x}_j(t), \lambda_j(t), \hat{u}_j(t), \lambda_0, t\right) = \max_{u \in U} H_{q_j^{\#}}\left(\hat{x}_j(t), \lambda_j(t), u, \lambda_0, t\right)$ for almost every $t \in I_{q_j}$ and $u \in U_{q_j}$.

Proposition 5.1.5 Transversality Conditions [67][68]

If (λ, λ_0) is an adjoint pair along $\Xi^{\#}$, then the adjoint pair will satisfy the transversality condition for E and φ if $\begin{bmatrix} \lambda_1(t_0) \\ -\lambda_{v^{\#}}(t_f) \\ h_v^{+} \\ -h_1^{-} \end{bmatrix} - \lambda_0 \cdot \nabla \varphi_{q_1^{\#}, q_v^{\#}}\left(\gamma_e\left(\Xi^{\#}\right)\right) \in K_e^{\perp}$, where $K_e^{\perp}$ is the polar of the Boltyanskii approximating cone to the set $E_{q_1^{\#}, q_v^{\#}}$ at $\gamma_e\left(\Xi^{\#}\right)$, where the quantities in the relationship are elements of $\mathbb{R}^{2n}$. Note the similarity between the transversality condition and the switching condition used in the definition of the adjoint pair. The main difference is the sign of the two values of the adjoint. Both of the conditions use the discrete behavior of the hybrid system to constrain the evolution of the adjoint equation.

Proposition 5.1.6 Non-triviality Condition [67][68]

If (λ, λ_0) is an adjoint pair along $\Xi^{\#}$, then the adjoint pair will satisfy the non-triviality condition if either $\lambda_0 \neq 0$ or at least one of the functions λ_j is not identically zero.

Proposition 5.1.7 Hamiltonian Value Condition [67][68]

If (λ, λ_0) is an adjoint pair along $\Xi^{\#}$ and $h_1 \ldots h_{v^{\#}}$ are constants that satisfy the Hamiltonian maximization condition, then the adjoint pair will satisfy the Hamiltonian value condition for the *fixed time* interval problem for every $j \in \{1, \ldots, v^{\#} - 1\}$:

1. if $t_j^{\#} - t_{j-1}^{\#} \in Interior\left(I_{q_j^{\#}}\right)$, then $h_j = h_{v^{\#}}$
2. if $t_j^{\#} - t_{j-1}^{\#}$ is the left endpoint of $I_{q_j^{\#}}$ and $I_{q_j^{\#}}$ is non-trivial, then $h_j \leq h_{v^{\#}}$
3. if $t_j^{\#} - t_{j-1}^{\#}$ is the right endpoint of $I_{q_j^{\#}}$ and $I_{q_j^{\#}}$ is non-trivial, then $h_j \geq h_{v^{\#}}$

If it is desired to solve a variable time problem instead of the fixed time interval problem, then if (λ, λ_0) satisfies the Hamiltonian value condition for the fixed time interval problem and $h_{v^{\#}} = 0$, then it satisfies the Hamiltonian value condition for the variable time problem. Note $h_{v^{\#}}$ is the Hamiltonian value for the last optimal location in the run of the hybrid system, where the value of the Hamiltonian at the other locations is determined by the Hamiltonian value condition.

Proof of SHMP

The precise proof of the SHMP for the case with classic differentials can be found in [67]. Just as in the proof of the PMP, Sussmann proves the necessary conditions using needle variations of a reference control, spatial variations in trajectory, and temporal variations in trajectory to form an endpoint set. He then assumes the reference trajectory is optimal, analyzes the endpoint set, and shows that the vector of improved cost is not contained in this endpoint set. Using these results he generates a set of necessary conditions for the control to be optimal.

Since the problem is hybrid the variation in the trajectory is much more complicated, because the variations need to be propagated along the hybrid trajectory to the endpoint set. Sussmann expands the definition of needle variation that is used in the PMP to the hybrid case, while maintaining the same switch sequence and time intervals as the reference trajectory. A needle variation in control will be denoted (ε, v) and is defined in [67].

A two location hybrid problem will be used to illustrate the steps required for proof of the SHMP. This is done to help simplify the notation for the equations required.

Let $\tilde{\xi}$ be the perturbed trajectory associated with the needle variation (ε, v). Then the endpoint cost map for the perturbed reference trajectory is

$$EC(\varepsilon, v) = \left\{ \left(\partial \underline{\tilde{\xi}}, C_L\left(\tilde{\xi}\right) \right) \right\} \tag{5.1.11}$$

where $\partial \underline{\tilde{\xi}}$ is the set of end conditions defined by

$$\partial \underline{\tilde{\xi}} = \left\{ \partial \tilde{\xi}, \partial \tilde{\xi}_1 \right\} \tag{5.1.12}$$

with $\partial \tilde{\xi} = \left(x_2\left(b_{\tilde{\xi}}\right), x_1\left(a_{\tilde{\xi}}\right), a_{\tilde{\xi}}, b_{\tilde{\xi}} \right)$ being the initial and final conditions and

$\partial \tilde{\xi}_1 = \left(x_1(\tau_1), x_2(t_2), \tau_1, t_2 \right)$ being the values of the trajectory at the pre and post switch points. Furthermore, $C_L\left(\tilde{\xi}\right)$ is the Lagrangian cost function associated with the perturbed reference trajectory.

Based on the assumptions for this problem, equation (5.1.11) is differentiable at 0 and the variation in endpoint map with respect to the reference trajectory, $DE(0)(\varepsilon, v)$, can be defined as

$$DE(0)(\varepsilon, v) = \left((w_1, v_2), (w_2, v_1), \alpha(\varepsilon, v)\right) \tag{5.1.13}$$

where w_q and v_q are the variation in the final and initial conditions respectively, and α is the variation in the Lagrangian cost associated with the perturbed trajectory, see [67] for derivation of α. Note that variation in the hybrid system's initial condition v_1 and the variation in the hybrid system's final condition w_2 are lumped together in the vector (w_2, v_1). These terms are lumped together to derive the general transversality condition for the hybrid system, whereas the vector (w_1, v_2) is associated with the switching condition for the hybrid system. Further, note that $\alpha(\varepsilon, v)$ does not define the total cost associated with the trajectory. It only represents the Lagrangian cost associated with moving the perturbed trajectory from its initial condition to its final condition. In the problem assumptions, other costs are associated with the initial and final conditions, as well as, switching from the first location to the second location (i.e. the switching costs are excluded from the equation).

Now define the endpoint and switch constraints on the trajectory for some trajectory ξ.

If the set $E = \{\mathrm{E}_{2,1}\}$ is an endpoint constraint for the system, then Ξ satisfies the endpoint constraint E, if $\partial\xi$ belongs to the set $\mathrm{E}_{2,1}$. Further, x satisfies the switching conditions for the system Σ if $\partial_1\xi$ belongs to the set $S_{1,2}$.

Next the cost associated with a trajectory, ξ, that satisfies the switch and endpoint constraints for the system can be defined. Let $L = \{L_1, L_2\}$ be the Lagrangian for the hybrid system Σ, then the Lagrangian cost functional, C_L is

$$C_L(\xi) = \int_{t_1}^{\tau_1} L_1(x(t), u(t), t)\,dt + \int_{t_2}^{\tau_2} L_2(x(t), u(t), t)\,dt \tag{5.1.14}$$

Furthermore, let $\Phi_{i,j}$ be cost associate with switching from one location to the next and Ψ be the cost associated with the endpoints of the trajectory. Then the total cost associated with the trajectory ξ is

$$\begin{aligned} J(x,u,t) = \Psi_{2,1}(\partial\xi) + \Phi_{1,2}(\partial_1\xi) + \int_{t_1}^{\tau_1} L_1(x(t), u(t), t)\,dt \\ + \int_{t_2}^{\tau_2} L_2(x(t), u(t), t)\,dt \end{aligned} \tag{5.1.15}$$

and for the optimal trajectory $\Xi^{\#}$, $J(\Xi^{\#}) = J(\hat{x}, \hat{u}, t)$

$$J(\hat{x}, \hat{u}, t) \le J(x, u, t) \tag{5.1.16}$$

Now that the cost associated with an admissible trajectory has been developed, the set of costs associated with all admissible trajectories can be collected in a Boltyanskii approximating cone.

Fix the two real functions, $\sigma_1, \sigma_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\sigma_1\left(x(\tau_1), x(t_1)\right)$ and $\sigma_2\left(x(t_0), x(\tau_2)\right)$ such that

$$\begin{aligned} &\sigma_1\left(x(\tau_1), x(t_1)\right) > 0 \\ &\sigma_2\left(x(t_0), x(\tau_2)\right) > 0 \end{aligned} \tag{5.1.17}$$

when $\left(x(\tau_1), x(t_1)\right) \neq \partial_1 \xi$ and $\left(x(t_0), x(\tau_2)\right) \neq \partial \xi$ and

$$\begin{aligned} &\sigma_1\left(x(\tau_1), x(t_1)\right) = 0 \\ &\sigma_2\left(x(t_0), x(\tau_2)\right) = 0 \end{aligned} \tag{5.1.18}$$

when $\left(x(\tau_1), x(t_1)\right) = \partial_1 \xi$ and $\left(x(t_0), x(\tau_2)\right) = \partial \xi$.

Let G be the set of points $\left(\left(x(\tau_1), x(t_1)\right), \left(x(t_0), x(\tau_2)\right), r\right) \in \left(\mathbb{R}^n \times \mathbb{R}^n\right) \times \left(\mathbb{R}^n \times \mathbb{R}^n\right) \times \mathbb{R}$ where $\left(x(\tau_1), x(t_1)\right) \in S_{1,2}$ and $\left(x(t_0), x(\tau_2)\right) \in S_{2,1}$ with the property

$$\begin{aligned} r \leq J\left(\Xi^{\#}\right) - \Phi_{1,2}\left(x(\tau_1), x(t_1)\right) - \Psi_{2,1}\left(\left(x(t_0), x(\tau_2)\right)\right) \\ - \sigma_1\left(x(\tau_1), x(t_1)\right) - \sigma_2\left(\left(x(t_0), x(\tau_2)\right)\right) \end{aligned} \tag{5.1.19}$$

Equation (5.1.19) is the set of system endpoint costs associated with all admissible pre- and post-switch states along the optimal trajectory. Note that the terms $\Phi_{1,2}$, $\Psi_{2,1}$, σ_1, and σ_2 are all greater than or equal to zero by definition, so

$$r \leq J\left(\Xi^{\#}\right) \tag{5.1.20}$$

which implies that r represents a lower cost than the optimal and the set G contains an infinite number of vectors with lower costs than the optimal.

Now let the set G^* be defined as the set

$$G^* = \left\{\partial_1\xi, \partial\xi, C_L\left(\Xi^\#\right)\right\} \tag{5.1.21}$$

and note that if P is the set of all possible needle variations, then G^* satisfies

$$G^* = EC\left(P\right) \cap G \tag{5.1.22}$$

Equation (5.1.22) says that the intersection of the set of all variations in end condition and the set of all admissible end conditions with cost less than the cost of the reference trajectory is the set of end conditions given by the assumed optimal control. As such, a Boltyanskii approximating cone, K, to the set G along the set G^* can be found that contains the vector of **improved** cost, with

$$K = \left\{K_1, K_2, r\right\} \tag{5.1.23}$$

where K_1 is the Boltyanskii approximating cone to the set S_{12} at the point $\partial_1\xi$, K_2 is the Boltyanskii approximating cone to the set S_{21} at the point $\partial\xi$, and r is a cost that is less than or equal to $J\left(\Xi^\#\right)$.

Using the problem assumptions and definition of the Boltyanskii approximating cone, K_1 and K_2 are going to have the following form

$$K_1 = \{K_1^0\} \times \{0\} \times \{0\}$$
$$K_2 = \{K_2^0\} \times \{0\} \times \{0\} \tag{5.1.24}$$

where K_1^0 is the Boltyanskii approximating cone for the set of possible final and initial conditions at the switch point $(x(\tau_1), x(t_2))$ and K_2^0 is the Boltyanskii approximating cone for the set of possible final and initial conditions at the system final and initial conditions $(x(\tau_2), x(t_1))$.

Then set K then is the set of all $((z_1(\tau_1), z_2(t_2)), (z_2(\tau_2), z_1(t_1)), r)$, such that $(z_1(\tau_1), z_2(t_2)) \in K_1^0$, $(z_2(\tau_2), z_1(t_1)) \in K_2^0$, and

$$r \le -\nabla\Phi_{12}(\partial_1\xi) \cdot ((z_1(\tau_1), z_2(t_2))) - \nabla\Psi_{12}(\partial\xi) \cdot (z_2(\tau_2), z_1(t_1)) \tag{5.1.25}$$

Note that the first term on the right hand side of equation (5.1.25) represents the cost associated with the variation in the pre- and post-switch points and the right hand term represents the cost associated with variation in the system's final and initial conditions.

Now that the set K has been defined, the set containing all of the end costs associated with the perturbed trajectory can be defined, $\hat{K}$, and a separation theorem can be applied resulting in the necessary conditions.

The interested reader can find the proof of the SHMP in [68], for this case and in [67] for the case where the dynamics and Lagrangian are non-smooth.

5.2 Riedinger's HMP

In [57][58][59] Riedinger utilizes a non-smooth maximum principle, like the one in [72], to synthesize optimal control solutions for hybrid systems. Riedinger develops a hybrid model and then presents the necessary conditions for the maximum principle for that model. The following section will summarize Riedinger's results.

5.2.1 Model

For a given finite set of discrete states $K = \{1, \dots, k\}$, there is an associated collection of continuous dynamics defined by the differential equations

$$\dot{x}(t) = f_k\left(x(t), u_k(t), t\right) \tag{5.2.1}$$

where

1. $k \in K$
2. the continuous state, x, takes its values in $\mathbb{R}^{n_k}$
3. the continuous control, u, takes its values in the control set U_k included in $\mathbb{R}^{m_k}$
4. the vector fields f_k are defined on $\mathbb{R}^{n_k} \times \mathbb{R}^{m_k} \times [a,b]$ for all $k \in K$

Further the discrete state, $k \in K$, is defined using the following transition function

$$k\left(t^+\right) = \phi\left(x\left(t^-\right), k\left(t^-\right), d(t), t\right) \tag{5.2.2}$$

and the discrete control $d(t)$ defined by $d:[a,b] \to D$ and D is a finite set. The function ϕ is a map that satisfies

$$\phi : X \times K \times D \times [a,b] \rightarrow K \tag{5.2.3}$$

Where $X \subseteq \mathbb{R}^{n_1+\ldots+n_k}$.

Now assume that there exist boundary conditions on the trajectory (x,t) of the form $C_{(k,k')}(x(t),t)=0$ that defines a discrete event for the system and jump functions $\Phi_{(k,k')}$ that reset the value of the state during a discrete event, where

$$x(t^+) = \Phi_{(k,k')}(x(t^-),t^-) \tag{5.2.4}$$

Note that this problem formulation allows for both controlled and autonomous switching. Controlled switching occurs when the controller generates a discrete event causing a transition. Autonomous switching occurs when the continuous state trajectory intersects a boundary or switching surface in the state space.

5.2.2 Hybrid Maximum Principle

Assume that the hybrid problem is the one presented in the previous section and let $[a,t_1,\ldots,b]$ and $[k_0,k_1,\ldots,k_m]$ be the sequence of switching times and the associated mode sequence associated to the control (u,d) over the time interval $[a,b]$. Then the cost associated with the control (u,d) can be defined as

$$J(u,d) = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} L_{k_i}(x(t),u(t),t) \cdot dt \tag{5.2.5}$$

Note that the cost is an indirect function of the discrete variable d through the Langrangian function, but is not a explicit function of the discrete variable.

Further, make the following assumptions:

1. the control domain U_k is a bounded subset of $\mathbb{R}^{m_k}$
2. the vector fields $f_k(x,u,t)$ and $L_k(x,u,t)$ are continuous on $\mathbb{R}^{n_k} \times \bar{U}_k \times [a,b]$ and are continuously differentiable with respect to the state variable and time variable
3. the boundaries ∂S_k are defined by a set of continuously differentiable equality constraints $\partial S_k := \{(x,t) : C_{(k,k')}(x(t),t) = 0\}$
4. for all $(k,k') \in K^2$, the functions $C_{(k,k')}$ and $\Phi_{(k,k')}$ are continuous and continuously differentiable

The optimal control problem is to find the control, $(\hat{u},\hat{d})$, that minimizes equation (5.2.5) subject to the constraints in the hybrid problem.

Before the necessary conditions are given, the Hamiltonian system needs to be defined. Let the Hamiltonian be defined for every $k \in K$ as

$$H_k(\lambda,\lambda_0,x,u,t) = \lambda(t) \cdot f_k(x,u,t) - \lambda_0 \cdot L_k(x,u,t) \tag{5.2.6}$$

where $\lambda_0 \geq 0$ and

$$\dot{x} = \frac{\partial H_k(\lambda,\lambda_0,x,u,t)}{\partial \lambda} \tag{5.2.7}$$

$$-\dot{\lambda} = \frac{\partial H_k(\lambda,\lambda_0,x,u,t)}{\partial x} \tag{5.2.8}$$

Now the RHMP can be presented.

Riedinger's Hybrid Maximum Principle (RHMP) [57]

If $\left(\hat{u},\hat{d}\right)$ is an admissible optimal control and $\left(\hat{x},\hat{k}\right)$ is the resulting state trajectory for equations (5.2.1), (5.2.2), (5.2.4), and (5.2.5), then there exists a piecewise absolutely continuous curve λ and constant $\lambda_0 \geq 0$, $\left(\lambda_0,\lambda\right) \neq \left(0,0\right)$ on $\left[a,b\right]$ such that:

1. the sextuplet $\left(\lambda,\lambda_0,\hat{x},\hat{k},\hat{u},\hat{d}\right)$ satisfies equations (5.2.7) and (5.2.8) almost everywhere
2. at time t for a given $\left(\lambda,\lambda_0,\hat{x},\hat{k}\right)$, the following maximum condition holds
$$H_k\left(\lambda,\lambda_0,\hat{x},\hat{u},t\right) = \sup_{u \in U_k} H_k\left(\lambda,\lambda_0,\hat{x},u,t\right) \tag{5.2.9}$$
3. at switching time t_i, $i = 0,\ldots,m$, a vector π_i exists such that the following transversality conditions are satisfied
   a. $\lambda\left(t_i^-\right) = \left(\dfrac{\partial \Phi_{(k_{i-1},k_i)}\left(x\left(t_i^-\right),t_i\right)}{\partial x}\right)^T \cdot \lambda^T\left(t_i^+\right) + \left(\dfrac{\partial C_{(k_{i-1},k_i)}\left(x\left(t_i^-\right),t_i\right)}{\partial x}\right)^T \cdot \pi_i$
   b. $H\left(t_i^-\right) = -\left(\dfrac{\partial \Phi_{(k_{i-1},k_i)}\left(x\left(t_i^-\right),t_i\right)}{\partial t}\right)^T \cdot \lambda^T\left(t_i^+\right) + H\left(t_i^+\right) + \left(\dfrac{\partial C_{(k_{i-1},k_i)}\left(x\left(t_i^-\right),t_i\right)}{\partial t}\right)^T \cdot \pi_i$
   c. when no boundary conditions exist for the switching time (i.e. a discrete event occurs), $\pi_i = 0$.

Riedinger actually proves his necessary conditions using the principle of dynamic programming. He starts from the final state, formulates the optimal control problem in terms of a terminal cost function, assumes the control is optimal, and derives the necessary conditions.

The inputs and resulting state trajectory that satisfy Riedinger's hybrid maximum principle are extremal controls. In order to compute the optimal control switching sequence for the controlled switch case, the necessary conditions and equation (5.2.9) can be used to compute the Hamiltonian for all of the feasible discrete states. From condition (a) of the RHMP the value of the adjoint doesn't change when the system switches locations under a controlled switch. Since the value of the adjoint is known for the feasible controlled switching locations, the Hamiltonian for these locations can be computed. The location that has the maximum value of the Hamiltonian over the admissible control set is the location that the controller switches the system too.

Because Riedinger allows for controlled discrete switching, he must pick a final location (along with the initial location) and build a map of all possible mode switches that can end in the final location at the desired final state. He then has to use a dynamic programming argument along with the HMP to calculate the optimal discrete switching path along the trajectory. However, the resulting optimal control is open-loop because he is solving for the control solution along a single reference trajectory.

5.3 Caines' HMP

In [61][63][64][65] Caines et al, utilize Riedinger's version [57][58][59] of the Hybrid Maximum Principle to develop a series of numerical algorithms to find local optimal switching schedules for hybrid systems. As will be shown here, Caines work is another specialized non-smooth maximum principle and will be summarized here for completeness.

First the model that Caines requires will be presented. The trajectory of the non-smooth system will be presented next with all of its associated assumptions. Finally, Caines' HMP will be presented and discussed.

5.3.1 Model

First the basic definitions describing the hybrid model and the motion of the hybrid trajectory will be introduced. Define a hybrid system to be the 5-tuple

$$\Sigma = \left(S \triangleq Q \times \mathbb{R}^n, I \triangleq E \times U, F, \Gamma, M\right) \tag{5.3.1}$$

Where:

1. S is the hybrid state space and is the product of the finite set of discrete states, Q, and the state space for the continuous dynamics, $\mathbb{R}^n$.
2. I is the hybrid admissible control set and is the product of the finite set of controlled and autonomous transition labels, E, and the set of all bounded measurable functions on some interval $\left[0, T_*\right)$, $T_* \le \infty$ taking values in U.
3. $F = Q \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is the indexed set of continuous dynamics, $\left\{ f_{q_i} \right\}_{q_i \in Q}$, describing the evolution of the continuous trajectory in between each discrete event.
4. $\Gamma = T \times E \to T$ is a time independent transition map defining the evolution of the discrete events.
5. $M = \left\{M_\alpha : \alpha \in P\right\}$, $P \subset Q \times Q$, $M_\alpha : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, $m = n-1$ is a collection of guards such that for $\alpha = \left(q_i, q_j\right)$, $\tilde{M}_\alpha = \left\{\left(t, x\right) : M_\alpha\left(t, x\right) = 0\right\}$ is a smooth n-dimensional sub-manifold of $\mathbb{R}^{n+1}$ and for all t, $\tilde{M}_\alpha\left(t\right) = \left\{x : M_\alpha(t, x) = 0\right\}$ is a $n-1$ dimensional switching sub-manifold of $\mathbb{R}^n$. It is further assumed that $\tilde{M}_\alpha\left(t\right) \cap \tilde{M}_\beta\left(t\right) = \varnothing$ for all $t \in \mathbb{R}$, $\alpha, \beta \in P$, $\alpha \ne \beta$.

Note that the model definition is straight forward except for the definition of the switching surfaces defining the autonomous discrete switching from the discrete state q_i

to q_j. The autonomous switching surfaces are defined to be surfaces in $\mathbb{R}^n$ of dimension $n-1$, that are parameterized by time. He further assumes that any two switching surfaces do not intersect at any specified time t.

Now define a hybrid event sequence as the finite or infinite sequence

$$(\tau,\sigma)=\left[(t_0,\sigma_0),(t_1,\sigma_1),(t_2,\sigma_2),\ldots\right] \tag{5.3.2}$$

where τ are the event times and σ are the discrete input events.

A switching sequence for the hybrid system is defined as a finite or infinite sequence

$$S_\Sigma=(\tau,q)=\left[(t_0,q_0),(t_1,q_1),(t_2,q_2),\ldots\right] \tag{5.3.3}$$

where τ are the event times and q_i are the discrete locations.

A hybrid switching schedule can also be defined as

$$S_Q=(q_0,q_1,q_2,\ldots) \tag{5.3.4}$$

Finally, the execution of a hybrid system e_Σ is the input trajectory (τ,q,u), defined over the interval $[t_0,T)$, together with a hybrid state trajectory (τ,q,x), defined over the interval $[t_0,T')\subset[t_0,T)$ which satisfy the following conditions:

1. Continuous Dynamics (CS) –
$$\frac{d}{dt}x_{q_j}(t)=f_{q_j}\left(x(t),u(t)\right),\ a.e.\ t\in\left[t_j,t_{j+1}\right) \tag{5.3.5}$$
2. Discrete Dynamics Controlled Switching (DSC) – In the hybrid execution consider the hybrid switching time t_i, $i\geq 1$, at which the left limit

$\lim_{t\uparrow t_i} x_{q_{i-1}}(t) = x^*(t_i)$ exists. A controlled discrete transition occurs at the controlled switching time $t = t_i$ if there exists a discrete control input $\sigma_i \in E_c$, $q_{i-1} \neq q_i$, and $x_{q_i}(t_i) = x^*(t_i)$, for which

$$\Gamma_c\left(q_{i-1}, \sigma_i(t_i)\right) \equiv \Gamma_c\left(q_{i-1}, \sigma_i\right) = q_i, \; (t_i, \sigma_i) \in (\tau, \sigma) \tag{5.3.6}$$

3. Discrete Dynamics Autonomous switching (DSU) – In the hybrid execution at the switching time t_i, $i \geq 1$, the limit from the left at t_i exists and satisfies $\lim_{t\uparrow t_i} x_{q_{i-1}}(t) = x^*(t_i)$. Let $M_{q_{i-1},q_i}(t,x) = 0$ define a switching manifold. A discrete transition, denoted Γ_u and is an element $\sigma_i \in \sum_u$, occurs at autonomous switching time t_i if

$$M_{q_{i-1},q_i}\left(t_i, x^*(t_i)\right) = 0, \; t_i \in \tau \tag{5.3.7}$$

Note that the Caines model has a set of fundamental assumptions associated with it.

They are

1. There exists $K_f < \infty$ and $L_f < \infty$, such that $\left\| f_{q_i}(x,u) \right\| \leq K_f$, $x \in \mathbb{R}^n$, $u \in U$, $q_i \in Q$ and $\left\| f_{q_i}(x_1,u) - f_{q_i}(x_2,u) \right\| \leq L_f \cdot \left\| x_1 - x_2 \right\|$, for $x_1, x_2 \in \mathbb{R}^n$, $u \in U$, $q_i \in Q$. (Lipschitz condition – The differential equations and hence the trajectories are well behaved.)
2. The matrix $\dfrac{\partial M_{q_i,q_j}(t,x)}{\partial x}$ has full rank for all $x \in \mathbb{R}^n$ and $q_i, q_j \in Q$.
3. $f_{q_i}(x,u_1)$ and $f_{q_j}(x,u_2)$ are transversal to $\tilde{M}_{q_i,q_j}$ for all $x \in \mathbb{R}^n$, $u_1, u_2 \in U$, and $q_i, q_j \in Q$.
4. $\left\| \dfrac{\partial M_{q_i,q_j}}{\partial x} \right\| \leq K_1 < \infty$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and for all $q_i, q_j \in Q$.

   $\left\| \dfrac{\partial M_{q_i,q_j}}{\partial t} \right\| \leq K_2 < \infty$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and for all $q_i, q_j \in Q$.
5. For every $q_i, q_j \in Q$ there exists $\sigma_{ij} \in E_c$ such that $\Gamma_c\left(q_i, \sigma_{ij}\right) = q_j$.
6. Non-Zeno condition – There exists $T_r > 0$ such that if t_s is a switching time then $t_0 < t_s - T_r$, $t_f > t_s + T_r$ and there is no other switching time in the interval $\left[t_s - T_r, t_s + T_r\right]$.
7. A switching time is controlled or autonomous, but not both.
8. The initial state $\left(x_0 \triangleq \left(x(t_0), q_0\right)\right) \in \sum$ is such that at initial time $t_0 \in \mathbb{R}$, $M_{q_0,q_j}(t_0, x_0) \neq 0$, $q_j \in Q$.

9. Let $L_{q_i}:\mathbb{R}^n \times U \to \mathbb{R}$ be a function defined for every $q_i \in Q$. There exists $K_L < \infty$ and $L_L < \infty$, such that $\left\| L_{q_i}\left(x,u\right) \right\| \le K_L$, $x \in \mathbb{R}^n$, $u \in U$, $q_i \in Q$ and $\left\| L_{q_i}\left(x_1,u\right) - L_{q_i}\left(x_2,u\right) \right\| \le L_L \cdot \left\| x_1 - x_2 \right\|$, for $x_1, x_2 \in \mathbb{R}^n$, $u \in U$, $q_i \in Q$.
10. Let $g:\mathbb{R}^n \to \mathbb{R}$ be a function that exists for the hybrid control problem, then there exists $K_g < \infty$ and $L_g < \infty$, such that $\left| g\left(x\right) \right| \le K_g$, $x \in \mathbb{R}^n$, and $L_g < \infty$. Also, $g \in C^k\left(\mathbb{R}^n : \mathbb{R}_+\right)$, $k \ge 1$.

The Caines assumptions are more general than the Riedinger assumptions because Caines only requires the dynamic constraints and the cost functional to be Lipschitz continuous in the state, where the Reidinger assumptions require them to be continuously differentiable in state and time. However, the Caines assumptions require a switch to either be autonomous or controlled and for Reidinger it doesn't matter. The Sussmann work allows for more general continuity assumptions on the dynamic constraints and the cost functional than Caines, but doesn't allow for controlled switching. The utility of the Caines' HMP is that the restrictions on the problem constraints are offset with the ability to study a larger class of hybrid problem by including controlled dynamic constraints. Note that two direct consequences of the assumptions are required for the Caines' HMP. First, the non-Zeno assumption (Assumption 6) coupled with fixing the end time t_f, requires that the number of discrete events has to be finite. This is true because the non-Zeno condition requires that an interval of time must exist between each discrete event, so when the final time is fixed, the number of intervals has to be finite.

An additional assumption is required for Caines work. They require local controllability of the continuous dynamics (See [63] Definition 5). Hence all admissible needle variations in control produce a set of trajectories that can be bounded by a tube.

5.3.2 Hybrid Maximum Principle

The Caines hybrid optimal control problem starts with a hybrid system defined by equation (5.3.1). Next let the functions $L_{q_i}(x,u)$, $q_i \in Q$, be a cost function that satisfies Assumption 10. Let $I_\Sigma = (S_\Sigma, u)$ be a hybrid input trajectory. Then for given initial and final times, t_0 and t_f respectfully, and hybrid state (q_0, x_0), the hybrid system Σ 's execution is well defined. Now define the hybrid cost function as

$$J_{I_\Sigma}(t_0, q_0, x_0) = \sum_{i=0}^{v} \int_{t_i}^{t_{i+1}} L_{q_i}\left(x(s), u(s)\right) ds + g\left(x\left(t_f\right)\right) \tag{5.3.8}$$

where $\dot{x}_{q_i}(t) = f_{q_i}\left(x(t), u(t)\right)$ for $i = 1 \ldots v$, $x\left(t_i^+\right) = \lim_{t \uparrow t_i} x_{q_{i-1}}(t)$ for $i = 1 \ldots v+1$, and $t_{v+1} = t_f$.

