ABSTRACT

Title of Thesis: FUSION FRAMES AND APPLICATIONS

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Fusion frames are emerging mathematical tools that provide a natural framework for performing data processing. In this article, we will discuss some properties of fusion frames and will show that they provide an extensive framework for modeling sensor networks.

FUSION FRAMES AND APPLICATIONS

by

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1. Introduction

Frames for a Hilbert space were first introduced by R. J. Duffin and A. C. Schaeffer and were used as a tool in the study of nonharmonic Fourier series [10]. The idea of Duffin and Schaeffer did not seem to attract the attention of people except in signal processing. In 1986, frames came to a turning point. In [9], Daubechies, Grossmann and Meyer pointed out that frames would be a useful and powerful tool in many areas of theoretical physics and applied mathematics. Since then the theory of frames began to be studied more widely. Because frames provide robust, stable, and non-unique representations of vectors, the theory of frames has grown rapidly during the last two decades with the development of several new applications where redundancy plays an important role (e.g., filter bank theory, sigma-delta quantization, signal and image processing, and wireless communications).

However, a number of new applications have emerged which can hardly be modeled naturally by one single frame system. Generally they share a common property that requires distributed processing. To handle these applications new approaches needed to be developed. In [6], Casazza and Kutyniok studied redundant subspaces for the purpose of easing the construction of frames by building them locally in (redundant) subspaces and then piecing them together by employing a special structure of the set of subspaces. This was referred to as a *frame of subspaces.* They realized that the idea could be far more reaching than that of building large frames from smaller local ones. The weighted and coherent subspace combination in such a mechanism is exactly what was needed in distributed and parallel processing for many fusion applications as mentioned above. They decided on a terminology of *fusion frames* since it reflected much more precisely the essence of the system studied and its applications. Some basic theory of *fusion frames* is studied in [6].

The organization of this article is as follows. The definition and basic properties of fusion frames and fusion frame systems will be given and relations between fusion frames and conventional frames will be studied in Section 2. In Section 3, Parseval fusion frames are presented. Section 4 is devoted to the sensor networks as an application of fusion frames.

2. Definitions and Basic Properties of Fusion Frames

Throughout this paper $\mathcal{H}$ is assumed to be a separable Hilbert space.

2.1. **Review of Frames.** A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$, if there exist $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2. \tag{1}$$

The constatns A and B are called a *lower* and *upper frame bound,* respectively. Those sequences which satisfy only the upper inequality in (1) are called *Bessel sequences.* A frame is *tight,* if $A = B$. If $A = B = 1$, it is called a *Parseval frame.* We call a frame $\{f_i\}_{i\in I}$ *uniform* (or *equal norm*), if we have $\|f_i\| = \|f_j\|$ for all $i, j \in I$. A frame is *exact,* if it ceases to be a frame whenever any single element is deleted from the sequence $\{f_i\}_{i\in I}$.

In the theory of frame an input signal is represented by a collection of scalar coefficients. The representation space employed is $\ell^2(I)$. In order to analyze a signal $f \in \mathcal{H}$, the *analysis operator* $T_{\mathcal{F}} : \mathcal{H} \to \ell^2(I)$ *given by* $T_{\mathcal{F}} f = \{\langle f, f_i\rangle\}_{i\in I}$ is applied. The associated *synthesis operator* is defined to be the adjoint operator $T^*_{\mathcal{F}} : \ell^2(I) \to \mathcal{H}$ by $T^*_{\mathcal{F}}(c) = \sum_{i\in I} c_i f_i$ for each sequence of scalars $c = \{c_i\}_{i\in I} \in \ell^2(I)$. By composing $T_{\mathcal{F}}$ and $T^*_{\mathcal{F}}$ we obtain the *frame operator* $S_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}$, $S_{\mathcal{F}} f = T^*_{\mathcal{F}} T_{\mathcal{F}} f = \sum_{i\in I}\langle f, f_i\rangle f_i$, which is a positive, self-adjoint and invertible operator. This provides the reconstruction formula

$$(2) \qquad f = S^{-1}_{\mathcal{F}} S_{\mathcal{F}}(f) = \sum_{i\in I}\langle f, f_i\rangle \tilde{f}_i = \sum_{i\in I}\langle f, \tilde{f}_i\rangle f_i \ \text{ for all } f \in \mathcal{H},$$

where $\tilde{f}_i = S^{-1}_{\mathcal{F}}(f_i)$. The sequence $\{\tilde{f}_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame* of $\{f_i\}_{i\in I}$. When $\mathcal{F}$ is a redundant frame, there exist infinitely many dual frames.

2.2. **Fusion Frames and Fusion Frame Systems.** The definition of a fusion frame shares many of the properties of frames, and thus can be viewed as a generalization of frames.

Definition 2.1. Let I be a countable index set, and let $\{W_i\}_{i\in I}$ be a family of closed subspaces $\{W_i\}_{i\in I}$ in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a *fusion frame,* if there exist constants $0 < C \le D < \infty$ such that

$$(3) \qquad C\|f\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \le D\|f\|^2 \ \text{ for all } f \in \mathcal{H},$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. We call C and D the *fusion frame bounds.* The family $\{(W_i, v_i)\}_{i\in I}$ is called a *C-tight fusion frame,* if in (3) the constants C and D can be chosen so that $C = D$, a *Parseval fusion frame* provided that $C = D = 1$ and an *orthonormal fusion basis* if $\mathcal{H} = \bigoplus_{i\in I} W_i$. Moreover, we call a fusion frame $\{(W_i, v_i)\}_{i\in I}$ *v-uniform,* if $v = v_i = v_j$ for all $i, j \in I$. We call $\{(W_i, v_i)\}_{i\in I}$ a *Bessel fusion sequence* with *Bessel fusion bound D,* if we only have the upper bound.

We need to consider a set of local frames and string together frames for each of subspaces to get a frame for the whole space.

Definition 2.2. Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i for each $i \in I$. Then we call $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ *a fusion frame system* for $\mathcal{H}$. C and D are the associated *fusion frame bounds* if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i\in I}$, and A and B are the *local frame bounds* if these are the common frame bounds for the *local frames* $\{f_{ij}\}_{j\in J_i}$for each $i \in I$. A collection of dual frames $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$ associated with the local frames will be called *local dual frames.*

The following theorem provides relations between properties of the associated fusion frame and the sequence of all local frame vectors.

Theorem 2.3 ([6], Theorem 3.2)**.** *For each $i \in I$, let $v_i > 0$, let W_i be a closed subspace of $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i with frame bounds A_i and B_i. Suppose that $0 < A = \inf_{i\in I} A_i \leq B = \sup_{i\in I} B_i < \infty$. Then the following conditions are equivalent.*

(i) $\{(W_i, v_i)\}_{i\in I}$ *is a fusion frame for $\mathcal{H}$.*
(ii) $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ *is a frame for $\mathcal{H}$.*

Furthermore, if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D, then $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for $\mathcal{H}$ with frame bounds AC and BD. Also if $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for $\mathcal{H}$ with frame bounds C and D, then $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

Proof. First, assume that $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D, and let $f \in \mathcal{H}$. Then by the definition of a fusion frame, we get (3). Since for each $i \in I$, $\{f_{ij}\}_{j\in J_i}$ is a frame for W_i with frame bounds A_i and B_i and $\pi_{W_i}(f) \in W_i$, we obtain

$$A_i \|\pi_{W_i}(f)\|^2 \leq \sum_{j\in J_i} |\langle \pi_{W_i}(f), f_{ij}\rangle|^2 \leq B_i \|\pi_{W_i}(f)\|^2 \tag{4}$$

and by (4)

$$\begin{aligned} &A \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq \sum_{i\in I} A_i v_i^2 \|\pi_{W_i}(f)\|^2 \leq \sum_{i\in I} v_i^2 \sum_{j\in J_i} |\langle \pi_{W_i}(f), f_{ij}\rangle|^2 \\ &= \sum_{i\in I} \sum_{j\in J_i} |\langle \pi_{W_i}(f), v_i f_{ij}\rangle|^2 \leq \sum_{i\in I} B_i v_i^2 \|\pi_{W_i}(f)\|^2 \leq B \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2. \end{aligned} \tag{5}$$

Since π_{W_i} is the orthogonal projection onto W_i, we have

$$\sum_{i\in I} \sum_{j\in J_i} |\langle \pi_{W_i}(f), v_i f_{ij}\rangle|^2 = \sum_{i\in I} \sum_{j\in J_i} |\langle f, \pi_{W_i}(v_i f_{ij})\rangle|^2 = \sum_{i\in I} \sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2. \tag{6}$$

Therefore, by (3), (5) and (6), we have

$$AC\|f\|^2 \leq \sum_{i\in I} \sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 \leq BD\|f\|^2.$$

Conversely, assume that $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds C and D, and let $f \in \mathcal{H}$. Then we have

$$C\|f\|^2 \le \sum_{i\in I}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 \le D\|f\|^2.$$