For the hybrid control problem defined above, let $g\left(x\left(t_f\right)\right) = 0$ and let I_Σ be the switching schedule and optimal control input that minimizes equation (5.3.8), then

1. there exists a continuous to the right, piecewise absolutely continuous adjoint λ, satisfying $\dot{\lambda} = -\frac{\partial H_{q_i}(x, \lambda, u)}{\partial x}$, *a.e.* $t \in \left(t_{s_i}, t_{s_{i+1}}\right)$, $i = 1 \ldots v$, where the following boundary conditions hold with $\lambda(t_0) \in \mathbb{R}_n$ and $\lambda\left(t_f\right) \in \mathbb{R}_n$
    a. if t_{s_i} is a controlled switching time then $\lambda\left(t_{s_i}^-\right) \equiv \lambda\left(t_{s_i}^+\right) \equiv \lambda\left(t_{s_i}\right)$
    b. if t_{s_i} is an autonomous switching time satisfying $M_{q_i, q_j}\left(t_{s_i}, x\left(t_{s_i}\right)\right) = 0$ then $\lambda\left(t_{s_i}^-\right) \equiv \lambda\left(t_{s_i}\right) \equiv \lambda\left(t_{s_i}^+\right) + \frac{\partial M_{q_i, q_j}\left(t_{s_i}, x\left(t_{s_i}\right)\right)}{\partial x} \cdot p$ for some $p \in \mathbb{R}^m$ where $q(t) = \begin{cases} q_i, \ t \in \left(t_{s_{i-1}}, t_{s_i}\right) \\ q_j, \ t \in \left(t_{s_i}, t_{s_{i+1}}\right) \end{cases}$.
2. the Hamiltonian minimization conditions are satisfied
    a. $H_{q_i}\left(x(t), \lambda(t), u(t)\right) \leq H_{q_i}\left(x(t), \lambda(t), u^*\right)$ for $u^* \in U$ and $t \in \left[t_{s_i}, t_{s_{i+1}}\right)$

b. $H_{q_i}\left(x(t),\lambda(t),u(t)\right) \leq H_{q_k}\left(x(t),\lambda(t),u(t)\right)$ for $k \in 1\ldots v$ and $t \in \left[t_{s_i}, t_{s_{i+1}}\right)$

3. the Hamiltonian value condition is satisfied. Let $H(t) \triangleq H_{q_i}(t)$, $t \in \left[t_{s_i}, t_{s_{i+1}}\right)$.

   a. If t_{s_i} is a controlled switching time then
   $H\left(t_{s_i}^-\right) \equiv H_{q_{i-1}}\left(t_{s_i}^-\right) \equiv H_{q_{i-1}}\left(t_{s_i}\right) = H_{q_i}\left(t_{s_i}\right) = H_{q_i}\left(t_{s_i}^+\right) \equiv H\left(t_{s_i}^+\right)$

   b. If t_{s_i} is an autonomous switching time satisfying $M_{q_i,q_j}\left(t_{s_i}, x\left(t_{s_i}\right)\right) = 0$ then
   $$H\left(t_{s_i}\right) \equiv H_{q_i}\left(t_{s_i}\right) = H_{q_i}\left(t_{s_i}^+\right) \equiv H\left(t_{s_i}^+\right) = H_{q_{i-1}}\left(t_{s_i}^-\right) + \frac{\partial M_{q_i,q_j}\left(t_{s_i}, x\left(t_{s_i}\right)\right)}{\partial x} \cdot p$$
   which equals $H\left(t_{s_i}^-\right) + \dfrac{\partial M_{q_i,q_j}\left(t_{s_i}, x\left(t_{s_i}\right)\right)}{\partial x} \cdot p$.

Now that the necessary conditions for Caines version of the Hybrid Maximum Principle have been presented, his algorithms for solving hybrid optimal control problems are briefly summarized.

5.3.3 Algorithms

Caines and Shaikh developed algorithms that utilize the necessary conditions and find the switching sequences that minimize the cost function in equation (5.3.8). The algorithms are limited to quadratic cost functions and linear dynamics.

In references [61][63][64][65] Caines developed three search algorithms. The first search algorithm is applied to problems where there are only autonomous switching surfaces. The inputs to the algorithm are the initial and final states and the initial and final times and the algorithm numerically computes the switching trajectory that minimizes the cost function. The second algorithm is applied to problems where the optimization problem deals with controlled switching only. For this problem the initial and final times and conditions are required as well as a fixed switching sequence. Given these inputs the algorithm computes the switch times that minimize the cost functional. The third algorithm is an extension of the second algorithm and uses a combinatorial approach to find an optimal switching sequence and switch times for the controlled switching optimal problem.

In [65] Caines applied discrete search methods to make the combinatorial algorithm more computationally efficient. Caines further used the hybrid optimal control problem definition to define zones of optimality, which can be used to reduce some optimal controlled switch problems to an autonomous switch surface problem, which is much more computationally efficient.

Chapter 6: Optimal Feedback Control of Hybrid Systems

The purpose of this section is to expand the results of the non-smooth necessary and sufficient conditions to provide a sufficient condition for the class of hybrid optimal control problems presented in Chapter 3. The necessary condition will be less general than the one given by Sussmann but is general enough to be applied to a wide variety of engineering problems and still have associated sufficient conditions.

In the derivation of the PMP and the non-smooth sufficient conditions, the dynamics of the system were required to satisfy a continuity assumption, which excluded direct application to hybrid optimal control problems. Further, the SHMP gave necessary conditions for hybrid optimal control problems with fixed switching sequences, but did not provide any sufficient conditions.

The purpose of this section is to utilize the non-smooth necessary and sufficient conditions presented in Chapter 4 to solve optimal feedback control problems for a subset of hybrid problems where the system constraints are autonomous and the hybrid problem has a fixed switching sequence. The resulting necessary conditions are not as general as the necessary conditions given by the SHMP because continuity of the dynamics is required for the derivation, but include sufficient conditions and can be applied to a large class of engineering problems.

The section can be outlined as follows; first the hybrid optimal control problem will be restated, the necessary and sufficient conditions will be presented in theorems, and finally the methodology for deriving the necessary and sufficient conditions will be presented and applied to the hybrid problem.

6.1 Problem Statement

Let the hybrid system, Σ, be defined as the 7-tuple

$$\Sigma = (Q, M, U, f, u, I, S) \tag{6.1.1}$$

that is given in Chapter 3, pg. 58.

Also, let there exist a control function $u(t)$, $t_0 \le t \le t_v$, that moves the trajectory from $(q_1, x(t_0))$ to $(q_v, x(t_v))$ where q_1 is the initial location, $x(t_0)$ is the initial condition for the state trajectory, q_v is the final location, and $x(t_v)$ is the final value of the state trajectory. Furthermore, let the cost associated with the control u be defined as

$$\begin{aligned} J(x(t_0), u, t_f - t_0) = \sum_{j=1}^{\nu} \int_{t_{j-1}}^{t_j} L_{q_j}(x(t), u(t), t)\, dt + \Phi_{q_0}(x(t_0)) \\ + \Phi_{q_v}(x(t_v)) + \sum_{j=1}^{\nu-1} \Phi_{q_j}\left(x(t_j^-), x(t_j^+)\right) \end{aligned} \tag{6.1.2}$$

where L_{q_j} is the Lagrangian for each location, Φ_{q_j} is the cost associated with switching from one location to the next, Φ_{q_0} and Φ_{q_v} are the costs associated with the initial and final conditions respectfully, and satisfies the assumptions in Chapter 3, pg. 50.

If u is an admissible control function for the hybrid system Σ with the associated cost $J\left(x\left(t_0\right),u,t_v-t_0\right)$, then the optimal control problem is to find the control that minimizes the cost and satisfies whichever of the following boundary conditions are part of the problem formulation

1. $x\left(t_v\right)\in\mathbb{R}^n$, i.e. the final value for the continuous trajectory is free.
2. $x\left(t_v\right)=x_v$, i.e. the final value for the continuous trajectory is fixed.
3. $x\left(t_v\right)\in g_v\left(x\left(t_v\right)\right)=0$, i.e. the final value for the continuous trajectory is constrained to a surface of values in the state space where the final time is either fixed or free.

Note that in the development of the necessary and sufficient conditions, the parameter I of the hybrid control problem will satisfy

$$I=\left\{\mathbb{R},\mathbb{R},\ldots,\mathbb{R}\right\} \tag{6.1.3}$$

so that the switching times are free. The case where the switching times are fixed or bounded can also be developed using the theory in [7] and the methodology given in this chapter, but is excluded here. For problems where the dynamic constraints are not autonomous, but vary with time in a regular way (as is the case under the control problem assumptions), the fixed time problem can be transformed into a free time problem by adding one more (clock) state to the dynamic constraint

$$\begin{aligned}\dot{x}\left(t\right)&=1\\ x\left(t_0\right)&=t_0\end{aligned} \tag{6.1.4}$$

to the control problem and to the switching surface,

$$x\left(t_s^-\right)=t_s \tag{6.1.5}$$

where t_s^- is the new problem's free switch time and t_s is the original problem's fixed switching time.

First Proposition 6.1.1 will be presented which describes the behavior of the viscosity solutions on a boundary which is required for the proof of the main theorem. Then the main theorem, Theorem 6.2.1 will be presented. Theorem 6.2.1 will be broken up into two parts. The first part gives the necessary and sufficient conditions for the initial and final locations and then the second part will give the necessary and sufficient conditions for the locations in between. The necessary and sufficient conditions are similar, but are broken up this way for notational convenience.

Proposition 6.1.1

Assume that there exists a subset, Ω, of $\mathbb{R}^n$ such that

$$\Omega = \Omega^1 \cup \Gamma \tag{6.1.6}$$

where Ω^1 is an open subset of Ω and Γ is a smooth surface in $\mathbb{R}^n$ that is the (possibly incomplete) boundary of Ω^1. Define $n(x)$ to be the unit vector normal to Γ at x, that lies in Ω^1 and $T(x)$ the tangent space to Γ at x. Also denote P_N as the orthogonal projection of $\mathbb{R}^n$ onto the space spanned by $n(x)$ and P_T as the orthogonal projection of $\mathbb{R}^n$ onto $T(x)$.

Let $u \in C(\Omega)$ and assume that its restriction u^1 to Ω^1 is a viscosity solution to the HJCB. Furthermore, assume that there exists a $t^+ > 0$ and vector q such that $u(x+t^+ \cdot q) \in \Omega^1$ and

$$\left.\frac{\partial u(x)}{\partial x}\right|^{+} = \lim_{t^+ \to 0} \frac{u(x+t^+ \cdot q) - u(x)}{t^+} \tag{6.1.7}$$

exists. Then for all $x \in \Gamma$ and continuous real valued Hamilton-Jacobi function F,

a. $F\left(x, u(x), P_T Du(x) + \left(P_N \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot n(x)\right) \cdot n(x)\right) \leq 0$

b. $F\left(x, u(x), P_T Du(x) + \left(P_N \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot n(x)\right) \cdot n(x)\right) \geq 0$

c. $F\left(x, u(x), P_T Du(x) + \left(P_N \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot n(x)\right) \cdot n(x)\right) = 0$

Proof of Proposition 6.1.1

When the hybrid problem is a free end time/fixed surface problem, Proposition 4.3.22 in Chapter 4 can be used to prove that u is a viscosity solution to the HJCB for every $x(t)$ that is an element of the reachable set $\Re$.

Pick a $y \in \Omega^1$, $x \in \Gamma$, $q \in \mathbb{R}^n$ and $t^+ > 0$ such that $u(x + q \cdot t^+) \in \Omega^1$. Also assume that the normal and tangential directions for $(x + t^+ \cdot q)$ parallel the tangential and normal directions of Γ at x. Since for all $y \in \Omega^1$ we have assumed u is differentiable, then

$$u(y)-u\left(x+t^{+}\cdot q\right)=\left(P_T\frac{\partial u\left(x+t^{+}\cdot q\right)}{\partial\left(x+t^{+}\cdot q\right)}+P_N\frac{\partial u\left(x+t^{+}\cdot q\right)}{\partial\left(x+t^{+}\cdot q\right)}\right)\cdot\left(y-\left(x+t^{+}\cdot q\right)\right)+o\left(\left|y-\left(x+t^{+}\cdot q\right)\right|\right) \tag{6.1.8}$$

Taking the limit of (6.1.8) as $t^{+}\to 0$ results in

$$u(y)-u(x)=\left(P_T\left.\frac{\partial u(x)}{\partial x}\right|^{+}+P_N\left.\frac{\partial u(x)}{\partial x}\right|^{+}\right)\cdot(y-x)+o(|y-x|) \tag{6.1.9}$$

Now, for all $u(y)$, $u\left(x+t^{+}\cdot q\right)\in\Omega^{1}$, $u\left(x+t^{+}\cdot q\right)$ will satisfy the definitions of viscosity sub- and super-solutions. Specifically the definition of viscosity sub-solution gives

$$u(y)-u\left(x+t^{+}\cdot q\right)\le\left(P_T p+P_N p\right)\cdot\left(y-x+t^{+}\cdot q\right)+o\left(\left|y-x+t^{+}\cdot q\right|\right) \tag{6.1.10}$$

Substituting equation (6.1.8) into (6.1.10) results in

$$\left(P_T\frac{\partial u\left(x+t^{+}\cdot q\right)}{\partial\left(x+t^{+}\cdot q\right)}+P_N\frac{\partial u\left(x+t^{+}\cdot q\right)}{\partial\left(x+t^{+}\cdot q\right)}\right)\cdot\left(y-\left(x+t^{+}\cdot q\right)\right)\le\left(P_T p+P_N p\right)\cdot\left(y-x+t^{+}\cdot q\right)+o\left(\left|y-x+t^{+}\cdot q\right|\right) \tag{6.1.11}$$

Since $u\left(x+t^{+}\cdot q\right)$ will satisfy the definition of viscosity sub-solution for all $t^{+}>0$ as $t^{+}\to 0$, we can take the limit of equation (6.1.11) as $t^{+}\to 0$ which results in

$$\left(P_T\left.\frac{\partial u(x)}{\partial x}\right|^{+}+P_N\left.\frac{\partial u(x)}{\partial x}\right|^{+}\right)\cdot(y-x)\le\left(P_T p+P_N p\right)\cdot(y-x)+o(|y-x|) \tag{6.1.12}$$

A similar result can be derived using the definition of the viscosity super-solution which is

$$\left(P_T \left. \frac{\partial u(x)}{\partial x} \right|^{+} + P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \right) \cdot (y-x) \geq (P_T p + P_N p) \cdot (y-x) + o(|y-x|) \quad (6.1.13)$$

In equations (6.1.12) and (6.1.13) the viscosity sub- and super-solutions have been projected onto the surface $x \in \Gamma$ through the limit argument. The proof of Proposition 4.3.15 can now proceed in an identical manner to Proposition 4.3.14.

Since the trajectory ends (or begins) at the surface, the surface serves as a boundary for the trajectory and $y = x + t \cdot \tau$ is not necessarily an element of $\Omega^1 \cup \Gamma$, so the method used to prove Proposition 4.3.14 does not directly apply. However, by problem assumption, all y such that $y = x + t \cdot n(x)$, $t > 0$, will be an element of Ω^1 so the proof can proceed.

Let $y = x + t \cdot n(x)$ for $t > 0$, then equation (6.1.12) implies

$$P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \cdot t \cdot n(x) \leq P_N p \cdot t \cdot n(x) + o(|y-x|) \quad (6.1.14)$$

and equation (6.1.13) implies

$$P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \cdot t \cdot n(x) \geq P_N p \cdot t \cdot n(x) + o(|y-x|) \quad (6.1.15)$$

Dividing both sides of equation (6.1.14) by t and taking the limit as $t \to 0$ results in

$$P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \cdot n(x) \le P_N\, p \cdot n(x) \tag{6.1.16}$$

Similarly equation (6.1.15) results in

$$P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \cdot n(x) \ge P_N\, p \cdot n(x) \tag{6.1.17}$$

Since by problem assumption equations (6.1.16) and (6.1.17) must be true and

$$P_N\, p = P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \tag{6.1.18}$$

Now pick $y = x + t_1 \cdot \tau + t_2 \cdot n(x)$, $t_1 \in \mathbb{R} \neq 0$, $t_2 > 0$, such that $y \in \Omega$, then equations (6.1.12) and (6.1.13) imply

$$\left(P_T \left. \frac{\partial u(x)}{\partial x} \right|^{+} + P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \right) \cdot \left(t_1 \cdot \tau + t_2 \cdot n(x) \right) \le$$

$$\left(P_T\, p + P_N\, p \right) \cdot \left(t_1 \cdot \tau + t_2 \cdot n(x) \right) + o\left(\left| t_1 \cdot \tau + t_2 \cdot n(x) \right| \right)$$

$$P_T \left. \frac{\partial u(x)}{\partial x} \right|^{+} \cdot t_1 \cdot \tau + P_N \left. \frac{\partial u(x)}{\partial x} \right|^{+} \cdot t_2 \cdot n(x) \le P_T\, p \cdot t_1 \cdot \tau + P_N\, p \cdot t_2 \cdot n(x) + o\left(\left| t_1 \cdot \tau + t_2 \cdot n(x) \right| \right)$$

(6.1.19)

and

$$P_T \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot t_1 \cdot \tau + P_N \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot t_2 \cdot n(x) \geq P_T\, p \cdot t_1 \cdot \tau + P_N\, p \cdot t_2 \cdot n(x) + o\left(\left|t_1 \cdot \tau + t_2 \cdot n(x)\right|\right) \tag{6.1.20}$$

Substituting equation (6.1.18) into equations (6.1.19) and (6.1.20) results in

$$P_T \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot t_1 \cdot \tau \leq P_T\, p \cdot t_1 \cdot \tau + o\left(\left|t_1 \cdot \tau + t_2 \cdot n(x)\right|\right) \tag{6.1.21}$$

and

$$P_T \left.\frac{\partial u(x)}{\partial x}\right|^{+} \cdot t_1 \cdot \tau \geq P_T\, p \cdot t_1 \cdot \tau + o\left(\left|t_1 \cdot \tau + t_2 \cdot n(x)\right|\right) \tag{6.1.22}$$

Finally since a positive and negative value of t_1 always exists such that $y \in \Omega$, equations (6.1.21) and (6.1.22) both imply

$$P_T\, p = P_T \left.\frac{\partial u(x)}{\partial x}\right|^{+} \tag{6.1.23}$$

which finishes the proof of the proposition. **Q.E.D.**

Note that this proposition doesn't provide an upper and lower bound for the sub- and super-differentials. Since the surface Γ provides a boundary, the state space on the "other" side of the surface doesn't provide any additional constraints on the problem.

<u>6.2 Theorem 6.2.1</u>

Let $\sum$ be a hybrid control system that satisfies the assumptions in Chapter 3, pg. 46-47, pg. 50.

First the necessary and sufficient conditions will be developed for the final location of the hybrid control system. Define $\lambda(t)$, $t_{v-1}^{+} < t < t_v$ as the solution

$$\begin{aligned} \frac{d\lambda(t)}{dt} &= -\lambda(t)\cdot\frac{\partial f_{q_v}(x,u)}{\partial x} \\ \lambda(t_v) &\in D^{+}\Phi_{q_v}(x(t_v)) \end{aligned} \tag{6.1.24}$$

then the control function $u(t)$, $t_{v-1}^{+} \le t \le t_v$, is optimal if and only if:

1. The Hamiltonian, $H_{q_v}(x(t),\lambda(t),u(t),\lambda_0) = \max_{\bar{u}\in U} H_{q_v}(x(t),\lambda(t),\bar{u}(t),\lambda_0)$, is maximized
2. and $\left(-\lambda(t), H_{q_v}(x(t),\lambda(t),u(t),\lambda_0)\right) \in D^{+}J_{c,q_v}^{*}(x(t),t_v - t)$ for all $t_{v-1}^{+} < t < t_v$
3. if $\lim\limits_{t^{+}\to 0}\dfrac{J_{c,q_v}^{*}(x(t_v)+t^{+}\cdot q, t_v - t_v) - J_{c,q_v}^{*}(x(t_v),t_v - t_v)}{t^{+}\cdot q}$ and

$$\left.\frac{\partial J_{c,q_v}^{*}(x(t_{v-1}^{+}),t_v - t_{v-1}^{+})}{\partial x(t_{v-1}^{+})}\right|^{+} = \lim_{t^{+}\to 0}\frac{J_{c,q_v}^{*}(x(t_{v-1}^{+})+t^{+}\cdot q, t_v - t_{v-1}^{+}) - J_{c,q_v}^{*}(x(t_{v-1}^{+}),t_v - t_{v-1}^{+})}{t^{+}\cdot q}$$

exist then $-\lambda(t_v) = P_T D\Phi_{q_v}(x(t_v)) + \left(P_N \left.\dfrac{\partial J_{c,q_v}^{*}(x(t_v),t_v - t_v)}{\partial x(t_v)}\right|^{+}\cdot n(x)\right)\cdot n(x)$ and

$$\begin{aligned} -\lambda(t_{v-1}^{+}) &= P_T DJ_{c,q_v}^{*}(x(t_{v-1}^{+}),t_v - t_{v-1}^{+}) + P_T D\Phi_{q_{v-1}}(x(t_{v-1}^{+}),x(t_v)) \\ &\quad + \left(P_N \left.\frac{\partial J_{c,q_v}^{*}(x(t_{v-1}^{+}),t_v - t_{v-1}^{+})}{\partial x(t_{v-1}^{+})}\right|^{+}\cdot n(x)\right)\cdot n(x) \end{aligned}$$

where $n(x)$ are the unit normal vectors to $S_{q_{v-1}}$ and S_{q_v} at $x(t_{v-1}^{+})$ and $x(t_v)$ pointing into the location.

where

$$H_{q_v}\left(x(t),\lambda(t),u(t),\lambda_0\right)=\left\langle f_{q_v}(x,u),\lambda(t)\right\rangle-\lambda_0\cdot L_{q_v}(x,u) \tag{6.1.25}$$

and

$$\begin{aligned}J^*_{c,q_v}\left(x(t),t_v-t\right)&=\inf_{\bar{u}\in U}J\left(x(t),\bar{u},t_v-t\right)\\J^*_{c,q_v}\left(x(t),t_v-t_v\right)&=\Phi_{q_v}\left(x(t_v)\right)\end{aligned} \tag{6.1.26}$$

Next, the necessary and sufficient conditions will now be developed for the initial location of the hybrid control system. Define $\lambda(t)$, $t_0<t<t_1^-$ as the solution

$$\begin{aligned}&\frac{d\lambda(t)}{dt}=-\lambda(t)\cdot\frac{\partial f_{q_1}(x,u)}{\partial x}\\&\lambda\left(t_1^-\right)\in D^+\left(\Phi_{q_1}\left(x\left(t_1^-\right),x\left(t_1^+\right)\right)+J^*_{c,q_2}\left(x\left(t_1^+\right),t_2^--t_1^+\right)\right)\end{aligned} \tag{6.1.27}$$

then the control function $u(t)$, $t_0\le t\le t_1^-$, is optimal if and only if:

4. The Hamiltonian, $H_{q_1}\left(x(t),\lambda(t),u(t),\lambda_0\right)=\max_{\bar{u}\in U}H_{q_1}\left(x(t),\lambda(t),\bar{u}(t),\lambda_0\right)$, is maximized
5. and $\left(-\lambda(t),H_{q_1}\left(x(t),\lambda(t),u(t),\lambda_0\right)\right)\in D^+\left(J^*_{c,q_1}\left(x(t),t_1^--t\right)\right)$
6. if $\lim_{t^+\to0}\dfrac{J^*_{c,q_1}\left(x\left(t_1^-\right)+t^+\cdot q,t_1^--t_1^-\right)-J^*_{c,q_1}\left(x\left(t_1^-\right),t_1^--t_1^-\right)}{t^+\cdot q}$ and

$$\left.\frac{\partial J^*_{c,q_1}\left(x(t_0),t_1^--t_0\right)}{\partial x(t_0)}\right|^+=\lim_{t^+\to0}\frac{J^*_{c,q_1}\left(x(t_0)+t^+\cdot q,t_1^--t_0\right)-J^*_{c,q_1}\left(x(t_0),t_1^--t_0\right)}{t^+\cdot q}\text{ exist}$$

then $-\lambda\left(t_1^-\right)=P_TDJ^*_{c,q_1}\left(x\left(t_1^-\right),t_1^--t_1^-\right)+\left(P_N\left.\dfrac{\partial J^*_{c,q_1}\left(x\left(t_1^-\right),t_1^--t_1^-\right)}{\partial x\left(t_1^-\right)}\right|^+\cdot n(x)\right)\cdot n(x)$

and

$$\begin{aligned}-\lambda(t_0)=&P_TDJ^*_{c,q_1}\left(x(t_0),t_1^--t_0\right)+P_TD\Phi_{q_0}\left(x(t_0)\right)\\&+\left(P_N\left.\frac{\partial J^*_{c,q_1}\left(x(t_0),t_1^--t_0\right)}{\partial x(t_0)}\right|^+\cdot n(x)\right)\cdot n(x)\end{aligned}$$

where $n(x)$ are the unit normal vectors to S_{q_1} and S_{q_0} at $x(t_1^-)$ and $x(t_0)$ pointing into the location.

where

$$H_{q_1}\left(x(t),\lambda(t),u(t),\lambda_0\right)=\left\langle f_{q_1}(x,u),\lambda(t)\right\rangle-\lambda_0\cdot L_{q_1}(x,u) \tag{6.1.28}$$

and

$$\begin{aligned} J^*_{c,q_1}\left(x(t),t_1^- - t\right) &= \inf_{\bar{u}\in U} J\left(x(t),\bar{u},t_1^- - t\right) \\ J^*_{c,q_1}\left(x(t_1^-),t_1^- - t_1^-\right) &= \Phi_{q_1}\left(x(t_1^-),x(t_1^+)\right)+J^*_{c,q_2}\left(x(t_1^+),t_2^- - t_1^+\right) \end{aligned} \tag{6.1.29}$$

Finally, the necessary and sufficient conditions for the rest of the locations of the hybrid system can be given. Define $\lambda(t)$, $t_{i-1}^+ < t < t_i^-$, for $i=2,\ldots,v-1$, as the solution to the dynamic system

$$\frac{d\lambda(t)}{dt}=-\lambda(t)\cdot\frac{\partial f_{q_i}(x,u)}{\partial x} \tag{6.1.30}$$

with final condition

$$\lambda(t_i^-)\in D^+\left(\Phi_{q_i}\left(x(t_i^-),x(t_i^+)\right)+J^*_{c,q_{i+1}}\left(x(t_i^+),t_{i+1}^- - t_i^+\right)\right) \tag{6.1.31}$$

If for all $i=2\ldots v-1$, $D^+\left(\Phi_{q_i}\left(x(t_i^-),x(t_i^+)\right)+J^*_{c,q_{i+1}}\left(x(t_i^+),t_{i+1}^- - t_i^+\right)\right)\neq\varnothing$, then the control function $u(t)$, $t_{i-1}^+\le t\le t_i^-$ is optimal for all $i=2\ldots v-1$ if and only if:

7. The Hamiltonian, $H_{q_i}\left(x(t),\lambda(t),u(t),\lambda_0\right)=\max_{\bar{u}\in U} H_{q_i}\left(x(t),\lambda(t),\bar{u}(t),\lambda_0\right)$, is maximized
8. and $\left(-\lambda(t),H_{q_i}\left(x(t),\lambda(t),u(t),\lambda_0\right)\right)\in D^+J^*_{c,q_i}\left(x(t),t_i^- - t\right)$

9. if $\lim\limits_{t^+\to 0} \dfrac{J^*_{c,q_i}\left(x\left(t_i^-\right)+t^+\cdot q, t_i^- - t_i^-\right) - J^*_{c,q_i}\left(x\left(t_i^-\right), t_i^- - t_i^-\right)}{t^+\cdot q}$ and

$$\left.\frac{\partial J^*_{c,q_i}\left(x\left(t_{i-1}^+\right), t_i^- - t_{i-1}^+\right)}{\partial x\left(t_{i-1}^+\right)}\right|^+ = \lim_{t^+\to 0}\frac{J^*_{c,q_i}\left(x\left(t_{i-1}^+\right)+t^+\cdot q, t_i^- - t_{i-1}^+\right) - J^*_{c,q_i}\left(x\left(t_{i-1}^+\right), t_i^- - t_{i-1}^+\right)}{t^+\cdot q}$$

exist then

$$-\lambda\left(t_i^-\right) = P_T DJ^*_{c,q_i}\left(x\left(t_i^-\right), t_i^- - t_i^-\right) + \left(P_N \left.\frac{\partial J^*_{c,q_1}\left(x\left(t_i^-\right), t_i^- - t_i^-\right)}{\partial x\left(t_i^-\right)}\right|^+ \cdot n(x)\right)\cdot n(x) \text{ and}$$

$$-\lambda\left(t_{i-1}^+\right) = P_T DJ^*_{c,q_i}\left(x\left(t_{i-1}^+\right), t_i^- - t_{i-1}^+\right) + P_T D\Phi_{q_i}\left(x\left(t_{i-1}^+\right), x\left(t_i^-\right)\right)$$
$$+\left(P_N \left.\frac{\partial J^*_{c,q_i}\left(x\left(t_{i-1}^+\right), t_i^- - t_{i-1}^+\right)}{\partial x\left(t_{i-1}^+\right)}\right|^+ \cdot n(x)\right)\cdot n(x)$$

where $n(x)$ are the unit normal vectors to S_{q_i} and $S_{q_{i-1}}$ at $x\left(t_i^-\right)$ and $x\left(t_{i-1}^+\right)$ pointing into the location.

where

$$H_{q_i}\left(x(t), \lambda(t), u(t), \lambda_0\right) = \left\langle f_{q_i}(x,u), \lambda(t)\right\rangle - \lambda_0 \cdot L_{q_i}(x,u) \tag{6.1.32}$$

and

$$\begin{aligned} J^*_{c,q_i}\left(x(t), t_i^- - t\right) &= \inf_{\bar{u}\in U} J\left(x(t), \bar{u}, t_i^- - t\right) \\ J^*_{c,q_i}\left(x(t), t_i^- - t_i^-\right) &= \Phi_{q_i}\left(x\left(t_i^-\right), x\left(t_i^+\right)\right) + J^*_{c,q_{i+1}}\left(x\left(t_i^+\right), t_{i+1}^- - t_i^+\right) \end{aligned} \tag{6.1.33}$$

Finally under the problem assumptions given in Chapter 3 pg. 46-47, pg. 50, the following condition is true for every location.

10. $H_{q_i}\left(x(t), -\lambda(t), u(t), \lambda_0\right)$ is constant for all $t_{i-1}^+ \le t \le t_i^-$.

In Theorem 6.1.2, the relationship between the super-differential and the cost-to-go function has a different sign than the work in Chapter 4. This is because the Hamiltonian used here has opposite sign of the one used in Chapter 4.

Proof of Theorem 6.2.1

The necessary and sufficient conditions for the hybrid control problem are going to be derived by recasting the hybrid problem into a series of local non-hybrid control problems and applying the theory given in Chapter 4. A method that is in the spirit of Bellman's Principle of Dynamic Programming will be used to decompose the hybrid problem into a series of local non-hybrid optimal control problems. The optimal control will be computed for location q_v first and the corresponding optimal cost-to-go to the final state will be calculated along the surface of initial conditions for the last location. Since the mapping of the state from location q_{v-1} to q_v is given through the problem definition, the equivalent optimal cost-to-go to the final state, $J^*_{c,q_{v-1}}$, can be calculated along the surface of final conditions for the next location q_{v-1} in reverse time. Now a new local non-hybrid optimal control problem can be developed for location q_{v-1} where $J^*_{c,q_{v-1}}$ is added as a terminal cost to the cost function. The optimal control is computed for this new problem and the process is repeated until the solution is computed for all locations. For each local non-hybrid control problem, a set of necessary and sufficient conditions are developed that get grouped together to form the necessary and sufficient conditions for the entire hybrid optimal control problem.

The first step in deriving the necessary and sufficient conditions is to develop a non-hybrid control problem that captures the behavior of the last location of the hybrid optimal control problem.

Let the location be equal to q_v and assume that there exists a control function $u(t)$, $t_{v-1}^{+} \le t < t_v$ that transfers the state from $x\left(t_{v-1}^{+}\right) \in S_{q_{v-1},q_v}$ to $x(t_v)$ with cost

$$J\left(x\left(t_{v-1}^{+}\right),u,t_v - t_{v-1}^{+}\right) = \int_{t_{v-1}^{+}}^{t_v} L_{q_v}\left(x(t),u(t)\right)\cdot dt + \Phi_{q_v}\left(x(t_v)\right) \qquad (6.1.34)$$

and satisfies the constraints

$$\dot{x}(t) = f_v(x,u) \qquad (6.1.35)$$

Now under the assumptions for the control problem, Theorem 3.44 in [7] applies.