Since for each $i \in I$, $\{f_{ij}\}_{j\in J_i}$ is a frame for W_i with frame bounds A_i and B_i, and $\pi_{W_i}(f) \in W_i$, we get

$$\begin{aligned} A\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2 &\le \sum_{i\in I} v_i^2 A_i\|\pi_{W_i}(f)\|^2 \le \sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{W_i}(f), v_i f_{ij}\rangle|^2 \\ &= \sum_{i\in I}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 \le D\|f\|^2 \end{aligned}$$

and

$$\begin{aligned} C\|f\|^2 &\le \sum_{i\in I}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{W_i}(f), v_i f_{ij}\rangle|^2 \\ &\le \sum_{i\in I} v_i^2 B_i\|\pi_{W_i}(f)\|^2 \le B\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2. \end{aligned}$$

These imply

$$\frac{C}{B}\|f\|^2 \le \sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2 \le \frac{D}{A}\|f\|^2.$$

Therefore, $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$. □

2.3. **Fusion Frame Operators.** In the theory of fusion frame an input signal is represented by a collection of vector coefficients that represent the projection onto each subspace. For the definition of operators for a fusion frame, we need the following notation and Lemma 2.4.

For each family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$, we define the space $(\sum_{i\in I}\oplus W_i)_{\ell_2}$ by

$$\left(\sum_{i\in I}\oplus W_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in I} | f_i \in W_i \text{ and } \sum_{i\in I}\|f_i\|^2 < \infty\right\}$$

with inner product given by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle.$$

Lemma 2.4. *Let $\{(W_i, v_i)\}_{i\in I}$ be a Bessel fusion sequence for $\mathcal{H}$. Then, for each sequence $\{f_i\}_{i\in I} \in (\sum_{i\in I}\oplus W_i)_{\ell_2}$, the series $\sum_{i\in I} v_i f_i$ converges unconditionally.*

Proof. Let $f = \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{\ell^2}$. Fix $J \subset I$ with $|J| < \infty$ and let $g = \sum_{i\in J} v_i f_i$. Then since $f_i \in W_i$, we have

$$\|\sum_{i\in J} v_i f_i\|^4 = \left(\langle g, \sum_{i\in J} v_i f_i\rangle\right)^2 = \left(\sum_{i\in J} v_i \langle \pi_{W_i}(g), f_i\rangle\right)^2 \leq \left(\sum_{i\in J} v_i \|\pi_{W i}(g)\|\|f_i\|\right)^2$$

$$\leq \sum_{i\in J} v_i^2 \|\pi_{W_i}(g)\|^2 \sum_{i\in J} \|f_i\|^2 \leq D\|g\|^2 \sum_{i\in J} \|f_i\|^2 \leq D\|\sum_{i\in J} v_i f_i\|^2 \|f\|^2.$$

Hence,

$$\|\sum_{i\in J} v_i f_i\|^2 \leq D\|f\|^2.$$

It follows that $\sum_{i\in I} v_i f_i$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$ ([13], page 392, Theorem 4.3.12). □

Definition 2.5. Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. Then the *analysis operator*

$$T_{\mathcal{W}} : \mathcal{H} \longrightarrow \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2}$$

is defined by

$$T_{\mathcal{W}}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}.$$

By the definition of the analysis operator $T_{\mathcal{W}}$, we can easily show that the *synthesis operator* $T^*_{\mathcal{W}}$, which is defined to be the adjoint operator, is given by

$$T^*_{\mathcal{W}} : \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} \to \mathcal{H} \text{ with } T^*_{\mathcal{W}}(f) = \sum_{i\in I} v_i f_i \text{ for all } f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2}.$$

Theorem 2.6. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$.*
(ii) *$T_{\mathcal{W}}$ is an isomorphism.*
(iii) *$T^*_{\mathcal{W}}$ is bounded, linear and onto.*

Proof. (i) ⇔ (ii): Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ and let $f \in \mathcal{H}$. Then by the definition of $T_{\mathcal{W}}$, we have

$$\|T_{\mathcal{W}}(f)\|^2 = \|\{v_i \pi_{W_i}(f)\}_{i\in I}\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2. \tag{7}$$

Since $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$, there is a constant $M > 0$ such that

$$\frac{1}{M}\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \|T_{\mathcal{W}}(f)\|^2 \leq M\|f\|^2.$$

Therefore $T_{\mathcal{W}}$ is an isomorphism.
Conversely, suppose $T_{\mathcal{W}}$ is an isomorphism. Then this claim follows immediately by (7).
(ii) $\Leftrightarrow$ (iii) holds in general for each operator on a Hilbert space. □

Definition 2.7. Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. Then the *fusion frame operator* $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(f) = T^*_{\mathcal{W}} T_{\mathcal{W}}(f) = T^*_{\mathcal{W}}(\{v_i \pi_{W_i}(f)\}_{i\in I}) = \sum_{i\in I} v_i^2 \pi_{W_i}(f).$$