Assume $D^{+}\Phi_{q_v}\left(x(t_v)\right) \neq \varnothing$, and define $\lambda(t)$, $t_{v-1}^{+} < t < t_v$ as the solution

$$\begin{aligned} &\frac{d\lambda(t)}{dt} = -\lambda(t)\cdot\frac{\partial f_{q_v}(x,u)}{\partial x} \\ &\lambda(t_v) \in D^{+}\Phi_{q_v}\left(x(t_v)\right) \end{aligned} \qquad (6.1.36)$$

then the control function $u(t)$ is optimal if and only if:

1. The Hamiltonian, $H_{q_v}\left(x(t),\lambda(t),u(t),\lambda_0\right) = \max_{\overline{u}\in U} H_{q_v}\left(x(t),\lambda(t),\overline{u}(t),\lambda_0\right)$, is maximized
2. and $\left(-\lambda(t), H_{q_v}\left(x(t),\lambda(t),u(t),\lambda_0\right)\right) \in D^{+}J^{*}_{c,q_v}\left(x(t),t_v - t\right)$

where

$$H_{q_v}\left(x(t),\lambda(t),u(t),\lambda_0\right) = \left\langle f_{q_v}(x,u),\lambda(t)\right\rangle - \lambda_0 \cdot L_{q_v}(x,u) \qquad (6.1.37)$$

and

$$J^*_{c,q_v}\left(x(t),t_v-t\right)=\inf_{\bar{u}\in U} J\left(x(t),\bar{u},t_v-t\right)$$
$$J^*_{c,q_v}\left(x(t),t_v-t_v\right)=\Phi_{q_v}\left(x(t_v)\right) \tag{6.1.38}$$

Furthermore, if the final state is an element of the set S_{q_v} and the value function is differentiable with respect to x, then it must satisfy the boundary condition

$$-\lambda(t_v)=P_T DJ^*_{c,q_v}\left(x(t),t_v-t_v\right)+\left(P_N \left.\frac{\partial J^*_{c,q_v}\left(x(t),t_v-t_v\right)}{\partial x}\right|^{+}\cdot n(x)\right)\cdot n(x) \tag{6.1.39}$$

for all $x\in S_{q_v}$, where $\left.\frac{\partial u(x)}{\partial x}\right|^{+}$ is the direction derivative of u at x, and $n(x)$ is the unit normal vector to S_{q_v} pointing into the set and

$$-\lambda\left(t^+_{v-1}\right)=P_T DJ^*_{c,q_v}\left(x(t),t_v-t^+_{v-1}\right)+\left(P_N \left.\frac{\partial J^*_{c,q_v}\left(x(t),t_v-t^+_{v-1}\right)}{\partial x}\right|^{-}\cdot n(x)\right)\cdot n(x) \tag{6.1.40}$$

for all $x\in S_{q_{v-1}}$, where $\left.\frac{\partial u(x)}{\partial x}\right|^{-}$ is the direction derivative of u at x, and $n(x)$ is the unit normal vector to $S_{q_{v-1}}$ pointing into the set. For the two boundary conditions, pointing into the set means that when the state is an element of the surface of initial conditions, the following must be true

$$sign\left(\left\langle f\left(x\left(t^+_{v-1}\right),u\left(t^+_{v-1}\right)\right),n\left(x\left(t^+_{v-1}\right)\right)\right\rangle\right)\geq 0 \tag{6.1.41}$$

and when the state is an element of the surface of final conditions the following equation must be true

$$sign\left(\left\langle f\left(x\left(t_{v-1}^{-}\right),u\left(t_{v-1}^{-}\right)\right),n\left(x\left(t_{v-1}^{-}\right)\right)\right\rangle\right)\leq 0 \tag{6.1.42}$$

Note that conditions (1) and (2) provide necessary and sufficient conditions for the optimal control in location q_v. Since the optimal control is given by conditions (1) and (2), the optimal cost-to-go from any $x\left(t_{v-1}^{+}\right)$ to $x\left(t_v\right)$ can be computed, so with some work the feedback control solution for location q_v is also computed. As such, the optimal cost-to-go from any initial state $x\left(t_{v-1}^{+}\right)$ to final state $x\left(t_v\right)$, $J_{c,q_v}^{*}\left(x\left(t_{v-1}^{+}\right),t_v-t_{v-1}^{+}\right)$, can be computed.

Now that the feedback control solution is computed for location q_v, the next step is to develop and solve a local non-hybrid optimal control problem for location q_{v-1}. Assume that there exists a control function $u\left(t\right)$, $t_{v-2}^{+}\leq t<t_{v-1}^{-}$ that transfers the state from $x\left(t_{v-2}^{+}\right)\in S_{q_{v-2}}$ to $x\left(t_{v-1}^{-}\right)$ with cost

$$\begin{aligned} J\left(x\left(t_{v-2}^{+}\right),u,t_{v-1}^{-}-t_{v-2}^{+}\right)=\int_{t_{v-2}^{+}}^{t_{v-1}^{-}}L_{q_{v-1}}\left(x\left(t\right),u\left(t\right)\right)\cdot dt+\Phi_{q_{v-1}}\left(x\left(t_{v-1}^{-}\right),x\left(t_{v-1}^{+}\right)\right)\\ +J_{c,q_v}^{*}\left(x\left(t_{v-1}^{+}\right),t_v-t_{v-1}^{+}\right)\end{aligned} \tag{6.1.43}$$

and satisfies the constraint

$$\dot{x}\left(t\right)=f_{v-1}\left(x,u\right) \tag{6.1.44}$$

Note that the inclusion of the optimal cost-to-go to the final state $x\left(t_v\right)$ in equation (6.1.43) completely captures the effect of a variation in optimal trajectory in location q_{v-1}

to the final location and state $x(t_\nu)$. As such, this local non-hybrid control problem is equivalent to solving the part of the hybrid problem that contains locations $q_{\nu-1}$ and q_ν.

The new non-hybrid optimal control problem satisfies the assumptions of required for Theorem 4.3.11 and it can be applied to this optimal control problem.

Assume $D^+\left(\Phi_{q_{\nu-1}}\left(x\left(t_{\nu-1}^-\right), x\left(t_{\nu-1}^+\right)\right) + J_{c,q_\nu}^*\left(x\left(t_{\nu-1}^+\right), t_\nu - t_{\nu-1}^+\right)\right) \neq \varnothing$, and define $\lambda(t)$, $t_{\nu-2}^+ < t < t_{\nu-1}^-$ as the solution

$$\begin{aligned} &\frac{d\lambda(t)}{dt} = -\lambda(t)\cdot\frac{\partial f_{q_{\nu-1}}(x,u)}{\partial x} \\ &\lambda\left(t_{\nu-1}^-\right) \in D^+\left(\Phi_{q_{\nu-1}}\left(x\left(t_{\nu-1}^-\right), x\left(t_{\nu-1}^+\right)\right) + J_{c,q_\nu}^*\left(x\left(t_{\nu-1}^+\right), t_\nu - t_{\nu-1}^+\right)\right) \end{aligned} \tag{6.1.45}$$

then the control function $u(t)$ is optimal if and only if:

1. The Hamiltonian, $H_{q_{\nu-1}}\left(x(t),\lambda(t),u(t),\lambda_0\right) = \max_{\bar{u}\in U} H_{q_{\nu-1}}\left(x(t),\lambda(t),\bar{u}(t),\lambda_0\right)$, is maximized
2. and $\left(-\lambda(t), H_{q_{\nu-1}}\left(x(t),\lambda(t),u(t),\lambda_0\right)\right) \in D^+ J_{c,q_{\nu-1}}^*\left(x(t), t_{\nu-1}^- - t\right)$

where

$$H_{q_{\nu-1}}\left(x(t),\lambda(t),u(t),\lambda_0\right) = \left\langle f_{q_{\nu-1}}(x,u), \lambda(t)\right\rangle - \lambda_0 \cdot L_{q_{\nu-1}}(x,u) \tag{6.1.46}$$

and

$$\begin{aligned} &J_{c,q_{\nu-1}}^*\left(x(t), t_{\nu-1}^- - t\right) = \inf_{\bar{u}\in U} J\left(x(t), \bar{u}, t_{\nu-1}^- - t\right) \\ &J_{c,q_{\nu-1}}^*\left(x(t), t_{\nu-1}^- - t_{\nu-1}^-\right) = \Phi_{q_{\nu-1}}\left(x\left(t_{\nu-1}^-\right), x\left(t_{\nu-1}^+\right)\right) + J_{c,q_\nu}^*\left(x\left(t_{\nu-1}^+\right), t_\nu - t_{\nu-1}^+\right) \end{aligned} \tag{6.1.47}$$

Furthermore, if $\lim_{t^+\to 0} \dfrac{J^*_{c,q_{v-1}}\left(x\left(t^-_{v-1}\right)+t^+\cdot q, t^-_{v-1}-t^-_{v-1}\right)-J^*_{c,q_{v-1}}\left(x\left(t^-_{v-1}\right), t^-_{v-1}-t^-_{v-1}\right)}{t^+\cdot q}$ and

$\left.\dfrac{\partial J^*_{c,q_{v-1}}\left(x\left(t^+_{v-2}\right), t^-_{v-1}-t^+_{v-2}\right)}{\partial x\left(t^+_{v-2}\right)}\right|^+ = \lim_{t^+\to 0} \dfrac{J^*_{c,q_{v-1}}\left(x\left(t^+_{v-2}\right)+t^+\cdot q, t^-_{v-1}-t^+_{v-2}\right)-J^*_{c,q_{v-1}}\left(x\left(t^+_{v-2}\right), t^-_{v-1}-t^+_{v-2}\right)}{t^+\cdot q}$ exist

then

$$-\lambda\left(t^-_{v-1}\right)=P_T DJ^*_{c,q_{v-1}}\left(x\left(t^-_{v-1}\right), t^-_{v-1}-t^-_{v-1}\right) + \left(P_N \left.\frac{\partial J^*_{c,q_{v-1}}\left(x\left(t^-_{v-1}\right), t^-_{v-1}-t^-_{v-1}\right)}{\partial x\left(t^-_{v-1}\right)}\right|^+ \cdot n(x)\right)\cdot n(x) \tag{6.1.48}$$

and

$$-\lambda\left(t^+_{v-2}\right)=P_T DJ^*_{c,q_{v-1}}\left(x\left(t^+_{v-2}\right), t^-_{v-1}-t^+_{v-2}\right)+P_T D\Phi_{q_{v-1}}\left(x\left(t^+_{v-2}\right), x\left(t^-_{v-1}\right)\right) + \left(P_N \left.\frac{\partial J^*_{c,q_{v-1}}\left(x\left(t^+_{v-2}\right), t^-_{v-1}-t^+_{v-2}\right)}{\partial x\left(t^+_{v-2}\right)}\right|^+ \cdot n(x)\right)\cdot n(x) \tag{6.1.49}$$

where $n(x)$ are the unit normal vectors to $S_{q_{v-1}}$ and $S_{q_{v-2}}$ at $x\left(t^-_{v-1}\right)$ and $x\left(t^+_{v-2}\right)$ pointing into the location.

As before, conditions (1) and (2) provide necessary and sufficient conditions for the optimal control in location q_{v-1} and the optimal cost-to-go from any $x\left(t^+_{v-2}\right)$ to $x\left(t^-_{v-1}\right)$, $J^*_{c,q_{v-1}}\left(x\left(t^+_{v-2}\right), t^-_{v-1}-t^+_{v-2}\right)$, can be computed.

Applying this decomposition of the hybrid optimal control problem in a series of local non-hybrid optimal control problems and applying Theorem 4.3.11 from Chapter 4 provides the general necessary and sufficient conditions for the hybrid optimal control problem.

Since the constraints on the problem are assumed autonomous, another condition can be added to the problem. Corollary 4.3.13 provides the following:

3. If $u(t)$, $t_{i-1}^{+} \le t \le t_i^{-}$, is the optimal control function, then for all t $H_{q_i}\left(x(t),\lambda(t),u(t),\lambda_0\right)$ is constant.

In summary, conditions (1) and (2) provide the framework for the necessary and sufficient conditions for the general hybrid optimal control problem, equations (6.1.48) and (6.1.49) provide the framework for the boundary conditions for the adjoint when the differentiability assumption is met, and condition (3) provides the framework for the constraint on the Hamiltonian and finishes the proof of Theorem 6.1.2. **Q.E.D.**

Chapter 7: Examples

Two examples will be presented in this chapter to demonstrate the application of the optimal control analysis tools developed in Chapter 6. The first example is a hybrid problem that combines the standard double integrator and harmonic oscillator optimal control problems commonly found in the optimal control literature. This is a simple problem that demonstrates the complexities of hybrid problems. The second example utilizes a simple tire and ground interface model to study the traction control problem. This example demonstrates how non-linear practical engineering problems can be analyzed using hybrid tools

7.1 Introduction to First Problem

The purpose of this section is to apply the methods developed in Chapter 6 to analyze a simple hybrid feedback control problem. The hybrid control problem is a two-node hybrid automaton that utilizes the dynamics from the well-known double integrator and harmonic oscillator optimal control problems. The solution to the control problem will be the control that minimizes the time to the origin. The double integrator and harmonic oscillator problems are used because the optimal control for the minimum time to the origin for each problem is well known and is studied extensively in optimal control texts [6].

The solution to both the individual minimum time to the origin optimal control problems for the double integrator and harmonic oscillator will be given. Providing the solution to each individual optimal control problem serves two purposes. First, parts of each solution will be directly applied to the hybrid control problem. Second, the result of the hybrid problem will be compared to the optimal solution of each individual problem in its respective region of the state space to demonstrate how the hybridization of the problem changes the feedback control. As such the feedback optimal control solution is required for each individual control problem.

The hybrid optimal control problem will be presented first. The solution to the minimum time to the origin problem for the individual double integrator and harmonic oscillator problems will be given next. Finally, the hybrid control problem will be examined and the solution presented.

7.1.1 Hybrid Control Problem Definition

The hybrid control problem will now be given. The problem statement is as follows:

> *Problem Statement:* Given the hybrid system in Figure 7.1 determine the feedback control strategy to drive the state, from any initial condition, to the origin in the minimal amount of time subject to the following conditions:

1. $x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$, where $x(t_0)$ can take any value in the state space
2. The initial location q is determined by the initial state and satisfies the following rule

a. If $x_{10} < \alpha \cdot x_{20}$, $q = q_1$
b. If $x_{10} > \alpha \cdot x_{20}$, $q = q_2$
c. If $x_{10} = \alpha \cdot x_{20} \wedge \alpha \cdot x_{20} \leq 0$, $q = q_1$
d. If $x_{10} = \alpha \cdot x_{20} \wedge \alpha \cdot x_{20} > 0$, $q = q_2$

3. $|u(t)| \leq 1$, for all t

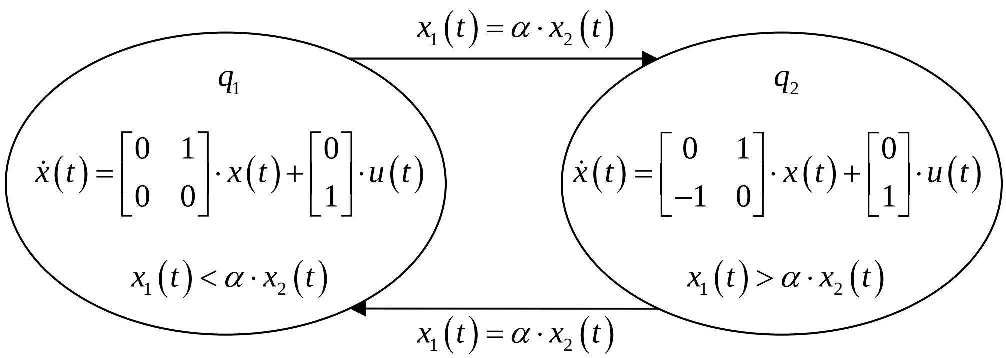

Figure 7.1: Hybrid automaton of example problem.

The system depicted in Figure 7.1 is a hybrid automaton. Per the figure, this hybrid system has two locations q_1 and q_2, with associated continuous time dynamics. In location q_1, the continuous time dynamics is the double integrator and while in location q_2, the dynamics driving the system is the harmonic oscillator. Two switching surfaces exist for this problem that are superimposed on each other. Each switching surface is the line $x_1 = \alpha \cdot x_2$, for an $\alpha \in \mathbb{R}$ such that $0 \leq \alpha < \infty$. If the system is in location q_1 and the trajectory intersects the switching surface, the system switches to location q_2. Conversely, if the system is in location q_2 and the trajectory intersects the switching surface, the system switches to location q_1.

Note that the solution to this problem is required to be global and to be a feedback solution.

To solve this global optimization problem, the work from Chapter 6 will be used. The solution will start at the origin and work backwards in time to reconstruct the optimal sequence of trajectories that is required to force the initial condition to the origin. To ensure that the trajectories are optimal, the necessary and sufficient conditions given in Chapter 6 will be verified.

Now that the hybrid control problem is given, it will be recast in the Sussmann form. Sussmann defines a hybrid control system as:

$$\Sigma = \left(Q, M, U, f, u, I, S\right) \tag{7.1.1}$$

where each component will be defined below.

Q is a finite set and is the number of locations where the continuous time dynamics change. In this problem $Q = \{q_1, q_2\}$ since there are two locations. M is a family of smooth manifolds indexed by q_i that define the state space for each location. For this problem the state is constrained for each location so

$M = \left\{\left(x_1, x_2\right) \in \mathbb{R}^2 : x_1 \le \alpha \cdot x_2, \left(x_1, x_2\right) \in \mathbb{R}^2 : x_1 \ge \alpha \cdot x_2\right\}$ $M = \left\{M_{q_1}, M_{q_2}\right\}$ where

$$\begin{aligned} M_{q_1} &= \left(x_1, x_2\right) \in \mathbb{R}^2 : x_1 \le \alpha \cdot x_2 \\ M_{q_2} &= \left(x_1, x_2\right) \in \mathbb{R}^2 : x_1 \ge \alpha \cdot x_2 \end{aligned} \tag{7.1.2}$$

The third component of the hybrid model is U which is the control space for each location. For this problem, there is one control input, so $U = \{\mathbb{R}, \mathbb{R}\}$. f is a family indexed by q_i that maps the product $M_q \times U_q$ into the tangent bundle of M_q such that $f_q(x,u) \in T_x M_q$ for every $(x,u) \in M_q \times U_q$. f defines the dynamic constraints for the state variables in each location, so for this problem $f = \{\dot{x}_{q_1}, \dot{x}_{q_2}\}$, where $\dot{x}_{q_i}$ is defined in Figure 7.1. u is a family indexed by q_i consisting of the set of admissible controls for each location. From the problem definition the set of admissible controls is the same for both locations and is $u = \{|u| \le 1, |u| \le 1\}$. I is a family of sub-intervals of $\mathbb{R}$, that give freedom to include bounds on the switching time in the hybrid model. Given the nature of the dynamics for the hybrid problem, each location's intersection of the switching set will be bounded in time, so $I = \{[t_{\min}, t_{\max}]_{q_1}, [t_{\min}, t_{\max}]_{q_2}\}$. Finally, S is a subset of the location space and state space, indexed by q_i that defines the switching set for the hybrid system. Using Figure 7.1, $S = \{S_{q_1,q_2}, S_{q_2,q_1}\}$ where $S_{q_1,q_2} = \{(x_1, x_2, x_1, x_2) : \alpha \cdot x_2 \le x_1\}$ and $S_{q_2,q_1} = \{(x_1, x_2, x_1, x_2) : \alpha \cdot x_2 \ge x_1\}$.

In order to use the work of Chapter 6, the hybrid problem Σ needs to be separated into two separate control problems. The reason is that no matter what final location, q_1 or q_2, is chosen, a set of states exist such that the origin is not reachable with any admissible control and the optimal cost-to-go function is discontinuous. However, if Σ is separated into two separate control problems with appropriate assumptions, the solution to the entire problem can be computed by means of the results in Chapter 6.

7.1.2 Double Integrator

Now that the hybrid control system has been defined, the minimum time to the origin control for each set of local dynamics will be reviewed, as these properties will be helpful in the analysis of hybrid control system.

The double integrator dynamics govern the continuous dynamics of location q_1. [6] performs extensive analysis on the solution of the minimum time to origin control problem for the double integrator. A short synopsis of the results will be provided here.

The differential equations of motion are:

$$\dot{x}_{q_1}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t) \tag{7.1.3}$$

The optimal control problem will find the control that minimizes the time to the origin from the state $x = (x_1, x_2) \in \mathbb{R}^2$ subject to the dynamic constraints given in (7.1.3) with the control $|u(t)| \leq 1$.

The necessary and sufficient conditions from Chapter 6 can be applied to this problem almost everywhere. Along the surface

$$x_1(t) = -\frac{1}{2} \cdot |x_2(t)| \cdot x_2(t) \tag{7.1.4}$$

the sufficient conditions don’t apply because the value function is continuous but not differentiable. The sufficient condition in Chapter 6 assumes that when the value function is continuous but not differentiable, it is at a single point and not along a trajectory. However, the Maximum Principle still applies and can be used along with the problem assumptions to prove the control is optimal.

First, the properties of the Hamiltonian will be studied. Recall from Chapter 6 that the Hamiltonian is defined as

$$H_i\left(x,\lambda_i,u,\lambda_0\right)=\left\langle \lambda_i\left(t\right),\, f_i\left(x,u\right)\right\rangle-\lambda_0\cdot L_i\left(x,u\right) \tag{7.1.5}$$

Where:

1. $\lambda\left(t\right)$ is the value of the adjoint at time t
2. f_i is the function describing the state equation constraint in location i
3. λ_0 is a constant used to define normality of the system
4. L_i is the Lagrangian for the system

The Hamiltonian for this control problem is

$$H\left(x,\lambda_1,u,\lambda_0\right)=\lambda_1\left(t\right)\cdot x_2\left(t\right)+\lambda_2\left(t\right)\cdot u\left(t\right)-1 \tag{7.1.6}$$

Where $\lambda_1\left(t\right)=\begin{bmatrix}\lambda_1\left(t\right)\\ \lambda_2\left(t\right)\end{bmatrix}$, λ_0 is chosen to equal 1 without loss of generality

and the adjoint equation satisfies

$$\dot{\lambda}\left(t\right)=-\frac{\partial H}{\partial x} \tag{7.1.7}$$

everywhere the differential exists.

Substituting equation (7.1.6) into equation (7.1.7) and simplifying results in

$$\dot{\lambda}(t) = \begin{bmatrix} 0 \\ -\lambda_1(t) \end{bmatrix} \tag{7.1.8}$$

Assuming that the initial condition for the adjoint equation is

$$\lambda(t_0) = \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \end{bmatrix} \tag{7.1.9}$$

and solving equation (7.1.8) with initial condition (7.1.9) gives

$$\lambda(t) = \begin{bmatrix} \lambda_1(t_0) \\ \lambda_2(t_0) - \lambda_1(t_0) \cdot t \end{bmatrix}, \; t_0 \le t < \overline{t} \tag{7.1.10}$$

The necessary conditions given in Chapter 6 require that the Hamiltonian is maximized, which means that the optimal control will satisfy

$$u^*(t) = \operatorname{sgn}\left(\lambda_2(t_0) - \lambda_1(t_0) \cdot t\right) \tag{7.1.11}$$

Since the optimal control must satisfy $|u(t)| \le 1$, then equation (7.1.11) implies

$$u^*(t) = \begin{cases} 1 \\ -1 \end{cases} \tag{7.1.12}$$

almost everywhere and the optimal control will be constant almost everywhere and switch sign at most one time. As such, equation (7.1.3) can be integrated when $u^*(t) = 1$

$$\begin{aligned} x_1(t) &= x_{10} + x_{20} \cdot (t - t_0) + \frac{1}{2} \cdot (t - t_0)^2 \\ x_2(t) &= x_{20} + (t - t_0) \end{aligned} \tag{7.1.13}$$

and

$$
\begin{aligned}
x_1(t) &= x_{10} + x_{20}\cdot(t-t_0) - \frac{1}{2}\cdot(t-t_0)^2 \\
x_2(t) &= x_{20} - (t-t_0)
\end{aligned}
\tag{7.1.14}
$$

for $u^*(t) = -1$.

Equations (7.1.13) and (7.1.14) imply that the optimal control requires that the optimal trajectory will only intersect the origin if it is traveling along one of two surfaces. If the initial condition satisfies

$$
x_{10} = -\frac{1}{2}\cdot|x_{20}|\cdot x_{20},\ x_{20} > 0 \tag{7.1.15}
$$

then the control $u^*(t) = -1$, $t_0 \le t < t_f$, will move the initial condition (x_{10}, x_{20}) to $(0,0)$ in time

$$
t_f = x_{20} \tag{7.1.16}
$$

Conversely, if the initial condition satisfies

$$
x_{10} = -\frac{1}{2}\cdot|x_{20}|\cdot x_{20},\ x_{20} < 0 \tag{7.1.17}
$$

then the control $u^*(t) = 1$, $t_0 \le t < t_f$, will move the initial condition (x_{10}, x_{20}) to $(0,0)$ in time

$$
t_f = -x_{20} \tag{7.1.18}
$$

When the initial condition does not satisfy equation (7.1.15) or (7.1.17), then the necessary condition implies that the control will switch once. If the initial condition satisfies

$$x_{10} < -\frac{1}{2} \cdot |x_{20}| \cdot x_{20} \tag{7.1.19}$$

then the control $u^*(t) = 1$, $t_0 \le t < t_s$ and control $u^*(t) = -1$, $t_s \le t < t_f$ where t_s is the time the state trajectory intersects the surface given in (7.1.15), will move the initial condition (x_{10}, x_{20}) to $(0,0)$ in time

$$t_f = -x_{20} + \sqrt{-4 \cdot x_{10} + 2 \cdot x_{20}^2} \tag{7.1.20}$$

Conversely, if the initial condition satisfies

$$x_{10} > -\frac{1}{2} \cdot |x_{20}| \cdot x_{20} \tag{7.1.21}$$

then the control $u^*(t) = -1$, $t_0 \le t < t_s$ and control $u^*(t) = 1$, $t_s \le t < t_f$ where t_s is the time the state trajectory intersects the surface given in (7.1.17), will move the initial condition (x_{10}, x_{20}) to $(0,0)$ in time

$$t_f = x_{20} + \sqrt{4 \cdot x_{10} + 2 \cdot x_{20}^2} \tag{7.1.22}$$

When the initial condition satisfies

$$x_{10} = -\frac{1}{2} \cdot |x_{20}| \cdot x_{20} \tag{7.1.23}$$

either $u(t) = 1$ or $u(t) = -1$, for $t_0 \le t < t_f$, are the only admissible controls identified by the necessary conditions that produce a path along the trajectory to the origin.

There is a trick that enables one to apply the HJCB sufficient conditions to prove the candidate control is optimal. It is to divide the original problem into two parts. Part one is to prove that the optimal control for initial conditions satisfying

$$x_{10} = -\frac{1}{2} \cdot |x_{20}| \cdot x_{20} \tag{7.1.24}$$

with $x_{20} > 0$ is $u^*(t) = -1$ for all t, and for initial conditions satisfying equation (7.1.24) with $x_{20} < 0$ is $u^*(t) = 1$ for all t. The proof is easy. Any change in control causes the trajectory to miss the origin and cross the x_1-axis, which will take more time to reach the origin. Part two is to solve the original minimum time problem but not with the origin as the target set, but the new target set

$$S = \left\{ (x_1, x_2) : x_1 = -\frac{1}{2} \cdot |x_2| \cdot x_2 \right\} \tag{7.1.25}$$

The solution to this problem is obvious from the preceding analysis and has a smooth cost-to-go function. Thus, the necessary and sufficient conditions apply. The result is a globally optimal feedback control.

A phase plane plot of the optimal trajectories can be found in Figure 7.2. Note that the trajectories that are thin are the trajectories associated with the control $u^*(t) = -1$ and the trajectories that are thick are the trajectories associated with the control $u^*(t) = 1$.

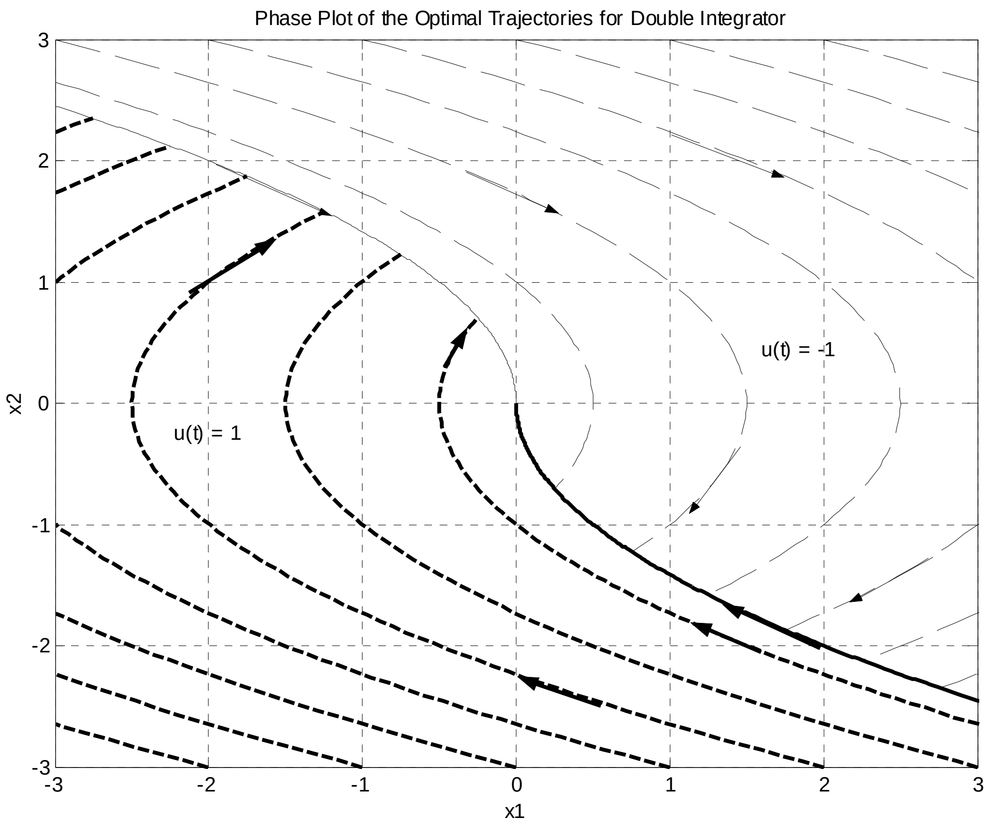

Figure 7.2: Phase Plot of the Optimal Trajectories for the Double Integrator Problem

The general feedback solution that minimizes the time to the origin from any initial condition is:

- $u^*(t) = 1$ for

$$x_1 \leq \frac{1}{2} \cdot x_2^{\ 2} \wedge x_1 < -\frac{1}{2} \cdot x_2^{\ 2} \tag{7.1.26}$$

- $u^*(t) = -1$ for

$$x_1 > \frac{1}{2} \cdot x_2{}^2 \wedge x_1 \geq -\frac{1}{2} \cdot x_2{}^2 \qquad (7.1.27)$$

so the optimal trajectory will follow one of two paths based on the initial condition:

1. The trajectory starts with $u^*(t) = 1$ until it intersects the $x_1 = -\frac{1}{2} \cdot x_2{}^2$ surface and then the control input switches to $u^*(t) = -1$ until it hits the origin
2. The trajectory starts with $u^*(t) = -1$ until it intersects the $x_1 = \frac{1}{2} \cdot x_2{}^2$ surface and then the control input switches to $u^*(t) = 1$ until it hits the origin.

7.1.3 Harmonic Oscillator

The harmonic oscillator dynamics govern the motion of the trajectories in location q_2. As with the double integrator, [6] performs extensive analysis on the solution of the minimum time to origin control problem and a short synopsis of the results is provided for reference.

The equations of motion for this system are:

$$\dot{x}_{q_2}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t) \qquad (7.1.28)$$

The optimal control problem will find the control that minimizes the time to the origin from the state $x = (x_1, x_2) \in \mathbb{R}^2$ subject to the dynamic constraints given in (7.1.28) with the control $|u(t)| \leq 1$.

The necessary and sufficient conditions from Chapter 6 can be applied to this problem almost everywhere. Along the surface

$$\left(x_1(t)-u\right)^2 + x_2^2(t) = 1 \tag{7.1.29}$$

where

$$u = \begin{cases} 1, & x_2(t) < x_1(t) \wedge x_2(t) \le 0 \\ -1, & x_2(t) > x_1(t) \wedge x_2(t) \ge 0 \end{cases} \tag{7.1.30}$$

the sufficient conditions don't apply because the value function is continuous but not differentiable along the entire trajectory. However, the Maximum Principle still applies and can be used along with the problem assumptions to prove the control is optimal.