Proposition 2.8. *Let $\{(W_i, v_i, \mathcal{F}_i = \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$, and let $\widetilde{\mathcal{F}_i} = \{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$ be associated local dual frames. Then the associated fusion frame operator $S_{\mathcal{W}}$ can be written as*

$$S_{\mathcal{W}} = \sum_{i\in I} v_i^2 T^*_{\widetilde{\mathcal{F}_i}} T_{\mathcal{F}_i} = \sum_{i\in I} T^*_{\mathcal{F}_i} T_{\widetilde{\mathcal{F}_i}}.$$

Proof. Since $\pi_{W_i}(f) \in W_i$, $\pi_{W_i}(f) = \sum_{j\in J_i} \langle \pi_{W_i}(f), f_{ij}\rangle \tilde{f}_{ij}$. Therefore, for all $f \in \mathcal{H}$ we have

$$S_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2 \pi_{W_i}(f) = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, f_{ij}\rangle \tilde{f}_{ij} = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, \tilde{f}_{ij}\rangle f_{ij}.$$

Since $T_{\mathcal{F}_i}(f) = \{\langle f, f_{ij}\rangle\}_{j\in J_i}$ and $T^*_{\widetilde{\mathcal{F}_i}}(\{\langle f, f_{ij}\rangle\}_{j\in J_i}) = \sum_{j\in J_i} \langle f, f_{ij}\rangle \tilde{f}_{ij}$, the result immediately. □

Proposition 2.8 provides that the fusion frame operator can be expressed in terms of local frame operators $T_{\mathcal{F}_i}$ and $T^*_{\widetilde{\mathcal{F}_i}}$ or $T^*_{\mathcal{F}_i}$ and $T_{\widetilde{\mathcal{F}_i}}$.

Proposition 2.9. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with frame bounds C and D. Then the fusion frame operator $S_{\mathcal{W}}$ is a positive, self-adjoint, invertible operator on $\mathcal{H}$ with*

$$C \cdot Id \leq S_{\mathcal{W}} \leq D \cdot Id. \tag{8}$$

Proof. For any $f \in \mathcal{H}$, we have

$$\langle S_{\mathcal{W}}(f), f\rangle = \langle \sum_{i\in I} v_i^2 \pi_{W_i}(f), f\rangle = \sum_{i\in I} v_i^2 \langle \pi_{W_i}(f), f\rangle = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2,$$

which implies that $S_{\mathcal{W}}$ is a positive operator. We further compute

$$\langle Cf, f\rangle = C\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \langle S_{\mathcal{W}}(f), f\rangle \leq \langle Df, f\rangle.$$

This shows that $C \cdot Id \leq S_{\mathcal{W}} \leq D \cdot Id$ and hence $S_{\mathcal{W}}$ is an invertible operator on $\mathcal{H}$. Furthermore, for any $f, g \in \mathcal{H}$ we have

$$\langle S_{\mathcal{W}}(f), g\rangle = \sum_{i\in I} v_i^2 \langle \pi_{W_i}(f), g\rangle = \sum_{i\in I} v_i^2 \langle f, \pi_{W_i}(g)\rangle = \langle f, S_{\mathcal{W}}(g)\rangle.$$

Thus $S_{\mathcal{W}}$ is self-adjoint. □

This provides the reconstruction formula from the collected and preprocessed data $\pi_{W_i}(f)$, $i \in I$.

Proposition 2.10. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame operator $S_{\mathcal{W}}$ and fusion frame bounds C and D. Then we have the reconstruction formula*

$$f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i}(f) \quad \textit{for all } f \in \mathcal{H}.$$

Proof. Since $S_{\mathcal{W}}$ is invertible, for all $f \in \mathcal{H}$ we have

$$f = S_{\mathcal{W}}^{-1} S_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i}(f).$$

□

2.4. **Analysis of the Fusion Frame Bounds.** Since the exact values of the fusion frame bounds will be important for determining the rate of convergence for reconstruction algorithms, we will show how to compute the optimal fusion frame bounds.