As before, the properties of the Hamiltonian will be studied. The Hamiltonian for this problem will satisfy

$$H\left(x,\lambda,u,\lambda_0\right) = \lambda_1(t)\cdot x_2(t) - \lambda_2(t)\cdot x_1(t) + \lambda_2(t)\cdot u(t) - 1 \tag{7.1.31}$$

and the differential equations describing the evolution of the adjoint equations can be calculated. As before, the adjoint equation will satisfy

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} \tag{7.1.32}$$

everywhere the differential exists.

Evaluating equation (7.1.32) results in

$$\dot{\lambda}(t) = \begin{bmatrix} -\lambda_2(t) \\ \lambda_1(t) \end{bmatrix} \quad (7.1.33)$$

Assuming that the initial conditions for the adjoint equations are

$$\lambda(t_0) = \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \end{bmatrix} \quad (7.1.34)$$

and solving the differential equation (7.1.33) with initial condition (7.1.34) gives

$$\lambda(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \lambda_1(t_0) \\ \lambda_2(t_0) \end{bmatrix}, \ \overline{t} \le t < t_f \quad (7.1.35)$$

The necessary conditions given in Chapter 6 require that the Hamiltonian is maximized, which means that the optimal control will satisfy

$$u^*(t) = \operatorname{sgn}\left(-\sin(t) \cdot \lambda_1(t_0) + \cos(t) \cdot \lambda_2(t_0)\right) \quad (7.1.36)$$

Equation (7.1.36) implies that the set of admissible controls will switch periodically as t increases. As such, equation (7.1.28) can be integrated resulting in

$$\begin{aligned} x_1(t) &= (x_{10} - u) \cdot \cos(t - t_0) + x_{20} \cdot \sin(t - t_0) + u \\ x_2(t) &= -(x_{10} - u) \cdot \sin(t - t_0) + x_{20} \cdot \cos(t - t_0) \end{aligned} \quad (7.1.37)$$

where either $u^*(t) = -1$ or $u^*(t) = 1$. The resulting trajectories can now be plotted for various initial conditions under constant input as depicted in Figure 7.3. Notice that the results are circles centered at the point $\begin{bmatrix} \operatorname{sgn}(u) \\ 0 \end{bmatrix}$ where the circle radii depend on the initial conditions, and the movements of the trajectories are clockwise about the center of the circle (the solid circles are the ones that have an edge that passes through the origin).

Since the trajectories are clockwise about the center of the circles, the trajectories that satisfy equations (7.1.29) and (7.1.30) define the set of initial conditions that reach the origin under constant control, $u^*(t)=1$ or $u^*(t)=-1$.

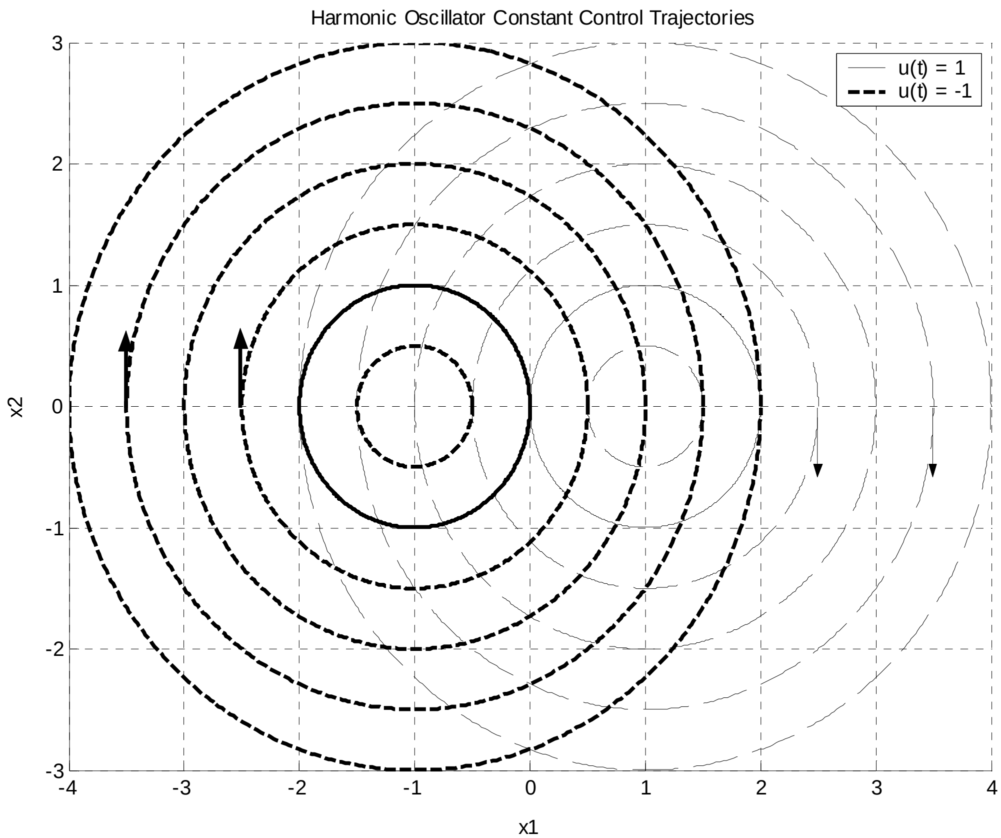

Figure 7.3: Phase plot of the harmonic oscillator system under constant control.

The feedback control law that minimizes the time to the origin is very complicated to state in equation form, but can easily be visualized. Figure 7.4 depicts the optimal solution to the harmonic oscillator problem. In the region depicted by the thin line the control $u^*(t)=1$ is optimal and in the region depicted by the thick line the control $u^*(t)=-1$ is optimal.

The same trick that was applied to the double integrator problem can be used to prove the proposed control is a global feedback solution to the optimal control problem. Again the original problem will be divided into two parts. Part one is to prove that the optimal control for initial conditions satisfying

$$x_{20} = -\sqrt{1-\left(x_{10}-1\right)^2} \tag{7.1.38}$$

with $0 \le x_{10} \le 2$ is $u^*\left(t\right) = -1$ for all t, and for initial conditions satisfying equation

$$x_{20} = \sqrt{1-\left(x_{10}+1\right)^2} \tag{7.1.39}$$

with $-2 \le x_{10} \le 0$ is $u^*\left(t\right) = 1$ for all t. The proof is easy. Any change in control causes the trajectory to miss the origin and cross the x_1-axis, which will take more time to reach the origin. Part two is to solve the original minimum time problem but not with the origin as the target set, but the new target set

$$S = \begin{cases} \left(x_1, x_2\right) : 0 \le x_1 \le 2, x_2 = -\sqrt{1-\left(x_1-1\right)^2} \\ \left(x_1, x_2\right) : -2 \le x_1 \le 0, x_2 = \sqrt{1-\left(x_1+1\right)^2} \end{cases} \tag{7.1.40}$$

The solution to this problem is obvious from the preceding analysis and has a smooth cost-to-go function. Thus, the necessary and sufficient conditions apply. The result is a globally optimal feedback control.

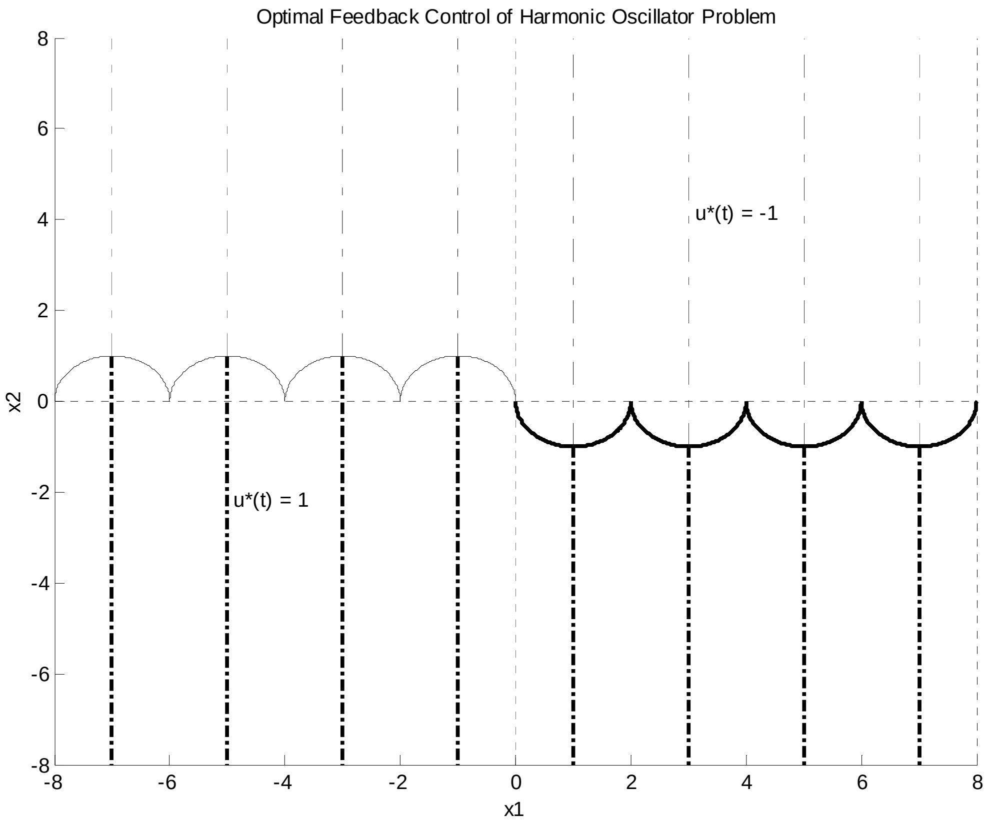

Figure 7.4: Feedback optimal control solution for the harmonic oscillator.

7.1.4 Problem Solution

Before the hybrid problem will be analyzed, the work of the previous section can be used to provide a logical "guess" for the optimal feedback control for the hybrid problem. Using the phase portraits in Figure 7.2 and Figure 7.4 and taking the optimal trajectory from the appropriate region of the state space, it is reasonable to conjecture that the phase portrait of the optimal feedback control may look like the one in Figure 7.5 when $\alpha = 1$. But as will be shown later the discrete part of the hybrid problem increases the complexity of the solution and hence changes the regions of constant control.

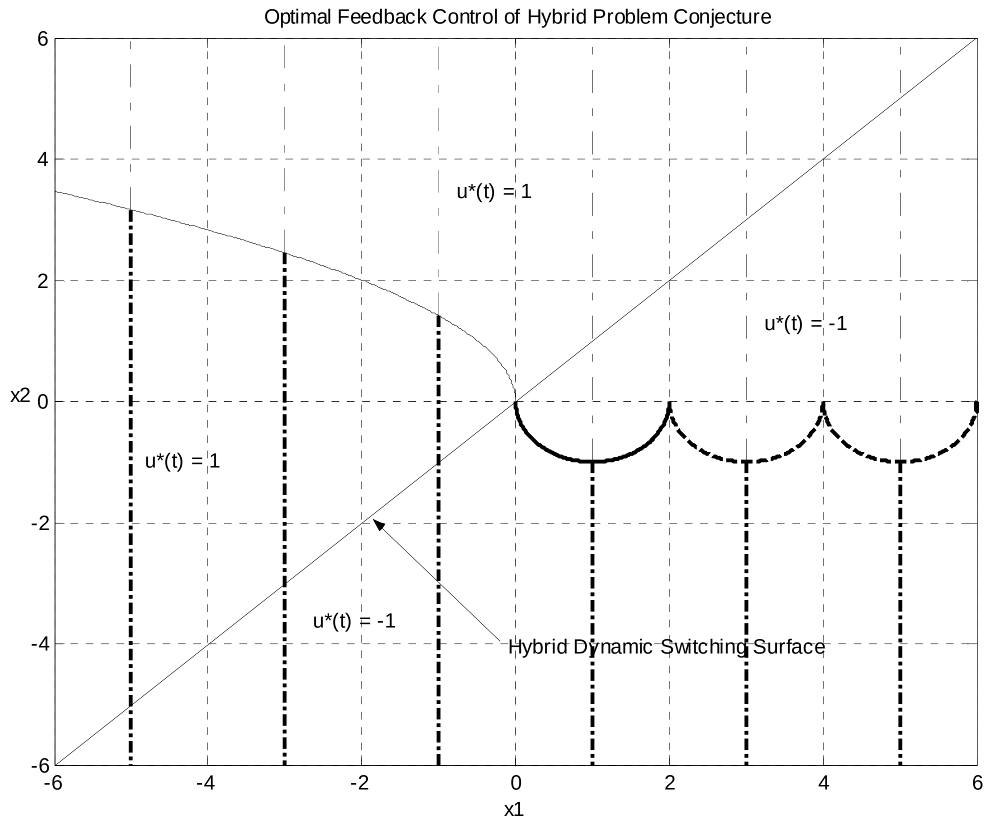

Figure 7.5: Guess of feedback control for the hybrid system.

In Figure 7.5, the thick lines represent the region of the state space where the optimal control is $u(t)=1$ and the thin lines indicate the optimal control is $u(t)=-1$. Further, the solid lines represent the two trajectories that intersect the origin and the direction of the trajectories is the same as those depicted in Figure 7.2 and Figure 7.4. However, it will be shown that the control is not optimal.

Optimal trajectory analysis

The purpose of this problem is to solve the minimum time to the origin for the hybrid automaton given in Figure 7.1 for any initial condition. In order to solve this problem, the necessary conditions given in Chapter 6 will be used to narrow down the possible control inputs for the optimal trajectory and then the sufficient conditions will be used to verify that the generated feedback control is indeed optimal.

First the hybrid problem will be decomposed into a series of optimal control problems with the appropriate boundary conditions. Once these optimal control problems and associated boundary conditions are given, the necessary and sufficient conditions of Chapter 6 are applied to each local optimal control problem providing the solution to the problem.

To solve the hybrid control problem depicted in Figure 7.1 using the work in Chapter 6, two separate control problems need to be solved because the origin is reachable from either side of the switch surface. Also, there are regions of the state space on both sides of the switch surface that cannot reach the origin. As such, the problem will be further broken up into its appropriate regions so that proper optimal control problems can be formed and the work of Chapter 6 will apply.

The solution to the problem is going to proceed as follows:

1. Generate a local optimal control problem that contains the desired target set (and associated cost-to-go function) with the appropriate initial boundary condition and apply the work in Chapter 6.
2. Generate a new optimal cost-to-go function along the initial boundary condition defined in step 1 that embodies the solution to step 1 and call it the desired target set for the next local optimal control problem.
3. This procedure will be repeated until the solution for the entire state-space is found. The correct hybrid optimal control problem will be formed that will satisfy the assumptions of the work in Chapter 6.

Both optimal control problems will be solved concurrently because as the problem is solved, the solutions will be used to form the next local optimal control problem. The first local optimal control problem will find the control that minimizes the time to the origin from any state in location q_1 that can reach the origin while remaining in location q_1. The second optimal control problem will find the control that minimizes the time to the origin from any state in location q_2 that can reach the origin while remaining in location q_2. The third optimal control problem will find the control that minimizes the time to the switch surface from the rest of the states in location q_2 that do not reach the origin and still remain in location q_2. Finally a fourth optimal control problem will find the control that minimizes the time to the switch surface for the set of states in location q_1 that cannot reach the origin while the trajectory remains in location q_1. The solutions for the four optimal control problems will be pieced together to form the solution to the complete hybrid problem.

Two additional assumptions will be made on the hybrid problem. The first assumption is that $\alpha = 1$. This simplification of the control problem allows for demonstration of the work in this dissertation without over complicating solution. Considering non-unity α causes the resulting equations to become very long and the extra complexity doesn't add any novelty to the results. The second assumption applies to the behavior of the "zero" trajectories. As will be shown, there is a unique surface in each location that provides a set of states that can reach the origin under constant control. It is going to be assumed initially that when a trajectory hits one of these two surfaces, the optimal control will switch and the optimal trajectory will follow that surface to the origin. This assumption is temporary and will be removed at the end of the analysis.

Region $q = q_1$, zero reachable states

The first step in solving the hybrid problem is to find the control that minimizes the time to the origin for the set of states that can reach the origin while remaining in location q_1. Note that the state space for location q_1 will be split into the region that can reach the origin and the region that cannot. These regions are a byproduct of the optimal control problem and will be discussed later. The local optimal control problem can now be formulated.

Assume that the initial state satisfies $x_0 \in \Re_1$, where $\Re_1$ is the set of states that can reach $x(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $t_f < \infty$ defined by the dynamical system

$$\dot{x}_{q_1}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t) \tag{7.1.41}$$

where the trajectory $x(t)$, $t_0 \le t \le t_f$, remains in q_1 and the control satisfies $u(t) \in [-1,1]$. The optimal control problem will find the control $u^*(t)$ that minimizes the time to the origin and satisfies the dynamic constraints given in (7.1.41), subject to the following conditions:

1. $x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in M_{q_1}$ such that $x_{10} \le x_{20}$

2. $|u(t)| \le 1$, for all $t_0 \le t \le t_f$
3. $L(x,u,t) = 1$
4. $\Phi(x) = 0$

In this region the solution to the optimal control problem is similar to the double integrator, so the result will be given directly and the proof omitted. The resulting optimal control and region of the state space where it applies is plotted in Figure 7.6. The thick lines indicate the control $u^*(t) = 1$ is optimal and the thin line indicates the control $u^*(t) = -1$ is optimal.

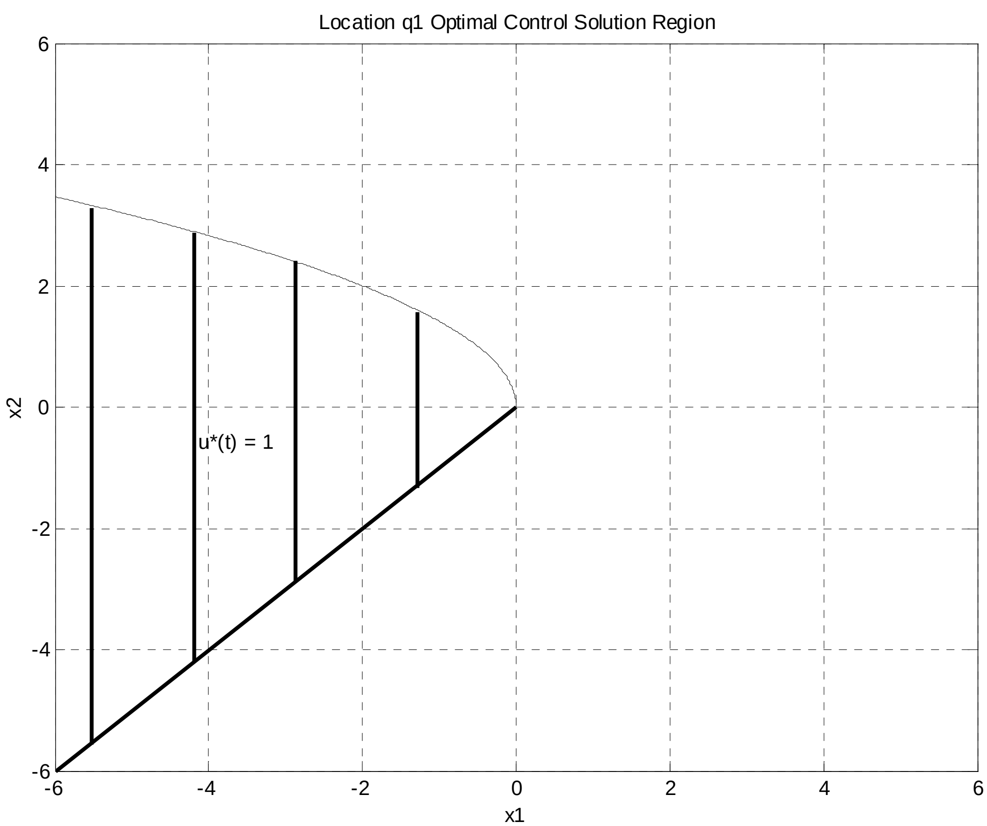

Figure 7.6: Region for optimal control solution for $q = q_1$**.**

Region $q = q_2$, zero reachable states

The second step in solving the hybrid optimal control problem is solving the local optimal control problem that finds the optimal control that minimizes the time to the origin while remaining in location q_2. Just as in the last section, there is a set of initial states in location q_2 that reach the origin and a set of states that do not reach the origin. These regions will be clearly defined as the analysis progresses.

The local optimal control problem will now be defined. Assume that the initial state satisfies $x_0 \in \Re_2$, where $\Re_2$ is the set of states that can reach $x(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $t_f < \infty$ defined by the dynamical system

$$\dot{x}_{q_2}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t) \tag{7.1.42}$$

where the trajectory $x(t)$, $t_0 \le t \le t_f$, remains in q_2 and the control satisfies $u(t) \in [-1,1]$. The optimal control problem will find the control $u^*(t)$ that minimizes the time to the origin and satisfies the dynamic constraints given in (7.1.42), subject to the following conditions:

1. $x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in M_{q_2}$ such that $x_{10} \ge x_{20}$
2. $|u(t)| \le 1$, for all $t_0 \le t \le t_f$
3. $L(x,u,t) = 1$
4. $\Phi(x) = 0$

As in the previous section the solution in this region is similar to the harmonic oscillator problem so the result will be given directly and the proof omitted.

The optimal control region is plotted for location $q = q_2$ in Figure 7.7. The thick lines represent the control $u^*(t) = 1$ and the thin lines represent the control $u^*(t) = -1$.

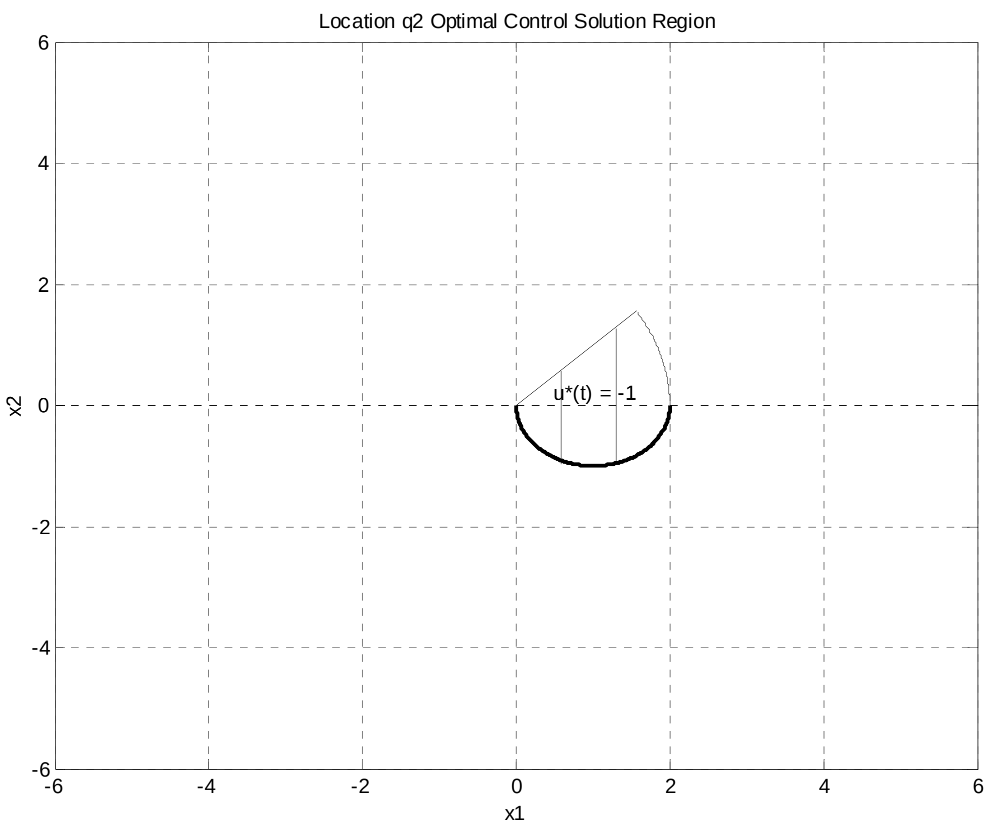

Figure 7.7: Control region for location $q = q_2$.

Region $q = q_3$, $x_1(t) \le x_2(t)$

The third step in solving the hybrid optimal control problem is solving the optimal control problem that finds the optimal control that minimizes the time to the surface

$$S_3 = \{(x_1, x_2) : x_1 = x_2,\ x_2 < 0\} \tag{7.1.43}$$

while remaining in location M_{q_2}.

The optimal control problem can now be defined. Assume that the initial state satisfies $x_0 \in \Re_3$, where $\Re_3$ is the set of states that can reach $x(t_f) \in S_3$, $x_1(t_f) = x_1(t_f) = x_{qf}$, $t_f < \infty$, for all $x_{qf} < 0$ defined by the dynamical system

$$\dot{x}_{q_2}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t) \tag{7.1.44}$$

where the trajectory $x(t)$, $t_0 \le t \le t_f$, remains in q_2 and the control satisfies $u(t) \in [-1,1]$. The optimal control problem is to find the control $u^*(t)$ that minimizes

$$J = \Phi\left(x(t_f)\right) + \int_0^{t_f} L(x,u,t) \cdot dt \tag{7.1.45}$$

subject to the dynamic constraints of equation (7.1.44) and the following conditions:

1. $x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in M_{q_2}$ such that $x_{10} \ge x_{20}$
2. $|u(t)| \le 1$, for all $t_0 \le t \le t_f$
3. $L(x,u,t) = 1$
4. $\Phi\left(x(t_f)\right) = -x_{qf} + \sqrt{-4 \cdot x_{qf} + 2 \cdot x_{qf}^2}$
5. $x(t_f) \in S_3$

Notice that in this local optimal control problem, the final cost is the cost-to-go to the origin in location q_1 from the surface S_3. Applying the necessary conditions of Chapter

6 to this new control problem gives that the optimal control will maximize the Hamiltonian and the optimal control will satisfy

$$u^*(t) = \operatorname{sgn}\left(-\sin(t)\cdot\lambda_1(t_0) + \cos(t)\cdot\lambda_2(t_0)\right) \tag{7.1.46}$$

or the optimal control will take a value from the set

$$u^*(t) = \begin{cases} 1 \\ -1 \end{cases} \tag{7.1.47}$$

and that the set of admissible controls will be constant but will switch periodically as t increases.

The necessary conditions require that the optimal control switch periodically. The first step in solving this problem is assuming that the optimal control is $u^*(t) = 1$ "close" to the switching surface, calculating the cost-to-go, and proving that the HJCB PDE is satisfied. Next the adjoint will be calculated along the switching surface. Finally, the adjoint will be computed in reverse time to calculate the time the control switches (when $\lambda_2\left(t_s^-\right) = 0$). Once the switch time is calculated, the cost-to-go will be calculated, and the sufficient condition will be checked along the resulting trajectories, proving the control is optimal.

Note that the nomenclature t^- will indicate the time variable in reverse time, so $t_0^- \le t^- \le t_f^-$ where $t_0^- = t_f$ and $t_f^- = 0$. The proposed optimal control will be $u^*\left(t^-\right) = 1$ for $0 \ge t^- > t_s^-$ and $u^*\left(t^-\right) = -1$ for $t_s^- < t^- \le t_f^-$.

The first step is to calculate cost-to-go function near the surface of final conditions. Since the control is assumed positive, the trajectory to the surface of final conditions satisfies

$$\begin{aligned} x_{qf} &= (x_{10}-1)\cdot\cos(t)+x_{20}\cdot\sin(t)+1 \\ x_{qf} &= -(x_{10}-1)\cdot\sin(t)+x_{20}\cdot\cos(t) \end{aligned} \tag{7.1.48}$$

Where x_{10} and x_{20} are elements of $\Re_3$. When $t \to t_f$, equation (7.1.48) can be linearized giving

$$\begin{aligned} x_{qf} &= (x_{10}-1)+x_{20}\cdot t+1 \\ x_{qf} &= -(x_{10}-1)\cdot t+x_{20} \end{aligned} \tag{7.1.49}$$

Solving equation (7.1.49) for t results in

$$t = \frac{x_{20}-x_{10}}{x_{20}+x_{10}-1} \tag{7.1.50}$$

Equation (7.1.50) provides the cost-to-go to the surface of final conditions for an initial condition "close" to the surface of final conditions using control $\hat{u}(t)=1$ and the optimal cost-to-go is calculated as

$$\hat{J}_c\left(x(t_0),t_f-t_0\right) = -\frac{(-x_{10}+1)\cdot(x_{20}-x_{10})}{x_{20}+x_{10}-1} - x_{20} + \Phi\left(x_{qf}\right) \tag{7.1.51}$$

or

$$\hat{J}_c\left(x\left(t_0\right),t_f-t_0\right)=-\frac{\left(-x_{10}+1\right)\cdot\left(x_{20}-x_{10}\right)}{x_{20}+x_{10}-1}-x_{20}+$$
$$\left(2\cdot\left(\frac{\left(-x_{10}+1\right)\cdot\left(x_{20}-x_{10}\right)}{x_{20}+x_{10}-1}+x_{20}\right)^2-4\cdot x_{10}-4\cdot\frac{x_{20}\cdot\left(x_{20}-x_{10}\right)}{x_{20}+x_{10}-1}\right)^{\frac{1}{2}}$$
(7.1.52)

Note that equation (7.1.52) is differentiable with respect to x_{10} and x_{20}, so as the trajectory gets “close” to the switching surface, $t_0 \to t_f$, $x_{10} \to x_1\left(t_f\right)$, $x_{20} \to x_2\left(t_f\right)$, and

$$\frac{\partial\hat{J}_c\left(x\left(t_0\right),t_f-t_0\right)}{x\left(t_0\right)}=\begin{bmatrix}-\dfrac{-2\cdot x_{qf}^2+4\cdot x_{qf}+x_{qf}\cdot\beta-2}{\beta\cdot\left(-1+2\cdot x_{qf}\right)}\\ -\dfrac{-2\cdot x_{qf}^2+2\cdot x_{qf}+\beta\cdot x_{qf}-\beta}{\beta\cdot\left(-1+2\cdot x_{qf}\right)}\end{bmatrix} \tag{7.1.53}$$

where

$$\beta=\sqrt{2\cdot x_{qf}^2-4\cdot x_{qf}} \tag{7.1.54}$$

Now the HJCB PDE requires that

$$\sup_{u\in U} H\left(x,-\frac{\partial\hat{J}_c\left(x\left(t_0\right),t_f-t_0\right)}{\partial x\left(t_0\right)},1,1\right)=\frac{\partial\hat{J}_c\left(x_1\left(t_0\right),t_f-t_0\right)}{\partial x_1\left(t_0\right)}\cdot x_{20}$$
$$-\frac{\partial\hat{J}_c\left(x_2\left(t_0\right),t_f-t_0\right)}{\partial x_2\left(t_0\right)}\cdot x_{10}+\frac{\partial\hat{J}_c\left(x_2\left(t_0\right),t_f-t_0\right)}{\partial x_2\left(t_0\right)}-1$$
(7.1.55)

Substituting equation (7.1.53) into equation (7.1.55), letting $x_{10}, x_{20} \to x_1\left(t_f\right), x_2\left(t_f\right)$, and simplifying results in

$$\sup_{u \in U} H\left(x, -\frac{\partial \hat{J}_c\left(x\left(t_0\right), t_f - t_0\right)}{\partial x\left(t_0\right)}, u, 1\right) = \left(-1+u\right) \cdot \frac{\left(x_{qf} - 1\right) \cdot \left(-2 \cdot x_{qf} + \beta\right)}{\beta \cdot \left(-1 + 2 \cdot x_{qf}\right)} = 0 \qquad (7.1.56)$$

which proves that the control $\hat{u}\left(t\right) = 1$ is optimal "close" to the switching surface. Note that the purpose in applying the necessary and sufficient conditions "close" to the switching surface is so that the transversality condition can be calculated and equated to the final value of the adjoint (not to prove the control optimal along the entire trajectory which will be done later).