Theorem 2.11. *Let $\{W_i\}_{i\in I}$ be closed subspaces in $\mathcal{H}$, let $\{v_i\}_{i\in I}$ be positive numbers, and let $S_{\mathcal{W}}$ denote the fusion frame operator associated with $\{(W_i, v_i)\}_{i\in I}$. Then the following conditions are equivalent.*

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a fusion frame with fusion frame bounds C and D.*
(ii) *We have $C \cdot Id \le S_{\mathcal{W}} \le D \cdot Id$.*

Proof. (i) implies (ii) by (8).
Now assume (ii). Let $T_{\mathcal{W}}$ denote the analysis operator associated with $\{(W_i, v_i)\}_{i\in I}$. Since $S_{\mathcal{W}} = T_{\mathcal{W}}^* T_{\mathcal{W}}$ and hence $\|S_{\mathcal{W}}\| = \|T_{\mathcal{W}}^* T_{\mathcal{W}}\| = \|T_{\mathcal{W}}\|^2$, for any $f \in \mathcal{H}$ we obtain

$$\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \|T_{\mathcal{W}}(f)\|^2 \le \|T_{\mathcal{W}}\|^2 \|f\|^2 = \|S_{\mathcal{W}}\| \|f\|^2 \le D\|f\|^2.$$

Also, for all $f \in \mathcal{H}$,

$$\|T_{\mathcal{W}}(f)\|^2 = \langle T_{\mathcal{W}}^* T_{\mathcal{W}}(f), f\rangle = \langle S_{\mathcal{W}}(f), f\rangle = \langle S_{\mathcal{W}}^{\frac{1}{2}}(f), S_{\mathcal{W}}^{\frac{1}{2}}(f)\rangle = \|S_{\mathcal{W}}^{\frac{1}{2}}(f)\|^2 \ge C\|f\|^2.$$

□

3. Parseval Fusion Frames

Parseval frames play an important role in abstract frame theory, since they are extremely useful for applications. Parseval fusion frames are also of particular importance due to their advantageous reconstruction properties. Therefore in this section we study characterizations of Parseval fusion frames.

Corollary 3.1. *For each $i \in I$ let $v_i > 0$, let W_i be a closed subspace of $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a Parseval frame sequence in $\mathcal{H}$. Then the following conditions are equivalent.*

(i) $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ *is a Parseval frame for $\mathcal{H}$.*
(ii) $\{(W_i, v_i)\}_{i\in I}$ *is a Parseval fusion frame for $\mathcal{H}$.*

Proof. This follows immediately from Theorem 2.3 □

We can also characterize Parseval fusion frames in terms of their frame operators in a similar manner as in frame theory.

Proposition 3.2. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(i) $\{(W_i, v_i)\}_{i\in I}$ *is a Parseval fusion frame for $\mathcal{H}$.*
(ii) $S_{\mathcal{W}} = Id$.

Proof. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. By Proposition 2.9, (i) implies (ii). Conversely, suppose that $S_{\mathcal{W}} = Id$. Then for all $f \in \mathcal{H}$ we have

$$f = S_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2 \pi_{W_i}(f) = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle \pi_{W_i}(f), e_{ij}\rangle e_{ij} = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, e_{ij}\rangle e_{ij}.$$

This yields

$$\|f\|^2 = \left\langle \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, e_{ij}\rangle e_{ij}, f \right\rangle = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2.$$

□

Proposition 3.3. *Let $\mathcal{W}_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(i) $\{W_i\}_{i\in I}$ *is an orthonormal basis of subspaces for $\mathcal{H}$.*
(ii) $\{W_i, 1\}_{i\in I}$ *is a 1-uniform Parseval fusion frame for $\mathcal{H}$.*

Proof. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i.
(i)⇒(ii). Suppose that $\{W_i\}_{i\in I}$ is an orthonormal basis of subspaces for $\mathcal{H}$. Then $\{e_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\mathcal{H}$. This implies

$$\|f\|^2 = \langle \sum_{i\in I} \sum_{j\in J_i} \langle f, e_{ij}\rangle e_{ij}, f\rangle = \sum_{i\in I} \sum_{j\in J_i} |\langle f, e_{ij}\rangle|^2$$

and since $\pi_{W_i}(f) \in W_i$, we have

$$\|\pi_{W_i}(f)\|^2 = \langle \pi_{W_i}(f), f\rangle = \langle \sum_{j\in J_i} \langle f, e_{ij}\rangle e_{ij}, f\rangle = \sum_{j\in J_i} |\langle f, e_{ij}\rangle|^2$$

for all $f \in \mathcal{H}$. Therefore,

$$\|f\|^2 = \sum_{i\in I} \|\pi_{W_i}(f)\|^2.$$

This implies (ii).
(ii)⇒(i). Suppose that (ii) holds. Then by the above equation, for all $f \in \mathcal{H}$ we have

$$\|f\|^2 = \sum_{i\in I} \|\pi_{W_i}(f)\|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle e_{ij}, f\rangle|^2$$

and $\|e_{ij}\| = 1$ for all $i \in I$, $j \in J_i$, which shows that $\{e_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\mathcal{H}$. This implies (i). □

Definition 3.4. A family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$ is called *complete,* if

$$\overline{\text{span}}_{i\in I}\{W_i\} = \mathcal{H}.$$

The following theorem gives a characterization of Parseval fusion frames in terms of projections from larger spaces.