Now evaluation of the transversality condition of Chapter 6 along the switching surface results in

$$\lambda\left(t_f\right) = \begin{bmatrix} \dfrac{-2 \cdot x_{qf}^2 + 4 \cdot x_{qf} + x_{qf} \cdot \beta - 2}{\beta \cdot \left(-1 + 2 \cdot x_{qf}\right)} \\ \dfrac{-2 \cdot x_{qf}^2 + 2 \cdot x_{qf} + \beta \cdot x_{qf} - \beta}{\beta \cdot \left(-1 + 2 \cdot x_{qf}\right)} \end{bmatrix} \qquad (7.1.57)$$

Furthermore, in a previous section for this optimal control problem it was shown that the adjoint equation satisfies

$$\lambda\left(t\right) = \begin{bmatrix} \cos\left(t\right) & \sin\left(t\right) \\ -\sin\left(t\right) & \cos\left(t\right) \end{bmatrix} \begin{bmatrix} \lambda_1\left(t_0\right) \\ \lambda_2\left(t_0\right) \end{bmatrix} \qquad (7.1.58)$$

which can be solved in reverse time giving

$$\begin{bmatrix} \lambda_1(t^-) \\ \lambda_2(t^-) \end{bmatrix} = \begin{bmatrix} \cos(t^-) & -\sin(t^-) \\ \sin(t^-) & \cos(t^-) \end{bmatrix} \begin{bmatrix} \lambda_1(t_f) \\ \lambda_2(t_f) \end{bmatrix} \tag{7.1.59}$$

The HJCB PDE can be written and can be evaluated using a Symbolic solver as

$$\sup_{u \in U} H(x, \lambda, 1, 1) = \lambda_1(t^-) \cdot x_2(t^-) - \lambda_2(t^-) \cdot x_1(t^-) + \lambda_2(t^-) - 1 = 0 \tag{7.1.60}$$

which proves the control $u^*(t^-) = 1$ for $t_f \le t^- < t_s^-$ as desired.

Next, equation (7.1.59) can be solved for t^- when $\lambda_2(t^-) = 0$ resulting in

$$t^- = \arctan\left(\frac{\lambda_2(t_f)}{-\lambda_1(t_f)} \right) \tag{7.1.61}$$

where equation (7.1.61) is evaluated using the four quadrant arctangent function. Substituting equation (7.1.53) into equation (7.1.61) and correcting for the quadrant results in

$$t^- = \pi - \arctan\left(\frac{-2 \cdot x_{qf}^2 + 2 \cdot x_{qf} + \beta \cdot x_{qf} - \beta}{-2 \cdot x_{qf}^2 + 4 \cdot x_{qf} + x_{qf} \cdot \beta - 2} \right) \tag{7.1.62}$$

for all $x_{qf} < 0$. To check the result, equation (7.1.59) can be evaluated for $x_{qf} = -1$ and $0 \le t^- \le 2 \cdot \pi$ and is graphed in Figure 7.8.

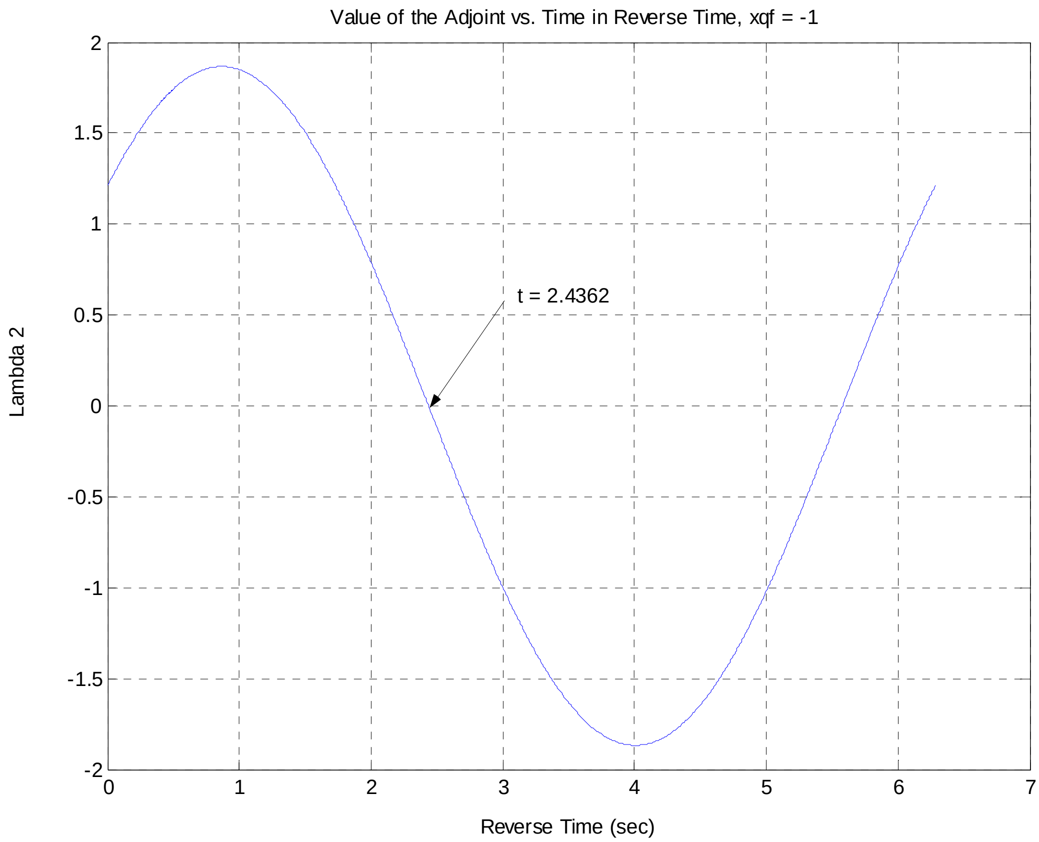

Figure 7.8: Value of $\lambda_2\left(t^-\right)$ for $0 \le t^- \le 2\cdot\pi$ and $x_{qf} = -1$.

Furthermore, equation (7.1.62) can be graphed for a range of x_{qf} and is shown in Figure 7.9. Note that when $x_{qf} = -1$ both plots give the same switch time verifying that equation (7.1.62) is true.

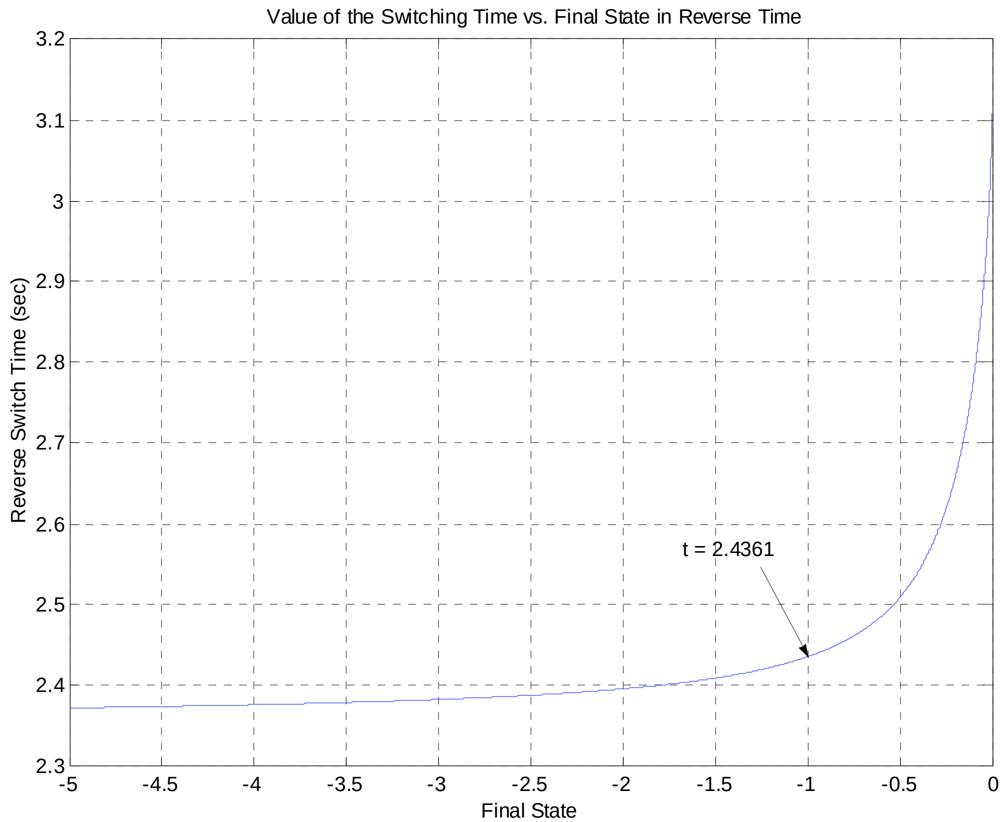

Figure 7.9: Switch time t_s^- versus final state x_{qf}.

When $t^- > t_s^-$, the control switches to $u^*(t^-) = -1$. Evaluating the HJCB PDE is more difficult, because the value of the state at the switching time is required as a function of the final condition. However the symbolic solver can solve this problem and prove the control is optimal.

The HJCB PDE for this time segment is

$$\sup_{u \in U} H(x,\lambda,u,1) = \lambda_1(t^-)\cdot x_2(t^-) - \lambda_2(t^-)\cdot x_1(t^-) + \lambda_2(t^-)\cdot u(t^-) - 1 \qquad (7.1.63)$$

When $u^*(t^-) = -1$, the value of $x(t^-)$ and $\lambda(t^-)$ satisfy

$$
\begin{aligned}
x_1\left(t_2^-\right) &= \left(x_1\left(t_s^-\right) - u\right)\cdot\cos\left(t_2^-\right) - x_2\left(t_s^-\right)\cdot\sin\left(t_2^-\right) + u \\
x_2\left(t_2^-\right) &= \left(x_1\left(t_s^-\right) - u\right)\cdot\sin\left(t_2^-\right) + x_2\left(t_s^-\right)\cdot\cos\left(t_2^-\right) \\
\lambda_1\left(t_2^-\right) &= \lambda_1\left(t_s^-\right)\cdot\cos\left(t_2^-\right) - \lambda_2\left(t_s^-\right)\cdot\sin\left(t_2^-\right) \\
\lambda_2\left(t_2^-\right) &= \lambda_1\left(t_s^-\right)\cdot\sin\left(t_2^-\right) + \lambda_2\left(t_s^-\right)\cdot\cos\left(t_2^-\right)
\end{aligned}
\tag{7.1.64}
$$

where $x\left(t_s^-\right)$ and $\lambda\left(t_s^-\right)$ are the values at the switching surface defined by time t_s^- (equation (7.1.62)) and $t_s^- \le t_2^- < t_f^-$. Again using the symbolic solver and the previous work, equation (7.1.64) can be rewritten as

$$
\begin{aligned}
x_1\left(t_2^-\right) &= \left(\alpha_1\left(x_{qf}\right) + 1 - u\right)\cdot\cos\left(t_2^-\right) - \alpha_2\left(x_{qf}\right)\cdot\sin\left(t_2^-\right) + u \\
x_2\left(t_2^-\right) &= \left(\alpha_1\left(x_{qf}\right) + 1 - u\right)\cdot\sin\left(t_2^-\right) + \alpha_2\left(x_{qf}\right)\cdot\cos\left(t_2^-\right) \\
\lambda_1\left(t_2^-\right) &= \gamma\left(x_{qf}\right)\cdot\cos\left(t_2^-\right) \\
\lambda_2\left(t_2^-\right) &= \gamma\left(x_{qf}\right)\cdot\sin\left(t_2^-\right)
\end{aligned}
\tag{7.1.65}
$$

Where $\alpha_1\left(x_{qf}\right)$, $\alpha_2\left(x_{qf}\right)$, and $\gamma\left(x_{qf}\right)$ are complicated functions of x_{qf}. Substitution of equation (7.1.65) into equation (7.1.63) and simplifying results in

$$
H\left(x,\lambda,u,\lambda_0\right) = \alpha_2\left(x_{qf}\right)\cdot\gamma\left(x_{qf}\right) - 1 = 0 \tag{7.1.66}
$$

where the terms $\alpha_2\left(x_{qf}\right)$ and $\gamma\left(x_{qf}\right)$ are

$$
\gamma\left(x_{qf}\right) = \frac{\sqrt{\begin{matrix} 8\cdot x_{qf}^4 - 24\cdot x_{qf}^3 - 8\cdot x_{qf}^3\cdot\beta + 28\cdot x_{qf}^2 + 16\cdot x_{qf}^2\cdot\beta - 16\cdot x_{qf} + 2\cdot x_{qf}^2\cdot\beta^2 \\ -8\cdot x_{qf}\cdot\beta - 2\cdot x_{qf}\cdot\beta^2 + \beta^2 + 4 \end{matrix}}}{\beta\cdot\left(-1 + 2\cdot x_{qf}\right)} \tag{7.1.67}
$$

and

$$\alpha_2\left(x_{qf}\right)=\frac{T_1\cdot x_{qf}-\left(x_{qf}-1\right)\cdot T_2}{\sqrt{T_1^2+T_2^2}} \tag{7.1.68}$$

where

$$\begin{aligned} T_1 &= -2\cdot x_{qf}^2+4\cdot x_{qf}+x_{qf}\cdot\beta-2 \\ T_2 &= -2\cdot x_{qf}^2+2\cdot x_{qf}+x_{qf}\cdot\beta-\beta \end{aligned} \tag{7.1.69}$$

And proves the control is optimal.

Finally as before, the region of the state space that this optimal control solution applies is given in Figure 7.10. The thick lines indicate the control $u^*\left(t\right)=1$ is optimal and the thin lines indicate the control $u^*\left(t\right)=-1$.

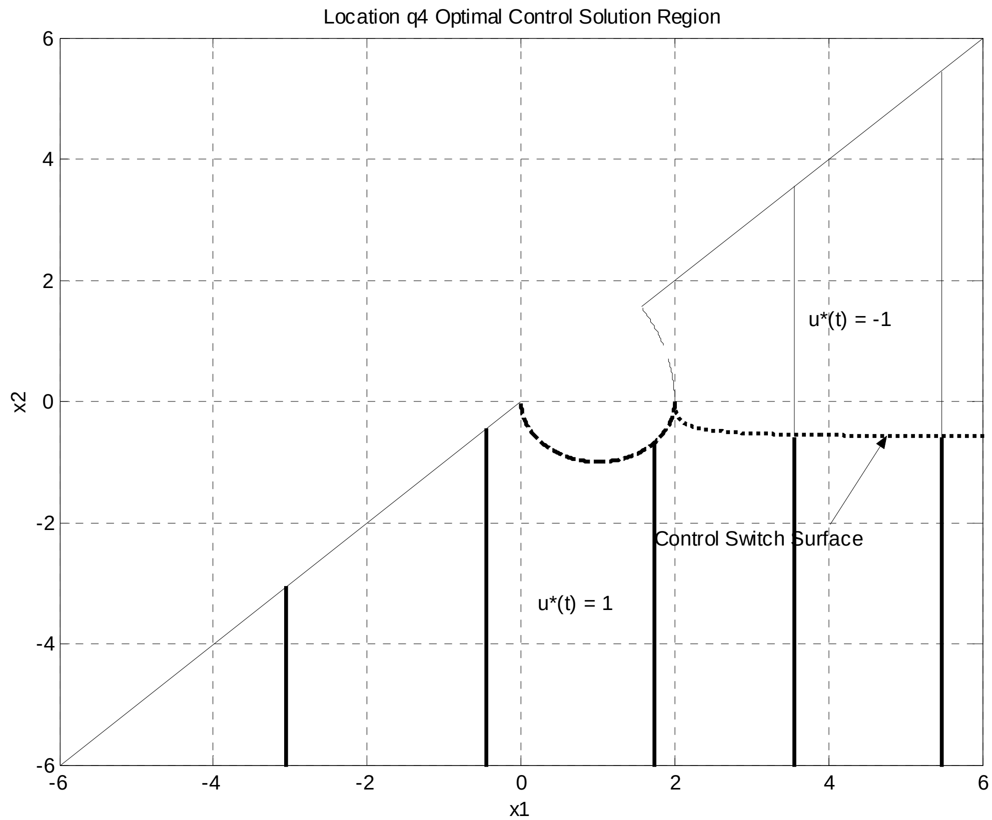

Figure 7.10: Control region for location $q = q_3$.

Region $q = q_4$**,** $x_1(t) \geq x_2(t)$

The final step in solving the hybrid optimal control problem is solving the local optimal control problem that finds the control that minimizes the time to the surface

$$S_4 = \{(x_1, x_2) : x_1 = x_2,\ x_2 > 0\} \tag{7.1.70}$$

while remaining in location M_{q_1}.

Assume that the initial state $x_0 \in \Re_4$, where $\Re_4$ is the set of states that satisfy $x\left(t_f\right) \in S_4$,

$x_1\left(t_f\right) = x_1\left(t_f\right) = x_{qf}$, $t_f < \infty$ defined by the dynamical system

$$\dot{x}_{q_1}\left(t\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot x\left(t\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u\left(t\right) \tag{7.1.71}$$

where the trajectory $x\left(t\right)$, $t_0 \le t \le t_f$, remains in M_{q_1}, and the control satisfies $u\left(t\right) \in \left[-1,1\right]$. The optimal control problem will find the control $u^*\left(t\right)$ that minimizes the time to the surface given by equation (7.1.70) and satisfies the dynamic constraints given in (7.1.71), subject to the following conditions:

1. $x\left(t_0\right) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in M_{q_1}$ such that $x_{10} \le x_{20}$
2. $\left|u\left(t\right)\right| \le 1$, for all $t_0 \le t \le t_f$
3. $L_1\left(x,u,t\right) = 1$
4. $\Phi_1\left(x_{qf}\right) = CTG_{origin}\left(x_{qf}\right)$ where $CTG_{origin}\left(x_{qf}\right)$ is given in Figure 7.11.
5. $x\left(t_f\right) \in S_4$

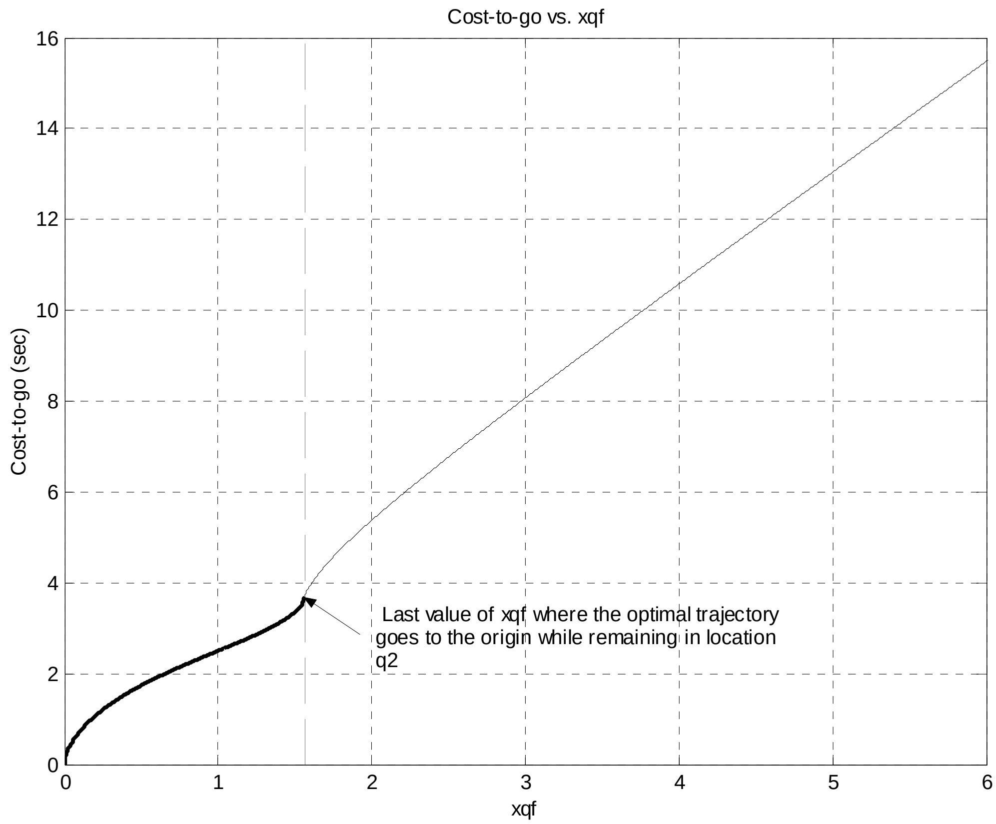

Figure 7.11: Cost-to-go to the origin from the surface $x_1(t_f) = x_2(t_f) = x_{qf} > 0$.

Figure 7.11 is the cost-to-go to the origin from the surface $x_1(t_f) = x_2(t_f) = x_{qf} > 0$ that is based on the analysis of regions q_2 and q_3. Note that the function is a continuously increasing function of $x(t_f)$, but is not differentiable at the point $x_1(t_f) = x_2(t_f) = -\frac{1}{2} + \sqrt{\frac{17}{4}}$. Because of the complexity of the cost-to-go function for $x_1(t_f) = x_2(t_f) > -\frac{1}{2} + \sqrt{\frac{17}{4}}$ a closed form solution was not computed, but is not necessary for the proof of the optimal control. Just the properties of the $CTG_{origin}(x_{qf})$ given by the figure are required.

The first step in solving this local optimal control problem is to evaluate the necessary conditions to find the candidate optimal controls. Just as in location $q = q_1$ evaluating the necessary condition requires that the optimal control satisfy

$$u^*(t) = \operatorname{sgn}\left(\lambda_2(t_0) - \lambda_1(t_0) \cdot t\right) \tag{7.1.72}$$

or the optimal control will take a value from the set

$$u^*(t) = \begin{cases} 1 \\ -1 \end{cases} \tag{7.1.73}$$

and can switch at most one time.

An argument similar to the one presented in the proof of the double integrator and harmonic oscillator problems can be used to the prove that the optimal control "close" to the surface $x_1(t_f) = x_2(t_f) = x_{qf}$ is $u^*(t) = -1$, and will be omitted here.

So as $t \to t_f$, the optimal cost-to-go is

$$\hat{J}_c\left(x(t_0), t_f - t_0\right) = CTG_{xqf}\left(x(t_0)\right) + CTG_{origin}\left(x_{qf}\right) \tag{7.1.74}$$

where $CTG_{xqf}\left(x(t_0)\right)$ is the cost-to-go to the surface $x_1(t_f) = x_2(t_f) = x_{qf}$ using the control $u^*(t) = -1$ and $CTG_{origin}\left(x_{qf}\right)$ is given in Figure 7.11. Since "close" to the surface $x_1(t_f) = x_2(t_f) = x_{qf}$ the optimal control is $u^*(t) = -1$, the cost-to-go to the surface x_{qf} can be calculated by solving the following equation for t_f

$$x_1(t_f) = x_{qf} = x_{10} + x_{20} \cdot t_f - \frac{1}{2} \cdot t_f^2$$
$$x_2(t_f) = x_{qf} = x_{20} - t_f \qquad (7.1.75)$$

which results in

$$t_f = CTG_{xqf}\left(x(t_0)\right) = (1 + x_{20}) - \sqrt{1 + 2 \cdot x_{10} + x_{20}^2} \qquad (7.1.76)$$

Notice that equation (7.1.76) is only a function of the initial condition and not time. Since equation (7.1.74) is continuous and differentiable almost everywhere, the necessary and sufficient conditions developed in Chapter 6 will apply to this local optimal control problem.

Because of the complexity of the function $CTG_{origin}(x_{qf})$, calculating the variation of $CTG_{origin}(x_{qf})$ is very difficult and will not be done to prove the candidate control is optimal. Properties of Figure 7.11 will be used in conjunction with the necessary conditions to prove the claim.

To prove the control $u^*(t) = -1$ is optimal for the entire region, the closed form solution for the trajectory and adjoint equations will be used in conjunction with the fact that the Hamiltonian is zero. The Hamiltonian will be zero because equation (7.1.74) is not an explicit function of the initial time so the work in Chapter 6 requires it to be zero..

Now working in reverse time from the surface of final conditions the state trajectory and adjoint can be computed in reverse time with the following equations

$$
\begin{aligned}
x_1\left(t^-\right) &= x_{qf} - x_{qf} \cdot t^- - \frac{1}{2}\cdot\left(t^-\right)^2 \\
x_2\left(t^-\right) &= x_{qf} + t^-
\end{aligned}
\qquad (7.1.77)
$$

and

$$
\begin{aligned}
\lambda_1\left(t^-\right) &= \lambda_1\left(t_f\right) \\
\lambda_2\left(t^-\right) &= \lambda_2\left(t_f\right) + \lambda_1\left(t_f\right)\cdot t^-
\end{aligned}
\qquad (7.1.78)
$$

Furthermore, the Hamiltonian for this problem with $u^*(t) = -1$ is

$$
H\left(x, \lambda_1, u, \lambda_0\right) = \lambda_1\left(t^-\right)\cdot x_2\left(t^-\right) - \lambda_2\left(t^-\right) - 1 = 0 \qquad (7.1.79)
$$

Substituting equation (7.1.78) into (7.1.79) and rearranging results in

$$
\lambda_2\left(t_f\right) = \lambda_1\left(t_f\right)\cdot x_{qf} - 1 \qquad (7.1.80)
$$

and substituting back into equation (7.1.78) gives

$$
\lambda_2\left(t^-\right) = \lambda_1\left(t_f\right)\cdot\left(x_{qf} + t^-\right) - 1 \qquad (7.1.81)
$$

or

$$
\lambda_2\left(t^-\right) = \lambda_1\left(t_f\right)\cdot x_2\left(t^-\right) - 1 \qquad (7.1.82)
$$

Equation (7.1.81) implies that if $\lambda_1\left(t_f\right) \le 0$, then the control $u^*\left(t^-\right) = -1$ is optimal for all $t^- \ge 0$ which is the desired result. Since the work in Chapter 6 applies to this local optimal control problem, the necessary and sufficient conditions require that the HJCB PDE evaluates to zero and that

$$\lambda(t) = -\frac{\partial \hat{J}_c\left(x(t_0), t_f - t_0\right)}{\partial x(t_0)} \tag{7.1.83}$$

where

$$\begin{aligned}
\hat{J}_c\left(x(t_0), t_f - t_0\right) &= (1 + x_{20}) - \sqrt{1 + 2 \cdot x_{10} + x_{20}^2} + CTG_{origin}\left(x_{qf}\right) \\
x_1\left(t_f\right) = x_{qf} &= x_{10} + x_{20} \cdot t_f - \frac{1}{2} \cdot t_f^2 \\
x_2\left(t_f\right) = x_{qf} &= x_{20} - \left((1 + x_{20}) - \sqrt{1 + 2 \cdot x_{10} + x_{20}^2}\right) = -1 + \sqrt{1 + 2 \cdot x_{10} + x_{20}^2}
\end{aligned} \tag{7.1.84}$$

and $CTG_{origin}\left(x_{qf}\right)$ is given in Figure 7.11. Everywhere the differential exists, equation (7.1.84) can be differentiated with respect to the initial condition resulting in

$$\frac{\partial \hat{J}_c\left(x(t_0), t_f - t_0\right)}{\partial x(t_0)} = \frac{\partial\left((1 + x_{20}) - \sqrt{1 + 2 \cdot x_{10} + x_{20}^2}\right)}{\partial x(t_0)} + \frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} \cdot \frac{\partial x_{qf}}{\partial x(t_0)} \tag{7.1.85}$$

The first term in equation (7.1.85) can be computed as

$$\frac{\partial\left((1 + x_{20}) - \sqrt{1 + 2 \cdot x_{10} + x_{20}^2}\right)}{\partial x(t_0)} = \begin{bmatrix} -\dfrac{1}{\sqrt{1 + 2 \cdot x_{10} + x_{20}^2}} \\ 1 - \dfrac{x_{20}}{\sqrt{1 + 2 \cdot x_{10} + x_{20}^2}} \end{bmatrix} \tag{7.1.86}$$

and the second term is given by

$$\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} \cdot \frac{\partial x_{qf}}{\partial x(t_0)} = \frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} \cdot \begin{bmatrix} \dfrac{1}{\sqrt{1 + 2 \cdot x_{10} + x_{20}^2}} \\ \dfrac{x_{20}}{\sqrt{1 + 2 \cdot x_{10} + x_{20}^2}} \end{bmatrix} \tag{7.1.87}$$

Substituting equations (7.1.87) and (7.1.86) into equation (7.1.85) results in

$$\frac{\partial \hat{J}_c\left(x(t_0), t_f - t_0\right)}{\partial x(t_0)} = \begin{bmatrix} \dfrac{1}{\sqrt{1 + 2\cdot x_{10} + x_{20}^2}} \cdot \left(\dfrac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} - 1 \right) \\ 1 + \dfrac{x_{20}}{\sqrt{1 + 2\cdot x_{10} + x_{20}^2}} \cdot \left(\dfrac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} - 1 \right) \end{bmatrix} \tag{7.1.88}$$

which implies that

$$\begin{aligned} \lambda_1\left(t_f\right) &= -\frac{1}{\left(1 + x_{qf}\right)} \cdot \left(\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} - 1 \right) \\ \lambda_2\left(t_f\right) &= -1 - \frac{x_{20}}{\left(1 + x_{qf}\right)} \cdot \left(\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} - 1 \right) \end{aligned} \tag{7.1.89}$$

Further, substituting equation (7.1.89) into equation (7.1.81) results in

$$\lambda_2\left(t^-\right) = -\frac{1}{\left(1 + x_{qf}\right)} \cdot \left(\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} - 1 \right) \cdot x_2\left(t^-\right) - 1 \tag{7.1.90}$$

As $t^- \to 0$, $x_2\left(t^-\right)$ is positive and decreasing, where $x_2\left(t^-\right) \to x_{qf}$, and $u^*\left(t\right) = -1$ (by problem assumption), so equation (7.1.90) implies that

$$\left(\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}} - 1 \right) > 0 \tag{7.1.91}$$

As $t^- \to \infty$ the value of equation (7.1.91) will not change because it is the variation of the final cost along the surface of final conditions, $x_2\left(t^-\right)$ will remain positive, and the optimal control must be $u^*\left(t\right) = -1$. Repeating this analysis for all $x_{qf} > 0$, requires that the proposed control in this region must be $u^*\left(t^-\right) = -1$. The HJCB PDE can now be evaluated to prove that the proposed control is optimal. The HJCB PDE for this problem is

$$H\left(x,-\frac{\partial \hat{J}_c\left(x\left(t_0\right),t_f-t_0\right)}{\partial x\left(t_0\right)},u,\lambda_0\right)=\lambda_1\left(t^-\right)\cdot x_2\left(t^-\right)-\lambda_2\left(t^-\right)-1 \qquad (7.1.92)$$

Substituting equations (7.1.77) and (7.1.78) into equation (7.1.92) and simplifying results in

$$H\left(x\left(t_0\right),-\frac{\partial \hat{J}_c\left(x\left(t_0\right),t_f-t_0\right)}{\partial x\left(t_0\right)},u,\lambda_0\right)=-\frac{x_{20}}{\sqrt{1+2\cdot x_{10}+x_{20}^2}}\cdot\left(\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}}-1\right)+1$$
$$+\frac{x_{20}}{\sqrt{1+2\cdot x_{10}+x_{20}^2}}\cdot\left(\frac{\partial CTG_{origin}\left(x_{qf}\right)}{\partial x_{qf}}-1\right)-1=0 \qquad (7.1.93)$$

which finishes the proof of optimality.

Note that Figure 7.11 can be used to remove the assumption of the optimal control for the initial condition $x_q=-\frac{1}{2}+\sqrt{\frac{17}{4}}$, in the analysis of location $q=q_2$. Figure 7.11 shows that $CTG_{origin}\left(x_{qf}\right)$ is a continuous function of x_{qf} that is differentiable everywhere except at the point $x_{qf}=-\frac{1}{2}+\sqrt{\frac{17}{4}}$. In locations q_2 and q_3 Figure 7.11 is the cost-to-go to the origin from the initial condition x_q. Since it is continuous and differentiable almost everywhere the work in Chapter 6 applies and can be used prove the control $u^*\left(t\right)=-1$ is optimal as desired.

Figure 7.12 depicts the control region for location $q = q_4$.

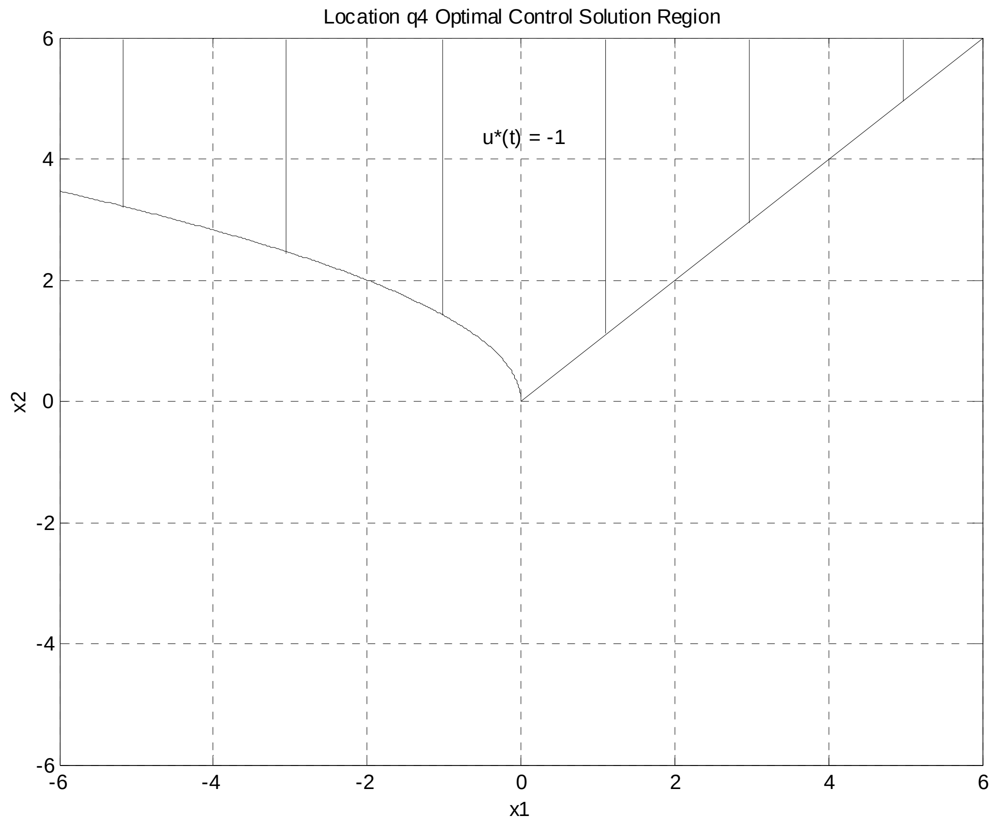

Figure 7.12: Control region for location $q = q_4$.

Feedback Control of the Hybrid Problem

Now the state space has been completely divided into regions where the optimal control has been computed, a global feedback control solution has been developed that solves the minimum time to the origin for the hybrid problem given any initial condition. The resulting solution is plotted in Figure 7.13.

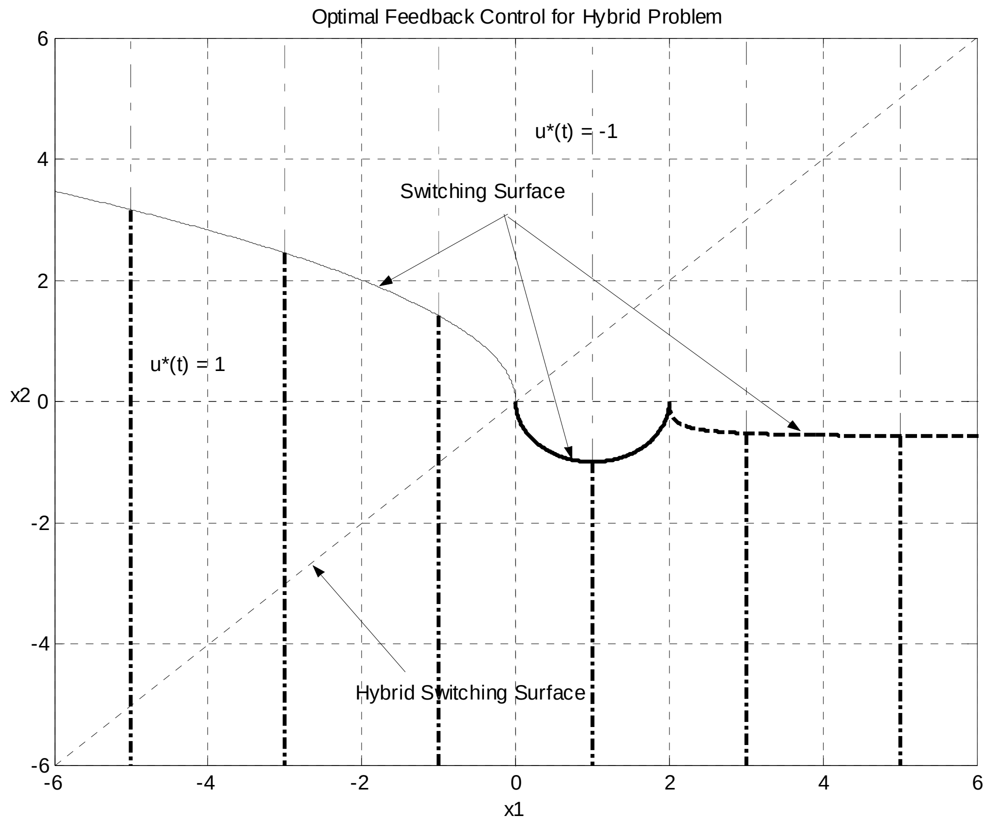

Figure 7.13: Regions of constant input that solve the hybrid time to origin optimal control problem.

Notice in Figure 7.13 that the state space is split into two regions according to a complex switching surface. Anything above the switching surface has a constant control of $u^*(t) = -1$ and anything below has a constant control of $u^*(t) = 1$. The feedback control law can be stated explicitly as the following:

- $u^*(t) = -1$ if the following conditions hold:

$$\begin{aligned} &x_2(t) > \sqrt{-2x_1(t)},\ \forall \mathrm{x}_1(t) \le 0 \\ &x_2(t) > -\sqrt{1-\left(x_1(t)-1\right)^2},\ \forall 0 \le \mathrm{x}_1(t) \le 2 \\ &x_2(t) > x_{1s}(t),\ \forall x_1(t) = x_{1s}(t) \end{aligned} \tag{7.1.94}$$

- $u^*(t) = 1$ if the following conditions hold:

$$
\begin{aligned}
&x_2(t) \le \sqrt{-2x_1(t)},\ \forall \mathrm{x}_1(t) \le 0 \\
&x_2(t) \le -\sqrt{1-\left(x_1(t)-1\right)^2},\ \forall 0 \le \mathrm{x}_1(t) \le 2 \\
&x_2(t) < x_{1s}(t),\ \forall x_1(t) = x_{1s}(t)
\end{aligned}
\tag{7.1.95}
$$

The hybrid version of the necessary and sufficient conditions was not used in solving the optimal control problem because the problem was decomposed into a series of smooth optimal control problems. Obviously, the specific local optimal control problems were formulated carefully so the state space was divided correctly. It was more appropriate to solve the problem as a series of smooth local optimal control problems instead of trying to properly formulate the two hybrid control problems.

7.2 Traction Control Problem

The purpose of this problem is to demonstrate the application of the necessary and sufficient conditions given in Chapter 6 to the drag racing and hot-rodder optimal control problems. The drag racing problem finds the control strategy that minimizes the time to traverse a specified distance from a stopped starting condition and the hot-rodder problem minimizes the time to traverse a specified distance from a stopped starting and ending condition. First the problem will be given and the optimal control will be derived using non-hybrid methods. Second the problem will be decomposed into a hybrid optimal control problem and the work in Chapter 6 will be applied to verify the solution.

The purpose of the drag racer problem is to drive a straight ¼ mile strip in minimum time from a standing start. Drag racing is a popular sport in the United States with many variants and vehicle classes with serious money at stake [30]. The best racer in 2005, who won the "World Championship," traversed the quarter mile drag strip in 4.496 seconds with a speed of 324.36 mph at the finish. The leading money winner received $1,141,500 for his efforts during the year.

The purpose of the hot-rodder problem is to drive a pre-specified distance in minimum time where the initial and final velocity of the vehicle is zero. An example of the hot-rodder problem would be two cars racing from traffic light to traffic light along a city street.

It is intuitively clear that the optimal control for the drag racing and hot-rodder problems is not bang-bang. If it were, there would be no need for traction control systems and ABS brakes. This also supplies the intuition for the drag racer. The limit on acceleration is often not the engine; it is the friction interaction between the tires and the road. A drag racer that spins his tires during the race will lose to one that does not.

The purpose of this section is to introduce the vehicle model used for the analysis, introduce the standard friction models encountered in the literature, and provide a review of the literature related to this problem.

7.2.1 Vehicle Model

For longitudinal vehicle motion analysis, the automobile is typically represented by a simple two wheel bicycle model [35], where a torque is applied to one or both tires producing longitudinal force that accelerates or decelerates the vehicle. The model used in this paper will assume that only the rear tire produces the longitudinal force acting on the vehicle and that the weight transfer associated with the acceleration process can be neglected. Making these assumptions allows the problem to be reduced from the longitudinal bicycle model to a quarter vehicle model with a single tire interacting with the ground and a lumped mass representing the quarter vehicle mass acting at the center-of-gravity of the wheel. This type of longitudinal vehicle model is common in the literature and can be found in [22][40][52][70]. Including the extra information in the problem can be done, but doesn't add any novelty to the analysis, only complexity.

Figure 7.14 is a schematic representing the single wheel longitudinal acceleration model.

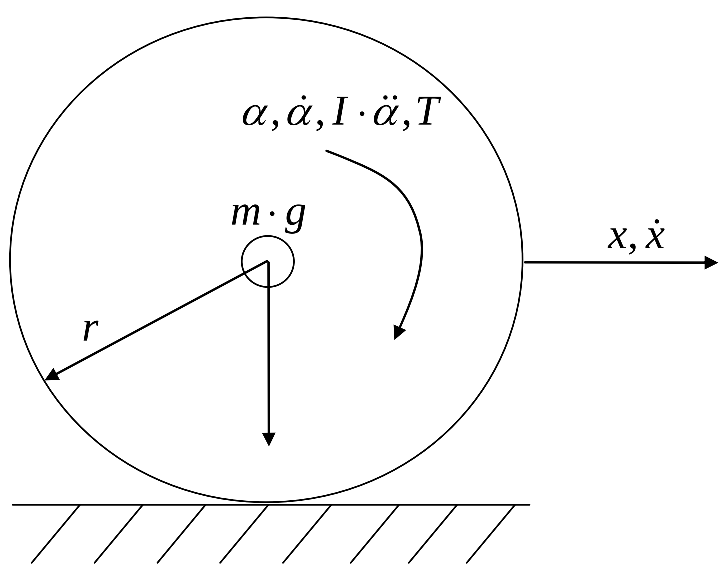

Figure 7.14: Single Wheel Acceleration Model

Using Figure 7.14, the dynamics describing the acceleration of the wheel can be written as

$$\begin{aligned} m \cdot \ddot{x}(t) &= \mu\big(S(t)\big) \cdot F_n \\ I \cdot \ddot{\alpha}(t) &= -r \cdot \mu\big(S(t)\big) \cdot F_n + T(t) \end{aligned} \tag{7.2.1}$$

where $\dot{x}$ is the longitudinal velocity of the center of the wheel, $\dot{\alpha}$ is the angular velocity of the wheel, T is the torque applied to the wheel, r is the effective radius of the tire at the tire-to-ground interface point, F_n is the normal force acting on the tire, m is the quarter vehicle mass, I is the rotating moment of inertia of the tire, and $\mu\big(S(t)\big)$ is the coefficient of friction with respect to the slip where the slip S is defined as

$$S(t) = \begin{cases} \dfrac{r \cdot \dot{\alpha}(t) - \dot{x}(t)}{\dot{x}(t)}, & \text{Braking and Slipping, } S < 0 \\ \dfrac{r \cdot \dot{\alpha}(t) - \dot{x}(t)}{r \cdot \dot{\alpha}(t)}, & \text{Acceleratingand Slipping, } S > 0 \end{cases} \tag{7.2.2}$$

Note that the term slip does not mean that the tire tread is slipping relative to the ground. When a torque is applied to the tire, the tread elements around the tire contact patch deform, but do not immediately move horizontally relative to the ground. Because of the deformation, the effective circumference of the tire changes and the angular velocity of the wheel must change to maintain the no horizontal movement condition at the tire-to-ground interface. As the torque applied to the wheel is increased the slip increases until the tire-to-ground interface can no longer counteract the torque. Since the ground cannot counteract the control torque, the tire contact patch has a different longitudinal velocity than the ground, and the wheel angular velocity either increases or decreases rapidly depending on the direction that the torque is applied. Soon thereafter the wheel spins if accelerating and locks if braking.

7.2.2 Friction Models

The most common tire-to-ground friction model found in the literature is a steady-state model based on the work in [53]. The steady-state friction model assumes that the coefficient of friction between the tire and ground is a function of the slip.

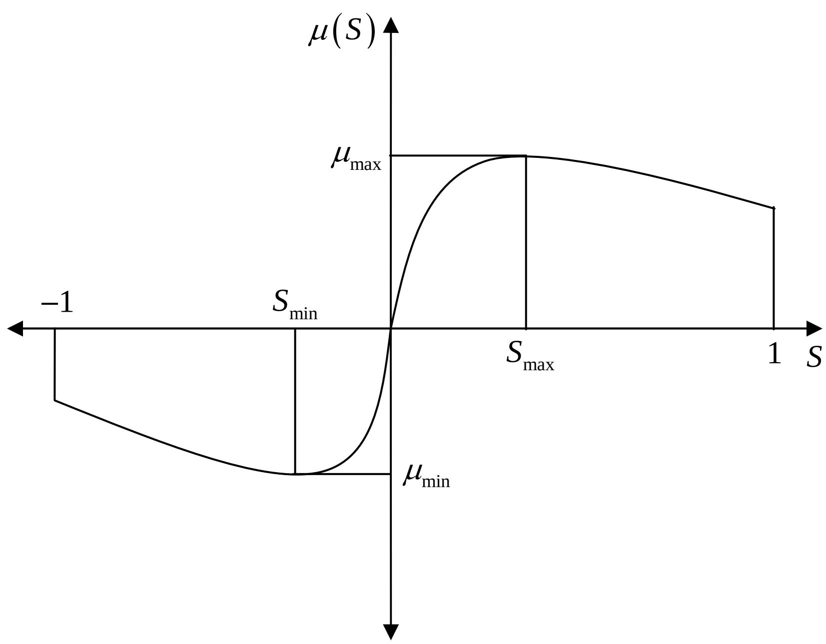

Figure 7.15: Sample Friction versus Slip Curve

Figure 7.15 is a sample coefficient of friction versus slip curve similar to the ones found in [53], and has some distinct features: friction is a continuous function of the slip for $-1 < S < 1$, attains its maximum at some slip, S_{max}, where $0 < |S_{max}| < 1$, attains its minimum at some slip, $S_{\min}$, where $0 < |S_{\min}| < 1$, and enters the pure sliding mode when $|S(t)| = 1$. Further when braking, the slip and the coefficient of friction is negative and when accelerating the slip and the coefficient of friction is positive.

The steady-state friction model presented is a good model to perform analysis with because it is relatively simple. Unfortunately, the force generation process is a dynamic not a static event, so the steady-state friction model does not account for dynamic events.

In [20] a dynamic friction model was introduced that is based on the point LuGre friction model. This model uses a first order non-linear filter with the relative velocity between the tire and ground as its input and the friction force is its output. Let $v_r = r \cdot \dot{\alpha} - \dot{x}$, then the friction force is

$$F_r = \left(\sigma_0 \cdot z + \sigma_1 \cdot \dot{z} + \sigma_3 \cdot v_r\right) \cdot F_n \tag{7.2.3}$$

where z is the solution of the non-linear differential equation

$$\dot{z} = v_r - \sigma_0 \cdot \frac{|v_r|}{\mu_c + \left(\mu_s - \mu_c\right) \cdot e^{-\left|\frac{v_r}{v_s}\right|^{1/2}}} \tag{7.2.4}$$

and σ_0, σ_1, σ_2, μ_c, and μ_s are constants relating to the fundamental properties of the tire and v_s is the Striebeck constant. With properly tuned parameters, this model reproduces the friction versus slip curve in steady-state, and provides good correlation with experimental data. For a detailed derivation of this model see [20].

The main disadvantages of this model are that accurate results require proper tuning of the constant parameters and it is much more complicated than the steady-state friction versus slip curve. Since the slip curve is not known a priori, is not really measurable, and not really correct, it is reasonable to use a simpler model for the problem analysis. In this analysis, a friction model that is simpler than both the steady-state friction model and the dynamic friction model is going to be used. Since the magnitude of I is much less than the magnitude of m in Figure 7.14, the torque applied to the wheel will act to accelerate and decelerate the wheel much faster than it will change the longitudinal velocity of the vehicle. As such, the transition from no slipping to slipping will happen very quickly, which can be seen in the example worked in [70]. Because of this behavior, a hybrid model can be used that embodies the standard stick/slip friction model, where the friction coefficient is $\mu_{\max}$ when the wheel is not sliding relative to the ground and $\mu_s < \mu_{\max}$ when the wheel is sliding with respect to the ground.

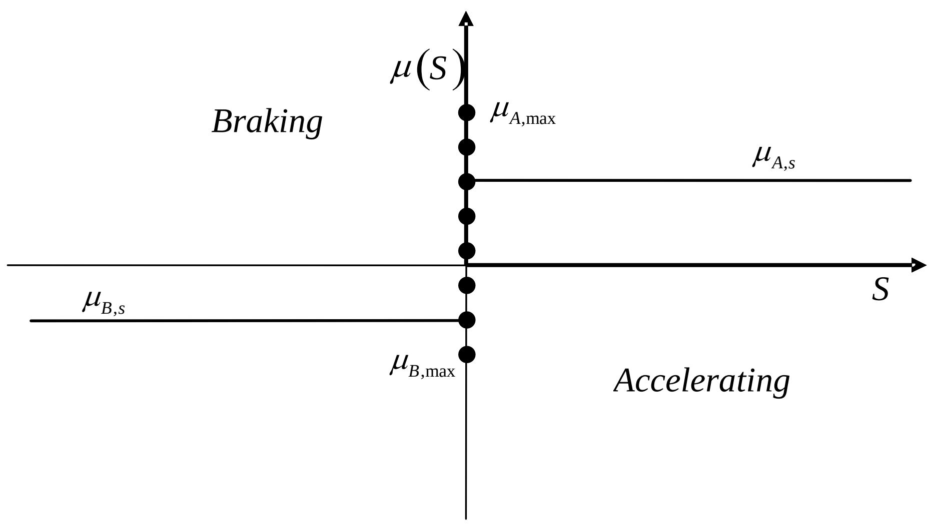

Figure 7.16: Friction versus slip curve for the optimal control problems

Figure 7.16 uses a standard elementary definition of friction where the coefficient of friction is constant when no slip occurs and switches instantaneously to another constant value when slipping begins. A conceptually simple hybrid system model for the vehicle with three discrete modes can be used to model the tire-to-ground interaction. The three nodes of the hybrid system are: no slipping, wheel accelerating and slipping, and wheel decelerating and slipping. This model is technically more difficult to optimize because of the non-differentiable dynamics on the transition curves between nodes. One justification for using this model is that a hybrid system model can be formulated in such a way that non-smooth necessary and sufficient conditions can be utilized to find the optimal feedback control for the drag racing and hot-rodder problems. Another is that the more detailed smoother model is neither an accurate description of the dynamics nor measurable in real time, which is highlighted by the actual ABS algorithms implemented on vehicles that only use wheel angular acceleration as the feedback variable [42].

7.2.3 Optimal Traction Control

Many papers exist in the literature that provide algorithms for optimal traction control for the model developed in the previous section. The Anti-lock Braking System (ABS) problem was the first problem addressed in the literature. Since acceleration control is a similar problem, the literature began addressing traction control in general, where traction control consists of braking and acceleration control. [12][22][31][40][51][52] present various control algorithms based on the steady-state friction versus slip curve that try to maximize the coefficient of friction between the tire and the ground. [31] uses sliding mode control techniques to solve the ABS problem. Using sliding mode techniques is common, but the novelty of the algorithm in [31] is that they assume the friction versus slip curve is unknown. [51] introduces an optimal fuzzy logic control algorithm to solve the ABS problem. The fuzzy algorithm is based on the steady-state friction versus slip curve where the fuzzy parameters are optimized using a genetic algorithm technique to maximize the coefficient of friction. [52] introduces a hybrid ABS control technique based on an estimation of the friction versus slip curve. This technique is unique because it exploits the fact that the friction versus slip curve is continuous and has a single maximum. [22] also utilizes the steady-state friction versus slip curve to develop a hybrid traction controller. They decompose the continuous problem into a four node hybrid automaton, where the dynamics of each node are defined based on the value of the time derivative of the slip and the switching rule is defined by the value of the slip. Controllers are then developed for each node of the automaton. The resulting control algorithm uses the slip to determine the appropriate controller to maximize the coefficient of friction between the tire and ground.

Note that these problems and techniques are not specific to vehicles with rubber-to-ground interfaces. [33] applies fuzzy logic methods to optimize traction control for a train system based on the steady-state friction model and a metal-to-metal interface.

The basic underlying assumption for these algorithms is that the optimal solution of the traction control problem is to maximize this coefficient of friction. In [70] the authors apply PMP [6][7][55] to the steady-state ABS problem to prove optimality of the control. There are two concerns with using the PMP. First, the PMP only provides necessary conditions for optimality. This can be problematic because a control can satisfy the necessary conditions and not be optimal. Second, the PMP only provides the open-loop control. For applications, the feedback solution is actually required. Because of these two limitations of the PMP, it is more desirable to solve the problem using a sufficient condition that generates an optimal feedback control instead of a necessary condition.

7.2.4 Solution Outline

The purpose of this section is to provide a precise formulation of the drag racing and hot-rodder problems, to provide feedback solution to both of them, and to prove these solutions are optimal. These optimal control problems are slightly different from the one given in [70] but the results of this work can be directly applied to solve the problem given in [70]. The rest of the section is organized in the following way. First the two optimal control problems will be defined. Then a candidate for the optimal feedback control will be proposed for each problem. The hybrid optimal control problem will then be developed. Finally, the work in Chapter 6 will be used to prove the candidate feedback control is optimal. In order to apply the work in Chapter 6, an assumption will be made initially for the hybrid problem and then a proof that the assumption is valid will be given.

7.2.5 Problem Formulation

The purpose of this section is to formulate the optimal control problems. The drag racing problem and the hot-rodder problem are similar and both optimal control problems will be given here. The drag racing problem will be developed first, then the hot-rodder problem will be given, and finally some comments about both control problems will be presented.

Simplified Dynamics

Figure 7.16 depicts the friction model that will be used in solving the drag racing and hot-rodder optimal control problems. Since a simplified friction model is being used, the dynamics governing the longitudinal motion of the vehicle are slightly different from the dynamics given in (7.2.1). Assume that the condition $T(t) \le T_{i,s}$ implying the wheel doesn't lose traction. Then the dynamics describing the motion of the vehicle are

$$\begin{aligned} \ddot{x}(t) &= r \cdot \ddot{\alpha}(t) \\ \ddot{\alpha}(t) &= \frac{T(t)}{I + m \cdot r^2} \end{aligned} \tag{7.2.5}$$

where

$$T_{i,s} = \frac{F_n \cdot \mu_{i,\max} \cdot \left(m \cdot r^2 + I\right)}{m \cdot r} \tag{7.2.6}$$

and $\mu_{i,\max}$ is given in Figure 7.16 for $i = A$ for acceleration and $i = B$ for braking.

When the wheels lose traction, the dynamics are given by

$$\begin{aligned} \ddot{x}(t) &= \frac{F_n}{m} \cdot \mu_{i,s} \\ \ddot{\alpha}(t) &= -\frac{r \cdot F_n}{I} \cdot \mu_{i,s} + \frac{T(t)}{I} \end{aligned} \tag{7.2.7}$$

where

$$\mu_{i,s} = \begin{cases} \mu_{A,s}, & \dot{x} < r \cdot \dot{\alpha} \\ \mu_{B,s}, & \dot{x} > r \cdot \dot{\alpha} \end{cases} \tag{7.2.8}$$

and $T(t)$ is the torque applied to the wheel. Now that the simplified system dynamics have been given, the drag racer and hot-rodder problems can be given.

Drag Racing Problem

Drag racing is a popular motor sport where the goal of the race is to traverse a one quarter or one eighth mile straight section of track in the shortest time. Two cars race at a time and the car that passes the finish line first wins the race. The drag racing problem can be simplified and studied using optimal control theory. Assume that the race car has only one gear and the dynamics given in (7.2.5) and (7.2.7) represent the motion of the vehicle over the time interval, $0 \le t \le t_f$. Further assume that the race car engine and brakes can instantaneously produce any torque in the set $\left[T_{B,\max}, T_{A,\max}\right]$, where $T_{A,\max}$ is a real positive number representing the maximum engine acceleration torque and $T_{B,\max}$ is a real negative number representing the maximum braking torque. Further, assume that at initial time $t_0 = 0$, the initial conditions for the race car are $x(0) = 0$, $\dot{x}(0) = 0$, $\alpha(0) = 0$, and $\dot{\alpha}(0) = 0$. The problem ends when $x(t) = 0.25$ miles, which is the terminal condition. Letting t_f represent the time at which $x(t) = 0.25$ miles, the performance criteria is

$$J\left(T(t)\right) = \int_0^{t_f} 1 \cdot dt \tag{7.2.9}$$

The problem is to find a $T(t)$, $0 \le t \le t_f$, that minimizes equation (7.2.9) subject to the constraints (7.2.5), (7.2.7), and the boundary conditions. In fact a more general problem is solved: Given any initial state, find the feedback control that minimizes equation (7.2.9) subject to the given dynamics and constraints.

The solution to the drag racing problem is the feedback control defined over the interval of time, $0 \le t \le t_f$, that satisfies the following equation

$$T(x,\dot{x},\dot{\alpha}) = \begin{cases} T_{A,\max}, & \dot{x} > r \cdot \dot{\alpha} \\ T_{A,s}, & \dot{x} = r \cdot \dot{\alpha} \\ T_{B,\max}, & \dot{x} < r \cdot \dot{\alpha} \end{cases} \tag{7.2.10}$$

where $T_{A,s}$ is given in (7.2.6). Note that the torques in (7.2.10) are constant and have the following properties

$$\begin{aligned} T_{A,\max} &\ge T_{A,s} \\ T_{B,\max} &\le T_{B,s} \end{aligned} \tag{7.2.11}$$

and equation (7.2.10) depends only on x, $\dot{x}$, and $\dot{\alpha}$.

The hybrid maximum principle [67] can be used to show that equation (7.2.10) is a candidate open-loop control for the drag racing problem with fixed initial condition. Note that there is a region of the state space where the optimal control is not unique. When the wheel is spinning and the drag racer is “close enough” to the target distance, d, any control will be optimal. In this region, there is not enough time to move to the non-sliding state before the target distance is reached. Further, since the torque applied to the wheel does not affect the velocity of the drag racer, it does not affect the performance. So, while in this region any torque applied to the wheel is optimal, so the candidate solution still applies.

Hot-rodder Problem

The hot-rodder problem is very similar to the drag racing problem. Hot-rodding is an artificial problem that is more complicated than the drag racing problem. The hot-rodding problem will be defined as the problem of traveling a fixed distance from a standing start to a dead stop in minimum time. Again assume that the car has one gear and the dynamics given in (7.2.5) and (7.2.7) define the vehicle's motion over the interval of time $0 \le t \le t_f$. Further, assume that the engine and brake can instantaneously produce the torque, $T(t) \in \left[T_{B,\max}, T_{A,\max}\right]$ and assume the initial conditions for the state variables are $x(0)=0$, $\dot{x}(0)=0$, $\alpha(0)=0$, and $\dot{\alpha}(0)=0$.

Let d represent the fixed distance to be traveled and assume that at time t_f, the state variables have the end conditions $x(t_f)=d$, $\dot{x}(t_f)=0$, $\dot{\alpha}(t_f)=0$, and $\alpha(t_f)$ is free. Further, let the cost (or performance criterion) associated with going from the initial condition to the final state be

$$J(T) = \int_{t_0}^{t_f} 1 \cdot dt \tag{7.2.12}$$

Then the control problem is to find the $T(t)$, $t \in [t_0, t_f]$ that minimizes equation (7.2.12) subject to the constraints given by equations (7.2.5) and (7.2.7) and the boundary conditions. Again, the solution developed here is more general than the one required to solve the problem, because the feedback control will be calculated from any initial condition. Note that the only difference between the drag-racer and hot-rodder problem is the boundary conditions. In particular, the hot-rodder has to stop at the end of the course while the drag racer can continue past the endpoint.

The solution to the hot-rodder problem is more complicated than the solution to the drag racing problem. Because this problem is a stop-go-stop problem, there is an acceleration phase followed by a braking phase.

Let time t_b, $0 < t_b < t_f$, be the time at which the acceleration phase ends and the braking phase begins. Then for the interval of time $0 \le t < t_b$ the solution is a constant control defined by

$$T(x, \dot{x}, \dot{\alpha}) = \begin{cases} T_{A,\max}, & \dot{x} > r \cdot \dot{\alpha} \\ T_{A,s}, & \dot{x} = r \cdot \dot{\alpha} \\ T_{B,\max}, & \dot{x} < r \cdot \dot{\alpha} \end{cases} \tag{7.2.13}$$

and for the interval of time $t_b \le t \le t_f$, the solution is a constant control that satisfies

$$T(x, \dot{x}, \dot{\alpha}) = \begin{cases} T_{A,\max}, & \dot{x} > r \cdot \dot{\alpha} \\ T_{B,s}, & \dot{x} = r \cdot \dot{\alpha} \\ T_{B,\max}, & \dot{x} < r \cdot \dot{\alpha} \end{cases} \tag{7.2.14}$$

where $T_{A,s}$ and $T_{B,s}$ satisfy (7.2.6) and the torques given in (7.2.13) and (7.2.14) satisfy (7.2.11).

Since equations (7.2.13) and (7.2.14) give an optimal feedback control, given initial conditions for the hot-rodder problem, it is straightforward to compute the time, t_b, that the vehicle begins the braking phase of the trajectory by integrating equations (7.2.5) and applying the boundary conditions and solving for t_b. Specifically

$$t_b = \sqrt{\frac{d}{\frac{r \cdot T_{A,s}}{2 \cdot \left(I + m \cdot r^2\right)} \cdot \left(1 + \frac{T_{A,s}}{\left|T_{B,s}\right|}\right)}} \tag{7.2.15}$$

when $x(0) = 0$, $\dot{x}(0) = 0$, and $\dot{\alpha}(0) = 0$.

As with the drag racing problem, a set of states in the state space exist where the solution is not well behaved. For this problem, a set of states exist where the optimal control does not exist because a trajectory does not exist that satisfies the boundary condition requiring the vehicle to stop at $x(t_f) = d$. Because $x(t) \le d$ for $t_0 \le t \le t_f$, the vehicle cannot drive past the light and back up to the final condition, so there is a region of the state space where no optimal solution exists.

Comments

The two problems given above are fundamentally different from the pure ABS problem given in [70], because the time is being minimized instead of the distance. The PMP can be used to solve the optimal control problem in [70], but it only provides and open-loop control. Further, since the problem has been formulated as a hybrid control problem, the standard PMP does not apply. Non-smooth versions of the maximum principle have been derived, for example see [67], but still only provide an open-loop solution to the control problem and necessary conditions for optimality.