Theorem 3.5. *For a complete family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$ and positive weights $\{v_i\}_{i\in I}$, the following conditions are equivalent.*

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$.*
(ii) *There exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$, an orthonormal basis $\{e_j\}_{j\in J}$ for $\mathcal{K}$, a partition of $\{J_i\}_{i\in I}$ of J, and isometries $L_i : \text{span}_{j\in J_i}\{e_j\} \to W_i$, $i \in I$ such that*

$$P = \sum_{i\in I} v_i L_i$$

is an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

Proof. (i)⇒(ii). For every $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Since $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$, by Corollary 3.1, $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathcal{H}$. By Theorem 3.4 in [3], there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with an orthonormal basis $\{\widetilde{e}_{ij}\}_{i\in I, j\in J_i}$ so that the orthogonal projection P of $\mathcal{K}$ onto $\mathcal{H}$ satisfies

$$P(\widetilde{e}_{ij}) = v_i e_{ij}, \quad i \in I, j \in J_i.$$

Setting $K_i = \text{span}_{j\in J_i}\{\widetilde{e}_{ij}\}$, the map

$$L_i = \frac{1}{v_i} P|_{K_i} : K_i \to W_i$$

is an isometry for all $i \in I$, and

$$P = \sum_{i\in I} v_i L_i$$

is an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.
(ii)⇒(i). Since $P = \sum_{i\in I} v_i L_i$ is an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$,

$\{Pe_j\}_{j\in J}$ is a Parseval frame for $\mathcal{H}$. Further, since $L_i := v_i^{-1}P|_{K_i} : K_i \to W_i$ is an isometry, it follows that $\{v_i^{-1}Pe_j\}_{j\in J_i}$ is an orthonormal basis for W_i, $i \in I$. Applying these observations and denoting by P_i the orthogonal projection onto W_i, for all $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I} v^2\|P_i f\|^2 &= \sum_{i\in I} v_i^2 \Big\| \sum_{j\in J_i} \Big\langle f, \frac{1}{v_i}Pe_j\Big\rangle \frac{1}{v_i}Pe_j\Big\|^2 \\ &= \sum_{i\in I} v_i^2 \sum_{j\in J_i} \Big|\Big\langle f, \frac{1}{v_i}Pe_j\Big\rangle\Big|^2 \\ &= \sum_{i\in I}\sum_{j\in J_i} |\langle f, Pe_j\rangle|^2 \\ &= \|f\|^2.\end{aligned}$$

Thus $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame. □

Corollary 3.6. *For a family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$ and positive weights $\{v_i\}_{i\in I}$, the following conditions are equivalent.*

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$.*

(ii) *There exists a Parseval frame $\{e_{ij}\}_{i\in I, j\in J_i}$ for $\mathcal{H}$ such that $\{v_i^{-1}e_{ij}\}_{j\in J_i}$ is an orthonormal basis for W_i for all $i \in I$.*

Lemma 3.7. *Let W_1, W_2 be closed non-trivial subspaces of $\mathcal{H}$, and let v_1, $v_2 > 0$. The following conditions are equivalent.*

(i) *$\{(W_1.v_1), (W_2, v_2)\}$ is a Parseval fusion frame for $\mathcal{H}$.*

(ii) *Either we have $W_1 \perp W_2$ and $v_1 = v_2 = 1$ or we have $W_1 = W_2 = \mathcal{H}$ and $v_1^2 + v_2^2 = 1$.*

Proof. (i)⇒ (ii). First we assume that $W_2 \neq \mathcal{H}$. Fix some $g \perp W_2$. Then, by (i),

$$\|g\|^2 = v_1^2\|\pi_{W_1}(g)\|^2 + v_2^2\|\pi_{W_2}(g)\|^2 = v_1^2\|\pi_{W_1}(g)\|^2 \leq v_1^2\|g\|^2,$$

hence $v_1^2 \geq 1$. But for all $f \in W_1$, we have

$$\|f\|^2 = v_1^2\|f\|^2 + v_2^2\|\pi_{W_2}(f)\|^2 \geq v_1^2\|f\|^2.$$

This implies $v_1^2 \leq 1$, and therefore $v_1 = 1$. Now for all $f \in W_1$,

$$\|f\|^2 = \|f\|^2 + v_2^2\|\pi_{W_2}(f)\|^2.$$

This shows that $W_2 \perp W_1$, and $v_2 = 1$ follows immediately.