The second thing to note is that the standard sufficient conditions derived from the HJCB PDE also do not apply to these problems because they require that the cost-to-go function be differentiable everywhere in the state space [6][7]. Of course, the integral equation of dynamic programming does still apply. Because of the discontinuity associated with the model, the cost-to-go function is continuous, but not differentiable everywhere. It is straightforward to prove that the cost-to-go function for the ABS problem in [70] is also continuous but not differentiable everywhere.

7.2.6 Optimal Solution

The purpose of this section is to prove that (7.2.10), (7.2.13), and (7.2.14) are the optimal controls for the drag racer and hot-rodder problems. Since the PMP is not applicable and the standard sufficient conditions do not apply, more general necessary and sufficient conditions are required for the proof of optimality. The hybrid control problem will be reformulated in such a way that the work in Chapter 6 applies. As in the previous example, the hybrid optimal control problem will be broken up into a series of local optimal control problems and the work of Chapter 6 will be applied.

Hybrid Model

The first step in proving the control is optimal is to model the system in hybrid form. The simplified friction model given earlier will be used as the basis of the hybrid model. In order to make the hybrid system easier to analyze, a coordinate transformation will be performed on the dynamic constraints given in equations (7.2.5) and (7.2.7).

Define a set of new state variables for the problem to be

$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \\ y_3 &= r \cdot \dot{\alpha} - \dot{x} \end{aligned} \tag{7.2.16}$$

When the tire is not sliding (i.e. $\dot{x}(t) = r \cdot \dot{\alpha}(t)$, $y_3(t) = 0$) the dynamics describing the system can be written as

$$\begin{aligned} \dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= r \cdot \frac{T(t)}{I + m \cdot r^2} \\ \dot{y}_3(t) &= 0 \end{aligned} \tag{7.2.17}$$

where $T_{B,s} \leq T(t) \leq T_{A,s}$, and when the tire has lost traction (i.e. $\dot{x}(t) \neq r \cdot \dot{\alpha}(t)$, $y_3(t) \neq 0$), the dynamics describing the system can be written as

$$\begin{aligned} \dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= \frac{F_n}{m} \cdot \mu_{i,s} \\ \dot{y}_3(t) &= \frac{r \cdot T(t)}{I} - F_n \cdot \mu_{i,s} \cdot \left(\frac{I + m \cdot r^2}{m \cdot I} \right) \end{aligned} \tag{7.2.18}$$

Where

$$\mu_{i,s} = \begin{cases} \mu_{A,s}, & \dot{x} < r \cdot \dot{\alpha} \\ \mu_{B,s}, & \dot{x} > r \cdot \dot{\alpha} \end{cases} \tag{7.2.19}$$

The input to the problem is the torque acting on the wheel which is bounded by the engine torque in the positive torque direction and is bounded by the maximum frictional torque produced by the brakes in the negative torque direction.

This new model can be cast into a problem in hybrid automaton form. The hybrid problem consists of three locations where the first location describes the behavior of the system when the wheel is not slipping, the second location describes the behavior of the system where the wheel is slipping and $r \cdot \dot{\alpha}(t) > \dot{x}(t)$, and the third location describes the behavior of the system where the wheel is slipping and $r \cdot \dot{\alpha}(t) < \dot{x}(t)$. The hybrid automaton is given in Figure 7.17.

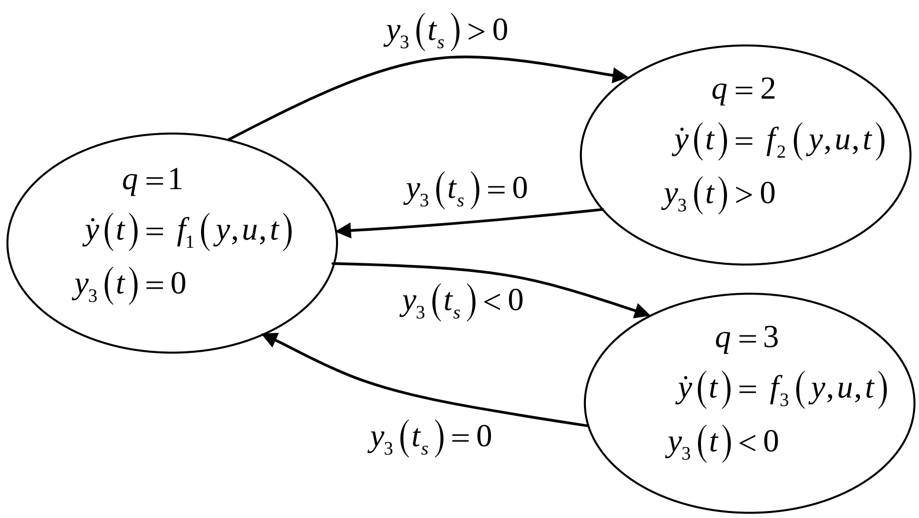

Figure 7.17: Hybrid model for the friction phenomenon between the tire and ground.

The dynamic constraints given in Figure 7.17 are not identical to equations (7.2.17) and (7.2.18). When the system is in node $q = 2$ or $q = 3$, then f_2 and f_3 are equal to equation (7.2.18) with $i = A$ and $i = B$ respectfully. However when the system is in node $q = 1$, f_1 is not equal to equation (7.2.17) because the equations are not valid when $T_{A,\max} \geq T(t) > T_{A,s}$ or $T_{B,\max} \leq T(t) < T_{B,s}$. If the constraint on the torque $T_{B,s} \leq T(t) \leq T_{A,s}$ is not met (which is possible), the condition on $\dot{y}_3$ allows for the system to transition out of node $q = 1$. It is this behavior of the system that makes the control singular when the system is in node $q = 1$ and not bang-bang.

Note that the hybrid model is not a well behaved model because it can exhibit Zeno behavior. Assume that the system is in location $q = 2$ and the control torque is such that $y_3(t) \to 0$. When $y_3(t) = 0$, the system will transition to location $q = 1$, but if the applied torque does not satisfy $T_{B,s} \leq T(t) \leq T_{A,s}$, the system will instantaneously transition into either location $q = 2$ or $q = 3$. For example assume that control torque is determined by the following feedback control strategy

$$T(t) = \begin{cases} T_{B,\max}, & y_3(t) \geq 0 \\ T_{A,\max}, & y_3(t) < 0 \end{cases} \tag{7.2.20}$$

then the system will transition between locations $q = 1$, $q = 2$, and $q = 3$ an infinite number of times in finite time.

Another characteristic of the hybrid model is that when the system is in location $q = 1$ the input torque directly affects the longitudinal position of the vehicle, but when the system is in location $q = 2$ or $q = 3$, the input torque doesn't affect the longitudinal position or velocity of the vehicle. In these locations, the input torque only controls the wheel speed which controls when the system switches to location $q = 1$, if ever. So when in location $q = 1$, the input torque controls the longitudinal states of the system and when in location $q = 2$ or $q = 3$ the input torque controls when the system will switch back to location $q = 1$.

Finally, in this form, the work of Chapter 6 doesn't apply. Because the theory has not been developed for controlled jumps, it cannot be applied to this hybrid model. When the system is in location $q = 1$ and the control torque satisfies $T_{B,s} \leq T(t) \leq T_{A,s}$, the system will jump to either $q = 2$ or $q = 3$. Hence the control torque will produce a controlled jump of the system.

Note that an assumption will be added to the hybrid model and verified, so that the work of Chapter 6 can be applied. It is going to be assumed that once the system enters location $q = 1$, it will stay there until the final state is reached. This assumption will restrict the model so that the controlled jump is not allowed, as well as, remove the Zeno behavior.

7.2.7 Proof of the Drag Racing Problem

The proof of the feedback control solution to the drag racing problem will be given first because it is the easier of the two control problems.

Location q=1 Analysis

The first step in solving the hybrid optimal control problem is to compute the feedback control for the system in node $q = 1$. For this analysis it will be assumed that once the system enters node $q = 1$, it will stay there until the final state is reached. This assumption will be verified later. In the previous section, it was noted that $f_1(y,u,t)$ does not equal equation (7.2.17). However the node assumption requires that $y_3(t) = 0$ for all $t_0 \le t \le t_f$, which requires the control torque to satisfy $T_{B,s} \le T(t) \le T_{A,s}$. As such f_1 reduces to

$$f_1(y,u,t) = \begin{bmatrix} y_2(t) \\ r \cdot \dfrac{T(t)}{I + m \cdot r^2} \end{bmatrix} \tag{7.2.21}$$

where $\dot{y}_3(t)$ can be eliminated from equation (7.2.17) and the dimension of the state space reduced from dimension 3 to dimension 2.

Now that the dynamic constraint for this problem has been defined, the optimal control problem can be given. Let the system start with the initial condition

$$\begin{aligned} 0 \le y_1(t_0) < d \\ y_2(t_0) \ge 0 \end{aligned} \tag{7.2.22}$$

and move to the final condition

$$\begin{aligned} y_1(t_f) = d \\ y_2(t_f) \in \mathbb{R} \end{aligned} \tag{7.2.23}$$

with the cost

$$J(x,u) = \int_{t_0}^{t_f} 1 \cdot dt \tag{7.2.24}$$

The optimal control problem finds the control function $u(\tau),\ t_0 \le \tau < t_f$, that will minimize equation (7.2.24) while satisfying the dynamic constraints in equation (7.2.21) and boundary conditions given in equations (7.2.22) and (7.2.23).

The first step in solving the optimal control problem is to compute the Hamiltonian for node $q=1$. The Hamiltonian for this problem is

$$H_1 = \lambda_{1,1}(t) \cdot y_2(t) + \lambda_{2,1}(t) \cdot r \cdot \frac{T(t)}{I + m \cdot r^2} - 1 \tag{7.2.25}$$

The necessary conditions requires that the Hamiltonian is maximized by the optimal control, so equation (7.2.25) implies that

$$T(t) = \begin{cases} T_{A,s}, & sign(\lambda_{2,1}(t)) > 0 \\ T_{B,s}, & sign(\lambda_{2,1}(t)) < 0 \\ u, & sign(\lambda_{2,1}(t)) = 0 \end{cases} \tag{7.2.26}$$

where $T_{A,s} \le u \le T_{B,s}$ is some unknown admissible control.

Furthermore, the necessary conditions require that the adjoint vector $\lambda(t)$ is the solution to the vector differential equations

$$\begin{aligned} \dot{\lambda}_{1,1} &= 0 \\ \dot{\lambda}_{2,1} &= -\lambda_{1,1}(t) \end{aligned} \tag{7.2.27}$$

with initial conditions $\lambda_{1,1}(t_0)$ and $\lambda_{2,1}(t_0)$.

Let $\lambda_{1,1}(t_0) = \lambda_{10,1}$ and $\lambda_{2,1}(t_0) = \lambda_{20,1}$, then equation (7.2.27) can be integrated resulting in

$$\begin{aligned} \lambda_{1,1}(t) &= \lambda_{10,1} \\ \lambda_{2,1}(t) &= \lambda_{20,1} - \lambda_{10,1} \cdot (t - t_0) \end{aligned} \tag{7.2.28}$$

Equations (7.2.28) and (7.2.26) imply that the optimal control is either constant or piecewise constant with only one switch.

The transversality conditions associated with the necessary conditions imply

$$\begin{aligned} &\lambda_{1,1}\left(t_f\right) \in \mathbb{R} \\ &\lambda_{2,1}\left(t_f\right) = 0 \end{aligned} \tag{7.2.29}$$

since $y_1\left(t_f\right) = d > 0$ is fixed and $y_2\left(t_f\right)$ is free. Note that equation (7.2.28) can be substituted into equation (7.2.29) resulting in

$$\begin{aligned} &\lambda_{1,1}\left(t_f\right) = \lambda_{10,1} \in \mathbb{R} \\ &\lambda_{2,1}\left(t_f\right) = \lambda_{20,1} - \lambda_{10,1} \cdot \left(t_f - t_0\right) = 0 \Rightarrow \lambda_{20,1} = \lambda_{10,1} \cdot \left(t_f - t_0\right) \end{aligned} \tag{7.2.30}$$

which proves that it is necessary that the optimal control is constant.

Let the control, $T(t) = T$. The constraint equations given in (7.2.21) can then be integrated resulting in

$$\begin{aligned} &y_1(t) = y_1(t_0) + y_2(t_0) \cdot (t - t_0) + \frac{1}{2} \cdot \frac{T \cdot r}{I + m \cdot r^2} \cdot (t - t_0)^2 \\ &y_2(t) = y_2(t_0) + \frac{T \cdot r}{I + m \cdot r^2} \cdot (t - t_0) \end{aligned} \tag{7.2.31}$$

Since the candidate optimal control is constant and equal to either $T_{A,s}$ or $T_{B,s}$, and $y_1\left(t_f\right) = d > 0$, then the constant control that satisfies the constraint and equation (7.2.31) must be positive and, an optimal control (if it exists) must satisfy $T(t) = T_{A,s}$.

The sufficient conditions will now be used to verify that the control function $T(t) = T_{s,A}$ is the optimal control for all $t_0 < t < t_f$. The non-smooth sufficient conditions require that the optimal control satisfy the HJCB PDE whenever the optimal cost-to-go function

($J_c^*(t,x)$) is differentiable in a region that includes both the initial and final states and the neighborhood of all those in between. Thus,

$$\frac{\partial J_c^*(t_0)}{\partial t_0} - H\left(x, -\frac{\partial J_c^*(t_0)}{\partial x}, T\right) = 0 \tag{7.2.32}$$

where $J_{c,1}^*$ is the cost (i.e. time) associated with moving the trajectory from $y(t_0)$ to $y(t_f)$ using the optimal control. When T is constant, equation (7.2.31) can be solved for $(t_f - t_0)$ using the boundary conditions yielding

$$(t_f - t_0) = \frac{-y_2(t_0) + \sqrt{y_2(t_0)^2 + 2\cdot\gamma_1\cdot(d - y_1(t_0))}}{\gamma_1} = J_{c,1}^*(y(t_0)) \tag{7.2.33}$$

where $d > 0$,

$$\gamma_1 = \frac{r\cdot T_{A,s}}{I + m\cdot r^2} \tag{7.2.34}$$

and $T_{A,s}$ is the constant torque.

Now, equation (7.2.33) is differentiable and not an explicit function of t_0 so

$$\begin{aligned}
&\frac{\partial J_{c,1}^*(y(t_0))}{\partial t_0} = 0 \\
&-\frac{\partial J_{c,1}^*(y(t_0))}{\partial y_1(t_0)} = \frac{1}{\sqrt{y_2(t_0)^2 + 2\cdot\gamma_1\cdot(d - y_1(t_0))}} \\
&-\frac{\partial J_{c,1}^*(y(t_0))}{\partial y_2(t_0)} = -\frac{1}{\gamma_1}\cdot\left[\frac{y_2(t_0)}{\sqrt{y_2(t_0)^2 + 2\cdot\gamma_1\cdot(d - y_1(t_0))}} - 1\right]
\end{aligned} \tag{7.2.35}$$

and the HJCB equation evaluates to

$$\underbrace{\frac{\partial J_{c,1}^{*}\left(y\left(t_0\right)\right)}{\partial t_0}}_{0}-\underbrace{\frac{y_2\left(t_0\right)}{\sqrt{y_2\left(t_0\right)^2+2\cdot\gamma_1\cdot\left(d-y_1\left(t_0\right)\right)}}-\frac{1}{\gamma_1}\cdot\left[\frac{y_2\left(t_0\right)}{\sqrt{y_2\left(t_0\right)^2+2\cdot\gamma_1\cdot\left(d-y_1\left(t_0\right)\right)}}-1\right]\cdot\gamma_1}_{0}-1=0 \tag{7.2.36}$$

which proves that the control $T(t)=T_{A,s}$ is optimal in the region that contains any initial state that satisfies equation (7.2.22) and is the feedback control for location $q=1$.

Location q=2/ $q=3$ **Analysis**

The next step in solving the hybrid optimal control problem is to analyze the behavior of the system while in node $q=2$ or $q=3$. These two nodes have similar behavior so the optimal control for node $q=2$ will be computed and the optimal control for location $q=3$ will be given directly.

When the system is in node $q=2$, the tire is slipping relative to the ground, $y_3\left(t_0\right)>0$, and the dynamics of the system are given by

$$\begin{aligned}\dot{y}_1\left(t\right)&=y_2\left(t\right)\\ \dot{y}_2\left(t\right)&=\frac{F_n}{m}\cdot\mu_{A,s}\\ \dot{y}_3\left(t\right)&=\frac{r\cdot T\left(t\right)}{I}-F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\end{aligned} \tag{7.2.37}$$

Where $T_{B,\max} \le T(t) \le T_{A,\max}$, and $\mu_{A,s}$ is positive and is the coefficient of sliding friction between the tire and the ground.

The local optimal control problem can now be given for this location. Assume the state trajectory starts at the initial condition

$$\begin{aligned} &0 \le y_1(t_0) < d \\ &y_2(t_0) \ge 0 \\ &y_3(t_0) > 0 \end{aligned} \tag{7.2.38}$$

and moves to the final condition

$$\begin{aligned} &y_1(t_s) < d \\ &y_2(t_s) \in \mathbb{R} \\ &y_3(t_s) = 0 \end{aligned} \tag{7.2.39}$$

with cost

$$J(x,u) = \int_{t_0}^{t_s} 1 \cdot dt + g\left(y_1(t_s), y_2(t_s)\right) \tag{7.2.40}$$

where

$$g\left(y_1(t_s), y_2(t_s)\right) = J^*_{c,1}\left(y(t_s)\right) = \frac{-y_2(t_s) + \sqrt{y_2(t_s)^2 + 2 \cdot \gamma_1 \cdot \left(d - y_1(t_s)\right)}}{\gamma_1} \tag{7.2.41}$$

Then the optimal control problem will be to find the control function $u(\tau),\ t_0 \le \tau < t_s$, that minimizes the cost given in equation (7.2.40) satisfying the dynamic constraints given in equation (7.2.37) and boundary conditions in (7.2.38) and (7.2.39).

Note that the final condition on y_1 has been restricted to any value less than d. Also it is required that $y_3(\tau)=0$ for some $\tau < t_f$. This is required because in this node, the longitudinal position and velocity are independent of the control as is obvious from equation (7.2.37). As such, if the system cannot switch to node $q=1$ before $y_1(t_s)=d$, the optimal control is not unique and any admissible control is optimal.

The necessary conditions for optimality are as follows. First the Hamiltonian for this problem is

$$H_2 = \lambda_{1,2}(t)\cdot y_2(t) + \lambda_{2,2}(t)\cdot \frac{F_n}{m}\cdot \mu_{A,s} + \lambda_{3,2}(t)\cdot\left[\frac{T(t)}{I} - F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right] - 1 \quad (7.2.42)$$

Defining $\gamma_2(T)$ as

$$\gamma_2(T) = \left[\frac{T(t)}{I} - F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right] \quad (7.2.43)$$

and substituting into equation (7.2.42) results in

$$H_2 = \lambda_{1,2}(t)\cdot y_2(t) + \lambda_{2,2}(t)\cdot \frac{F_n}{m}\cdot \mu_{A,s} + \lambda_{3,2}(t)\cdot\gamma_2(T) - 1 \quad (7.2.44)$$

Using the definition of the adjoint variable, equation (7.2.44) can be used to compute

$$\begin{aligned} \dot{\lambda}_{1,2} &= 0 \\ \dot{\lambda}_{2,2} &= -\lambda_{1,2}(t) \\ \dot{\lambda}_{3,2} &= 0 \end{aligned} \tag{7.2.45}$$

which implies

$$\begin{aligned} \lambda_{1,2}(t) &= \lambda_{10,2} \\ \lambda_{2,2}(t) &= \lambda_{20,2} - \lambda_{10,2} \cdot (t - t_0) \\ \lambda_{3,2}(t) &= \lambda_{30,2} \end{aligned} \tag{7.2.46}$$

The necessary conditions require that the Hamiltonian is maximized for almost every time $t_0 \le t \le t_s$, so the control that maximizes the Hamiltonian is

$$T(t) = \begin{cases} T_{A,\max}, & \lambda_{3,2}(t) > 0 \\ T_{B,\max}, & \lambda_{3,2}(t) < 0 \\ T_{B,\max} \le u \le T_{A,\max}, & \lambda_{3,2}(t) = 0 \end{cases} \tag{7.2.47}$$

where u is some unknown admissible control. Note that as long as $\lambda_{30,2} \ne 0$, the optimal control is constant and is either $T(t) = T = T_{A,\max}$ or $T(t) = T = T_{B,\max}$.

Since $y_3(t) > 0$, the logical choice for the constant optimal control is $T(t) = T_{B,\max}$ because it will drive $y_3(t) \to 0$.

Assume that the optimal control is equal to the constant $T(t) = T_{B,\max}$, then equation (7.2.37) can be integrated to find

$$
\begin{aligned}
y_1(t) &= y_1(t_0) + y_2(t_0)\cdot(t-t_0) + \frac{1}{2}\cdot\frac{F_n}{m}\cdot\mu_{A,s}\cdot(t-t_0)^2 \\
y_2(t) &= y_2(t_0) + \frac{F_n}{m}\cdot\mu_{A,s}\cdot(t-t_0) \\
y_3(t) &= y_3(t_0) + \left(\frac{r\cdot T_{B,\max}}{I} - F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right)\cdot(t-t_0)
\end{aligned}
\tag{7.2.48}
$$

and the third equation in (7.2.48) can be rearranged and evaluated at t_s resulting in

$$
(t_s - t_0) = \frac{y_3(t_s) - y_3(t_0)}{\left(\frac{r\cdot T_{B,\max}}{I} - F_n\cdot\mu_{i,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right)}
\tag{7.2.49}
$$

Since the constraint associated with switching time $t = t_s$ requires that $y_3(t_s) = 0$, then equation (7.2.49) can be written as

$$
(t_s - t_0) = -\frac{y_3(t_0)}{\left(\frac{r\cdot T_{B,\max}}{I} - F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right)}
\tag{7.2.50}
$$

and the optimal cost-to-go function from $y(t_0)$ to $y(t_s)$ using the candidate optimal control is

$$
J_{c,2}^{*}\left(y(t_0)\right) = -\frac{y_3(t_0)}{\gamma_2} + \frac{-y_2(t_s) + \sqrt{y_2(t_s)^2 + 2\cdot\gamma_1\cdot\left(d - y_1(t_s)\right)}}{\gamma_1}
\tag{7.2.51}
$$

where

$$
\begin{aligned}
\gamma_1 &= \frac{r\cdot T_{A,s}}{I+m\cdot r^2} \\
\gamma_2 &= \left(\frac{r\cdot T_{B,\max}}{I} - F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right)
\end{aligned}
\tag{7.2.52}
$$

The value of $y_1(t_s)$ and $y_2(t_s)$ can be computed from equation (7.2.48) resulting in

$$\begin{aligned} y_1(t_s) &= y_1(t_0) - y_2(t_0)\cdot\frac{y_3(t_0)}{\gamma_2} + \gamma_4\cdot y_3(t_0)^2 \\ y_2(t_s) &= y_2(t_0) + \gamma_3\cdot y_3(t_0) \end{aligned} \tag{7.2.53}$$

where

$$\begin{aligned} \gamma_3 &= -\frac{F_n}{m}\cdot\frac{\mu_{A,s}}{\gamma_2} \\ \gamma_4 &= \frac{1}{2}\cdot\frac{F_n}{m}\cdot\mu_{A,s}\cdot\left(\frac{1}{\gamma_2}\right)^2 \end{aligned} \tag{7.2.54}$$

Finally equation (7.2.53) can be substituted into equation (7.2.51) to calculate the optimal cost-to-go as

$$\begin{aligned} J^*_{c,2}\left(y(t_0)\right) = &-\frac{y_3(t_0)}{\gamma_2} - \frac{\left(y_2(t_0)+\gamma_3\cdot y_3(t_0)\right)}{\gamma_1} \\ &+\frac{\sqrt{\left(y_2(t_0)+\gamma_3\cdot y_3(t_0)\right)^2 + 2\cdot\gamma_1\cdot\left(d-\left(y_1(t_0)-y_2(t_0)\cdot\frac{y_3(t_0)}{\gamma_2}+\gamma_4\cdot y_3(t_0)^2\right)\right)}}{\gamma_1} \end{aligned} \tag{7.2.55}$$

Note that equation (7.2.51) is differentiable when the initial state satisfies the boundary condition given in equation (7.2.38) and produces a trajectory such that the end condition in equation (7.2.39) is satisfied. So for any $(t_s - t_0) > 0$, $y_3(t_0)$ will be

$$y_3(t_0) = -(t_s - t_0)\cdot\left(\frac{r\cdot T_{B,\max}}{I} - F_n\cdot\mu_{A,s}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right)\right) \tag{7.2.56}$$

and $y_1(t_0)$ and $y_2(t_0)$ are required to satisfy

$$\begin{aligned} &0 \le y_1(t_0) < d + y_2(t_0) \cdot \frac{y_3(t_0)}{\gamma_2} - \gamma_4 \cdot y_3(t_0)^2 \\ &y_2(t_0) \ge 0 \end{aligned} \qquad (7.2.57)$$

which define a region of the state space where all possible initial conditions must lie.

Differentiating equation (7.2.51) results in

$$-\frac{\partial J_{c,2}^*}{\partial y_1(t_0)} = \frac{1}{\sqrt{\left(y_2(t_0) + \gamma_3 \cdot y_3(t_0)\right)^2 + 2 \cdot \gamma_1 \cdot \left(d - y_1(t_0) + y_2(t_0) \cdot \frac{y_3(t_0)}{\gamma_2} - \gamma_4 \cdot y_3(t_0)^2\right)}}$$

$$-\frac{\partial J_{c,2}^*}{\partial y_2(t_0)} = \frac{1}{\gamma_1} \cdot \left[1 - \frac{y_2(t_0) + \gamma_3 \cdot y_3(t_0) + \gamma_1 \cdot \frac{y_3(t_0)}{\gamma_2}}{\sqrt{\left(y_2(t_0) + \gamma_3 \cdot y_3(t_0)\right)^2 + 2 \cdot \gamma_1 \cdot \left(d - y_1(t_0) + y_2(t_0) \cdot \frac{y_3(t_0)}{\gamma_2} - \gamma_4 \cdot y_3(t_0)^2\right)}}\right]$$

$$-\frac{\partial J_{c,2}^*}{\partial y_3(t_0)} = \frac{1}{\gamma_2} + \frac{\gamma_3}{\gamma_2} - \frac{1}{\gamma_1} \cdot \frac{\left(y_2(t_0) + \gamma_3 \cdot y_3(t_0)\right) \cdot \gamma_3 + \left(\frac{y_2(t_0)}{\gamma_2} - 2 \cdot \gamma_4 \cdot y_3(t_0)\right) \cdot \gamma_1}{\sqrt{\left(y_2(t_0) + \gamma_3 \cdot y_3(t_0)\right)^2 + 2 \cdot \gamma_1 \cdot \left(d - y_1(t_0) + y_2(t_0) \cdot \frac{y_3(t_0)}{\gamma_2} - \gamma_4 \cdot y_3(t_0)^2\right)}}$$

(7.2.58)

The non-smooth sufficient condition requires that the optimal control satisfy the HJCB equation everywhere it exists. In order to compute the HJCB equation, the Hamiltonian function needs to be computed. Substituting equations (7.2.58) into (7.2.46) and then into (7.2.44) and simplifying gives

$$H_2\left(y(t_0),-\frac{\partial J^*_{c,2}}{\partial y(t_0)},T\right)=\frac{y_3(t_0)\cdot(\gamma_3+2\cdot\gamma_2\cdot\gamma_4)}{\sqrt{\left(y_2(t_0)+\gamma_3\cdot y_3(t_0)\right)^2+2\cdot\gamma_1\cdot\left(d-y_1(t_0)+y_2(t_0)\cdot\frac{y_3(t_0)}{\gamma_2}-\gamma_4\cdot y_3(t_0)^2\right)}}$$

(7.2.59)

Finally, using the definition of γ_2, γ_3, and γ_4, equation (7.2.59) results in

$$H_2\left(y(t_0),-\frac{\partial J^*_{c,2}}{\partial y(t_0)},T\right)=0 \qquad (7.2.60)$$

which is the desired result. Note that equation (7.2.60) is true for any $0\le t_0<t_s<t_f$ and finishes the proof.

So for node $q=2$, $T(t)=T_{B,\max}$ is the optimal solution for any state for which the state $y_3(t_f)=0$ is reachable. When in node $q=3$, an identical method is used to prove that the control $T(t)=T_{A,\max}$ is optimal and finishes the proof of optimality.

Assumption Justification

The proof the optimal control required the assumption that once the system reached node $q=1$ the trajectory would stay in node $q=1$. This assumption will now be justified.

First assume that the system starts in location $q=1$ at time t_0, with $y_1(t_0)=\alpha_1<d$ and $y_2(t_0)=\alpha_2\ge 0$, and stays in node $q=1$ until $y_1(t)=\alpha_3\le d$. The optimal control analysis for a trajectory in node $q=1$ showed that the optimal control is $T(t)=T_{A,s}$ for

$t_0 \le t \le t_f$. The position and velocity of the vehicle for any time, t, $t_0 \le t \le t_f$, is given by equation (7.2.31) resulting in

$$\begin{aligned} y_1(t) &= y_1(t_0) + y_2(t_0)\cdot(t-t_0) + \frac{1}{2}\cdot\frac{T_{A,s}\cdot r}{I+m\cdot r^2}\cdot(t-t_0)^2 \\ y_2(t) &= y_2(t_0) + \frac{T_{A,s}\cdot r}{I+m\cdot r^2}\cdot(t-t_0) \end{aligned} \tag{7.2.61}$$

but since

$$T_{A,s} = \frac{F_n\cdot\mu_{A,\max}}{r}\cdot\left(\frac{I+m\cdot r^2}{m\cdot I}\right) \tag{7.2.62}$$

equation (7.2.61) can be rewritten as

$$\begin{aligned} y_1(t) &= y_1(t_0) + y_2(t_0)\cdot(t-t_0) + \frac{1}{2}\cdot\frac{F_n}{m}\cdot\mu_{A,\max}\cdot(t-t_0)^2 \\ y_2(t) &= y_2(t_0) + \frac{F_n}{m}\cdot\mu_{A,\max}\cdot(t-t_0) \end{aligned} \tag{7.2.63}$$

Now assume that the system is in node $q = 2$ or node $q = 3$ with the same initial condition. The position and velocity for this node is independent of the optimal control and can be calculated from equation (7.2.48) resulting in

$$\begin{aligned} y_1(t) &= y_1(t_0) + y_2(t_0)\cdot(t-t_0) + \frac{1}{2}\cdot\operatorname{sgn}\left(y_3(t)\right)\cdot\frac{F_n}{m}\cdot\mu_{i,s}\cdot(t-t_0)^2 \\ y_2(t) &= y_2(t_0) + \operatorname{sgn}\left(y_3(t)\right)\cdot\frac{F_n}{m}\cdot\mu_{i,s}\cdot(t-t_0) \end{aligned} \tag{7.2.64}$$

Since

$$\mu_{i,s} < \mu_{A,\max} \tag{7.2.65}$$

by definition, $y_{1,1}(t) > y_{1,2}(t)$ for all $t > t_0$. As such if the system is in any node other than $q = 1$ for some interval of time, the distance traveled will always be less than if the system stayed in node $q = 1$ for that interval of time, so the final node will always be $q = 1$ justifying the assumption.

7.2.8 Proof of Hot-rodder Problem

The hot-rodder problem is a similar problem to the drag racing problem except that the end condition is different. The purpose of the hot-rodder problem is to minimize the amount of time required to travel a prescribed distance. Thus the initial and final velocities must be zero. In the development of the solution to the drag racing problem, it was shown that a region of the state space existed where the optimal control was not unique. The hot-rodder problem has a similar region, except that in this region the final state is not reachable, so an optimal control will not exist. An example is when a person wants to stop at a stop light and they wait too long to apply the brakes, they will not be able to stop and will pass through the stop light.