If $W_2 = \mathcal{H}$, towards a contradiction assume that $W_1 \neq \mathcal{H}$. Fix $g \perp W_1$. Then

$$\|g\|^2 = v_1^2\|\pi_{W_1}(g)\|^2 + v_2^2\|\pi_{W_2}(g)\|^2 = v_2^2\|g\|^2,$$

hence $v_2^2 = 1$. Now for $g \in W_1$, we obtain

$$\|g\|^2 = v_1^2\|\pi_{W_1}(g)\|^2 + v_2^2\|\pi_{W_2}(g)\|^2 = (v_1^2 + 1)\|g\|^2.$$

But this can only be true if $v_1 = 0$, a contradiction. Thus $W_1 = \mathcal{H}$. Now $v_1^2 + v_2^2 = 1$ follows from

$$\|f\|^2 = v_1^2\|f\|^2 + v_2^2\|\pi_{W_2}(f)\|^2 = (v_1^2 + v_2^2)\|f\|^2 \quad \text{for all} f \in \mathcal{H}.$$

(ii)⇒(i). This is obvious. □

4. Sensor Networks

The theory of fusion frame is a generalization of frame theory that is more acceptable to applications where two-stage signal/data analysis is required. Although fusion frames are usually used to model general distributed processing applications, in this section we will focus on the role of fusion frames in sensor networks fields.

4.1. **(Wireless) Sensor Networks.**

Wireless (micro)sensor networks have been identified as one of the most important technologies of the 21st century.

Current and potential applications of sensor networks include: military sensing, physical security, air traffic control, traffic surveillance, video surveillance, industrial and manufacturing automation, distributed robotics, environmental monitoring, and building and structures monitoring. The sensors in these applications may be small or large, and the networks may be wired or wireless. However, ubiquitous wireless networks of microsensors most likely offer the most potential in changing the world of sensing.

Some of the most powerful benefits of a distributed network are due to the integration of information gleaned from multiple sensors that merge into a larger worldview not detectable by any single sensor alone. For example, consider a sensor network whose goal is to detect a stationary phenomenon P. P might be a region of a water table that has been polluted within a field of chemical sensors. Each individual sensor might be very simple, capable only of measuring chemical concentration and thereby detecting where or not it is within P. However, by fusing data from all sensors, *combined with the knowledge of the sensors' positions,* the complete network can describe more than just a set of locations covered by P; it can also compute P's size, shape, speed, and so forth. The whole of the information has become greater than the sum of it's parts: the network can deduce the size and shape of P even though it does not have a "size" or "shape" sensor. Nearly every sensor network does this type of data fusion.

4.1.1. *Distributed signal processing.* For decades, the signal processing community has devoted much research and attention to the seamless integration of signals from multiple sources. It is often called *data fusion.* Such applications include signal enhancement(noise reduction), source localization, process control, and source coding. It would seem to be a natural match to implement such algorithms into distributed sensor networks, and there has been great interest in doing so.

Signal processing is a crucial building block in sensor networks. For example, responsible for target tracking, signal identification, and classification. All of these are processes that resolve low-level sensor signals into higher level sensors. In sensor networks, the challenge is to get the best signal processing results given the bandwidth and computational constraints of the sensing platform.

4.1.2. *Future.* In sensor networks, many unique challenges arise in ensuring the security of sensor nodes and the data they generate. For example, the fact that sensors are embedded in the environment presents a problem, in that, the physical security of the nodes making up the network cannot be assured. This can make security measures significantly different than that of internet servers. In sensor networks, attackers may modify node hardware, replace it with malicious counterparts, or fool sensors into making observations that do not accurately reflect the environment. Algorithms for ensuring network-wide agreements are crucial in detecting attacks because we can no longer assume the security of individual nodes, or the data they generate.

The limited resources on the smallest sensor nodes also can pose challenges. Many encryption schemes are impractically resource-intensive, consuming far more energy, memory, and computational time than would be required to send a raw, unprotected data value. In addition, protection of data from eavsedropping en-route – particularly important in wireless networks – traditionally implies end-to-end encryption. That is, data is encrypted as soon as it is created, transmitted through the network, and finally received by a secured server where decryption keys can be stored without danger of exposure. Unfortunately, finite energy drives the need for in-network processing in sensor networks, which confounds the traditional security schemes. Nodes inside the network can not perform application-specific processing on encrypted data. Decrypting the data at each node implies decryption keys are stored at each node; unfortunately, the node hardware itself is exposed, and can not be assumed to be out of reach of the attacker.

As with many types of information technology that developed throughout history, sensor networks also raise important questions about the privacy of individuals. Certain aspects of privacy have gradually eroded due to various forces – for example, the tracks we leave behind by using credit, the ubiquity of surveillance cameras, and the seeming omniscience of Internet seach engines. Sensor networking, similarly, is a technology that can be used to enrich and improve our lives, or in turn, evlove into an invasive tool. As sensor networks become more widespread, they will become an important point to consider in the continuing debate between public information and private lives.

4.2. **Modeling Sensor Networks.** A common sensor network strategy is to divide the network into subgroups where each sensor communicates its data to a single node within the cluster. Central nodes communicate the summarized data from their clusters to a more central location for collective processing. In frame theory terminology, the frame elements are divided into groups that span individual subspaces. The signal is reconstructed within each subspace and communicated to a central location for final reconstruction. Due to both the unpredictable nature of low-cost devices and its situations, certain local sensor systems can be less reliable than others. While facing the task of combining local subspace information coherently, one also has to consider weighting the more reliable sets of substation information against the suspected less reliable ones. Consequently, the coherent combination mechanism often requires weighted structures.

To model a sensor network, we associated each sensor with a vector f_{ij} $(j \in J_i, i \in I)$ in $\mathbb{R}^M$ that quantified how it measured the environment. This vector models the point-spread function of the sensor. The sensor measurement of a signal f is given by the inner product $\langle f, f_{ij} \rangle$. If all of these measurements were available at a single central location, reconstruction of the environmental signal f would be a straightforward application of frame theory.

Because of the prohibitive communication constraints of the typical sensor nodes under consideration, we focus on a common wireless sensor network paradigm where sensors are grouped into clusters. Specifically, the sensors are grouped so that for each group J_i with $i \in I$ the sensors $\{f_{ij}\}_{j \in J_i}$ belong to that cluster. Each sensor is able to communicate its information to a collection point assigned to the relevant cluster, which may simply be a fixed or rotating assignment of a sensor node within the group. Remembering that the vectors $\{f_{ij}\}_{j \in J_i}$ form a frame for their closed linear span(i.e., for W_i), we can obtain a partial reconstruction of the signal by the "normal" frame reconstruction inside W_i. More precisely, we obtain $\sum_{j \in J_i} \langle f, f_{ij} \rangle S_i^{-1} f_{ij} = \pi_{W_i}(f)$, where S_i is the local frame operator for $\{f_{ij}\}_{j \in J_i}$. Therefore, each collection point can use a frame theoretic approach to reconstruct the orthogonal projection of the (environmental) signal onto the subspace spanned by the sensors in the cluster.

After collecting data from the individual sensors in the cluster and performing a local subspace reconstruction, the collection point transmits this information to a master collection point (central station) for the whole sensor network. The central station is tasked with reconstructing the signal f from the processed data it receives from each cluster collection point. This reconstruction now depends entirely on the set of subspaces, i.e., only on the fusion frame itself. The choice of the local frame vectors does not interfere anymore.

References

[1] C. Y. Chong and S. P. Kumar, *Sensor Networks: Evolution, Opportunities, and Challenges*, Proc IEEE, 2003.

[2] R. Calderbank, P.G. Casazza, A. Heinecke, G. Kutyniok and A. Pezeshki, *Fusion frames: Existence and construction*, submitted (2009).

[3] P. G. Casazza, *A frame theory primer for the Kadison-Singer problem*, Preprint.

[4] P. G. Casazza, *The art of Frame theory*, Taiwanese Journal of Math. 4 No.2 (2000).

[5] P. G. Casazza, D. Han and D. R. Larson, *Frames for Banach spaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Contemp. Math. vol. 247, (1999) 149-182.

[6] P. G. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math. vol. 345, Am. Math. Soc., Providence, RI (2004), 87-113.

[7] P. G. Casazza, G Kutyniok, S. Li, and C. J. Rozell, *Modeling sensor networks with fusion frames*, Wavelets XII (San Diego, CA, 2007), 67011M-167011M-11, SPIE Proc. 6701, SPIE, Bellingham, WA, 2007.

[8] P. G. Casazza, G. Kutyniok, and S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. 25 (2008), 114-132.

[9] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. 27 (1986), 1271-1283.

[10] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341-366.

[11] M. H. Faroughi and R. Ahmadi, *Some properties of C-fusion frames*, Truk. J. Math vol. 33 (2009), 1-23.

[12] S. K. Kaushik, *A generalization of frames in banach spaces*, J. Comtemp. Math. Anal. vol. 44 (2009), 212-218.

[13] R. E. Megginson, *An introduction to Banach space theory*, Springer-Verlag New York, 1998.

[14] A. Pezeshki, G. Kutyniok and R. Calderbank, *Fusion frames and robust dimension reduction*, 42nd Annual Conference on Information Sciences and Systems (CISS), Princeton University, Princeton, NJ, 2008.

[15] C. S. Raghavendra, K. M. Sivalingam and T. Znati, *Wireless sensor networks*, Springer Science+Business Media,LLC , 2004.