The solution to the hot-rodder problem will proceed in a manner identical to the drag racing problem. First, the feedback control will be calculated for the final node and then the optimal cost-to-go will be evaluated along the switching surface between locations and the new local optimal control problem will be analyzed. As with the drag racing problem, it will be assumed that the final location is $q = 1$. The justification of the assumption for the drag racing problem applies to this problem here, so will be omitted from this section.

Node q=1 Analysis

The first step in solving the hybrid optimal control problem is to compute the feedback control for the system in node $q=1$. For this analysis it will be assumed that once the system enters node $q=1$, it will stay there until the final state is reached. Just as in the drag racer problem, the node assumption requires that $y_3(t)=0$ for all $t_0 \le t \le t_f$, and the control torque satisfy $T_{B,s} \le T(t) \le T_{A,s}$. The dynamic constraint for the problem will then reduce to

$$f_1(y,u,t)=\begin{bmatrix} y_2(t) \\ r\cdot\dfrac{T(t)}{I+m\cdot r^2} \end{bmatrix} \tag{7.2.66}$$

Now that the dynamic constraint for this problem has been defined, the optimal control problem can be given. Let the system start with the initial condition

$$\begin{aligned} &0 \le y_1(t_0) < d \\ &y_2(t_0) \ge 0 \end{aligned} \tag{7.2.67}$$

and move to the final condition

$$\begin{aligned} y_1(t_f) &= d \\ y_2(t_f) &= 0 \end{aligned} \tag{7.2.68}$$

with the cost

$$J\left(x,u\right)=\int_{t_0}^{t_f}1\cdot dt \tag{7.2.69}$$

The optimal control problem will find the control function $u\left(\tau\right),\ t_0 \le \tau < t_f$, that will minimize equation (7.2.69) while satisfying the dynamic constraints in equation (7.2.66) and boundary conditions given in equations (7.2.67) and (7.2.68). Note that this problem is identical to the minimum time to the origin problem for the double integrator. As such the optimal feedback solution is well known and was given in the previous example problem. The optimal solution will consist of a surface in the state space that will reach the final condition given in equation (7.2.68) with the control $u\left(t\right)=T_{B,s},\ t_b \le t \le t_f$, which satisfies

$$y_1\left(t_b\right)=-\frac{1}{2}\cdot y_2^2\left(t_b\right)+d \tag{7.2.70}$$

Further, when the initial condition satisfies the constraint

$$y_1\left(t_0\right)<-\frac{1}{2}\cdot y_2^2\left(t_0\right)+d \tag{7.2.71}$$

the optimal control will be $u\left(t\right)=T_{A,s},\ t_0 \le t < t_b$. However, if the initial condition satisfies

$$y_1\left(t_0\right)>-\frac{1}{2}\cdot y_2^2\left(t_0\right)+d \tag{7.2.72}$$

Then the final condition is not reachable, and the optimal cost-to-go is infinite. Theoretically, a way to make the final condition reachable by the entire state space is for the model to allow the vehicle to overshoot the final vehicle position $y_1(t_b) = d$, put the vehicle in reverse, and back up to the final condition. However, this violates the underlying problem definition and will be excluded here.

Figure 7.18 depicts the region of the state-space that can reach the final condition.

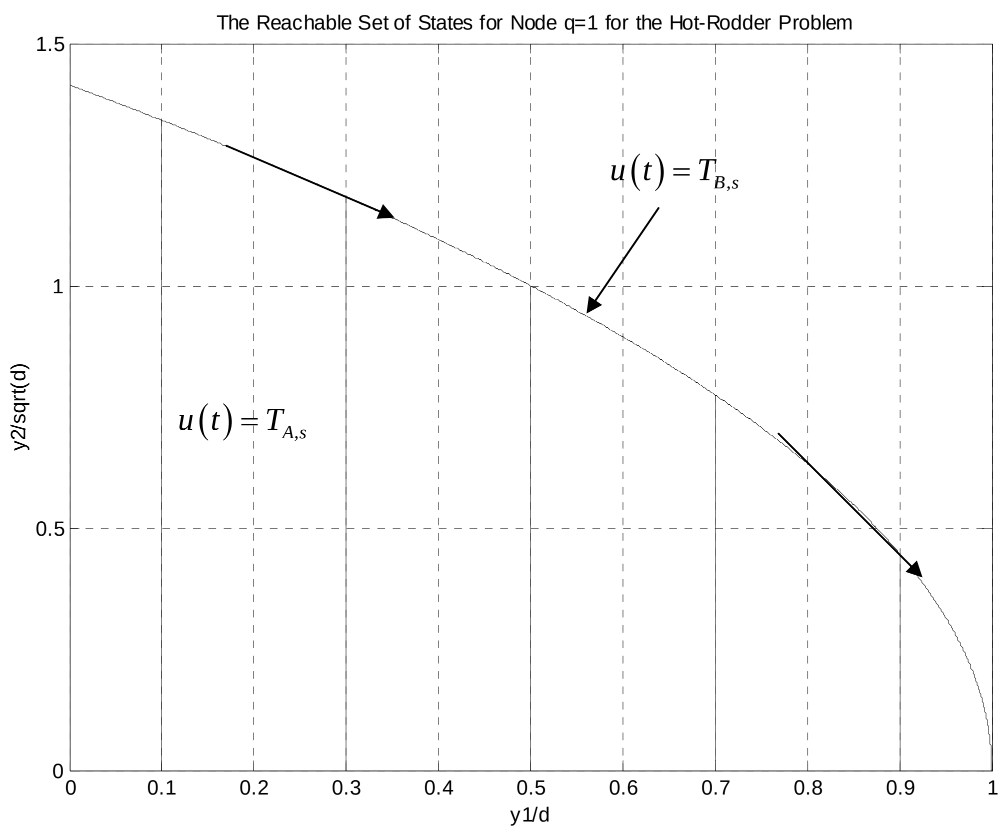

Figure 7.18: Set of states that can reach the final state for node $q = 1$.

The optimal control given above will be at most piecewise continuous, so the state trajectory will satisfy

$$
\begin{aligned}
y_1(t) &= y_1(t_0) + y_2(t_0)\cdot(t-t_0) + \frac{1}{2}\cdot\frac{T_{i,s}\cdot r}{I+m\cdot r^2}\cdot(t-t_0)^2 \\
y_2(t) &= y_2(t_0) + \frac{T_{i,s}\cdot r}{I+m\cdot r^2}\cdot(t-t_0)
\end{aligned}
\tag{7.2.73}
$$

for constant control $u(t) = T_{i,s}$, $i = A, B$, so the cost-to-go to the final condition can be calculated. First assume the state trajectory satisfies equation (7.2.70), then the cost-to-go to the final condition from any value along the braking surface is

$$
(t_f - t_b) = -\frac{y_2(t_b) + \sqrt{y_2^2(t_b) - 2\cdot\gamma_1\cdot y_1(t_b)}}{\beta_1}
\tag{7.2.74}
$$

where

$$
\beta_1 = \frac{T_{B,s}\cdot r}{I+m\cdot r^2} < 0
\tag{7.2.75}
$$

Further, if the state trajectory satisfies equation (7.2.71), then system is in the acceleration phase and cost-to-go to the braking surface is

$$
(t_b - t_0) = -\frac{y_2(t_0)}{\beta_2} + \frac{\sqrt{(\beta_2+1)\cdot\left(-2\cdot\beta_2\cdot y_1(t_0) + 2\cdot\beta_2\cdot d + y_2^2(t_0)\right)}}{\beta_2\cdot(\beta_2+1)}
\tag{7.2.76}
$$

where

$$
\beta_2 = \frac{T_{A,s}\cdot r}{I+m\cdot r^2} > 0
\tag{7.2.77}
$$

Finally, the cost-to-go to the final condition from any initial condition in the reachable portion of the state space will be

$$J_c^*\left(y_1\left(t_0\right), y_2\left(t_0\right)\right) = -\frac{\left(\sqrt{\beta_3} + \sqrt{\left(1-\beta_1\right)\cdot\beta_3}\right)}{\beta_1} - \frac{y_2\left(t_0\right)}{\beta_2} + \frac{\sqrt{\beta_3}}{\beta_2} \tag{7.2.78}$$

where

$$\beta_3 = \frac{-2\cdot\beta_2\cdot y_1\left(t_0\right) + 2\cdot\beta_2\cdot d + y_2^2\left(t_0\right)}{\beta_2 + 1} \tag{7.2.79}$$

The HJCB equation will not be evaluated here because the previous work proved that the proposed control is optimal. However equation (7.2.78) was derived because it is required for the analysis of nodes $q=2$ and $q=3$.

Node q=2/q=3 Analysis

Just as in the drag racer problem, the analysis of nodes $q=2$ and $q=3$ is very similar, so the analysis will be performed for node $q=2$ and the results will be presented for node $q=3$. The optimal control problem is very similar to the drag racing problem but will be given here for completeness.

When the system is in node $q=2$, the tire is slipping relative to the ground, $y_3\left(t_0\right) > 0$, and the dynamics of the system are given by

$$\begin{aligned}\dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= \frac{F_n}{m} \cdot \mu_{A,s} \\ \dot{y}_3(t) &= \frac{r \cdot T(t)}{I} - F_n \cdot \mu_{A,s} \cdot \left(\frac{I + m \cdot r^2}{m \cdot I} \right)\end{aligned} \tag{7.2.80}$$

with $T_{B,\max} \le T(t) \le T_{A,\max}$, and $\mu_{A,s}$ is positive and is the coefficient of sliding friction between the tire and the ground.

Assume the state trajectory starts at the initial condition

$$\begin{aligned}&0 \le y_1(t_0) < d \\ &y_2(t_0) \ge 0 \\ &y_3(t_0) > 0\end{aligned} \tag{7.2.81}$$

and moves to the final condition

$$\begin{aligned}&y_1(t_s) < d \\ &y_2(t_s) \le \sqrt{-2 \cdot \left(y_1(t_s) - d \right)} \\ &y_3(t_s) = 0\end{aligned} \tag{7.2.82}$$

with cost

$$J(x,u) = \int_{t_0}^{t_s} 1 \cdot dt + g\left(y_1(t_s), y_2(t_s) \right) \tag{7.2.83}$$

where

$$g\left(y_1(t_s), y_2(t_s) \right) = -\frac{\left(\sqrt{\beta_3} + \sqrt{(1-\beta_1) \cdot \beta_3} \right)}{\beta_1} - \frac{y_2(t_s)}{\beta_2} + \frac{\sqrt{\beta_3}}{\beta_2} \tag{7.2.84}$$

and

$$\beta_3 = \frac{-2 \cdot \beta_2 \cdot y_1(t_s) + 2 \cdot \beta_2 \cdot d + y_2^2(t_s)}{\beta_2 + 1} \tag{7.2.85}$$

Then the optimal control problem will be to find the control function $u(\tau),\ t_0 \leq \tau < t_s$, that minimizes the cost given in equation (7.2.84) satisfying the dynamic constraints given in equation (7.2.80) and boundary conditions in (7.2.81) and (7.2.82).

Also it will be assumed that $t_s < t_f$. Note that the restrictions on final conditions are made so that the state trajectory can reach the final conditions for the hybrid optimal control problem and the optimal control is unique.

The analysis for this problem is identical to that for the drag racer and the necessary conditions show that the candidate optimal control is $u(t) = T_{B,\max}$, and the associated cost-to-go function to the surface of final conditions is

$$J_{c,2}^{*}\left(y(t_0)\right) = -\frac{y_3(t_0)}{\gamma_2} - \frac{\left(\sqrt{\beta_3} + \sqrt{(1-\beta_1)\cdot\beta_3}\right)}{\beta_1} - \frac{y_2(t_s)}{\beta_2} + \frac{\sqrt{\beta_3}}{\beta_2} \tag{7.2.86}$$

where

$$\gamma_2 = \left(\frac{r \cdot T_{B,\max}}{I} - F_n \cdot \mu_{A,s} \cdot \left(\frac{I + m \cdot r^2}{m \cdot I}\right)\right) \tag{7.2.87}$$

It is obvious from equations (7.2.85) and (7.2.86) that the optimal cost-to-go is differentiable everywhere with respect to $y(t_0)$. Following the same method as the one used in the solution of the drag racer, $J^*_{c,2}(y(t_0))$ can be differentiated with respect to $y(t_0)$ and can be used to show the HJCB is equal to zero for all initial states. The mathematics for the hot-rodder problem are much more complicated than the drag racer, so the actual equations will not be given here. A symbolic solver can be used along with the equations (7.2.86) and those in the drag racer proof to prove the control is optimal. The proof of node $q = 3$ follows an identical process.

7.2.9 Conclusions

The solutions to the drag racer and hot-rodder problems were presented here through the application of hybrid modeling and analysis tools. The work in Chapter 6 was indirectly applied to the hybrid optimal control problem to prove the feedback controls presented were optimal. Any switching sequence of nodes that end with node $q = 1$ can be cast into a hybrid optimal control problem where the necessary and sufficient conditions given in Chapter 6 will prove that the control is optimal. Furthermore, the local values of the Hamiltonian, adjoint, and optimal cost-to-go functions will also be the same. The method given here used the same technique used to derive the hybrid necessary and sufficient conditions to solve the easier local optimal control problems and project the feedback solution backwards in time.

The real drag racer and hot-rodder problems are more complicated than the problems solved here. The wheel-to-ground interaction is unknown and changing, so the controller must estimate the torque at which the wheels start to spin. This may be simpler with these problems than in the ABS case because of the way drag racing is actually done. Two cars race together and the first car to cross the finish line advances to the next race. The competition is organized as a conventional tournament, thus a competitor can use the early races to estimate the maximum torque while running at a torque he or she knows is below the maximum, which is an example of an iterative learning control problem. The hot-rodder problem is similar except requiring the vehicle to be stopped at the final condition adds complexity to the system. Not only does the driver need to perform multiple runs to estimate the maximum accelerating and braking torque, the driver needs to estimate the reachable set of states so that the vehicle stops at the required point.

Chapter 8: Summary and Recommendations for Future Work

Hybrid systems are combinations of continuous and discrete time systems that are becoming more commonplace in the world today. With the miniaturization of computer systems it is becoming easier to embed them into controlled physical systems. A subset of control synthesis tools are provided by optimal control theory. In particular, the necessary conditions for optimality of control are given by various Maximum Principles and the sufficient conditions are given by using Bellman's Principle of Dynamic Programming and through evaluation of the non-smooth HJCB equation.

In this dissertation, analysis tools were developed for optimal feedback control synthesis of hybrid systems. The necessary and sufficient conditions for the hybrid problem were developed using a non-smooth form of the HJCB equation and viscosity solution theory. Viscosity solution theory provides enough generality so that the necessary and sufficient conditions can be applied to a large class of engineering problems. The non-smooth necessary and sufficient conditions were generalized to hybrid systems through a method that is similar to Bellman's Principle of Dynamic Programming. The switching sequence was defined for the entire run of the model and the final node was identified. A local optimal control problem was generated for the final node and the feedback solution was computed using the non-smooth necessary and sufficient conditions. Next a local optimal control problem was generated for the next to last node. The non-smooth necessary and sufficient conditions are used to compute the feedback control where the end-cost constraint for the local control problem is the cost-to-go to the final condition in the final node. This process is repeated until the feedback solution for the entire hybrid optimal problem is computed. Using this method, general necessary and sufficient conditions can be developed that will compute the feedback optimal control strategy for the hybrid problem. Two example problems were analyzed using this technique and the optimal feedback controls were computed. These two example problems provide a reference for applying the methods developed here and demonstrated the complexity associated with solving hybrid optimal control problems.

In order to continue the research presented in this dissertation several paths of research could be followed. First, the necessary and sufficient conditions could be generalized further by adding tools that allow for discontinuous value functions. The Viscosity solution work in [7], Chapter 5 provides one method for this generalization. Second, numerical algorithms that embody the hybrid optimal control tools presented in this dissertation could be developed. The advantage of the numerical algorithms is that they could provide a convenient means for calculating optimal feedback controls for complex hybrid problems. Solving these problems by hand can sometimes be done, but as demonstrated with the example problems in this dissertation, the analysis for simple problems can get very complicated. Third, the analysis tools in this dissertation could be expanded to allow for controlled discrete switching. The tools developed here only apply to problems where the switching occurs autonomously through surfaces in the state-space, but with some work, the tools could be expanded to allow for non-autonomous discrete switching. Non-autonomous discrete switching between nodes could be evaluated since the cost-to-go to the final condition is always known through the computation of the feedback optimal control.

Appendix A

Mathematical Preliminaries

Throughout the development of the analytical tools used to study the hybrid optimal control problem, several mathematical tools are going to be used to prove properties of the optimal control problems. This appendix will introduce the concepts of an admissible control and perturbation equations.

Admissible Controls

In order to develop the necessary conditions for optimal controls, the set of admissible controls is required. Loosely, the set of admissible controls is the set of all possible controls for which the optimal control problem is valid. All candidate control functions, $u(t) \in U$ for $t_0 \le t \le t_f$, must be admissible in order to be compared with a candidate for the optimal control.

[55], let D be some class of controls, then a control is admissible if it belongs to D, where D satisfies the following three conditions and t can take any value from the set $t_0 \le t \le t_f$:

1. All controls $u(t)$, for all t such that $t_0 \le t \le t_f$, which belong to D are measurable in t. Further the set of points $u(t)$, for every t such that $t_0 \le t \le t_f$,

which belong to D are bounded, which means the set of points has a compact closure in $\mathbb{R}^m$.

2. If $u(t)$ is admissible, v is an arbitrary point in the control region U, and t_1 and t_2 are numbers such that $t_0 \le t_1 \le t_2 \le t_f$, then the following control is also admissible (NOTE the $\wedge$ is the set "and" operator):

$$u^*(t) = \begin{cases} v, & t_1 \le t \le t_2 \\ u(t), & t_0 \le t \le t_1 \wedge t_2 \le t \le t_f \end{cases} \tag{9.1.1}$$

3. If the interval $t_0 \le t \le t_f$ is broken up into a finite number of subdivisions, where for each subdivision $u(t)$ is admissible, then the control $u(t)$ over the whole time interval is admissible.

Given the definitions above, the most general class of controls D which are admissible is the class of controls that are measurable and bounded. This class contains every other possible class of admissible controls as a subclass. The most specific class of admissible controls is the set of all piecewise constant controls and is contained in every other class of admissible controls.

Perturbation Equations

The purpose of this section is to develop a set of equations that describe a trajectory that is perturbed from a reference trajectory through variations in initial condition and variations in time. Because this section is fundamental to the development of the optimal control analytical tools, the perturbation theory will be developed in detail through a series of mathematical concepts.

The first mathematical concept that will be given is $o(\varepsilon)$. A function $f(\varepsilon)$, where $f:\mathbb{R}\to\mathbb{R}$, will be described as order $o(\varepsilon)$, or $f(\varepsilon)=o(\varepsilon)$, if the following relation is true

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\varepsilon}=0 \tag{9.1.2}$$

The definition of $o(\varepsilon)$ is a special case of the following definition given in [48] (pg. 113).

Definition 9.1.1:

A function $f(x)=o(g(x))$ as $x\to x_0$, if for every positive constant M

$$|f(x)|\le M\cdot|g(x)| \tag{9.1.3}$$

whenever x is sufficiently close to x_0.

The concept of $o(\varepsilon)$ is important in the first order approximation of a function because if the vector y is sufficiently close to the vector x,

$$y=x+\varepsilon\cdot\xi_0+o(\varepsilon) \tag{9.1.4}$$

where ε is a positive scalar sufficiently close to 0, ξ_0 is any vector, and the function f at x is once continuously differentiable, then the first order approximation to the function f at y can be computed using the Taylor Series Expansion of equation (9.1.4) about x

$$f(y) = f(x) + \frac{\partial f(x)}{\partial x} \cdot (\varepsilon \cdot \xi_0) + o(\varepsilon) \tag{9.1.5}$$

Equation (9.1.5) will be very useful in studying spatial and temporal perturbations of trajectories that are defined by some initial condition and dynamic equation.

A series of mathematical properties of continuous time systems will now be given. These properties form the basis of the perturbation equations that are used to derive the necessary conditions that the optimal control problems must satisfy. Proofs of these properties are included because books and papers on optimal control generally claim these properties are true but do not provide proofs, and the correct proofs are difficult to find in the literature. The first property is a precise definition of Lipschitz continuity.

Definition 9.1.2, [11]:

A vector valued function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz continuous in $x \in \mathbb{R}^n$ uniformly in $t \in \mathbb{R}$, if and only if for all $t \in [t_0, t_f]$ there exists a constant $h > 0$ such that

$$|f(x,t) - f(y,t)| \leq h \cdot |x(t) - y(t)| \tag{9.1.6}$$

for all $x, y \in \mathbb{R}^n$ and $t \in [t_0, t_f]$.

The next set of mathematical concepts, provide properties of solutions of differentiable functions.

Lemma 9.1.3, [11]:

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying the inequality

$$\dot{\sigma}(t) \leq K \cdot \sigma(t) \tag{9.1.7}$$

for all $t \in [t_0, t_f]$, where K is a constant. Then

$$\sigma(t) \leq \sigma(t_0) \cdot e^{K \cdot (t - t_0)} \tag{9.1.8}$$

for all $t \in [t_0, t_f]$.

Lemma 9.1.3 implies the next Corollary.

Corollary 9.1.4, [11]:

If $\sigma(t_0) = 0$ in Lemma 9.1.3 and $\sigma(t) \geq 0$, then $\sigma(t) = 0$.

The proof of Corollary 9.1.4 comes directly from Lemma 9.1.3.

Now that Lemma 9.1.3 and Corollary 9.1.4 have been given, they can be used to provide a uniqueness result on solutions to non-linear differential equations.

Theorem 9.1.5, Uniqueness of Solutions, [11]:

Assume that

$$\frac{d\left(x(t)\right)}{dt} = f\left(x(t),t\right) \tag{9.1.9}$$

where $f:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. If f is Lipschitz continuous in x uniformly in t for all $t \in \left[t_0, t_f\right]$, then there is at most one solution $x(t)$ to the differential equation given in (9.1.9), with initial condition $x(t_0) = c$.

Now that uniqueness of solutions to equation (9.1.9) has been given, an existence theorem can be presented that guarantees the existence of solutions to a differential equation.

Theorem 9.1.6, Existence of Solutions, [11]:

Suppose that the function $f\left(x(t),t\right)$, given in Theorem 9.1.5, is defined and continuous in the closed domain $|x-c| \le K$, $|t-t_0| \le T$, and satisfies a Lipschitz condition in the closed domain. Let $M = \sup\left|f\left(x(t),t\right)\right|$ in this closed domain, then equation (9.1.9) has a unique solution satisfying $x(t_0) = c$ that is defined on the smaller interval $|t-t_0| \le \min\left(T, \frac{K}{M}\right)$.

Now that existence and uniqueness of solutions of equation (9.1.9) have been given, continuity of the solutions can now be given.

Theorem 9.1.7, Continuity, [11]:

Let $x(t)$ and $y(t)$ be any two solutions to equation (9.1.9), where $f(x(t),t)$ is continuous and satisfies a Lipschitz condition with constant L. Then for all $t \in (t_0, t_f)$

$$|x(t) - y(t)| \leq e^{L \cdot |t - t_0|} \cdot |x(t_0) - y(t_0)| \tag{9.1.10}$$

Theorem 9.1.7 now implies a very important Corollary that proves the continuous nature of solutions on their initial conditions.

Corollary 9.1.8, Continuous Dependence of Solutions on Initial Conditions, [11]:

Let $x(t, x_0)$ be the solution to

$$\frac{d(x(t, x_0))}{dt} = f(x(t, x_0), t) \tag{9.1.11}$$

with initial condition $x(t_0, x_0)$. Assume that Theorem 9.1.7 is satisfied and let the functions $x(t, x_0)$ be defined for all x_0 and t_0, such that $|x_0 - x_0^0| \leq K$ and $|t - t_0| \leq T$.

Then

1. $x(t, x_0)$ is a continuous function of both variables.
2. if $x_0 \rightarrow x_0^0$, then $x(t, x_0) \rightarrow x(t, x_0^0)$ uniformly for $|t - t_0| \leq T$

Theorem 9.1.7 and Corollary 9.1.8, can be used to derive an equation that approximates the perturbation in a trajectory associated with a perturbation in initial condition.

Theorem 9.1.9, [11] (pg. 123):

Let the vector function f be of class C^1, where C^1 is the class of differentiable functions and the differential f is continuous, and let $x(t,x_0)$ be the solution to equation (9.1.11) with the initial condition $x(t_0,x_0)$. Then $x(t,x_0)$ is a differentiable function of the components of x_0.

Theorem 9.1.9 implies the following corollary.

Corollary 9.1.10 [11]:

If $x(t,x_0)$ is a solution to equation (9.1.11) with initial condition $x(t_0,x_0)=x_0$, and if each component of f is of class C^1, then $\frac{\partial x(t,x_0)}{\partial x_0}$ is a solution to the perturbation equation

$$\frac{d}{dt}\left(\frac{\partial x(t,x_0)}{\partial x_0}\right)=\frac{\partial f\left(x(t,x_0),t\right)}{\partial x}\cdot\frac{\partial x(t,x_0)}{\partial x_0} \tag{9.1.12}$$

Corollary 9.1.10 provides the framework for the derivation of the equation that describes the perturbation in a reference trajectory with respect to a variation in initial condition.

Theorem 9.1.11 [55]

Let $x(t)$, $t\in\left[t_0,t_f\right]$, be the solution to the differential equation

$$\dot{x}(t)=f\left(x,u,t\right) \tag{9.1.13}$$

with initial condition $x(t_0)$ and control $u(\tau)$, $t_0 \le \tau \le t$. Furthermore, let f be Lipschitz continuous in x uniformly in u and t for all $u(\tau)$, $t_0 \le \tau \le t$, and $t \in [t_0, t_f]$, and assume f is differentiable with respect to x along the trajectory $x(t)$.

If

$$y(t_0) = x(t_0) + \varepsilon \cdot \xi(t_0) + o(\varepsilon) \tag{9.1.14}$$

where $\varepsilon > 0$ and $\xi(t_0) \in \mathbb{R}^n$ is a constant vector, then for ε sufficiently small, the solution $y(t)$ to equation (9.1.13) with initial condition $y(t_0)$ and control function $u(\tau)$, $t_0 \le \tau \le t$, is

$$y(t) = x(t) + \varepsilon \cdot \delta x(t) + o(\varepsilon) \tag{9.1.15}$$

where $\delta x(t)$ is the solution to the differential equation

$$\frac{d(\delta x(t))}{dt} = \frac{\partial f(x,u,t)}{\partial x} \cdot \delta x(t) \tag{9.1.16}$$

with initial condition $\xi(t_0)$.

The next concept is to study the effect of a temporal variation of a reference trajectory that is defined by some initial condition and dynamic equation. Let $u(t)$, $a < t < b$, be an arbitrary measurable function on the open interval (a,b). A point, $t_r \in (a,b)$ is a regular point of the function $u(t)$, $a < t < b$, if the following relationship is satisfied for every neighborhood $O \subset U$ of $u(t_r)$ [55]:

$$\lim_{mes(I)\to 0} \frac{mes\left(u^{-1}(O)\cap I\right)}{mes(I)} = 1 \qquad (9.1.17)$$

where I is an arbitrary interval that contains t_r, mes is the Lebesgue measure, and $u^{-1}(O)$ is the set of points $t \in (a,b)$ for which $u(t) \in O$. Note that points, t, where $u(t)$ is either continuous or has isolated jumps are regular points. A simple example of a point t that would not be a regular point of $u(t)$ would be if there is an interval of time $Q = [t, t_1]$ where $u(t)$ doesn't exist. Since an arbitrary interval I exists where the set $u^{-1}(O)\cap I$ is bigger than the set I, the limit in equation (9.1.17) is not satisfied.

The definition of a regular point provides the following proposition:

Proposition 9.1.12

Let $u(t)$ be an arbitrary measurable function on the interval $a < t < b$, then almost every point of the interval $a < t < b$ is a regular point for the function $u(t)$.

The reader is referred to the references note on page 77 of [67] for the proof of the proposition.

Now the first order expansion approximation of an integral equation with respect to a temporal variation can be given. Let $g\left(u\left(t\right),t\right)$ be a real continuous function of $t \in \left(a,b\right)$ and $u \in U$, and $u\left(t\right)$, $a < t < b$ is a bounded measurable function where $u\left(t\right) \in U$. Then if t_r is a regular point of $u\left(t\right)$, the first order Taylor Series Expansion can be written as

$$\int_{t_r+\varepsilon p}^{t_r+\varepsilon q} g\left(u\left(t\right),t\right)\cdot dt = \varepsilon \cdot \left(q-p\right)\cdot g\left(u\left(t_r\right),t_r\right)+o\left(\varepsilon\right) \qquad (9.1.18)$$

where p and q are arbitrary real numbers [55].

Theorem 9.1.11 and Proposition 9.1.12 are the two tools that are essential in deriving the Maximum Principle. These two tools provide a means of describing the first order variation in the reference trajectory associated with temporal and spatial variations. Because the variation in the reference trajectory associated with a variation in control, is specific to the type of variation in the control, it will not be given here, but presented in Maximum Principle section.

Appendix B

Geometric Concepts

The second concept that is required for the development of the work in this dissertation is geometric properties of smooth functions and some geometric properties of a system of equations will be defined. The definitions can all be found in [55].

Define a hypersurface in $X \subseteq \mathbb{R}^n$ to be the set, S, of all points that satisfy the relation:

$$f\left(x\right)=0 \tag{10.1.1}$$

where $f:\mathbb{R}^n \to \mathbb{R}$. All $x \in S$ such that

$$\frac{\partial f\left(x\right)}{\partial x_1}=0,\ldots,\frac{\partial f\left(x\right)}{\partial x_n}=0 \tag{10.1.2}$$

are singular points of x and all $x \in S$ such that

$$\frac{\partial f\left(x\right)}{\partial x_1}\neq 0,\ldots,\frac{\partial f\left(x\right)}{\partial x_n}\neq 0 \tag{10.1.3}$$

are non-singular points. Next define a smooth hypersurface as any hypersuface such that $f\left(x\right)$ is continuously differentiable with respect to x for all $x \in S$ and contains no singular points. Furthermore, if equation (10.1.1) is linear, then the hypersurface is called a hyperplane.

Now, if x_0 is an arbitrary point of a smooth hypersurface, S, defined by equation (10.1.1), then the gradient of f at x_0 is a normal vector of S at point x_0. If S is a smooth hypersurface with x_0 as one of its points and has a normal vector at x_0, then the hyperplane formed by adding the vector x_0 to any vector perpendicular to the normal vector is the tangent hyperplane to S at x_0. Any vector that lies in the tangent hyperplane and emanates from x_0 is a tangent vector.

Next, let $S_1, \ldots, S_k$ be smooth hypersurfaces defined by the following k equations:

$$\begin{gathered} f_1(x) = 0 \\ \vdots \\ f_k(x) = 0 \end{gathered} \tag{10.1.4}$$

The intersection M of all of the hypersurfaces is called an $(n-k)$-smooth manifold in X if all $x \in M$ satisfy every equation in equation (10.1.4) and the vectors $\nabla f_1(x), \ldots, \nabla f_k(x)$ are linearly independent for all $x \in M$ (∇ is the gradient operator) which is equivalent to the matrix

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k(x)}{\partial x_1} & \cdots & \frac{\partial f_k(x)}{\partial x_n} \end{bmatrix} \tag{10.1.5}$$

having full rank (i.e. rank of k).

If equations (10.1.4) are linear, then M is called an $(n-k)$-dimensional hyperplane of the space X . Let L_i be the tangent hyperplanes of their corresponding hypersurfaces S_i at point x . The intersection of the L_i , $i = 1 \dots k$, is an $(n-k)$-dimensional hyperplane defined as the tangent hyperplane of M at x . Similarly, a tangent vector to M is any vector emanating from x that lies in the tangent plane with the added constraint that it must be orthogonal to all of the gradient vectors $\nabla f_1(x), \dots, \nabla f_k(x)$.

Finally, let $x_0 \in M$ be some arbitrary point on manifold M . A vector emanating from x_0 is a tangent vector of M if and only if it is tangent to any smooth curve which lies in M and passes through x_0 .

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